Breaking free from the stability dogma: Samuelson and the multiplier-accelerator model over the years

Michaël Assous and Mauro Boianovsky

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Michaël Assous, Université Lumière Lyon-2, Triangle Michael.Assous@univ-lyon2.fr

Mauro Boianovsky, Universidade de Brasilia mboianovsky@gmail.com

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Abstract. On the occasion of the centennial of his mentor Alvin Hansen, Paul Samuelson published in 1988 a modified version of his seminal 1939 multiplier-accelerator model with the specific aim to address aspects of Hansen's secular stagnation hypothesis. The "Keynes-Hansen-Samuelson" model (or KHS, as he called it) was built in order to provide an analysis of the effects of population growth on the trajectory of the economy. Several changes were then made. Instead of difference equations and a tight accelerator as in his 1939 model, Samuelson deployed differential equations and a flexible accelerator in order to produce a nonlinear limit cycle in the tradition of Richard Goodwin, as well as a life-cycle saving hypothesis. Despite Samuelson's strong claims for the analytical contributions of his 1988 paper, it has – in sharp contrast with the 1939 model – received only scant attention by macroeconomists and historians of economics alike. Samuelson's 1988 paper was his last published macroeconomic model, along the lines of his long established tradition of non-optimizing macro-dynamics. Our paper provides a close reading of Samuelson 1988, together with a discussion of how it historically links up with business cycle models advanced by John Hicks, Nicholas Kaldor, Roy Harrod and especially Goodwin, among others. Moreover, it investigates to what extent Samuelson's 1988 failure to attract a large readership has to do with the fact that macroeconomists' modelling strategy of endogenous business cycles changed sharply in the 1980s and after.

Keywords: stagnation, Samuelson, flexible accelerator, local instability, Hansen

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1. Samuelson's original take on stability, cycles and exponential growth

Paul A. Samuelson often argued that his 1939 multiplier-accelerator model – sometimes regarded as one of the first mathematical endogenous business cycle model – had its origins in Alvin Hansen's attempt to explain the 1937-38 American recession. With the aim to dynamize a basic Keynesian model,[1] Hansen developed a determinate numerical example - assuming a propensity to consume equal to 0,5 and a coefficient of acceleration equal to 2 - and concluded that following a rise in autonomous demand (public spending or private investment), national income would not reach a new equilibrium but would eventually slide into recession. This is where Samuelson (1939a) came in. Reducing Hansen's analysis to a second-order difference equation, his role consisted in showing that Hansen's example would in fact generate self–sustained cycles. As he would recollect,

> At once I made the inference that the drop in income which had so struck Hansen was not the end of the story. Quite by chance, he had picked numerical values which were on the razor's edge that yielded perpetual oscillations, with no damping and no exploding. In other words, if he had continued his numerical example far enough, his downturn too would have come to an end; and he would have been able to generate a succession of never-ending expansions and contractions (Samuelson 1959: 183).

But it was not the end of it. Undertaking a comprehensive analysis of the algebraic structure of Hansen's model, Samuelson also highlighted how, for different values of the marginal propensity to consume and the coefficient of acceleration, various dynamic behaviours could be generated.[2] Four movements, reflecting the

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[1] Oskar Lange's 1938 Keynesian model was a common reference to Hansen and Samuelson, who both met him after his move to the US. See Assous and Lampa (2014) and Backhouse (2017, chapter 18) on Samuelson's and Lange's correspondence about its properties.

[2] In that respect, Samuelson's 1939 multiplier-accelerator model was tightly connected to the works of his fellow econometricians. He viewed it as "a useful introduction to the mathematical theory of [Jan Tinbergen's] work" (Samuelson,

stability of the economy, were eventually pointed out: stable movements comprising monotonic and cyclical convergence towards stationary equilibrium and unstable movements comprising monotonic and cyclical divergence from stationary equilibrium.[3]

In the case of a stable movement, Samuelson (1939a) concluded that a single impulse – like a rise in public spending – would have a transitory effect, reinforcing Hansen's doubt that the government just had to "prime the pump" before balancing its budget to induce private investment. In addition, if the accelerator proved important for the trajectory of the system and its stability, it did not influence the final level attained (Samuelson 1939b), a result which seemed to vindicate Keynes's lack of interest in the acceleration principle: "From the long-run point of view Keynes was partially justified in ignoring the acceleration principle completely. The average level of the system is independent of its operation, depending rather upon the level of investment outlets" (Samuelson, 1939b: 795).[4]

In the end, it turned out that it was only in the unstable cases that an initial change in government expenditures could lead to ever increasing levels of consumption, induced investment and income. Only then "[a] constant level of governmental expenditure will result in an ever increasing national income, eventually approaching a compound interest rate of growth" (Samuelson, 1939a: 77 and 1940: 502). That corresponded to Region D in Samuelson's (1939a) stability diagram. But Samuelson viewed this possibility as an extreme case, characterized by a particularly high sensitivity of expectations to income – it was more likely that private enthusiasm would soon peter out and a downturn would occur once the acceleration coefficient has been reduced (Samuelson, 1940: 502–503).

1939a: 78) as well as an example of the importance, repeatedly underlined by Frisch in the early 1930s, of having a unified argument accounting for the turning points of the phases of the business cycle (Samuelson, 1939b: 785, 789). See Assous and Carret (2022, chapters 7 and 8) on the connection between Samuelson and early econometricians.

[3] In each case, stability referred to movement with respect to the stationary state, as the dynamic equation was derived under the assumption that the macroeconomic equilibrium condition is verified at each point of time.

[4] See Assous and Carret (2022 sections 7.4 and 8.3) for a detailed account of the various properties of Samuelson's multiplier-accelerator model.

This was all more likely to happen if, in accordance with Keynes's idea (also shared by Harrod), the marginal propensity to consume tended to fall, causing an expanding economy to move away from regions of instability, another blow to the possibility of "pure pump priming." Of course, changes in the propensity to consume, while they would prevent cumulative upward movement, would, on the other hand, intensify downward movements. But Samuelson thought that such movements would be avoided because of the existence of a lower bound on net investment. In his discussion, he also raised the possibility that for certain values of the propensity to consume and the coefficient of acceleration - but without working out the mathematics behind the analysis - there would probably exist a "periodic motion of definite amplitude" which would be approached regardless of initial conditions (Samuelson, 1939b: 795), or in modern terms, a limit cycle. This case was though quite like stable cases that excluded the possibility of endogenous cyclical growth.

The present paper is set out to discuss Samuelson's multiplier-accelerator models from the perspective of what he eventually described as the "stability dogma", which played a key role in his 1947 *Foundations*, partly under the influence of Ragnar Frisch's (1933) approach to macro-dynamics. Our paper discusses carefully Samuelson's 1988 article on the multiplier-accelerator interaction, when, instead of his 1939 linear approach à la Frisch, he put forward a nonlinear model which broke with the "stability dogma". Again, like the 1939 articles, Samuelson (1988) aimed at clarifying Hansen's claims, this time with a focus on secular stagnation. Despite Samuelson's strong claims for the analytical contributions of his 1988 paper, it has – in sharp contrast with the 1939 model – received only scant attention by macroeconomists and historians of economics alike.

Old business cycle literature, previous to Samuelson (1939a, b) and Frisch (1933), had been groping toward nonlinear endogenous cycle verbal models, where prosperity created conditions for economic depression as the economy hit its full employment « ceiling » and vice-versa when it reached the « floor ». Such an approach was non-mathematical and focused on the explanation of the upper and lower « turning points » of the business cycle (see Haberler 1946 chapter 10 for a classic treatment). Frisch (1933) put forward instead a system of linear equations with shocks (« impulses ») that brought about fluctuations through « propagation mechanisms » implied by various time adjustments[5]. As put by Olivier Blanchard (2000: 1383), Samuelson's (1939a, b) analysis of the multiplier-accelerator « reinforced » Frisch's point, so that their convenience and easy mapping to the data "quickly led to the dominance of linear models with shocks as the basic approach to fluctuations, and alternative nonlinear approaches largely faded from the scene" (Blanchard, op. cit.) – even after Richard Goodwin (1951) managed to build a nonlinear business cycles model on the basis of the multiplier-accelerator interaction.

Samuelson (1939b: 788) challenged the hitherto prevailing notion – which he associated with J.M. Clark, Gottfried Haberler, Alvin Hansen and Roy Harrod, among others – that mechanisms akin to the multiplier-accelerator interaction could only bring about a cyclical downturn due to the full-employment ceiling or perverse price-cost movements caused by bottlenecks. Instead, Samuelson (1939b: 792) claimed that his mathematical model was more general than the nonlinear verbal approach, since, even without any bottlenecks, the expansion would always come to an end for certain values of the marginal propensity to consume and the coefficient of acceleration, as mentioned above. As Samuelson (1955: 313, n. 3) would recall, the great merit of fully determinate linear model was to provide a possibility to account for all phases of the business cycles.

