Solving the Life-Cycle Model with Labour Income Uncertainty: Some Implications of Income Volatility for Consumption Plan

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Abstract

We derive a generalised version of the Ramsey-type consumption function when labour income is assumed to follow the standard geometric Brownian motion, and show how the propensity to consume might depend on its drift and diffusion parameters. This enables us to explain the circumstance in which precautionary savings can arise when a risk averse consumer faces income uncertainty, and to resolve the main consumption puzzles: excess smoothness and excess sensitivity of consumption relative to income and its insensitivity to the real interest rate. Our results also show how labour income uncertainty could explain the existence of a subsistence level of consumption and, in that context, shed light on Kuznets' paradox regarding constancy of the average propensity to consume in the long run. Finally, we find that using the subjective rate of time preference as the sole measure of a consumer's impatience to consume could be misleading when the path of labour income is volatile.

Keywords: life-cycle model; income volatility; geometric Brownian motion; risk aversion; precautionary savings; excess sensitivity; excess smoothness; Kuznets' paradox

JEL Classification Codes: C61, D11, E21, D91; D80.

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1. Introduction

This paper fills a gap in the theoretical life-cycle/permanent-income literature by deriving a closed-form solution to the standard intertemporal utility maximisation problem of a risk averse representative consumer whose labour income path is assumed to be volatile. Zeldes (1989) firmly acknowledged this gap and its possible consequences for interpreting empirical evidence. He used a numerical solution method to derive an accurate approximation to the solution of the stochastic optimisation problem which he found to have different implications from that obtained under certainty equivalence. His solution provided explanations for the main (inter-related) puzzles regarding the excess sensitivity and excess smoothness of consumption, as well as the emergence of precautionary saving behaviour, which Deaton (1987) had signalled when questioning the empirical validity of the life-cycle model. Given its importance, precautionary saving behaviour has been intensively studied in the literature and there is now theoretical consensus on the positive (negative) effect of rising income uncertainty on savings (consumption). Formal theoretical work on precautionary savings behaviour dates back to Leland (1968) and Sandmo (1970), and it has kept evolving steadily. More recently, Deaton (1991) and Carroll (1992, 1994) made considerable headway by showing that prudent consumers' propensity to consume out of income would be smaller the more volatile is the path of income, concluding that precautionary motives are likely to trigger wealth accumulation when consumers perceive a rise in uncertainty regarding their income profile.[1] Evidence on this phenomenon had already started to appear in the literature in the late 1980s – e.g., Campbell (1987) examined the predictive power of saving for declines in labour income, and Skinner (1988) investigated consequences of having a riskier occupation for savings – and the question still continues to generate interesting empirical work. Lugilde et al. (2017) provide a useful summary of the results reported in the literature and conclude that the existing evidence remains inconclusive. Perhaps, to a greater extent, their conclusion resonates the earlier concern raised by Carroll and Kimball (2008) that "*researchers have not yet achieved consensus on how the wide variety of survey and empirical evidence should be integrated with theory*." Notwithstanding this concern, however, there exists sufficiently robust evidence to render the phenomenon important. For instance:

- Caballero (1990) defines a measure of precautionary savings by decomposing consumption and uses it in conjunction with evidence obtained from US data to explain the puzzling empirical evidence regarding excess-growth, excess-smoothness, and excess-sensitivity of consumption.[2]

[1] Theoretical work mainly focussed on issues such as extending the time horizon of the consumer and examining the role of risk aversion properties of the utility function and the size of elasticity of intertemporal substitution in determining the impact of uncertainty; see, e.g. Kimball (1990, 1991) and Kimball and Weil (2009).

[2] Caballero also obtains a closed-form solution using a constant absolute risk averse utility function and setting the subjective rate of time preference equal to the real interest rate. However, he incorporates the income uncertainty in retrospect via augmenting the implied Euler equation he obtains from utility maximisation with an income process which is approximated by an MA equation. Solving these jointly then yields his so-called closed-form solution. His approach does not allow the consumer to take account of the information on income volatility at the optimisation stage and does not, therefore, follow the conventional principles of stochastic dynamic programming.

- Carroll (1994) uses US data from Consumer Expenditure Survey (for 1960-1961 participants) and Panel Study of Income Dynamics (1969-1985 period) and finds that consumers who face greater income uncertainty consume less.
- Kazarosian (1997) uses the National Longitudinal Survey of Older and Young Men (US Bureau of Labour Statistics) over the period 1966-1981 and finds evidence for a strong precautionary motive: "*A doubling of uncertainty increases the ratio of wealth to permanent income by 29%.*" Carroll and Samwick (1998) report very similar results.
- Hahm and Steigerwald (1999) use a US panel data on individual forecasters to construct a time-series measure of income uncertainty which they find to be positively correlated with aggregate saving rate.
- Engen and Gruber (2001) simulate a stochastic life-cycle model and find that reducing income risk by providing a more generous unemployment insurance scheme (similar to typical U.S. schemes) reduces a median worker's precautionary savings. This result is then confirmed by data on expected unemployment benefit replacement rates and financial assets held by households in the Survey of Income and Program Participation (based on US Census data).
- Gourinchas and Parker (2002) use a life cycle model to estimate the optimal consumption expenditures and saving behaviour with existence of labour income uncertainty. They find that optimal consumption significantly changes over the lifecycle.
- Murata (2003) uses self-reported measures of income uncertainty (drawn from Japanese household data for those in their 30s) and finds that uncertainty concerning public pension benefits stimulates precautionary savings, mainly among either nuclear-family households or households that do not receive income transfers from parents.
- Pozzi (2005) uses quarterly aggregate US data over the period 1952-2001 and finds evidence that favours a negative correlation between consumer-specific income risk and consumers' expenditure.
- Garz (2014) uses time-series data for Germany over the period 2001-2009 and finds that favourable news stimulate consumer spending.
- Christelis et al. (2020) use survey data from a representative sample of Dutch households and find that expected consumption risk is positively correlated with being self-employment and with having risky income, and is negatively correlated with age.

In Section 2 we set up and solve the stochastic life-cycle optimisation problem for a risk averse consumer and interpret the results. We derive a generalised version of the Ramsey-type consumption function when income follows the standard geometric Brownian motion, and find that propensity to consume depends on its drift and volatility parameters. This generalisation is then used to explain why precautionary savings, or in general underspending, can arise when the consumer faces uninsurable labour income uncertainty, and clarifies why, in fact, it is not feasible to expect a firm empirical consensus that precautionary saving is an inevitable response to uncertainty. Our solutions, for the optimal level and path of consumption, also resolve the main consumption puzzles noted in the literature regarding the excess smoothness of consumption relative to income, the excess sensitivity of consumption to income beyond that predicted by the life-cycle/permanent-income hypothesis, and the failure to find a positive and statistically significant explanatory role for the real interest rate as the main determinant of consumption growth. Finally, our results also show how labour income uncertainty could explain the existence of a subsistence level of consumption and, in that context, clarify an early consumption puzzle pointed out by Kuznets' regarding the conflict between his observation that the average propensity to consume had remained remarkably constant over a long period, and the early evidence (based on estimation of the absolute income hypothesis model ascribed to Keynes) that implied the average propensity to consume should decline the short-run as income rises. The existence of a subsistence level of consumption, which arises because of income uncertainty, also implies that using the subjective rate of time preference as the sole measure of a consumer's impatience to consume could be misleading: ceteris paribus, a higher degree of impatience characterised by a larger subjective rate of time preference does not necessarily lead to a higher consumption level when labour income path is sufficiently volatile. Section 3 offers a brief conclusion.

