Capital Commitment *

Elise Gourier
ESSEC Business School and CEPR

Ludovic Phalippou
University of Oxford, Said Business School

Mark M. Westerfield
University of Washington, Foster Business School

December 2022

Abstract

Twelve trillion dollars are allocated to private market funds that require outside investors to commit to transferring capital on demand. We show within a novel dynamic portfolio allocation model that ex-ante commitment has large effects on investors' portfolios and welfare, and we quantify those effects. Investors are under-allocated to private market funds and are willing to pay a larger premium to adjust the quantity committed than to eliminate other frictions, like timing uncertainty and limited tradability. Perhaps counter-intuitively, commitment risk premiums increase with secondary market liquidity and they do not disappear even if investments are spread over many funds.

JEL Classification: G11, G12

Keywords: Capital commitment, private equity, commitment risk, liquidity premium

---

*Email: elise.gourier@essec.edu, ludovic.phalippou@sbs.ox.ac.uk, and mwesterf@uw.edu. We would like to thank Andrew Ang, Michael Brandt, John Campbell, Catherine Casamatta, Joost Driessen, Michael Ewens, Valentin Haddad, Francis Longstaff, Stefan Nagel (the Editor), Lubos Pastor, Giorgia Piacentino, Morten Sorensen, Jos van Bommel, Marno Verbeek, Luis Viceira, the Associate Editor and an anonymous referee for feedback and suggestions on our project. We also thank participants to the NBER LTAM, the EFA, the SITE, the UCLA Fink Center, the Southern California Private Equity, the ILB Rising Talents in Finance and Insurance, the Luxembourg Asset Pricing and the Amsterdam Netspar conferences, as well as participants to the McGill, Toulouse School of Economics, HEC Paris and Tilburg seminars for comments. This research has been supported by a Netspar grant and a grant from the "Institut Europlace de Finance and Labex Louis Bachelier".

Conflict-of-interest disclosure statement

Elise Gourier

I have nothing to disclose.

Ludovic Phalippou

I have received compensation for several consulting projects, keynote talks, and in-company executive education courses. Financial organizations that paid me more than $10,000 over the last three years are: Blackrock (Blackrock Investment Institute), JP Morgan Chase (Equities Structuring), Institutional Investors, Fernweh group, and Natixis Investment Managers (Institutional Sales). I do not expect any of these organizations to be affected by the results in this paper. I have no other potential conflicts to disclose.

Mark Westerfield

I have nothing to disclose.

1 Introduction

Institutional investors' exposure to private market funds amounts to over $12 trillion.[1] These funds span a wide range of investments from real estate to leveraged buyouts, private debt and venture capital. A defining feature of private market funds, irrespective of their focus, is that they require investors to commit capital to fund managers before it is used ("called"), and thus to relinquish control over their portfolio allocation. Capital commitments are large, having trebled since 2008 to a total of $3.2 trillion, and the average delay between commitments and calls is significant at about three years. This paper quantifies the effect of these ex-ante capital commitments on portfolio allocation decisions and investors' welfare.

We solve the dynamic portfolio optimization problem of a risk-averse investor with an infinite horizon and access to stocks, bonds, and private equity funds. We model investments in Private Equity (PE) from the investor's perspective, taking as given the features and other key institutional details of PE contracts. At time 0, the investor commits a positive amount to a PE fund; they do not know when the capital will be called or when investment proceeds will be distributed. We use Poisson processes to model the stochastic timing of capital calls and distributions. The first jump triggers the capital call, which is when the investor transfers the committed amount to the fund manager, or defaults on their commitment. If the investor makes the transfer, the capital is invested by the fund manager. The second jump of the Poisson process marks the time at which the fund manager distributes the proceeds from the fund back to the investor. Then, a new capital commitment can be made to a new PE fund, and the process is repeated.

Our model incorporates both strategic default – the investor can skip a capital call at the cost of lost future opportunities – and access to a secondary market – the investor can sell the claim on their invested capital. We further extend the model along two dimensions. First, we allow for liquidity cycles in which private and public return moments co-vary with call and distribution intensities. Second, we let the investor access multiple PE funds to quantify the benefits of diversification across both cash flows and liquidity shocks. We study the limiting case of an economy with an infinity of PE funds.

We conduct a unique and thorough calibration of our model. We jointly estimate the set of parameters that best capture the empirically observed speed of capital calls and the cross-sectional distribution of fund performance as measured by the Kaplan-Schoar Public Market Equivalent (PME). We believe this is the first time a comprehensive structural model of PE

[1] As is customary, the $12 trillion figure represents the sum of the Net Asset Value of all existing funds and of all committed but uncalled capital ("dry powder"). This information is from the Preqin Pro website, as of November 2022.

investments is brought to the data, delivering a quantitative and empirically implementable portfolio choice approach to evaluate these investments. We show that our relatively stylized model can closely and simultaneously match the whole empirical distribution of both capital calls and fund performance. Our calibration of the one-fund model leads to an optimal PE commitment of 5.2% of wealth. With an infinity of PE funds, at the steady state, 21.9% of wealth is allocated to PE.[2]

Our model design allows us to define and decompose the liquidity frictions related to commitment. The delay between a capital commitment and call is stochastic, and we call the associated risk *commitment-timing* risk. Next, because of public market movements while waiting for the capital call, the amount of capital called *as a fraction of wealth* is stochastic;[3] we label the risk associated with this friction *commitment-quantity* risk. For both risks, we define the welfare cost as the one-off amount of wealth the investor would give up to remove the risk from the economy. Similarly, the return premium is the permanent PE return loss the investor would accept to remove the risk.

Our central result is that the cost of *commitment-quantity* risk is large. The investor is willing to pay 1.25% of their initial wealth (i.e., 24% of their optimal PE commitment), to switch to an economy without this liquidity friction. Similarly, they accept a permanent loss (i.e., return premium) of 1.10% of PE returns. This large cost of *commitment-quantity* risk is explained by the cost of being away from the target PE allocation during the holding period. If the PE allocation becomes too large relative to liquid wealth, consumption decreases as the investor cannot consume out of their PE stakes. The commitment period adds time during which the PE allocation may drift away from the optimum. To avoid the possibility of the PE allocation becoming too large, the investor under-commits to PE: on average, they would like to nearly double their PE allocation at the time of capital call.

Our results, therefore, offer a novel rationalization for the increased offering of co-investment opportunities to investors by fund managers at the time of capital call. Co-investment opportunities are valuable options to increase PE exposures post commitment, whereas the literature mostly presents them as tools to reduce fees.

In contrast to commitment-quantity risk, *commitment-timing* risk carries a cost close to zero. In fact, this cost can be negative for large values of the subjective discount factor.

[2] Aggregate asset allocation across endowments and foundations as of March 2020 according to the Bank of New York Mellon Corporation was 31.1% in listed equity, 16% in fixed income, 18.8% in hedge funds, and 16.9% in private equity. From https://www.pionline.com/interactive/larger-endowments-foundations-lean-private-equity-allocations

[3] During the time period between commitment and capital call, investors face stock volatility, causing the fraction of wealth committed to be sub-optimal at the time of the capital call. For example, following a decline in stock prices during the commitment period, the amount called is larger than the optimal amount.

Intuitively, the investor receives a discounted utility gain from the capital call, and exponential discounting is a convex function of time, making a deterministic time of capital call less valuable than a stochastic time.

We also find that, given the investor's under allocation to PE, it is almost never optimal for the investor to strategically default before or upon a capital call. Similarly, the secondary market, as a tool to liquidate PE positions, is relatively unimportant to the investor. As the investor cannot sell partial stakes in a fund,[4] they rarely find themselves with such excess holdings that they would like to sell. In the model with an infinite number of funds, partial sales are allowed, but the investor is better off stopping their commitments to new funds, thereby rapidly decreasing the PE allocation, rather than using the secondary market.