However, what Samuelson saw as the strength of his 1939 cycle model came to be perceived by some as its weakness. Upon describing that model in some detail and calling it a "brilliant" achievement (Haberler 1946: 473-77), Haberler (1949: 85) would complain: give any "sophomore a couple of lags and initial conditions and he will construct systems which display regular, damped or explosive oscillation … as desired." The dependence of the quantitative behaviour of Samuelson's 1939 model upon the values of the coefficients of the variables deployed in the equations became increasingly seen as problematic. It indicated to Kydland and Prescott (1991: 165), for instance, that "pure theory was not providing sufficient discipline", which brought into the picture the statistics discipline provided by Tjalling Koopmans

[5] See Carret (2021) for a detailed account of the mathematical properties of Frisch's 1933 macrodynamic model. At about the same time, Tinbergen had developed a nonlinear macrodynamic model which he used to shed new light on several policy problems: wage changes, government expenditure and its relation to pump-priming, and the regulation of purchasing power. See Assous and Carret (2022b and 2023).

and others. Even Keynesian economists such as Tobin (1983: 195) observed how Samuelson did not fully recognize at the time the defect of linear models of the business cycle, in the sense that they either explode or die out except for singular values of the parameters.

Another feature of Samuelson's 1939 multiplier-accelerator model – shared by macroeconomics in general until the early 1970s – was that it provided a typical example of a "dynamic system that can in no useful sense be related to a maximum problem", as Samuelson (1972: 258) pointed out in his Nobel Lecture. Instead, the 1939 model was solved through a mathematical analysis of stability conditions, by analysing the stability regions corresponding to the possible roots of the quadratic equation that formed the dynamic system's characteristic equation (see also Samuelson 1959: 184; 1972: 258). That distinction – between optimization problems in microeconomics on one hand and the study of the dynamical properties of aggregative systems under the assumption of stability – became the hallmark of Samuelson's (1947) *Foundations* encapsulated by the "Correspondence Principle" between statics and dynamics.

Samuelson (1947: 284) regarded the Correspondence Principle as a continuation and further elaboration of the "revolution" from static to dynamic modes started by Frisch (1933) (see also Boianovsky 2020). He shared with Frisch the view that the economy is a naturally stable system, which, unless disturbed from the outside, always remains around an equilibrium state.[6] That stability postulate was part of Frisch's view of damped propagation mechanisms, with cyclical oscillations caused by exogenous shocks to a stable equilibrium structure (see also Punzo 2009: 93). Linear mathematics suited the stability postulate well, as distinct from nonlinear mathematics later applied to self-sustained fluctuations by Samuelson (1988) under Richard Goodwin's influence.

The so-called Harrod-Domar model – particularly in Harrod's version – shared with Samuelson (1939a, b) the notion that the dynamic path is determined by the interaction between the multiplier and the accelerator. Samuelson did not develop or

[6] In the meantime, Samuelson thought that a system in which money wages would respond to unemployment was highly likely to be unstable. See Assous and Carret (2020) on Samuelson's early take on instability of full employment equilibria.

anticipate that growth model, as he was concerned, under Frisch's influence, with damped-root stability. Samuelson (1974: 10; see also 1955: 312-23; 1972: 259; 1988: 17) recollected how he and Metzler "fell into the dogma … that all economic business cycle models" should be dynamically stable, in the sense of having "damped roots". Moreover, the "dogma" was inspired by the behavior of the American economy in 1933-40, when it seemed incapable of "self-fulfilling bootstrap returns to prosperity" (Samuelson 1974:10). Looking back on this from the vantage-point of four decades later, Samuelson regretted that, under the influence of the "stability dogma", he had suppressed "development of the Harrod-Domar exponential growth aspects that kept thrusting themselves on anyone who worked with accelerator–multiplier systems" (Samuelson, 1974: 10). But some did not join him on that path. Eager to show how one could make the transition from Samuelson's cyclical model to Evsey Domar's and Roy Harrod's growth analyses, economists like Thomas Schelling, Sydney Alexander and Richard Goodwin (all of them with links to Harvard University) came with new insights about economic growth and business cycles that Samuelson eventually tackled upfront in his 1988 Hansen anniversary article.

2. Questioning the stability dogma in the context of the rise of growth economics

Domar joined Harvard in 1941 as a graduate student and discovered its seminars there, alongside his PhD supervisor, who was none other than Hansen. After attending for three years the Fiscal Policy seminar that Hansen had conducted at Harvard since 1938, he presented his famous paper on the burden of public debt and economic growth. Besides providing an analysis of debt sustainability, his concern was to highlight the conditions under which an economy could grow steadily. Given the duality of investment – both as a component of aggregate demand (the higher investment, the higher aggregate demand) but also as a component of aggregate supply (the higher investment, the greater the productive capacity and aggregate supply), he came to develop an ingenuous model combining the accelerator and multiplier mechanisms.

On the demand side, the maintenance of macroeconomic equilibrium (saving equals investment), requires that the rate of change of national income is proportional to the rate of change of investment

$$\frac{dY}{dt} = \frac{1}{\alpha}\frac{dI}{dt} \quad (1)$$

α being the propensity to save, Y the national income and I investment.

On the supply side, investment increases potential output of the economy. Deriving P as the level of output Y under full employment of labor and σ the the "potential social average productivity of investment" for the whole economy, one gets

$$\frac{dP}{dt} = \sigma I \quad (2)$$

As long as full employment is maintained, productive capacity and income must grow at the same rate, which leads to Domar's ([1946] 1957: 75) "fundamental equation":

$$\sigma I = \frac{1}{\alpha}\frac{dI}{dt} \quad (3)$$

from which was deduced the "required rate of increase in investment", i.e. the rate of growth of investment ensuring that all the additional production resulting from the increase in productive capacity is sold to consumers and businesses. Because of its simplicity, such an equation proves to only generate exponential growth trajectories showing that investment and income must grow at the rate r = $\alpha\sigma$ for continuous maintenance of full employment income.[7]

Domar's model did not intend to describe the actual growth path, but only the equilibrium one, with little attention to whether the economy will follow that path – that is, no stability analysis. Moreover, the model equation did not assume a priori causality, but a relationship between the three variables *r, α* and *σ* necessary for full employment growth (see Boianovsky 2017a). From Domar's standpoint, the economy can only be in equilibrium if it is growing. Samuelson's (1948: 361-62) survey of « dynamic process analysis » used Domar's ([1944] 1957; [1946] 1957) debt and growth models to illustrate the application of differential equations to formal features of continuous economic processes, whose solution is an exponential

[7] As Samuelson (1988) pointed out, Tinbergen (1937) had reached the same conclusion in his review of Harrod's 1936 *Trade Cycle*.

function. Domar ([1948] 1957: 111) stressed that, like in Harrod, Hansen and the whole family of multiplier-accelerator models, the assumption that the capital-output ratio must have some constancy was essential – otherwise, investment opportunities are unlimited and the "problem of capital accumulation" he tried to solve through his growth model did not exist in the first place.

Domar ([1947] 1957: 94, n. 13) mentioned the similarity between his "output-capital" ratio σ and Samuelson's (1939a) accelerator coefficient, but noticed critically that the latter deployed not the ratio of income to capital but of (lagged) consumption to capital (or its reciprocal).[8] He referred to Samuelson's Region D, but was sceptical of its relevance in assuring steady growth: "It is possible to construct a theoretical model in which investment and income continuously reinforce each other. But such model is liable to be unstable and can hardly be relied on as a means of achieving a continuous prosperity" (Domar [1948] 1957: 112). What worried Domar most, in his 1948 treatment of the "problem of capital accumulation", was the explanation of the "end of the prosperity", that is, the upper turning point.

Domar shared the then prevailing notion that capital over-accumulation is caused by failure of income to grow at some required rate. However, he rejected the hypothesis – ascribed by him to Hansen, Harrod, Kaldor, Hicks and others – that the failure of the economy to grow at the required rate was due to the *inability* of income so to grow. According to the view Domar criticized, the required rate could not be physically achieved or sustained, since, as the economy approached its full-employment ceiling, the consequent fall of induced investment would bring about a reduction of actual growth rate in the downswing through the accelerator mechanism. Those authors perceived "over-accumulation" as the result of excessive propensity to save in relation to the capital requirements decided by technological progress and the growth of the labour force – in Harrod's terms, an excess of the warranted rate over the "natural" growth rate.

Under Domar's assumption that the required growth rate could be achieved, the propensity to save was excessive in relation to the volume of investment as

[8] Samuelson's (1939a) deployment of the relation between consumption (instead of income) growth and induced investment was mathematically significant, since otherwise the model would generate a much more complex third order difference equation.

determined by “existing institutional conditions”, not in relation to the growth potential of the economy (Domar [1948] 1957: 118). That was the basis for Domar’s ([1946] 1957) claim that confident expectations, generated by government’s assurance of future growth through fiscal policy, would induce private investment decisions in a scale that would bring about the required growth rate – as defined by Domar’s formula – and by that justify the expectations, without putting the guarantee to test. That was Domar’s proposal of a “guaranteed growth rate” in order to stabilize the business cycle (see Boianovsky 2021).