2. Optimal life-cycle consumption with uncertain labour income

The standard generalisation of the basic version of Ramsey-type intertemporal utility maximisation problem with a stochastic exogenous path of labour income is as follows: at any time t, the infinitely-lived representative consumer chooses her consumption stream $\{c_s\}_{s\geq t}$ to maximise her expected discounted life-time utility,

$$E_t\left[\int_{s=t}^{\infty} e^{-\rho(s-t)}u(c_s)ds\right], \tag{1}$$

subject to her wealth accumulation constraint

$$dA_s = (rA_s + y_s - c_s)ds. \tag{2}$$

All variables are in real terms, E_t is the expectations operator based on information available at time t, $u(\cdot)$ is the function measuring instantaneous consumption utility and satisfies the standard concavity properties, ρ is a constant time-preference rate used to discount future utility, A is the stock of accumulated savings (non-human, or financial, wealth) and its initial level A_t is known, r is the constant real rate of return on savings, and y is labour income which we assume to follow the geometric Brownian motion

$$dy_s = \mu y_s ds + \sigma y_s d\omega_s. \tag{3}$$

$\{\omega_s\}_{s\geq t}$ is the standard Wiener process; $\omega_s = \int_t^s d\omega_s$ where $d\omega_s \sim N(0, ds)$ is the independently distributed income innovation, and $\mu < r$ and $\sigma > 0$ are constant parameters respectively representing the drift and volatility of income. Therefore, future labour income evolves according to[3]

$$y_s = y_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(s-t)+\sigma\omega_s}, \; s > t, \tag{4}$$

[3] This is the continuous-time analogue of the discrete-time case in which $\ln y_s$ follows random walk with a drift.

and, given the initial income y_t, the distribution of future income is known and μ represents the growth rate of expected income: $E_t y_s = y_t e^{\mu(s-t)},\ s > t.$

A closed-form solution to the above problem can prove useful in understanding how income volatility affects the optimal level and path of consumption, in particular explaining certain discrepancies between data-based evidence and theoretical implications that are typically derived under the certainty equivalence assumption. However, as noted by Zeldes (1989), such a solution for the case where $u(c)$ is of constant relative risk aversion form does not exist in the literature and, to the best of our knowledge, still remains unavailable.

The standard way to solve the above problem is to use the corresponding Hamiltonian-Jacobi-Bellman equation

$$\rho V \ = \ \max_c \left\{ u(c) + (rA + y - c)V_A + \mu y V_y + \frac{1}{2}\sigma^2 y^2 V_{yy} \right\}, \tag{5}$$

where V is the implied value function to be specified and V_A, V_y and V_{yy} are its partial derivatives. The first order condition with respect to consumption is $V_A = u'(c)$. Therefore, under the appropriate concavity assumption, $c = {u'}^{-1}(V_A)$ which can be used to eliminate c from (5) and obtain a second order partial differential equation whose solution will then enable finding the implied consumption function, namely $c(A, y) = {u'}^{-1}\big(V_A(A, y)\big)$. The common approach to solving the problem is to guess a consistent functional form for the value function $V(A, y)$ – based on $u(c)$ and the anticipated properties of $c(A, y)$ – and then find the specific conditions under which it satisfies (5). This approach, however, does not seem to yield a solution in the case we are considering here.[4] As a result, a solution that generalises, for the existence of income growth and volatility, the standard Ramsey-type consumption function which is obtained under certainty where y_t remains constant and $\sigma = \mu = 0$, that is

$$c_t = \frac{\rho}{r}(rA_t + y), \tag{6}$$

has not yet been found. Below we offer such a solution based on the simplest relative risk averse preferences by assuming[5] $u(c) = \ln c$. Thus, we use the implied first order condition, $c^{-1} = V_A$, to eliminate c from (5) and rewrite it as

$$\rho V = \ln V_A^{-1} + (rA + y)V_A + \mu y V_y + \frac{1}{2}\sigma^2 y^2 V_{yy} - 1. \tag{7}$$

A 'generic' solution for (7) can be assumed to have the form[6]

$$V = \frac{A + \varphi y}{\kappa y^{\alpha}} + \lambda \ln y + \ \pi, \tag{8}$$

[4] The problem we are trying to solve here is very similar to the stochastic portfolio model outlined in Merton (1971) but assigns the uncertainty to the exogenous labour income stream instead.

[5] Using $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$ instead, where $\gamma > 0$ is the risk-aversion parameter and $1/\gamma$ is the elasticity of intertemporal substitution, will generalise preferences but will also considerably complicate the derivation of a closed-form solution. We are therefore using the special case of the latter by setting $\gamma = 1/\gamma = 1$.

[6] This is chosen on the grounds that any solution to (7) ought to be consistent with the nonlinear relationships that could exist between c and y given (3) as well as satisfying (2) and the implied life-time budget constraint. In particular, the usual constant elasticity function κy^{α} is used to reflect the steady state functional form.

where $\alpha, \varphi, \lambda, \pi$ and κ are constant parameters that depend on the parameters of the model (μ, σ, r, ρ) and on the initial conditions regarding A_t and y_t.

The corresponding generic solution for consumption therefore is

$$c = \kappa y^{\alpha}, \tag{9}$$

which is obtained using the first order condition $c^{-1} = V_A$ together with the partial derivative of (8) with respect to A, namely

$$V_A = \frac{1}{\kappa y^{\alpha}}. \tag{10}$$

Totally differentiating (9) using Ito's Lemma gives $dc = \frac{\partial c}{\partial y} dy + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 c}{\partial y^2} ds$ which, upon substitution for the partial derivatives $\frac{\partial c}{\partial y} = \alpha \frac{c}{y}$ and $\frac{\partial^2 c}{\partial y^2} = \alpha(\alpha - 1)\frac{c}{y^2}$ and for dy from (3), yields

$$dc = \mu_c c ds + \sigma_c c d\omega. \tag{11}$$

Thus, at any time t, consumption evolves according to the geometric Brownian motion in (11) where $\mu_c = \alpha\mu + \frac{1}{2}\alpha(\alpha - 1)\sigma^2$ and $\sigma_c = \alpha\sigma$. This is the continuous-time analogue of the discrete time optimal consumption path proposed by Hall (1978) who concluded that consumption is likely to follow random walk with drift. The main advantage of deriving it in this way, in addition to explicitly relating the drift and volatility of consumption to those of labour income, is that we are now able to obtain a solution for the level of the path. To do so, note that the future level of consumption obeys

$$c_s = c_t e^{\left(\mu_c - \frac{\sigma_c^2}{2}\right)(s-t) + \sigma_c \omega_s}, s \geq t. \tag{12}$$

Note that later in the next section, we show the situations become more complicated for consumption since there are two characteristic roots for α. The behaviour of consumption becomes $c = \kappa_1 y^{\alpha_1} + \kappa_2 y^{\alpha_2}$. We then proceed with special cases in the next section to discuss the dynamics of consumption and labour income uncertainty.

