What do financial markets say about the exchange rate?*

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Abstract

Financial markets play two roles with implications for the exchange rate: they accommodate risk-sharing and act as a source of shocks. In prevailing theories, these roles are seen as mutually exclusive and individually face challenges in explaining exchange rate dynamics. However, we demonstrate that this is not necessarily the case. We develop an analytical framework that characterizes the link between exchange rates and finance across all conceivable market structures. Our findings indicate that full market segmentation is not necessary for financial shocks to explain exchange rates. Moreover, risk-sharing can have a significant role without leading to the traditional puzzles associated with the macro disconnect. We identify plausible market structures where both roles coexist, addressing challenges faced when examined separately.

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Introduction

A wide body of evidence indicates that exchange rate movements have minimal or no correlation with macroeconomic aggregates (Meese and Rogoff, 1983; Obstfeld and Rogoff, 2001). This finding has spurred researchers to seek the sources of these fluctuations elsewhere. Naturally, many have turned their attention to financial markets. While this approach has yielded many insights, it also presents its own set of challenges. In this paper, we conduct a general analysis of how the financial sector of an equilibrium model interacts with the exchange rate. This perspective clarifies the root cause of the main challenges to existing theories of this interaction, and allows us to identify frameworks that overcome these challenges.

We focus on the duality between two roles of financial markets in the determination of the exchange rate:

- Financial markets are where sharing of macroeconomic risks across countries takes place. Households use financial claims to line up their marginal rates of substitution, and this determines the exchange rate that smoothes out certain macro shocks (e.g, Backus and Smith, 1993, Cole and Obstfeld, 1991).[1]

- Financial markets are also a source of shocks to the exchange rate. For example, macroeconomic or financial shocks to financial institutions intermediating international trading affect the exchange rate (e.g., Gabaix and Maggiori, 2015, and Itskhoki and Mukhin, 2021, Jiang, Krishnamurthy, Lustig, and Sun, 2022).

The literature usually adopts market structures, that is, a combination of assumptions about which assets are traded and who trades them, in which only one of these

[1]By households, we generally refer to the local representative agents.

roles is emphasized. Each of these roles runs into significant challenges. On the one hand, models of risk-sharing typically assume integrated markets, complete markets, or both. Complete means that every Arrow-Debreu claim is available, integrated means that everyone can trade with each other. This rich market structure leads to a tight connection of the exchange rate with the macroeconomy, at odds with the classic evidence of the disconnect, or, put differently, imposing strong constraints on difficult-to-measure aspects of macroeconomic dynamics. On the other hand, models of financial shocks typically focus on limited and segmented market structures. These assumptions are at odds with the existence of widely accessed local markets and of many global multi-market intermediaries.

While the literature has explored specific market structures beyond these two extremes, general results have been elusive, as each case seemingly requires a separate analysis. The first contribution of this paper is to provide an analytical framework which characterizes the link between the exchange rate and financial markets across all possible market structures. We fully map out when and how the two roles of finance shape the properties of the exchange rate.

Given the extant literature, the dual roles of financial markets might appear mutually exclusive. Moreover, the two challenges which we have highlighted seem inherent to each role of the financial markets. Our second contribution is to show that, while there is a tension between the two roles, both conjectures are incorrect. In particular, we highlight plausible market structures in which risk-sharing and financial shocks jointly determine the exchange rate, and address both challenges.

Specifically, we demonstrate that extreme market segmentation is not a prerequisite for shocks in the financial sector to have a substantial impact on the exchange rate.

Even if households trade assets in their respective countries and a few risky assets in common, all with global intermediaries, there is still a lot of flexibility for financial sector shocks to determine the exchange rate.

Next, we show that the disconnect puzzle persists when markets are incomplete but integrated, or intermediated but complete. However, this conclusion is not intrinsic to all structures with a considerable extent of macroeconomic risk-sharing. Specifically, this challenge can disappear if markets are both intermediated and incomplete, even when many risks are still shared in such structures.

Finally, our framework leads to an empirical method to quantify how the properties of asset returns discipline the exchange rate in any given market structure. We implement this approach using data on stocks and bonds. Their lack of strong correlation across countries and with the exchange rate identifies a scope for solving both challenges simultaneously. We conclude that a market structure in which households in each country trade their local stocks and bonds with a global intermediary features both roles without their challenges, and hence is particularly promising for building models of the exchange rate.

A useful starting point for our analytical results is the case of complete and integrated markets. In this setting, risk-sharing between local households completely pins down the exchange rate, which must equal the difference between their (log) intertemporal marginal rates of substitution (IMRSs) m and m^*:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1}, \tag{1}$$

where Δs is the log home currency depreciation rate (see, e.g., Backus, Foresi, and Telmer, 2001).

This expression highlights why complete and integrated markets often struggle with the macro disconnect: many models relate the discount factors m and m^*, and hence the exchange rate, to macroeconomic aggregates. This issue implies a mismatch between several moments of the model-based and the empirical exchange rate: the volatility (Brandt, Cochrane, and Santa-Clara, 2006), cyclicality (Backus and Smith, 1993), and risk premium (Fama, 1984) puzzles. Another implication of the relation in equation (1) is that the associated risk-sharing leaves no room for the second role of financial markets: even if financial frictions are modeled, they have no impact on the depreciation rate above and beyond what can be learned from households' marginal utility.

Our framework generalizes this baseline case to all possible market structures including deviations from market completeness, market integration, or both simultaneously. We assume that households in each country trade a potentially distinct set of assets in their local currency. That is, Euler equations hold with respect to each country's household IMRS for these assets. We use IMRSs to consider the implications of our analysis in the context of equilibrium models. However, our theoretical results do not require equilibrium-grounded IMRSs and equally apply to any generic pair of local stochastic discount factors (SDF).

We summarize the remainder of what happens in international financial markets by the assumption of no arbitrage. This corresponds to assuming that interactions between all players in international markets result in an "international" SDF which may or may not coincide with one of the two local SDFs. In the latter case, it could be the discount factor of a global intermediary.

In this setting, we characterize *all* restrictions imposed on the exchange rate based on the households' discount factors, thereby producing a general version of the relation in equation (1). These restrictions take a simple form that can be summarized in two relations. First, innovations to the depreciation rate coincide with innovations in the relative discount factor across countries when projected on risks that both home and foreign households can trade and, therefore, share:

$$\text{proj}(\widetilde{\Delta s}_{t+1} \,|\epsilon^g_{t+1}) = \text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} \,|\epsilon^g_{t+1}), \tag{2}$$

with ϵ^g the set of globally traded risks. Second, we show that the expected depreciation rate is similar to the one under complete markets when traded asset returns span the exchange rate; otherwise, it is unconstrained by local discount factors. These two results exhaust all possible restrictions imposed by local SDFs on the exchange rate, i.e. they are necessary and sufficient for precluding international arbitrage opportunities. In general, these constraints do not pin down the exchange rate completely. That leaves space for the second role of financial markets, as a source of financial shocks, to determine the remainder of the exchange rate.

To address the conjectures about the interaction between the two financial roles and the challenges associated with them, we conduct the following exercise. We fix households' IMRSs m and m^*.[2] We assume that these discount factors are such that applying equation (1) leads to the exchange rate puzzles. We use equation (2) to study the equilibrium behavior of the exchange rate implied by various choices of market structure. This path is complementary to the large literature which fixes

[2]For example, with CRRA preferences, $m_{t+1} = -\rho - \gamma \Delta c_{t+1}$. So, given knowledge of the preference parameters and consumption data, m_{t+1} is a fixed observable quantity. We do not distinguish between real and nominal IMRS when connecting theory to evidence because the difference between foreign and domestic inflation rates has a minimal effect on the exchange rate variation in developed economies and over short horizons (in contrast to Campbell, Viceira, and White, 2003).

financial markets to be complete and integrated, while varying assumptions about preferences (CRRA, habits, or recursive utility) and aggregate dynamics (random walk, long-run risks, or disasters).

We ask whether the currency puzzles remain in market structures other than complete and integrated. Examining equation (2) reveals when risk-sharing puts as tight a restriction on the exchange rate as equation (1). Specifically, if households can use assets to trade exposure to the depreciation rate and to innovations in their relative discount factors $m^* - m$, these two variables become globally traded risks. Then, the projections in equation (2) disappear and this relation coincides with equation (1), which leads to the currency puzzles.

We show that even if only one of the two variables is spanned by globally traded risks, the puzzles remain. This general principle manifests in two distinct scenarios. In the first scenario, markets are incomplete and integrated. Then, the exchange rate is globally traded as soon as both households can trade risk-free bonds in both countries (the case considered in Lustig and Verdelhan, 2019). In the second scenario, markets allow to trade all macro risks, a relaxed version of market completeness, and are intermediated. In this case, innovations in relative marginal utility are globally traded. Taken together, these results show that relaxing market incompleteness alone or market integration alone does not solve the puzzles.

At the other extreme, we ask which market structures are not constrained by household risk-sharing at all. This lack of constraints is a central ingredient in typical models focused on financial shocks. These models start from extreme segmentation: households can only trade their respective risk-free bonds, and intermediaries engage in the carry trade only. They then have the freedom to choose sources of financial shocks — frictions to intermediaries, noise traders, etc. — to pin down realistic exchange rate dynamics. Equation (2) reveals the precise source of this freedom. Because households cannot trade any risks in common, ϵ^g_{t+1} is empty, and the projections in equation (2) are degenerate. As a result, there are no restrictions on the exchange rate.

We demonstrate that the lack of global shocks and the flexibility of financial shocks that comes with it survive in a larger set of market structures. First, intermediaries can be sophisticated and trade an arbitrarily large set of assets. Second, households can trade a risky asset in common — a contrast to the case of trading risk-free bonds. Finally, households in the two countries can each trade many local assets, as long as the risks in the two countries are not related. These market structures present a more compelling scenario compared to the extreme models with minimal or no trading, as commonly discussed in the literature.

Of course, there are many local assets in each country and at least some of their returns are correlated across countries. This suggests that there might be some global sources of risk that can be traded by all investors. We show that such situations do not necessarily take us back to the puzzles. Risk-sharing can play a substantial role without removing all the flexibility of financial shocks to determine the exchange rate. This occurs when there are some global shocks in equation (2), but they do not explain 100% of the variation in either the depreciation rate or the relative discount factor.

We show how to quantify the magnitude of global risks empirically given a choice of market structure. For this purpose, we consider a market structure in which investors in each country can trade a wide set of stock indices (the market, value-growth, and industry portfolios) and sovereign bonds with maturities ranging from 2 to 10 years. We focus on G-10 countries, U.S. vs foreign on a bilateral basis, from 1988 to 2022 at a monthly frequency.

First, we evaluate whether the depreciation rate can be spanned by these asset returns. The answer is no: the largest spanning regression R^2 is 45% for Canada (vs the U.S.), the lowest is 25% for Switzerland. Thus, unspanned shocks play an important role in the variation of the exchange rate, enriching the results of Chernov and Creal (2023) which were established for the case of the international yield curves.

Next, we quantify global shocks using two methods. First, we use canonical correlation analysis to find maximally correlated portfolios in a pair of countries. Second, we use shocks that are commonly used as global in the literature: the Volatility Index (VIX), the Global Financial Cycle (GFC, Miranda-Agrippino and Rey, 2020), and the Excess Bond Premium (EBP, Gilchrist and Zakrajsek, 2012). Regardless of the method, global shocks contribute mildly to the variation in exchange rates: most countries have no more than 10% of FX variation explained by global shocks. Thus, the evidence supports the conclusion that, in this market structure, the combination of risk-sharing and financial shocks, with a more prominent role for the latter, can explain exchange rate movements.

Naturally, this empirical analysis leaves some questions unanswered. Our conclusions depend on the choice of spanning assets. While we find the intermediated market structure with local stock and bond trading appealing, one can follow our approach and revisit the quantification of spanning and global shocks in any structure they prefer. Similarly, one could construct these decompositions inside of a specific equilibrium model. Further, while we show that financial shocks naturally play a major role in determining the exchange rate, our analysis focused on asset prices alone as a source of discipline. More direct evidence on the trading of these financial players is necessary to establish this conclusion tightly.

Related literature. We derive general restrictions on the exchange rate given properties of financial markets in each of the two countries. As such, we follow the tradition of Hansen and Jagannathan (1991). We similarly apply these restrictions to simple moments of exchange rates, such as their relatively low volatility and weak relation to business cycles. The key difference is that the international setting leads to a preponderant role of the financial market structure in determining these constraints.

Closely related to our empirical findings, Chernov and Creal (2023) emphasize the inability of bonds to span exchange rates and propose an affine term structure model with martingale shocks to the SDF, which affect the exchange rate but not bond prices.

Departures from complete markets in the context of currency puzzles are explored by Lustig and Verdelhan (2019). They consider a special case where the exchange rate is spanned because each country's investor can trade the other country's risk-free bond.[3] That makes it difficult to capture volatility and cyclicality puzzles jointly. Marin and Singh (2023) revisit this framework and argue that the path towards the resolution of the cyclicality puzzle lies via a model in which domestic households are heterogenous in their access to foreign assets. Jiang, Krishnamurthy, Lustig, and Sun (2022) consider a similar incomplete-market setting with international access

[3] In this sense, such an assumption leads to the same conclusions as the full market integration, which is explicitly assumed by Maurer and Tran (2021) and Sandulescu, Trojani, and Vedolin (2021). These authors do not investigate the puzzles associated with the macro disconnect.

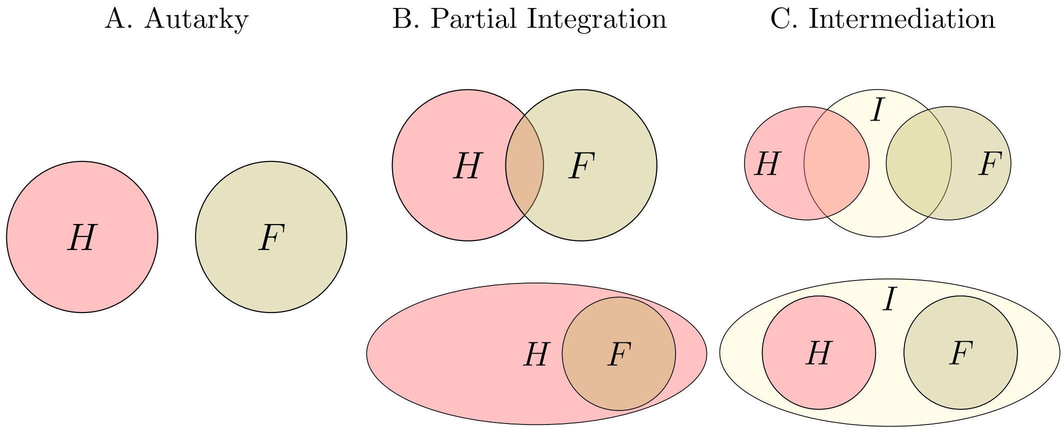
to trading in risk-free bonds but complemented by safe asset demand for dollar bonds. This feature leads to wedges in the Euler equations. One implication of these wedges is that the exchange rate is affected by the convenience yield in addition to risks spanned by the SDFs. Jiang, Krishnamurthy, and Lustig (2023) reach related conclusions in a more general setting, which does not posit SDF dynamics. Our restrictions are based solely on assets whose Euler equations hold, thereby providing sufficient conditions for the absence of international arbitrage opportunities. Lewis and Liu (2022) construct wedges in the exchange rate by considering the households' IMRS measured using observed macro and asset return data and their counterfactual versions under complete markets, with the objective of measuring the welfare effects of such wedges.

Our focus on market structure complements the literature that maintains the complete market assumption, but varies assumptions about preferences and aggregate dynamics. Some prominent examples of this line of work include Verdelhan (2010) (habits), Colacito and Croce (2011) (long-run risk), and Farhi and Gabaix (2016) (disasters), among many others. We also connect to the literature on exchange rate models with intermediation. This literature is examplified by Gabaix and Maggiori (2015), Gourinchas, Ray, and Vayanos (2022), Greenwood, Hanson, Stein, and Sunderam (2022), and Itskhoki and Mukhin (2021).

1 Framework

We introduce our framework to represent and analyze a variety of financial market structures.

Figure 1: Examples of Market Structures



The figure illustrates different market structures. H and F are the set of assets invested in by the home and foreign household. Panel A corresponds to financial autarky. Panel B corresponds to partial integration, symmetric or asymmetric. Panel C corresponds to an intermediated market, with an intermediary I trading some or all assets.