It did not take long for Schelling to explore the possibility to bridge Domar’s analysis with Samuelson’s. As a member of the cohort of economists of Harvard and also a PhD student of Hansen, Schelling partially overlapped with Domar. At the time, Domar worked at the Federal Reserve Board in Washington. It is highly likely that Schelling attended the seminars Domar organized (together with Hansen) from 1943 to 1946 on macroeconomic policy and interacted with him about his growth models. Schelling (1947) was in particular attracted to Domar’s guaranteed growth rate, which he saw as shedding new light on Harrod’s (1939) involved notion of the “warranted growth rate”

In order to indicate the closeness of the "multiplier-accelerator" approach to Domar’s and Harrod’s approaches (Schelling 1947 was the first to refer to the « Harrod-Domar model »), Schelling examined a first model in which consumption is assumed to operate with a one period lag, and the investment demand function is of the form $I_t = \beta(Y_t - Y_{t-1})$ (with β being the acceleration coefficient, with the difference with Samuelson’s model that investment does not depend on consumption but on income). As long as β is greater than one, he argued that the economy would grow at a compound rate which is “approximately the Harrod-Domar $\alpha\sigma$” (Schelling 1947: 872). The problem, Schelling argued, is that the steady growth rate thus defined would prove to be unstable. This is because any chance discrepancy between aggregate output and aggregate demand which may occur will cause either income to increase indefinitely (in case of an excess aggregate demand) or to decrease indefinitely (in case of excess aggregate supply), all within one time period. Drawing a parallel with the Keynesian cross diagram, Schelling concluded: “In fact, a value of β greater than unity will necessarily yield a "negative multiplier," indicating the same kind of instability as would result from a marginal propensity to consume greater than unity." (Schelling 1947: 873)

The issue then arises: "May not there be more resiliency in the system than we have supposed, so that disturbances will be cushioned?" (Schelling 1947: 875). To examine it, Schelling built a second model in which consumption operates with no time lag but investment is now geared to past investment only, with $I_t = \beta(Y_{t-1} - Y_{t-2})$ in which investment no longer depends on current income. As long as the propensity to consume remained lower than 1, stability at any point of time was then shown to be ensured. But, as in Samuelson's original model, steady growth will now be possible only if the two roots of the characteristic equations are real and greater than one, which is the case only if $\beta > 4(1-\alpha)$ where α is the propensity to consume. So, apart from the fact that the growth rate thus generated by any combination of coefficients α and β is "rather frightening," any demand disturbance will have a permanent effect on the growth rate, generating thus a new form of instability (Schelling 1948: 876), as already pointed out by Samuelson. Schelling eventually sided with Samuelson and dismissed the growth solutions, thus reinforcing the so-called stability dogma. "Our conclusion then is that growth via induced investment is unreliable, that a stable dynamic equilibrium cannot be got from acceleration" (Schelling 1948: 875-76). As Schelling noted, that was related to Domar's paradoxical result that excess investment causes capital shortage whereas low investment brings about excess capacity (see Boianovsky 2021). It would in fact be up to Sidney Alexander (1949) to qualify Schelling's view.

Like Schelling, Alexander (1949: 174) noted how Samuelson (1939a, b) had shown how the interaction between the accelerator and the marginal propensity to consume could lead to a variety of time patterns of national income – including steady growth – depending on their numerical values. Alexander's point was that neither Schelling nor Samuelson had sufficiently clarified the nature of the exponential solutions of their models. It was clear to all that any linear difference equation has as many components as there are lags. What remained to be discussed was the relationship that existed between those two components, a point he analysed on the basis of the following numerical example. Consider the case in which the propensity to consume is equal to 0.95 and the coefficient of acceleration is equal to 2.1, one gets the following equation for Y_t:

$$Y_t = 0{,}95Y_{t-1} + 2{,}1(Y_{t-1} - Y_{t-2}) \text{ (4)}$$

The solution to (2) can be seen to depend both on the “initial conditions," that is, the initial sequence of incomes that start off the series, which must be given, and the values assigned to the propensity to consume and the coefficient of acceleration. For $Y_{t-2} = 100$ and $Y_{t-1} = 105$, one can show that the economy will keep growing steadily at 5%. But what if the initiating series is growing at a rate slightly over 5%? One must then conclude that the economy will move away from its growth path. If, for instance, for unchanged Y_{t-2}, we start off the series with $Y_{t-1} = 106$ instead of 105, then, Y_t will be equal to 113,3 instead of 110,25 and will permanently move away from its initial growth path, eventually reaching a growth path of 100%. This is because the second root of the characteristic equation of equation (2) ($r^2 - 2{,}95 + 2 = 0$) - which is equal to 1 - will eventually dominate the first one which is equal to 0,05. This becomes obvious by simply factoring the general solution of (2) by $(2{,}00)^t$ where K_1 and K_2 are determined by initial conditions.

$$Y_t = (2{,}00)^t \left[K_1 \frac{(1{,}05)^t}{(2{,}00)^t} + K_2\right] \text{ (5)}$$

As t increases, one can see that the ratio $\frac{(1{,}05)^t}{(2{,}00)^t}$ will tend to zero, making the behaviour of Y_t dependent only on the dynamic of $(2{,}00)^t$. “If, however, the initial series is growing at a rate just below 5% a year, K_2 will be small but negative, and the rate of growth will steadily diminish and will eventually reach a very rapid rate of decline” (Alexander 1949: 179).

This led Alexander to draw three conclusions. First, the minor rate of growth turns out to be a “critical dividing line between initial rates that will lead to an ever-increasing rate of growth approaching the dominant growth factor, and initial rates that will lead to declining rates of growth, and eventually to rapid decline.” (Alexander 1949: 179). Second, “The frequently observed tendency for growth to be cumulative holds only for rates between minor and dominant rates. Within this range, the more investment there is the faster income will grow and the more investment will be required. Outside the range between minor and dominant rates, a given rate of growth or decline cannot sustain itself.” (Alexander 1949: 180) Third, “as the income or investment period is shortened, the dominant rate of growth increases without limit. So, as the income or investment period approaches zero the dominant rate becomes infinitely great. But if either period is assumed to be zero and completely ignored, the infinitely large dominant rate of growth is lost from sight. Only the minor rate of growth will then be noticed. The recognition of the instability of the rate of growth thus discovered is an implicit reflection of the operation of the (hidden) infinitely large rate of growth. We may conclude that it is highly misleading to assume that expenditure instantaneously becomes income. The income period is an important characteristic of our economic system which must not be assumed away" (Alexander 1949: 180).

So, the challenge for explaining steady growth eventually comes down to figuring out whether, for plausible values of the coefficient of acceleration, moderate dominant rates may be possible (Alexander 1949: 178). This brought Alexander to argue that for the same value of the propensity to consume, as the acceleration coefficient increases, the minor rate increases and the dominant rate decreases until a threshold value is reached (equal in this example to 1,497). At this point, "all possibility of steady growth disappears and the dominant component of Y_t becomes cyclical. The corresponding minimum dominant rate of growth is over 22% a year" (Alexander 1949: 178). "Consequently, under the assumed conditions, values of the [propensity to consume] which are all different from unity permit only extremely rapid steady dominant rates. Values of the accelerator that do not generate such high dominant rates will lead to cycles rather than to steady growth. For very small values of the accelerator B, the cycles disappear, and only a steady growth approach toward zero is possible for each component of Y_t." (Alexander 1949: 178)

The effort, by Schelling, Alexander and others, to solve Harrod's and Domar's growth instability problems by going back to Samuelson's 1939 multiplier-accelerator model attracted significant attention from the late 1940s to mid 1950s, as illustrated by Hamberg's (1956) contemporary survey. Hamberg agreed with Schelling and Alexander about the accelerator as an unreliable generator of steady growth and that the source of long period trends in investment and income – and the reason for their oscillations – should be sought in autonomous investment. Solow (1957) charged Hamberg (and the rest of the contemporary growth-instability literature) for overlooking the role of the substitution between capital and labour in reaction to relative price changes. Indeed, capital deepening is conspicuously absent from Schelling and Alexander, as it was from Samuelson (1939a, b; 1940) and even

Samuelson (1988) later on.[9] As argued by Punzo (2009: 97), the failure to produce a theory of self-sustained endogenously generated dynamics combining growth and fluctuations eventually led to Solow's (1956) distinct approach, which "liberated growth from the embrace with fluctuations."

Like Schelling and Alexander, Goodwin, was concerned with the way to account for growth oscillations within the context of the multiplier-accelerator interaction, with limited success (see Punzo 2009). He was the more likely to understand what Samuelson meant when he referred to the stability dogma. Since the publication of Frisch's 1933 Cassel paper, as Goodwin argued in the introduction to his 1951 *Econometrica* article, economists had faced the following "unpleasant dilemma": either assuming that the economy is unstable and likely to explode or collapse, or is stable but kept alive by outside forces. The only way to avoid that dilemma was to build a new class of determinate models displaying nonlinearities. The first reason is clear. Only such models - except for Hansen's razor-edge solution - made it possible to account for self-sustained oscillations. The second is less obvious. The great merit of such models, Goodwin argued, was their ability to be "frequency converters" revealing that any steady force acting on the system (like the steady change in technical progress) would change the period of the movement but not its amplitude. Similarly, Goodwin claimed, that approach might more easily account for the impact of "historical events" whose effects would prove to prolong or shorten the boom or depression but without changing the trajectory of the economy (1951: 8).