Provided that the transversality condition $\lim_{s\to\infty} A_s e^{-rs} = 0$ holds, the expected life-time budget constraint implied by (2) is

$$\int_0^{\infty} E_t c_s e^{-r(s-t)} ds = A_t + \int_0^{\infty} E_t y_s e^{-r(s-t)} dt \tag{13}$$

which, using (4) and (12) and assuming $\mu_c < r$, yields

$$\frac{c_t}{r - \mu_c} = A_t + \frac{y_t}{r - \mu}.$$

This result can then be rewritten in the form of a conventional consumption function as

$$c_t = \theta(rA_t + \beta y_t), \tag{14}$$

where $\theta = 1 - \mu_c/r$ is the propensity to consume from the 'drift-adjusted' current total income and $\beta = (1 - \mu/r)^{-1}$ is the scale factor that adjusts the 'actual' current total income,

$(rA_t + y_t)$, for the drift in labour income, μ.[7] To find the condition under which the consumer saves, we use (14) and $\theta = 1 - \mu_c/r$ to rewrite (2) as

$$\frac{dA_t}{dt} = \mu_c A_t + \left(\frac{\mu_c - \mu}{r - \mu}\right) y_t. \tag{15}$$

Provided that $A_t > 0$, $\frac{dA_t}{dt} > 0$ always holds when $\mu_c \geq \mu > 0$, but when $\mu_c < \mu$ the consumer will only save if her income is below a certain level, namely $y_t < (r - \mu)(\mu/\mu_c - 1)^{-1} A_t$.[8]

To complete our derivation of a closed-form solution, below we shall derive an explicit expression for α in terms of the parameters (μ, σ, r, ρ). This will enable explaining how uncertainty affects the rate of growth of expected consumption and the propensity to consume, μ_c and θ. However, even before doing so, the results so far can illustrate how precautionary saving emerges, and explain why there may not in fact be any puzzle regarding the statistically established evidence that consumption is smoother than labour income and is likely to show sensitivity to current income beyond that captured by the effect of income innovation.

Precautionary saving is tantamount to consumers reducing their consumption when they perceive a rise in their income volatility. We therefore examine how σ^2 affects θ using,

$$\frac{\partial \theta}{\partial \sigma^2} = -\frac{1}{r}\frac{\partial \mu_c}{\partial \sigma^2} = -\frac{1}{r}\left[\frac{1}{2}\alpha(\alpha - 1) + \left(\left(\mu - \frac{1}{2}\sigma^2\right) + \alpha\sigma^2\right)\frac{\partial \alpha}{\partial \sigma^2}\right], \tag{16}$$

which shows that precautionary saving requires the expression in square brackets on the right-hand-side of the above to be positive. As we shall see below, plausible circumstances exist in which $\frac{\partial \theta}{\partial \sigma^2} > 0$, hence precautionary saving is not a necessary reaction to rising income volatility. This clarifies the reason why, as pointed in the introduction, no firm consensus in the empirical literature on the saving behaviour has emerged in the literature.

The excess smoothness anomaly was highlighted by observing that whilst the rational expectations version of the life-cycle/permanent-income hypothesis, RE-LC-PIH, requires consumption to be more volatile than labour income, data clearly show the opposite. For instance, as Campbell and Deaton (1989) explain, given any plausible value of the real interest rate, theory predicts the standard deviation of changes in consumption, $\hat{\sigma}_c$, to be larger than that of the innovation in labour income, $\hat{\sigma}$, by a factor that exceeds 1.75. They report $\hat{\sigma} = 25.2$, and $\hat{\sigma}_c = 27.3$ for total consumers' expenditure and $\hat{\sigma}_c = 12.4$ for non-durables and services, which is indeed puzzling. However, within our framework too consumption volatility is proportional to labour income volatility and the relationship is $\sigma_c = \alpha\sigma$. As we shall see below, $|\alpha| < 1$ can emerge under plausible conditions implying that consumption should in fact be smoother than labour income.

[7] $\mu_c < r$ also implies $0 < \theta < 1$; conditions under which this assumption holds are further discussed below. When $\sigma = 0$ but $\mu < r$, equation (14) offers a straightforward generalisation of the Ramsey-type consumption function under certainty, namely $c_t = \frac{\rho}{r}(rA_t + \beta y_t)$, compared to equation (6) which corresponds to $\sigma = \mu = 0$. It is straightforward to verify that this result can be obtained directly as the solution to the dynamic optimisation problem defined by equations (1)-(3) when $\sigma = 0$ but $\mu < r$.

[8] The assumption that $A_t > 0$, which we shall maintain throughout the paper, is particularly relevant in this context since we relate to the literature in which the representative consumer portrays the aggregate behaviour. This is a common assumption in Ramsey-Cass-Koopmans type neoclassical growth models where A_t represents physical capital and A is determined by a production function.

The RE-LC-PIH also predicts that consumption is a martingale, implying that the change in consumption is unpredictable, and that transitory consumption is proportional to income innovation. This is not consistent with the established econometric evidence which finds consumption to respond to current income beyond that predicted by the RE-LC-PIH. Flavin (1981) provided evidence on the existence of positive correlation between the change in consumption and the lagged change in income and this persistent correlation, reported also by other studies, became known as the excess sensitivity puzzle in the literature. To some extent, this puzzle can be explained by the role of income drift in the consumption function. Rewriting equation (14) as $c_t = \theta(rA_t + y_t) + \theta(\beta - 1)y_t$ shows that the excess sensitivity phenomenon emerges since $\beta - 1 = \frac{\mu}{r-\mu} > 0$ when $0 < \mu < r$ regardless of uncertainty. In other words, what is being picked up by the above-mentioned correlation could simply be due to the impact of the drift in labour income which has not been taken into account explicitly when formulating the underlying regression equations. The planned level of consumption is usually thought to be proportional to the expected life-time wealth $\left(A_t + \int_0^\infty E_t y_s e^{-r(s-t)} ds\right)$ in the RE-LC-PIH models, or is related to the actual income $(rA_t + y_t)$ in the conventional empirical studies of consumption. Neither approach takes an explicit account of the role of drift in y. Hence, depending on the regression model being used, disregarding the effect of labour income drift could results in an omitted variable problem or a measurement problem which would manifest itself in significant past income effect.