Yet, we find that secondary market liquidity and commitment risk are complements. The direct effect of making the secondary market liquid is a small welfare gain and an increase in PE allocation. The indirect effect is an increase of the welfare cost and return premiums associated with commitment-quantity risk. Thus, our model implies that the development of a PE secondary market increases the investor's willingness to alleviate commitment-quantity risk, rather than satiate that desire.

Our results remain unchanged with our calibrated liquidity cycles. In our calibration, the bad state features lower returns and higher volatility for both public and private equity, higher correlation between public and private equity, longer average commitment and holding periods, and a larger haircut on the secondary market. However, if private and public equity returns are no longer both low in the same liquidity state, any switch in the liquidity state increases the investor's desire to adjust their PE exposure, which exacerbates commitment-quantity risk. In this case, the return premium of commitment-quantity risk significantly increases in both states.

Increasing the number of funds generates moderate diversification benefits. The investor accepts a permanent PE return reduction of 0.86% to go from one PE fund to two PE funds. This is *less* than the premium associated with commitment-quantity risk. Furthermore, going from one fund to two funds hardly decreases the return premium of commitment-quantity risk, from 1.10% to 0.79%, and access to an infinity of funds brings it down to 0.74%. Although the investor is able to smooth both cash flow shocks and investment timing – PE cash flow risk and investment timing risk are idiosyncratic – two effects hinder diversification. First, commitment-quantity risk is driven by the denominator of the commitment-to-wealth ratio – commitments are constant, but the investor's liquid wealth is volatile – and the denominator is the same for all funds. Thus, even with an infinity of funds, the investor cannot

[4] We discuss how our assumptions on the private equity secondary market match practice in Section 2.5.

diversify commitment-quantity risk away. Second, investing in multiple funds creates the potential for a *funding mismatch*, or funding externality. Increasing investment across multiple funds means using distributions from earlier funds to meet later capital calls. However, if one fund distributes late and another calls early, the investor may be short of liquid assets. Thus, capital commitment remains relevant, even when the investor has access to multiple funds.

We build on a literature that studies the optimal portfolio choice problem in the presence of illiquid assets. Illiquidity is often defined as the inability to trade an asset during a given period of time, e.g., Longstaff (2001); Kahl et al. (2003); Longstaff (2009); Gârleanu (2009); Dai et al. (2015). Recent papers model the specific illiquidity features of private equity funds. In Sorensen et al. (2014), a (single) private equity fund is acquired at time 0, hence the capital is immediately invested, but this investment cannot be traded. The fund is liquidated at maturity T, which is finite and known ex ante. In Ang et al. (2014), an illiquid asset cannot be traded during stochastic periods of time. They illustrate how trading illiquidity can create funding illiquidity, and the resulting portfolio effects and welfare costs are found to be large. Dimmock et al. (2019) allow the agent to liquidate their positions in the illiquid asset on a secondary market at a cost and evaluate the "endowment model" used by some institutions that invest in alternative assets. Bollen and Sensoy (2021) extend the analysis of Sorensen et al. (2014) by allowing for a secondary market for partnership interests.

These papers focus on illiquidity by constraining an investor to hold an illiquid asset over a period of time, which is either deterministic or stochastic.[5] Capital committed to the illiquid asset is immediately invested. Thus, the central feature of private market funds, ex ante capital commitment, is not modelled.

In a contemporaneous paper, Giommetti and Sorensen (2020) model a private equity portfolio in which capital is gradually called and distributed from a composite private equity fund. Capital commitments are therefore implicitly embedded in their model. Their central finding is that the optimal allocation to private equity is not sensitive to risk aversion due to the nature of private equity funds' illiquidity. They also find that this result depends on the liquidity of the secondary market.

[5] Korteweg (2019) and Korteweg and Westerfield (2021) provide surveys of this literature.

2 Institutional Setup

2.1 Investment Vehicles

Private market investing spans the following investment strategies: Leveraged Buy-Out, Venture Capital, Growth Equity, Private Debt, and Real Assets (real estate, infrastructure, timber, natural resources). In this subsection, we detail the three broad routes that institutional investors have to invest in private markets.[6]

2.1.1 Blind Pool Funds

Most of the investments in private markets are made via finite-life closed-ended blind pools of capital, which are structured as private limited partnerships and simply referred to as funds. A company (e.g., KKR & Co. Inc.) acts as the General Partner (GP) for the fund (e.g., KKR XII), and capital is provided by the Limited Partners (LPs). LP interests in a fund cannot be traded; but they can be transferred to another investor with the consent of the GP. The commitment is 'blind' in the sense that LPs do not have a say about whether an investment should be made or not. A fund is a pool of ten to twenty investments.

During a fund-raising period that spans three to eighteen months, a GP seeks capital for its fund. LPs bear a significant due diligence cost to decide whether or not they commit capital (Da Rin and Phalippou (2017)); if they do, they agree to provide cash on demand to the fund, up to their committed amount during a pre-specified "investment period." When the GP ends its fundraising, it has its "final close", and the year this occurs is called the fund vintage year. The time between capital commitment and deployment ("capital calls") is quite long as the time frame spans both the fundraising period and the investment period. This investment approach is sometimes called Commitment-and-Drawdown.

GPs are highly specialized agents who devise a value-add plan for each investment. They are said to pursue a 'buy-to-sell' strategy, i.e. their main objective is to increase the asset value and sell as soon as this value-add can be cashed in. LPs have no say on the timing of asset sales, just as they have no say about the timing of capital calls.

2.1.2 Solo Investing

Solo investments – as coined by Fang et al. (2015) – are 'direct' ownership stakes taken by asset owners into companies. This route is common for assets that do not require a

[6] See also Korteweg and Westerfield (2021) and Phalippou (2021) for a survey of the institutional details associated with private market fund investor issues and the associated academic literature.

value-add plan (so-called 'core' assets); for example, New Hampshire's Great North Woods (Yale Endowment), London O2 arena (Trinity College Cambridge). However, Fang et al. (2015) also identify some solo investments among the LBO and VC investments of the large institutional investors in their dataset.

With solo investments, institutional investors target an amount of capital ex-ante, search for an opportunity, and eventually deploy the capital. The time between commitment and deployment varies but it may take a few months. These investments are typically intermediated by specialized agents (e.g., generalist real estate brokers such as Savills in the UK), do not require a value-add plan and are buy-and-hold investments (not buy-to-sell).

2.1.3 Discretionary vehicles

Discretionary investments are buy-to-sell investments that are proposed separately by intermediaries to prospective asset owners. The intermediaries devise and implement a value-add plan, and the prospective owner does costly due diligence each time before deciding to opt in or to pass. There are two main sub-categories of discretionary vehicles:[7]

Pledge funds: Fund participants pledge to contribute capital to a series of investments, but have the right to opt out of specific investments. These structures are observed more in certain regions (e.g. in Asia), for certain types of assets (e.g. real estate, venture capital), and with less established fund managers.

Fund co-investments: Fund co-investment opportunities allow LPs to add capital to a deal when the GP makes a capital call. They are restricted to LPs that are already in the fund; this effectively gives LPs the opportunity to take a greater stake in some of the fund's investments. Pre-2008, co-investment invitations were limited to large LPs; but since then, they are widespread.

