Stochastic Sequential Screening

Li, Hao
UBC

Xianwen Shi
University of Toronto

January 2023

Stochastic Allocation: Rationing and “Haggling”

- Rationing is common in practice.
- “Haggling” in marketplace – interpreted as random allocation by Riley and Zeckhauser (1983) – is also common.
- Randomization helps relax the incentive constraints in the static environment (Myerson 1981, Bulow and Roberts 1989).

- Relatively little is known about the role of randomization in the dynamic setting.

Stochastic Allocation: Rationing and “Haggling”

- Rationing is common in practice.
- “Haggling” in marketplace – interpreted as random allocation by Riley and Zeckhauser (1983) – is also common.
- Randomization helps relax the incentive constraints in the static environment (Myerson 1981, Bulow and Roberts 1989).

- Relatively little is known about the role of randomization in the dynamic setting.

First-Order Approach and Deterministic Mechanism

- We know little because almost all papers in the (dynamic) mechanism design literature adopt the first-order approach.
  - form a relaxed problem by retaining only local downward IC.
  - impose some regularity conditions so that the solution to the relaxed problem has sufficient monotonicity to ensure global IC.
- Under these regularity conditions, optimal mechanisms are deterministic.

First-Order Approach and Deterministic Mechanism

- We know little because almost all papers in the (dynamic) mechanism design literature adopt the first-order approach.
  - form a relaxed problem by retaining only local downward IC.
  - impose some regularity conditions so that the solution to the relaxed problem has sufficient monotonicity to ensure global IC.
- Under these regularity conditions, optimal mechanisms are deterministic.

Regularity and Monotonicity

- Regularity conditions are mild in static but not in dynamic setting.
- Static setting
  - static virtual surplus is increasing in agent's type,
  - $\Rightarrow$ allocation monotone in type, which is also necessary for IC.
- Dynamic setting
  - dynamic virtual surplus is increasing in ex ante and ex post types,
  - $\Rightarrow$ allocation is monotone is both ex ante and ex post types.

Problem: Monotonicity in ex ante type is not necessary; average monotonicity in ex ante type is.

Regularity and Monotonicity

- Regularity conditions are mild in static but not in dynamic setting.
- Static setting
  - static virtual surplus is increasing in agent's type,

  $\Rightarrow$ allocation monotone in type, which is also necessary for IC.
- Dynamic setting
  - dynamic virtual surplus is increasing in ex ante and ex post types,

  $\Rightarrow$ allocation is monotone is both ex ante and ex post types.

Problem: Monotonicity in ex ante type is not necessary; average monotonicity in ex ante type is.

Regularity and Monotonicity

- Regularity conditions are mild in static but not in dynamic setting.
- Static setting
  - static virtual surplus is increasing in agent's type,
  - $\Rightarrow$ allocation monotone in type, which is also necessary for IC.
- Dynamic setting
  - dynamic virtual surplus is increasing in ex ante and ex post types,
  - $\Rightarrow$ allocation is monotone is both ex ante and ex post types.

Problem: Monotonicity in ex ante type is not necessary; average monotonicity in ex ante type is.

This Paper

- Methodology
  - we propose a method to deal with failure of regularity.
  - we exploit necessity of average monotonicity of allocations.
  - our method is valid whenever local IC implies global IC.
- Results:
  - show when and how randomization can increase revenue in a sequential screening setting.
  - provide sufficient (and necessary) conditions for optimal mechanisms to be stochastic ($\Rightarrow$ new sufficient condition for optimal mechanisms to be deterministic).
  - characterize optimal stochastic contracts.

This Paper

- Methodology
  - we propose a method to deal with failure of regularity.
  - we exploit necessity of average monotonicity of allocations.
  - our method is valid whenever local IC implies global IC.
- Results:
  - show when and how randomization can increase revenue in a sequential screening setting.
  - provide sufficient (and necessary) conditions for optimal mechanisms to be stochastic ($\Rightarrow$ new sufficient condition for optimal mechanisms to be deterministic).
  - characterize optimal stochastic contracts.

