Inference in Direct Multi-Step and Long Horizon Forecasting Regressions

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Introduction

This paper proposes and evaluates a new method for inference in direct multi-step and long horizon forecasting regressions, using the heteroskedasticity and autocorrelation consistent (HAC) covariance estimator in West (1997). I present two empirical applications to illustrate the relevance of the approach: the first uses the variance risk premium to forecast currency returns, and the second uses inflation to forecast the equity premium.

Contribution

I model the autocorrelation process of the residuals to estimate the covariance matrix. This results in a substantial improvement in the efficiency and accuracy of the estimates, relative to conventional HAC estimation that imposes no structure on the autocorrelation process of the residuals.

Contribution

I present a unified framework for inference in both direct multi-step and long horizon forecasts, by taking advantage of the fact that the error terms in a long horizon forecast are equal to the sum of the error terms from direct multi-step forecasts.

Direct Multi-Step and Long Horizon Forecasts

$$\boldsymbol{y}_{t+h} = \boldsymbol{a}^{(d,h)} + \boldsymbol{A}_1^{(d,h)}\boldsymbol{y}_{t-1} + \cdots + \boldsymbol{A}_p^{(d,h)}\boldsymbol{y}_{t-p} + \boldsymbol{e}_{t+h}^{(d,h)}$$

$$\sum_{s=0}^{h} \boldsymbol{y}_{t+s} = \boldsymbol{a}^{(l,h)} + \boldsymbol{A}_1^{(l,h)}\boldsymbol{y}_{t-1} + \cdots + \boldsymbol{A}_p^{(l,h)}\boldsymbol{y}_{t-p} + \boldsymbol{e}_{t+h}^{(l,h)}$$

Autocorrelation Process

$$\boldsymbol{y}_t = \boldsymbol{a}^{(d,0)} + \boldsymbol{A}_1^{(d,0)} \boldsymbol{y}_{t-1} + \cdots + \boldsymbol{A}_p^{(d,0)} \boldsymbol{y}_{t-p} + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{e}_{t+h}^{(d,h)} = \boldsymbol{\epsilon}_{t+h} + \sum_{j=0}^{h-1} \boldsymbol{A}_1^{(d,j)} \boldsymbol{\epsilon}_{t+h-1-j}$$

$$\boldsymbol{e}_{t+h}^{(l,h)} = \boldsymbol{\epsilon}_{t+h} + \sum_{j=0}^{h-1} \boldsymbol{B}_1^{(l,j)} \boldsymbol{\epsilon}_{t+h-1-j}$$

$$\boldsymbol{B}_1^{(l,j)} \equiv \boldsymbol{I}_n + \sum_{k=0}^{j} \boldsymbol{A}_1^{(d,k)}$$

Covariance Matrix of OLS Estimates: $\boldsymbol{V}^{(d,h)}$

$$\boldsymbol{V}^{(d,h)} = \left(\mathrm{E}[\boldsymbol{X}_{t-1}\boldsymbol{X}'_{t-1}]\right)^{-1} \boldsymbol{S}^{(d,h)} \left(\mathrm{E}[\boldsymbol{X}_{t-1}\boldsymbol{X}'_{t-1}]\right)^{-1}$$

$$\boldsymbol{S}^{(d,h)} \equiv \boldsymbol{\Gamma}_0^{(d,h)} + \sum_{j=1}^{h} \left(\boldsymbol{\Gamma}_j^{(d,h)} + \boldsymbol{\Gamma}_j^{(d,h)\prime}\right)$$

$$\boldsymbol{S}^{(d,h)} = \mathrm{E}[\boldsymbol{d}_{t+h}^{(d,h)} \boldsymbol{d}_{t+h}^{(d,h)\prime}]$$

$$\boldsymbol{d}_{t+h}^{(d,h)} \equiv \left(\boldsymbol{X}_{t-1} + \sum_{j=0}^{h-1} \boldsymbol{X}_{t+j} \boldsymbol{A}_1^{(d,j)}\right) \epsilon_{t+h}$$

