Auditability in School Choice

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I. Motivation

Many cities around the world use a centralized system to allocate public school seats. In a centralized system, applicants report a ranking over schools, and schools specify priority scores for applicants. The assignment is determined by some algorithm or a rule that takes these rankings and scores as inputs. Applicants know their own rankings and scores, but they do not necessarily observe each others' rankings and scores. Because of this incomplete information, a centralized admission system may be vulnerable to undetected errors or fraud. That is, if the school district implements a different assignment than the one specified by the rule for a given market, then participants may not be able to detect this deviation. In this paper, we introduce an indexed-based auditability measure for an assignment rule that captures how hard or easy it is for groups of participants to detect adversarial deviations. More specifically, the **auditability index** of an assignment rule is the size of the smallest group of applicants whose private information alone is sufficient to detect an adversarial deviation. We find a striking contrast between the auditability properties of prominent assignment rules. On one extreme, the Serial Dictatorship and Immediate Acceptance rules have an auditability index of only two, meaning that for any deviation from the assignment rule there will always be just two applicants whose private information alone is sufficient to detect this deviation. On the other extreme, the Deferred Acceptance rule has an auditability index equal to the number of applicants, meaning that some deviations may go undetected unless participants observe all rankings and scores.

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II. Related Literature

The need for transparent school assignment rules has been highlighted by Parag A. Pathak (2017) and Meg Benner and Ulrich Boser (2018). In allocation problems, auditability and transparency notions have been formalized by Mohammad Akbarpour and Shengwu Li (2020), Rustamdjan Hakimov and Madhav Raghavan (2020), and Markus Möller (2022). These papers study deviations that can be detected by a single individual. In general, this is a demanding requirement if no further information is revealed by the designer. Consequently, those papers assume some private communication or public information in their models.

In our paper we do not study any explicit form of information sharing. Instead, we develop an index-based *measure* for quantifying auditability, and comparing assignment rules. The analysis is complementary to the previous and concurrent works on the topic. Our measure proves tractable and informative in various social choice environments. Aram Grigoryan and Markus Möller (2023) studies the problem in a general setup.

III. Preliminaries

Consider the standard school choice setup (Atila Abdulkadiroğlu and Tayfun Sönmez, 2003; David Gale and Lloyd S. Shapley, 1962). There is a finite set of applicants A, and a finite set of schools $S, |S| \geq 2$. Each school $s \in S$ has a capacity $q_s \in \mathbb{N}_+$, which indicates how many applicants can be assigned there. We assume that $\sum_{s \in S} q_s = |A|$. Each applicant $a \in A$ has a preference list P_a, which is a strict linear order over S, and a priority score $r_{as} \in \mathbb{R}_+$ at each school s. No two applicants share a priority score at a given school, that is, for any $a, a' \in A$ and $s \in S$, we have $a \neq a' \implies r_{as} \neq r_{a's}$. Let $r_a := (r_{as})_{s \in \mathcal{S}}$. We refer to the pair (P_a, r_a) as the type of applicant a, and we denote it by θ_a. For a subset of applicants $A' \subseteq A$, let $\theta_{A'} := (\theta_a)_{a \in A'}$ and $\theta_{-A'} = (\theta_a)_{a \in A \setminus A'}$. We call the types' profile of all applicants $\theta := (\theta_a)_{a \in A}$ a school choice problem, or simply, a problem. We denote the space of all problems by Θ.

An assignment $\mu : A \to S$ is a mapping satisfying $\left|\mu^{-1}(s)\right| = q_s$ for all $s \in S$. Let $\mathcal{M}$ denote the space of all assignments. An assignment rule $\varphi : \Theta \to \mathcal{M}$ specifies an assignment for each problem.

We will analyze three prominent assignment rules.

- ***Serial Dictatorship (SD) rule.*** *Applicants are ranked according to some commonly known 'dictatorial' order. Initially, each school s is available and has a capacity q_s.*
  Step $k \geq 1$. The k-th applicant in the order is selected as the next dictator, and she is assigned to her most preferred available school. The school's capacity is reduced by one. If the school has zero capacity (after the reduction), it become unavailable. If all schools are unavailable, the procedure terminates. Otherwise, we proceed to Step $k+1$.
- ***Immediate Acceptance (IA) rule.*** *Initially, each school s is available and has a capacity q_s.*
  Step $k \geq 1$. Each applicant is considered at her k-th most preferred school. Each available school accepts the highest priority score applicants up to its capacity. Each school's capacity is reduced by the number of accepted applicants. Schools with a zero capacity become unavailable. If there are no available schools, the procedure terminates. Otherwise, we proceed to Step $k+1$.
- ***Deferred Acceptance (DA) rule.*** *Initially, no applicant is rejected by any school.*
  Step $k \geq 2$. Each applicant is considered at her most preferred school out of those that have not rejected her yet. Each school tentatively accepts highest priority applicants up to its capacity. The remaining individuals are rejected by the corresponding school. If there are no rejections at this step, the procedure terminates and the tentative assignments are finalized. Otherwise, we proceed to Step $k+1$.