Related Papers

- Classical static mechanism design: Myerson (1981) and Riley and Zeckhauser (1983)
- Stochastic sequential screening: Courty and Li (2000), Krahmer and Kovac (2016), Krähmer and Strausz (2017), Battaglini and Lamba (2019), Krasikov and Lamba (2021)
- Most related paper: Bergemann, Casto and Weintraub (2020)
  - ex post participation constraint
  - binary ex ante types

Related Papers

- Classical static mechanism design: Myerson (1981) and Riley and Zeckhauser (1983)
- Stochastic sequential screening: Courty and Li (2000), Krahmer and Kovac (2016), Krähmer and Strausz (2017), Battaglini and Lamba (2019), Krasikov and Lamba (2021)

- Most related paper: Bergemann, Casto and Weintraub (2020)
  - ex post participation constraint
  - binary ex ante types

Model Setup

- Sequential screening with one seller, one buyer and one object:
  - buyer's ex ante type $\theta \in \{H, M, L\}$, with probability ϕ_θ, is a noisy signal of his true value (ex post type) $\omega \in [\underline{\omega}, \overline{\omega}]$;
  - first-order stochastic dominance: $F_H(\omega) \leq F_M(\omega) \leq F_L(\omega)$ for all ω;
  - seller's reservation value $c \in (\underline{\omega}, \overline{\omega})$;
  - direct mechanism: allocation rule $x_\theta(\omega)$, payment rule $t_\theta(\omega)$.

Remark: We need at least three ex ante types because deterministic contracts are proved to be optimal with binary ex ante types.

- Timing of the game:
  1. buyer observes θ; seller proposes $\{x_\theta(\omega), t_\theta(\omega)\}$; buyer reports $\hat{\theta}$;
  2. buyer observes ω and reports $\hat{\omega}$; outcome $\left(x_{\hat{\theta}}(\hat{\omega}), t_{\hat{\theta}}(\hat{\omega})\right)$ is chosen.

Model Setup

- Sequential screening with one seller, one buyer and one object:
  - buyer's ex ante type $\theta \in \{H, M, L\}$, with probability ϕ_θ, is a noisy signal of his true value (ex post type) $\omega \in [\underline{\omega}, \overline{\omega}]$;
  - first-order stochastic dominance: $F_H(\omega) \leq F_M(\omega) \leq F_L(\omega)$ for all ω;
  - seller's reservation value $c \in (\underline{\omega}, \overline{\omega})$;
  - direct mechanism: allocation rule $x_\theta(\omega)$, payment rule $t_\theta(\omega)$.

Remark: We need at least three ex ante types because deterministic contracts are proved to be optimal with binary ex ante types.

- Timing of the game:
  1. buyer observes θ; seller proposes $\{x_\theta(\omega), t_\theta(\omega)\}$; buyer reports $\hat{\theta}$;
  2. buyer observes ω and reports $\hat{\omega}$; outcome $(x_{\hat{\theta}}(\hat{\omega}), t_{\hat{\theta}}(\hat{\omega}))$ is chosen.

Model Setup

- Sequential screening with one seller, one buyer and one object:
  - buyer's ex ante type $\theta \in \{H, M, L\}$, with probability ϕ_θ, is a noisy signal of his true value (ex post type) $\omega \in [\underline{\omega}, \overline{\omega}]$;
  - first-order stochastic dominance: $F_H(\omega) \leq F_M(\omega) \leq F_L(\omega)$ for all ω;
  - seller's reservation value $c \in (\underline{\omega}, \overline{\omega})$;
  - direct mechanism: allocation rule $x_\theta(\omega)$, payment rule $t_\theta(\omega)$.

Remark: We need at least three ex ante types because deterministic contracts are proved to be optimal with binary ex ante types.

- Timing of the game:
  1. buyer observes θ; seller proposes $\{x_\theta(\omega), t_\theta(\omega)\}$; buyer reports $\hat{\theta}$;
  2. buyer observes ω and reports $\hat{\omega}$; outcome $\left(x_{\hat{\theta}}(\hat{\omega}), t_{\hat{\theta}}(\hat{\omega})\right)$ is chosen.