1.1 Market structure

We consider settings with two representative households, h for home, and f for foreign. Each household can trade a set of assets, H and F, respectively. Those sets can contain subsets of local assets and foreign assets converted to local currency. Figure 1 demonstrates some examples. For instance, in autarky H contains domestic stocks and bonds, while F contains the foreign ones. When markets are integrated, H and F contain identical assets but expressed in respective currencies, e.g., H may include a domestic sovereign bond and a foreign equity index converted to domestic currency, while F contains domestic bond converted to foreign currency and foreign equity index. If markets are complete, H and F contain the full set of Arrow-Debreu securities expressed in respective currencies.

Further, we consider a set I of assets traded in international markets. Assets can be included in this set for two reasons. First, it could be that home and foreign households trade some assets in common, as in the partially integrated cases above. Then, either h or f can be considered as an international arbitrageur, with $I = H$ or $I = F$, respectively. Second, it could be that financial intermediaries trade across borders even if households do not, as in the examples in panel C of Figure 1. In this case, I are the assets from H and F that intermediaries can trade.

Our main result is that, in this large family of market structures, restrictions on the exchange rate coming from risk-sharing between households are detemined by the properties of returns in $H \cap I$ expressed in domestic currency and returns in $F \cap I$ expressed in foreign currency. To continue our examples, if markets are partially integrated and $I = H$, then $H \cap I = H$ are the assets traded by the domestic household, $F \cap I = F \cap H$ are the assets traded by both households. In intermediated markets, $H \cap I$ is the set of assets traded both by the domestic household and the intermediaries; ditto for $F \cap I$.

The base assets in the set $H \cap I$ have log returns $\boldsymbol{r}_{t+1} = (r_{1,t+1}, \ldots, r_{N,t+1})$. We assume this collection includes a risk-free asset with return r_{ft} in home currency known at time t. We consider all feasible portfolios that can be constructed from these assets. The corresponding set of (log) returns is $\boldsymbol{r}_{p,t+1} = \{r_{p,t+1} | \exists \boldsymbol{w}_t \in \mathbb{R}^N : \boldsymbol{w}_t' \boldsymbol{\iota} = 1, r_{p,t+1} = \log(\boldsymbol{w}_t' \exp(\boldsymbol{r}_{t+1}))\}$. Furthermore, we assume that asset returns are log-normal, that is, $\boldsymbol{r}_{t+1}$ are multivariate normal, $MVN(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$. Similarly, the returns of base assets in $F \cap I$ are $\boldsymbol{r}^*_{t+1}$ in foreign currency, log-normal of size N^*, and contain a foreign-currency risk-free rate of r^*_{ft} . The corresponding set of portfolio returns is $\boldsymbol{r}^*_{p,t+1}$. Throughout the paper, we use the Campbell and Viceira (2002) approximation for log portfolio excess returns in the relevant derivations as described in Appendix A.

1.2 Pricing Assumptions

We introduce two sets of assumptions, which enable us to characterize how risk-sharing between households constrains the behavior of the exchange rate.

Local Euler equations. We specify valuation mechanisms by each representative household with a given SDF m at home and m^* abroad. These SDFs value assets as follows.

Assumption 1. *The domestic (log) stochastic discount factor m_{t+1} prices all assets in H in domestic currency. In particular, it satisfies the Euler equation:*

$$\forall r_{t+1} \in \boldsymbol{r}_{p,t+1} : \quad E_t \left[\exp(m_{t+1} + r_{t+1})\right] = 1. \tag{3}$$

Similarly, the foreign log SDF m^*_{t+1} prices all assets in F in foreign currency, and

$$\forall r^*_{t+1} \in \boldsymbol{r}^*_{p,t+1} : \quad E_t \left[\exp(m^*_{t+1} + r^*_{t+1})\right] = 1. \tag{4}$$

Recall that $\boldsymbol{r}_{p,t+1}$ ($\boldsymbol{r}^*_{p,t+1}$) is the set of feasible portfolio returns constructed from assets in $H \cap I$ ($F \cap I$). Thus (3) and (4) require only pricing of assets in sets $H \cap I$ and $F \cap I$, respectively. These Euler equations are all that is needed for our formal results. Nevertheless, in many economic environments it is reasonable to assume that the same home and foreign SDFs price all assets in H and F, respectively. Assumption 1 can be viewed as the definition of local financial market equilibrium that we use in our analysis.[4]

[4]Note that equilibrium in the financial market may involve borrowing or short-sale constraints,

We focus on situations with log-normal SDFs. The Euler equations imply that expected excess returns are proportional to the covariance with the stochastic discount factors. In our log-normal setting, this corresponds to:

$$\forall r_{t+1} \in \boldsymbol{r}_{p,t+1}: \quad E_t(r_{t+1}) + \frac{1}{2} var_t(r_{t+1}) = r_{ft} - cov_t(m_{t+1}, r_{t+1}), \tag{5}$$

$$\forall r^*_{t+1} \in \boldsymbol{r}^*_{p,t+1}: \quad E_t(r^*_{t+1}) + \frac{1}{2} var_t(r^*_{t+1}) = r^*_{ft} - cov_t(m^*_{t+1}, r^*_{t+1}). \tag{6}$$

These Euler equations are the point of contact of the economy with financial markets, and hold irrespective of the remainder of the economic environment. For example, with CRRA utility, $m_{t+1} = -\gamma \Delta c_{t+1}$ where γ is the coefficient of risk aversion and c_t is log aggregate domestic consumption, which could be exogenous as in an endowment economy, or allowed to change endogenously.

International arbitrage. So far none of our assumptions involve the exchange rate. In order to analyze how it interacts with local Euler equations, one has to take a stand on how international markets operate. To isolate the risk-sharing role of these markets, we make a minimal assumption about them: there are no arbitrage opportunities for assets in I.

Specifically, the set of returns in I combines the domestic and foreign set of international returns converted to the domestic currency.[5] Following our notations, international portfolios are generated by the base assets $\boldsymbol{r}^I_{t+1} = (\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1} + \Delta s_{t+1})$,

infrequent portfolio adjustment, or convenience yield on certain assets. In all such cases, some Euler equations do not always hold with equality, and in our analysis this simply requires redefining sets H and F to exclude such assets (for a given time period t). In this case, conditions (3) and (4) can be thought of as definitions of sets H and F rather than an assumption.

[5] Our conclusions are unchanged if we focus on international arbitrage in foreign currency.

where Δs_{t+1} is the log home currency depreciation rate. We denote the set of international portfolios generated by these base assets by $\boldsymbol{r}^{I}_{p,t+1}$.

Assumption 2. *There are no arbitrage opportunities in the set of international returns $\boldsymbol{r}^{I}_{p,t+1}$, that is:*

$$\forall r_{p,t+1} \in \boldsymbol{r}^{I}_{p,t+1}: \; var_t(r_{p,t+1}) = 0 \quad \Rightarrow \quad E_t(r_{p,t+1}) = r_{ft}. \tag{7}$$

In words, any portfolio that has no risk must earn the risk-free rate of return.[6] This is equivalent to the existence of an international SDF m^I. For example, this could be the discount factor of one of the households or of an international arbitrageur. However, unlike for households, at this stage we do not assume any knowledge of this SDF beyond its existence. This does not imply that the unique characteristics of international arbitrageurs are insignificant. Rather, we view them as integral to the second role of financial markets, which is serving as a source of shocks.[7]

1.3 Global, local and unspanned shocks

Intuitively, returns are affected by a collection of shocks, some of which are local to each economy, $\boldsymbol{\epsilon}_{t+1}$ or $\boldsymbol{\epsilon}^*_{t+1}$, while others are common to both, i.e. global shocks $\boldsymbol{\epsilon}^g_{t+1}$. We use tilde to denote the innovation (or shock) to any variable x, that is $\widetilde{x}_{t+1} \equiv$

[6]In our log-normal setting, condition (7) is equivalent to the absence of arbitrage opportunities. In more general settings, it is a necessary condition for no arbitrage.

[7]It might be tempting to replace both representative households by the arbitrageur in Assumption 2, but such an approach is mostly vacuous as it effectively considers twice the same investor. Mechanically the conversion of an intermediary's SDF from domestic to foreign currency is $m^{I*} = m^I + \Delta s$, irrespective of market structure — an accounting relation, not an equilibrium relation.

$x_{t+1} - E_t x_{t+1}$.[8] Next, we define the set of globally-traded shocks, or global shocks for short.

Definition 1. *The set of global shocks is* $\boldsymbol{\epsilon}^g_{t+1} = \{\epsilon^g_{t+1} | \exists \boldsymbol{\lambda} \in \mathbb{R}^N, \boldsymbol{\lambda}^* \in \mathbb{R}^{N^*} : \epsilon^g_{t+1} = \boldsymbol{\lambda}' \widetilde{\boldsymbol{r}}_{t+1} = \boldsymbol{\lambda}^{*\prime} \widetilde{\boldsymbol{r}}^*_{t+1}\}$.

Global shocks can be traded by local investors in their local currency in both countries. Formally, this means the coincidence of two properties. First, such a shock must affect returns in the two countries. Second, investors must have access to a trading strategy in each country that isolates the shock from other sources of risk. Appendix B shows how to construct a basis of this space from the covariance matrix of $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. Local shocks $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\epsilon}^*_{t+1}$ are the residuals of return innovations, $\widetilde{\boldsymbol{r}}_{t+1}$ and $\widetilde{\boldsymbol{r}}^*_{t+1}$, after controlling for global shocks.

Global shocks can arise because of common underlying economic shocks (e.g., productivity) that determine returns in both countries as long as such shocks can be replicated by investors in the two countries. Alternatively, global shocks can emerge without common fundamental shocks as a result of asset trading across countries — either directly by households or via an intermediary.

As an example, consider partially integrated markets such as the ones in Figure 1B. Imagine that $\boldsymbol{r}_{t+1} = (r_{ft}, r_{1,t+1}, r_{2,t+1}, r^*_{ft} + \Delta s_{t+1}, r^*_{1,t+1} + \Delta s_{t+1})$ and $\boldsymbol{r}^*_{t+1} = (r^*_{ft}, r^*_{1,t+1}, r^*_{2,t+1}, r_{ft} - \Delta s_{t+1}, r_{1,t+1} - \Delta s_{t+1})$. In such a setting, the domestic investor can construct a portfolio with excess return $r^*_{1,t+1} - r^*_{ft}$ by buying the foreign risky asset 1 and by selling the foreign risk-free asset, both converted into domestic currency. Similarly, the foreign investor can construct a portfolio with excess return $r_{1,t+1} - r_{ft}$. As a result, both $\widetilde{r}_{1,t+1}$ and $\widetilde{r}^*_{1,t+1}$ are in the set of global

[8] Note that $var_t(\widetilde{x}_{t+1}) = var_t(x_{t+1})$ and we use this notation interchangeably.

shocks $\boldsymbol{\epsilon}^g_{t+1}$.[9] Furthermore, here the FX risk $\widetilde{\Delta s}_{t+1}$ is also a global shock: it can be traded by both households through their respective carry trades (see Appendix A for a precise definition of the carry trade).

Finally, we refer to any other sources of variation orthogonal to asset returns $(\widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})$, or equivalently orthogonal to local and global shocks $(\boldsymbol{\epsilon}^g_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})$, as unspanned shocks.

Exchange rate depreciation. We can use this taxonomy of shocks to decompose the innovation to the depreciation rate as follows:

$$\widetilde{\Delta s}_{t+1} = g_{t+1} + \ell_{t+1} + u_{t+1}, \tag{8}$$

where g_{t+1} is a linear combination of global shocks $\boldsymbol{\epsilon}^g_{t+1}$, ℓ_{t+1} is a linear combination of both types of local shocks, $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\epsilon}^*_{t+1}$, and u_{t+1} is unspanned.

Thus, there are four components to the exchange rate depreciation Δs_{t+1}. The first is the conditional expectation $E_t \Delta s_{t+1}$. Then, there are two types of shocks spanned by assets, a global component g_{t+1} and a local component ℓ_{t+1}. Finally, there can be unspanned shocks u_{t+1}. This decomposition plays a central role in our characterization of restrictions on the behavior of the exchange rate.

Relatedly, one can construct the spanned components directly from asset returns:

$$\widetilde{\Delta s}_{t+1} = \widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} + u_{t+1}, \tag{9}$$

[9] A practical example of such global shocks arises in the context of commodity (e.g., oil) futures denominated in different currencies, or stocks of the same company traded in jurisdictions with different currencies (e.g., Royal Dutch Shell).

where $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ are the returns on two portfolios with the largest R^2 for explaining the exchange rate.[10] Mechanically, the residual coincides with the unspanned component u_{t+1} in equation (8). If this unspanned component is equal to 0, the depreciation rate is spanned by asset returns, and the difference between the shocks to returns on the two portfolios replicates the exchange rate shock exactly.

2 The general risk-sharing view of exchange rates

In this section, we characterize the restrictions on the behavior of the exchange rate imposed by the absence of international arbitrage and given the properties of returns on traded assets, $\boldsymbol{r}$ and $\boldsymbol{r}^*$, and local SDFs m and m^* that price them. We show that Assumptions 1 and 2 impose two sets of necessary restrictions on the depreciation rate: one on the shocks to the depreciation rate $\widetilde{\Delta s}_{t+1}$, and another on the expected depreciation rate $E_t \Delta s_{t+1}$.

We demonstrate that in a complete market setting these two sets of restrictions lead to the well-known asset market view of exchange rates and the puzzles that come with it. Subsequent analysis spells out the implications of these restrictions for a much larger set of market structures and revisits the puzzles in light of these results. All the proofs are in Appendix C. Appendix D proves the sufficiency of our key results: if the two sets of restrictions hold, Assumption 2 about the absence of international arbitrage opportunities is valid. Appendix E derives exact non-linear versions of the results, which includes the case of disasters.

[10] Formally, the portfolios maximize $R^2 = 1 - var_t(\Delta s_{t+1} - (\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1}))/var_t(\Delta s_{t+1})$.This pair of portfolios is not unique when global shocks are present. All of our results hold for any such pair.

2.1 Exchange rate shocks

We show that the component of the depreciation rate that loads on global shocks, g_{t+1} from equation (8), must coincide with the component of the difference of SDFs that loads on global shocks.

Proposition 1. *Under Assumptions 1 and 2,*

$$proj(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = proj(\widetilde{\Delta s}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = g_{t+1}. \tag{10}$$

Said differently, start from the pair local SDFs and regress them on all global shocks. The predicted value of this regression is equal to the global component of the exchange rate, g_{t+1}:

$$m^*_{t+1} - m_{t+1} = g_{t+1} + v_{t+1} \qquad \text{with} \quad v_{t+1} \perp \boldsymbol{\epsilon}^g_{t+1}. \tag{11}$$

What is missing from Proposition 1 is just as important as what is there. Local financial markets do not impose any restrictions on the component of the depreciation rate loading on either local shocks $(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})$ or its unspanned component u_{t+1}. Thus, in general, financial markets impose less restrictions on the exchange rate as compared with complete and integrated markets.

How does the absence of arbitrage lead to this result? In complete markets, local and foreign investors must agree on the price of all payoffs after conversion to a common currency: $cov_t(m_{t+1}, r_{t+1}) = cov_t(m^*_{t+1} - \Delta s_{t+1}, r_{t+1})$ for every r_{t+1}. Proposition 1 comes from a generalization of this result. To preclude arbitrage opportunities, local and foreign investors must only agree on the price of risks that they both trade — the global shocks.

Without a change of currency, the argument is standard: the international arbitrageur can buy the global shock ϵ^g_{t+1} in the home market and sell it in the foreign market (e.g., Chen and Knez, 1995). Because this portfolio is riskless, the two risk premia must coincide, $cov_t(m_{t+1}, \epsilon^g_{t+1}) = cov_t(m^*_{t+1}, \epsilon^g_{t+1})$. In Appendix A.2, we show that this logic extends to the case with currency conversion, and no arbitrage requires the so-called quanto adjustment $cov_t(\Delta s_{t+1}, \epsilon^g_{t+1})$ to expected returns. This implies that the comovement of the depreciation rate with global shocks must be the same as that of the relative SDFs, $cov_t(m^*_{t+1} - m_{t+1}, \epsilon^g_{t+1}) = cov_t(\Delta s_{t+1}, \epsilon^g_{t+1})$. Conversely, for shocks that are not traded by both investors, it is impossible to construct candidate arbitrage portfolios that relate pricing in the two markets (see Appendix D).