The excess sensitivity evidence is further explained within our framework by noting the implications of uncertainty. To see this we obtain, from (12),

$$E_t c_s = c_t e^{\mu_c (s-t)}, \qquad s \geq t, \tag{17}$$

which shows that, conditional on the current consumption c_t, the expected future consumption, $E_t c_s$, is affected by the determinants of consumption drift, $\mu_c = \alpha\mu + \frac{1}{2}\alpha(\alpha - 1)\sigma^2$, in particular μ and σ. This becomes clear below where we derive explicit expressions for α under different scenarios which show that, for any given level of income volatility, the drift in income is likely to be one of the main determinants of $E_t[c_s/c_t]$, $s > t$. Within the framework developed here, therefore, the excess sensitivity puzzle – that arose because of statistically significant correlation between changes in consumption and anticipated changes in income (and hence with past changes in income) – might simply be due to a spurious interpretation of evidence. What is captured by this 'additional correlation' is simply the effect of the drift in labour income in determining the drift in consumption. Taking this explanation into account in empirical work can in fact improve the quality of evidence if the regression equations depicting the paths of consumption and income are jointly specified and the cross-equation restrictions that link their drift and volatility coefficients are correctly identified and exploited as additional information to gain efficiency. This task, however, lies beyond the scope of this paper.

2.1 Derivation of an explicit solution

We now derive an explicit solution for α in terms of the parameters (μ, σ, r, ρ) in order to explain how uncertainty affects consumption drift and volatility and the propensity to consume, μ_c, σ_c and θ respectively. Substituting into (7) for the partial derivatives of (8), i.e. (10) and

$$V_y = -\frac{\alpha(A + \varphi y)}{\kappa y^{1+\alpha}} + \frac{\varphi}{\kappa y^{\alpha}} + \frac{\lambda}{y}$$

and

$$V_{yy} = \frac{\alpha(1+\alpha)(A + \varphi y)}{\kappa y^{2+\alpha}} - \frac{2\alpha\varphi}{\kappa y^{1+\alpha}} - \frac{\lambda}{y^2}$$

and rearranging the resulting equation, we obtain

$$\left[\frac{\sigma^2\alpha(1+\alpha)}{2} - \alpha\mu + r - \rho\right] A\kappa^{-1}y^{-\alpha} + \left[\frac{\sigma^2\alpha(\alpha-1)}{2} - \rho + \frac{1}{\varphi} + \mu - \alpha\mu\right]\varphi\kappa^{-1}y^{-\alpha-1}$$
$$+ [\alpha - \rho\lambda]\ln y + \left[\ln\kappa - \rho\pi - \frac{1}{2}\lambda\sigma^2 + \lambda\mu - 1\right] = 0.$$

For the above equation to hold, all the square brackets should be equated to zero, when $A > 0$ and $y > 0$, and solved to find $(\alpha, \varphi, \lambda, \pi, \kappa)$ in terms of (μ, σ, r, ρ). The first two square brackets can be solved to determine φ and α. It turns out that the consistency of the solution requires $\varphi = (r - \mu + \alpha\sigma^2)^{-1}$, which can be used to eliminate φ from the above equation and rearrange it as

$$\left[\frac{\sigma^2\alpha(1+\alpha)}{2} - \alpha\mu + r - \rho\right](Ay + (r - \mu + \alpha\sigma^2)^{-1})\kappa^{-1}y^{-\alpha-1} + [\alpha - \rho\lambda]\ln y$$
$$+ \left[\ln\kappa - \rho\pi - \frac{1}{2}\lambda\sigma^2 + \lambda\mu - 1\right] = 0. \tag{18}$$

Equating the first bracket with zero results in a quadratic equation in α whose roots are

$$\alpha = \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 - \frac{2}{\sigma^2}(r - \rho)}. \tag{19}$$

Let α_S and α_L denote the small and the large root respectively, which will be real and distinct if $\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 - \frac{2}{\sigma^2}(r - \rho) > 0$. In this case the solution in (9) will be $c = \kappa_S y^{\alpha_S} + \kappa_L y^{\alpha_L}$ where κ_S and κ_L will be determined as explained above. However, we need to rule out the possibility that $\alpha_S < 0 < \alpha_L$ because it implies that starting from a given initial situation (y_0, c_0), c tends to a large value both as y tends to a large value and as it tends to a small value. Therefore, behavioural consistency requires choosing either α_S or α_L when $\alpha_S < 0 < \alpha_L$ – on some theoretical grounds – or imposing either $0 \leq \alpha_S < \alpha_L$ or $\alpha_L < \alpha_L \leq 0$ on the roots in (19) to maintain asymptotic consistency. Moreover, making the two roots distinct severely limits analytical tractability as well as requiring to tie $\frac{\mu}{\sigma^2} - \frac{1}{2} > 0$ with $r > \rho$ if $0 \leq \alpha_S < \alpha_L$, and $\frac{\mu}{\sigma^2} - \frac{1}{2} < 0$ with $r < \rho$ if $\alpha_L < \alpha_L \leq 0$, which is too restrictive. We shall therefore limit our solutions to three more special cases of (19):

- $\rho < r$ and the roots overlap;

- $\rho = r$ and there are two real distinct roots of which one is zero;
- $\rho > r$ and $\frac{\mu}{\sigma^2} - \frac{1}{2} = 0$, hence there are two distinct roots, but we choose the positive root.

Below, we first derive the solutions in each case and then provide a comparison between them and discuss their implications.

Case 1: $\boldsymbol{\rho \equiv \rho_1;\ \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 - \frac{2}{\sigma^2}(r - \rho_1) = 0}$ ***and*** $\boldsymbol{\frac{2}{3} r < \rho_1 < r}$

In this case $\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 = \frac{2}{\sigma^2}(r - \rho_1) > 0$. Thus, $\alpha \equiv \alpha_1 = \frac{\mu}{\sigma^2} - \frac{1}{2}$ and the gap between the consumer's subjective rate of time preference and the real interest rate is determined by the scale and diffusion coefficients of labour income process,

$$\rho_1 = r - \frac{\sigma^2}{2}\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2. \tag{20}$$

The path of consumption and its evolution are described by equations similar to (11) and (12), and substituting for α_1 in the expression for μ_c, it follows that $\mu_{c_1} = \frac{3\sigma^2}{2}\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2$, and hence in this case the propensity to consume is

$$\theta_1 = 1 - \frac{3\sigma^2}{2r}\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2. \tag{21}$$

$\theta_1 < 1$ always holds but $\theta_1 > 0$, required for consistency, will hold if $\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 < \frac{2r}{3\sigma^2}$, restricting the range of the time preference rate in this case to $\frac{2}{3} r < \rho_1 < r$.

