Time Trumps Quantity in the Market for Lemons*

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January 3, 2024

Abstract

We consider a dynamic adverse selection model where privately informed sellers of divisible assets can choose how much of their asset to sell at each point in time to competitive buyers. With commitment, delay and lower quantities are equivalent ways to signal higher quality. Only the discounted quantity traded is pinned down in equilibrium. With spot contracts and observable past trades, there is a unique and fully separating path of trades in equilibrium. Irrespective of the horizon and the frequency of trades, the same welfare is attained by each seller type as in the commitment case. When trades can take place continuously over time, each type trades all of its assets at a unique point in time. Thus, only delay is used to signal higher quality. When past trades are not observable, the equilibrium only coincides with the one with public histories when trading can take place continuously over time.

*We thank Andres Almazan, Diego Moreno, Andy Skrzypacz, Tom Wiseman and Pavel Zryumov for comments and suggestions. We also thank seminar audiences at Bocconi, Durham, UC3M, UT Austin, Finance Theory Webinar and BSE Summer Forum and SAET. Moreira acknowledges FAPERJ and CNPq for financial support. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. Fuchs gratefully acknowledges the support from ERC Grant 681575, Grant PGC2018-096159-B-I00 financed by MCIN/ AEI/10.13039/501100011033, and Comunidad de Madrid (Spain), grant EPUC3M11 (V PRICIT) and grant H2019/HUM-5891. Gottardi acknowledges the support from the British Academy Grant SRG20-201337.

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1 Introduction

In adverse selection settings, it is well understood that the seller of an asset can use higher retained ownership or delay its trades to signal higher quality when only one of such instruments is available. How does signaling take place when both instruments are available? We show that with linear preferences, under long-term commitment to a path of trades, the only relevant consideration is the discounted value of the total quantity sold. Either delay or fractional trade (or a combination of both) can equivalently be used to achieve separation. Instead, when only spot contracts are allowed, in equilibrium delay is exclusively used to signal quality. Time trumps quantity as a signal because the irreversibility of time provides a form of endogenous commitment which cannot be achieved with retained ownership. Once I have waited for a given amount of time to sell my asset I cannot go back and sell it earlier. Instead, an implicit promise not to sell more of the asset at a future date can always be broken.

Our paper unites under one framework the literature on dynamic adverse selection with that of signalling with quantities. Concretely, we study a model with a privately informed seller of a perfectly divisible asset facing a competitive fringe of buyers who have a higher common value for the asset. The valuations for the asset are linear in quantity for both buyers and sellers. Trading can take place at a pre-specified set of times. We first consider a benchmark case in which there is commitment to future trades and exclusive dealings. We show that all that really matters is the discounted quantity sold and the average per period price obtained. Thus, although all equilibria satisfying the D1 refinement are separating and there is a unique present discounted total quantity $Q^*(c)$ sold by each seller type c in any equilibrium, the exact path of trades is not pinned down (see Theorem 1). Importantly, the value of $Q^*(c)$ and the welfare of each type of seller is the same regardless of the frequency with which sellers can trade. This is in contrast to the findings of Fuchs and Skrzypacz (2019) (henceforth FS) which study the case of an indivisible asset and show that the frequency of trade can have important welfare implications. The reason for the differing results is that, when the asset is non-divisible, the set of values that $Q(c)$ can take is determined by how frequently you can trade. With the cardinality of the set increasing as the interval between trades shrinks. With a restricted set of discounted quantities available to signal with, types cannot fully separate and some pooling ensues. Instead, with a fully divisible asset and commitment, the change in frequency of trade has no impact on the set of possible discounted quantities the seller can offer, and thus there are no welfare effects on equilibrium.

When trade occurs via exclusive spot contracts, thus with potential re-contracting at future dates, we must consider two cases: 1) Public Trading: all past trades by the seller are observable and thus prices at a given date depend not just on the current quantity offered but also on the observed history of trades and 2) Private Trading: buyers do not observe past trades and thus prices only depend on the quantity offered in the current period and calendar time.

With public trading, when the market is open at discrete times $0, \Delta, 2\Delta, \ldots, T$ there is a unique equilibrium, where each type of seller trades in at most two consecutive periods, exhausting its supply (except for those only trading at T). The equilibrium is again separating and, for all T and Δ, the present discounted quantity traded, $Q(c)$ coincides with $Q^*(c)$ from the commitment benchmark. Thus, sellers' welfare is also the same for any trading frequency (as with commitment and in contrast to the case where the asset is indivisible). As trading frequency increases, i.e., $\Delta \to 0$, we get that the measure of types trading at a given time goes to zero. That is, trade can be characterized in the continuous trading limit by a function $\tau(c)$ in which each type c trades its full supply at date $\tau(c)$ such that $Q^*(c) = exp(-r\tau(c))$, where $r > 0$ is the interest rate. Thus, time and not quantity is used to signal the quality of the seller's asset. Allowing for continuous trading we show there is no discountinuity in the limit.

With private trades, the frequency of trade is relevant again. While we can still sustain the same equilibrium as with public trading when agents can trade continuously, this is no longer true when we consider a set of discrete trading times. The reason is that when trading is continuous, when a type trades, it trades its full supply. Thus, on path, every type has either a full unit or nothing left. Instead, with discrete time trading, partial trades also need to be used to attain $Q^*(c)$. As a result, at any given trading time, with public trades, there will be two prices for any given quantity (one price for a seller that has not sold any quantity in the previous period and a different price for the seller that also traded in the previous period). With private trades this conditioning on past trades is not possible. Thus, the same equilibrium cannot be sustained. Note that the quantity traded at any given date is in principle a sufficiently rich instrument to screen sellers' types, but requires commitment (or the observability of trades at earlier dates); without commitment it proves ineffective. Hence the reason why the set of trading dates matters is quite different from the one pointed out by FS. Another way to understand our result is that we get an additional source of private information, the residual supply.

The shape of the residual supply, which evolves endogenously, is critical because it affects which trades a given type can enter. To see this, suppose the residual supply were strictly upward sloping at some given time. That is, higher quality types have strictly more of the asset to sell. In such a situation, trade could take place efficiently. Although low types would like to pretend to be higher types to get a better price, they do not have the necessary quantity to imitate the higher types. In general, when past trades are not observable the supply varies endogenously over time across types, at each date we have a market with multidimensional private information. Identifying which are the binding incentive constraints and characterizing the properties of the equilibrium with private trades in a discrete setting is therefore quite challenging. Despite this we are able to characterize the properties of equilibria when there are two trading dates (See Proposition 3). The equilibria with unobservable past trades may exhibit pooling and we show that welfare is not generally ranked across the two information regimes (See Corollary 2). This contrasts several findings in the literature which we discuss further below.

Beyond establishing an important difference between the strength of signalling via time vis a vis quantity our paper shows how many of the results in the literature are overturned once we jointly consider divisibility and dynamics. Furthermore, we uncover important interactions between the observability and frequency of trade. Our findings suggest that when designing exchanges or regulation, there are important additional considerations to account for.

Related literature

Our work intersects several strands of the literature on trade with adverse selection. First we have the dynamic models of a market for an indivisible asset that build on the static model of Akerlof (1970). Janssen and Roy (2002) presented the first analysis of the dynamic case and FS provided a comprehensive analysis of this case, discussed in the previous section. Second, and closely related are the models with an indivisible asset (or worker) which build on Spence (1973). This literature (which is focused on the discrete type case) is concerned with the assumption in Spence's model that the student can commit to go to school for some given amount of time. This is important because if only good types go to school (and assuming schooling is not productive), after the first day of class, firms would potentially want to make offers to this student. While Nöldeke and van Damme (1990) show that the commitment equilibrium can be implemented with public offers and uniquely so under a refinement, Swinkels (1999) shows that this no longer holds true when the offers are private. With private offers, the unique equilibrium has immediate trade with both types and thus is more efficient. Our benchmark commitment case (see Section 3) roughly corresponds to the analysis in Spence (1973). The analysis of spot trading with observable trades (see

Section 4) to that of Nöldeke and van Damme (1990) and that of private trades (see Section 5) to that of Swinkels (1999). Though there are three important differences in our setup: (1) In our setting there is more adverse selection so full trade at a pooling price is not possible.[1] (2) We consider an environment with a continuum of types.[2] (3) The asset in our case is fully divisible.

In our setting the payoffs of the commitment equilibrium are always obtained when trades are publicly observable, and, more surprisingly, also when they are privately observable provided trade is continuous. Additional contributions to the public vs. private offer case analysis include Hörner and Vieille (2009) and Fuchs et al. (2016). The latter shows that private offers are more efficient since buyers will be more willing to make higher offers when rejecting them cannot be used by sellers to obtain higher offers in the future. The privacy of previous traded quantities, which we study in this paper, works quite differently. It creates a two dimensional private information problem which, might lead to pooling when trades cannot take place continuously. Also, in contrast to Hörner and Vieille (2009), we can have that private trading induces less trade in equilibrium than public trading.

The dynamic bargaining literature for an indivisible durable asset is also related. In these models, the uninformed party starts by making unattractive offers and screens the informed party by only slowly improving the offers as time goes by. The time between offers plays an important role since it represents the power of commitment of the uninformed party. In the limit, as offers can be made continuously (i.e. no commitment power) the Coase conjecture forces kick-in (Coase (1972)) and the uninformed party loses any ability to profitably screen. When seller and buyer valuations of the asset are independent of each other, the limit of the unique stationary equilibrium exhibits immediate trade with all types pooling. Instead, with correlated values (as in this paper) there is slow screening over time.[3] Gerardi et al. (2022) study the role of asset divisibility in this context with two types. They show divisibility matters when there are decreasing gains from trade in the quantity traded of the asset, not when gains from trade are constant, the situation considered in this paper. We show that when the asset is divisible the commitment equilibrium is obtained also with spot trades, regardless of the frequency of trading (when trades are public).

The seminal work by Leland and Pyle (1977) showed in a static setting with a divisible asset that the seller can retain part of the asset to signal his higher type and obtain a higher price. This idea has lead to a large body of work in finance. In particular, it was extended by DeMarzo and Duffie (1999) to a security design problem, where the firm can choose not only the fraction of its cash-flow it retains, but how this is split in the different contingencies between the firm and outside investors. Yet, implicit in this literature is the idea that the owner is committed not to access the market again. As we show, this is very important since the temptation to sell the rest of the asset undermines the ability to signal via retention.[4]

There is also a literature that studies adverse selection in the context of directed search models (see Gale (1992) and Guerrieri et al. (2010)). These models show, in the context of markets for an indivisible object and a single trading date, that perfect separation occurs via the probability of trade associated to the different price offers made by buyers. High price markets attract few buyers relative to sellers and thus for a given seller there is a low probability of trade. Conversely, low price markets have many buyers relative to sellers so the sellers can sell

[1] This is a common assumption of the vast literature that builds on Akerlof (1970).

[2] There are some interesting considerations between the cardinality of the type space and that of the signalling space. Since we will be varying the signalling space we find it useful to fix the type space to be as rich as possible. In addition, this allows for a more direct comparison with FS and Leland and Pyle (1977).

[3] See Ausubel et al. (2002) for an excellent survey of the earlier literature and Fuchs and Skrzypacz (2022) for an overview of more recent work.

[4] See Li (2022) for some related work along these lines.

with very high probability.[5] Williams (2021) (see also Guerrieri et al. 2010) allows for divisibility of the asset in this environment and shows that separation occurs exclusively via probability of trade, not quantity traded. The rationale is based on the fact that using the trading probability allows to save on trading costs. Relative costs do not play a role in our result that only time of trade, not quantity traded, is used to separate seller types. Instead, the key factor we highlight is the differential commitment embedded in these instruments.

Finally, the unobservability of past trades introduces an element of non-exclusivity in the contracts being traded and may prevent perfect separation in equilibrium. In this regard our paper is also related to Attar et al. (2011), Kurlat (2016), Auster et al. (2021) and Asriyan and Vanasco (2021). While these papers focus on non-exclusivity within a period, we discuss non-exclusivity across periods. Thus, the time between periods can also be interpreted as a measure of commitment to exclusivity.