2.2 Why are blind pool fund structures dominant?

The vast majority of private market investments are made via blind pool funds. In dollar terms, Lerner et al. (2021) report that private market investments are split 93% in blind

[7]The label 'discretionary vehicles' was coined by Lerner et al. (2021); they define it as follows: "co-investments into individual companies by one or more LPs; solo investments by LPs in previously private capital-financed companies; pledge fund structures where transactions are funded by the LP on a deal-by-deal basis (sometimes raised by groups that have encountered poor performance and who encountered difficulties raising a traditional fund); co-investment or overage funds that are raised alongside a main fund; and co-sponsored transactions between LPs and GPs. We also include co-investment funds raised by funds-of-funds and other intermediaries." Note that some funds allow investors to add capital regularly over time; they could also be considered discretionary vehicles (e.g. Tiger Global).

pool funds and 7% in discretionary vehicles; but they exclude real assets, private debt, and solo investments. Solo investments are at best as large as discretionary investments.[8] Blind pools are intermediated and pool capital commitments.

Intermediaries are ubiquitous in financial markets and their existence has been justified by two features. First is transaction cost minimization: pooling capital across multiple agents reduces the per unit cost due to the presence of fixed costs. Second is the information advantages of specialized investors, which is probably significant in private markets. These benefits are counter-balanced by agency frictions.[9] This tradeoff is consistent with the finding that most direct investments are core real asset investments, which are seen as the least complex investments, whereas nearly all LBO investments, which are generally perceived as more complex, are done via a specialized intermediary.

Axelson et al. (2009) argues that ex-ante capital commitments can be a second best optimal contracting solution. In their model, fund managers have skills in identifying and managing potentially profitable investments, but as they have limited liability, they have an incentive to overstate the quality of potential investments when they raise financing. This agency problem is minimized when capital is committed ex ante to finance a number of future projects rather than when capital is raised on a deal-by-deal basis. This model, therefore, provides a rationale for why investors would accept a fund structure despite the cost associated to it.

In addition, in practice, deal-by-deal structures suffer from severe shortcomings: "the fund manager will not have existing contractual commitments from investors, that can be called down at very short notice, and this can affect the ability of the manager to commit to underlying transactions in a timely manner. Clearly, entering into a binding underlying purchase contract cannot be finalized until the necessary capital has been raised, as this would give rise to a risk of a breach of contract if the funding cannot ultimately be obtained. If the fund manager is competing against another potential purchaser for an asset, and such other purchaser already has guaranteed funding in place, the vendor may prefer to deal with such other purchaser."[10] As Fang et al. (2015) conclude: "In sum, the different approaches to private equity investing — the traditional intermediated partnership vs. direct investing — present a tradeoff between cost and investment quality."

[8] In the dataset of Fang et al. (2015), co-investments alone are larger than solo investments, but real assets and private debt are excluded.

[9] GPs deploy capital too quickly at market peaks (Axelson et al., 2013), exit investments prematurely (Barrot, 2017), invest sub-optimally near the end of investment period (Arcot et al. (2015), Degeorge et al. (2016); and LPs need to provide GPs with "liquidity insurance" in bad times (Lerner and Schoar (2004)).

[10] Source: `https://www.harneys.com/hubs/offshore-funds/the-art-of-the-deal-by-deal/`.

2.3 Capital Calls and Distributions

We now discuss some of the specific institutional features of intermediated blind pools. A capital call is made by the GP on the LPs and is in connection to either a fee payment or an investment. The timing of capital calls is uncertain; LPs only know an ex-ante specified investment period, during which most capital calls should occur. The length of the investment period depends on the investment type. Leveraged buyout funds typically have a five year investment period, with some capital called afterwards for fee payments or follow-on investments in existing portfolio companies. For venture capital funds, the investment period is typically longer to allow for large sums to be invested in later stage rounds for successful portfolio companies. To reduce the frequency of capital calls, GPs often pool some of them and bridge-finance using credit facilities with LP commitments as collateral.

The capital distribution period is flexible, spanning the entire life of the fund, including an overlap with the investment period. When an investment is exited, the payout is distributed to LPs and cannot be recycled to make a new investment, but there are some exceptions.

Funds' life is set to ten years but there are multiple circumstances under which funds obtain extensions. Most funds are not fully liquidated in their twelfth year and beyond, which shows that there is no hard deadline in practice.[11] There is a wide dispersion in the number of contemporary fund commitments held by institutional investors. At the high end, CalPERS, which is one of the most active PE investors, reports a total of 311 commitments to PE over the last 23 years, i.e., about 65 simultaneously active commitments. At the lower end, many small Endowments and Family Offices only have one or two active commitments. Using Preqin data, we find that the (log) of the number of private equity funds increases on a one-to-one basis with the (log) allocation to private equity (million of USD); non-tabulated, see also Cavagnaro et al. (2019). Thus, institutions differ greatly in their degree of diversification within PE, with many institutions facing lumpy stochastic capital calls.

2.4 Defaulting on Commitments

The stated penalties for default, specified in limited partnership agreements, are high (Banal-Estañol et al. (2017)). Penalties include forfeiture of some or all existing investments in the fund, and impossibility to invest in subsequent funds. Perhaps as a result, default is rare. We do not know of any major PE investor that has defaulted on their PE commitment. There is also anecdotal evidence that LPs are willing to take significant and costly actions

[11] Barrot (2017) finds that the type of investments is nonetheless influenced by the fund age. Earlier in the fund's life, investments tend to be in younger companies at an earlier stage of their development.

to avoid default. These costly actions include, for example, redeeming capital from other investments despite low overall liquidity, selling their fund stakes on the secondary market at large discounts, and issuing high yield bonds.

To illustrate the extent and cost of the default avoidance strategies, here is a typical account of how PE investors fared during the 2008 crisis: "A growing set of limited partners find themselves short on cash amid the financial crisis – and thus are scrambling for ways to make good on undrawn obligations to private equity vehicles. Among those in the same boat: Duke University Management, Stanford Management, University of Chicago and University of Virginia... Brown, whose $2.3 billion endowment has a 15% allocation for private equity products, is apparently thinking about redeeming capital from hedge funds to raise the money it needs to meet upcoming capital calls from private equity firms... Carnegie, a $3.1 billion charitable foundation, is also in a squeeze. Its managers have been calling on commitments faster than expected, while distributions from older funds have slowed down, creating a cash shortfall. As for Duke, the university's endowment has been named as one of the players most likely to default on private equity fund commitments. That partly explains a massive secondary-market offering that the school floated last month, as it sought to raise much-needed cash and get off the hook for undrawn obligations by unloading most of its $2 billion of holdings in the sector... Some of the bigger investors are considering tapping credit facilities to meet near-term capital calls."[12]

2.5 The Secondary Market

Before 2006, the secondary market for fund stakes was quasi non-existant due to contractual restrictions on transfers (see Lerner and Schoar (2004)). This market then grew quickly from an annual turnover of $10 billion in 2006 to over $100 billion in 2021. Yet, $100 billion volume still represents less than 1% of the $12 trillion allocated to private market funds.[13]

Nadauld et al. (2019) report an average discount to the reported Net Asset Value (NAV) of 13.8% (9% since 2010). In addition, and importantly, Nadauld et al. (2019) show that there are few transactions of funds during their investment period. There are very few, if any, transactions before the fund has started to invest; i.e., it is rare to observe the sale of a pure capital commitment.

[12]From the magazine 'Private Equity Insider' in its issue of November 5, 2008. See also Barron's, 6/29/2009,"The Big Squeeze"; Forbes, 10/24/2009, "Did Harvard Sell At the Bottom?"; Institutional Investor, 11/4/2009, "Lessons Learned: Colleges Lose Billions in Endowments."