Incentive and Participation Constraints

- Period-two IC$_\theta$: for all $\omega, \hat{\omega}$,

$$\omega x_\theta(\omega) - t_\theta(\omega) \geq \omega x_\theta(\hat{\omega}) - t_\theta(\hat{\omega}).$$

- Period-one IC$_{\theta\hat{\theta}}$: for all $\hat{\theta} \neq \theta$,

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(\omega x_\theta\left(\omega\right) - t_\theta\left(\omega\right)\right) f_\theta\left(\omega\right) d\omega \geq \int_{\underline{\omega}}^{\overline{\omega}} \left(\omega x_{\hat{\theta}}\left(\omega\right) - t_{\hat{\theta}}\left(\omega\right)\right) f_\theta\left(\omega\right) d\omega.$$

- Period-one (interim) IR$_\theta$:

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(\omega x_\theta\left(\omega\right) - t_\theta\left(\omega\right)\right) f_\theta\left(\omega\right) d\omega \geq 0.$$

Buyer's ex post payoff may fall below 0.

Seller's Problem

Seller chooses and commits to mechanism $\{x_\theta(\omega), t_\theta(\omega)\}$ to solve

$$\max_{(x_\theta, t_\theta)} \sum_{\theta=H,M,L} \phi_\theta \int_{\underline{\omega}}^{\overline{\omega}} (t_\theta(\omega) - c x_\theta(\omega)) f_\theta(\omega) \, d\omega$$

subject to IC$_\theta$, IC$_{\theta\hat{\theta}}$ and IR$_\theta$, for all $\theta, \hat{\theta} \in \{H, M, L\}$.

Simplified Problem

- As usual, no distortion at the top with $x_H(\omega) = \mathbb{1}\{\omega \geq c\}$.
- Choose weakly increasing $x_M(\omega)$ and $x_L(\omega)$ to maximize

$$\phi_M \int_{\underline{\omega}}^{\overline{\omega}} x_M(\omega)\delta_M(\omega) f_M(\omega) d\omega + \phi_L \int_{\underline{\omega}}^{\overline{\omega}} x_L(\omega)\delta_L(\omega) f_L(\omega) d\omega,$$

subject to an average monotonicity condition MON_{ML}:

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_M(\omega) - x_L(\omega)\right)\left(F_L(\omega) - F_M(\omega)\right) d\omega \geq 0,$$

where $\delta_\theta(\omega)$ are virtual surplus functions.

Optimal Deterministic Mechanism

- A mechanism with $x_\theta(\omega) = \mathbb{1}\{\omega \geq k_\theta\}$ is optimal iff $c \leq k_M \leq k_L$.
- Let (k_L, k_M) be the deterministic solution to the simplified problem:

$$\max_{k_M \leq k_L} \int_{k_M}^{\overline{\omega}} \phi_M \delta_M(\omega) f_M(\omega) d\omega + \int_{k_L}^{\overline{\omega}} \phi_L \delta_L(\omega) f_L(\omega) d\omega.$$

  Then we have

$$(k_L, k_M) = \begin{cases} (\hat{k}_L, \hat{k}_M) & \text{if} \quad \hat{k}_M \leq \hat{k}_L \\ (\hat{k}, \hat{k}) \text{ with } \hat{k} \in (\hat{k}_L, \hat{k}_M) & \text{if} \quad \hat{k}_M > \hat{k}_L \end{cases}$$

  where $\hat{k}_M$, $\hat{k}_L$ are unique maximizers of the two terms, respectively.

- Each (k_L, k_M) corresponds to an optimal deterministic mechanism.

Optimal Deterministic Mechanism

- A mechanism with $x_\theta(\omega) = \mathbb{1}\{\omega \geq k_\theta\}$ is optimal iff $c \leq k_M \leq k_L$.
- Let (k_L, k_M) be the deterministic solution to the simplified problem:

$$\max_{k_M \leq k_L} \int_{k_M}^{\overline{\omega}} \phi_M \delta_M(\omega) f_M(\omega) d\omega + \int_{k_L}^{\overline{\omega}} \phi_L \delta_L(\omega) f_L(\omega) d\omega.$$

Then we have

$$(k_L, k_M) = \begin{cases} (\hat{k}_L, \hat{k}_M) & \text{if} \quad \hat{k}_M \leq \hat{k}_L \\ (\hat{k}, \hat{k}) \text{ with } \hat{k} \in (\hat{k}_L, \hat{k}_M) & \text{if} \quad \hat{k}_M > \hat{k}_L \end{cases}$$

where $\hat{k}_M$, $\hat{k}_L$ are unique maximizers of the two terms, respectively.