2 Model

There is a seller with a durable and divisible asset for sale. We index the seller type by the quality, $c \in [0,1]$, of the asset they own. The seller is fully informed about the quality of asset, while buyers are uninformed and only know that the quality of the asset is drawn from a distribution $F(c)$ with density $f(c) > 0$. Assets are perfectly divisible and the seller holds one unit of it.[6] An asset of quality c is worth c per unit to the seller and $v(c)$ to any potential buyer. We assume $v(c) > c$ for all $c \in [0,1)$, $v(1) = 1$, $v(\cdot)$ is continuously differentiable and $v'(c) > 0$. That is, there are strict gains from trade with all seller types with the exception of the highest and types are ordered so that higher types have assets that are better for both sellers and the buyers.[7]

Buyers and sellers can trade at an exogenously given set of trading dates, given by $\varsigma \subseteq [0,\infty)$, at which the market is open. We use this notation to be able to capture different possibilities, from one shot, static trade $\varsigma = \{0\}$ to continuous trading $\varsigma = [0,\infty)$. At each trading date $t \in \varsigma$ two or more buyers are present. Each buyer is only active in the market at a single trading date.[8] A seller's strategy (defined more carefully later) is a choice over the path of trades to carry out.

All players are risk neutral, have linear preferences over the quantity of the asset they hold and discount the benefits and costs of future trades at the rate r. For the seller this means that, if she expects to sell an amount $q(t)$ at date t with per unit price $p(t)$, her utility from that trade, in present expected terms, is:

$$e^{-rt}q(t)\left(p(t) - c\right).$$

Similarly, the utility of buyers for such trade would be:

$$q(t)E\left[v(c) - p(t)\right],$$

where the expectation is taken according to the buyers' belief described below.

[5] The properties of the dynamics of trade obtained in random search models with adverse selection are shaped by rather different forces, given by the interaction between bargaining and competition (see, e.g., Moreno and Wooders (2016)).

[6] We consider the case where sellers are endowed with different amounts of assets in Section 6.

[7] The assumption $v(1) = 1$ can be easily relaxed. The fact that the highest type has no benefits from trade helps to simplify the analysis.

[8] This assumption only allows to simplify notation when there is commitment to future deliveries or past trades are observable. When instead past trades are not observable it precludes buyers from conditioning their offers on their own private trade histories. See Lee (2023) for a study of the case where buyers are long lived in a simpler setting.

We will study different market arrangements in terms of information available to buyers over past trades and the seller's ability to commit to future trades. Formal equilibrium definitions for each case will be provided in the sections below.

In particular, we will consider the following cases:

(i) *Commitment to long term contracts*: at time 0 the seller offers an exclusive contract that precludes any further trades and specifies the amount of the asset to be transferred from the seller to the buyer at every date $t \in \varsigma$. Buyers compete by making price offers to enter the contract.

(ii) *Short term (spot) contracts with observable trades*: at every trading date $t \in \varsigma$ the seller decides how much of its residual supply to offer for sale at t, buyers' offers for this quantity can be conditioned on all previous trading activity i.e. on all past trades and rejected offers.

(iii) *Short term (spot) contracts with unobservable trades*: at every trading date $t \in \varsigma$ the seller decides how much of its residual supply to offer for sale at t and buyers' offers for this quantity can be only conditioned on the current quantity offered and calendar time.

3 A Benchmark: Long term Contracts with Commitment

If traders at $t = 0$ can fully commit to long term contracts and cannot trade additional contracts at subsequent dates after entering a contract, then effectively we can study the problem as one in which contracts are only traded at $t = 0$. All transactions (current and future) are determined by the contract entered at the initial date. Let us denote by ω a generic contract parties can enter at $t = 0$. We say the contract is *feasible* for a seller if it specifies a total amount to be delivered that is compatible with the seller's endowment: $\int_{t\in\varsigma} dq(t;\omega) \leq 1$, where the integral is with respect to the positive measure $q(\cdot;\omega)$ on ς. We assume the seller can only offer feasible contracts. Let Ω denote the space of feasible contracts.

Given the linearity of preferences, we can focus without loss of generality on the case where the price of a generic contract ω is paid by the buyer at the initial date $t = 0$. Regarding the transactions that are prescribed, all that matters for the utility of the seller and the buyers is the total discounted amount to be delivered $Q(\omega) = \int_{t\in\varsigma} e^{-rt} dq(t;\omega)$. Each of the i buyers (of which there are $N \geq 2$), observes the offered contract ω and makes a bid. Given the buyers' bidding strategies $P_i(\omega)$, let $P(\omega) = \max_i P_i(\omega)$ denote the highest price, in per unit terms, the seller expects to receive from offering contract ω. The utility gain of entering contract ω at unit price $P(\omega)$ for a seller of type c, $W(c,\omega)$, is then:

$$Q(\omega)\left(P(\omega) - c\right) = W(c,\omega).$$

Let $U(c)$ denote the maximal payoff attainable by a type c seller when trading in the market, given the prices implied by the buyers' strategies.

$$U(c) = \max_{\omega\in\Omega} W(c,\omega). \tag{1}$$

In equilibrium, given her type, the seller chooses to offer a contract $\omega(c)$, buyers observe the offered contract ω, update their beliefs, and make offers $P_i(\omega)$. The seller then accepts the highest offer $P(w)$ or rejects them all. It is worth noting that the fact that the seller is assumed to move first, makes this a signalling rather than a screening model and helps us avoid the existence problems in competitive screening models first pointed out by Rothschild and Stiglitz (1976) and discussed in Section 4.2 of Attar et al. (2009) for the divisible case with linear preferences and a continuum of types. Of course, as is well known, signalling games can have multiple equilibria if we allow for arbitrary off-path beliefs. Therefore, we will restrict off-path beliefs with an adaptation of the D1 refinement introduced by Cho and Kreps (1987) for a setting with a continuum of types proposed by Ramey (1996).[9] Since throughout we are only interested in equilibria that satisfy this refinement we will include the restrictions of off-path beliefs directly as part of our equilibrium definitions. More formally:

Definition 1 (Sequential Equilibrium with Commitment) *A sequential equilibrium satisfying D1 in the market with commitment is given by: a choice of contract to offer by each seller type, $\omega^*(c)$, a system of buyers' beliefs specifying for each contract $\omega \in \Omega$ and each seller type c the probability measure $\mu(c;\omega)$ that contract ω is offered by type c, a bidding strategy for each buyer $P_i(\omega)$ and an acceptance rule regarding the highest offer received $P(\omega)$,*[10] *such that:*
(i) Sellers optimize: $\omega^*(c) \in \arg\max_{\omega\in\Omega} W(c,\omega)$.
(ii) On-path beliefs: Given $\omega^*(c)$, let Ω^a denote the set of contracts offered by some type on path. Then, for all $\omega \in \Omega^a$ buyers' beliefs are formed using Bayes rule.[11]
(iii) Off-path beliefs: for each contract not offered on path $\omega \in \Omega/\Omega^a$, buyers are restricted to place zero posterior weight on a type c whenever for any buyers' response $P(\omega)$ there exists another type c' that has a stronger incentive to deviate from equilibrium, in the sense that type c' would strictly prefer to deviate for any $P(\omega)$ that would give type c only a weak incentive to deviate.
(iv) Taking other buyers' strategies as given, each buyer i makes an offer $P_i(\omega)$ to maximize his expected payoff conditional on his beliefs.

It is worth noting that the specification of the off-path beliefs in *(iii)* is effectively similar to the conditions imposed in competitive models with adverse selection, such as Azevedo and Gottlieb (2017), Bisin and Gottardi (2006), Dubey and Geanakoplos (2002), as well as in competitive search equilibria, see Guerrieri et al. (2010) and Eeckhout and Kircher (2010). In line with this literature, in a working paper version of our paper we derived essentially the same results using a competitive equilibrium notion where buyers and sellers take contracts and prices as given.

Given $\omega^*(c)$, let $K(\omega)$ denote the set of types that offer contract ω.

Lemma 1 *Any price $P(\omega)$ that is accepted with positive probability must be such that the buyer breaks even: $E[v(c)\,|c \in K(\omega)] = P(\omega)$.*

We characterize next the equilibria in this case, which constitutes an important benchmark result.

Theorem 1 *When sellers can commit to long term contracts, for all ς, all sequential equilibria that satisfy D1 are perfectly separating: that is, for any contract that is offered in equilibrium, there is a unique seller type who offers it. The utility level attained by each seller type c, $U^*(c)$, is the same across all these equilibria, and so is the total discounted quantity traded by this type, $Q^*(\omega^*(c))$. More specifically, we have:*

$$Q^*(\omega^*(c)) = \exp\left[-\int_0^c \frac{v'(x)}{v(x)-x}dx\right]$$

[9] If we considered a discrete set of types then we could instead use the intuitive criterion to restrict off-equilibrium beliefs and it would also lead us to select the least cost separating equilibrium (Riley 1979).

[10] We write for simplicity the definition for the case where each type of seller chooses one contract, though in equilibrium it is possible that sellers of a given type select different contracts, when they are indifferent among them. This is not relevant with a continuum of types.

[11] When a zero measure set of types picks a given ω, and thus Bayes rule cannot be not directly applied, we only require that the support of the beliefs be a subset of said types.

strictly decreasing in c, and the per-unit equilibrium price is $P^*(\omega(c)) = v(c)$, for all c.

The formal proof can be found in the Appendix. The idea is the following. The single crossing property implies that if two types c' and c'' found optimal to offer the same contract ω then all types $c \in [c', c'']$ must also optimally choose contract ω. Then one can show by contradiction that there cannot be any pooling in equilibrium. This follows since the highest type in the pool $\tilde{c}$, would find it attractive to deviate and trade a slightly lower quantity. If this alternative quantity is on-path, by single-crossing it must be traded by a seller of a higher type than that of any seller in the pool and hence its price must be strictly higher given the buyers' must break even condition, so there would be a profitable deviation for the $\tilde{c}$ seller. Similarly, if this quantity were off-path, since for any P higher types loose less from trading less, condition (iii) implies that the off-path belief must be that the quantity is traded by a type equal or higher than $\tilde{c}$. Thus, given such belief, in equilibrium its price must again be strictly higher than that of the pool and thus there would exist a profitable deviation for type $\tilde{c}$.

Once we know the equilibrium is separating, we can solve for the traded quantities by considering the incentive compatibility constraints in a mechanism where we ask the seller to report its type and promise a quantity and unit price as a function of said report (see Mailath (1987)). Prices are pinned down by the breakeven condition for buyers. From this we obtain a differential equation for $Q(\tilde{c})$ that guarantees that sellers do not want to misreport their types. To pin it down we require an initial condition. Since there are strict gains from trade and type 0 cannot be belived to be anything lower than 0, it must be that this type trades efficiently i.e. $Q(0) = 1$. In any separating equilibrium in which it would be supposed to trade less, it can benefit from deviating to trading its full unit. Note that this is also the least cost separating equilibrium (see Riley (1979)) which was the focus of the analysis of Leland and Pyle (1977). Solving the differential equation we get $Q^*(\omega^*(c))$ which we will denote simply by $Q^*(c)$.

Remark 1 *Note that the equilibrium only pins down the value of $Q^*(c)$, but any contract specifying a sequence of trades such that the total discounted amount is equal to $Q^*(c)$ constitutes a contract that may be traded by type c in equilibrium. Hence the amount $Q^*(c)$ could be obtained by trading the full unit at a specific time or a smaller amount only at $t = 0$. With long-term commitment there is no sense in which screening via different quantities traded by different types at a given date or via the same quantities traded by various types at different dates prevails. We should add that, if we also allowed for contracts with stochastic deliveries, probabilities of trade would also yield the same properties as they could be used to generate the same discounted, expected amount of total trades. In search models, the probability of trade induced by market tightness indeed plays a similar role to delay or partial trade. The equivalence also relies on all types having the same supply. We discuss this further in Section 6.*

Example 1 *To further illustrate Remark 1, consider the case in which $v(c) = 1/2 + c/2$. In this case the equilibrium level of total discounted trades of an arbitrary seller type c is:*

$$Q^*(c)) = \exp\left[-\int_0^c \frac{v'(x)}{v(x) - x} dx\right] = 1 - c.$$

Thus, for any ς we have an equilibrium allocation as in Leland and Pyle (1977) where there is only trade in the first trading date and all the separation is via quantities/retention: $q(0; \omega^*_{LP}(c)) = 1 - c$ and $q(t; \omega^*_{LP}(c)) = 0$. Letting $\varsigma = [0, \infty)$ we can also have a payoff equivalent equilibrium allocation in which all types sell their full unit when they trade, and as in FS separation is achieved because they trade at different times: $q(t; \omega^*_{FS}(c)) = 1 \quad if \quad e^{-rt} = 1 - c$ and 0 otherwise.