[13]For turnover data, see https://www.jefferies.com/CMSFiles/Jefferies.com/Files/IBBlast/Jefferies-Global-Secondary-Market-Review.pdf

Sellers are LPs who transfer their entire stake in a fund: partial sales are rare. Buyers are specialized intermediaries managing dedicated vehicles that are structured as blind pool funds. These buyers raise equity from asset owners and borrow capital to buy fund stakes on the secondary market. LPs usually buy stakes on the secondary market through these intermediaries rather than directly, with additional charges and delays. Hence, secondary markets mostly allow for downward adjustments in private market allocations. For upward adjustments, investors need to use discretionary or solo investments.

3 Model

We model investment portfolios that combine private equity with liquid risky and riskless assets. Our setup is designed to capture important institutional details from Section 2, and it allows for one, two, or an infinity of private equity funds.

3.1 The Liquid Assets

There are two liquid assets in the economy that can be rebalanced continuously at no cost: a risk-free bond, which captures the fixed-income market, and a risky stock, which captures the public equity market.[14]

The price B_t of the bond appreciates at constant rate r,

$$dB_t = rB_t dt, \tag{1}$$

and the stock price P_t follows a geometric Brownian motion,

$$\frac{dP_t}{P_t} = \mu\, dt + \sigma dZ_t^L, \tag{2}$$

where Z_t^L is a standard Brownian motion associated with liquid public markets, μ is the return drift, and σ is the return volatility.

The investor's liquid wealth W_t is the sum of their holdings in the stock and bond.

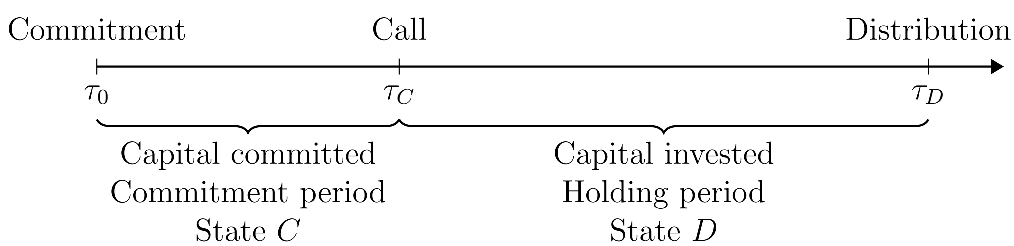
[14] We consider an information structure that obeys standard assumptions. There exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting the vector of four independent Brownian motions $Z_t = (Z_t^L, Z_t^{PE}, Z_t^{1\perp}, Z_t^{2\perp})$, and two independent Poisson processes $M_t = (M_t^1, M_t^2)$. Some stochastic processes may be unused, depending on the number of private equity funds. $\mathcal{P}$ is the corresponding measure and $\mathcal{F}$ is a right-continuous increasing filtration generated by $Z \times M$. Following Dybvig and Huang (1988) and Cox and Huang (1989), we restrict the set of admissible strategies to those that satisfy the standard integrability conditions. All policies are appropriately adapted to $\mathcal{F}_t$.

3.2 Modelling Private Equity

Investors can allocate capital to private equity funds. We begin by describing the entire investor's problem for one fund and then extend the model to include multiple funds.

As illustrated in Figure 1, the fund manager collects capital commitments from the investor at time τ_0, calls the committed capital and invests it at time τ_C, and then distributes the value of the investment at time τ_D. We call the *commitment period* the time period $[\tau_0, \tau_C)$ (or the fund is in state C), and the *holding period* the time period $[\tau_C, \tau_D)$ (the fund is in state D). After the fund distribution, the investor makes their next commitment and the process repeats to infinity.

Figure 1: Timeline of a fund's life.



At time τ_0, the investor commits a positive amount $X_{\tau_0} \geq 0$ to the private equity fund. This commitment is a promise to make capital available when the manager calls it at time τ_C. The commitment X_{τ_0} cannot be changed after time τ_0, meaning $dX_t = 0$ until the committed capital is called and invested at τ_C. During the commitment period $[\tau_0, \tau_C)$, fees are paid out of the investor's liquid wealth to the fund manager at rate $fX_{\tau_0}dt$, but no investment is made.

We use a Poisson process to model the timing of capital transfers between the investor and the fund manager. The process has intensity λ_C during the commitment period. A jump triggers the capital call and the end of the commitment period, at which time the investor transfers X_{τ_0} of liquid wealth to the fund manager.

In a slight abuse of notation, we denote by X_t the amount of capital committed between times τ_0 and τ_C, and we use X_t again to refer to the net-of-fee amount of capital invested in the fund after τ_C.

After capital is transferred and invested, the value of the private equity asset, net of all fees, evolves as a geometric Brownian Motion, with drift ν, and volatility ψ:[15]

$$\frac{dX_t}{X_t} = \nu dt + \psi dZ_t^X, \tag{3}$$

where $dZ_t^X = \rho_L dZ_t^L + \sqrt{1-\rho_L^2} dZ_t^{1\perp}$ and $Z^{1\perp}$ is the idiosyncratic shock associated with the fund. This specification implies that the correlation between public and private equity is ρ_L, and the beta of a private equity fund is

$$\beta = \rho_L \frac{\psi}{\sigma}. \tag{4}$$

We again use the Poisson process to model the timing of capital distributions. During the holding period $[\tau_C, \tau_D)$, the intensity of the Poisson process is λ_D, and a jump triggers capital distribution. The private equity investment is fully exited, and the investor receives the value of the fund, $X_{\tau_D -}$. The Poisson process resets, and the investor is immediately able to make a new capital commitment to a new private equity fund.[16]

In our setup, the uncertainty around capital calls and distributions is modelled with two random times for the private equity fund, τ_C and τ_D. These two random times represent two sources of market incompleteness. Even if the liquid asset and the private equity fund had fully correlated returns, the investor would not be able to hedge the risk coming from the random times and the market would still be incomplete.

Our model with one fund relies on two assumptions that ensure analytical tractability. First, there is a single capital call equal to the committed amount, as opposed to having capital calls spread across the investment period. Second, there is a single payout.

Below, we generalize this basic model in two important directions. First, we allow for multiple funds, including an infinite-fund limit. Second, we allow for private equity cycles in which parameters, including returns and waiting times, are allowed to vary over time. In all cases, our representation allows us to use numerical methods based on Markov chain approximations to solve the ODEs and PDEs associated with the portfolio allocation problem.

Our model setup is flexible enough to allow for the existence of a secondary market. We assume that during the holding period, the investor can sell their invested private equity on a secondary market, receiving αX_t. $0 \leq \alpha \leq 1$, and $1-\alpha$ is the haircut. After the sale, the

[15]To keep the model parsimonious, during the holding period we do not model the management fee and carried interest separately from returns. Instead, we assume that the net-of-fees value X_t follows a geometric Brownian motion.

[16]Without a pledge to private equity, the investor's opportunities are constant, so it is never optimal for the investor to wait to commit.

investor waits until the end of the fund's life τ_D, then makes a new commitment, starting the process over.

The investor can strategically default on their capital commitment either at the time of the call or any time before. The consequences of default are that the investor does not turn over the capital and stops paying the associated fee, but the investor is banned from accessing private equity in the future – which is a realistic feature (see Section 2.4).

3.3 The Investor's Problem

The investor continuously rebalances their liquid wealth between the two liquid assets, and consumes out of liquid wealth at rate $c_t = C_t/W_t$. We denote with θ_t the fraction of liquid wealth allocated to stocks, so the evolution of the investor's liquid wealth is given by:

$$\frac{dW_t}{W_t} = (r + (\mu - r)\,\theta_t - c_t)\,dt - \mathbb{1}_{S=C} f \frac{X_{\tau_0}}{W_t} dt + \theta_t \sigma dZ_t^L - \frac{dI_t}{W_t} \tag{5}$$

where dI_t denotes any transfer between liquid wealth and illiquid wealth. Throughout the paper we will use $\mathbb{1}$ as an indicator variable. Thus $\mathbb{1}_{S=C} f \frac{X_{\tau_0}}{W_t}$ denotes fees that are paid during the commitment period.