2.2 Expected depreciation rate

We turn to restrictions on the behavior of the expected depreciation rate. These restrictions depend on the relation of the exchange rate with asset returns. Start from the projection of the exchange rate on asset returns, represented by two portfolio $r_{p,t+1}$ and $r^*_{p,t+1}$ as in equation (9). Recall that when $r_{p,t+1}$ and $r^*_{p,t+1}$ span the exchange rate, the unspanned component u_{t+1} is equal to 0. We define δ_t as the difference of the two portfolios' expected returns:

$$\delta_t \equiv \left[r_{ft} - cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2} var_t(r_{p,t+1}) \right] - \left[r^*_{ft} - cov_t(m^*_{t+1}, r^*_{p,t+1}) - \frac{1}{2} var_t(r^*_{p,t+1}) \right]. \quad (12)$$

The following proposition relates the behavior of the expected depreciation rate to spanning of the exchange rate and this quantity, which only depends on asset returns and local SDFs.

Proposition 2. *The expected depreciation rate is pinned down if and only if the exchange rate is spanned by asset returns, that is when $u_{t+1} = 0$. In this case, it is:*

$$E_t \Delta s_{t+1} = \delta_t = \underbrace{r_{ft} - r^*_{ft}}_{UIP} - \underbrace{cov_t(m_{t+1}, \Delta s_{t+1})}_{exchange\ rate\ risk\ premium} - \underbrace{\frac{1}{2} var_t(\Delta s_{t+1})}_{convexity} + \theta_t, \tag{13}$$

where $\theta_t = cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1})$. This quantity collapses to $\theta_t = 0$ when the exchange rate is spanned by global shocks.

The most important implication of Proposition 2 is that it delineates two cases: either local market pricing determines expected depreciation exactly, or it says nothing about it. The expected depreciation rate is closely related to the risk premium for exchange rate risk. Exposure to this risk can be obtained by engaging in the carry trade. This risk premium is pinned down by pricing in local financial markets only if the international arbitrageur can use locally traded assets to perfectly offset this risk. Therefore, the absence of arbitrage has no bearing on this quantity if the exchange rate is not spanned by asset returns, that is, $u_{t+1} \neq 0$.

Spanned exchange rate. When the exchange rate is spanned, the international arbitrageur uses the two local markets to price the exchange rate risk. Hence, the two local SDFs play a role in the expected depreciation rate. This insight explains the presence of the novel adjustment term θ_t in equation (13) relative to the standard complete market formula (with $\theta_t = 0$). It also leads to a symmetric expression to equation (13) which emphasizes the foreign SDF m^*_{t+1}:

$$\delta_t = r_{ft} - r^*_{ft} - cov_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) + \theta^*_t, \tag{14}$$

with $\theta^*_t = cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{p,t+1})$.

It is only when the local investors are able to replicate the exchange rate on their own that their individual Euler equations are enough to obtain the expected depreciation. If the home (foreign) investor can trade both spanning portfolios, then $\theta_t = 0$ ($\theta^*_t = 0$), and the standard complete market formula for the home (foreign) investor holds. For example, this situation occurs in settings in which the home investor acts as an international arbitrageur. For both home and foreign investors to price the exchange rate risk, they must be able to trade it, that is, the exchange rate is a global shock.

Unspanned exchange rate. When the exchange rate is not spanned by traded assets, its expectation can deviate from this formula by an arbitrary wedge,

$$E_t \Delta s_{t+1} = \delta_t + \psi_t. \tag{15}$$

This flexibility might lead to implausibly large trading profits for the international investor. One can be more informative about these deviations ψ_t by imposing a condition that is stronger than the absence of arbitrage (Assumption 2).

Assumption 3. *(No quasi-arbitrage) There is an upper bound B on Sharpe ratios in international markets:*

$$\forall r^I_{p,t+1} \in \boldsymbol{r}^I_{p,t+1}: \quad \left| E_t(r^I_{p,t+1}) + \frac{1}{2} var_t(r^I_{p,t+1}) - r_{ft} \right| \leq B \sqrt{var_t(r^I_{p,t+1})}. \tag{16}$$

This assumption restricts the Sharpe ratio of trades in international markets. Such bounds have a long tradition in finance, going back to Cochrane and Saa-Requejo (2000), Kozak, Nagel, and Santosh (2020), and Ross (1976). Intuitively, it can be motivated by the view that if trades that are too profitable emerged in equilibrium, new financial institutions would step in to take advantage of them. Under this view we obtain the following condition.

Proposition 3. *Under Assumption 3, the wedge* ψ_t *in the expected depreciation rate must satisfy:*

$$\left|\psi_t + \frac{1}{2}var_t(u_{t+1})\right| \leq B\sqrt{var_t(u_{t+1})} \equiv B\sqrt{(1-R^2)var_t(\Delta s_{t+1})}, \tag{17}$$

where R^2 *is the R-squared in the regression of* Δs_{t+1} *on* $\boldsymbol{r}_{t+1}$ *and* $\boldsymbol{r}^*_{t+1}$.

This proposition limits possible expected depreciations in the case of an unspanned exchange rate. It indicates that deviations from the risk premium in the spanned case are bounded by the volatility of unspanned shocks.

3 The dual role of financial markets

These general results inform our understanding of how finance interacts with the exchange rate beyond the standard market structures studied in the literature. In this section, we apply these results to address three questions suggested by the properties of these standard market structures. Does risk-sharing necessarily lead to the currency puzzles? Is extreme segmentation necessary financial markets to be a source of shocks to the exchange rate? Are these two roles of financial markets mutually exclusive? We answer all three questions by the negative. Furthermore, we characterize which features of the market structure offer solutions to these challenges.

We conduct the following exercise. We fix the households' IMRSs m and m^*. We assume that our selected IMRSs are such that the exchange rate's cyclicality, volatility, and risk premium are counterfactual under complete and integrated markets — the currency puzzles. We ask how those features of exchange rate dynamics behave under alternative market structures and, in particular, whether the puzzles can be solved.

Guided by Propositions 1 and 2, we study the implications for volatility and cyclicality separately from the currency risk premium. First, we explore where the problems with volatility and cyclicality come from when markets are complete. Then, we address our three main questions one by one for these moments. Finally, in Section 3.5, we focus on the restrictions that affect the currency risk premium.

3.1 Complete and integrated markets

The case of complete and integrated markets is the relevant benchmark for our discussion as it is the standard setting under which the currency puzzles occur. Financial markets are complete when investors have access to the full set of Arrow-Debreu securities in both markets. Financial markets are integrated when both households can trade all available securities. In this setting, $\boldsymbol{\epsilon}^g_{t+1}$ spans all possible risks. Then Proposition 1 implies

$$\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} = \widetilde{\Delta s}_{t+1}. \tag{18}$$

Innovations to the depreciation rate must equal innovations to the difference of stochastic discount factors, completely pinning down exchange rate shocks.

This result leads to two puzzles about the behavior of the exchange rate. First, consider the variance of the depreciation rate:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m^*_{t+1} - m_{t+1}) \\ &= var_t(m^*_{t+1}) + var_t(m_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}). \end{aligned} \tag{19}$$

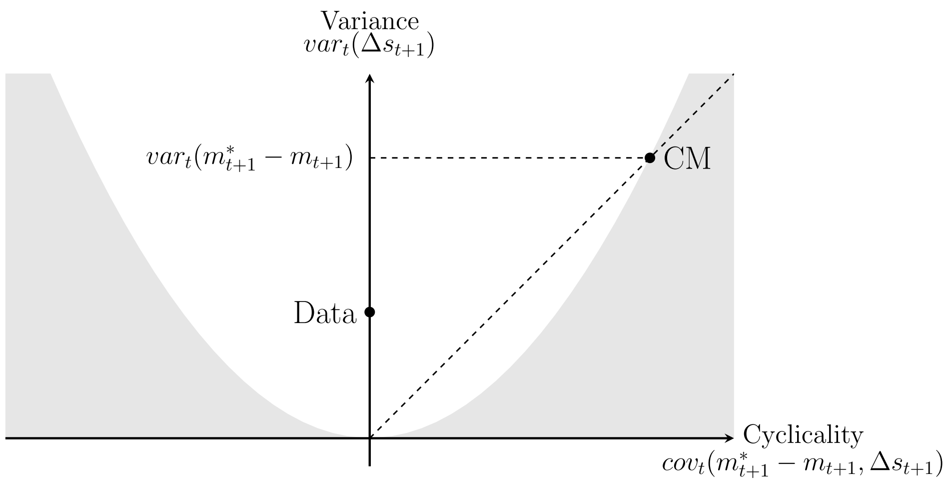
Brandt, Cochrane, and Santa-Clara (2006) argue that this equation leads to the volatility puzzle, with the exchange rate being not volatile enough. Typically observed Sharpe ratios on domestic assets imply highly volatile IMRSs, much more so than exchange rate depreciation. The mild correlation of macroeconomic quantities across countries suggests that the IMRSs are not correlated enough for the last term of equation (19) to offset this high variance and obtain realistic exchange rate risk.

Further, equation (18) also implies

$$var_t(\Delta s_{t+1}) = cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}), \tag{20}$$

and $corr_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) = 1$. Changes in exchange rates must be perfectly correlated with changes in relative marginal utilities of the domestic and foreign households, that is, the home currency depreciates in relatively good times for home investors. As pointed out by Backus and Smith (1993), this implication is counter-

Figure 2: Proposition 1 in complete and integrated markets



The figure illustrates implications of the complete market setting, labeled as CM, for the properties of depreciation rates. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The grey area represents the infeasible combinations of volatility and criticality of depreciation rates due to the Cauchy-Schwarz inequality.

factual for various measures of good times, leading to the cyclicality puzzle.

We introduce a visualization of these puzzles which we will revisit for other market structures. Figure 2 demonstrates the tension in capturing volatility, on the vertical axis, and cyclicality, on the horizontal axis, at once. The point labeled 'CM' shows the prediction of the complete and integrated market setting. The variance of the exchange rate is equal to $var_t(m^*_{t+1} - m_{t+1})$. Equation (20) also implies that the complete markets case is on the 45-degree line. The point labeled 'Data' is a stylized representation of our assumption that the selected m and m^* lead to the puzzles when markets are complete and integrated: the exchange rate is not as volatile as predicted by the model, and not correlated with the relative discount rates. Finally, the gray area represents combinations of volatility and cyclicality that are never mathematically feasible because of the Cauchy-Schwarz inequality.[11]

The distance between Data and CM is the essence of the volatility and cyclicality puzzles. In what follows, we ask how market structures other than complete and integrated markets lead to different predictions and allow to rationalize the evidence. This path is complementary to the large literature making progress on the puzzles by altering preferences or aggregate dynamics, but maintaining the assumptions that markets are complete and integrated.

3.2 When does risk-sharing lead to the currency puzzles?

We first characterize a set of market structures that deviate from the case of complete and integrated markets, but share the cyclicality and volatility puzzles.

In light of equation (10) from Proposition 1, the key feature of complete and integrated markets is that the projections of the relative IMRS $m^* - m$ and the depreciation rate Δs on global shocks are these quantities themselves. This occurs because all shocks are globally traded. We saw that the puzzles are a mechanical implication of this property. We now show that the puzzles arise even if only one of the two projections is the quantity itself, and that this corresponds to two economically meaningful families of market structures.

Start with the case when assets in each country span the relative IMRS $m^* - m$. This situation is close to market completeness: households in each country are presumed

[11]The Cauchy-Schwarz inequality implies that

$$cov_t^2(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) \leq var_t(\Delta s_{t+1}) \cdot var_t(m^*_{t+1} - m_{t+1}).$$

The grey area on the chart indicates all combinations that violate this inequality and are therefore infeasible.

to be able to trade each other's shocks to marginal utility. This case frequently arises in models with a small number of common macro risks that are traded in both countries. RBC models would fall into this category. One could also consider various settings popular in finance, such as habits, long-run risk, or rare disasters as long as they feature a small set of shocks that is traded in both of the two countries. However, this situation does not require integrated markets. For example, it can arise when households cannot trade with each other, $H \cap F = \emptyset$, and the markets are intermediated as long as households in each of the countries have access to a set of assets that is sufficiently rich.

In this case, equation (10) simplifies to

$$\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} = \text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = \text{proj}(\widetilde{\Delta s}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = g_{t+1}. \tag{21}$$

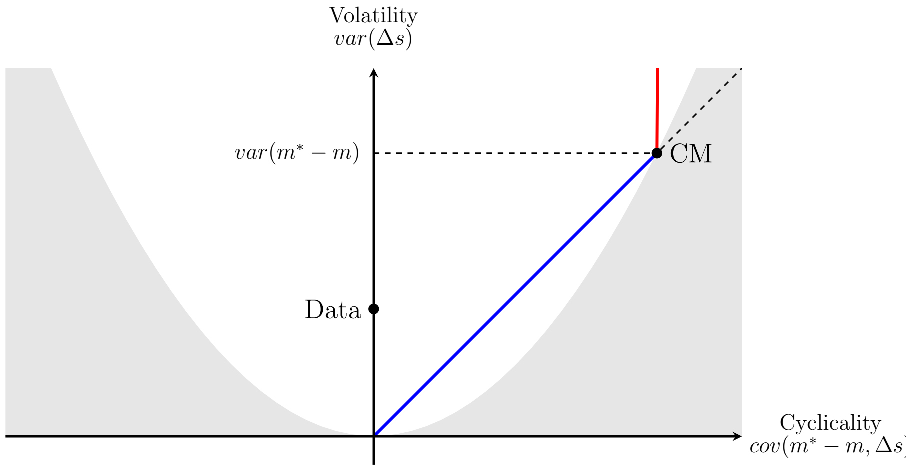
The projection of the exchange rate innovations on global shocks, that is, its global component, is equal to the difference in shocks in the IMRS. That is, a regression of the exchange rate depreciation on the difference of log IMRS yields a coefficient of 1.

This setting deepens the volatility puzzle and leaves the cyclicality unchanged. Specifically, equation (21) implies:

$$var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\ell_{t+1} + u_{t+1}) \geq var_t(m^*_{t+1} - m_{t+1}).$$

In words, the volatility of the depreciation rate can be greater than that of the

Figure 3: Proposition 1 and risk-sharing



The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The point labeled CM represents the complete market setting where $var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})$. The grey area represents the infeasible combinations of volatility and cyclicality of depreciation rates due to the Cauchy-Schwarz inequality. We consider a scenario when financial assets span the IMRS of households. The red line shows the feasible variance-cyclicality combinations in such a scenario. We also consider a scenario when the exchange rate is spanned by global shocks. The blue 45° line shows the feasible variance-cyclicality combinations in that case.

relative IMRS. Equation (21) also implies:

$$\begin{aligned} cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) &= cov_t(g_{t+1}, g_{t+1} + \ell_{t+1} + u_{t+1}) \\ &= var_t(g_{t+1}) = var_t(m^*_{t+1} - m_{t+1}). \end{aligned}$$

Cyclicality is the same as in the complete and integrated benchmark. We depict this situation in Figure 3 via a vertical line emanating from CM.

The second case is when the exchange rate can be traded directly by households, or when it can be spanned by the traded shocks. This situation arises when markets are integrated but not necessarily complete. Equation (10) simplifies to

$$\text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}|\boldsymbol{\epsilon}^g_{t+1}) = \text{proj}(\widetilde{\Delta s}_{t+1}|\boldsymbol{\epsilon}^g_{t+1}) = \widetilde{\Delta s}_{t+1}. \tag{22}$$

The projection of the difference in shocks in the IMRS on global shocks is equal to the innovations in the depreciation rate. That is, a regression of the difference of log IMRS on the exchange rate depreciation yields a coefficient of 1.

As a result, one can make only limited progress in addressing the exchange rate puzzles. Specifically,

$$var_t(\Delta s_{t+1}) = var_t(\text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}|\boldsymbol{\epsilon}^g_{t+1})) \leq var_t(m^*_{t+1} - m_{t+1}),$$

which potentially alleviates the volatility puzzle. As regards the cyclicality puzzle, the covariance of the depreciation rate with the SDF differential must be still equal to the variance of the exchange rate,

$$cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) = cov_t(\Delta s_{t+1}, \text{proj}(m^*_{t+1} - m_{t+1}|\boldsymbol{\epsilon}^g_{t+1})) = var_t(\Delta s_{t+1}).$$

Therefore, just like in the complete and integrated markets case, there is a cyclicality puzzle.[12] Figure 3 summarizes these constraints on the cyclicality and volatility of the depreciation rate: the exchange must be on the 45-degree line segment between the origin and the complete markets point.