- Each (k_L, k_M) corresponds to an optimal deterministic mechanism.

Optimal Deterministic Mechanism

- A mechanism with $x_\theta(\omega) = \mathbb{1}\{\omega \geq k_\theta\}$ is optimal iff $c \leq k_M \leq k_L$.
- Let (k_L, k_M) be the deterministic solution to the simplified problem:

$$\max_{k_M \leq k_L} \int_{k_M}^{\overline{\omega}} \phi_M \delta_M(\omega) f_M(\omega) d\omega + \int_{k_L}^{\overline{\omega}} \phi_L \delta_L(\omega) f_L(\omega) d\omega.$$

  Then we have

$$(k_L, k_M) = \begin{cases} (\hat{k}_L, \hat{k}_M) & \text{if} \quad \hat{k}_M \leq \hat{k}_L \\ (\hat{k}, \hat{k}) \text{ with } \hat{k} \in (\hat{k}_L, \hat{k}_M) & \text{if} \quad \hat{k}_M > \hat{k}_L \end{cases}$$

  where $\hat{k}_M$, $\hat{k}_L$ are unique maximizers of the two terms, respectively.

- Each (k_L, k_M) corresponds to an optimal deterministic mechanism.

Toward Sufficient Condition for Randomization

- Failure of regularity ($\hat{k}_M > \hat{k}_L$) is necessary for randomization, but is not sufficient, because it can be cheaper, in satisfying MON$_{ML}$

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_M\left(\omega\right) - x_L\left(\omega\right)\right)\left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega \geq 0,$$

  to set $k_L = k_M = \hat{k}$ than to randomize.

- If we can construct a stochastic mechanism that is more profitable than the deterministic solution $\hat{k}$, then optimal mechanisms must be stochastic.

Toward Sufficient Condition for Randomization

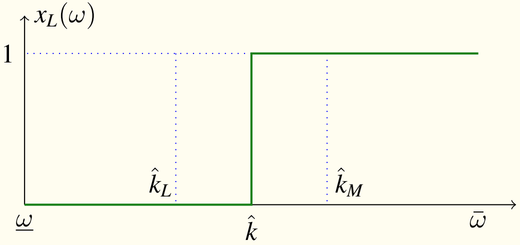
- Failure of regularity ($\hat{k}_M > \hat{k}_L$) is necessary for randomization, but is not sufficient, because it can be cheaper, in satisfying MON$_{ML}$

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(x_M\left(\omega\right) - x_L\left(\omega\right)\right)\left(F_L\left(\omega\right) - F_M\left(\omega\right)\right) d\omega \geq 0,$$

  to set $k_L = k_M = \hat{k}$ than to randomize.

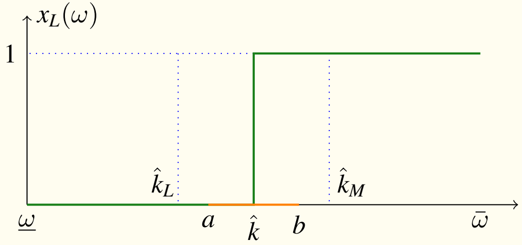
- If we can construct a stochastic mechanism that is more profitable than the deterministic solution $\hat{k}$, then optimal mechanisms must be stochastic.

Idea of Construction



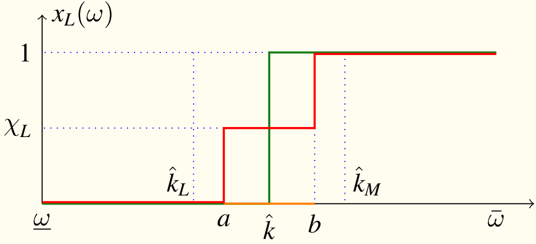
- Start from deterministic solution $k^M = k^L = \hat{k}$;
- Fix some interval $[a, b] \ni \hat{k}$ for type L (or type M);
- Set $x_L(\omega) = \chi_L \in (0, 1)$ for all $\omega \in [a, b]$ to bind MON_{ML}: $\int_{\hat{k}}^{b}(F_L(\omega) - F_M(\omega))d\omega = \chi_L \int_a^b (F_L(\omega) - F_M(\omega))d\omega$;
- Check if gain of (virtual) surplus over $[a, \hat{k}]$ exceeds loss over $[\hat{k}, b]$: $\int_a^{\hat{k}} \chi_L \phi_L \delta_L(\omega) f_L(\omega) d\omega \geq \int_{\hat{k}}^{b} (1 - \chi_L)\phi_L \delta_L(\omega) f_L(\omega) d\omega$.