The value function is given by

$$F(W_t, X_t, S_t) = \max_{\{\theta, X, c\}} \mathrm{E}_t \left[\int_t^{\infty} e^{-\delta(u-t)} U(C_u) du \right], \tag{6}$$

subject to (3) and (5). We use δ to denote the subjective discount factor and $S_t = \{C, D\}$ to denote the state. The investor has a standard power utility, i.e., $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$, with $\gamma > 1$. Given our assumptions, $W > 0$ and $X \geq 0$ a.s. We extend our results to Epstein-Zin utility in Online Appendix C.

At $t = \tau_C$, and if the investor has not defaulted on their commitment, we have $dI_t = dI_{\tau_C} = X_{\tau_0}$; i.e. committed capital is called, and X_{τ_0} is transferred out of liquid wealth to private equity. The state changes from $S = C$ (commitment period) to $S = D$ (holding period), and the value function jumps discretely from $F(W, X, S = C)$ to $F(W - X, X, S = D)$.

The investor strategically defaults if the welfare value of the standard Merton problem ($F^{Merton}(W)$), which is our model with access to the liquid stock and bond but without the private equity sector, exceeds the continuation value with PE. The investor defaults before a capital call if $F(W, X, S = C) < F^{Merton}(W)$ and upon a capital call (at τ_C) if

$F(W-X, X, S=D) < F^{Merton}(W)$.

During the holding period, the investor can sell their stakes on the secondary market, and they do so if $F(W, X, S=D) < F(W+\alpha X, 0, S=D)$.

When $t = \tau_D$ is reached, we have $dI_{\tau_D} = -X_{\tau_D -}$; i.e. capital is paid out and X_{τ_D} is transferred from private equity to liquid wealth. Then, the investor chooses their level of committed capital to the new fund, the state changes from D to C. The value function jumps discretely from $F(W, X, S=D)$ to $\max_{X'} F(W+X, X', S=C)$. At all other times, $dI_t = 0$.

The investor's value function, optimal consumption and allocation solve the Hamilton-Jacobi-Bellman (HJB) equation given in Appendix A.1. Because the utility function is homothetic and the return processes have constant moments, it must be the case that the value function F is homogenous of degree $1-\gamma$ in total wealth. We use V to denote total wealth and ξ is the fraction of total wealth either committed or invested, so that

$$V = W + X\mathbb{1}_{S=D} \tag{7}$$

$$\xi = \frac{X}{V}. \tag{8}$$

Since $W > 0$, we require $\xi < 1$.

Thus, the investor's value function can be written as the product of a power function of total wealth and a function of the wealth composition:

$$F(W, X, S) = V^{1-\gamma} H(\xi, S). \tag{9}$$

The optimal commitment is given by the following Proposition:

Proposition 1 *The investor's value function can be written as in (9), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(\xi, S=C)$, which exists.*

The function H is characterized by the set of ODEs shown in Online Appendix A.1, and our method for generating numerical results is detailed in Online Appendix B.

3.4 The Illiquidity Frictions

The model just presented (Sections 3.2 and 3.3) is our **baseline economy (Economy 0)**, i.e. an economy where all liquidity frictions are present. We now define five other economies, each of which corresponds to a situation where one or more of the frictions of our baseline model are modified. These modifications allow us to assess theoretical counter-factuals and isolate the impact of the various private equity investment frictions. The ODEs and PDEs that characterize the solutions to the investor's problem in these five economies are given in the Online Appendices A.2 to A.6.

Economy 1: Deterministic call time, choose quantity when committing.

The first friction is that the time of capital call is unknown. We turn off this *commitment-timing risk* by making the call time deterministic. The agent commits capital at τ_0, but instead of waiting for a random delay of expected length $\frac{1}{\lambda_C}$, capital is called deterministically at $\tau_C = \tau_0 + \frac{1}{\lambda_C}$. Thus we maintain the average delay but remove the uncertainty about this delay.

Economy 2: Stochastic call time, choose quantity when called

Next, we turn off *commitment-quantity risk*. In the baseline economy, committed capital is fixed, but liquid wealth evolves randomly before the committed capital is called. Thus, *the relative size of the commitment changes*, and investors do not know the fraction of their wealth that they will be required to turn over to the PE fund. To turn this commitment-quantity risk off, we let the investor choose the quantity invested in private equity when the capital is called, instead of at commitment time. So, the commitment-to-wealth ratio ξ^* is chosen at call time τ_C rather than commitment time τ_0. This change also removes any risk of default, but the timing of the capital call remains stochastic.

Economy 3: Deterministic call time, choose quantity when called

If neither commitment-timing nor commitment-quantity risk is present, there is a commitment delay but *no commitment risk*.

Economy 4: Immediate private equity access

Absent commitment risk, our model still features a *commitment delay*. The investor needs to wait until call time to access the private equity returns: their active investment time is the holding period of the fund. This restriction can be lifted and the commitment period time brought to zero. Hence, at time τ_D, the investor freely allocates capital between bonds, stocks and private equity and directly enters the fund's holding period.

Economy 5: Deterministic payout time

Economies 1-4 constitute a peeling back of the institutional details associated with commitment risk. We remove the commitment-timing and quantity risks separately and then together, and then we remove the commitment delay as well. For comparison, we also consider making the stochastic distribution time deterministic. In this economy, the investor's capital is called at random time τ_C, but the holding period duration is deterministic, with $\tau_D = \tau_C + \frac{1}{\lambda_D}$. Thus we maintain the average holding period but remove the risk. This economy still contains commitment risk.

3.5 Economies with several PE funds

We can extend the baseline model presented in Sections 3.2 and 3.3 to include more than one PE fund. In this sub-section, we describe the economy with two or an infinity of PE funds. Analyzing economies with multiple PE funds is important for at least four reasons.

First, with multiple PE funds, LPs can diversify cash flow risk. Each fund has idiosyncratic risk, and a standard diversification intuition indicates than an investor would prefer to spread their wealth across multiple assets. Second, investors reduce the lumpiness of capital calls and distributions. However, any shock to public equity is a shock to all commitment-to-wealth ratios through the denominator. So while diversifying across funds allows for multiple smaller commitments and smoother call timing, the total outstanding commitment-to-wealth ratio remains volatile. Third, investors can potentially use distributions from earlier investments to fund later investments. Doing so smoothes out their quantity invested and increases the active investment time – the time during which capital is invested and earning returns in PE. However, there can also be a funding mismatch: the risk that earlier distributions will be late or insufficient to fund capital calls. Fourth, in contrast to the one-fund case, investors can partially sell their PE holdings on the secondary market.

3.5.1 Liquidity Diversification: Investing in Two Funds

As in the one-fund case, the investor chooses an optimal commitment X^i to fund i after this fund distributes its previous round of capital. If the investor defaults on one fund before or upon capital call, we assume that they also default on the other fund, if that other fund is in its commitment period. If the second fund has been called already, then the investor sells it on the secondary market. In both cases, they lose access to PE and their investment opportunity set reduces to the liquid stock and bond, as in the one-fund model.