Case 2: $\boldsymbol{\rho \equiv \rho_2 = r}$

This case is quite common in the literature and is usually used as a benchmark scenario under certainty with constant labour income since it implies that at any time the consumer simply consumes the whole of her actual income, $c_t = rA_t + y_t$ where y_t, c_t and A_t remain constant. To derive the corresponding solution under uncertainty, we impose $\rho = r$ on (19) which leads to two possible solutions, $\bar{\alpha}_2 = 0$ and $\alpha_2 = \frac{2\mu}{\sigma^2} - 1$. The existence of two distinct solutions requires generalising the generic solution for consumption to $c = \bar{\kappa}_2 + \kappa_2 y^{\alpha_2}$, and then following the same approach used above we write the path of the variable component of consumption, $c_2 = (c - \bar{\kappa}_2)$, as in equation (11), that is

$$dc_2 = \mu_{c_2} c_2 ds + \sigma_{c_2} c_2 d\omega,$$

where now $\mu_{c_2} = \alpha_2 \mu + \frac{1}{2}\alpha_2(\alpha_2 - 1)\sigma^2$ and $\sigma_{c_2} = \alpha_2 \sigma$. Thus, consumption evolves according to

$$(c_s - \bar{\kappa}_2) = (c_t - \bar{\kappa}_2) e^{\left(\mu_{c_2} - \frac{\sigma_{c_2}^2}{2}\right)(s-t) + \sigma_{c_2}\omega_s}, \quad s \geq t,$$

which, together with (4), can be substituted in the expected life-time budget constraint which is now written as

$$\int_0^\infty \bar{\kappa}_2 e^{-r(s-t)} ds + \int_0^\infty E_t(c_s - \bar{\kappa}_2) e^{-r(s-t)} ds = A_t + \int_0^\infty E_t y_s e^{-r(s-t)} dt,$$

and solved to obtain

$$\frac{\bar{\kappa}_2}{r} + \frac{c_t - \bar{\kappa}_2}{r - \mu_{c_2}} = A_t + \frac{y_t}{r - \mu}.$$

The above can then be rewritten in the form of the consumption function as

$$c_t = \bar{c}_2 + \theta_2(rA_t + \beta y_t), \tag{22}$$

where $\bar{c}_2 = \frac{\bar{\kappa}_2 \mu_{c_2}}{r}$, $\beta = (1 - \mu/r)^{-1}$ as before scales labour income for drift, and now the marginal propensity to consume is $\theta_2 = 1 - \mu_{c_2}/r$. Thus, using $\mu_{c_2} = \alpha_2 \mu + \frac{1}{2}\alpha_2(\alpha_2 - 1)\sigma^2$, and $\alpha_2 = 2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)$, we obtain $\mu_{c_2} = 4\sigma^2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2$ and

$$\theta_2 = 1 - \frac{4\sigma^2}{r}\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2, \tag{23}$$

which is similar to θ_1 in (21). In particular, $\theta_2 \leq 1$ always holds but now $\frac{\mu}{\sigma^2} - \frac{1}{2} < \frac{2\sqrt{r}}{\sigma}$ is required to ensure that $\theta_2 > 0$ holds.

Case 3: $\boldsymbol{\rho \equiv \rho_3;\ \frac{\mu}{\sigma^2} = \frac{1}{2}}$ ***and*** $\boldsymbol{r < \rho_3 < 2r}$

This final case portrays a relatively impatient consumer – since $\rho_3 > r = \rho_2 > \rho_1$ – whose income process is a special case of (4), namely $y_s = y_t e^{\sigma \omega_s}$ and $E_t y_s = y_t e^{\mu(s-t)}$. Imposing $\frac{\mu}{\sigma^2} = \frac{1}{2}$ on (19) yields $\alpha = \pm\sqrt{\frac{2}{\sigma^2}(\rho_3 - r)}$, but allowing for both roots leads to behavioural inconsistency explained above. We therefore rule out the negative root on the grounds that $\sigma_c = \alpha\sigma$ should hold and, *a priori*, income innovation is thought to have a positive impact on consumption.[9]

Substituting from $\frac{\mu}{\sigma^2} = \frac{1}{2}$ and $\alpha_3 = \sqrt{\frac{2}{\sigma^2}(\rho_3 - r)}$ into $\mu_c = \alpha\mu + \frac{1}{2}\alpha(\alpha - 1)\sigma^2$ and $\sigma_c = \alpha\sigma$ it follows that $\mu_{c_3} = \rho_3 - r$ and $\sigma_{c_3}^2 = 2(\rho_3 - r)$. Thus, in this case the Brownian motion describing the evolution of consumption has identical properties as that of income – since now $\frac{\mu_{c3}}{\sigma_{c_3}^2} = \frac{\mu}{\sigma^2} = \frac{1}{2}$. Consumption evolves according to $c_s = c_t e^{\sigma_{c3}\omega_s}$, and the level of the path is identical to (14). Using the expression for μ_{c_3} to evaluate $\theta = 1 - \mu_c/r$, the propensity to consume in this case is

[9] In Case 1 and 2 the sign of α could not be fixed *a priori* and would be determined by the size of income drift relative to its volatility parameter via $\frac{\mu}{\sigma^2} - \frac{1}{2}$; $\alpha < 0$ would result if the growth rate of expected income was sufficiently: $\mu < \frac{1}{2}\sigma^2$.

$$\theta_3 = 2 - \frac{\rho_3}{r}, \tag{24}$$

which satisfies $0 < \theta_3 < 1$ if $r < \rho_3 < 2r$ holds.

2.2 Comparing the solutions in the three cases

A comparison between solutions to three cases outlined above can provide further explanation of excess smoothness of consumption relative to income, reveal additional aspects of the impact of income volatility on consumer's saving behaviour and, by doing so, clarify whether the subjective rate of time preference is an accurate measure of consumers' impatience to consume when they face income uncertainty. These solutions also go some way in explaining two other important empirical anomalies that have been noted in the literature: Kuznets' (1942) paradox on constancy of the average propensity to consume over a long time span; and, Hall's (1988) empirical evidence that shows the elasticity of intertemporal substitution is unlikely to be positive.

2.2.1 Smoothness of consumption

To check the smoothness of consumption relative to income, we use the ratio of their volatility implied in the three cases:

Case 1: $\frac{\sigma_{c_1}}{\sigma} = \alpha_1 = \frac{\mu}{\sigma^2} - \frac{1}{2}$, hence c will smoother than y if $\left|\frac{\mu}{\sigma^2} - \frac{1}{2}\right| < 1$

Case 2: $\frac{\sigma_{c_2}}{\sigma} = \alpha_2 = \frac{2\mu}{\sigma^2} - 1$, hence c will smoother than y if $\left|\frac{\mu}{\sigma^2} - \frac{1}{2}\right| < \frac{1}{2}$

Case 3: $\frac{\sigma_{c_3}}{\sigma} = \alpha_3 = \sqrt{\frac{2}{\sigma^2}(\rho_3 - r)}$, hence c will smoother than y if $\rho_3 - r < \frac{1}{2}\sigma^2$

The condition in Case 3 is more subjective since it depends how one interprets the (positive) gap between the subjective rate of time preference and the interest rate. Nevertheless, its implications are worth highlighting: (i) for a given value of σ, the smaller is this gap the less volatile is consumption; and (ii) consumption will be smoother than income if the gap is less than 50% of income volatility. The conditions in the other two cases 2 is more stringent and can be confronted with data. For instance, examining US data we found that these conditions do not hold for compensation of employees which happens to be rather smooth. However, the sample mean and standard deviation of rate of growth of income of self-employed are 0.064475 and 0.93626, respectively, and the standard error of an AR(1) regression fitted to the rate of growth of series is 0.78118, which satisfy the condition.[10]

2.2.2 Under-spending and precautionary saving behaviour

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[10] We used annual data for period 1984-2020, from Federal Reserve Bank of St. Louis, available at: https://fred.stlouisfed.org. Data on wages and salaries were constructed by multiplying the real GDP series (GDPC1) with share of labour compensation in GDP (LABSHPUSA156NRUG). Data on income of self-employed were obtained by deflating income before tax from self-employment (CXUSFEMPINCLB1202M) by consumers' price index (CPALTT01USM657N_NBD19600101).