We could also have cases in which both time and quantity are used. Letting $\varsigma = \{0, \Delta\}$, we can have:

$$q(0;\omega^*(c)) = \begin{cases} 1 - \frac{c}{1-e^{-r\Delta}} & if \quad c \leq e^{-r\Delta} \\ 0 & if \quad c > e^{-r\Delta} \end{cases}$$

$$q(\Delta;\omega^*(c)) = \begin{cases} \frac{c}{1-e^{-r\Delta}} & if \quad c \leq e^{-r\Delta} \\ \frac{1-c}{1-e^{-r\Delta}} & if \quad c > e^{-r\Delta} \end{cases}.$$

Note in particular that in the last specification we have two contracts that prescribe the delivery of the same quantity at $t = \Delta$ but have different (unit) prices since they feature the delivery of different quantities at $t = 0$. A similar property can obtain even without commitment as long as past trades are observable, as we will show next.[12] *Instead, when past trades are not observable this will no longer be possible. This case will be analyzed in Section 5.*

4 Spot Contracts with Observable Trades

We turn next to the case in which sellers are unable to commit to their future transactions. Buyers at any given time t can observe and condition their bids on all the seller's previous trades. We first analyze this case in discrete time. That is, we restrict ς to be $\varsigma = \{0, \Delta, 2\Delta, .., T\}$, where $T \leq \infty$. It will be also convenient to use $q_t \in [0, 1]$ to indicate the quantity offered for trade at time t.[13] The important difference with respect to the commitment case is that we need to specify a price for each feasible q_t for all $t \in \varsigma$. Any quantity offered at t must be feasible, i.e. not greater than the residual supply of a seller given all the previous trades. Sellers can only offer one q_t at a given t. Importantly, as we said, the per unit bids at time t, in addition to the current quantity q_t, are allowed to depend on all the past history of offered quantities $q^{t-\Delta} = \{q_0, q_\Delta, ...q_{t-\Delta}\}$, the history of the max of the buyers' bids which we denote $p^{t-\Delta} = \{p_0, p_\Delta, ...p_{t-\Delta}\}$ and the history of acceptance decisions encoded by $a^{t-\Delta} = \{a_0, a_\Delta, ...a_{t-\Delta}\}$. Together they constitute the trading history in the observable case. We denote it by $h^{t-\Delta} = \left\{q^{t-\Delta}, p^{t-\Delta}, a^{t-\Delta}\right\}$ for $t > 0$ with $h^{t-\Delta} = \varnothing$ for $t = 0$. We will denote buyer $i's$ bidding strategy by $p_i\left(h^{t-\Delta}, q_t\right)$, and let $p\left(h^{t-\Delta}, q_t\right)$ denote the highest offer for quantity q_t after history $h^{t-\Delta}$.[14]

Since sellers cannot commit to a sequence of trades, their quantity choice and their acceptance decisions must be optimal at every t given $h^{t-\Delta}$. We must therefore consider the whole sequence of offered quantities $\{q_t\}_{t\in\varsigma}$. At all times t, after history $h^{t-\Delta}$, given the buyers' strategies, the problem of a type c seller, can be written recursively as follows:

$$U\left(c, h^{t-\Delta}\right) = \max_{q_t, a_t} a_t\left(h^{t-\Delta}, q_t, p\left(h^{t-\Delta}, q_t\right)\right) q_t\left(p\left(h^{t-\Delta}, q_t\right) - c\right) + e^{-r\Delta} U\left(c, h^t\right) \tag{2}$$

$$s.t. \quad \sum_{s=0}^{t} q_s \leq 1$$

where q_t and $a_t\left(h^{t-\Delta}, q_t, p\left(h^{t-\Delta}, q_t\right)\right)$ denote the seller's offered quantity and acceptance strategy. We assume that there is an infinitesimal opportunity cost of offering a positive quantity for sale. Thus, sellers would only offer $q_t > 0$ if they plan to accept the offer they expect to receive given $p\left(h^{t-\Delta}, q_t\right)$. This allows us to use a more efficient notation for on-path behavior since the seller's acceptance strategy $a_t\left(h^{t-\Delta}, q_t, p\left(h^{t-\Delta}, q_t\right)\right)$ will always be 1 and we can ignore some irrelevant source of multiplicity that does not affect equilibrium trades. In addition,

[12] Without commitment, the seller is still free to choose its trades at future times $s > t$. Thus, the key difference is that prices at t can be conditioned on past quantities but not conditioned on future quantities.

[13] For most results we do not need trades to be equally spaced but this assumption helps us to reduce notation.

[14] No additional equilibria arise if we allowed for mixed strategies by buyers. Thus, for convenience, we will not develop explicit notation to account for them.

this makes the problem more clearly a signalling problem in which there is meaning in the offered quantity. To further simplify our notation we will use p_t as a shorthand for $p\left(h^{t-\Delta}, q_t\right)$ when convenient. Let $q_t^*(c)$ denote the on-path optimal quantity choice of type c at time t, after the sequence of trades $h^{t-\Delta}$. Let $K\left(h^{t-\Delta}, q_t\right)$ denote the set of types that optimally choose to offer quantity q_t after history $h^{t-\Delta}$ on-path.

Definition 2 (Equilibrium with Observable Trades) *Given ς, a sequential equilibrium that satisfies D1 in the market with observable trades is given by: an offer strategy for the seller $q_t\left(h^{t-\Delta}\right)$, a system of buyers' beliefs specifying for each $\left\{h^{t-\Delta}, q_t\right\}$ a probability measure over the types offering q_t, a bidding strategy for each buyer $p_i\left(h^{t-\Delta}, q^t\right)$ and an acceptance strategy for the seller $a\left(h^{t-\Delta}, q_t, p_t\right)$ for each $h^{t-\Delta}$ and feasibly offered q_t at every $t \in \varsigma$, such that:*
(i) Sellers optimize: (2) holds, for all c and at every $h^{t-\Delta}$;
(ii) For all on-path offered quantities, buyers' beliefs are determined by Bayes rule[15]*;*
(iii) Off-path beliefs: for all $\left\{h^{t-\Delta}, q_t\right\}$ such that $K\left(h^{t-\Delta}, q_t\right) = \emptyset$, the buyers are restricted to place zero posterior weight on a type c whenever for a given buyers' response there exists another type c' that has a stronger incentive to deviate from equilibrium, in the sense that type c' would strictly prefer to deviate for any buyer response that would give type c only a weak incentive to deviate. We assume that if buyers at time t deviate, future buyers' beliefs can only have positive support on types $c \in K\left(h^{t-\Delta}, q_t\right)$.
(iv) Taking the other buyers' and the seller's strategies as given, each buyer i present at t makes an offer $p_i\left(h^{t-\Delta}, q_t\right)$ to maximize his expected payoff conditional on his beliefs.

It is important to note that in this dynamic setting the use of the D1 refinement (iii) must not only contemplate the current off-path q_t but also the subsequent off-path offered quantities to follow. When determining the off-path price at t, the buyers' belief's about such future trades will clearly play a role. Thus, it is useful to note that, in terms of discounted quantities, there are two important cases, one in which the seller is expected to sell the rest of its residual supply at the next trading period and thus maximize the amount traded; and the other in which the seller is never expected to trade again, thus minimizing the amount traded. Given the single crossing property, the former case is particularly relevant since it provides the worst case scenario from the seller's perspective in terms of buyers' beliefs. That is, the worst the buyers' can possibly believe about the seller is that the seller would try to sell all its residual supply the next trading round. Since when trying to construct an equilibrium it is useful to place very negative beliefs for off-path actions, if we cannot sustain an equilibrium even under the conjectured belief that the seller would trade the remaining supply the next period, there is no other off-equilibrium conjectured path of future trades that would sustain it. Thus, if we find a unique equilibrium under these beliefs, then we know that there cannot be any other equilibrium.

In the dynamic setting another issue is that, for a given history $h^{t-\Delta}$, on-path, some types would not have the required residual supply to offer the off-path quantity q_t. Yet, we allow buyers to "question" the whole path and thus consider the whole set of sellers' types when considering who might have deviated. This would generally allow for harsher off-path beliefs which facilitates multiplicity of equilibria. Thus, our choice, strengthens our uniqueness results. Our results would also hold if, when forming beliefs after a seller deviation, buyers only considered the subset of types that have the required residual supply on-path. Lastly, note that the last part of (iii) implies that if after a given history $\left\{h^{t-\Delta}, q_t\right\}$ buyers at time t believe they are facing a unique type c, but they make off-equilibrium offers, then future buyers would not change their beliefs about the seller if the seller rejected p_t.

[15] When a zero measure set of types picks a given Q, and thus Bayes rule cannot be not directly applied we only require that the support of the beliefs must be a subset of said types.

4.1 Characterization of Equilibria

First, we show that buyers must break even on separating trades.

Lemma 2 *For any T and $\Delta > 0$, there exists a sufficiently large number of buyers $N > 0$ such that every on-path (or off-path) separating trade must satisfy the buyers break-even condition.*

As noted in Nöldeke and van Damme (1990), even for finite T, the zero-profit result might be more subtle to establish when a set of types pools on a given quantity. For our analysis it is sufficient to establish that zero-profit holds for separating offers. For this case, standard Bertrand logic delivers the result. When when T is finite $N = 2$ suffices. When considering the $T = \infty$ case, we will assume that N is large enough to rule out collusive equilibria.[16]

We show next that, with observable trades, there is a unique equilibrium for any Δ and T. We begin by establishing two important properties. First, Lemma 3 shows that every individual trade must be separating and thus every trade is priced by the unique type entering that trade. Second, Lemma 4 shows that there is also full separation with regard to the present discounted value of the total quantity traded $Q(c)$.

Lemma 3 (Separation in q) *For every $\Delta > 0$ and T, in all observable trade equilibria, every on-path nonzero trade must be fully separating.*

This result highlights the main difference between time and quantity and is essential. The intuition is that, by choosing a path of trades where total discounted trades are lower and trades are postponed to the maximal extent possible, no lower type of seller would find it profitable to follow part of this path of trades. Thus, it is possible to credibly signal your type. More concretely, supposing that the pooling quantity was $Q(c'')$ the deviating trades would be:

$$\begin{aligned} e^{-r\tau} q_\tau + e^{-r(\tau+\Delta)}(1 - q_\tau) &= Q(c'') - \varepsilon \quad \text{if } \tau \leq T - \Delta \\ e^{-rT} q_T &= Q(c'') - \varepsilon \quad \text{otherwise} \end{aligned}$$

Importantly, this trading strategy is robust to negative beliefs. The worse that can be thought of you if you are selling q at t is that at $t + \Delta$ you will sell your residual supply. The seller is indeed following that strategy. Since the seller is using the time already past without trading, which is observable, instead of a simple promise not to sell in the future, this is always credible and buyers must be convinced that it is indeed a higher type. So, sellers can effectively signal their types and separate from others. As we will see below in Proposition 1 this same force shapes the trading pattern prevailing in the unique equilibrium which arises with this market arrangement.

Lemma 4 (Separation in Q) *For every $\Delta > 0$ and T, in all observable trade equilibria, the total discounted quantity $Q(c)$ traded by any type c must be strictly decreasing in c.*

To see this must be true not first that IC prevents $Q(c)$ from being increasing in c (since it would imply the seller can trade more and at a higher price by claiming to be a higher type). *Next*, since we showed that all trades are separating we know that $P(c) = v(c)$ but then $Q(c)$ cannot be constant. To see this, consider a set of types $[c', c'']$ with the same total discounted trades $\bar{Q}$. Then all of them would clearly want to choose the set of trades entered by type c'' which feature higher prices and thus would yield a strict improvement.

Using the two lemmas above we can now show there is a unique path of trades compatible with equilibrium in which, as we anticipated, each seller postpones its trades as much as possible.

[16] Note that $T = \infty$ does not play any special role for our results in this Section. Thus, we could work with a finite large T (or take a sequence as $T \to \infty$).