IV. Auditability

We now introduce our auditability notion. For a problem θ and an assignment $\mu \neq \varphi(\theta)$, we say that a non-empty set of applicants $A' \subseteq A$ *detects* the deviation μ, if for any $\theta_{-A'}$, we have that

$$(\mu(a))_{a \in A'} \neq \big(\varphi(\theta_{A'}, \theta_{-A'})\big)_{a \in A'}.$$

An auditability index of the assignment rule is the size of the smallest group that can detect an adversarial deviation. Formally, the *auditability index* of φ is

$$\max_{\theta \in \Theta, \mu \neq \varphi(\theta)} \min \big\{|A'| : A' \subseteq \text{ detects } \mu\big\}.$$

We denote the auditability index of φ by $\#\varphi$.

V. Results

Theorem 1. *$\#SD = \#IA = 2$ and $\#DA = |A|$.*

PROOF:

We compute the auditability index for each assignment rule separately.

SD. Consider an SD rule with some arbitrary dictatorial order.

Let θ be an arbitrary problem and an arbitrary assignment $\mu \neq SD(\theta)$. Since θ and μ are arbitrary, to prove that $\#SD = 2$, it sufficient to find two applicants a_1 and a_2 that detect the deviation μ.

Let a_1 be the applicant with the highest rank in the dictatorial order for whom $\mu(a_1) \neq SD(\theta)(a_1) := s$. Let $\bar{A}$ be the set of individuals who have higher ranks in the dictatorial order than a_1 and let

$$\bar{S} := \{s \in S : SD(\theta)^{-1}(s) \not\subseteq \bar{A}\}.$$

That is, $\bar{S}$ is the set of schools that are available at step $|\bar{A}|+1$ of the SD implementation.

By definition, $\mu(a) = SD(a)$ for all $a \in \bar{A}$. Hence, $\bar{S} = \{s \in S : \mu^{-1}(s) \not\subseteq \bar{A}\}$. Therefore, $\mu(a_1) \in \bar{S}$.

Consider the SD implementation at problem θ. Since a_1 is assigned to school $SD(\theta)(a_1)$ at step $|\bar{A}| + 1$ of the SD implementation when all schools in $\bar{S}$ are available, it should be that $SD(\theta)(a_1)$ is a_1's most preferred school in $\bar{S}$. Since $\mu(a_1) \in \bar{S}$, we have that $SD(\theta)(a_1) P_{a_1} \mu(a_1)$.

Consider the applicant a_2 with $\mu(a_2) = SD(\theta)(a_1)$. Note that $a_2 \notin \bar{A} \cup \{a_1\}$, hence a_2 has a lower rank in the dictatorial order than a_1.

So far we established that, (i) $\mu(a_2) = SD(\theta)(a_1) P_{a_1} \mu(a_1)$, and (ii) a_2 has a lower rank in the dictatorial order than a_1. This can never happen under the SD implementation for any problem. Hence, $\{a_1, a_2\}$ detects the deviation $\mu \neq SD(\theta)$. This completes the proof for SD.

IA. Let θ be an arbitrary problem and consider an arbitrary assignment $\mu \neq IA(\theta)$. To prove that $\#IA = 2$, it sufficient to find two applicants a_1 and a_2 that detect the deviation μ.

Let k be the first step in the IA implementation such that some school accepts an applicant a_1 for whom $IA(a_1) \neq \mu(a_1)$. Let $\bar{A}$ be the set of individuals who are accepted by a school before step k of the IA implementation, and let

$$\bar{S} := \{s \in S : IA(\theta)^{-1}(s) \not\subseteq \bar{A}\}.$$

That is, $\bar{S}$ is the set of schools that are available at step $k = |\bar{A} + 1|$ of the IA implementation.

By definition, $\mu(a) = IA(a)$ for all $a \in \bar{A}$. Hence, $\bar{S} = \{s \in S : \mu^{-1}(s) \not\subseteq \bar{A}\}$. Therefore, $\mu(a_1) \in \bar{S}$.

Consider the IA implementation at problem θ. Since school $IA(\theta)(a_1)$ accepts a_1 at step k of the IA implementation, when all schools in $\bar{S}$ are available, it should be that $IA(\theta)(a_1)$ is a_1's most preferred school in $\bar{S}$. Since $\mu(a_1) \in \bar{S}$, we have that $IA(\theta)(a_1) P_{a_1} \mu(a_1)$.