Covariance Matrix of OLS Estimates: $\boldsymbol{V}^{(l,h)}$

$$\boldsymbol{V}^{(l,h)} = \left(\mathrm{E}[\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}']\right)^{-1}\boldsymbol{S}^{(l,h)}\left(\mathrm{E}[\boldsymbol{X}_{t-1}\boldsymbol{X}_{t-1}']\right)^{-1}$$

$$\boldsymbol{S}^{(l,h)} \equiv \boldsymbol{\Gamma}_0^{(l,h)} + \sum_{j=1}^{h}\left(\boldsymbol{\Gamma}_j^{(l,h)} + \boldsymbol{\Gamma}_j^{(l,h)'}\right)$$

$$\boldsymbol{S}^{(l,h)} = \mathrm{E}[\boldsymbol{d}_{t+h}^{(l,h)}\boldsymbol{d}_{t+h}^{(l,h)'}]$$

$$\boldsymbol{d}_{t+h}^{(l,h)} \equiv \left(\boldsymbol{X}_{t-1} + \sum_{j=0}^{h-1}\boldsymbol{X}_{t+j}\boldsymbol{B}_1^{(l,j)}\right)\epsilon_{t+h}$$

Simulation

$$y_{1,t} = a_1 + A_{1,11} y_{1,t-1} + \epsilon_{1,t}$$

$$y_{2,t} = a_2 + A_{1,21} y_{1,t-1} + \epsilon_{2,t}$$

$$\begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \sim \mathrm{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \right]$$

Simulation

$$y_{1,t+h} = a_1^{(d,h)} + A_{1,11}^{(d,h)} y_{1,t-1} + e_{1,t+h}^{(d,h)}$$

$$\sum_{s=0}^{h} y_{1,t+s} = a_1^{(l,h)} + A_{1,11}^{(l,h)} y_{1,t-1} + e_{1,t+h}^{(l,h)}$$

$$y_{2,t+h} = a_2^{(d,h)} + A_{1,21}^{(d,h)} y_{1,t-1} + e_{2,t+h}^{(d,h)}$$

$$\sum_{s=0}^{h} y_{2,t+s} = a_2^{(l,h)} + A_{1,21}^{(l,h)} y_{1,t-1} + e_{2,t+h}^{(l,h)}$$

Simulation Results Summary

	$T = 1260$ & $H = 20$			
	RMSE	90% CP	95% CP	99% CP
$A^{(d,h)}_{1,11}$	0.547	0.131	0.158	0.233
$A^{(l,h)}_{1,11}$	0.476	0.272	0.325	0.394
$A^{(d,h)}_{1,21}$	0.636	0.274	0.327	0.240
$A^{(l,h)}_{1,21}$	0.384	0.041	0.055	0.051
	$T = 240$ & $H = 5$			
	RMSE	90% CP	95% CP	99% CP
$A^{(d,h)}_{1,11}$	0.426	0.376	0.403	0.406
$A^{(l,h)}_{1,11}$	0.410	0.372	0.374	0.349
$A^{(d,h)}_{1,21}$	0.765	0.077	0.084	0.095
$A^{(l,h)}_{1,21}$	0.557	0.054	0.047	0.045

Variance Risk Premium and Euro Futures Return

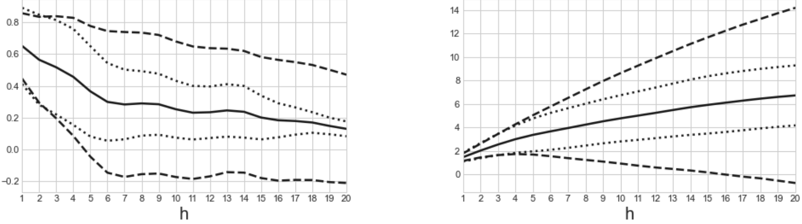
$$VP_{t+h} = a_1^{(d,h)} + A_{1,11}^{(d,h)} VP_{t-1} + e_{1,t+h}^{(d,h)}$$

$$\sum_{s=0}^{h} VP_{t+s} = a_1^{(l,h)} + A_{1,11}^{(l,h)} VP_{t-1} + e_{1,t+h}^{(l,h)}$$