Goodwin's work on nonlinear business cycles, together with contributions by Kaldor (1940) and Hicks (1950), eventually indicated to Samuelson the limits of the "stability dogma." Donald Gordon (1955) challenged the empirical and theoretical validity of Samuelson's (1947) key assumption that the real world is dynamically

[9] As Samuelson put it in a letter of 1 July 2002 to Craufurd Goodwin, regarding some refereeing work he was then doing for *History of Political Economy*, "in this important *trend* area, there is à la Cassel and Solow a distinctly *neoclassical* role for investment (as both cause and effect) that is 180 degrees from Keynes-Harrod-Samuelson 1930s investment role as merely a way of generating purchasing-power stimulus – to the neglect of its role as capital formation (capital deepening) to raise full-employment productivity within a Say's Law world where pump-priming is not 'needed.'"

stable. Gordon (1955: 308) pointed out that "recent theories of the business cycle … suggest that actual economic variables may possess no stable equilibrium values over the observable range, yet the values observed may all be points on stable functions." As Samuelson (1955: 313) remarked, Gordon was referring to auto-relaxation business cycle models of the kind proposed by Kaldor, Hicks and especially Goodwin, based on local instability at their stationary levels, and featuring limited oscillations due to nonlinearities. That was distinct from Samuelson's linear multiplier-accelerator model. Gordon's point was that – instead of Samuelson's (1947, p. 5) claim that actual observations are either points of dynamically stable or unstable equilibrium, which makes the latter very unlikely to be observed – what we may actually observe, as implied by the mentioned business cycle models, are neither. As Samuelson acknowledged,

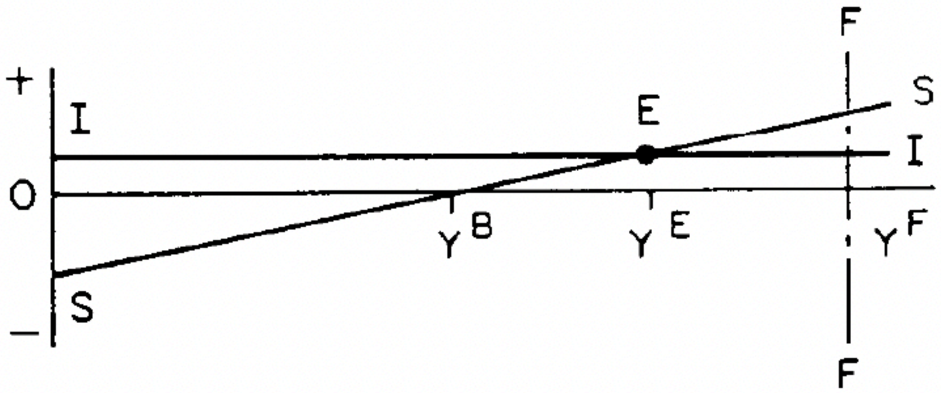
> Well, maybe the system is unstable. That is one possibility, and as Gordon is cogently pointing out, many of the cobweb cycles and auto-relaxation trade cycle theories of such moderns as Kaldor, Goodwin, Hicks, and others are squarely based on the notion of a system that is locally unstable at its stationary levels so that it oscillates — but because of such nonlinear elements as full-employment ceilings, capacity limitations, impossibility of disinvesting faster than at certain limiting rates, the system oscillates with a preferred finite amplitude (Samuelson 1955: 313).

It would finally take Samuelson (1988) three decades to formulate a nonlinear model of the business cycle, as he attempted to make sense of Hansen's notion of secular stagnation while paying a last tribute to his old mentor's centennial.

3. Samuelson's last macroeconomic model: secular stagnation and endogenous cyclical growth

On the occasion of Alvin Hansen's centennial, Samuelson (1988) produced his last contribution to macroeconomic dynamics, an extended reformulation of his first 1939 multiplier-accelerator model (see also the Appendix below for a more detailed formal treatment of Samuelson's 1988 KHS model). At a time when macroeconomics had moved toward optimization (with frequent references to his 1958 overlapping generation model), he made the choice to go against the trend, stick to macro-dynamics and start from the model which marked his early career.

The 1988 article begins by examining the working of a "Model T Keynesian system" in which income is the sole determinant of saving so that as long as capital widens to match population growth, equilibrium is shown to be set at any time at the intersection of a rising SS saving schedule with a horizontal II investment curve (Samuelson 1988: 6-7).



Model T Keynesian system and the invariance of the saving propensity (Source: Samuelson 1988: 7)

At point E, one can see that equilibrium happens at level Y^E below full employment equilibrium Y^F. That was reminiscent of the famous Keynesian cross diagram Samuelson had introduced in 1939. A characteristic of that model, Samuelson (1988: 7) argued, is to account for Simon Kuznets's 1941 finding that the saving/income ratio is an invariant function of the ratio Y^E to Y^F or of actual income Y (as long as $I = S$) to full employment income Y^*. Assume, for instance that, following a significant rise in population, the II curve has shifted upwards. Because full employment income and investment rise by the same amount, no change in the saving/income ratio will occur while the economy will keep growing smoothly with unchanged chronic unemployment. This can be formally expressed in the following way

$$\frac{S}{Y} = f\left(\frac{Y}{Y^*}\right),\ 0 < [Yf(y)]' < 1 \ (6)$$

where $[Yf(y)]'$ is the marginal propensity to save which shows that the fraction of income saved is a function of the ratio of actual income to full employment income.

Along those lines, the system is dynamized in accordance with a "tight accelerator mechanism," which indicates that, in absence of "feasible deepening" of capital, and no autonomous investment, investment is proportional to the rate of change of actual income (Samuelson 1988: 8):

$$\dot{K} = \frac{dK}{dt} = \beta\dot{Y} \ (7)$$

where the numerical value of β ("Harrod's 'relation') depends on the time unit used to measure income and saving rates.

As long as equilibrium between saving and investment is maintained, we have

$$f\left(\frac{Y}{Y^*}\right)Y = \beta\dot{Y}(8)$$

with $y = Y/Y^*$, it turns out that the rate of growth of actual income is equal to $f(y)/\beta$.

On the assumption that full-employment income grows exponentially at the ('natural') rate n, which reflects mere population growth and neutral labour saving technical change, Samuelson (1988: 9) derived the equilibrium level $\bar{y}$ for which actual income keeps pace with population, that is $\bar{y} = f^{-1}(n\beta)$. Therefore, if the rate of population slackens off permanently, the long run rate level of underemployment eventually increases by the same proportion.

On that basis, Samuelson argued that Hansen had figured out a way to dynamize Keynes's 1936 analysis to a situation in which the level of full employment is not constant but is changing over time, an idea that Keynes would have developed in his 1937 Galton lectures published the same year in the *Eugenics Review*. So, Samuelson (1988) chose to name it the Keynes-Hansen-Samuelson (KHS), which, he argued, is different from most macroeconomic models in several ways.

In Harrod's (alleged) model in which the propensity to save is constant, there is no possibility to "have a determination solution for equilibrium Y/Y^*" and for relative unemployment (Samuelson 1988: 9). This is because the equality between saving and investment is possible for any value of $\bar{y}$ and thus any level of unemployment. “A determinate theory of relative employment” would hence be lacking in Harrod’s model (Samuelson 1988: 10).

The KHS model would also crucially differ from the Solow-Meade neoclassical growth model in which the whole population is assumed to be fully employed and y is assumed to be permanently equal to 1. When the production function takes the neoclassical form $Y^* = Q(K, L) = LQ(K/L)$ and the investment-saving condition is permanently met at full employment, we have

$$\dot{K} = f(1)LQ(K/L) \quad (9)$$

Writing K/L as Z, we have $\dot{K}/K = f(1)\, L/K\, Q(K/L)$ or $\dot{K}/K = f(1)(Q(Z)/Z)$ from which it results that

$$\dot{Z}/Z = \dot{K}/K - \dot{L}/L = f(1)[(Q(Z)/Z)] - n \quad (10)$$

which means that Z converges to a steady state value of $\bar{Z}$ (root of the equation $\dot{Z}/Z = 0$) defined by

$$n = f(1)/\left[\left(\bar{Z}/Q(\bar{Z})\right)\right] \quad (11)$$

That is distinct from the equations for both the KHS and Harrod’s warranted-growth-rate versions.

In accordance with the notion of a neoclassical synthesis he put forward in the 1950s and early 1960s – as long as one assumes that Keynesian forces of effective demand are managed by economic policy so as to ensure full employment – Samuelson did not fail to emphasize that both models complement each other: the Solow-Meade model showing how the economy behave when $y = 1$ and factors of production are substitutable. It is precisely because of this complementarity that Samuelson could state that “Keynesianism as a tool of analysis” has for a time superseded early Keynesianism “as a depression ideology”: “The old King is dead; long live the new King of the neoclassical synthesis’” (Samuelson 1988: 18).

The KHS model would also differ significantly from Kaldor’s [1955-56] full-employment model or from the “‘Cambridge’ long-term generalization of that model” (Samuelson undated) in which changes to the income distribution are, in

Samuelson's view, assumed to automatically stabilize the full employment growth path – thereby ultimately and dangerously reducing the case for Keynesian government interventions.

What the KHS model has in common with Harrod is to display an unstable growth rate resulting from "positive feedback" effects (Samuelson 1988: 11). Assume for instance that the economy is on the right of the break-even point (intersection between the SS and II curves in figure 1). This means that y has increased, which will cause a rise in the propensity to save, but, due to the accelerator, the economy will keep expanding thus the gap between the rate of growth of actual income and the rate of growth of full employment income until full employment is reached. Inversely, if the economy is on the left of the break-even point, the departure from $\underline{\bar{y}}$ will be self-aggravating until "the unemployment rate soars toward 100 percent!" (Samuelson 1988: 11). So, Samuelson concludes, "the KHS model is seen, transiently, to possess some of Harrod's razor's-edge pathology" (Samuelson 1988: 10). But, due to the fact that y cannot exceed 1 and the saving propensity is flexible, such instability proves to be only local.