Proposition 1 (Uniqueness with Observable Trades) *For every $\Delta > 0$ and T there is a unique observable trade equilibrium. The equilibrium is fully separating, all sellers trade in at most two consecutive periods and, if a seller's first trade occurs before T, its total sales exhaust its supply. Furthermore, for all c the total discounted quantity $Q^*(c)$ and payoff $U^*(c)$ are the same as in the commitment equilibrium.*

The unique pattern of trades is given by the fact that if a seller c did not exhaust its supply in two consecutive periods, there would be a profitable deviation for a lower type $c' < c$. Suppose c traded for the first time at time t and amount $q_t(c)$. Then, let type c' be such that $q_t(c) + \delta(1 - q_t(c)) = Q^*(c')$. Type c' could immitate c at t receving a higher price than it otherwise would and it could trade the rest of its supply the next period at a fair price of $v(c')$.

It is worth pointing out that the equilibrium payoffs are independent of the values of Δ and T. This is in stark contrast to the non-divisible asset case studied by FS. With non-divisible assets, the set of attainable values of total discounted quantity traded Q is constrained to $\{0, e^{-r\Delta}, e^{-2r\Delta}, ..., e^{-Tr\Delta}\}$ and thus depends on Δ and T. Once we allow for q_t to take any value in $[0,1]$ this limitation is circumvented and the set of attainable values of Q is $[0,1]$, for all Δ and T.

To illustrate our results, consider again the situation considered in Example 1 where $v(c) = \frac{1+c}{2}$ and, as we saw, we have:

$$Q^*(c) = \exp\left[-\int_0^c \frac{v'(x)}{v(x) - x} dx\right] = 1 - c.$$

With two trading dates, $\varsigma = \{0, \Delta\}$, with observable trades the unique equilibrium pattern of trades is the following:

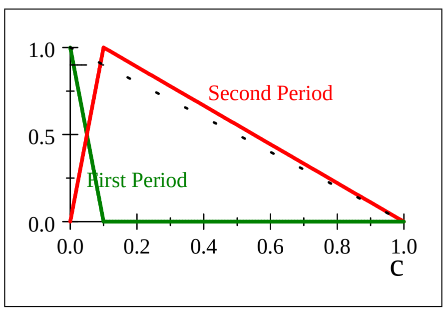

Figure 1

$$q_0(c) = \begin{cases} 1 - \frac{c}{1-e^{-r\Delta}} & c \leq 1 - e^{-r\Delta} \\ 0 & c > 1 - e^{-r\Delta} \end{cases}$$

$$q_\Delta(c) = \begin{cases} \frac{c}{1-e^{-r\Delta}} & c \leq 1 - e^{-r\Delta} \\ \frac{(1-c)}{e^{-r\Delta}} & c > 1 - e^{-r\Delta}. \end{cases}$$

When $e^{-r\Delta} = 0.9$, in the first period only types $c \in [0, 0.1]$ trade (as described by the green line in Figure 1). These types then trade their residual supply at $t = \Delta$ while types $c > 0.1$ only trade $\frac{1-c}{0.9}$ in the second period (see the red line). Importantly, at $t = \Delta$ there are two contracts being traded for any given quantity in $[0,1]$, one by types that had not traded in the first period which will command a high price and one by types that had traded already and which will be priced lower: $p(0,q) > p(1-q,q)$.[17] Note that, although we started with a uniform supply for all sellers, after $t = 0$ we endogenously get a non-uniform residual supply. This creates difficulties in the unobservable case, where we cannot condition prices on previous trades.

[17] As established in Proposition 1, this property is true for general T, Δ, for all $t > 0$.

Although for every $\Delta > 0$ all quantities $q \in [0,1]$ can be traded at every date t, in equilibrium each type trades at most twice and the measure of types trading at a given $t < T$ is decreasing in Δ. Indeed, as we show next, in the limit as Δ goes to 0 the measure of types trading at a given date $t < T$ goes to 0 and effectively only the time of trade is used by sellers to signal their type.

Theorem 2 *For $T = \infty$, the limit as $\Delta \to 0$ of the sequence of observable trade equilibrium allocations converges to an allocation where each type c trades all its supply at a unique and distinct moment in time $\tau(c)$, satisfying $e^{-r\tau(c)} = Q^*(c)$, where $Q^*(c)$ is the commitment equilibrium value.*

Note that the above result is also true for finite T. In this case, it is still true that every type trades only in one date, but only the seller types c such that $\tau(c) \leq T$ trade their entire supply. Every other type of seller (for whom $\tau(c) > T$) would trade at the final date T a different quantity $q_T(c) < 1$ such that $e^{-rT} q_T(c) = Q(c)$.

4.2 Equilibria with Continuous Trading

We show in this section that the result established in Theorem 2 can also be obtained if we work directly in the limit, with trading occurring in continuous time, i.e., $\varsigma = [0, \infty)$. Thus we can say there is no discontinuity in the limit as $\Delta \to 0$. When the market is continuously open to trade, conditional on trading at t, the subsequent dates at which a seller can trade is an open set. This poses some technical problems and so we will assume that conditional on trading at some t the seller must wait some amount of time $\lambda > 0$ before it can trade again. We are interested in particular in the limit as $\lambda \to 0$.

Note first that Lemmas 3 and 4 can easily be extended to this case. Indeed, the argument is cleaner since the seller does not need to split its trade over two periods and thus the buyers do not need to consider any future actions by the seller.

Thus, the equilibrium value of the total discounted quantity traded $Q(c)$ is still unique and coincides with the value when trading is discrete $Q^*(c)$. Indeed, there is an equilibrium with the same allocation as in the limit described in Theorem 2, in which each type trades its full unit at a unique moment in time $\tau(c)$ such that $e^{-r\tau(c)} = Q^*(c)$, with $Q^*(c)$ as in the commitment equilibrium.

Corollary 1 *When $\varsigma = [0, \infty)$ for any $\lambda > 0$, there exists an equilibrium where each seller type trades all its supply at a unique moment in time $\tau(c)$ satisfying: $e^{-r\tau(c)} = Q^*(c)$. This is also the limit of all equilibria when $\lambda \to 0$.*

Note however that for $\lambda > 0$ the equilibrium is not unique. In particular, there is also an equilibrium in which λ plays exactly the same role as Δ did before and the equilibrium is exactly as described in the previous section. Yet, as $\lambda \to 0$, Theorem 2 implies that all the equilibria concide and only time is used to signal. It is also worth pointing out that the equilibrium coincides with the one described by FS for the case of a non-divisible asset and continuous trading. Thus, in equilibrium, time trumps quantity as a way to signal when both instruments are available.

5 Spot Contracts with Unobservable Trades

As we observed in the example with two trading dates, in the equilibrium with observable trades, for the same spot quantity in the second period there will be two different bids depending on the previous period trade. Low prices when the seller also traded in the previous period and high prices when it had not. When past trades are not observable by buyers, this differentiation is not possible as prices can only be contingent on the spot quantity being offered. We show in this section (see Proposition 2) that with discrete trading such restriction makes it impossible, without observability, to replicate the equilibrium payoffs obtained with commitment as well as with public trading. The fact that past trades are unobservable introduces a second dimension of private information, the seller's residual supply. This makes an explicit construction of equilibria for arbitrary T and $\Delta > 0$ quite hard. For the case with only two trading dates, we can explicitly characterize an equilibrium, and show it features partial pooling.

When $T = \infty$ and $\Delta = 0$ we are able to establish the existence of an equilibrium and to characterize it. This equilibrium is separating and coincides with the equilibrium with observable trades and continuous trading described in Corollary 1, where each type trades its full unit at a single point in time. Thus, sellers can again use only the time of trade, not quantity to signal their type.

5.1 Equilibrium Definition

We present here a definition of equilibrium that allows for the possibility that trade occurs at any point in time, i.e., that $\varsigma = [0, \infty)$. As in Section 4.2, we maintain the assumption that the seller must wait at least $\lambda > 0$ between two trades and also that there is an infinitesimal per-unit cost of putting the goods for sale. This latter assumption is again notationally convenient in not having to worry about on-path rejected offers. Since, prices are independent of the history of the game it is convenient consider the seller's strategy for a given pricing path $p(q)$ which determines prices for all quantities at all dates. A seller's strategy can be described by a sequence of discrete trading dates and quantities traded $\{R(c,p), q(c,p)\}$ a such that $R(c) \subset \varsigma$ and $\sum_{t \in R} q(\cdot) \leq 1$. Given prices, we say a trading strategy $\{R(c), q(c)\}$ is optimal for type c if:

$$\sum_{t \in R(c)} e^{-rt} q_t(c)\left(p_t\left(q_t(c)\right) - c\right) \geq \sum_{t \in \tilde{R}} e^{-rt} \tilde{q}_t\left(p_t\left(\tilde{q}_t\right) - c\right) \tag{3}$$

for all admissible trading strategies $\left\{\tilde{R}, \tilde{q}\right\}$. The associated payoff is again denoted $U(c)$.

Let $\{R(c,p), q(c,p); c \in [0,1]\}$ denote a profile of trading strategies for all types. Given such a profile we can then define $K(q_t, t)$ as the set of types that on-path offer quantity q_t at time t. As before, we will restrict our attention to equilibria that satisfy the D1 refinement. This is now even more complex to apply than in the observable case. When buyers now observe an off-path quantity at time t when forming their belief not only they must consider potential future trades by the seller but, in addition, now they also must contemplate the possibility that the seller deviating at t might have also deviated in the past.

Definition 3 (Equilibrium with Unobservable Trades) *When $\varsigma \subseteq [0, \infty)$ an equilibrium with unobservable past trades that satisfies the D1 refinement is given by a profile of strategies $\{R(c,p), q(c,p); c \in [0,1]\}$, prices offers for each buyer present at t, $p_i(q,t)$ for all q and t and buyers' beliefs such that:*

(i) Sellers optimize: (3) *is satisfied, for all c.*

(ii) On-path beliefs: for q, t such that $K(q,t) \neq \varnothing$, buyers' beliefs are determined using Bayes' rule.

(iii) Off-path beliefs: for all q and t such that $K(q,t) = \varnothing$, the buyers are restricted to place zero posterior weight on a type c whenever there is another type c' that has a stronger incentive to deviate from equilibrium, in the sense that type c' would strictly prefer to deviate for any buyer response that would give type c only a weak incentive to deviate.

(iv) Taking other buyer's and the seller's strategies as given, each buyer i present at t makes an offer $p_i(q,t)$ to maximize his expected payoff conditional on his beliefs.

The common assumption made in the literature that new buyers are active in the market in every given period is important here. Otherwise, as shown by Lee (2023) in the context of a simpler setting, buyers would have an incentive to experiment and use mixed strategies. Instead, when only being active in one period, they face a completely static problem from their perspective since the continuation game does not depend on their actions. As a result, they compete a la Bertrand leading to no profits and on-path prices must satisfy:

$$p_i\left(q,t\right)=E\left[v\left(c\right)|c\in K\left(q,t\right)\right]$$

In general, this is not sufficient to guarantee an equilibrium since we need to verify that the buyers could not offer some $\tilde{p}>p_i\left(q,t\right)$ that would be profitable. That is, that the profit as a function of the offered price must be negative for all prices above the candidate equilibrium price.[18] This is satisfied since, at any given time, the seller first chooses the quantity *or* to stay out ($q_t=0$) and thus sellers cannot attract higher types by making an off-equilibrium higher offer. The analysis of private offer equilibria with an alternative timing (more akin to screening rather than signalling), where every period buyers first make an offer and sellers then decide whether to trade or not is analyzed in Fuchs et al. (2015). As shown there, even for the indivisible asset case, the equilibrium characterization is quite intractable so we will leave that case for future work. Furthermore, in the divisible case, when buyers can additionally choose the q_t on which they bid we have a competitive screening problem and run into non-existence issues.

Lastly, it worth noting that, in determining off-equilibrium beliefs with unobservability upon observing an off-path q_t, buyers can not only contemplate future trades by this seller but they can also assume that the residual supply was actually already traded in the past. The most pesimistic belief in this case would be that the seller sold $1-q_t$ at time zero. This makes it harder for a seller to signal at time $t>0$ by reducing q_t. As we will show in Section 5.3, it is more effective for separation for the seller to wait sufficiently long and sell its full unit when it trades. For this reason, having an infinite horizon and continuous trading will be important. We show next how separation fails when trading opportunities are restricted.