[12] Having said that, the correlation between relative discount factors in the domestic and foreign economies and depreciation rate is less than perfect:

$$corr_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = \frac{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}{\sqrt{var_t(m^*_{t+1} - m_{t+1}) \cdot var_t(\Delta s_{t+1})}} \leq \frac{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}{var_t(\Delta s_{t+1})} = 1.$$

Such a situation does not even require full market integration as long as households in both countries can trade risk-free assets of both countries, that is, partial market integration. Lustig and Verdelhan (2019) consider this specific case (their Assumption 2) and our conclusions concur with theirs. We note that allowing the intermediary associated with the SDF m^I from Assumption 2 to trade both risk-free bonds instead of households leads to different conclusions as the exchange rate is no longer spanned by the assets traded by the households, and $\text{proj}(\widetilde{\Delta s}_{t+1}|\boldsymbol{\epsilon}^g_{t+1}) \neq \widetilde{\Delta s}_{t+1}$.

Taken together, the results of this section imply that the exchange rate is constrained by the properties of the households' IMRS as long as there is a single departure from the complete and integrated markets. If one modifies who can trade assets by allowing for imperfect market integration or intermediation as long as available set of assets is rich, then we end up in the spanned IMRS scenario. If one limits which assets can be traded, while ensuring market integration for these specific assets, then we find ourselves in the spanned exchange rate scenario. As Figure 3 summarizes, these cases result in the joint volatility-cyclicality puzzles.

While these cases imply that the puzzles are a lot more robust than in complete and integrated markets, it is also important to emphasize that they do not encompass all situations with substantial risk-sharing. We will see in Section 3.4 that risk-sharing is not always associated with the puzzles.

3.3 Can financial shocks impact the exchange rate without extreme segmentation?

We now turn to the other extreme: structures that shut down the risk-sharing role of financial markets and focus on their second role as a source of shocks. Workhorse models of this type feature extreme segmentation by assumption. Households can trade risk-free assets of their own country only, while the intermediary trades both of those assets and bears currency risk. By appropriately choosing shocks affecting the financial sector, these models can generate a large family of exchange rate dynamics.

The extreme form of market segmentation assumed in those models might not be appealing. In practice, households trade more than one asset and intermediaries participate in more than one market. We ask whether one can consider additional markets in the intermediated setup without losing the empirical flexibility afforded by extreme segmentation.

To answer this question we can represent the workhorse models in our framework. Because households trade no risky assets, there are no global shocks. Equation (10) of Proposition 1 becomes degenerate, that is, $0 = 0$. There are no restrictions on exchange rate innovations that are associated with the households. In this case, any point within the white cone in Figure 3 is feasible, potentially resolving the puzzles. While our propositions are silent on the details of the financial sector, Appendix F presents a fully specified model, showing which foundations allow to reach each of those points. Intuitively, shocks to the ability of institutions to intermediate international financial trade — such as financial health shocks or demand shocks from other investors — pass through to the exchange rate. The dynamics of those shocks shape exchange rate dynamics.

The more general insight is immediate: as long as the market structure maintains the degeneracy of the projection, exchange rate dynamics remain flexible. One can add trading opportunities as long as they do not create global shocks. Two such dimensions are particularly helpful in moving towards a more realistic market structure. First, because the trading opportunities of intermediaries do not affect global shocks, these intermediaries can be as sophisticated and active in as many markets as one wants. Second, households can trade rich sets of their respective local assets as long as their returns are not related enough to create global shocks.

3.4 Can the two roles of financial markets coexist?

So far, we have focused on each role of financial markets in isolation. For the risk-sharing role, this corresponds to situations with a very large share of global shocks. In contrast, there are no global shocks when financial markets act as a source of fluctuations. While we have argued that each of these extremes encompasses substantially more market structures than complete markets and extreme segmentation respectively, these remain special cases. One can naturally conceive situations in which the role of global shocks is intermediate. For example, going back to our discussion in the previous section with households trading a rich set of local assets, it could be that these assets have some common shocks across countries. In such models, the two roles of financial markets coexist in shaping the exchange rate.

Going along with this reasoning, we can ask how small must the role of global shocks be in order to resolve the cyclicality and volatility puzzle. To answer this question, we revisit the graphic representation of the two puzzles introduced in Figure 2.

Taking the amount of global shocks, $var_t(g_{t+1})$, as given, Proposition 1 implies the following trade-off between cyclicality and volatility

$$\overbrace{var_t(\Delta s_{t+1})}^{\text{volatility}} \geq var_t(g_{t+1}) + \frac{\left(\overbrace{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}^{\text{cyclicality}} - var_t(g_{t+1})\right)^2}{var_t(m^*_{t+1} - m_{t+1}) - var_t(g_{t+1})}. \tag{23}$$

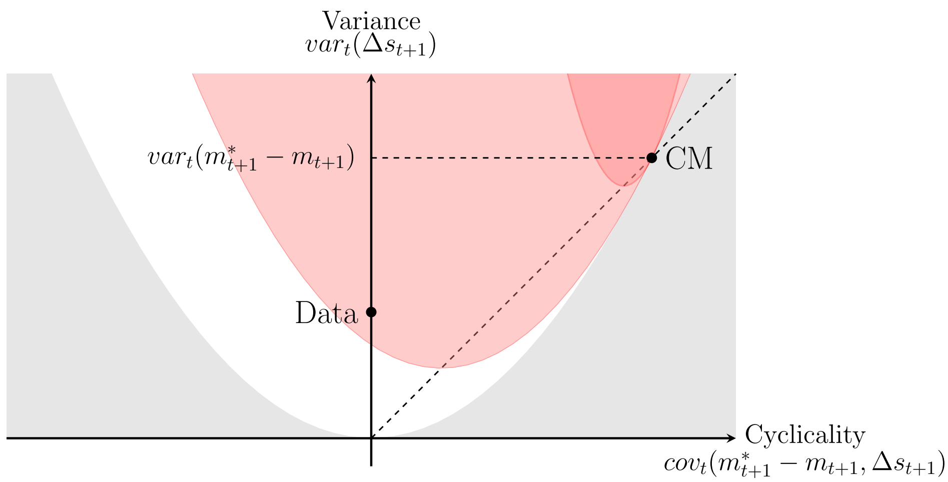
Figure 4 visualizes this relation.[13] The minimum variance of the exchange rate is $var_t(g_{t+1})$; it is attained when the cyclicality has the same value, $cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = var_t(g_{t+1})$. This point corresponds to the vertex of the cone, which lies on the segment of the 45-degree line between the origin and the complete markets point. This contribution of global shocks is the consequence of risk-sharing.

Shocks from financial markets add to the volatility of the exchange rate above this base level — Appendix F spells out a foundation for this mechanism even in presence of risk-sharing. When this additional source of variation comoves with the relative IMRS, cyclicality is also altered. The inequality in equation (23) quantifies how much it can change. The red cones in the figure represent this relation.

As the contribution of global shocks diminishes, two effects occur, both facilitating solving the puzzles. Moving from the darker to the lighter red cone illustrates this transition. First, risk-sharing puts a smaller lower bound on exchange rate volatility — the cone moves down. Second, there is more space shocks from financial market to alter cyclicality — the cone widens. The lighter cone contains the "Data" point. This indicates that it is possible to rationalize exchange rate dynamics by combining the two roles.

[13]This relation is a consequence of the Cauchy-Schwartz inequality applied to the non-global components of the exchange rate and SDF differential, $\Delta s_{t+1} - g_{t+1}$ and $m^*_{t+1} - m_{t+1} - g_{t+1}$, respectively.

Figure 4: The impact of global shocks on variance and cyclicality of exchange rates



Note: The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate (see text). The point labeled Data is a stylized representation of the evidence regarding the depreciation rates; the point labeled CM represents the complete market setting. The grey area represents infeasible combinations of volatility and cyclicality due to the Cauchy-Schwarz inequality.

These observations lead to a significant conclusion regarding the two roles of financial markets. Our findings suggest that extreme market segmentation is not necessary to realistically represent various aspects of the exchange rate. Instead, the presence of risk-sharing can be an effective mechanism, restricting the properties of the exchange rate in meaningful ways. Therefore, the two roles of financial markets can have a joint impact on the exchange rate. This joint effect helps overcome the limitations that arise in models that consider each role in isolation, namely the over-reliance of exchange rate dynamics on macroeconomic factors and the excessive segmentation in market participation among households and intermediaries.

3.5 The currency risk premium

The currency risk premium is another moment that often leads to challenges for settings with complete and integrated markets. In this structure, the expected depreciation rate is:

$$E_t \Delta s_{t+1} = r_{ft} - r^*_{ft} \; - cov_t(m_{t+1}, \Delta s_{t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) \tag{24}$$

The second term, a premium for currency risk, generates deviations from uncovered interest parity (UIP), a well-documented empirical feature. However, standard international models struggle with generating the empirically observed magnitude of currency risk premium simultaneously with addressing the cyclicality and volatility puzzles.

Proposition 2 indicates a sharp delineation between market structures regarding the currency risk premium. On the one hand, if the exchange rate is spanned by asset returns, $u_{t+1} = 0$, expected depreciation is given by equation (13). Because this expression is close to equation (24), the currency risk premium puzzle arises in such market structures as well. On the other hand, if the exchange rate is not spanned by asset returns, expected depreciation can deviate arbitrarily from this tight risk-sharing relation.

When does spanning occur in the market structures discussed in the previous sections? Let us consider a few examples. In integrated markets, as in Section 3.2, the exchange rate is explained by global shocks. Because global shocks are constructed from asset returns, the exchange rate is fully spanned. Therefore, these structures, which already wrestle with the volatility and cyclicality puzzles also face the risk premium puzzle.

In market structures with less risk-sharing, like those of Sections 3.3 and 3.4, the exchange rate may or may not be spanned. For example, lack of spanning occurs naturally if local asset returns such as stocks and bonds are not affected by financial sector shocks that transmit to the exchange rate. In this type of economy, all three puzzles can be solved simultaneously. In contrast, an economy in which one of the investors can engage in the carry trade but the other one only invests in the local risk-free asset features spanning but no global shocks; the risk premium puzzle remains.

4 Empirical Analysis

In this section we investigate empirically the tightness of constraints on the exchange rate associated with risk-sharing. Specifically, we measure the two key inputs to our restrictions: spanning and the importance of global shocks. These measures depend on return data and the specifics of a market structure, and they do not require knowledge of the IMRSs or risk premia. In some cases no empirical analysis is needed, e.g., when the exchange rate is one of the traded assets, it is automatically spanned. We consider a structure where an empirical investigation is warranted.

We assume that markets are intermediated and that households in each country can trade a broad collection of local assets. We limit the asset set in each country to sovereign bonds and various stock portfolios of that country. For this market structure, we first demonstrate that exchange rates appear to have a large component u_{t+1} unspanned by the returns of traded assets. Then, we provide methods to characterize global shocks. We find that they explain a relatively small share of exchange rate fluctuations. Both of these exercises lead to the conclusion that, for the market structure and the data we consider, the effects of risk-sharing are present, but play a modest role.

4.1 Data

We consider countries corresponding to G10 currencies between 2/1988 and 12/2022. We consider Germany as the representative country for the euro. Prior to the introduction of the euro, we use the German Deutschemark and splice these series together beginning in 1999. Our analysis focuses on the monthly frequency. We obtain exchange rates from WM/Reuters. Government bond yields are from each country's central bank websites. Monthly bond returns are computed from bond yields using a second-order Taylor approximation. We obtain equity indices from Morgan Stanley Capital International (MSCI). For each country, 10 different industry indices and 3 different style equity indices (Large + Mid Cap, Value, Growth) are sourced. Risk-free rates are approximated by dividing the 1-year yield by 12.

4.2 Is the exchange rate spanned?

Motivated by Proposition 2, we ask whether the depreciation rate is spanned by combination of domestic and foreign asset returns. We implement regressions of the form:

$$\Delta s_{t+1} = \alpha + \beta' \boldsymbol{r}_{t+1} + \beta^{*\prime} \boldsymbol{r}^{*}_{t+1} + u_{t+1}. \tag{25}$$

Here the residual u_{t+1} is a direct estimate of the unspanned component of the depreciation rate in equation (8). We report the adjusted R^2 of these regressions. Exact spanning corresponds to an R^2 of 1. Furthermore, Proposition 3 highlights that R^2 is an appropriate measure of economic distance to the case of perfect spanning.[14]

Table 1 reports the results. We always report the results for the combination of assets in the United States and another country. Each column in the table corresponds to that other country. Each row reflects a particular combination of assets used in the regression. Broadly speaking, we consider bonds and equities separately and in combination. Within each asset class, we zoom in on various individual contributions.

Major asset classes do not span exchange rates. When looking at all assets together, the R^2s range from 25% for Switzerland to 45% for Canada (in each case combined with the U.S.). Most of the explanatory power comes from the equity side. For example in the case of Canada, the combination of market, value, growth and industry returns explain 42% of variation in the depreciation rate. While the market alone gets to some substantial amount of variation — 27% for Canada —, the addition of industry returns is particularly informative. Consistent with the evidence in Chernov and Creal (2023), bond returns only explain a modest amount of variation in exchange rates: between 0.2% and 7% for the 10-year bond alone, and between 7.2% and 14% for the combination of bonds at all maturities.

We refer to the observation that asset returns do not span changes in exchange rates as the *financial exchange rate disconnect*. While the R^2s we obtain from regressions on asset returns are meaningfully larger than their counterpart with real

[14] Campbell, Serfaty-De Medeiros, and Viceira (2010) focus on currency hedging of equity and bond portfolios, so they essentially implement reverse regressions with a focus on the sign and significance of the associated betas. The documented insignificant betas for bond portfolios are suggestive of low R^2.

Table 1: Spanning of depreciation rates by asset returns – R^2

Dependent Variable	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Bonds									
10Y	0.25	0.33	7.49	5.36	4.73	1.05	4.79	4.01	0.92
All Maturities	7.23	7.89	15.72	10.15	13.66	5.67	13.95	11.52	13.65
Stocks									
Mkt	21.67	26.56	6.96	4.44	11.24	16.56	16.20	12.34	12.71
Mkt + Value/Growth	21.60	27.98	6.75	5.06	12.47	17.16	15.91	12.71	13.68
Mkt + Value/Growth + Ind.	35.07	41.61	18.55	22.78	29.41	24.53	24.00	19.61	26.88
Bond + Equity	36.74	45.05	26.79	29.13	36.64	27.95	30.62	25.28	33.80
N	419	395	419	419	406	419	414	419	419

Notes: The table reports the adjusted R^2 of a regression of the depreciation rate on various subsets of asset returns, as in equation (25). Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is a different country's currency relative to the U.S. dollar. The first row uses only 10-year bonds, while the second entertains maturities between 2 and 10 years, obtained from various central banks. The next three row successively add various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. The final row considers all assets simultaneously.

quantities, these magnitudes are much too small for leading to meaningful theoretical implications. Taking the strictest definition of absence of arbitrage, only a value of 1 leads to the relevance of Proposition 2. According to Proposition 3, even the largest numbers we measure imply a bound for the expected depreciation that is only $\sqrt{1-0.45} = 67\%$ of the bound with an R^2 of 0, not much tighter. Thus, observing the properties of returns on other assets is not informative about the expected currency depreciation rates.

The flipside of this conclusion is that the unspanned component of the depreciation rates, u_{t+1}, is large. In the context of models of intermediated markets, this result offers more flexibility in capturing realistic currency risk premium. As we discussed in section 3.5, partially integrated markets still imply tight restrictions on the currency premium because Proposition 2 holds.

4.3 Identifying global shocks

In this section we quantify the importance of global shocks $\boldsymbol{\epsilon}^{g}_{t+1}$, which play the key role in Proposition 1. We do so using two empirical approaches. The undirected approach uses canonical correlation analysis (CCA) to identify these shocks from the asset return data. The directed approach starts from candidates for global shocks such as global macro and financial variables proposed in the literature.

4.3.1 Undirected approach

The CCA procedure finds a US and a foreign portfolio of asset returns consisting of $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^{*}_{t+1}$, respectively, such that they have the highest correlation possible in sample. Next, conditional on finding this pair, the procedure looks for the next maximally correlated pair of portfolios that are orthogonal to their first pair. And so on.