Idea of Construction



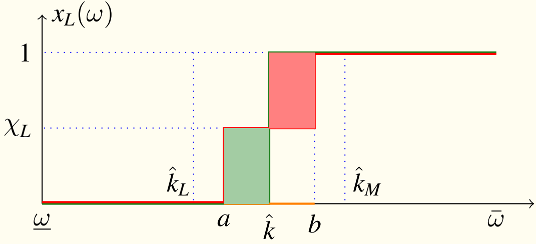
- Start from deterministic solution $k^M = k^L = \hat{k}$;
- Fix some interval $[a, b] \ni \hat{k}$ for type L (or type M);
- Set $x_L(\omega) = \chi_L \in (0, 1)$ for all $\omega \in [a, b]$ to bind MON_{ML}:
  $\int_{\hat{k}}^{b}(F_L(\omega) - F_M(\omega))d\omega = \chi_L \int_a^b (F_L(\omega) - F_M(\omega))d\omega$;
- Check if gain of (virtual) surplus over $[a, \hat{k}]$ exceeds loss over $[\hat{k}, b]$:
  $\int_a^{\hat{k}} \chi_L \phi_L \delta_L(\omega) f_L(\omega) d\omega \geq \int_{\hat{k}}^{b}(1 - \chi_L)\phi_L \delta_L(\omega) f_L(\omega) d\omega$.

Idea of Construction



- Start from deterministic solution $k^M = k^L = \hat{k}$;
- Fix some interval $[a,b] \ni \hat{k}$ for type L (or type M);
- Set $x_L(\omega) = \chi_L \in (0,1)$ for all $\omega \in [a,b]$ to bind MON$_{ML}$: $\int_{\hat{k}}^{b}(F_L(\omega) - F_M(\omega))d\omega = \chi_L \int_a^b (F_L(\omega) - F_M(\omega))d\omega$;
- Check if gain of (virtual) surplus over $[a,\hat{k}]$ exceeds loss over $[\hat{k},b]$: $\int_a^{\hat{k}} \chi_L \phi_L \delta_L(\omega) f_L(\omega)d\omega \geq \int_{\hat{k}}^{b}(1-\chi_L)\phi_L\delta_L(\omega)f_L(\omega)d\omega$.

Idea of Construction



- Start from deterministic solution $k^M = k^L = \hat{k}$;
- Fix some interval $[a, b] \ni \hat{k}$ for type L (or type M);
- Set $x_L(\omega) = \chi_L \in (0, 1)$ for all $\omega \in [a, b]$ to bind MON_{ML}: $\int_{\hat{k}}^{b}(F_L(\omega) - F_M(\omega))d\omega = \chi_L \int_a^b (F_L(\omega) - F_M(\omega))d\omega$;
- Check if gain of (virtual) surplus over $[a, \hat{k}]$ exceeds loss over $[\hat{k}, b]$: $\int_a^{\hat{k}} \chi_L \phi_L \delta_L(\omega) f_L(\omega) d\omega \geq \int_{\hat{k}}^{b}(1 - \chi_L)\phi_L \delta_L(\omega) f_L(\omega) d\omega$.

Surplus-to-Rent Ratio

- The randomization in the construction improves revenue if

$$\frac{\int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_a^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega} \geq \frac{\int_{\hat{k}}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_{\hat{k}}^b (F_L(\omega) - F_M(\omega)) d\omega}.$$

- For any $w' < w''$, define the average surplus-to-rent ratio for θ as

$$R_\theta(w', w'') = \frac{\int_{w'}^{w''} \phi_\theta \delta_\theta(\omega) f_\theta(\omega) d\omega}{\int_{w'}^{w''} (F_L(\omega) - F_M(\omega)) d\omega}.$$