5.2 Non-equivalence

For any value of $\Delta>0$ and $T>0$, that is, for any market with multiple discrete trading dates, the value of total discounted equilibrium trades $Q^*\left(c\right)$ with observable trades cannot be attained as an equilibrium with unobservable trades. We will use the case where there are two trading dates $\varsigma=\{0,\Delta\}$ to illustrate this claim. As we will show below the problem, relative to the observable case, is that at $t=\Delta$ the price for quantity q_Δ can only depend on q_Δ. Thus we can no longer separate the types that are trading $\{0,q_\Delta\}$ from the ones that are trading $\{1-q_\Delta,q_\Delta\}$. Since in the separating equilibrium with observable trades all quantities $q\in[0,1]$ are traded at $t=0$ by some low type seller and their entire residual supply, also spanning $(0,1)$, is traded at Δ, there is no possibility for high types to attain the same level of trades as at $Q^*\left(c\right)$ without pooling with lower types.

Proposition 2 *The (unique) level of discounted trades in the equilibrium with observable trades cannot be attained in an equilibrium with unobservable trades when $\varsigma=\{0,\Delta\}$ and $c\in[0,1]$.*

The non-equivalence result extends to an arbitrary number of trading dates as long as $\Delta>0$. The issue is that, in the unique equilibrium allocation with observable trades characterized in Proposition 1, at any $t>0$, two

[18] This is also a consideration for the public case but a more relavant concern with private offers.

distinct types trade the same quantity. Since these two types traded different amounts in the past and we can condition the current terms of trade on the history of past trades, with different prices for each of them. But, when past trades are unobservable, this conditioning is not possible, and hence the same pattern of trades can no longer be supported in equilibrium.

Proposition 2 shows that with private trades there can be no equilibrium yielding the same payoffs as when trades are observable. We establish next the existence of an equilibrium when there are two trading dates and characterize its properties: with private trades we have pooling.

Proposition 3 *Suppose that, for some $\alpha \in (0,1)$, $v(c) \geq \alpha(c-1)+1$ in a neighborhood of 1.*[19] *Then there exists an equilibrium with unobservable trades where the pattern of transactions is as follows. There are two threshold types $0 \leq c_0 < c_1 < 1$ such that $Q^*(c_0) > e^{-r\Delta}$ and:*

(i) low types $c \in [0,c_0]$) trade as in the equilibrium with public trades: $q_0(c) = \frac{Q^*(c)-e^{-r\Delta}}{1-e^{-r\Delta}}$ and $q_\Delta(c) = 1-q_0(c)$ at prices $p_0(c) = p_\Delta(c) = v(c)$;

(ii) intermediate types $c \in [c_0,c_1]$ pool on the trade of the entire supply in the second period at the price $\overline{p}_\Delta = E\left[v(c)|c \in [c_0,c_1]\right] = c_1$;

(iii) high types $c \in [c_1,1]$) do not trade.

Moreover, the beliefs regarding the trade of any off-path quantity in the first and the second periods are the Dirac measure concentrated on type c_0.

The formal proof of this proposition can be found in the Appendix. Comparing Propositions 1 and 3 we see that the patterns of trade at the equilibria with observable and with private trades are quite different. It is then relatively easy to construct examples in which one information environment yields higher expected welfare in equilibrium than the other and vice versa:

Corollary 2 *The expected welfare with observable trades and unobservable trades is not generally ranked.*

To establish the result we again revisit the specification used for Example 1 with $v\left(c\right) = \frac{1+c}{2}$ and trading only at 0 and Δ. When types are uniformly distributed and for $e^{-r\Delta} \geq 3/4$ the equilibrium with unobservable trades features no trading in the first period and pooling in the second: types $c \in [0,2/3]$ trade their entire supply in the second period (i.e. $c_0 = 0$ and $c_1 = 2/3$). Prices are $p_0\left(q\right) = e^{-r\Delta}\frac{2}{3}$ and $p_\Delta\left(q\right) = 2/3$ for all q. One can then compute the expected welfare of sellers, equal to $e^{-r\Delta}\frac{4}{18}$ and compare its value with the one for the equilibrium with observable trades characterized in Example 1, equal to $\frac{1}{6}$. Recall that buyers always make zero expected profits in equilibrium, hence expected welfare only depends on the sellers' payoff.

We see that when $e^{-r\Delta} = 3/4$ welfare in both information regimes coincides. Noticing that welfare in the observable trades case is independent of $e^{-r\Delta}$ and the cost of delaying trades to the second period is decreasing in $e^{-r\Delta}$, for any $e^{-r\Delta} > 3/4$ we get that sellers achieve a higher expected payoff at the unobservable trade equilibrium.

In general the trade-off that determines the welfare comparison is that (when $e^{-r\Delta} \geq 3/4$) with unobservable trades types between $c = 1-e^{-r\Delta}$ and $c = 2/3$ trade more while types $c > 2/3$ do not trade at all. This observation allows us to find other specifications, where some mass is moved from lower types to types close to 1, for which observable trading is preferable. For example, let

$$f\left(c\right) = \begin{cases} 0.9 & for\ c < 0.9 \\ 1.9 & for\ c \geq 0.9 \end{cases}.$$

[19] This assumption precludes that the function $v(c)$ is tangent to the 45^0 degree line at $c = 1$. This is always true if $v(c)$ is a concave function.

Note that this change does not modify the pattern of trade in either equilibrium but it reduces the expected welfare of the unobservable trading case relative to the observable trade one. Hence it is easy to verify that for $e^{-r\Delta} = 0.75$ welfare is now higher when trades are observable.

5.3 Characterization of Equilibria with Continuous Trading

As shown in Theorem 2 and Corollary 1, when $T = \infty$, in the limit $\Delta \to 0$ with observable trades each type sells its full unit at a unique point in time. This suggests that the difficulties described in Proposition 2 might not arise when trading can take place continuously over an infinite horizon, i.e., when $\varsigma = [0, \infty)$. We explore this case next.

Similarly to what we did for the observable case, we start by showing that, when $\varsigma = [0, \infty)$, in any equilibrium with unobservable trades: (1) each individual trade must be fully separating; and (2) we must have full separation in terms of the total discounted quantity traded $Q(c)$. The force inducing separation is the same we highlighted in the commitment case. Since the seller's preferences satisfy the single crossing property, higher types would be willing to signal their type by committing to trading a lower Q. Of course, without commitment, and in particular when past trades are not observable and thus prices cannot condition on past trades, this can be hard to replicate. As highlighted in the previous section, when time is discrete, the need to rely on both quantities and time to separate makes it impossible to replicate the equilibrium with observable trades once trades are unobservable. Yet, when $\varsigma = [0, \infty)$ we can generate any $Q \in (0, 1)$ by trading the full unit with the right amount of delay. The crucial aspect is that then, when the seller trades, it trades its full unit. Thus, it does not require buyers to trust the seller will not sell (or potentially have sold) any additional amounts. Therefore, delay combined with full trade provides a credible way to signal your type that is robust to the unbservability of past trades, in the sense that there is no room for buyers to form beliefs about what might happen with the rest of the seller's supply.

In other words, an endogenous commitment can still be generated if, whenever a type trades, it trades its full supply. This effectively amounts to a commitment never to trade at any other date. This is the key why time trumps quantity as the way to signal sellers' types in a dynamic setting.

To formalize our result we begin by showing that the properties established in Lemmas 3 and 4 for the observable trade case extend to the situation where past trades are not observable.

Lemma 5 (Separation in q, Private) *For $\varsigma = [0, \infty)$, in an equilibrium with unobservable trades all transactions with $q > 0$ must be fully separating.*

The proof proceeds along similar steps of that of Lemma 3. Importantly, since time is continuous, the seller can signal/separate by selling its full supply at some given τ such that the discounted quantity sold is ε less than it was supposed to trade in the pool. Since it sells all its quantity at once, there is no consideration when forming beliefs about neither future nor past trades.

Lemma 6 (Separation in Q, Private) *For $\varsigma = [0, \infty)$, in any equilibrium with unobservable trades the total discounted quantity $Q(c)$ traded by type c must be strictly decreasing in c.*

The proof follows the same steps as the proof of Lemma 4 and is thus omitted.

Proposition 4 *For $\varsigma = [0, \infty)$, with private trading there is a unique equilibrium level of total discounted trades $Q(c)$ for each seller type c and this level coincides with the one of the equilibrium with commitment.*

Proof. Lemmas 5 and 6 imply that the seller's problem can be written as:

$$U(c) = \max_{\tilde{c}} Q(\tilde{c})(v(\tilde{c}) - c)$$

and we require that $Q(\tilde{c})$ is such that choosing $\tilde{c} = c$ is optimal for the type c seller, for all c. This problem is the same as the one we had in the equilibrium with commitment. Like in that case, from the first-order condition for the seller we obtain a differential equation characterizing $Q(c)$. This implies that $Q(c)$ is uniquely pinned down up to a constant. But since we must have $Q(0) = 1$, $Q(c)$ is then uniquely pinned down. ■

The last remaining step is to characterize the path of trades that implements the equilibrium $Q^*(c)$. For this, recall $\tau(c)$ denotes the time such that, if a type were to trade its full unit at this date, it would attain the commitment equilibrium value $Q^*(c)$. Formally:

$$e^{-r\tau(c)} = Q^*(c).$$

Since $Q^*(c)$ is unique and strictly decreasing, $\tau(c)$ is strictly increasing and we can define $C(t)$ as the inverse of $\tau(c)$. To understand why only using time to signal is an equilibrium let us follow a guess and verify argument. Suppose that indeed each seller type c trades its full unit at $\tau(c)$ and let the supporting prices be $p(q, \tau(c)) = v(c)$ for all $q \in (0, 1]$. With these prices the seller's solution of when and how much to sell is clear-cut: at any date it will be optimal to sell the full unit or nothing at all. Since the trading date $\tau(c)$ solves the sellers' problem with commitment it must be optimal also in the present environment. The off-equilibrium part of the prices is sustained by the belief that type c such that $\tau(c) = t$ is the type of seller who is most willing to deviate to trading a partial amount at t. At the above prices this type of seller is indifferent between this deviation, together with the trade of the rest of its supply the next instant and its equilibrium trades. Which shows:

Theorem 3 *For $\varsigma = [0, \infty)$, there exists an equilibrium with private trade where the present discounted value of total trades for each seller is the same as in the commitment equilibrium $Q^*(c)$ and separation takes place by the timing of the trades, i.e. type c sells its entire supply at time $\tau(c)$. As $\lambda \to 0$, prices are given by $p(q, \tau(c)) = v(c)$ for all $q \in (0, 1]$ and any off-path partial trade at $\tau(c)$ is attributed to type c.*

6 Non-Uniform Supply

The assumption of a uniform supply across all seller types is natural in many situations. For example, the one considered by Leland and Pyle (1977) where an entrepreneur is selling part of its firm. This is the case at the time in which the trading process starts and also along this process as long as past trades are observable. As already discussed, when instead past trades are unobservable, the residual supply available at any date $t > 0$ is endogenous as it depends on past trades and hence may no longer be uniform and varies with the seller types. In other cases it is even natural to assume the supply is different for different types. For example, if we consider informed hedge funds trading a particular asset, it might be impossible to know the exact amount of the asset they hold. In those situations we face a problem of multidimensional private information. A full analysis of the non-uniform supply case is beyond the scope of this paper. In what follows we illustrate how some of our results may change when the supply is non-uniform.

When the amount of the supply available to each type c at the beginning of the trading process is given by $s(c) \in [0, 1]$ and varies with c, the characterization of the equilibria with observable trades provided in Proposition 1 and Theorem 2 no longer holds true. The key to understand this is that with uniform supply the relevant IC constraints that limit the trades occurring in the equilibrium are the local upward constraints: that is, the ones ensuring that any given type $c \in [0,1)$ does not wish to imitate the trades of type $c+\varepsilon$ for a small $\varepsilon > 0$. As shown, in equilibrium, separation is achieved by having higher types trade less, albeit at higher prices. But when $s(c)$ is not constant, different types may not be able to trade the same quantity. Hence, a local upward constraint might not be binding because type c might not be able to supply the quantity needed to imitate type $c+\varepsilon$. This is best illustrated by the fact that, despite the presence of adverse selection, efficient trade may be attainable with a non-uniform supply.