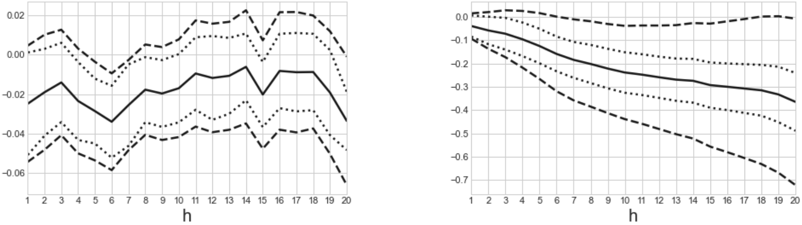
$$r_{t+h} = a_2^{(d,h)} + A_{1,21}^{(d,h)} VP_{t-1} + e_{2,t+h}^{(d,h)}$$

$$\sum_{s=0}^{h} r_{t+s} = a_2^{(l,h)} + A_{1,21}^{(l,h)} VP_{t-1} + e_{2,t+h}^{(l,h)}$$

(a) 95% Confidence Intervals: $A^{(d,h)}_{1,11}$ **and** $A^{(l,h)}_{1,11}$



(b) 95% Confidence Intervals: $A^{(d,h)}_{1,21}$ **and** $A^{(l,h)}_{1,21}$



Inflation and Equity Premium

$$Inf_{t+h} = a_1^{(d,h)} + A_{1,11}^{(d,h)} Inf_{t-1} + e_{1,t+h}^{(d,h)}$$

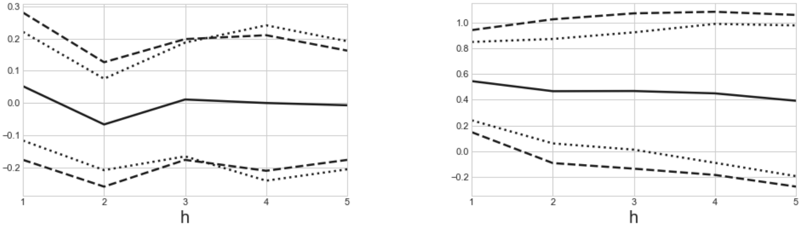
$$\sum_{s=0}^{h} Inf_{t+s} = a_1^{(l,h)} + A_{1,11}^{(l,h)} Inf_{t-1} + e_{1,t+h}^{(l,h)}$$

$$r_{t+h} = a_2^{(d,h)} + A_{1,21}^{(d,h)} Inf_{t-1} + e_{2,t+h}^{(d,h)}$$

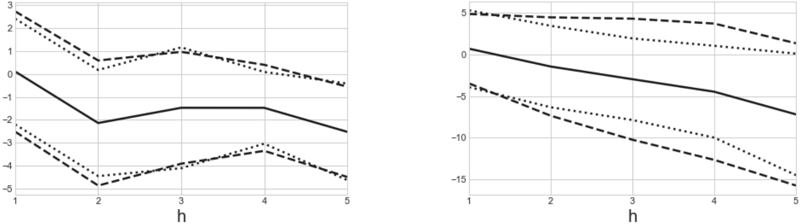
$$\sum_{s=0}^{h} r_{t+s} = a_2^{(l,h)} + A_{1,21}^{(l,h)} Inf_{t-1} + e_{2,t+h}^{(l,h)}$$

Inflation and Equity Premium

(a) 95% Confidence Intervals: $A^{(d,h)}_{1,11}$ and $A^{(l,h)}_{1,11}$



(b) 95% Confidence Intervals: $A^{(d,h)}_{1,21}$ and $A^{(l,h)}_{1,21}$



References I

West, Kenneth D. 1997. Another heteroskedasticity-and autocorrelation-consistent covariance matrix estimator. *Journal of Econometrics*, **76**(1-2), 171–191.

Thank you!