To examine the implications of our results for saving, we list in Table 1 below the expressions for $\frac{dA_t}{dt}$ and $\frac{\partial\theta}{\partial\sigma^2}$ which are obtained by evaluating equations (15) and (16) for each case. As columns 2 and 3 of Table 1 show, precautionary saving – defined as a reduction in consumption in response to a rise in income volatility – is not an inevitable outcome. In Cases 1 and 2, what matters is how the volatility parameter compares to drift. With a relatively large value of $\frac{\mu}{\sigma^2}$, the consumer perceives her expected income to be growing sufficiently so as to feel reassured against rising income volatility. The opposite occurs when $\frac{\mu}{\sigma^2}$ is relatively small, in which case the consumer feels the need to save more to be prepared for rainy days ahead when she perceives an increase in income volatility. In the same vein, this result explains why elderly/retired consumers whose income is not expected to grow might tend to underspend. Finally, as Cases 2 and 3 show, when $\frac{\mu}{\sigma^2} = \frac{1}{2}$ holds volatility does not affect consumption via propensity to consume, but its impact is exerted through the scale factor $\beta = (1 - \mu/r)^{-1}$ since μ is tied to σ^2.

Table 1. Impact of labour income volatility on propensity to consume and saving

Case	Impact of σ^2 on μ_c	Impact of σ^2 on $\theta = 1 - \frac{\mu_c}{r}$	Saving
1*	$\frac{\partial\mu_{c_1}}{\partial\sigma^2} = \frac{3}{2}\left(\frac{1}{4} - \frac{\mu^2}{\sigma^4}\right)$	$\frac{\partial\theta_1}{\partial\sigma^2} < 0 (> 0)$ if $\frac{\mu}{\sigma^2} < \frac{1}{2}\left(> \frac{1}{2}\right)$	$\frac{dA_t}{dt} = 3(r - \rho_1)A_t + \left(\frac{3(r-\rho_1)-\mu}{r-\mu}\right)y_t$
2	$\frac{\partial\mu_{c_2}}{\partial\sigma^2} = 4\left(\frac{1}{4} - \frac{\mu^2}{\sigma^4}\right)$	$\frac{\partial\theta_2}{\partial\sigma^2} < 0 (> 0)$ if $\frac{\mu}{\sigma^2} < \frac{1}{2}\left(> \frac{1}{2}\right)$ $\frac{\partial\theta_2}{\partial\sigma^2} = 0$ if $\frac{\mu}{\sigma^2} = \frac{1}{2}$	$\frac{dA_t}{dt} = 4\sigma^2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 A_t + \left(\frac{4\sigma^2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 - \mu}{r-\mu}\right)y_t$
3	$\frac{\partial\mu_{c_3}}{\partial\sigma^2} = 0$	$\frac{\partial\theta_3}{\partial\sigma^2} = 0$	$\frac{dA_t}{dt} = (\rho_3 - r)A_t + \left(\frac{\rho_3 - r - \mu}{r-\mu}\right)y_t$

* In Case 1 $\frac{\partial\theta_1}{\partial\sigma^2} = 0$ is ruled out because $\frac{\mu}{\sigma^2} - \frac{1}{2} \neq 0$. Also in Case 1, σ^2 affects savings since ρ_1 depends on it via equation (20). The sign of $\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)$ determines the asymptotic properties of y_s in the sense that $\lim_{s\to\infty} y_s \to 0 (\to \infty)$ if $\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) < 0 (> 0)$ since $\lim_{s\to\infty}\left(\mu - \frac{\sigma^2}{2}\right)(s - t)$ dominates $\lim_{s\to\infty} \sigma\omega_s$ in probability.

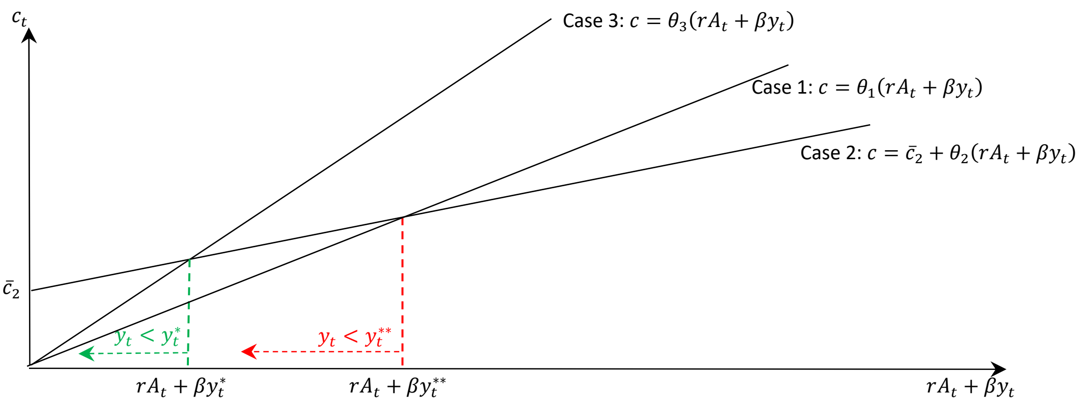
With regard to the saving decision in general, as column 4 of Table 1 shows, provided that $A_t > 0$ the consumer will always save if she perceives that her expected income is not growing sufficiently: it is straightforward to verify that $\frac{dA_t}{dt} > 0$ holds when $\mu \leq 3(r - \rho_1)$, $\mu \leq \frac{\sigma^2}{4}$ (or $\mu \geq \sigma^2$), and $\mu \leq \rho_3 - r$, respectively in the three cases. If μ violates these conditions, then the consumer saves if she perceives her income to be relatively small, for instance, in Case 3 when $\mu > \rho_3 - r > 0$ then $\frac{dA_t}{dt} > 0$ if $y_t < \frac{(\rho_3 - r)(r-\mu)}{r+\mu-\rho_3}A_t$. In fact, the result for Case 3 is rather interesting because it shows that circumstances exist under which a relatively impatient consumer ($\rho_3 > r$ in this case) saves when she faces income uncertainty. This result could not be obtained under certainty since a consumer with $\rho > r$ does not save.[11]

[11] The equivalent consumption function under certainty is $c_t = \left(\frac{\rho}{r}\right)(rA_t + \beta y_t)$ – see footnote 7. It therefore follows that $\frac{dA_t}{dt} = \left(1 - \frac{\rho}{r}\right)(rA_t + y_t) - \left(\frac{\rho}{r}\right)(\beta - 1)y_t < 0$ when $\frac{\rho}{r} > 1$ since $\beta > 1$.