Proposition 5 (First Best) *Suppose $s(c)$ is strictly increasing. Then, for any set of trading dates ς, regardless of whether past trades are observable or not, or of whether there is commitment, the unique equilibrium features $q_0(c) = s(c)$ for all c. That is, we have full, instantaneous trade and the first-best welfare level is thus attained.*

Proof. Since only types $c' \geq c$ can trade the contract $s(c)$ this contract must be priced at least $v(c)$. Since this is true for all quantities, each type obviously chooses to trade immediately all its supply. Thus trades will be perfectly separating and efficient. ■

It is also worth highlighting that the results obtained in the previous sections, that both with commitment or with observable trades equilibrium payoffs are independent of the set of trading opportunities ς, may no longer hold when $s(c)$ is not uniform. To illustrate this point, consider the supply function described in Figure 2.

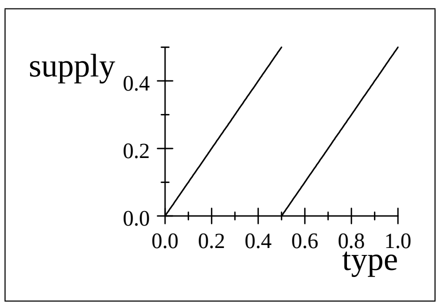

Figure 2

If we had only one trading date, $\varsigma = \{0\}$, types $c \geq 0.5$ would not be able to trade without pooling with the lower types with their same supply and they may then prefer to forgo trading altogether. This happens, for instance, when $v(c) = \frac{1}{3} + \frac{2}{3}c$, since at the pooling price $\frac{1}{2}v(c) + \frac{1}{2}v(c+\frac{1}{2}) = \frac{1}{2} + \frac{2}{3}c < c + 1/2$ for all $c \in [0, 0.5]$. Thus, type $c + \frac{1}{2}$, who has the same supply as c, prefers not to trade at this price. The situation has clear analogies with the one encountered in the previous section when past trades are not observable: seller types are multidimensional, hence the quantity traded at any point in time is not sufficient to screen them.

Instead, with a sufficiently larger set of trading dates, sellers have an additional instrument to signal their type, the time at which they trade. For instance, if $\varsigma = [0, \infty)$, separating equilibria again exist, in which types $c < 0.5$ trade their entire supply $s(c)$ at $t = 0$ and types $c \geq 0.5$ wait until some time τ and only then trade $q_\tau(c) = s(c)$.[20] The value of τ must be sufficiently large that each type $c < 0.5$ prefers to trade at the lower price $v(c)$ at $t = 0$ rather than wait until τ to trade at the higher price $v(c + 0.5)$.

Thus, with non-uniform supply the equilibrium payoffs depend on the available trading dates ς even with observable trades. Furthermore, in the situation described above the surplus generated by equilibrium trades is

[20] It is easy to verify that the beliefs for off equilibrium trades at earlier dates $t < \tau$ put all weight on lower types $c < 0.5$. given those beliefs and the associated prices, buyers do not want to trade at those dates.

clearly lower when $\varsigma = \{0\}$ than when $\varsigma = [0, \infty)$. This is because in the latter case there is a higher level of (discounted) trades for all types, since also high types trade fully, albeit with delay τ. Note however that, for a different $v(c)$, we could also have that the ex-ante expected surplus generated in equilibrium when $\varsigma = \{0\}$ is higher than when $\varsigma = [0, \infty)$. For this to happen, the gains from trade would have to be sufficiently large so that high types are willing to pool with low types with the same supply when $\varsigma = \{0\}$, yet they separate when $\varsigma = [0, \infty)$. Pooling then occurs with $\varsigma = \{0\}$. Signalling by lowering the amount traded is not attractive since the upward sloping supply implies that lower quantities actually trade at lower prices. As in Corollary 2 in Section 5, depending on parameter values the separating equilibrium may feature a higher or lower welfare than the equilibrium with some pooling trades.

Remark 2 *When $s(c)$ does not take the same value for all c, allowing contracts to use probabilities of trade between 0 and 1 for any given quantity (as in the equilibrium of markets with directed search) offers an additional instrument, besides the quantity traded, to screen seller types. In the situation discussed above, where sellers' supply is as in Figure* **??** *and $\varsigma = \{0\}$, the equilibrium is again fully separating if contracts can specify not only the quantity traded but also the probability of trade, with two active markets for each quantity: a low price market with probability 1 of trade and a high price market with a probability of trade $\pi^* = e^{-r\tau} < 1$.*[21] *The value of π^* is sufficiently low that the types $c < 1/2$ prefer to trade for sure at a low price than with probability π^* at a high price.*[22] *As discussed above, the welfare effects of enlarging the contract space are ambiguous.*

7 Conclusions

We have shown in this paper that time dominates quantity as an instrument to signal high quality. Low types trade early while high types trade at later times to signal the higher quality of the asset they are selling. The basic intuition for this result is straightforward. Once you have waited a certain amount of time to trade your full supply, you do not need to further restrict your future actions in any way to convince your counterparties the quality of your asset is high. Instead, if you only make a partial trade, in order to credibly signal you have a high quality asset, it must be believed that you won't trade your residual supply at some time in the future. Since there continue to be gains from trade, it is not possible in equilibrium to avoid succumbing to the temptation to sell the residual quantity. Thus, without commitment, a promise not to sell more in the future is not credible. We also showed that, even though the set of available trading dates does not matter when past trades are publicly observable, this is no longer true with unobservable trades. If trade happens at discrete intervals, sellers are no longer able to perfectly signal their types.

There are several promising avenues for future research. Our analysis highlighted the importance of the consequences for equilibrium outcomes of the presence of two dimensional private information, where both the quality of the asset and the amount of the asset held are only known to the seller. A characterization of the equilibria in that case could offer novel, interesting insights. In those environments, or in the situations with unobservable trade considered in Section 5, where private information over the residual supply endogenously arises, one could also explore the consequences of allowing for stochastic deliveries (in a model with search or by adding this feature in the specification of contracts). This would expand the set of available trade opportunities, beyond quantities

[21] In a directed search equilibrium, these probabilities of trade for sellers are generated by having relatively few sellers and many buyers in the first market and vice-versa in the second market.

[22] Williams (2021) shows in fact, in a directed search model, that when the privately known type of the seller is two dimensional, perfect screening can be achieved in equilibrium using jointly the quantity traded and the probability of trade.

and time of trade, and so provide sellers with an additional instrument with which to signal their types. Another interesting question is how the introduction of curvature in agents' utilities, so that the size of the gains from trade depends on the total quantity traded, affects the properties of equilibria with dynamic trading.

8 References

Akerlof, G. (1970). The Market for Lemons: Quality Uncertainty and the Market Mechanism. *Quarterly Journal of Economics* 84, 488–500.

Asriyan, V. and V. Vanasco (2021). Security Design in Non-Exclusive Markets with Asymmetric Information. Working paper.

Attar, A., T. Mariotti, and F. Salanié (2009). Non-exclusive competition in the market for lemons. TSE Working paper series.

Attar, A., T. Mariotti, and F. Salanié (2011). Nonexclusive competition in the market for lemons. *Econometrica* 79(6), 1869–1918.

Auster, S., P. Gottardi, and R. Wolthoff (2021). Simultaneous Search and Adverse Selection. Working Paper.

Ausubel, L., P. Cramption, and R. Deneckere (2002). Bargaining with incomplete information. *Handbook of Game Theory with Economic Applications* 3, 1897–1945.

Azevedo, E. and D. Gottlieb (2017). Perfect Competition in Markets With Adverse Selection. *Econometrica* 85(1), 67–105.

Bisin, A. and P. Gottardi (2006). Efficient Competitive Equilibria with Adverse Selection. *Journal of Political Economy* 114(3), 485–516.

Coase, R. (1972). Durability and Monopoly. *Journal of Law and Economics* 15(1), 143–149.

Daley, B. and B. Green (2012). Waiting for News in the Market for Lemons. *Econometrica* 80(4), 1433–1504.

DeMarzo, P. and D. Duffie (1999). A liquidity-based model of security design. *Econometrica* 67(1), 65–99.

Dubey, P. and J. Geanakoplos (2002). Competitive pooling: Rothschild-stiglitz reconsidered. *Quarterly Journal of Economics* 117(4), 1529–1570.

Eeckhout, J. and P. Kircher (2010). Sorting and decentralized price competition. *Econometrica* 78(2), 539–574.

Fuchs, W., A. Öery, and A. Skrzypacz (2016). Transparency and Distressed Sales under Asymmetric Information. *Theoretical Economics* 11(3), 1103–1144.

Fuchs, W. and A. Skrzypacz (2019). Costs and benefits of dynamic trading in a lemons market. *Review of Economic Dynamics* 33, 105–127.

Fuchs, W. and A. Skrzypacz (2022). Dynamic Bargaining with Private Information. *Palgrave* Macmillan.

Gale, D. (1992). A Walrasian theory of markets with adverse selection. *Review of Economic Studies* 59, 229–255.

Gerardi, D., L. Maestri, and I. Monzón (2022). Bargaining over a Divisible Good in the Market for Lemons. *American Economic Review* forthcoming.

Guerrieri, V., R. Shimer, and R. Wright (2010). Adverse selection in competitive search equilibrium. *Econometrica* 78(6), 1823–1862. 6, 11

Hörner, J. and N. Vieille (2009). Public vs. Private Offers in the Market for Lemons. *Econometrica* 77(1)(December), 29–69.

Janssen, M. C. W. and S. Roy (2002). Dynamic Trading in a Durable Good Market with Asymmetric Information. *International Economic Review* 43(1), 257–282.

Kurlat, P. (2016). Asset markets with heterogeneous information. *Econometrica* 84(1), 33–85.

Lee, J. (2023). Price Experimentation in Confidential Negotiations. Working paper.

Leland, H. E. and D. H. Pyle (1977). Informational Asymmetries, Financial Structure, and Financial Intermediation. *Journal of Finance* 32(2), 371–387.

Li, Q. (2022). Security Design without Verifiable Retention. *Journal of Economic Theory* 200.

Mailath, G. J. (1987). Incentive Compatibility in Signaling Games with a Continuum of Types. Econometrica, 55(6), 1349–1365.

Moreno, D. and J. Wooders (2016). Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention. *Theoretical Economics* 11, 601–639.

Nöldeke, G. and E. van Damme (1990, jan). Signalling in a Dynamic Labour Market. *Review of Economic Studies* 57(1), 1–23.

Rothschild, M. and J. Stiglitz (1976). Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *Quarterly Journal of Economics* 90(4), 629–649.

Spence, M. (1973). Job market signaling. *Quarterly Journal of Economics* 87, 355–374.

Swinkels, J. M. (1999). Education Signaling with Preemptive Offers. *Review of Economic Studies* 66, 949–970.

Williams, B. (2021). Search, Liquidity, and Retention: Screening Multidimensional Private Information. *Journal of Political Economy* 129(5), 1487–1507.

Appendix

8.0.1 Benchmark:

Proof of Lemma1. This follows from the fact that under full commitment the game is essentially static, buyers have common value for the asset and are symmetrically uninformed about the sellers type. ■

Proof of Theorem 1

Fix a commitment equilibrium and let $U(c)$ be the seller's value function. By Lemma 9 in the Online Appendix, $U(c)$ is a convex function and the envelope condition implies that $U'(c) = -Q(c)$ almost everywhere, which implies that $Q(\cdot)$ must be non-increasing. We will first show that the equilibrium is separable. Suppose that there is a pooling set, i.e., a non-degenerated interval $[c', c'']$. Hence, since the equilibrium price must the average of the buyers' value on this interval, $P(c'') + \delta < v(c'')$,for some $\delta > 0$. Let type c'' deviate to a slightly lower quantity $Q(c'') - \epsilon$. If $Q(c'') - \epsilon$ is on the equilibrium path, it must be priced by a type higher than c''; otherwise, $Q(c'') - \epsilon$ is off the equilibrium path and D1 would imply that the support of the beliefs must be contained in the interval $(c'', 1]$. Hence, c'' will face a price of at least $v(c'')$ for quantity $Q(c'') - \epsilon$. Making $\epsilon \to 0$, this deviation ensures an extra profit of at least $\delta Q(c'')$, which is a contradition.

It is straightforward that D1 also implies that $Q(0) = 1$. By Mailath (1987) (see Theorem 2 and its corollary), there is a unique separating equilibrium that must satisfy

$$U(c) = \int_c^1 Q(x)dx = [P(Q(c)) - c]\, Q(c) = (v(c) - c)Q(c),$$

since $U(1) = (v(1) - 1)\, Q(1) = 0$. Taking the derivative we have

$$-Q(c) = (v'(c) - 1)Q(c) + (v(c) - c)Q'(c)$$

or

$$(v(c)-c)Q'(c)+v'(c)Q(c)=0. \tag{4}$$

Solving the ODE in (4) with initial condition $Q(0)=1$, we get the solution in the statement of the Theorem.