According to Definition 1, global shocks would manifest themselves as innovations to portfolios with perfect correlation. In that case, Proposition 1 implies that projections of the depreciation rate and the difference in the SDFs on the global shocks coincide. In the data, even the largest correlation could be less than 1. So, in practice we would have to use an ad-hoc cut-off to decide which portfolios are sufficiently close to each other to constitute a measure of a global shock.

Table 2: Maximally correlated shocks across asset markets

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Rank 1	75.27	89.82	83.07	75.01	79.47	64.31	78.33	82.95	85.87
Rank 2	65.00	85.06	74.17	64.43	63.49	53.95	65.72	62.62	78.70
Rank 3	61.16	83.44	66.70	58.71	57.14	41.73	59.57	60.41	73.55
Rank 4	57.04	78.79	64.90	51.31	45.86	35.98	55.55	56.12	68.02
Rank 5	51.01	76.82	52.80	46.81	41.74	31.44	49.63	52.32	65.85
Rank 6	41.67	70.79	44.19	46.62	33.59	25.33	38.94	46.83	62.21
Rank 7	34.19	62.84	42.30	41.94	26.88	22.99	38.20	41.16	55.83
Rank 8	31.57	56.20	36.66	39.57	25.80	14.58	33.82	35.18	51.39
N	419	395	419	419	406	419	414	419	419

*N*otes: The table reports the correlation in % between the maximally correlated portfolios of asset returns between the U.S. and each country. The successive pairs of portfolio are orthogonal to each other, and obtained by canonical correlation analysis. Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is for a different country's assets relative to the U.S. assets. The assets include government bonds of maturities between 2 and 10 years (obtained from various central banks) and various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios (from MSCI).

Table 2 reports the results. Each column represents a foreign country. For a given country, each row reports the canonical correlation between the assets of that country and the US assets, reported in order of importance, starting from the largest.

The values of the largest correlations range from 64% for New Zealand to 90% for Canada. In some cases lower ranked correlations are similar to the largest one, like for Canada or the UK. In other cases, the magnitude of correlation drops off quickly, e.g., for New Zealand or Norway. Strictly speaking, the evidence suggests that there are no global shocks amongst the assets that we consider.

As alluded to earlier, we can be more generous with interpreting the evidence in Table 2 and assign a value of 1 to each estimated correlation that is above a certain threshold. We choose the value of 60% as such a threshold. We denote the matrix of foreign portfolio weights by $\boldsymbol{w}^*$; if there is only one global shock, this is a vector. We ask how much variation in the depreciation rate is explained by global shocks. We implement regressions of the form:

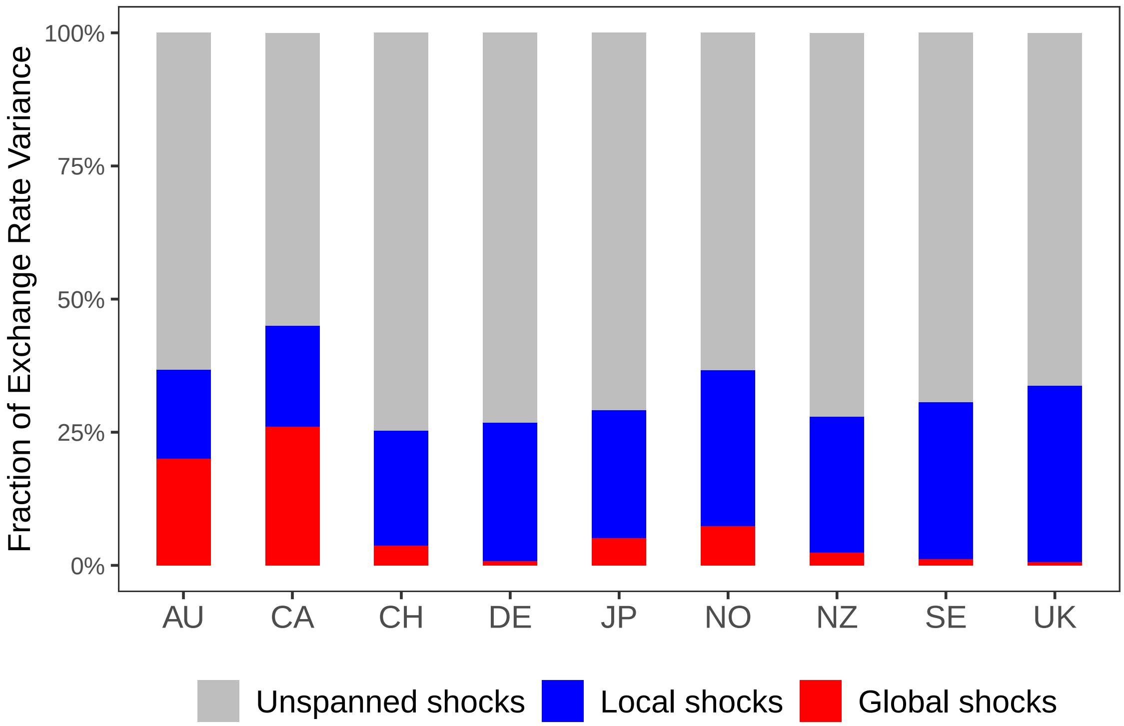
$$\Delta s_{t+1} = \alpha + \beta^{g\prime}(\boldsymbol{w}^{*\prime}\boldsymbol{r}^*_{t+1}) + \varepsilon_{t+1}. \tag{26}$$

The R^2 of such a regression is the fraction of variance in exchange rate explained by global shocks. Because we assume that the corresponding domestic portfolio is perfectly correlated with its foreign counterpart, we do not include it in the regression. The regression residual is a direct estimate of the contribution of local and unspanned shocks to the depreciation rate, $\varepsilon_{t+1} = \ell_{t+1} + u_{t+1}$.

Combining with the results of regression (25), we can decompose variation in the depreciation rate into the contribution of global, local, and unspanned shocks. Specifically, we have $var(\beta^{g\prime}(\boldsymbol{w}^{*\prime}\boldsymbol{r}^*_{t+1}))$ for global shocks, and $var(\varepsilon_{t+1}) - var(u_{t+1})$ for local shocks. Figure 5 reports these quantities as fraction of the variation in depreciation rate; the contributions mechanically add up to 1.

For all currencies, at least half of the variation in exchange rates is unspanned by asset returns — the financial disconnect we have already noted. Global shocks contribute up to 25% to variation in the depreciation rates (e.g., Australia and Canada). These estimates should be seen as an upper bound on the role of global shocks; remember that estimated global shocks include any pair of portfolios with correlation above 60%, far from the strict Definition 1.

Figure 5: Decomposition of exchange rate innovations, undirected



Notes: The figure reports the fraction of variance in exchange rates explained by globally traded shocks, local shocks, and shocks that are not spanned by asset returns. Each bar is a different country's currency relative to the U.S. dollar. Global shocks are measured via returns of the assets that we use in our analysis using CCA.

4.3.2 Directed approach

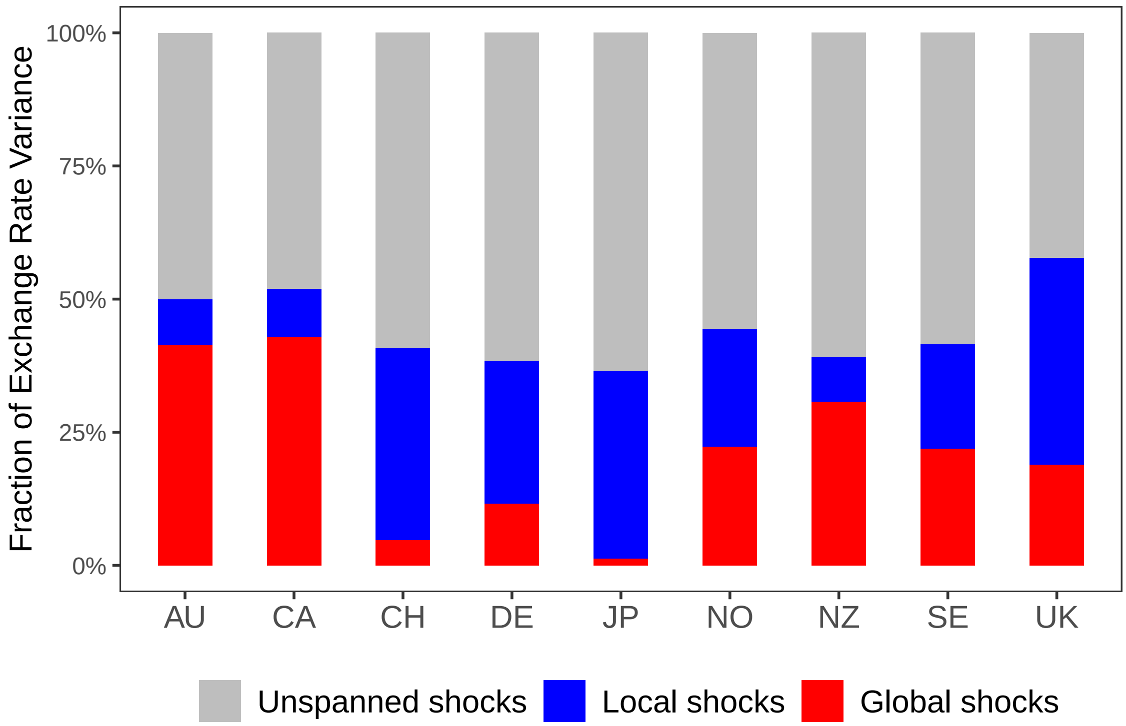
Instead of being agnostic about the nature of global shocks we rely on macroeconomic research and assume that they are known. Specifically, we take VIX, GFC (Miranda-Agrippino and Rey, 2020), and EBP (Gilchrist and Zakrajsek, 2012) as such shocks. This approach requires a strong assumption that portfolios of traded assets in each economy can span these shocks.

For each country, we regress its depreciation rate vs USD on these measures of global shocks. The R^2 from such a regression produce the fraction of the exchange rate variation due to global shocks. Next, we implement the regression in Equation (25) where the set of returns is complemented by the three global shocks to obtain the unspanned component. Naturally, it is going to be smaller than that in the previous section. The knowledge of the variation due to global and unspanned shocks delivers the variation due to local shocks.

Figure 6 reports the resulting decomposition of the variation in the exchange rate into the three types of shocks. The directed approach delivers somewhat larger contribution of global shocks, but qualitatively the conclusions are unchanged. The unspanned shocks represent the largest share of shocks. Contribution of the global shocks is the largest for Australia and Canada, which approach 50%.

Just like the financial disconnect leads to weak restrictions about the expected depreciation rate, the relatively modest role of global shocks implies weak restrictions about exchange rate risks. The flipside of this conclusion is that the evidence corresponds to the setting of Section 3.4, which allows for sizable exposure to local shocks relative to global shocks. In its turn, this feature is capable of resolving the cyclicality and volatility puzzles jointly. Given that partially integrated markets still impose tight restrictions on the currency risk premium, the intermediated market structure appears to be the most promising avenue for describing the equilibrium behavior of the exchange rate.

Figure 6: Decomposition of exchange rate innovations, directed



*N*otes: The figure reports the fraction of variance in exchange rates explained by globally traded shocks, local shocks, and shocks that are not spanned by asset returns. Each bar is a different country's currency relative to the U.S. dollar. Global shocks are measured via VIX, GFC, and EBP.

4.4 Relation to returns-based SDFs

Because of our discussion of currency puzzles, we have focused on the IMRS in the earlier sections. Our Propositions apply to any SDFs, even the one inferred from no-arbitrage models, which are estimated using asset prices, or directly from the returns data in a model-free fashion. In such context, our results allow to address the following question. Suppose a researcher is given a set of domestic and foreign asset returns without the knowledge of market structure that generated them and without labels, i.e., a domestic asset could be a foreign asset converted to home currency. What can SDFs constructed from these returns say about the exchange rate?

This is a substantial generalization of the extant returns-based analysis where market integration is assumed (e.g., Maurer and Tran, 2021, Sandulescu, Trojani, and Vedolin, 2021), or the international SDF m^I is estimated (e.g., Chernov and Creal, 2023). In all of these cases, the exchange rate is a traded asset, and, thus, the estimated SDFs do not impose restrictions on the exchange rate. That is in contrast to IMRS-based conclusions when the exchange rate is spanned.

5 Conclusion

In this paper, we propose a general framework for understanding how financial markets determine the behavior of exchange rates. Our theory accommodates many settings: complete or incomplete markets, arbitrary forms of market integration, or situations in which international financial trade happens through intermediaries. We characterize restrictions on the behavior of exchange rates due to the absence of international arbitrage. These restrictions can be summarized by two conditions that share the simplicity of the complete market result while having richer implications.

We use these results to study many different market structures, which leads to new insights on the interaction of financial markets and the exchange rate. First, we show that the puzzles arising in settings with complete and integrated markets are still present when markets are either incomplete or imperfectly integrated. However, they can disappear without giving up on risk-sharing by relaxing both of these features. Second, we demonstrate that financial markets can be a source of shocks to the exchange rate without the extreme market segmentation featured in standard models emphasizing this mechanism. Finally, we show that the two roles of financial markets, facilitating risk-sharing and transmitting financial shocks, are not mutually exclusive and both can play an important role in shaping exchange rate dynamics and avoid the currency puzzles.

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Appendix

A Portfolio algebra

A.1 Portfolio approximation

To maintain tractability, we follow Campbell and Viceira (2002) and approximate the log portfolio excess returns relative to a risk-free rate r_{ft}:

$$\begin{aligned} r_{p,t+1} - r_{ft} &= \log\left(\boldsymbol{w}_t' e^{\boldsymbol{r}_{t+1} - r_{ft}\boldsymbol{\iota}}\right) \\ &\approx \boldsymbol{w}_t'(\boldsymbol{r}_{t+1} - r_{ft}\boldsymbol{\iota}) + \frac{1}{2}\boldsymbol{w}_t' \operatorname{diag}(\boldsymbol{\Sigma}_t) - \frac{1}{2}\boldsymbol{w}_t'\boldsymbol{\Sigma}_t\boldsymbol{w}_t, \end{aligned} \tag{27}$$

where $\boldsymbol{\Sigma}_t$ is the variance-covariance matrix of log returns. This approximation allows us to represent portfolios returns as linear combination of log returns. Importantly, it is stable by recombination, leading to the same result when applied in two steps or all at once for a portfolio of portfolios. The approximation becomes exact as time becomes continuous and the underlying data-generating process for returns converges to a purely diffusive stochastic process.

A.2 Two international portfolios

Two international portfolios are useful for the derivation of our main results.

Carry trade. One zero-cost portfolio, often referred to as carry, entails taking long and short positions in related assets:

$$R_{\text{carry},t+1} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t. \tag{28}$$

Traditionally, the traded assets are taken to be domestic and foreign risk-free (one-period) bonds. But carry does not have to be limited by that. For instance, Lustig, Stathopolous, and Verdelhan (2019) consider long-term bonds. More generally, one could use any pair of assets that are close to each other, e.g., $corr_t(r_{t+1}, r^*_{t+1}) \approx 1$.

The key characteristic of the carry trade is that it exposes the arbitrageur to currency risk.

Lemma 1. *The conversion from foreign to home returns in the carry portfolio introduces exposure to currency risk,* $\widetilde{r}_{\text{carry},t+1} = \widetilde{r}_{t+1} - \widetilde{r}^*_{t+1} + \widetilde{\Delta s}_{t+1}$.

Proof. We map the zero-cost portfolio (28) into the log approximation of a funded portfolio in equation (27) by adding a position in the risk-free asset:

$$R_{p,t+1} \equiv R_{\text{carry},t+1} + R_{f,t} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t + R_{f,t}.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to USD, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the domestic risk-free asset with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the log gross return relative to the risk-free rate $R_{p,t+1}/R_{f,t}$:

$$\begin{aligned} r_{\text{carry},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= r_{t+1} - r^*_{t+1} - \Delta s_{t+1} + cov_t(r_{t+1} - r^*_{t+1} - \Delta s_{t+1}, r^*_{t+1} + \Delta s_{t+1}). \end{aligned} \tag{29}$$

Thus, the shocks to the exchange rate have an impact on the portfolio performance. ■

Differential carry. That carry is exposed to currency risk prompts us to consider another zero-cost portfolio, labeled as differential carry, which is long one unit of the domestic asset, and short one unit of the foreign asset, financed at the respective risk-free rates:

$$R_{\text{diff},t+1} = (R_{t+1} - R_{ft}) - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t. \tag{30}$$

Intuitively, this portfolio does not introduce additional currency exposure because, in contrast to carry, only the foreign excess return is converted to USD. We demonstrate this formally in the following lemma.