Conventionally, the size of the subjective rate of time preference ρ which discounts the future utility of consumption is used to characterise consumers' degree of impatience to consume. In other words, it is usually thought that, ceteris paribus, at any given time a consumer with a larger ρ will plan to consume more. This certainly holds under certainty, where $c_t = \frac{\rho}{r}(rA_t + \beta y_t)$ – see footnote 7. We can now examine whether this also holds under uncertainty. First consider two consumers that are identical in every respect except their ρ: one with $\rho > r$ as in Case 3 and the other with $\rho = r$ as in Case 2. It might be somewhat puzzling to find that the consumption level of the consumer in Case 2 initially exceeds that in Case 3 until income reaches a certain threshold, regardless of the fact that propensity to consume in the latter case were larger. This is depicted in Figure 1 where the existence of a constant intercept $\bar{c}_2$ in Case 2 explains this outcome: for given (A_t, r, μ, σ), we find that $\theta_3(rA_t + \beta y_t) > \bar{c}_2 + \theta_2(rA_t + \beta y_t)$ holds only after y_t exceeds y_t^*.

Next, consider comparing two identical consumers except that one has $\rho < r$ as in Case 1 and the other has $\rho = r$ as in Case 2. We therefore expect to find the consumption level of the latter to exceed that of the former – when $\frac{\mu}{\sigma^2} - \frac{1}{2} \neq 0$, so that the two cases are comparable. But this result happens to hold only at relatively lower income levels only: as shown in Figure 1, again for a given (A_t, r, μ, σ), the existence of a constant intercept $\bar{c}_2$ in Case 2 implies that $\bar{c}_2 + \theta_2(rA_t + \beta y_t) > \theta_1(rA_t + \beta y_t)$ is satisfied but only while $y_t < y_t^{**}$. It is straightforward to verify that since $\theta_1 > \theta_2$ at some point the consumption level in Case 1 will exceed that of Case 2 as income becomes sufficiently large. This outcome, which could not arise under certainty, cautions against assuming that, ceteris paribus, a consumer who discounts her future utility more will always consume more now.

Figure 1. Comparing consumption levels between Case1 and Case 2



$y_t^* = \frac{(\rho_3 - r)(r-\mu)A_t}{4\sigma^2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2 + r - \rho_3}$; $y_t^{**} = \frac{3(r-\mu)A_t}{5}$; $\theta_3 > \theta_2$ requires $r < \rho_3 < r + \frac{3\sigma^2}{2}\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2$; $\theta_1 > \theta_2$ always holds.

It is worthwhile to point out at this juncture that the consumption function in Case 2 which has a constant intercept can also explain one of the oldest puzzles in the literature that was first highlighted by Kuznets (1942). Early estimates of the propensity to consume were based on estimating the linear relationship between aggregate consumers' expenditure C and aggregate (disposable) income Y, that is $C = a + bY$. This relationship, which is known as the *absolute income hypothesis*, was inferred from Keynes' explanations regarding the likely behaviour of the *marginal* and the *average propensity to consume*, $MPC = \partial C/\partial Y = b$ and $APC = C/Y = b + a/Y$, respectively. As Keynes had expected, estimates of b were found to be close to unity (around 0.75), those of a were positive too, and both estimates were statistically significant. Kuznets pointed out that whereas $a > 0$, and hence $MPC < APC$, would imply a declining C/Y as income grew, his reconstructed historical national accounts data, spanning over almost a century, indicated that C/Y had remained remarkably constant over that period. Attempts to explain this apparent empirical anomaly, later known as Kuznets' paradox, led to a fascinating body of work on the theory and econometrics of the consumption function amongst which the work of Duesenberry (1949) remains outstanding. At the time, his proposed *relative income hypothesis* was overshadowed by the *permanent income hypothesis* and the *life cycle hypothesis* which explicitly drew on microeconomic foundations of consumer behaviour. But, by distinguishing between the short-run and the long-run consumption-income relationships, his theory provided an interesting and unique approach to explaining Kuznets' paradox.[12] In his view, there was no paradox since while the long-run relationship between consumption and income should exhibit unit income elasticity hence satisfying $MPC = APC$ as established by Kuznets, the short-run relationship was likely to be flatter, exhibiting $MPC < APC$. In this way, a 1% increase in income raises consumption by 1% in the long-run but by less than 1% in the short-run. Duesenberry provided a behavioural interpretation for the constant intercept in the short-run consumption function which linked it to its long-run counterpart and required the intercept to shift when income rose relative to some income standard based on past income profile in the aggregate time series context. At the representative household level, Duesenberry used cross section data and replaced the past income with some income standard that the household sets based on the social hierarchy with which it identifies itself.

A literal interpretation of Duesenberry's relative income effect is tantamount to making the short-run Keynesian consumption function dynamic, for instance $C_t = a_{sr}\,\Delta_k Y_t + b_{sr} Y_t$, where the lag length k is chosen to represent the reference horizon and the magnitude of $\Delta_k Y_t$ determines the shift in the short-run function. In the long-run, $\Delta_k Y_t / Y_t$ is constant and so is C_t / Y_t. However, when the short-run function is modified in this way it can no longer correspond to the level of consumption path that is derived within the standard utility maximisation framework. For instance, within the life-cycle/permanent income framework implies that at any given time the level of a consumer's planned consumption, or permanent income, is simply the annuity she derives from her net worth (the present value of human and financial wealth). Shift in actual consumption are therefore its deviations around planned consumption which are caused by labour income innovations (or transitory income, which is a

[12] A phenomenon that later heavily influenced the development of the co-integration approach in econometrics. His explanations also revealed the importance of confronting evidence from time series data with that based on cross-section panel and survey data, and generalising consumers' preferences to allow for phenomena such as habits, altruism, envy, etc.

white noise disturbance term and hence completely unpredictable) – see, e.g., Flavin (1981) for details. Despite this, some post permanent income studies have continued to ascribe a behavioural role to the constant intercept. For instance, in a study of post WWII aggregate US data, Blinder (1975) found that, contrary to the prediction of the absolute income hypothesis, the C_t/Y_t ratio had risen as income rose. He argued that this phenomenon could be explained by the relative income hypothesis where the shifts in the short-run consumption function were due to changes in the composition of the labour force. Musgrove (1980), who further developed this idea by showing how the average propensity to consume is affected by income distribution, argued that propensities of a household unit's consumption are related to its income above a subsistence level, the whole of which is consumed. Interpreted in this way, therefore, within our framework $\bar{c}_2$ is a constant level of consumption that the consumer can afford regardless of her labour income and will be able to maintain when her labour income is zero. Thus, upon imposing $y_s = 0$ for all $s \geq t$, the budget constraint becomes $\bar{c}_2 + \theta_2 r A_t = r A_t$, from which we obtain

$$\bar{c}_2 = (1 - \theta_2) r A_t, \quad s \geq t. \tag{25}$$

According to our solution in Case 2, therefore, in the short-run, $\sigma > 0$ and its variations rotate and shift $c_t = \bar{c}_2 + \theta_2(rA_t + \beta y_t)$. In the long-run, $\sigma = 0$ which implies that $\bar{c}_2 = 0$, $\theta_2 = 1$, and $c_t = rA_t + \beta y_t$ that is compatible with the life-time budget constraint. In this way, the 'mechanics' of Duesenberry's explanation of Kuznets' paradox holds here too as the short-run function slides along the long-run function, but the driving force is the change in trend and volatility of labour income.