It is for this very reason that Samuelson thought that the KHS model provided a way to account for Goodwin's (1951) and Hicks' (1950) as well as Kaldor's (1940) nonlinear business cycles. Let us replace equation (1) by a flexible accelerator and write the dynamic equation for change in current income as:

$$\lambda \frac{dY}{dt} = \left(\frac{dK}{dt} - f(y)Y\right) (12)$$

where λ is a positive constant which denotes the inverse of the speed of adjustment, $\frac{dK}{dt} - f(y)$ represents the deviation between aggregate demand and aggregate output and

$$\frac{dK}{dt} = max\left[\frac{(\beta Y - K)}{\epsilon}; \delta K\right],\ O \preccurlyeq \epsilon \ll 1\ (13)$$

where ϵ is a rate-of-adjustment parameter, which makes K changes with respect to desired capital. Once (12)-(13) is rewritten with ratio-to-trend variables $Y/Y^* = y$ and $K/Y^* = k$, one eventually gets a timeless system of dimension 2 in which the state variables are y and k.

$$\dot{k} = H(k, y; n)\ (14)$$

$$\lambda\dot{y} = (H(k,y;n) - n)\frac{k}{y} - f(y) - \lambda n = G(k,y;n) \quad (15)$$

Samuelson does not proceed to a detailed analysis of the properties of this system, only providing a figure illustrating the working of the process over a complete limit cycle. As population grows exponentially at the given rate n, the system features a moving equilibrium of $\bar{y}$ determined by n. The horizontal straight line $F'F''$ describes the ceiling due to resource limitations, i.e., the maximal growth path. An actual trajectory starting at a point where y is off its stationary state (with y above $\bar{y}$ and k/y below $\bar{k}/\bar{y}$) is described in the following way: as the population is rising, the system, in accordance with the acceleration principle, is exploding. Actual income therefore rises with a growth rate higher than full employment output. Once the ceiling is reached, the growth rate of income is limited by that of the resource capacity. As Y grows that slowly, K/Y becomes excessive until $\dot{K}$ turns negative and the system is sent in a self-aggravating way below its break-even point. Eventually, k/y is so low that $f(y)$ becomes large relative to β, which generates a positive $\dot{k}$. Following this, the system features again a positive growth rate of y until reaching the full employment ceiling.

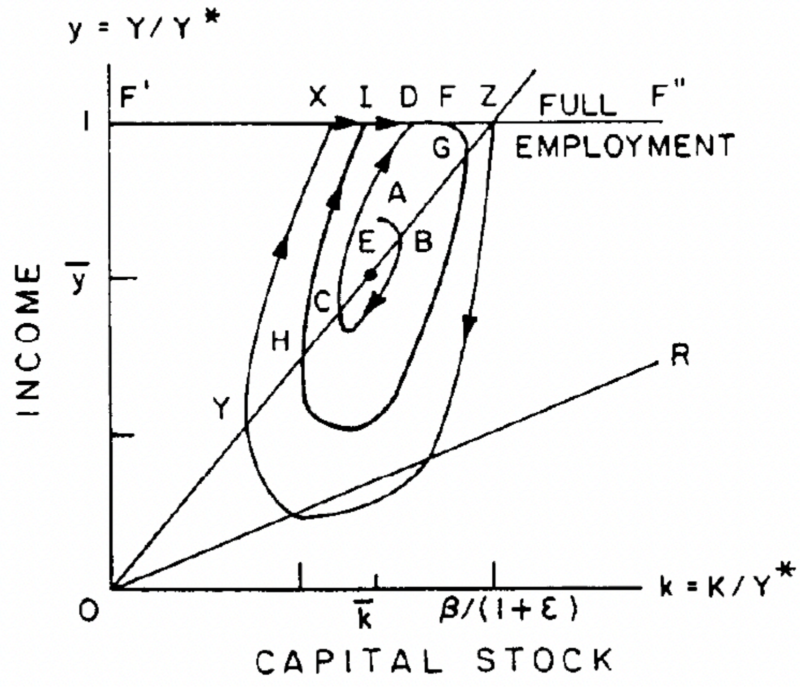


Figure 2: Hansen Limit-Cycle (Source: Samuelson 1988: 13)

Samuelson’s KHS model has the advantage that it is no longer necessary to specify the parameter values that lead to self-sustained oscillations. In addition, it allows accounting simultaneously for growth and cycles. However, because it relies on the assumption of an exogenous rate of growth of the population, as long as that rate is no longer changing, the economy starts cycling around a static position.

On that basis, Samuelson concluded that “all that Kaldor (1951) found lacking in Hicks (1950) is achieved by KHS” (Samuelson 1988:14). First, it solves the limitations of linear systems by providing a way to account for self-sustained oscillations. Second, it is derived from a flexible accelerator, the value of which varies in the course of the cycles, falling off in the course of the downswing and becoming null once full employment has been reached. But, unlike Hicks, Samuelson does not need to assume the existence of a floor resulting from autonomous investment to account for low turning points. In the KHS model, as y adjusts downwards, $f(y)$ becomes infinitely low. The bottom line is that the KHS model does not rely on the introduction of any arbitrarily assumed bounds. Third, as in Hicks’ model, cyclical oscillations are treated as deviations around a rising long-term trend of output but while Hicks explained the trend by growing autonomous investment, Samuelson explained it by the rise in population.

For those reasons, Samuelson claimed that the KHS model “adds Harrodian trends to the effective-demand system of Keynes, but in a less *ad hoc* fashion than the Hicks [1950] bald axiomatizing of autonomous exponential investment approach” (Samuelson undated). Of course, Samuelson added, one can complain that the KHS model is not derived from rigorous microeconomics foundations. “But even worse defects must be acknowledged; sometimes [systems derived from proper microeconomic foundations] lack proper macroeconomic foundations also” (Samuelson 1988: 12).

4. The persistence and many lives of the multiplier-accelerator model

In a letter dated 11 February 1997, addressed to one of the authors (Boianovsky), Samuelson wrote:

> For your interest, I enclose a 1988 reprint few have noticed. This Keynes-Hansen-Samuelson non-linear limit cycle captures the empirical content of the 1936 Harrod, 1930s Kalecki, 1940 Kaldor, 1940s Goodwin, 1950 Hicks cyclical model, avoiding certain infelicities and omissions; and it enabled me to discern (50 years later!) that decelerating population growth, at the same time that it lowered the acceleration-principle investment propensity, also lowered (by virtue of Modigliani's lifecycle theory of saving) the propensity to save. In principle, *Prosperity and Depression* would agree with its spirit.

Hence, nearly 10 years after its publication, Samuelson was painfully aware that, unlike his 1939 articles, the 1988 KHS model did not attract a large readership, despite his strong claims regarding its analytical achievements. Such claims for what he named the Keynes-Hansen-Samuelson multiplier–accelerator model may be found in the article, where he stated that, "among other virtues", it "provides a nice fulfilment of the limit-cycle paradigm sough by Kaldor, Goodwin, and Hicks" (Samuelson 1988: 4). The fact that it came out in the first issue of a not well-known Japanese journal probably contributed to the negligible impact of that article.[10]

The modern debate about secular stagnation, started mainly by Lawrence Summers around 2013, brought back an interest on Hansen's original ideas on the subject (see e.g. Summers 2015; Backhouse and Boianovsky 2016). Except for Pagano and Sbracia's (2014: 31, n. 38) brief reference to Samuelson's (1988) formalization of Hansen's secular stagnation, the KHS model has gone unnoticed in

[10] The inaugural issue of *Japan and the World Economy*, edited by Ryuzo Sato, included as well articles by other prominent economists, such as Paul Krugman and Hal Varian.

the recent revival of interest in secular stagnation. Before that, Rostow (1998: 213) – while discussing the Japanese stagnation of the 1990s in Hansenian terms – mentioned Samuelson's (1988) modelling of Keynes' and Hansen's intuitions about the perverse effects of lower population growth. The closest discussion of the KHS model so far has been provided by Vellupillai (2019: 357; 362-63) as part of his chapter on Samuelson's macroeconomics. Vellupillai discusses the assumptions leading to the KHS limit-cycle fluctuations, while remarking that the growth path remains exogenous in that model despite Samuelson's intention to provide a mechanism of cyclical growth.

Summers and others have associated secular stagnation to a negative Wicksellian natural rate of interest and to the zero lower bound on the short-term nominal interest rate (see Boianovsky 2017b). The rate of interest, however, was hardly mentioned by Samuelson (1988), who followed along Hansen's assumption that there is "no feasible deepening of capital in the system" (8) and made investment demand a function of the accelerator only. As for savings, Samuelson (1988: 14-16) examined the effects on savings of a reduced population growth in a life-cycle model à la Modigliani. Hans Neisser (1944) had criticized Hansen for overlooking that a declining rate of population growth implies an aging population, and hence a lower saving propensity. Against Neisser, Samuelson (1988) showed that, in the expanded KHS model, population deceleration lowers investment propensity more than saving propensity, which confirmed the intuitions of Keynes and Hansen.[11]

Samuelson (1988: 16-17) dealt with capital deepening while discussing the Solow-Meade neoclassical growth model featuring Say's Law. The neoclassical growth model was part and parcel of the "neoclassical synthesis" famously advocated by Samuelson since the 1950s (see section 3 above). However, the "synthesis", to some extent, contributed to explain why the KHS multiplier-

[11] Recent efforts to model secular stagnation tackle the problem that a negative rate of discount makes the inter-temporal maximization problem of the representative agent intractable. The alternative is the introduction of heterogeneous agents in the form of overlapping generation models of saving, of the kind advanced by Samuelson back in 1958 (see Eggertsson, Mehrotra and Robbins 2019).

accelerator model "got lost in the shuffle of history," as concerns about long-term unemployment became less pressing in the post-war era and textbooks "lagged behind the scouting parties at the frontier of research" (Samuelson 1988: 17-18). As Samuelson (1964a: 743; 1964b: 341-42) pointed out, the capital deepening mechanism of the neoclassical synthesis presupposes that the capital-output ratio is flexible, against the view – argued by Hansen, Eisner, Domar and others– that it is very nearly a constant. As described by Samuelson (1964a: 743), Hansen's view (carried over to the KHS model) implied that "any attempt to accumulate capital beyond the rate required by the annual growth in output will soon be unsuccessful." That was apparently the case in 1956-57, when the American equipment boom was followed by investment sluggishness and excess capacity. "A final judgement is not yet possible" of which assumption is closer to reality, Samuelson (1964b: 342) observed.