Consider the applicant a_2 with $IA(\theta)(a_2) \neq \mu(a_2) = IA(\theta)(a_1)$. Note that $a_2 \notin \bar{A} \cup \{a_1\}$, or equivalently, a_2 is not accepted to any school before step k of the IA implementation. Since $IA(\theta)(a_1)$ was available at step k of the IA implementation, it should be that a_2 was note considered at that school before step k of the IA implementation (since otherwise, she would have been accepted by $IA(\theta)(a_1)$). Hence, a_2 prefers $IA(\theta)(a_1)$ weakly less than her k-th most preferred school. Consider cases:

(i) $IA(\theta)(a_1)$ is a_2's k-th most preferred school. In that case, since $IA(\theta)(a_1)$ accepts a_1 and does not accept a_2 at step k of the IA implementation, it should be that a_1 has a higher priority score at $IA(\theta)(a_1)$ than a_2. If both a_1 and a_2 rank $IA(\theta)(a_1)$ as their k-th most preferred school, and a_1 has a higher priority score at $IA(\theta)(a_1)$ than a_2, then the IA implementation can never yield an outcome μ where $\mu(a_2) = IA(\theta)(a_1)$, and $IA(\theta)(a_1) P_a \mu(a_1)$. Hence, $\{a_1, a_2\}$ detects the deviation $\mu \neq IA(\theta)$.

(ii) a_2 prefers $IA(\theta)(a_1)$ strictly less than her k-th most preferred school. Then, in the IA implementation a_1 will be considered at $IA(\theta)(a_1)$ at an earlier step than a_2, and the IA implementation can never yield an outcome μ where $\mu(a_2) = IA(\theta)(a_1)$, and $IA(\theta)(a_1) P_a \mu(a_1)$. Hence, $\{a_1, a_2\}$ detects the deviation $\mu \neq IA(\theta)$. This completes the proof for IA.

DA. To prove that $\#DA = |A|$, we need to construct a problem θ and a deviation $\mu \neq DA(\theta)$, such that no proper subset of applicants detects this deviation.

The result is immediate when $|A| = 1$. In what follows, assume that $|A| \geq 2$. Pick arbitrary two applicants $a_1, a_2 \in A$ and two schools $s_1, s_2 \in S$. Consider an arbitrary mapping $f : A \setminus \{a_1, a_2\} \to S$ satisfying $f^{-1}(s_1) = q_{s_1} - 1, f^{-1}(s_2) = q_{s_2} - 1$, and $f^{-1}(s) = q_s$ for all $s \in S \setminus \{s_1, s_2\}$.

Consider the following problem $\theta = (P_a, r_a)_{a \in A}$:

- $s_1 P_{a_1} s_2 P_{a_1} s$ and $s_2 P_{a_2} s_1 P_{a_2} s$ for all $s \in S \setminus \{s_1, s_2\}$,
- $f(a) P_a s$ for all $a \in A \setminus \{a_1, a_2\}$ and for all $s \in S \setminus \{f(a)\}$,
- $r_{a_1 s_1} < r_{a s_1} < r_{a_2 s_1}$ and $r_{a_2 s_2} < r_{a s_2} < r_{a_1 s_2}$ for all $a \in A \setminus \{a_1, a_2\}$,
- $r_{a f(a)} > r_{a' f(a)}$ for all $a \in A \setminus \{a_1, a_2\}$ and all $a' \in A \setminus \{a\}$,
- otherwise, the preferences and priority scores are arbitrary.

Under $DA(\theta)$, all applicants are assigned to their most preferred schools. That is, $DA(\theta)(a_1) = s_1, DA(\theta)(a_2) = s_2$, and $DA(\theta)(a) = f(a)$ for all $a \in A \setminus \{a_1, a_2\}$. Now consider the deviation $\mu \neq DA(\theta)$ that differs from $DA(\theta)$ by that $\mu(a_1) = s_2$ and $\mu(a_2) = s_1$.

Let $A' \subsetneq A$ be an arbitrary proper subset of applicants. We will show that A' does not detect the deviation. Consider cases:

(i) $a_1 \notin A'$ or $a_2 \notin A'$. Without loss of generality, suppose $a_1 \notin A'$. Consider the problem $\tilde{\theta}$ that differs from θ by only that $s_2 P'_{a_1} s$ for all $s \in S \setminus \{s_2\}$. Then, $DA(\theta_{A'}, \tilde{\theta}_{-A'}) = DA(\tilde{\theta}) = \mu$. Thus, A' does not detect the deviation μ.

(ii) $a_1, a_2 \in A'$. Consider an arbitrary $a \in A \setminus \big(A' \cup \{a_1, a_2\}\big)$. Such an a exists because $A' \subsetneq A$ and $a_1, a_2 \in A'$. Consider the problem $\tilde{\theta}$ that differs from θ by only that $s_1 P_a s$ for all $s \in S \setminus \{s_1\}$. Then, $DA(\theta_{A'}, \tilde{\theta}_{-A'}) = DA(\tilde{\theta}) = \mu$. Thus, A' does not detect the deviation μ. This completes the proof for DA.

VI. Conclusion

In this paper we formalize an auditability notion for school assignment rules. Using an index-based measure for auditability, we establish that prominent assignment rules drastically differ with respect to this measure. The analysis potentially highlights the importance of addressing trust and transparency considerations in centralized admission systems.

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Mathematical Appendix