Finally, our results also throw light on interpreting the role of the real interest rate in determining the evolution of consumption; a relationship about which there has been somewhat of a puzzle in the more recent literature. According to the rational expectations version of the life-cycle hypothesis, the effect of the real interest rate on consumption growth should be determined by the size of the elasticity of intertemporal substitution. Deaton (1987) expressed some concern regarding the observed persistent growth of real consumption even during high inflation periods when the real interest rate was negative. Hall (1988), who stressed the role of elasticity of intertemporal substitution, reported evidence based on estimating the regression equation he derived from the life-cycle theory[13] which revealed that the size of elasticity would be unlikely to exceed 0.1, and it is in fact more likely to be much closer to zero. Although these findings might be somewhat puzzling, our results can provide some explanation for them. From (12), the equation for the rate of growth of consumption is

$$d\ln c_s = \mu_c ds + \sigma_c d\omega_s, \; s \geq t. \tag{26}$$

Recall that $\mu_c = \alpha\mu + \frac{1}{2}\alpha(\alpha - 1)\sigma^2$ and $\sigma_c = \alpha\sigma$ where, as we have seen above, α is a coefficient of the generic solution which will, in general, depend on the parameters of the model

[13] This relationship is derived from the Euler equation which, in model with discrete time, is given by $E_t\left[\left(\frac{1+r_{t+1}}{1+\rho}\right)u'(c_{t+1})/u'(c_t)\right] = 1$. Postulating $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$, the implied regression equation is $\Delta\ln c_{t+1} = \beta_0 + \beta_1 r_{t+1} + \varepsilon_{t+1}$ where $\beta_1 = 1/\gamma$ is the elasticity of intertemporal substitution and ε is expected to behave as a white noise disturbance term under the rational expectations hypothesis.

amongst which are the real interest rate as well as the elasticity of intertemporal substitution.[14] Below are the growth rates of expected consumption, μ_c, in the three cases examined above are:

Case 1: $\mu_{c_1} = \frac{3\sigma^2}{2}\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 = 3(r - \rho_1)$

Case 2: $\mu_{c_2} = 4\sigma^2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2$

Case 3: $\mu_{c_3} = \rho_3 - r$

which clearly show that, depending on the circumstances, the role of the interest rate in determining growth of expected consumption might vary drastically. Accordingly, when we estimate the regression equation proposed by Hall (see footnote 14), the estimated coefficient of the real interest rate is likely to be akin to a reduced form coefficient which is scaled by parameters of the income process. Therefore, some caution ought to be exercised in interpreting this coefficient. In fact, to measure the impact of the real interest rate on consumption growth, it seems that one needs to jointly specify two regression equations, one for consumption path and one for labour income path, that explicitly allow for the coefficient of the real interest rate in the consumption equation to correctly embody the impact of drift and volatility of labour income. To some extent, this explains why Hall, and the studies that followed, did not obtain encouraging results when they tried to find a remedy by jointly modelling the paths of consumption and the real interest rate but disregarded income uncertainty.

3. Conclusion

The more recent literature on the consumption function, driven by the implications of the rational expectations version of life-cycle/permanent-income model, has revealed a number of puzzles that stem from conflicts between theory and evidence. There has been no shortage of work on trying to explain these puzzles, be it by remedying the underlying theory, mainly via introducing some modification of the existing model, or by improving the quality of evidence, for instance via augmenting the consumption regression equation with other variables, through adding regression equations for income and/or interest rate, or by means of using panel and/or survey date. Amongst these, it could be argued that generalising the optimisation framework to include a volatile but uninsurable labour income should take priority. However, when the representative consumer's utility function is of the constant relative risk aversion form, a closed-form solution to the life-cycle maximisation problem which takes account of income volatility has not yet been derived. As a result, empirical work is still based on various generalisations of the Euler equation despite the fact that numerical simulations of the model with stochastically evolving labour income have revealed the shortcomings of that approach.

[14] In our analysis we have used $u(c) = \ln c$ to enable obtaining a closed-form solution for consumption. Hence this parameter will not explicitly feature as a determinant of α. But since this argument does not require a closed-form solution for the level of consumption, it is possible to derive a similar result for $u(c) = (c^{1-\gamma} - 1)/(1 - \gamma)$ to illustrate the point more precisely.

In this paper we have offered a closed-form solution for the life-cycle maximisation problem of a representative consumer with constant relative risk aversion preferences whose labour income follows the standard geometric Brownian motion. Our proposed solution for the level of consumption generalises the Ramsey-type consumption function that is obtained under certainty by making the propensity to consume from current income a function of the drift and volatility parameters of the labour income process. We have shown that this generalisation explains how precautionary savings might emerge, why consumption could be smoother than income, how it turns out to exhibit sensitivity to current income beyond that predicted by the rational expectations version of the life-cycle/permanent-income model, and why we should not be so concerned about the observed insensitivity of consumption growth to the real interest rate. Two further interesting implications also emerge from our results:

(i) We show that circumstances exist in which the consumption function includes a constant intercept that characterises the subsistence level of consumption; an outcome that could not occur under certainty. The existence of an intercept which shifts with changes in the extent of income volatility and disappears when uncertainty about income ceases to exist is shown to explain Kuznets' paradox via the distinction between the short-run and the long-run versions of the consumption function: the mechanism is the same as the 'ratchet effect' proposed by Duesenberry but works through the shits in the intercept and slope of the short-run function which are determined by the drift and volatility coefficients of labour income process.

(ii) We also explain why caution should be exercised in interpreting the subjective rate of time preference as the sole measure of a consumer's impatience since we find, when comparing two consumers that are identical in every respect except their subjective rate of time preference, that under uncertainty it is no longer necessarily true that the consumer with a higher rate of time preference will always decide to consume more.

It is worthwhile to conclude by highlighting an empirical implication of our results. It is generally agreed that circumstances could be found in which any single regression equation which is used to estimate the parameters of a consumption function can correspond to a reduced form equation whose coefficients of interest are nonlinear functions of the parameters of a more general model. For instance, our results show that the drift and volatility parameters of the labour income process feature in determining the coefficients of a Keynesian-type consumption function or, even, of the Euler equation that governs the path of consumption. The implication of these findings, which might seem to amount to stating the obvious, cautions strongly against interpreting the coefficient estimates of the consumption regression equations unless it can be clearly established that such interpretations remain robust to disregarding the income generating equation and the possible cross equation restrictions. For instance, we highlighted how such a pitfall could arise when one estimates the standard Euler equation and interprets the coefficient of the real interest rate as a measure of the elasticity of intertemporal substitution.

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