Another reason why Samuelson did not develop the 1988 KHS model of secular stagnation before was that he "took it more or less as a dogma that our dynamic systems should be 'stable', in the sense of having damped rather than anti-damped characteristic roots" (Samuelson 1988: 17, n. 2). That was partly based on the perception that pump-priming did not seem to work in the New Deal (see also section 1 above).

> So, from 1937 on, I rejected the multiplier-accelerator explosive exponentials that kept thrusting themselves at me in my research notebooks. My effect on Hansen in this regard was baneful: fortunately he and Keynes manfully resisted any downplaying of the widening-of-capital phenomenon (Samuelson 1988: 17, n. 2).

Samuelson (1976: 30) recalled on other occasions how Hansen resisted "even my harmful influence" as he (Samuelson) kept insisting in the 1930s-40s "rightly but misleadingly" that there was no net stimulus from the accelerator "in a multiplier model that oscillates around a horizontal trend." That came out clearly in Samuelson's (1939b: 791; italics in the original) statement that "*the acceleration principle can determine the nature of the oscillations but not the average level of the system.*" In a letter to Samuelson, dated 27 February 1940 (held in the Samuelson Papers, Duke University), Hansen reacted critically to an unpublished paper written on that occasion by Samuelson on aspects of secular stagnation. Hansen charged Samuelson for missing the "real problem" of secular stagnation, that is, the transition from a rapidly growing society with high saving and investment to one in which they are low. It was about the transition from one equilibrium position to another, not a comparison of two stable equilibria. Hansen further criticised Samuelson (1939a, b) for assuming that investment depended on the level of consumption, which, in Hansen's view, determined only replacement capital. The real determinants of new investment, Hansen argued, "are new products, new ways of producing goods more economically, and growth."

It was only after the development of the Harrod-Domar model in the 1940s that Samuelson changed his mind about the accelerator as a key factor not just in business cycles but in economic growth as well. It is worth noting that economic growth – or secular stagnation for that matter – was not a topic of discussion in Samuelson's 1947 *Foundations*. Shortly after that, Samuelson (1948b: 361-62) provided one of the first expositions of the Harrod-Domar formula for the "exponential growth trends", based on the relation between income and the capital stock expressed by the "acceleration principle". Although not referring to Harrod or Domar, Samuelson (1948a) pointed out in the first edition of his *Economics* – upon discussing short-run economic oscillations along the lines of the multiplier-accelerator model – that "in the long run, if the system is growing because of population increase and technical progress, then the acceleration principle works primarily as a stimulating factor: growing national income causes extensive growth of capital, which in turn means brisk demand for investment and relatively low unemployment." That passage was kept in further editions.

Samuelson introduced in the 1964 edition of *Economics* a new section titled "Interactions of accelerator and multiplier" and a new chapter on growth economics. In his presentation of the Harrod-Domar growth models, Samuelson (1964a: 743, n. 3) remarked that the multiplier-accelerator interaction was then applied to the economic growth trend, rather than to the business cycle as deviations from that trend.[12] That was quite distinct from his 1939 approach. And so was his discussion of

[12] Investment appears in Domar's aggregate supply and demand functions, with asymmetrical effects: the former is a function of the (net) *level* of investment, while

the cyclical effects resulting from the collision of Harrod's warranted growth rate with the ceiling represented by the natural growth rate. The bouncing back of the system – which he associated to Hicks (1950) – became a key element of Samuelson's 1964 statement of the multiplier-accelerator business cycle model:

> But how can a system grow forever at 5 or 6 per cent if its labour force grows only at 1 or 2 per cent? It can't. The self-warranting expansion … must ultimately bump into the full-employment ceiling. Like a tennis ball … it is likely to bounce back from the full-employment ceiling into a recession. Why? Because the minute the system stops its fast growth, the accelerator dictates the end of the high investment supporting the boom. [Similarly] when output plummets downward rapidly, the acceleration principle calls for negative investment … greater than the rate at which machines can wear out. This wear-out rate puts a floor on how far disinvestment can push the economy below its break-even point. Bumping along such a basement floor means that eventually firms will work down their capital stock to the level called for by that low level of income; and now the acceleration principle calls for a termination of disinvestment! (Samuelson 1964a: 263).

This long quotation forcefully illustrates how, by the early 1960s, Samuelson was shifting toward a nonlinear explanation of the multiplier-acceleration interaction. Throughout the several editions of his *Economics*, including those co-authored with William Nordhaus from 1985-2010, Samuelson claimed that the multiplier-accelerator model (especially, but not only, in its nonlinear version) provided a

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the latter depends on its *rate of growth*. A higher investment level has a permanent effect on capacity, but a temporary one on income as the multiplier mechanism peters out (Domar [1947] 1957, p. 98). Hence, in order that sufficient demand is generated and capacity remains fully utilized it is necessary for investment (and income) to grow at a certain rate, determined by the equality between aggregate demand and supply.

suitable account of endogenous macroeconomic fluctuations from a Keynesian perspective. The 1939 linear multiplier-accelerator was briefly discussed in the *Foundations* (Samuelson 1947: 341-42), whereas nonlinear dynamics was reported as surrounded by “formal difficulties”. That explained why nonlinear business cycle models were still essentially “literary theories”, which he called “billiard table” theories (340). Samuelson (1947: 339) was aware that nonlinear systems had received attention in theoretical mechanics as “relaxation oscillations” and limit-cycle theory. It was only after Goodwin (1951), though, that such methods were applied to nonlinear business cycle models. As Samuelson (1988: 12) recalled, Goodwin taught physics at Harvard during the Second World War, under the influence of the French engineer Philippe LeCorbeiller, who perfected the Van der Pool–Raleigh limit-cycle theory of dynamics. In section 6, named “Goodwin auto-relaxation cycles”, Samuelson (1988) put forward what he called a Goodwin-Hicks-Kaldor nonlinear model in its KHS version (see section 3 above), so that, he claimed, those three economists’ “goals are realized” (Samuelson 1988: 13).

Hansen’s meaning of “secular stagnation” was closed connected to the nonlinear approach to economic fluctuations. According to Hansen (1947: 177), there was a tendency for investment to outrun in boom periods the requirements of technical progress and population growth, called temporary “saturation” of investment opportunities. The “amount of investment needed to maintain full employment has historically far exceeded the amount needed for growth and progress”, argued Hansen (1947: 177). Only in boom years had the amount of investment been adequate to provide for full employment, but “this amount of investment could not be maintained continuously without exceeding by far the requirements of growth and progress”, claimed Hansen (1947: 178). Such abrupt end of investment, amplified by the acceleration mechanism, was the “essential cause of depressions and unemployment,” concluded Hansen (ibid). If the “growth and progress” factors, and the values of the multiplier and of the accelerator coefficient, are weak, the economy is set for “secular stagnation,” with stillborn recoveries and recurring depressions caused by an excess of the secular propensity to save over the long-run maintainable rate of investment.[13]

Accordingly, (Hansen 1939: 4) defined the “essence of secular stagnation” as “sick recoveries which die in their infancy and depressions which feed on themselves and leave a hard and seemingly immovable core of unemployment”. That was consistent with Samuelson’s (1988: 16; italics in the original) remark that the KHS model described what would be the “*average* level of spontaneous unemployment throughout the business cycle that will occur in economies experiencing different natural rates of growth than in describing truly steady states of equilibrium.” Hence, secular stagnation should be interpreted from the perspective of the average performance of the economy over the business cycle.

By the late 1980s, when Samuelson’s KHS model came out, his 1939 multiplier-accelerator model, while retaining its status as a classic reference, had long lost its role as an influential account of economic fluctuations. Grandmont (1989: 279), for instance, mentioned Samuelson (1939a) as an example of expectations-driven endogenous deterministic cycles in Keynesian economics. According to Grandmont, since the 1980s, in contrast with earlier macroeconomic formulations, economists relied on “explicit modeling of the traders’ optimizing behavior”, which permitted analysis of “how expectations interact with the internal mechanisms of the economic system” to generate fluctuations. That was related to the fact that the “internal nonlinear dynamics” of the economy could generate complex periodic orbits, as well as to the existence of multiple stochastic equilibria produced by random factors (sunspots) that influence traders’ expectations.

It is, therefore, hardly surprising that Samuelson (1988) would fail to attract attention when the sort of nonlinear dynamics deployed by macroeconomists at the frontier of business cycle research clearly differed from the Goodwin-like mathematical foundations of that article. Samuelson was not totally oblivious to such developments. He noticed that “these days it is fashionable to complain that various macroeconomic systems – like the General Theory or the present KHS model – lack rigorous microeconomic foundations. Fair enough” (Samuelson 1988: 12, n. 1).