Lemma 2. *The conversion from foreign to US returns in the diff portfolio does not introduce additional exposure to currency risk,* $\widetilde{r}_{\text{diff},t+1} = \widetilde{r}_{t+1} - \widetilde{r}^*_{t+1}$.

Proof. We map the zero-cost portfolio (30) into a funded portfolio to use the approximation of equation (27):

$$R_{p,t+1} \equiv R_{\text{diff},t+1} + R_{f,t} = R_{t+1} - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to USD, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the foreign risk-free asset converted to USD, $R^*_{ft} \cdot S_{t+1}/S_t$, with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the relative log return:

$$\begin{aligned} r_{\text{diff},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= r_{t+1} - r_{ft} - (r^*_{t+1} - r^*_{ft}) - cov_t(r^*_{t+1}, \Delta s_{t+1}) + cov_t(r^*_{t+1}, r_{t+1} - r^*_{t+1}). \end{aligned} \tag{31}$$

Thus, only the covariance of the foreign return with the exchange rate has a material impact on portfolio performance, not the shocks to the exchange rate. ■

The disappearance of exchange rate risk for the diff returns is in part due to our portfolio approximation. In Appendix Section G, we confirm that this approximation is very tight empirically. We compare the excess returns on various stock portfolios and sovereign bonds in their origin currency and in converted currency. The correlation between the two monthly series is always around 99.9%. Also, see Daniel, Hodrick, and Lu (2017) (Online Appendix C) and Chernov, Dahlquist, and Lochstoer (2023) (Internet Appendix II).

B Global shocks

B.1 Identification and construction

We show how to identify a basis for the set of global shocks $\boldsymbol{\epsilon}^g_{t+1}$. We drop time indices for parsimony.

First, recall what canonical correlation analysis does.

Definition 2. *Canonical correlation analysis identifies pairs* $(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}^*_i)$ *for* $i = 1, \ldots, K$ *for some* K *such that:*

1. $\forall i \ var(\boldsymbol{\lambda}'_i \boldsymbol{r}) \neq 0$
2. $\forall i \ \boldsymbol{\lambda}'_i \boldsymbol{r} = \boldsymbol{\lambda}^{*\prime}_i \boldsymbol{r}^*$
3. $\forall i \neq j \ \boldsymbol{\lambda}'_i \boldsymbol{r} \perp \boldsymbol{\lambda}'_j \boldsymbol{r}$

4. $\forall r \in span(\boldsymbol{r}), r^* \in span(\boldsymbol{r}^*)$ *if* $\forall i \; r \perp \boldsymbol{\lambda}_i'\boldsymbol{r}$ *and* $r^* \perp \boldsymbol{\lambda}_i'\boldsymbol{r}$ *then* $r \neq r^*$

Then we show that this procedure identifies a basis of $\boldsymbol{\epsilon}^g$.

Lemma 3. *The collection* $(\boldsymbol{\lambda}_1'\boldsymbol{r}, \ldots \boldsymbol{\lambda}_K'\boldsymbol{r})$ *identified by canonical correlation analysis is a basis of* $\boldsymbol{\epsilon}^g$*.*

Proof. By point 2 of Definition 2, all the $\boldsymbol{\lambda}_i'\boldsymbol{r}$ are in $\boldsymbol{\epsilon}^g$. Thus, $span(\boldsymbol{\lambda}_1'\boldsymbol{r}, \ldots \boldsymbol{\lambda}_K'\boldsymbol{r}) \subset \boldsymbol{\epsilon}^g$.

Let us show the other direction. Assume that there exists $r \in \boldsymbol{\epsilon}^g$ such that $r \notin span(\boldsymbol{\lambda}_1'\boldsymbol{r}, \ldots \boldsymbol{\lambda}_K'\boldsymbol{r})$. We can orthogonalize r to all the $\boldsymbol{\lambda}_i'\boldsymbol{r}$ and obtain $\hat{r}$. Because $\hat{r}$ is a linear combination of r and $\boldsymbol{\lambda}_i'\boldsymbol{r}$ which are all in $\boldsymbol{\epsilon}^g$, it is also in $\boldsymbol{\epsilon}^g$, and therefore in $span(\boldsymbol{r})$ and $span(\boldsymbol{r}^*)$. By substituting $\hat{r}$ for both r and r^* in point 4 of Definition 2, we immediately obtain a contradiction. Therefore $span(\boldsymbol{\lambda}_1'\boldsymbol{r}, \ldots \boldsymbol{\lambda}_K'\boldsymbol{r}) \supset \boldsymbol{\epsilon}^g$; the two sets are equal. By point 3 of the CCA definition, $dim(span(\lambda_1'r, \ldots \lambda_K'r)) = K$, so $(\lambda_1'r, \ldots \lambda_K'r)$ is indeed a basis of $\boldsymbol{\epsilon}^g$. ■

Furthermore, we relate the dimension of $\boldsymbol{\epsilon}^g$ to the rank of covariance matrices of $\boldsymbol{r}$, $\boldsymbol{r}^*$, and the two combined.

Lemma 4. *The dimension of* $\boldsymbol{\epsilon}^g$ *is:*

$$dim(\boldsymbol{\epsilon}^g) = rank(var(\boldsymbol{r})) + rank(var(\boldsymbol{r}^*)) - rank(var(\boldsymbol{r}, \boldsymbol{r}^*)).$$

Proof. Observe that, by construction,

$$\begin{aligned} dim(span(\boldsymbol{r}, \boldsymbol{r}^*)) &= [dim(span(\boldsymbol{\epsilon}^g)) + dim(span(\boldsymbol{\epsilon}))] + \{dim(span(\boldsymbol{\epsilon}^*))\} \\ &= [dim(span(\boldsymbol{r}))] + \{dim(span(\boldsymbol{r}^*) - dim(\boldsymbol{\epsilon}^g))\}. \end{aligned}$$

Therefore,

$$dim(\boldsymbol{\epsilon}^g) = dim(span(\boldsymbol{r})) + dim(span(\boldsymbol{r}^*)) - dim(span(\boldsymbol{r}, \boldsymbol{r}^*)),$$

which yields the result. ■

B.2 Examples and counter-examples

It may be intuitively appealing to think about sources of common variation in domestic and foreign assets as global shocks. There is a critical difference between such intuition and the formal definition of global shocks, which requires replication of the exposure to such shock solely using assets of either country.

As an example, consider economies with N risky assets each, with all of these assets having exposure to a shock $\epsilon_{t+1} : \widetilde{r}_{i,t+1} = \alpha_i \epsilon_{t+1} + \beta_i \epsilon_{i,t+1}$, and $\widetilde{r}^*_{i,t+1} = \alpha^*_i \epsilon_{t+1}$, and all the shocks are orthogonal to each other. If $\beta_i = 0$ for at least one domestic asset i, then ϵ_{t+1} is a global shock. If none of the β_i is equal to zero, then ϵ_{t+1} is a global shock if $N \to \infty$ which allows a portfolio of $\widetilde{r}_{i,t+1}$ to isolate ϵ_{t+1} via diversification. If neither condition holds then ϵ_{t+1} is not a global shock despite affecting common variation in domestic and foreign assets.

C Derivation of the main results

C.1 Proof of Proposition 1

Consider one of the global shocks, ϵ^g_{t+1}. By definition 1, there exist two portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ such that $\epsilon^g_{t+1} = \widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1}$.

The differential carry portfolio of Lemma 2 is in $\boldsymbol{r}^I_{p,t+1}$. In this case, the portfolio has no risk because $\widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1}$. The shocks to foreign and domestic return perfectly offset each other. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. That is:

$$0 = E_t[r_{p,t+1} - r_{ft}] - E_t[r^*_{p,t+1} - r^*_{ft}] - cov_t(r^*_{p,t+1}, \Delta s_{t+1}) + cov_t(r^*_{p,t+1}, r_{p,t+1} - r^*_{p,t+1}).$$

The last term is equal to 0 because $r_{p,t+1} - r^*_{p,t+1}$ has no risk. We can replace the first two terms by covariances with the SDFs using the domestic and foreign Euler equations (5) and (6),

$$\begin{aligned} 0 = &-cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2} var_t(r_{p,t+1}) + cov_t(m^*_{t+1}, r^*_{p,t+1}) + \frac{1}{2} var_t(r^*_{p,t+1}) \\ &- cov_t(r^*_{p,t+1}, \Delta s_{t+1}). \end{aligned}$$

Remembering that both portfolio shocks are equal to ϵ^g_{t+1}, this expression simplifies to:

$$cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, \epsilon^g_{t+1}) = 0.$$

This equation is equivalent to

$$cov(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} - \widetilde{\Delta s}_{t+1}, \epsilon^g_{t+1}) = 0,$$

which under log-normality implies equation (10).

Because this result holds for any global shock, it must also hold in terms of multivariate projections on all global shocks $\boldsymbol{\epsilon}^g_{t+1}$. ■

C.2 Proof of Proposition 2

Consider the carry portfolio of Lemma 1 constructed with a pair of portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ which span the exchange rate (equation (9)). In this case, the portfolio has no risk because $\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} = \widetilde{\Delta s}_{t+1}$. The shocks to foreign and domestic return perfectly offset exchange rate risk. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. This corresponds to

$$0 = E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}).$$

The covariance term is equal to 0, because $r_{p,t+1} = r^*_{p,t+1} - \Delta s_{t+1}$ has no risk. We can replace expected returns using the domestic and foreign Euler equations (5) and (6):

$$\begin{aligned} E_t[\Delta s_{t+1}] = r_{ft} - cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2} var_t(r_{p,t+1}) \\ - r^*_{ft} + cov_t(m^*_{t+1}, r^*_{p,t+1}) + \frac{1}{2} var_t(r^*_{p,t+1}) = \delta_t \end{aligned}$$

We replace $\widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1} + \widetilde{\Delta s}_{t+1}$:

$$\begin{aligned} E_t[\Delta s_{t+1}] &= r_{ft} - r^*_{ft} - cov_t(m_{t+1}, \Delta s_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, r^*_{p,t+1}) \\ &\quad + \frac{1}{2} var_t(r^*_{p,t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) - \frac{1}{2} var_t(r^*_{p,t+1}) - cov_t(\Delta s_{t+1}, r^*_{p,t+1}) \\ &= r_{ft} - r^*_{ft} - cov_t(m_{t+1}, \Delta s_{t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) \\ &\quad + cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1}). \end{aligned}$$

This proves part b) of Proposition 2. If markets are fully integrated, all asset returns are global shocks, and proposition 1 implies that the last term in the equation above is equal to 0, part a) of the proposition. If the exchange rate is not spanned by asset returns, it is impossible to construct a trade with expected returns involving the expected depreciation rate that is risk-free. Therefore, no arbitrage imposes no restriction on the expected depreciation rate. ■

C.3 Proof of Proposition 3

Recall our decomposition of the depreciation rate into a spanned and unspanned components, $\Delta s_{t+1} = E_t(\Delta s_{t+1}) + g_{t+1} + \ell_{t+1} + u_{t+1}$. Because $g_{t+1} + \ell_{t+1}$ is spanned by asset returns, there exists $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ such that $\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} = g_{t+1} + \ell_{t+1}$. Using Lemma 1, we see that the risk of this portfolio is equal to $var_t(u_{t+1})$. We apply Assumption 3 to relate this risk to the expected return of the carry trade.

$$\left| E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) + \frac{1}{2} var_t(u_{t+1}) \right|$$
$$\leq B\sqrt{var_t(u_{t+1})}$$

Examining the terms in the left-hand-side, we have:

$$\begin{aligned} E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] &= \delta_t - E_t[\Delta s_{t+1}] = -\psi_t \\ cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) &= cov_t(-u_{t+1}, r_{p,t+1} + u_{t+1}) \\ &= -var_t(u_{t+1}) \end{aligned}$$

Plugging back into the inequality, we obtain:

$$|\psi_t + \frac{1}{2} var_t(u_{t+1})| \leq B\sqrt{var_t(u_{t+1})}.$$

■

D Propositions 1 and 2 are sufficient

We show that the results of Propositions 1 and 2 are not only necessary for the absence of international arbitrage — Assumption 2 — but also sufficient. Specifically we show the following.

Proposition 4. *If:*

1. *Assumption 1 holds,*
2. $E\left(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}|\epsilon^g_{t+1}\right) = E\left(\widetilde{\Delta s}_{t+1}|\epsilon^g_{t+1}\right)$,
3. (a) *Either* $\exists r^s_{p,t+1} \in \boldsymbol{r}_{p,t+1}, r^{s*}_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ *such that* $\widetilde{\Delta s}_{t+1} = \widetilde{r}^s_{p,t+1} - \widetilde{r}^{s*}_{p,t+1}$ *and*

$$E_t\left(\Delta s_{t+1}\right) = r_{f,t} - r^*_{f,t} - cov_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) + cov_t\left(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{p,t+1}\right),$$

   (b) *Or* $\forall r^s_{p,t+1} \in \boldsymbol{r}_{p,t+1}, r^{s*}_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}, \widetilde{\Delta s}_{t+1} \neq \widetilde{r}^s_{p,t+1} - \widetilde{r}^{s*}_{p,t+1}$

then there are no arbitrage opportunities in international markets, Assumption 2 holds.

Proof. We proceed by contradiction. Assume that there exists an international arbitrage:

$$\exists r^I_{p,t+1} \in \boldsymbol{r}^I_{p,t+1}, var_t(r^I_{p,t+1}) = 0 \text{ and } E_t\left(r^I_{p,t+1}\right) \neq r_{ft},$$

and denote $\boldsymbol{w}$ and $\boldsymbol{w}^*$ the set of weights of such a portfolio on $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. Remember that $1'\boldsymbol{w} + 1'\boldsymbol{w}^* = 1$. We consider the cases of 3a and 3b in turn.

Assume condition 3a holds. As a preliminary, note that this condition is equivalent to saying that a carry portfolio constructed with $r^s_{p,t+1}$ and $r^{s*}_{p,t+1}$ has no risk and no average return in excess of the risk-free rate. Consider the following portfolio: long $\boldsymbol{w}'\boldsymbol{r}_{t+1}$, long $(1'w^*)\, r^s_{p,t+1}$, long $w^{*\prime}\left(r^*_{t+1} + \Delta s_{t+1}\right)$, short $(1'w^*)\, r^{s*}_{p,t+1}$. Because we have added and subtracted the same total weights, the new weights still add up to 1, so this is still a portfolio. Because this portfolio combines two risk-free portfolio, our assumed arbitrage and the risk-free carry trade, its expected return is the sum of the two expected returns, $E_t\left(r^I_{p,t+1}\right)$. The total weight on foreign in the portfolio are $1'\boldsymbol{w}^* - 1'\boldsymbol{w}^* = 0$. Therefore, this trade is a differential carry portfolio. Because it has no risk, its home and foreign leg offset each other. They form a global shock. Applying condition 1 in the proposition and Lemma 2 leads immediately to the result that the portfolio return must equal the risk-free rate. This contradicts the assumption that $E_t\left(r^I_{p,t+1}\right) \neq r_{f,t}$.

Now assume that condition 3b holds. If $1'\boldsymbol{w}^* \neq 0$, then the arbitrage portfolio has a non-zero loading on Δs_{t+1} in addition to the home and foreign returns. Because the portofolio is riskless this implies that we can find a pair of home and foreign returns that spans the depreciation rate, a contradiction of condition 3b. If $1'\boldsymbol{w}^* = 0$, then the two legs of the portfolio in their home currency perfeclty offset each other. Their innovations constitute a global shock and applying condition 1 in the proposition jointly with Lemma 2 implies that the arbitrage portoflio has 0 expected return, a contradiction as well.

E Exact non-linear version of the propositions

Our proofs heavily rely on log-linearization of portfolio returns as described in Appendix A.1. In this section we address a question of how the propositions would change if the derivations are exact.