- For any ω, define the point surplus-to-rent ratio ratio at ω as

$$r_\theta(\omega) = \frac{\phi_\theta \delta_\theta(\omega) f_\theta(\omega)}{F_L(\omega) - F_M(\omega)}.$$

Surplus-to-Rent Ratio

- The randomization in the construction improves revenue if

$$\frac{\int_a^{\hat{k}} \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_a^{\hat{k}} (F_L(\omega) - F_M(\omega)) d\omega} \geq \frac{\int_{\hat{k}}^b \phi_L \delta_L(\omega) f_L(\omega) d\omega}{\int_{\hat{k}}^b (F_L(\omega) - F_M(\omega)) d\omega}.$$

- For any $w' < w''$, define the average surplus-to-rent ratio for θ as

$$R_\theta(w', w'') = \frac{\int_{w'}^{w''} \phi_\theta \delta_\theta(\omega) f_\theta(\omega) d\omega}{\int_{w'}^{w''} (F_L(\omega) - F_M(\omega)) d\omega}.$$

- For any ω, define the point surplus-to-rent ratio ratio at ω as

$$r_\theta(\omega) = \frac{\phi_\theta \delta_\theta(\omega) f_\theta(\omega)}{F_L(\omega) - F_M(\omega)}.$$

Optimality of Deterministic Mechanisms

Theorem

Optimal mechanism is deterministic if either one of the following two conditions holds:

1. $\hat{k}_M \leq \hat{k}_L$ *(regularity condition);*
2. $\hat{k}_M > \hat{k}_L$, $\hat{k}$ *is interior, and the following condition holds for all* θ*:*

$$\max_{\omega \leq \hat{k}} R_\theta(\omega, \hat{k}) \leq r_\theta(\hat{k}) \leq \min_{\omega \geq \hat{k}} R_\theta(\hat{k}, \omega). \quad (1)$$

Remark: (i) Condition (1) is satisfied if $r_\theta(\omega)$ is increasing and is violated if $r_\theta(\omega)$ is decreasing. (ii) Condition (1) is necessary for randomization when regularity fails.

Optimality of Deterministic Mechanisms

Theorem

Optimal mechanism is deterministic if either one of the following two conditions holds:

1. $\hat{k}_M \leq \hat{k}_L$ *(regularity condition);*
2. $\hat{k}_M > \hat{k}_L$, $\hat{k}$ *is interior, and the following condition holds for all* θ*:*

$$\max_{\omega \leq \hat{k}} R_\theta(\omega, \hat{k}) \leq r_\theta(\hat{k}) \leq \min_{\omega \geq \hat{k}} R_\theta(\hat{k}, \omega). \tag{1}$$

Remark: (i) Condition (1) is satisfied if $r_\theta(\omega)$ is increasing and is violated if $r_\theta(\omega)$ is decreasing. (ii) Condition (1) is necessary for randomization when regularity fails.

Optimality of Deterministic Mechanisms

Theorem

Optimal mechanism is deterministic if either one of the following two conditions holds:

1. $\hat{k}_M \leq \hat{k}_L$ *(regularity condition);*
2. $\hat{k}_M > \hat{k}_L$, $\hat{k}$ *is interior, and the following condition holds for all* θ*:*

$$\max_{\omega \leq \hat{k}} R_\theta(\omega, \hat{k}) \leq r_\theta(\hat{k}) \leq \min_{\omega \geq \hat{k}} R_\theta(\hat{k}, \omega). \quad (1)$$

Remark: (i) Condition (1) is satisfied if $r_\theta(\omega)$ is increasing and is violated if $r_\theta(\omega)$ is decreasing. (ii) Condition (1) is necessary for randomization when regularity fails.

Sufficient Conditions for Randomization

Define function $\alpha(\omega)$ as

$$F_M(\omega) = [1 - \alpha(\omega)] F_L(\omega) + \alpha(\omega) F_H(\omega).$$

Theorem

Suppose $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. Any optimal mechanism is stochastic if either of the following two conditions holds:
(i) $\alpha(\omega)$ is weakly decreasing and

$$\max_{\omega \leq \hat{k}} R_L(\omega, \hat{k}) > \min_{\omega \geq \hat{k}} R_L(\hat{k}, \omega); \tag{2}$$

(ii) $\alpha(\omega)$ is weakly increasing and

$$\max_{\omega \leq \hat{k}} R_M(\omega, \hat{k}) > \min_{\omega \geq \hat{k}} R_M(\hat{k}, \omega). \tag{3}$$

Monotonicity of $\alpha(\omega)$ ensures that the stochastic solution to the simplified problem corresponds to an optimal mechanism.