[13] This may be understood in terms of an excess of Harrod’s warranted growth rate over the natural growth rate.

However, his main concern was the lack of proper macrofoundations, in the sense that the assumption of a given price level cannot be sustained when the economy approaches its full employment ceiling. Even so, the KHS model, he claimed, could be still acceptable if the central-bank reaction to accelerating inflation was taken into account to produce a bounce-back from the full-employment ceiling. Samuelson stuck to his guns as he kept elaborating non-optimizing macroeconomic models along the methodological lines argued back in the *Foundations*. Hence, it is not unexpected that Samuelson's 1939 multiplier-accelerator model has become a topic of research for mostly heterodox economists (see e.g. Westerhoff 2006), who, however, have continued to overlook his 1988 nonlinear KHS model.

References

Assous, M., and R. Lampa, R. 2014. Lange's 1938 model: Dynamics and the "optimum propensity to consume". *The European Journal of the History of Economic Thought*, 21(5): 871–898.

Assous, M. and V. Carret. 2020. (In)Stability at the Cowles Commission (1939–1948). *The European Journal of the History of Economic Thought.* 27 (4): 582–605.

Assous, M. and V. Carret. 2022a. *Modeling Economic Instability: A History of Early Macroeconomics*. Cham: Springer

Assous, M.and V. Carret. 2022b. Moving dynamics beyond business cycles: Jan Tinbergen's first macrodynamic model (1934–1936). *European Journal of the History of Economic Thought.* https://doi.org/10.1080/09672567.2022.2123541

Assous, M.and V. Carret. 2021. The importance of multiple equilibria for economic policy in Jan Tinbergen's early works. *The European Journal of the History of Economic Though* 29 (3). https://doi.org/10.1080/09672567.2021.2019294

Backhouse, R.E. and M. Boianovsky. 2016. Secular stagnation: the history of a macroeconomic heresy. *European Journal of the History of Economic Thought.* 23: 946-70.

Backhouse, R. E. (2017). *Founder of modern economics: Paul A. Samuelson. Volume 1, Becoming Samuelson, 1915–1948*. New York: Oxford University Press.

Blanchard, O. 2000. What do we know about macroeconomics that Fisher and Wicksell did not? *Quarterly Journal of Economics.* 115: 1375-1409.

Boianovsky, M. 2017a. Modeling economic growth: Domar on moving equilibrium. *History of Political Economy.* 49: 405-36.

Boianovsky, M. 2017b. Wicksell, secular stagnation, and the negative natural rate of interest. *History of Economic Ideas*. 25: 37-61.

Boianovsky, M. 2021. Domar, expectations, and growth stabilization. *Cambridge Journal of Economics*. 45: 723-49.

Boianovsky, M. and K.D. Hoover (eds.). *Robert Solow and the development of growth economics.* Durham (NC): Duke University Press. Annual supplement to *History of Political Economy*, vol. 41.

Brown, E.C. and R.M. Solow (eds.). 1983*. Paul Samuelson and modern economic theory.* New York: Mc-Graw-Hill.

Carret, V**.** 2021. Fluctuations and growth in Ragnar Frisch's rocking horse model. *Journal of the History of Economic Thought* (forthcoming). Preprint at SocArXiv. doi:https://doi.org/10.31219/osf.io/69nsg. [Crossref], [Google Scholar]

Domar, E. [1944] 1957. The 'burden of debt' and the national income. *American Economic Review*. 34: 798-827. As reproduced in Domar (1957), pp. 35-69

Domar, E. [1946] 1957. Capital expansion, rate of growth, and employment. *Econometrica*. 14: 137-47. As reproduced in Domar (1957), pp. 70-82.

Domar, E. [1947] 1957. Expansion and employment. *American Economic Review*. 37: 34-55. As reproduced in Domar (1957), pp. 83-108.

Domar, E. 1957. *Essays in the theory of economic growth.* New York: Oxford University Press.

Eggertsson, G.B., N.R. Mehrotra and J.A. Robbins. 2019. A model of secular stagnation: theory and quantitative evaluation. *American Economic Journal: Macroeconomics.* 11: 1-48.

Frisch, R. 1933. Propagation problems and impluse problems in dynamic economics. In *Economic Essays in Honour of Gustav Cassel*, pp. 171-205. London: Allen & Unwin.

Goodwin, R. 1951. The nonlinear accelerator and the persistence of business cycles. *Econometrica*. 19: 1-17.

Gordon, D.F. 1955. Professor Samuelson on operationalism in economic theory. *Quarterly Journal of Economics*. 69: 305-10.

Grandmont, J.-M. 1985. On endogenous competitive business cycles. *Econometrica*. 53: 995-1045.

Grandmont, J.-M. 1989. Keynesian issues and economic theory. *Scandinavian Journal of Economics.* 91: 265-93.

Haberler, G. 1946. *Prosperity and Depression: a Theoretical Analysis of Cyclical Movements.*, 3rd. edition. Lake Success (NY): United Nations.

Haberler, G. 1949. Discussion. *American Economic Review*. 39: 84-88.

Hamberg, D. 1956. *Economic growth and instability*. New York: W.W. Norton.

Hansen, A. 1939, Economic Progress and Declining Population Growth, *American Economic Review*, 29(1), pp. 1-15.

Hansen, A. H. 1947. *Economic policy and full employment.* New York: McGraw-Hill.

Harrod, R. F. 1939. An essay in dynamic theory. *Economic Journal.* 49: 14-33.

Hicks, J. 1950. *A Contribution to the Theory of the Trade Cycle*. Oxford: Clarendon Press.

Keynes, J.M. 1937. Some economic consequences of a declining population. *Eugenics Review*. 29: 13-17.

Kydland, F. and E. Prescott. 1991. The econometrics of the general equilibrium approach to business cycles. *Scandinavian Journal of Economics.* 93: 161-78.

Neisser, H. 1944. The economics of a stationary population. *Social Research*. 11: 470-90.

Pagano, P. and M. Sbracia. 2014. The secular stagnation hypothesis: a review of the debate and some insights. Banca D'Italia. Working Paper # 231.

Punzo, L. 2009. A nonlinear history of cycle and growth theories. In M. Boianovsky and K.D. Hoover (eds.): 88-106.

Rostow, W.W. 1998. *The great population spike and after: reflections on the 21st century.* New York: Oxford University Press.

Samuelson, P.A. 1939a. Interactions between the multiplier analysis and the principle of acceleration. *Review of Economics and Statistics*. 21: 75-78.

Samuelson, P.A. 1939b. *A synthesis of the principle of acceleration and the multiplier.* Journal of Political Economy. 47: 786-97.

Samuelson, P.A. 1940. The theory of pump-priming reëxamined. *American Economic Review*. 30: 492-506.

Samuelson, P.A. 1947. *Foundations of Economic Analysis.* Cambridge (Mass.): Harvard University Press.

Samuelson, P.A. 1948a. *Economics: An Introductory Analysis*. New York: McGraw-Hill.

Samuelson, P.A. 1948b. "Dynamic Process Analysis." In H.S. Ellis (ed.) *A Survey of Contemporary Economics.* Volume 1, 352-78. Homewood, Ill.: Richard D. Irwin.

Samuelson, P.A. 1964a. *Economics: An Introductory Analysis.* Sixth edition. New York: McGraw-Hill.

Samuelson, P.A. 1964b. A brief survey of post-Keynesian developments. In R. Lekachman (ed.). *Keynes's General Theory: Report of Three Decades*, pp. 331-47. London: St. Martin's Press.

Samuelson, P.A. 1958. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy.* 66: 467-82.

Samuelson, P.A. 1972. Maximum principles in analytical economics. *American Economic Review*. 62: 249-62.

Samuelson, P.A. 1976. Alvin Hansen as a creative economic theorist. *Quarterly Journal of Economics*. 90: 24-31.

Samuelson, P.A. 1988. The Keynes-Hansen-Samuelson multiplier-accelerator model of secular stagnation. *Japan and the World Economy*. 1: 3-19.

Samuelson, P.A. Undated. A unique business cycle of determinate amplitude and period: the 1940 Kaldor limit cycle generalized to possess trend. Unpublished manuscript. Samuelson Papers. Box 136. Duke University.

Schelling, T. C. 1947. Capital growth and equilibrium. *American Economic Review.* 37: 864-76.

Solow, R.M. 1956. A contribution to the theory of economic growth. *Quarterly Journal of Economics.* 70: 65-94.

Solow, R.M. 1957. Review of Hamberg (1956). Econometrica. 25: 612-13.

Summers, L. 2015. Demand side secular stagnation. *American Economic Review.* 105: 60-65.

Tinbergen, J. 1937. Review of the trade cycle. *Weltwirtschaftliches Archiv,* 45, 89–91.

Tobin, J. 1983. Macroeconomics and fiscal policy. In Brown and Solow (eds.), pp. 189-201.

Velupillai, K. Vela. 2019. Paul Samuelson, the multiplier-accelerator model and Alvin Hansen, in *Paul Samuelson: Master of Modern Economics,* pp. 343-74, edited by R. Cord, R. Anderson and W. Barnett. London: Palgrave Macmillan.