E.1 A version of Proposition 1

Consider two portfolios, domestic with returns $R_{p,t+1}$ and foreign with returns $R^*_{p,t+1}$ such that their innovations coincide with one of the global shocks, that is, they can be represented as $R_{p,t+1} = \alpha_t + R^*_{p,t+1}$. The local Euler equations imply:

$$\begin{aligned} E_t(M_{t+1}R_{p,t+1}) &= 1, \\ E_t(M^*_{t+1}R^*_{p,t+1}) &= 1. \end{aligned}$$

The local Euler equations can be re-written as

$$E_t(R_{p,t+1}) = R_{ft} - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1}\right) \tag{32}$$

$$E_t(R^*_{p,t+1}) = R^*_{ft} - cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}, R^*_{p,t+1}\right). \tag{33}$$

Now consider an intermediary whose SDF expressed in the units of domestic currency,

M^I_{t+1}, satisfies the following Euler equations:

$$E_t(M^I_{t+1}R_{ft}) = 1, \tag{34}$$
$$E_t(M^I_{t+1}R^*_{ft}S_{t+1}/S_t) = 1, \tag{35}$$
$$E_t(M^I_{t+1}(R_{p,t+1} - R_{ft})) = 0, \tag{36}$$
$$E_t(M^I_{t+1}(R^*_{p,t+1} - R^*_{ft})S_{t+1}/S_t) = 0. \tag{37}$$

The intermediary trades the zero-cost differential carry portfolio:

$$\begin{aligned}
0 &= E_t\left(M^I_{t+1}\left[(R_{p,t+1} - R_{ft}) - (R^*_{p,t+1} - R^*_{ft})\cdot S_{t+1}/S_t\right]\right) \\
&= E_t\left(M^I_{t+1}\left[(R_{p,t+1} - R_{ft}) - (R_{p,t+1} - \alpha_t - R^*_{ft})\cdot S_{t+1}/S_t\right]\right) \\
&= E_t\left(M^I_{t+1}\left[(R_{p,t+1} - R_{ft})(1 - S_{t+1}/S_t) + (\alpha_t + R^*_{ft} - R_{ft})\cdot S_{t+1}/S_t\right]\right).
\end{aligned}$$

Replace the risk-free rates by the expressions from the local Euler equations (32) and (33), divide the equation by $E_t(M^I_{t+1})$, and define

$$\begin{aligned}
cov^I_t\left(\frac{S_{t+1}}{S_t}, R^*_{p,t+1}\right) &\equiv E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}(R^*_{p,t+1} - R^*_{ft})\frac{S_{t+1}}{S_t}\right) \\
&\quad - \underbrace{E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}(R^*_{p,t+1} - R^*_{ft})\right)}_{0}\cdot E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}\frac{S_{t+1}}{S_t}\right), \\
E^I_t\left(\frac{S_{t+1}}{S_t}\right) &\equiv E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}\frac{S_{t+1}}{S_t}\right) = \frac{R_{ft}}{R^*_{ft}}.
\end{aligned}$$

Then

$$0 = -cov^I_t\left(\frac{S_{t+1}/S_t}{E^I_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) + cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})} - \frac{M_{t+1}}{E_t(M_{t+1})}, R^*_{p,t+1}\right).$$

(We replace $R_{p,t+1}$ with $R^*_{p,t+1}$ in the cov^I_t term because of our assumption about $R_{p,t+1}$ and $R^*_{p,t+1}$.)

This expression implies

$$\begin{aligned}
&cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}-\frac{M_{t+1}}{E_t(M_{t+1})}-\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)},R^*_{p,t+1}\right)\\
&=\underbrace{cov^I_t\left(\frac{S_{t+1}/S_t}{E^I_t(S_{t+1}/S_t)},R^*_{p,t+1}\right)-cov_t\left(\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)},R^*_{p,t+1}\right)}_{W} \qquad (38)
\end{aligned}$$

As we noted in section A.1, the log approximation that we use in Proposition 1 becomes exact if time is continuous and the data-generating process converges to a pure diffusion. Under such scenario, the covariance in the equation above is observable, and, thus, has the same value with and without risk adjustment (via M^I_{t+1}). As a result, $W=0$. Also, each Arrow-Debreu claim makes the corresponding state global. For such global risk $W=0$.

Further, the projection result depends on the knowledge of intermediary's SDF, M^I_{t+1} via the term with cov^I_t. The log approximation relies only on the existence of such SDF, due to Assumption 2, and allows us to be agnostic about its actual values.

The first term in the second line is equal to $R^*_{ft}\cdot QRP_t$, where QRP_t is the quanto-implied risk premium of Kremens and Martin (2019). Its role in our paper is different from that of these authors. They use it to approximate the currency risk premium assigned by the intermediary, $R^*_{ft}E_t(S_{t+1}/S_t)-R_{ft}=-R_{ft}cov_t(M^I_{t+1},R^*_{ft}\cdot S_{t+1}/S_t)$. Here it measures the gap in projections of the relative discount factor and the depreciation rate on global risks.

E.2 A version of Proposition 2

Consider two portfolios, domestic with returns $R_{p,t+1}$ and foreign with returns $R^*_{p,t+1}$ such that their innovations span the exchnage rate, that is, they can be represented as $R_{p,t+1}=\alpha_t+R^*_{p,t+1}S_{t+1}/S_t$. The local Euler equations (32) and (33) hold for these portfolios. Also, we consider a (domestically funded) intermediary whose SDF, M^I_{t+1}, satisfies the following Euler equations:

$$E_t(M^I_{t+1}R_{p,t+1})=1,$$
$$E_t(M^I_{t+1}R^*_{p,t+1}S_{t+1}/S_t)=1.$$

First, we show that $\alpha_t = 0$. The intermediary can form a zero-cost carry portfolio:

$$0 = E_t\left(M^I_{t+1}\left[R_{p,t+1} - R^*_{p,t+1}S_{t+1}/S_t\right]\right) = \alpha_t E_t\left(M^I_{t+1}\right).$$

Therefore, expected return on the carry portfolio is equal to zero:

$$\begin{aligned}
0 &= E_t\left(R_{p,t+1} - R^*_{p,t+1}S_{t+1}/S_t\right) \\
&= R_{ft} - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1}\right) \\
&- E_t(R^*_{p,t+1})E_t(S_{t+1}/S_t) - cov_t(R^*_{p,t+1}, S_{t+1}/S_t) \\
&= R_{ft} - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1}\right) - cov_t(R^*_{p,t+1}, S_{t+1}/S_t) \\
&- \left[R^*_{ft} - cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}, R^*_{p,t+1}\right)\right] E_t(S_{t+1}/S_t),
\end{aligned}$$

where we substituted the local Euler equations (32) and (33) in lines 2 and 5, respectively. This equation implies the currency risk premium:

$$\begin{aligned}
R^*_{ft}E_t\left(\frac{S_{t+1}}{S_t}\right) - R_{ft} &= -cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R^*_{p,t+1}\frac{S_{t+1}}{S_t}\right) \\
&+ cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}, R^*_{p,t+1}\right) E_t\left(\frac{S_{t+1}}{S_t}\right) - cov_t\left(R^*_{p,t+1}, \frac{S_{t+1}}{S_t}\right) \\
&= -R^*_{ft}cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, \frac{S_{t+1}}{S_t}\right) - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, [R^*_{p,t+1} - R^*_{ft}]\frac{S_{t+1}}{S_t}\right) \\
&+ cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}E_t\left(\frac{S_{t+1}}{S_t}\right) - \frac{S_{t+1}}{S_t}, R^*_{p,t+1}\right) \\
&= \underbrace{-R_{ft}cov_t\left(M_{t+1}, R^*_{ft}\frac{S_{t+1}}{S_t}\right)}_{\text{complete markets}} \\
&+ \underbrace{cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})} - \frac{M_{t+1}}{E_t(M_{t+1})} - \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) E_t\left(\frac{S_{t+1}}{S_t}\right)}_{A} \\
&- \underbrace{cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, [R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}}{S_t} - E_t\left(\frac{S_{t+1}}{S_t}\right)\right]\right)}_{B},
\end{aligned}$$

where in the first line we take advantage of spanning and replace $R_{p,t+1}$ with $R^*_{p,t+1}S_{t+1}/S_t$; the third line is obtained from the first by adding and subtracting the leading term in line 3; the fourth line is obtained by combining the two terms in the second line; the 6th and 7th lines are obtained by adding and subtracting $cov_t(M_{t+1}/E_t(M_{t+1}), R^*_{p,t+1})$.

The term B in the seventh line is the domestic household's risk premium for quanto exposure and disappears in the log-normal approximation. Also, $B = 0$ if $R^*_{p,t+1}$ happens to be R^*_{ft}, that is, domestic household can trade foreign risk-free bond. The term A in the sixth line is equal to zero in this case as well.

Next, if financial markets are integrated then the innovation to $R^*_{p,t+1}$ is global shock. Then, equation (38) from the non-linear version of Proposition 1 implies that

$$A = W \cdot E_t(S_{t+1}/S_t).$$

As is the case for Proposition 1, the log approximation treats this term as close to zero.

It might appear that departure from log-normality in the case of integrated markets leads to two extra terms, A and B. In fact, when markets are integrated $A - B$ can be simplified to a term with a single source of departures from zero. Indeed, we obtain

$$\begin{aligned}\frac{A-B}{E_t(S_{t+1}/S_t)} &= cov^I_t\left(\frac{S_{t+1}/S_t}{E^I_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) - cov_t\left(\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) \\ &- cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})} - 1, [R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1\right]\right) \\ &= E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}[R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1\right]\right) \\ &- E_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}[R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1\right]\right).\end{aligned}$$

Thus, $A - B$ is close to zero when the intermediary pricing the global (quanto) risk the same way as the domestic household.

If there is no spanning, $R_{p,t+1} \neq \alpha_t + R^*_{p,t+1}S_{t+1}/S_t$, then it is impossible to find a risk-free strategy and derive restrictions on the currency risk premium.

F An Example of a Financial Sector

In this section, we study exchange rate dynamics within a complete specification of the financial sector. The goal of this analysis is two-fold. First, it illustrates the second role of financial markets as a conduit of transmission of shocks to the exchange rate. Second, we show that this second role complements the risk-sharing role: there exists a foundations for what happens in international financial markets that can justify any exchange rate process satisfying Propositions 1 and 2.

F.1 Setting

We maintain Assumption 1 about local financial markets. That is, the two fixed sets of returns $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$ are priced by the household discount factors m_{t+1} and m^*_{t+1}. Then, we posit the remainder of international markets in such a way that Assumption 2 is satisifed.

First, we assume that there are noise traders with exogenous demand, D_t^{noise}, for the carry trade. That is, each period, they go long D_t^{noise} of the home currency in the foreign risk free asset and go short the same amount in the domestic risk-free asset.

Second, we assume that, there are overlaping generations of intermediaries in the market. Each period, an intermediary enters the market with wealth $W_{0,t}$ and lives until the next period. The intermediary maximizes its utility of next period wealth,

$$E_t\left(\frac{1}{1-\gamma_t}W_{1,t+1}^{1-\gamma_t}\right).$$

The coefficient of relative risk aversion γ_t can be viewed as a stand-in for various frictions limiting the risk-bearing capacity of the intermediary (see Haddad and Muir, 2021 for a discussion of this interpretation). The intermediary has access to all assets in I, and takes their returns as given in its optimization problem. We denote by D_t^I the optimal position of the intermediary in the carry trade.

The presence of these intermediaries guarantees that Assumption 2 is satisfied: if there were an arbitrage opportunity, the intermediary would choose an infinitely large position in the corresponding portfolio, which would be incompatible with being in equilibrium.

Third, we include a market-clearing condition for the carry trade. Positions of noise traders and intermediaries must offset the imbalance of positions between households, that is, the net foreign assets NFA_t:

$$D_t^{\text{noise}} + D_t^I + NFA_t = 0. \tag{39}$$

Because we take the household SDFs as given, we do not need additional market-clearing conditions for assets other than the carry trade. We assume they are provided perfectly elastically to the intermediary. This also explains why we do not need to specify the demand from noise traders for these assets.

Finally, because all these relations hold period by period, we need a restriction at infinite horizon to close the model. We focus on a generic form of such a restriction originating from the combination of the budget constraint and the transversality condition:

$$\Delta s_{t+1} - \delta_t + (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \beta^j \delta_{t+j}\right] = (E_{t+1} - E_t)\left[\sum_{j=0}^{\infty} \beta^j \Delta x_{t+j+1}\right]. \tag{40}$$

Appendix F.3 derives this restriction.

Here, $\beta < 1$ is a linearization constant and Δx_{t+j+1} is a change a real quantity (for example, relative TFP across countries). The right-hand-side of the restriction is a present value of innovations to this quantity, which we denote $N_{x,t+1}$. By definition, $N_{x,t+1}$ is unpredictable. An alternative approach to the asymptotic behavior of the exchange rate which some researchers have used is to impose stationarity of the exchange rate. This case is encompassed in equation (40) by assuming $\Delta x_{t+1} = 0$ for all t and taking the limiting case of $\beta \to 1$.

F.2 Which exchange rate processes can be rationalized by financial shocks?

Because this model respects Assumptions 1 and 2, any equilibrium exchange rate process must satisfy Propositions 1 and 2. We now show the converse: there exist values for the inputs of the model — noise trader demand, intermediary risk aversion — that justify any exchange rate process satisfying the two propositions.

Proposition 5. *For any sequence of shocks $\{\zeta_t\}$ such that:*

1. *$proj(\zeta_{t+1}|\epsilon^g_{t+1}) = proj\left(m^*_{t+1} - m_{t+1}|\epsilon^g_{t+1}\right)$, and*

2. *ζ_{t+1} is not spanned by asset returns,*

there exist processes for noise trader demand $\{D_t^{noise}\}$ and intermediary risk aversion $\{\gamma_t\}$ such that an equilibrium exchange rate satisfies:

$$\widetilde{\Delta s}_{t+1} = \zeta_{t+1}, \forall t. \tag{41}$$

We prove this result in three steps. First, because of condition 2 in Proposition 5, Proposition 2 does not impose any restrictions on the expected depreciation δ_t period by period. Yet, it has to satisfy the infinite-horizon restriction (40). Thus, we derive a process for δ_t which satisfies this restriction at infinity in order to complete the candidate exchange process from equation (41). Second, we derive the intermediary demand D_t^I as a function of the characteristics of the exchange rate process. Finally, we clear the market.

Candidate depreciation rate satisfying equation (40). We guess that exchange rate dynamics follow the following structure:

$$\Delta s_{t+1} = \delta_t + \zeta_{t+1}, \tag{42}$$

$$\delta_{t+1} = \rho\delta_t + \theta(\zeta_{t+1} - N_{x,t+1}), \tag{43}$$

where $0 \leq \rho < 1$ and θ are two fixed parameters to be chosen.

Iterating forward the auto-regressive dynamics of δ_t gives the innovation

$$(E_{t+1} - E_t)[\delta_{t+j}] = \rho^{j-1}\theta[\zeta_{t+1} - N_{x,t+1}]. \tag{44}$$

We can then compute the present value of these innovations:

$$(E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \beta^j \delta_{t+j}\right] = \sum_{j=1}^{\infty} \beta^j \rho^{j-1}\theta[\zeta_{t+1} - N_{x,t+1}] \tag{45}$$

$$= \beta\theta\frac{1}{1-\beta\rho}[\zeta_{t+1} - N_{x,t+1}]. \tag{46}$$

The restriction at infinite horizon of equation (40) is satisfied if:

$$\frac{\beta\theta}{1-\beta\rho} = -1 \tag{47}$$

$$\Leftrightarrow \theta = -\frac{1-\beta\rho}{\beta}, \tag{48}$$

which pins down θ as a function of ρ.

The parameter ρ is free, and controls the persistence of expected depreciation shocks. As β and ρ approach 1, this exchange rate process converges to a random walk. Therefore, it becomes difficult to distinguish it from a random walk in finite samples. Choosing value of ρ close to 1 will therefore generate empirically realistic dynamics.