During each fund's holding period, returns follow (3), with the same expected return ν and volatility ψ. We assume that the Brownian motions that drive fund returns have correlation ρ_L with public market equities and correlation $\rho_{PE} > \rho_L$ with each other. Thus the returns of each fund i during its holding period are given by

$$\frac{dX_t^i}{X_t^i} = \nu dt + \psi dZ_t^i, \tag{10}$$

where $dZ_t^i = \rho_L dZ_t^L + \sqrt{\rho_{PE}^2 - \rho_L^2} dZ_t^{PE} + \sqrt{1 - \rho_{PE}^2} dZ_t^{i\perp}$, Z^L is the public market shock, Z^{PE} is a common PE shock, and $Z^{i\perp}$ is the idiosyncratic shock associated with fund i.

Even if the two funds are synchronized at some point in time, they will rapidly desynchronize because of the stochastic call and distribution timing. One should thus think of the steady state in this economy as one in which calls and distributions randomly overlap one another. The solution to this problem is given in Online Appendix A.7.

3.5.2 Limiting Case: Investing in an Infinity of Funds

Appealing to the law of large numbers, we assume that when there are an infinity of PE funds, PE funds make calls and distributions continuously and the investor makes commitments continuously as well. Since individual funds have commitment periods that are exponentially distributed with parameter λ_C, a fraction $\lambda_C dt$ of funds call capital over the interval dt. Similarly, a fraction $\lambda_D dt$ of funds make distributions over dt.

These assumptions imply that commitment-timing risk does not exist with an infinity of funds. However, commitment-quantity risk remains: the investor's commitments are called over time, and liquid wealth is fluctuating randomly.

Our two state variables are the aggregate capital committed to all PE funds (X_t^∞), and the aggregate invested amount (Y_t^∞). We label the investor's new commitments as $dJ_t \geq 0$. Then, extending (5) and (10), we have the following dynamics:

$$\frac{dX_t^\infty}{X_t^\infty} = \frac{dJ_t}{X_t^\infty} - \lambda_C dt \tag{11}$$

$$\frac{dY_t^\infty}{Y_t^\infty} = \lambda_C \frac{X_t^\infty}{Y_t^\infty} dt - \lambda_D dt + \nu dt + \psi^\infty dZ_t^\infty \tag{12}$$

$$\frac{dW_t}{W_t} = (r + (\mu - r)\,\theta_t - c_t)\,dt + \theta_t \sigma dZ_t^L - f\frac{X_t^\infty}{W_t} dt - \lambda_C \frac{X_t^\infty}{W_t} dt + \lambda_D \frac{Y_t^\infty}{W_t} dt. \tag{13}$$

where $dZ_t^\infty = \rho_L^\infty dZ_t^L + \sqrt{1 - \rho_L^{\infty 2}} dZ_t^{PE}$.

The parameters driving equation (12) are those of an equally weighted portfolio of PE funds, taking the limit as the number of funds in the portfolio goes to infinity. As with the two-fund case, we assume that the shocks of each PE fund have correlation ρ_L with public markets and ρ_{PE} with each other. Then we can calculate the volatility ψ^∞ and correlation with the stock market ρ_L^∞ of an equally weighted portfolio analytically:

$$\psi^\infty = \psi\sqrt{\rho_{PE}} \quad ; \quad \rho_L^\infty = \frac{\rho_L}{\sqrt{\rho_{PE}}}. \tag{14}$$

The investor maximizes their expected discounted utility as in (6), subject to the budget constraints (11)-(13) and the constraint that J_t is non-decreasing.

Next, we define the ratio of committed wealth to total wealth, and the ratio of invested illiquid wealth to total wealth to be

$$\pi_t \equiv \frac{X_t^\infty}{Y_t^\infty + W_t} \quad , \quad \xi_t \equiv \frac{Y_t^\infty}{Y_t^\infty + W_t} \tag{15}$$

As in the case of a finite number of funds, the investor's value function can be decomposed into the effect of total wealth and the effect of wealth composition on the continuation utility:

$$F^\infty(W, X^\infty, Y^\infty) = (W + Y^\infty)^{1-\gamma} H^\infty(\pi, \xi), \tag{16}$$

where the function $H^\infty(\pi, \xi)$ satisfies PDEs given in Online Appendix A.8. In contrast with the one-fund case, there is an optimal PE commitment for each level of PE investment.

Furthermore, the investor can default at any time, in which case they sell their aggregate investment on the secondary market and lose access to PE. They can also sell any fraction ω of their invested capital at any time on the secondary market, which they will do if $F^\infty(W, X^\infty, Y^\infty) < F^\infty(W + \alpha\omega Y^\infty, X^\infty, Y^\infty(1-\omega))$. This corresponds to selling complete positions in some subset of the infinity of funds.

We conduct our welfare analysis at the steady state of the aggregate investment Y_t^∞, i.e., when $E[dY_t^\infty] = 0$.[17] We denote this steady state investment by Y^∞. Using the laws of motion, the steady state aggregate commitment X^∞ is linked to the investment Y^∞ by:

$$\frac{X^\infty}{Y^\infty} = \frac{\lambda_D - \nu}{\lambda_C}, \tag{17}$$

Thus, the steady state ratio of committed to allocated capital is a simple function of the rates of calls and distributions and the mean PE return.

[17]One might also be interested in the steady state for the Y^∞/W ratio: $E[d\ln(Y_t^\infty/W_t)] = 0$. This change implies using a different location in $\{X, Y, W\}$ space to do the welfare analysis. The overall allocation to PE is slightly higher, with more capital committed and less invested, and the welfare and return premiums are almost the same.

Because commitment-timing risk is eliminated with an infinity of funds, our comparison of illiquidity frictions (from Section 3.4) collapses to the comparison between the baseline economy and Economy 2 in which the investor can choose their commitment at the time of a capital call. With an infinity of funds, Economy 2 allows an investor to immediately add to their PE assets (Y_t^∞) to reach the optimal level of invested capital. Equation (12) becomes:

$$\frac{dY_t^\infty}{Y_t^\infty} = \frac{dJ_t}{Y_t^\infty} - \lambda_D dt + \nu dt + \psi^\infty dZ_t^\infty \tag{18}$$

Because negative calls are not allowed, an investor above their optimal allocation must wait for distributions for their invested capital to decline, or they must sell a fraction of their invested capital on the secondary market at a discount.

3.6 Measuring the Cost of Illiquidity

We define two measures to quantify the costs of the different liquidity frictions described in Section 3.4. First, the **welfare cost to the investor** of any economy A with respect to any economy B, denoted $\zeta^{A,B}$, is the fraction of wealth the investor would be willing to pay at the time of commitment to switch from economy A to economy B while simultaneously adjusting their capital commitment. If $\zeta^{A,B} = 0$, then the investor is indifferent between the two economies. $\zeta^{A,B}$ is the solution to the following equation:

$$H^A(\xi^{A*}, S) = (1 - \zeta^{A,B})^{1-\gamma} H^B(\xi^{B*}, S), \tag{19}$$

where ξ^{A*} and ξ^{B*} denote, respectively, the optimal commitments in economies A and B. We evaluate welfare at the time the agent chooses the allocation, i.e., at time τ_0 for Economies 0, 1 and 5 and τ_C for Economies 2, 3 and 4.

Second, the **return premium** of any economy A with respect to economy B is the additional return of the PE funds that would be needed in economy A to make the investor indifferent between the two economies. The return premium applies to all PE funds, both current and future. If the investor is indifferent between economy B with PE expected returns ν and economy A with expected returns $\nu + \epsilon_{AB}$, then the return premium is ϵ_{AB}.

In the two-fund model, welfare costs and return premiums are computed from the value function evaluated at $\xi^1 = \xi^{1*}$ and $\xi^2 = \xi^{2*}$, i.e. assuming that commitments to both funds are optimal.