We claim that

$$Q(1)=\exp\left[-\int_0^1 \frac{v'(x)}{v(x)-x}dx\right]=0.$$

Indeed, notice that

$$\int_{1-\epsilon}^1 \frac{v'(x)}{v(x)-x}dx \geq \int_{1-\epsilon}^1 \frac{v'(x)}{1-x}dx \geq -\gamma\ln(1-x)|_{1-\epsilon}^1=\infty,$$

since $v'(x)\geq\gamma>0$ and $v(x)\in[x,1]$, for all $x\in[1-\epsilon,1]$ and for some $\gamma>0$.■

8.0.2 Observable Trades:

Proof of Lemma 2. If T is finite we can argue by backward induction. In the last period, for any $N>1$ this is just a static Bertrand competition game and we must have zero profits thus $p_T=E[v]$. Next, if in the future periods the price is fixed, i.e independent of the current buyer's actions, then again the game is essentially static from their perspective and if $p_t<E[v]$ they would always prefer to deviate. Note that our restriction on beliefs is such that on-path, if after a given history of offered quantities, beliefs were on a singleton type and buyers deviate in their price offers, future buyers do not update further on the seller's type. Thus, as needed in the above argument, future prices do not react to current offers. Thus establishing the induction step.
If $T=\infty$ for a small number of buyers one could in principle construct collusive type equilibria in which offers are below the expected value and if a buyer at time t were to deviate by offering ε more, the future buyers would offer the break even price at $t+1$ leading the seller to reject the higher offer at time t. Note however, that for any Δ, if we have a sufficiently large number of buyers, splitting the collusive profits would not be an equilibrium. This is because the buyer could offer a price sufficiently close to the competitive price (so waiting is not worth it for the buyers) and in this way it could get all the market share and make higher profits than with an infinitesimally small fraction of the collusive profits. For any off-path offered quantity, as long as the resulting off-path beliefs were degenerate, the same arguments apply. Formally, suppose on path only type c makes an offer q_t at time t. Then, for any Δ and ε both >0, there exists a number of buyers N such that it cannot be that in equilibrium $p_t+\varepsilon=v(c)$ with each buyer making a profit of $\frac{\varepsilon}{N}$. Note, that given the belief restrictions and that buyers cannot loose money, the highest price at $t+\Delta$ is $p_{t+\Delta}=v(c)$. Note that for any $\Delta>0$, $\varepsilon>0$ there exists a $\tilde{p}$ where $v(c)>\tilde{p}>p_t$ *such that* $e^{-r\Delta}(v(c)-c)<(\tilde{p}-c)$. That is, a buyer can find a price $\tilde{p}$ that would attract the seller for sure. This would lead to profits of $v(c)-\tilde{p}>\frac{\varepsilon}{N}$ for N sufficiently large which would imply a profitable deviation from the collusive equilibrium. ■

Proof of Lemma 3. We proceed by contradiction, suppose there is an equilibrium in which some pooling trade occurs. Consider the highest type c'' that ever participates in a pooling trade and without loss assume it happens in the first period.[23] Let $Q(c'')$ be the total discounted quantity traded by c'' in this equilibrium and $P(c'')$ be the average unit price paid. Note that $P(c'')<v(c'')$ since c'' pools with lower types for at least some trades, never pools with higher types, and buyers cannot incur losses on path.

[23] For ease of exposition we assume the set of types pooling includes its supreme but a similar argument can be constructed even if this property did not hold.

Consider instead a deviation for type c'' in which trades for the first time at time τ a quantity q_τ and all its residual supply immediately afterwards such that:

$$\begin{aligned} e^{-r\tau}q_\tau + e^{-r(\tau+\Delta)}(1-q_\tau) &= Q(c'') - \varepsilon \quad if\ \tau \leq\ T-\Delta \\ e^{-rT}q_T &= Q(c'') - \varepsilon \quad otherwise \end{aligned}$$

for some small $\varepsilon > 0$. Suppose first that these trades were on-path. Then, since they imply a smaller total discounted quantity, by the single crossing property they must be carried out by some type $c' > c''$ and thus their unit prices have to be strictly higher than $v(c'')$. But then, since ε can be arbitrarily small and the change in the price from switching from the original trading strategy with pooling to this one is bounded away from zero, deviating to this alternative strategy would be profitable for type c'', a contradiction.

Suppose next that the offer q_τ (following no trade at earlier dates) were not on the equilibrium path. Note that since type c'' is the highest type that pools, any $c > c''$ must be separating. If $c'' = 1$ then D1 implies that upon observing the deviation beliefs are degenerate and thus $p = v(c'')$. If instead $c'' < 1$ then c'' must be indifferent between trading its pooling quantity or separating by immitating type c''_+ a type arbitrarily close to but higher than c''.

$$Q(c'')(P(c'') - c'') = Q\left(c''_+\right)(v(c'') - c'')$$

Note that since $v\left(c''_+\right) > P(c'')$ in the candidate equilibrium there exists and ε *is such that* $Q\left(c''_+\right) < Q(c'') - \varepsilon < Q(c'')$. For ε small, the D1 refinement would imply that the posterior beliefs should be degenerate on c''. Then, by the fact these quantities would be traded at break-even prices it implies that for a small decrease in quantity type c'' can attain a discrete increase in prices and thus this would be a profitable deviation. ■

Proof of Proposition 1. The fact that the discounted quantity $Q(c)$ and payoff $U(c)$ for each seller type is the same as in the commitment case follows directly from Lemmas 3 and 4, since the seller's time 0 problem is equivalent to the one in the commitment case. To establish the unique pattern of trade we will proceed by contradiction. In search of a contradiction suppose $q_{t+\Delta}(c) < 1 - q_t(c)$. Suppose there exists a lowest $c' < c$ that would be indifferent between its equilibrium trades $Q^*(c')(v(c') - c')$ and $e^{-rt}(q_t(c)(v(c) - c') + \delta(1 - q_t(c))(v(c') - c'))$. Note that this implies for c' a tradeoff of a reduction of quantity for an increase in average price. This, together with the single crossing property, implies that there must then exist an $\varepsilon > 0$ such that type $\tilde{c} = c' + \varepsilon$ would actually be strictly better off trading $q_t(c)$, $(1 - (q_t(c)))$ thus, given the D1 restriction on beliefs, we cannot have an equilibrium with sufficiently low off-equilibrium prices such that no type would find it profitable to deviate. Returning to the existence of the type c'. Note that for type c there would be a strict improvement from trading its residual supply if it were to be believed to be still type c. The non-existence of such a c' would thus imply that for type 0 deviating to immitate type c at t and then trading its residual supply at $t + \Delta$ would be profitable thus, in itself, rule out the proposed equilibrium which does not exhaust trade in two periods. ■

8.0.3 Unobservable Trades:

Let $q_0^l(c)$ denote the lowest amount type c can trade at $t = 0$ and still attain a total discounted level of trades equal to $Q^*(c)$. Let c_l be such that $Q^*(c_l) = e^{-r\Delta}$ then for all $c < c_l$ it must be that $q_0^l(c) > 0$.

Lemma 7 *For $\varsigma = \{0, \Delta\}$, suppose the equilibrium with unobservable trades is fully separating, featuring a level of total discounted trades $Q^*(c)$ for each c as when trades are observable. Then, for all types $c < c_l$ it must be that $q(0, c) = q_0^l(c)$.*

Proof of Lemma 7. In search of a contradiction, suppose $q(0, c) > q_0^l(c)$ for some c. Then, there exists another type $c' < c$ such that $q(0, c) = q_0^l(c')$. This implies that c' can imitate c at $t = 0$ and sell quantity $q(0, c)$ for a higher price since $v(c) > v(c')$. Importantly, type c' can then trade its whole residual supply $1 - q_0^l(c')$ at $t = \Delta$ thus achieving the same value $Q(c')$. Furthermore, if the trade $1 - q_0^l(c')$ at $t = \Delta$ were on path, it must hold that $p\left(1 - q_0^l(c'), \Delta\right) \geq v(c')$ since no type lower than c' chooses trades of total amount $Q(c')$. Hence, in such a situation, c' can profitably deviate by imitating c at $t = 0$, a contradiction. If instead $1 - q_0^l(c')$ were off-path, the type of seller most willing to carry out those trades is again c' who would combine them with the trade of $q(0, c) = q_0^l(c')$. Thus, by the D1 refinement the price must be $v(c')$. Implying a profitable deviation would exist, again a contradiction. ■

Building on this result we can show that the equivalence result established in Proposition 1 when trades are observable no longer holds.

Proof of Proposition 2. We know the equilibrium with observable trades is separating and the values of total discounted trades $Q^*(c)$ by different seller types span the whole interval $[0, 1]$. To support the same level of total discounted trades in the unobservable case, given Lemma 7, all quantities in $[0, 1]$ must be traded at $t = 0$ and at $t = \Delta$ by types $c < c_l$. But then types $c > c_l$ cannot trade in either period without pooling with some lower type, hence cannot attain the level of trades $Q^*(c)$ at the price $v(c)$. ■

Proof of Proposition 3. For each $c \in [0, 1]$, let $\varphi(c)$ be the implicit solution of the equation

$$E[v(\tilde{c})|\tilde{c} \in [c, \varphi]] - \varphi = 0. \tag{5}$$

Lemma 11 in the Online Appendix shows that φ is well defined and provides its basic properties. Let $\bar{c}_1 = \varphi(0)$. Consider the two following cases:

(1) $e^{-r\Delta}\bar{c}_1 < v(0)$. We claim that in this case $c_0 \in (0, 1)$. Indeed, type c_0 must be indifferent between trading the public trade equilibrium allocation or the whole unit in the second period, i.e., c_0 must satisfy the following indifference condition:

$$e^{-r\Delta}(\varphi(c) - c) - U^*(c) = 0,$$

(where $U^*(c)$ denotes commitment equilibrium utility) which has at least the trivial solution $c = 1$. Notice that the left hand side (l.h.s.) of the previous equation becomes $e^{-r\Delta}\bar{c}_1 - v(0) < 0$ at $c = 0$. Hence, in order to obtain a non-trivial solution for the equation we need to show that the derivative of the l.h.s. is negative at $c = 1$, which is equivalent to

$$e^{-r\Delta}\left(\varphi'(1) - 1\right) + Q(1) = e^{-r\Delta}(\varphi'(1) - 1) < 0 \text{ or } \varphi'(1) < 1,$$

which holds by the assumption of the proposition and Lemma 12 in the Online Appendix. Therefore, there exists $c_0 \in (0, 1)$ such that

$$e^{-r\Delta}(\varphi(c_0) - c_0) = (v(c_0) - c_0)Q(c_0).$$

In particular, we have that $\overline{p}_\Delta := p_\Delta(0,1) = c_1 = \varphi(c_0)$. Since $\varphi(c_0) > v(c_0)$, we must have that $Q(c_0) > e^{-r\Delta}$.
Let $V(c)$ be the utility that type c gets at the proposal equilibrium allocation, i.e.,

$$V(c) = \begin{cases} U(c) & \text{if } c \in [0, c_0] \\ U(c_0) + e^{-r\Delta}(c_0 - c) = e^{-r\Delta}(c_1 - c) & \text{if } c \in [c_0, c_1] \\ 0 & \text{if } c \in [c_1, 1] \end{cases} .$$

Notice that V is the continuous pasting from the convex function U and a piecewise linear function with non-decreasing slopes, since $\dot{U}(c_0) = -Q(c_0) < -e^{-r\Delta}$. Hence, V is a convex function.
It is clear, given the specified pattern of trade of the various seller types at the two trading dates, that there is no profitable deviation given by a price posted for any of the equilibrium quantities, any higher price would generate losses.
To complete the construction of the equilibrium, we need to specify the off-path beliefs regarding trades $q_0 \in [0, q_0(c_0)]$ and $q_1 \in [q_\Delta(c_0), 1]$. We claim the set C' specifying the support of such beliefs, in line with D1 is a singleton, constructed on the basis of the fact that the continuation trades accompanying this deviation are such as to exhaust the seller's supply: thus, for any $q_0 \in [0, q_0(c_0)]$, $q_1 = 1 - q_0$. For any $q_0 \in [0, q_0(c_0)]$ and any p, let

$$\widetilde{c}_0 \in \arg\min_c V(c) - pq_0 + cq_0 - e^{-r\Delta}(p - c)(1 - q_0); c \in [0, 1]. \tag{6}$$