Intermediary demand. We derive the optimal portfolio choice of the intermediary. We apply the Campbell and Viceira (2002) approximation, summarized in Appendix A.1, for this portfolio problem. The approximation reduces the intermediary optimization problem to:

$$\max_{r_{p,t+1} \in \boldsymbol{r}^I_{p,t+1}} E_t(r_{p,t+1}) + \frac{1}{2}(1-\gamma_t)var_t(r_{p,t+1}) \tag{49}$$

We represent returns $\boldsymbol{r}^I_{p,t+1}$ by the risk-free asset and a basis $\boldsymbol{r}^b_{t+1}$ with mean $E_t(\boldsymbol{r}^b_{t+1})$, and covariance matrix $\Sigma_{b,t}$. Then the optimal vector of portfolio weights on the risky assets $\boldsymbol{w}_t$ is:

$$\boldsymbol{w}_t = \frac{1}{\gamma_t}\Sigma^{-1}_{b,t}\left(E_t(\boldsymbol{r}^b_{t+1}) - r_{f,t+1}\boldsymbol{\iota} + \text{diag}(\Sigma_{b,t})/2\right) \tag{50}$$

If the first risky asset with return $r^b_{1,t}$ is orthogonal to the other risky assets, we simply have:

$$\boldsymbol{w}_{1,t} = \frac{1}{\gamma_t var_t(r^b_{1,t+1})}\left(E_t(r^b_{1,t+1}) - r_{f,t+1}\boldsymbol{\iota} + var_t(r^b_{1,t+1})/2\right). \tag{51}$$

Motivated by this result, we construct a basis $\boldsymbol{r}^b_{t+1}$ such that the first asset's is orthogonal to other assets, and the position in it corresponds to the position in the carry trade. First, we can represent both the domestic and foreign asset spaces using assets that load on the local and global shocks. For the home country, returns on such assets are:

$$\begin{cases} r_{f,t} + \boldsymbol{\epsilon}^g_{t+1} - cov_t(m_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) - var_t(\boldsymbol{\epsilon}^g_{t+1})/2, \\ r_{f,t} + \boldsymbol{\epsilon}_{t+1} - cov_t(m_{t+1}, \boldsymbol{\epsilon}_{t+1}) - var_t(\boldsymbol{\epsilon}_{t+1})/2, \end{cases} \tag{52}$$

using the log-normal Euler equation (5). For the foreign country, we similarly have:

$$\begin{cases} r^*_{f,t} + \boldsymbol{\epsilon}^g_{t+1} - cov_t(m^*_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) - var_t(\boldsymbol{\epsilon}^g_{t+1})/2, \\ r^*_{f,t} + \boldsymbol{\epsilon}^*_{t+1} - cov_t(m^*_{t+1}, \boldsymbol{\epsilon}^*_{t+1}) - var_t(\boldsymbol{\epsilon}^*_{t+1})/2, \end{cases} \tag{53}$$

using the Euler equation (6). We convert the corresponding excess returns and combine them with the home assets to get all international assets, remembering that we need to add the carry trade converting the foreign risk-free asset in domestic currency, obtained as an application of equation (29) from Lemma 1 to a pair of risk-free assets. This gives a basis:

$$\begin{cases} r_{f,t} + \boldsymbol{\epsilon}^g_{t+1} - cov_t(m_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) - var_t(\boldsymbol{\epsilon}^g_{t+1})/2 \\ r_{f,t} + \boldsymbol{\epsilon}_{t+1} - cov_t(m_{t+1}, \boldsymbol{\epsilon}_{t+1}) - var_t(\boldsymbol{\epsilon}_{t+1})/2 \\ r_{f,t} + \boldsymbol{\epsilon}^g_{t+1} - cov_t(m^*_{t+1} - \Delta s_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) - var_t(\boldsymbol{\epsilon}^g_{t+1})/2 \\ r_{f,t} + \boldsymbol{\epsilon}^*_{t+1} - cov_t(m^*_{t+1} - \Delta s_{t+1}, \boldsymbol{\epsilon}^*_{t+1}) - var_t(\boldsymbol{\epsilon}^*_{t+1})/2 \\ r_{f,t} + (r_{f,t} - r^*_{f,t}) - var_t(\Delta s_{t+1}) - \Delta s_{t+1}. \end{cases} \tag{54}$$

Two sets of assets are exposed to the globally traded shocks ϵ^g_{t+1}, but the first assumption in the theorem ensures that they are exactly equivalent: $cov_t(m_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) = cov_t(m^*_{t+1} - \Delta s_{t+1}, \boldsymbol{\epsilon}^g_{t+1})$. Furthermore, because the shock to the exchange rate is not spanned by the other returns, the last asset is not spanned by the previous ones. Yet, it covaries with the rest of them. To ensure that the first asset in the basis is independent of the shocks other than the ones unique to the carry exposure, we can hedge out the exposure to these other shocks. To do so, start by regressing ζ_{t+1} on the returns in the first four rows of (54):

$$\begin{aligned} \zeta_{t+1} &= \boldsymbol{\beta}^{g\prime}_t \boldsymbol{\epsilon}^g_{t+1} + \boldsymbol{\beta}'_t \boldsymbol{\epsilon}_{t+1} + \boldsymbol{\beta}^{*\prime}_t \boldsymbol{\epsilon}^*_{t+1} + u_{t+1}, \\ \text{with } u_{t+1} &\perp (\boldsymbol{\epsilon}^g_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1}) \end{aligned} \tag{55}$$

Then, construct an excess return that is long the carry portfolio, long $\boldsymbol{\beta}^g_t$ of the assets replicating $\boldsymbol{\epsilon}^g_{t+1}$, long $\boldsymbol{\beta}_t$ of the assets replicating $\boldsymbol{\epsilon}_{t+1}$, and long $\boldsymbol{\beta}^*_t$ of the assets replicating $\boldsymbol{\epsilon}^*_{t+1}$:

$$
\begin{aligned}
r_{\text{hcarry},t+1} - r_{f,t} = &(r_{f,t} - r^*_{f,t}) - var_t(\Delta s_{t+1}) - u_{t+1} - \delta_t \\
&+ \boldsymbol{\beta}^{g\prime}_t \left(-cov_t(m_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) - var_t(\boldsymbol{\epsilon}^g_{t+1})/2\right) \\
&+ \boldsymbol{\beta}^{\prime}_t \left(-cov_t(m_{t+1}, \boldsymbol{\epsilon}_{t+1}) - var_t(\boldsymbol{\epsilon}_{t+1})/2\right) \\
&+ \boldsymbol{\beta}^{*\prime}_t \left(-cov_t(m^*_{t+1} - \Delta s_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) - var_t(\boldsymbol{\epsilon}^*_{t+1})/2\right) \\
&+ \frac{1}{2}[\boldsymbol{\beta}^{g\prime}_t var_t(\boldsymbol{\epsilon}^g_{t+1}) + \boldsymbol{\beta}^{\prime} var_t(\boldsymbol{\epsilon}_{t+1}) + \boldsymbol{\beta}^{*\prime} var_t(\boldsymbol{\epsilon}^*_{t+1})] \\
&+ \frac{1}{2} var_t(\Delta s_{t+1}) - \frac{1}{2} var_t(u_{t+1}) \qquad (56) \\
= &(r_{f,t} - r^*_{f,t}) - \frac{1}{2} var_t(\Delta s_{t+1}) - \frac{1}{2} var_t(u_{t+1}) - u_{t+1} - \delta_t \\
&- \boldsymbol{\beta}^{g\prime}_t cov_t(m_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) - \boldsymbol{\beta}^{\prime}_t cov_t(m_{t+1}, \boldsymbol{\epsilon}_{t+1}) \\
&- \boldsymbol{\beta}^{*\prime}_t cov_t(m^*_{t+1} - \Delta s_{t+1}, \boldsymbol{\epsilon}^g_{t+1}).
\end{aligned}
$$

We can apply equation (51) to this new portfolio that is by construction orthogonal to all the other assets. This implies:

$$
w_{\text{hcarry},t} = \frac{(r_{f,t} - r^*_{f,t}) - \frac{1}{2} var_t(\Delta s_{t+1}) - \delta_t - \left(\boldsymbol{\beta}^{g\prime}_t cov_t(m_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) + \boldsymbol{\beta}^{\prime}_t cov_t(m_{t+1}, \boldsymbol{\epsilon}_{t+1})\right)}{\gamma_t var_t(u_{t+1})}. \qquad (57)
$$

Noticing that this portfolio has a weight of 1 on the carry trade and all other assets do not have exposure to it, the weight on the carry trade in terms of original assets is identical, $w_{\text{carry},t} = w_{\text{hcarry},t}$. The intermediary demand is $D^I_t = W_{0,t} w_{\text{carry},t}$.

Market clearing. To have an equilibrium, all that is required is to satisfy the market-clearing condition (39) each period. This corresponds to:

$$
\begin{aligned}
D^{\text{noise}}_t = &-NFA_t \\
&- W_{0,t} \frac{(r_{f,t} - r^*_{f,t}) - \frac{1}{2} var_t(\Delta s_{t+1}) - \delta_t - \left(\boldsymbol{\beta}^{g\prime}_t cov_t(m_{t+1}, \boldsymbol{\epsilon}^g_{t+1}) + \boldsymbol{\beta}^{\prime}_t cov_t(m_{t+1}, \boldsymbol{\epsilon}_{t+1})\right)}{\gamma_t var_t(u_{t+1})}.
\end{aligned} \qquad (58)
$$

All the terms of the right-hand-side only depend on the properties of ζ_{t+1} and primitives of the model excluding D^{noise}_t. This concludes the proof: if we assume that D^{noise}_t is equal to this right-hand-side expression, our conjectured exchange rate is an equilibrium.

F.3 Deriving the transversality condition

We show how to derive the condition of equation (40). The transversality condition together with the sequence budget constraint within a country often gives the following relation:

$$b_t + \sum_{j=0}^{\infty} \beta^j s_{t+j} - \sum_{j=0}^{\infty} \beta^j x_{t+1} = 0, \forall t \tag{59}$$

Taking a first difference, this leads to:

$$\Delta b_t + \sum_{j=0}^{\infty} \beta^j \Delta s_{t+j} - \sum_{j=0}^{\infty} \beta^j \Delta x_{t+1} = 0. \tag{60}$$

We can take the conditional expectation of this expression from the point of view of date t and from the point of view of date $t+1$, respectively:

$$\Delta b_t + \Delta s_t - \Delta x_t + \beta E_t(\Delta s_{t+1} - \Delta x_{t+1}) + \sum_{j=2}^{\infty} \beta^j E_t(\Delta s_{t+j} - \Delta x_{t+j}) = 0 \tag{61}$$

$$\Delta b_t + \Delta s_t - \Delta x_t + \beta \Delta s_{t+1} - \Delta x_{t+1} + \sum_{j=2}^{\infty} \beta^j E_t + 1(\Delta s_{t+j} - \Delta x_{t+j}) = 0 \tag{62}$$

Subtracting the second relation from the first one gives:

$$\beta \left[\Delta s_{t+1} - \delta x_{t+1} - E_t(\Delta s_{t+1} - \Delta x_{t+1})\right] + (E_{t+1} - E_t) \left[\sum_{j=2}^{\infty} \beta^j \Delta s_{t+j} - \Delta x_{t+j}\right] \tag{63}$$

Rearranging this expression leads to

$$(\Delta s_{t+1} - \delta_t) + (E_{t+1} - E_t) \left[\sum_{j=1}^{\infty} \beta^j \Delta_{t+j}\right] = (E_{t+1} - E_t) \left[\sum_{j=0}^{\infty} \beta^j \Delta x_{t+k}\right], \tag{64}$$

the condition of equation (40).

Table 3: Correlation between excess returns converted in different currencies: foreign stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Market	99.88	99.91	99.93	99.96	99.88	99.89	99.91	99.94	99.94
Value	99.92	99.94	99.93	99.96	99.89	99.85	99.92	99.93	99.94
Growth	99.82	99.88	99.93	99.96	99.9	99.93	99.92	99.95	99.94
Oil, Gas, Coal	99.89	99.93	NA	99.96	99.92	99.92	99.93	NA	99.96
Basic Material	99.84	99.94	99.94	99.95	99.88	99.91	99.91	99.96	99.91
Consumer Discretionary	99.91	99.95	99.93	99.96	99.92	99.94	99.94	99.93	99.96
Consumer Products, Services	99.88	99.96	99.97	99.95	NA	NA	99.94	99.93	99.98
Industrials	99.90	99.91	99.94	99.95	99.89	99.92	99.92	99.94	99.94
Health Care	99.91	99.97	99.96	99.96	NA	99.91	99.93	99.96	99.97
Financials	99.92	99.95	99.94	99.96	99.89	99.93	99.91	99.93	99.92
TeleCom	99.92	99.95	99.96	99.96	99.92	99.84	99.93	99.94	99.96
Technology	99.91	99.88	99.96	99.96	99.86	NA	99.94	99.95	99.95
Utilities	99.93	99.91	99.94	99.97	NA	99.93	NA	99.95	99.97

*N*otes: The table reports the correlation (in %) between the excess return on various stock indices expressed in their home currency and converted to U.S. dollar. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

G Evaluating the portfolio approximation

We report the correlation (in %) between the excess return on various stock portfolios —Table 3— and bonds of different maturities —Table 5— in their origin currency and converted to U.S. dollars. Tables 4 and 6 start from the U.S. version of these portfolios and converts them to foreign currency. These correlations are pervasively extremely high, almost all over 99.9%.

Table 4: Correlation between excess returns converted in different currencies: U.S. stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US Market	99.88	99.94	99.95	99.96	99.87	99.90	99.92	99.94	99.94
US Value	99.90	99.95	99.96	99.96	99.87	99.91	99.92	99.95	99.95
US Growth	99.87	99.93	99.94	99.96	99.88	99.90	99.92	99.94	99.94
US Oil, Gas, Coal	99.90	99.96	99.97	99.98	99.92	99.92	99.94	99.96	99.96
US Basic Material	99.81	99.90	99.92	99.95	99.85	99.88	99.90	99.93	99.93
US Consumer Discretionary	99.91	99.95	99.95	99.96	99.9	99.91	99.92	99.95	99.95
US Consumer Products, Services	99.93	99.97	99.97	99.97	99.92	99.93	99.94	99.96	99.96
US Industrials	99.86	99.93	99.94	99.96	99.84	99.90	99.90	99.94	99.94
US Health Care	99.90	99.96	99.95	99.96	99.88	99.93	99.93	99.95	99.96
US Financials	99.91	99.95	99.95	99.94	99.87	99.93	99.91	99.92	99.94
US TeleCom	99.87	99.93	99.95	99.95	99.9	99.91	99.93	99.96	99.95
US Technology	99.88	99.93	99.94	99.96	99.89	99.91	99.92	99.94	99.94
US Utilities	99.84	99.92	99.94	99.96	99.85	99.88	99.91	99.96	99.94

*N*otes: The table reports the correlation (in %) between the excess return on various stock indices expressed in the U.S. dollars and converted to foreign currency. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

Table 5: Correlation between excess returns converted in different currencies: foreign bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
2Y Bond	99.86	99.97	99.92	99.97	NA	99.85	99.91	99.91	99.95
3Y Bond	99.86	99.97	99.92	99.97	99.91	NA	NA	99.93	99.96
4Y Bond	NA	99.97	99.93	99.97	NA	NA	NA	99.94	99.96
5Y Bond	99.87	99.97	99.93	99.97	99.91	99.85	99.91	99.93	99.96
6Y Bond	NA	99.96	99.93	99.97	NA	NA	NA	99.92	99.96
7Y Bond	NA	99.96	99.93	99.96	NA	NA	99.91	99.91	99.96
8Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.90	99.96
9Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.89	99.96
10Y Bond	99.87	99.96	99.93	99.96	99.91	99.88	99.91	99.88	99.96

*N*otes: The table reports the correlation (in %) between the excess return on government bonds of different maturity expressed in their home currency and converted to U.S. dollars. Bond returns are constructed from yields obtained from each country's central bank. Each column corresponds to a different country.

Table 6: Correlation between excess returns converted in different currencies: U.S. bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US 2Y Bond	99.9	99.95	99.95	99.97	99.91	99.93	99.95	99.93	99.96
US 3Y Bond	99.91	99.96	99.95	99.97	99.92	99.93	99.95	99.92	99.96
US 4Y Bond	99.92	99.96	99.94	99.96	99.92	99.94	99.95	99.91	99.96
US 5Y Bond	99.91	99.97	99.93	99.96	99.91	99.94	99.95	99.89	99.95
US 6Y Bond	99.91	99.97	99.93	99.96	99.89	99.94	99.94	99.88	99.95
US 7Y Bond	99.9	99.96	99.92	99.96	99.88	99.94	99.94	99.86	99.95
US 8Y Bond	99.89	99.96	99.91	99.96	99.86	99.93	99.93	99.85	99.95
US 9Y Bond	99.88	99.96	99.9	99.96	99.85	99.93	99.93	99.84	99.95
US 10Y Bond	99.88	99.96	99.9	99.96	99.84	99.93	99.92	99.83	99.94

*N*otes: The table reports the correlation (in %) between the excess return on U.S. government bonds of different maturity expressed in U.S.. dollars and converted to foreign currency. Bond returns are constructed from yields obtained from the Federal Reserve. Each column corresponds to a different country.