The two welfare measures should be interpreted differently. The welfare cost is a one-time payment to switch economies, so it is strongly increasing in the optimal PE allocation. In contrast, the return premium impacts the investor in proportion to the amount allocated to PE, so it is much closer to a *per unit* cost of illiquidity.

4 Model Calibration & Portfolio Allocation

In this section we provide a detailed calibration of private equity return dynamics. We calibrate for economies with one-, two-, and an infinite- number of PE funds and provide the resulting portfolio and consumption policies in the baseline economy.

4.1 Model Calibration

We use the past thirty years of data to calibrate our model (1991-2020). The average 3-month Treasury Bill is $r = 0.03$. The mean and volatility of the S&P 500 index log returns at monthly frequency are $\mu = 0.08$, and $\sigma = 0.15$. We use standard values for the investor risk aversion and discount factor: $\gamma = 4$ and $\delta = 0.05$. The discount for PE-funds secondary market sales is set to the average reported in Nadauld et al. (2019): 13.8%. Management fees during the commitment period are set to $f = 2\%$ of the committed amount (see Metrick and Yasuda (2010)). Our calibration of the private equity return dynamics uses the Preqin dataset with fund cash flows as of the end of year 2020. We select all US-focused private equity funds (venture capital, growth equity, leveraged buyout) raised between 1991 and 2015 (so that they have at least five years of investment activity).[18]

We construct two cumulative distribution functions for fund cash inflows: the empirical distribution and the model-implied distribution. The former is derived directly from the Preqin dataset. The latter is calculated analytically and verified with simulations.

In our model, PE-fund cash *inflows* consist of the management fees during the commitment period plus the investment capital call at time τ_C. Assume that \$1 is committed to each one of the N funds. The delay from τ_0 to τ_C has an exponential distribution. Therefore, at any time t, the proportion of funds across simulations that have not called yet is $e^{-\lambda_C t}$. The total fees paid by these funds is $Nfe^{-\lambda_C t}$. Imposing the law of large numbers, the cumulative cash inflow across is the the sum of the cumulative fees paid by each fund until its capital call, $\int_{\tau_0}^{t}(Nf)(e^{-\lambda_C t})dt = \frac{Nf}{\lambda_C}\left(1 - e^{-\lambda_C t}\right)$, plus the amount of capital already

[18]We select funds with a size of at least \$10 million, at least two capital calls, and at least two capital distributions. The resulting sample contains 1398 funds. Note that Preqin records all cash inflows into any given private equity fund, without a distinction between fee payments and capital invested.

called, $(1-e^{-\lambda_C t})N$. As $t \to \infty$, i.e. after all funds have exited their investments, the total cash inflows approach $N\left(\frac{f}{\lambda_C}+1\right)$. Thus, the cumulative cash inflow at time t, as a fraction of the total, is

$$\frac{\frac{Nf}{\lambda_C}\left(1-e^{-\lambda_C t}\right)+(1-e^{-\lambda_C t})N}{N\left(\frac{f}{\lambda_C}+1\right)} = 1-e^{-\lambda_C t},$$

which implies an exponential distribution with parameter λ_C.

We search for the λ_C that minimizes the least-square distance between the model-implied and empirical cumulative distributions. The best fit is obtained for $\lambda_C = 0.344$, which corresponds to an average commitment period of about three years. Panel A of Figure 1 shows that the two cumulative distribution functions are very close to one another, with an RMSE of 1.9×10^{-2}, which validates our modeling choice.

To calibrate cash *outflows*, we do not directly use the fund distributions observed in Preqin, as we do for the inflows. The reason is that it takes about fifteen years to observe the complete time series of fund distributions, and we would then be restricted to using a sample of funds raised before 2005. Instead, we use the same sample as above – funds raised up to 2015 – and match the distribution of their performance as of end of year 2020. To measure fund performance, we adopt the most common measure: the Kaplan-Schoar Public Market Equivalent (PME). That is, for each fund, we compute the present value of cash inflows and cash outflows, each discounted using the realized S&P 500 index returns, and we value unexited investments at their reported Net Asset Value (NAV).[19] As we are interested in the distribution of fund-level PMEs, we assign equal weight to all funds.

The model-implied PMEs are obtained by simulating the cash flows of 100,000 private equity funds using i) the return dynamics of equation (3), and ii) draws from Poisson distributions to trigger capital calls and distributions. There are four free parameters in our model: PE expected return (ν), PE volatility (ψ), the intensity of capital distributions (λ_D), and the correlation between private and public equity (ρ_L).

We choose the parameters $(\nu, \psi, \lambda_D, \rho_L)$ which minimize the least-square distance between

[19] NAVs result from subjective judgments about the appropriate valuation technique and input parameters for each portfolio company. Despite accounting rules and although NAVs have no direct impact on investors' wealth, fund managers may purposely smooth NAVs with the aim of facilitating investor relationship management (e.g. avoid negative return news), facilitating fund-raising (Barber and Yasuda (2017); Brown et al. (2019)), or because they believe that public market returns are excessively volatile. Crain and Law (2016) provide evidence that NAVs are quite accurate overall, although sluggish. Nadauld et al. (2019) show that some secondary market transactions are executed at prices that differ significantly from NAV. We have also calibrated the model using only funds with vintage year prior to 2012, i.e., using only funds that are past the normal liquidation age. All our results still hold.

the model-implied and the empirical cumulative distributions. The empirical and model-based cumulative distributions of PMEs are shown in Panel B of Figure 1. The two curves are remarkably close, with an RMSE of 3×10^{-2} for PME $\in [0.5, 2.5]$, and the best fit is obtained with a combination of a relatively high ν and ψ:

$$\nu = 14\%; \quad \psi = 33.5\%; \quad \lambda_D = 0.174; \quad \rho_L = 0.66$$

Our calibrated parameters produce an implied private equity β of 1.47:

$$\beta = \rho_L \frac{\psi}{\sigma} = 0.66 \left(\frac{0.335}{0.15} \right) = 1.47.$$

This β is nearly the same as the 1.43 estimate obtained by Ang et al. (2018), which uses a completely different methodology: Bayesian methods to extract the risk exposures that are most consistent with the observed panels of cash flows. Our calibration implies that the CAPM-alpha of PE is 3.6%, which is close to the beliefs of asset owners, who usually have a benchmark return for PE of 3 to 5 percent above that of public equity.

Our calibrated λ_D implies an average holding period of about 5.7 years, which is close to the median holding period of 5.3 years that Brown et al. (2020) report.[20]

In the two-fund case, we use the set of parameters described above for the dynamics of each fund, but we also need to calibrate the correlation between the two PE funds, ρ_{PE}. We draw 5000 portfolios of two funds each, taken randomly and with replacement. We calculate the PME for each of these portfolios, and match the PME distribution to its model-implied counterpart again minimizing the least-squared distance between distributions. We obtain a correlation between PE funds $\rho_{PE} = 0.68$.

We calibrate the infinite-fund problem by using analytic extensions of the parameters above. In the infinite-fund problem, the investor is continuously active and earning PE returns. We assume an equally weighted portfolio and take limits as the number of funds in the portfolio goes to infinity. The volatility and the correlation of the PE portfolio with the stock market can be calculated following equation (14):

$$\psi^{\infty} = \psi \sqrt{\rho_{PE}} = 0.335 \times \sqrt{0.68} \approx 0.276 \quad ; \quad \rho_L^{\infty} = \frac{\rho_L}{\sqrt{\rho_{PE}}} = \frac{0.66}{\sqrt{0.68}} \approx 0.80. \tag{20}$$

This correlation estimate matches the most commonly used estimate in practice, that of

[20]They do not report an average, but the skewness of the distribution indicates that it would be higher than the median.