Westerhoff, F.H. 2006. Samuelson's multiplier-accelerator model revisited. *Applied Economics Letters*. 13: 89-92.

Appendix

Derivation of $H(k, y; n)$ (equation 14)

$$\frac{\dot{K}}{Y^*} = max\left[\frac{(\beta y - k)}{\epsilon}; \delta k\right]$$

$$\frac{\dot{K}}{K}\frac{K}{Y^*} = max\left[\frac{(\beta y - k)}{\epsilon}; \delta k\right]$$

$$\left(\frac{\dot{k}}{k} - n\right)k = max\left[\frac{(\beta y - k)}{\epsilon}; \delta k\right]$$

$$\frac{\dot{k}}{k} = max\left[\frac{(\beta y k^{-1} - 1)}{\epsilon}; \delta\right] + n$$

$$\frac{\dot{k}}{k} = H(k, y; n)$$

Derivation of $G(k, y; n)$ (equation 15)

$$\lambda\frac{\dot{Y}}{Y} = \frac{\dot{K}}{Y} - f(y)$$

$$\lambda\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K}\frac{K}{Y} - f(y)$$

$$\lambda\left(\frac{\dot{y}}{y} + n\right) = \left(\frac{\dot{k}}{k} - n\right)\frac{K}{Y} - f(y)$$

$$\lambda\left(\frac{\dot{y}}{y} + n\right) = \left(\frac{\dot{k}}{k} - n\right)\frac{K}{Y^*}\frac{Y^*}{K} - f(y)$$

$$\lambda\left(\frac{\dot{y}}{y} + n\right) = \left(\frac{\dot{k}}{k} - n\right)\frac{k}{y} - f(y)$$

$$\lambda\left(\frac{\dot{y}}{y} + n\right) = \left(\frac{\dot{k}}{k} - n\right)\frac{k}{y} - f(y) - \lambda n$$

$$\lambda\left(\frac{\dot{y}}{y} + n\right) = (H(k, y; n) - n)\frac{k}{y} - f(y) - \lambda n$$

$$\lambda \dot{y} = (H(k, y; n) - n)\frac{k}{y} - f(y) - \lambda n = G(k, y; n)$$

Samuelson's KHS model re-examined

Consider the following slightly different system with the two state variables y and k.

$$\lambda \frac{\dot{y}}{y} = \left(\frac{\dot{k}}{k} - n\right)\frac{k}{y} - f(y) - \lambda n$$

$$\dot{y} = \frac{1}{\lambda}\left[\left(\frac{\dot{y}}{k} - n\right)k - f(y)y - \lambda n y\right]$$

$$\dot{y} = \frac{1}{\lambda}\left[\left(\frac{(\beta y k^{-1} - 1)}{\epsilon} - \delta\right)k - f(y)y - \lambda n y\right]$$

$$\dot{y} = \frac{1}{\lambda}\left[\frac{(\beta y - k)}{\epsilon} - \delta k - f(y)y - \lambda n y\right]$$

$$\dot{y} = \frac{1}{\lambda}\frac{\beta y}{\epsilon} - \frac{1}{\lambda}\frac{k}{\epsilon} - \frac{\delta}{\lambda}k - \frac{f(y)y}{\lambda} - \frac{\lambda n y}{\lambda}$$

$$\dot{y} = \left(\frac{1}{\lambda}\frac{\beta}{\epsilon} - \frac{f(y)}{\lambda} - n\right)y + \left(-\frac{1}{\lambda\epsilon} - \frac{\delta}{\lambda}\right)k$$

and

$$\dot{k} = \frac{1}{\epsilon}\beta y - \left(\delta + n + \frac{1}{\epsilon}\right)k$$

Or

$$\dot{y} = Ay + Bk$$

$$\dot{k} = Cy + Dk$$

with

$A = \left(\frac{1}{\lambda}\frac{\beta}{\epsilon} - \left(\frac{f(y)}{\lambda} + n\right)\right)$ whose sign is indeterminate

$B = -\left(\frac{1}{\lambda\epsilon} + \frac{\delta}{\lambda}\right) < 0$

$C = \frac{1}{\epsilon}\beta > 0$

$D = -\left(\delta + n + \frac{1}{\epsilon}\right) < 0$

As long as $A + D > 0$, the trace of the Jacobian matrix is positive and the stationary state is unstable. So, if one assumes that the trace changes sign and becomes negative for extremes values of y, one can apply the Poincarré-Bendixson theorem and show that there exist limit cycles.

Consider first the set of points (y, k) such that the capital stock does not change

$$\dot{k} = Cy + Dk = 0$$

Total differentiation yields

$$\frac{dk}{dy}_{\dot{k}=0} = -\frac{D}{C} > 0$$

Thus, the locus of all points in the set $\{(y, k) \text{ I } \dot{k} = 0\}$ is an upward sloping curve. Obviously, for all k above the curve $\dot{k} = 0$, investment decreases because of $B < 0$,hence $\dot{k} < 0$. In the same way, $\dot{k} > 0$ for all k below the curve for $\dot{k} = 0$

Secondly, the set of points (y, k) with $\dot{\underline{y}}$ is given by $y = A\dot{y} + Bk$. It follows that

$$\frac{dk}{dy}_{\dot{y}=0} = -\frac{B}{A}$$

Because $B < 0$, its sign depends, given the set of fixed parameters, on the relative magnitudes of $f(y)$ and β, both being positive by assumption. Recalling Samuelson's assumption, we have $B > 0$ for low as well as for high levels of y and $B < 0$ in the neighborhood of the stationary equilibrium. Hence, the curve for $\dot{y} = 0$ is negatively sloped for low and for high values of y and it is positively sloped for normal levels of y. Furthermore, because the stationary state is unique, the curve for $\dot{y} = 0$ intersects the curve for $\dot{k} = 0$ just once. In order to examine the direction of change of y , one can divide the plane into regions A, B, and C, characterized by $B < 0$, 0, and again > 0 respectively. In regions A and C income decreases (increases) for y to the right (left) of the curve $\dot{y} = 0$ because of $B > 0$. For points in region B, y increases (decreases) to the right (left) of the curve $\dot{y} = 0$. Altogether this leads to the phase portrait in the following figure. The curves are drawn such that the locus $\dot{k} = 0$ intersects the ordinate at $k_0 > 0$. The curve $\dot{y} = 0$ intersects the abscissa at $y_0 > \bar{y}$

and approaches the k- axis for $k \rightarrow +\infty$.

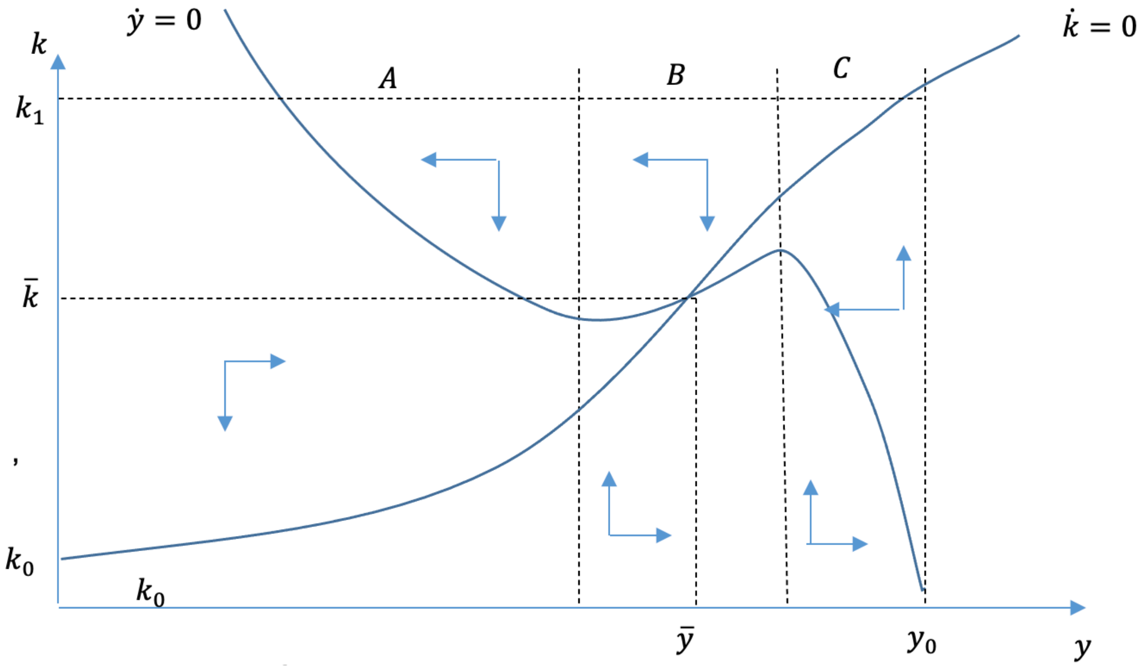

Figure 3: The Phase Portrait of Samuelson's KHS Model

Every trajectory starting in R^2 either is a limit cycle or approaches a limit cycle. The proof is a straightforward application of the Poincaré-Bendixson theorem.

Condition on the trace

$$\frac{1}{\lambda}\frac{\beta}{\epsilon} - \left(\frac{f(y)}{\lambda} + n\right) - \left(\delta + n + \frac{1}{\epsilon}\right) > 0$$

or

$$\beta > \lambda\epsilon\left[\left(\frac{f(y)}{\lambda} + n\right) + \left(\delta + n + \frac{1}{\epsilon}\right)\right]$$