Sufficient Conditions for Randomization

Define function $\alpha(\omega)$ as

$$F_M(\omega) = [1 - \alpha(\omega)] F_L(\omega) + \alpha(\omega) F_H(\omega).$$

Theorem

Suppose $\hat{k}_M > \hat{k}_L$ and $\hat{k}$ is interior. Any optimal mechanism is stochastic if either of the following two conditions holds:
(i) $\alpha(\omega)$ is weakly decreasing and

$$\max_{\omega \le \hat{k}} R_L(\omega, \hat{k}) > \min_{\omega \ge \hat{k}} R_L(\hat{k}, \omega); \tag{2}$$

(ii) $\alpha(\omega)$ is weakly increasing and

$$\max_{\omega \le \hat{k}} R_M(\omega, \hat{k}) > \min_{\omega \ge \hat{k}} R_M(\hat{k}, \omega). \tag{3}$$

Monotonicity of $\alpha(\omega)$ ensures that the stochastic solution to the simplified problem corresponds to an optimal mechanism.

Optimal Randomization: Preliminary Observations

- At most one level of stochastic allocation for types M and L.
- Randomization for at most one type.
- If $r_\theta(\omega)$ is strictly increasing in some interval, then no randomization can happen in any subinterval for type θ; if $r_\theta(\omega)$ is strictly decreasing on some interval, then allocation $x_\theta(\omega)$ cannot be deterministic with a threshold on the interval.

Optimal Randomization: Preliminary Observations

- At most one level of stochastic allocation for types M and L.
- Randomization for at most one type.
- If $r_\theta(\omega)$ is strictly increasing in some interval, then no randomization can happen in any subinterval for type θ; if $r_\theta(\omega)$ is strictly decreasing on some interval, then allocation $x_\theta(\omega)$ cannot be deterministic with a threshold on the interval.

Optimal Randomization: Characterization

Theorem

Suppose $r_\theta(\omega)$ is single-dipped for $\theta = L, M$, and appropriate boundary conditions are satisfied.

(i) if $r_{\theta'}(\omega)$ is strictly increasing, then the simplified problem is solved by a stochastic allocation for θ and a deterministic allocation for θ'.

(ii) if $r_{\theta'}(\omega)$ is strictly decreasing, then the simplified problem is solved by a deterministic allocation for θ and a stochastic allocation for θ'.

Again, monotonicity of $\alpha(\omega)$ can be imposed to ensure that the solution to the simplified problem corresponds to an optimal mechanism.

Optimal Randomization: Characterization

Theorem

Suppose $r_\theta(\omega)$ is single-dipped for $\theta = L, M$, and appropriate boundary conditions are satisfied.

(i) if $r_{\theta'}(\omega)$ is strictly increasing, then the simplified problem is solved by a stochastic allocation for θ and a deterministic allocation for θ'.

(ii) if $r_{\theta'}(\omega)$ is strictly decreasing, then the simplified problem is solved by a deterministic allocation for θ and a stochastic allocation for θ'.

Again, monotonicity of $\alpha(\omega)$ can be imposed to ensure that the solution to the simplified problem corresponds to an optimal mechanism.

Extension to Many Types

- Alignment condition (constant $\alpha(\omega)$): $\forall i \neq j$, $\forall i' \neq j'$, and $\forall \omega$,

$$\frac{f_i(\omega) - f_j(\omega)}{F_i(\omega) - F_j(\omega)} = \frac{f_{i'}(\omega) - f_{j'}(\omega)}{F_{i'}(\omega) - F_{j'}(\omega)}.$$

- Sufficient conditions for randomization can be found by the same construction, but they are no longer necessary.
- Ironing procedure for finding optimal stochastic mechanisms can be extended.

Conclusion

- We study randomization in a model of sequential screening.
- We provide necessary and sufficient conditions for randomization to be optimal.
- We characterize the optimal stochastic mechanism.