Similarly, for any $q_1 \in [q_\Delta(c_0), 1]$ and any p, let

$$\widetilde{c}_1 \in \arg\min_c V(c) - e^{-r\Delta}pq_1 + e^{-r\Delta}cq_1 - (p - c)(1 - q_1); c \in [0, 1] . \tag{7}$$

For $\widetilde{c}_0$ and $\widetilde{c}_1$ to solve these problems, they must satisfy the following first-order conditions:[24]

$$-V'_+(\widetilde{c}_0) \leq q_0 + e^{-r\Delta}(1 - q_0) \leq -V'_-(\widetilde{c}_0)$$

and

$$-V'_+(\widetilde{c}_1) \leq 1 - q_1 + e^{-r\Delta}q_1 \leq -V'_+(\widetilde{c}_1).$$

Notice that

$$V'_-(c) = V'_+(c) = \begin{cases} -Q(c) & \text{if } c \in [0, c_0) \\ -e^{-r\Delta} & \text{if } c \in (c_0, c_1) \\ 0 & \text{if } c \in (c_1, 1] \end{cases} ,$$

$$V'_-(c_0) = -Q(c_0), V'_+(c_0) = -e^{-r\Delta}, V'_-(c_1) = -e^{-r\Delta} \text{ and } V'_+(c_1) = 0.$$

Hence, by the convexity of V, $\widetilde{c}_0 = \widetilde{c}_1 = c_0$ if and only if

$$e^{-r\Delta} \leq q_0 + e^{-r\Delta}(1 - q_0) \leq Q(c_0) \text{ and } e^{-r\Delta} \leq 1 - q_1 + e^{-r\Delta}q_1 \leq Q(c_0).$$

These conditions are obviously equivalent and are satisfied for all $q_0 \in [0, q_0(c_0)]$.
Given that the off equilibrium path beliefs are concentrated at type $c = c_0$, the buyer's no deviation conditions boil down to verify that at the price $p_0(\cdot) = v(c_0)$ and $p_\Delta(\cdot) = v(c_0)$ type c_0 does not wish to deviate. This is clearly the case and the same is true for all other types.
(2) $e^{-r\Delta}\overline{c}_1 \geq v(0)$. Thus, at $c_0 = 0$, we have

$$e^{-r\Delta}E[v(c)|c \in [0, c_1]] \geq v(0),$$

[24] We are implicitly assuming that these solutions exist for the conjectured price functions. Indeed, for the constructed functions, this will be the case.

where $c_1 = \varphi(0) = \bar{c}_1$.

Let $V(c)$ be the utility that type c gets at the proposal equilibrium allocation, i.e.,

$$V(c) = \begin{cases} e^{-r\Delta}(\bar{c}_1 - c) & \text{if } c \in [0, \bar{c}_1] \\ 0 & \text{if } c \in [\bar{c}_1, 1] \end{cases},$$

which is obviously a convex function.

To complete the construction of the equilibrium, we need again to specify the off-path equilibrium beliefs associated to trades $q_0 \in [0,1]$ in the first period and $q_1 \in [0,1)$ in the second period. To construct the set C' specifying the support of these beliefs in line with the definition of D1, those beliefs must satisfy the same conditions given by the minimization problems (6) and (7) but with this new utility function $V(c)$. The first-order conditions of these minimization problems in c are:

$$-V'_+(\widetilde{c}_0) \leq q_0 + e^{-r\Delta}(1 - q_0) \leq -V'_-(\widetilde{c}_0)$$

and

$$-V'_+(\widetilde{c}_1) \leq 1 - q_1 + e^{-r\Delta}q_1 \leq -V'_-(\widetilde{c}_1).$$

Notice that

$$V'_-(c) = V'_+(c) = \begin{cases} -e^{-r\Delta} & \text{if } c \in (0, c_1) \\ 0 & \text{if } c \in (c_1, 1] \end{cases},$$

$$V'_-(0) = -\infty,\ V'_+(0) = -e^{-r\Delta},\ V'_-(c_1) = -e^{-r\Delta} \text{ and } V'_+(c_1) = 0.$$

By the convexity of V, to show that $\widetilde{c}_0 = \widetilde{c}_1 = 0$, it suffices that

$$e^{-r\Delta} \leq q_0 + e^{-r\Delta}(1 - q_0) \text{ and } e^{-r\Delta} \leq 1 - q_1 + e^{-r\Delta}q_1,$$

which are clearly satisfied for all q_0 and all q_1.

Finally, if the off equilibrium path beliefs are concentrated at type $c = 0$, the buyer's no deviation conditions boil down to check that type $c = 0$ does not want to deviate to any trade $q_0 \in [0,1]$ in the first period and $q_1 \in [0,1)$ in the second period, given the associated prices $p_0(\cdot) = v(0)$ and $p_\Delta(\cdot) = v(0)$, which is true given the specification of the equilibrium price and since $e^{-r\Delta}\bar{c}_1 \geq v(0)$. ■

Proof of Lemma 5. The proof proceeds along similar steps of that of Lemma 3. Consider the highest type c'' that ever participates in a pooling trade.[25] Let $Q(c'')$ be the total discounted quantity traded in equilibrium by type c'' and $P(c'')$ the average price paid. Note that $P(c'') < v(c'')$ since c'' pools with lower types for at least some trade, never pools with higher types, and buyers must break even. Consider instead a trade of the entire supply at time τ such that:

$$e^{-r\tau} = Q(c'') - \varepsilon,$$

for some small $\varepsilon > 0$. If such contract were on the equilibrium path, it must be traded by types higher than c'' since, by the single crossing property, if type c'' does not prefer this contract to its equilibrium trade no type below it would. Thus, the price of the contract would need to be greater than $v(c'')$. But then, since ε can be arbitrarily small, and the change in prices is discrete, this would be a profitable deviation for type c''. If instead such contract were not on the equilibrium path, by a very similar argument as in Lemma 3 we can show that the seller of type c'' is the one that is most willing to trade thus by the D1 refinement the price would be such that the c'' seller would have a profitable deviation. Thus, all trades must be separating. ■

[25] We assume for convenience that the set of types participating in some pooling trade includes its supremum. A similar argument can be constructed when this is not the case.

Online Appendix

Let $U : [0, 1] \to \mathbb{R}$ be any function. We say that D is a subgradient of U at c if

$$D(c - \tilde{c}) \geq U(c) - U(\tilde{c}),$$

for all $c, \tilde{c} \in [0, 1]$. We denote $\partial U(c)$ the set of all subgradients of U at c. The following lemma is useful.

Lemma 8 *Let $U : [0, 1] \to \mathbb{R}_+$ be a convex function. The following properties hold:*

(a) $\partial U(c)$ is an interval;

(b) $\partial U(c) \geq \partial U(\tilde{c})$, for all $c > \tilde{c}$;[26]

(c) $\partial U(c)$ is singleton if and only if U is differentiable at c;

(d) $\{c \in [0, 1]; \partial U(c)$ is not singleton$\}$ is countable.

The proof is a consequence of classical results of Convex Analysis.

Let Ω^b be the set of posted contracts (by the buyer). For all $\omega \in \Omega^b$, let $Q(\omega) = \sum_{t \in R} e^{-rt} q(t; \omega)$ denote the total discounted quantity traded and $P : \Omega^b \to \mathbb{R}_+$ the price function. Prices can be written simply as functions of Q, hence $P(\omega) = P(Q(\omega))$. Define the seller's value function $U : [0, 1] \to \mathbb{R}_+$ as

$$U(c) = \max_{\omega \in \Omega^b} (P(Q(\omega)) - c)Q(\omega).$$

The following lemma shows that U is convex and provides its envelope condition.

Lemma 9 *(Seller's value function). $U(c)$ is a non-increasing convex function.*
If $\omega(c) \in \arg\max_{\omega \in \Omega^b} (P(Q(\omega)) - c)Q(\omega)$, then $-Q(\omega(c)) \in \partial U(c)$.[27]

$U(\cdot)$ is a convex function because it is the maximum of the linear functions $\varphi(c|Q) = (P(Q) - c)Q$ indexed by $Q(\omega) : \omega \in \Omega^b$. The other properties are immediate.

Let us write, more compactly, $Q(c) = Q(\omega(c))$. The next lemma provides the link between the subgradient of U and the optimal seller's optimal contract.

Lemma 10 *(Characterization). $Q(c) \in -\partial U(c)$, for all c, if and only if $Q(c)$ is an optimal contract for the seller with type c, with price function $P(Q(c)) = \frac{U(c)}{Q(c)} + c$, for all c.*

Notice that $-Q(\tilde{c}) \in \partial U(\tilde{c})$ if and only if

$$-Q(\tilde{c})(\tilde{c} - c) \geq U(\tilde{c}) - U(c),$$

or, equivalently,

$$U(c) \geq P(Q(\tilde{c}))Q(\tilde{c}) - cQ(\tilde{c}),$$

for all $c, \tilde{c} \in [0, 1]$, i.e., $Q(c)$ is an optimal contract for the type-c seller with price function $P(Q(c)) = \frac{U(c)}{Q(c)} + c$.

Lemma 11 *$\varphi(c)$ is a well defined and increasing function such that $\varphi(c) \in (v(c), v(\varphi(c)))$, for all $c \in [0, 1)$, and $\varphi(1) = 1$.*

[26] $\partial U(c) \geq \partial U(\tilde{c})$ means that $q \geq \tilde{q}$, for all $q \in \partial U(c)$ and $\tilde{q} \in \partial U(\tilde{c})$.

[27] The reciprocal result is true when Ω^b is a closed set.

Let $c \in [0,1)$. If $\varphi = c$, the left hand side (l.h.s.) of (5) becomes $v(c) - c > 0$, and if $\varphi = 1$, it becomes $E[v(\widetilde{c})|\widetilde{c} \in [c,1]] - 1 < 0$. Therefore, by the intermediate value theorem, there exists $\varphi(c) \in (c,1)$ that solves the equation (5). Notice that the derivative of the left hand side (l.h.s.) of (5) with respect to φ evaluated at a solution $\varphi(c)$ is

$$(v(\varphi(c)) - \varphi(c)) \frac{f(\varphi(c))}{\int_c^{\varphi(c)} f(\widetilde{c}) d\widetilde{c}} - 1 < 0,$$

which implies that $\varphi(c)$ is uniquely determined. Moreover, since the l.h.s. of (5) is strictly increasing in c, $\varphi(c)$ must be strictly increasing in c. Finally, $\varphi(c)$ is an average of the increasing function $v(c)$ in the interval $[c, \varphi(c)]$ and, therefore, $\varphi(c) \in (v(c), v(\varphi(c)))$, for all $c \in [0,1)$, and $\varphi(1) = 1$.

Since $\varphi(c) > v(c) > c$, we have that $1 - \varphi(c) \leq 1 - c$, for all $c \in [0,1)$, which implies that $\varphi'(1) \leq 1$. The following lemma provides a condition that ensures $\varphi'(1) < 1$.

Lemma 12 *Suppose that there exists $\alpha \in (0,1)$ such that $v(c) \geq \alpha(c-1)+1$ in a neighborhood of* 1. *Then* $\varphi'(1) \leq \alpha < 1$.

Proof. By the mean value theorem for integrals, for each c, there exists $\widetilde{\varphi}(c) \in (c, \varphi(c))$ such that $\varphi(c) = v(\widetilde{\varphi}(c))$. Hence, for c sufficiently close to 1, we have that

$$\varphi(c) = v(\widetilde{\varphi}(c)) \geq \alpha(\widetilde{\varphi}(c) - 1) + 1,$$

which implies that $\alpha(1 - \widetilde{\varphi}(c)) \geq 1 - \varphi(c)$. Dividing both sides by $1 - c$ and making $c \to 1$, we get $\alpha\widetilde{\varphi}'(1) \geq \varphi'(1)$. Moreover, since $1 - \widetilde{\varphi}(c) \leq 1 - c$, we have that $\widetilde{\varphi}'(1) \leq 1$. Hence, $\varphi'(1) \leq \alpha < 1$. ■