Blackrock.[21]

Table 1 summarizes the parameter values that we use. It is remarkable that although we use a parsimonious model, we not only match the distributions of PMEs to their empirical counterparts, but also generate calibrated parameters that are in line with the literature. These results give us additional confidence in our model and the associated counter-factuals.

4.2 Portfolio Allocation in the Baseline Economy

4.2.1 Allocation to Private Equity

At time τ_0, the beginning of each PE fund's life, the investor chooses the optimal commitment to PE as a fraction of total wealth, ξ^*. Following Proposition 1, ξ^* is chosen so that it maximizes the value function that prevails during the commitment period. After commitment, fluctuations in liquid wealth make the committed amount as a fraction of total wealth, ξ_t, move away from ξ^*. At the time of the capital call and during the holding period, ξ_t denotes the amount invested in PE as a fraction of total wealth. We refer to ξ_t as the PE allocation in both cases.

Figure 2, Panel A shows the portion of the agent's value function related to their investment in PE, $H(\xi, S)$ from equation (9). The dashed (solid) line represents the function during the commitment (holding) period. The investor optimally chooses to commit $\xi^* = 5.2\%$ of wealth. The shape of the value function shows that during the commitment period the region around the optimal allocation is fairly flat: as long as the PE allocation remains between 0% and 10%, welfare does not vary much. If liquid wealth decreases so that the PE allocation grows beyond 10%, welfare decreases rapidly. The investor strategically defaults on their commitment if the PE allocation reaches 20.8% of wealth during the commitment period (black circle). The investor does not wait for a capital call before defaulting; they avoid fees by defaulting early, and the consequence is a permanent loss of access to private equity funds.

However, Panel B shows that the likelihood of reaching a 20.8% PE allocation is nearly zero, so default hardly ever happens. In fact, the PE allocation does not vary much during the commitment period and remains smaller than 7.4% with a 99% probability. Thus, strategic default is both very rare and also (as we show later) important for welfare. This matches the institutional details from Section 2.4.

[21] https://www.blackrock.com/institutions/en-us/insights/charts/capital-market-assumptions. The model used by Blackrock to generate this correlation is not publicly available, but they also use PE fund cash flows to infer the correlation.

When capital is called, the value function jumps up to the solid black line (Panel A). At that point in time, the optimal PE allocation is 9.1%, compared to an earlier optimal capital commitment of 5.2% (and a 99th percentile of 7.4%). Thus, the investor chooses an optimal commitment that results in a significant under-allocation to PE. This is despite the fact that liquid wealth drifts up on average, meaning that the investor's commitment as a fraction of wealth declines on average from 5.2% by the time capital is called.

The secondary market is hardly ever used to liquidate the PE position, even with a relatively small haircut. With a 13.8% discount, the investor sells their stakes on the secondary market only when the PE allocation ξ reaches 58.4% of wealth (black square on Panel A), which is a rare event (Panel B). These features are consistent with the institutional details in Section 2.5: secondary market volume is low relative to aggregate allocations.

4.2.2 Liquid Portfolio and Consumption Policies

Panel C of Figure 2 shows how the investor alters their consumption policy if they move too far from their optimal portfolio composition. During the commitment period (dashed line), consumption is relatively stable, varying between 4.5% and 4.6% of wealth. In contrast, during the holding period (solid line), consumption is higher at the optimal portfolio composition, but more sensitive to the PE allocation, dropping to 4.2% before the investor accesses the secondary market.

Large changes in consumption are likely to occur during the holding period and this consumption volatility is costly for the investor. For example, there is a 5.1% probability that the PE allocation increases from 9.1% to 26.1% with a corresponding drop in consumption from 4.6% to 4.5% of wealth. Similarly, there is a 2.4% probability that the PE allocation increases from 9.1% to 35.2% with a corresponding drop in consumption to 4.4% of wealth. These changes have a high welfare cost. The investor would be willing to pay 1.6% of their total wealth to go from a PE allocation of 4.5% back to their optimal allocation of 4.6%, and a welfare cost of 3.4% of wealth to get back from 4.4%.

Panel D describes the liquid asset allocation. The stock allocation fluctuates around 55.5% during the commitment period, i.e., 44.5% of the portfolio is invested in the bond. This large allocation to the bond would be only slightly smaller (44.2%) without PE. Recall (previous section) that strategic default occurs when the PE commitment reaches 20.8% of wealth. Thus, our investor does not allocate more to the liquid bond to avoid default.

However, the option to default creates a near convexity in the value function (Figure 4, Panel A). Thus, the investor takes more risk – tilts their allocation toward the liquid stock and away from the liquid bond – as they approach default (Figure 2, Panel D).

In contrast, during the PE holding period, the investor's allocation to public market equity declines strongly and monotonically with their PE exposure. This is simple hedging: PE and the liquid markets are correlated and an excess allocation to PE is associated with more volatile consumption. So, when public markets decline, the relative allocation to PE increases, and the investor responds by taking less risk with their liquid assets. The result is that when public markets decline, the investor responds by reducing their stock allocation.

Both consumption and liquid asset policies are consistent with private equity changing the concavity of the investor's value function. Capital calls are good news, and consumption is higher during the holding period than the commitment period. However, an unbalanced portfolio reduces the investor's welfare, and consumption is more sensitive to market movements during the holding period. The investor holds a portion of their liquid assets in the bond to finance their commitments, but gambles to avoid default. Once the investor's commitment has been called, they modify their stock holdings to control their overall investment risk exposure.[22]

4.3 The Illiquidity Stack

In this section, we quantify the impact of the different liquidity frictions on the investor's portfolio allocation and welfare by solving the investor's problem in the five economies described in Section 3.4. The objective is to understand which frictions cause the under-allocation to PE and how costly these frictions are for the investor.

4.3.1 Commitment-Timing Risk

We start by examining the difference between Economies 0 and 1, i.e., we turn off call timing risk by making the capital call time deterministic instead of stochastic. Results in Table 2 and Figure 3 show that eliminating timing risk has little impact on the optimal commitment, consumption policy, and stock allocation. The welfare cost and return premiums associated with commitment-timing risk are close to zero. In fact, Figure 10 shows that for larger values of the subjective discount factor δ, the welfare cost becomes negative, implying that the agent *prefers* uncertainty about the timing of capital calls then.

[22] Online Appendix C describes the portfolio allocation of an investor with recursive utility instead of power utility. We find that the rate of consumption increases for lower elasticity of intertemporal substitution, in line with the intuition of a weaker desire to smooth consumption. But most importantly, the changes in the allocations to the risky assets are negligible.

Two competing forces can explain the sign and magnitude of the commitment-timing risk premium. On the one hand, when capital is called, the value function jumps up, as shown by Panel A of Figure 2. A stochastic capital call time implies that there is uncertainty about the timing of this utility gain. The key fact is that the expected present value of a utility gain is *increasing* in uncertainty over its timing because discounting, $e^{-\delta t}$, is a convex function of time. Jensen's inequality implies $\mathbb{E}\left[e^{-\delta\tau_C}U\right] > e^{-\delta\mathbb{E}[\tau_C]}U$. This feature pulls the cost of commitment timing risk down, and the effect increases in δ.

On the other hand, a stochastic capital call time induces uncertainty in the ability of the investor to fund both the capital call and consumption. Because the PE allocation ξ varies over time in the commitment period (due to fluctuations in liquid wealth), the investor is more likely to default on the call or reduce consumption if the commitment period is longer.

Importantly, whether it is positive or negative, the cost of timing risk remains close to zero, of the order of 10^{-4}.

4.3.2 Commitment-Quantity Risk

Next, we compare Economies 0 and 2, i.e., we enable the investor to adjust their committed amount (upward or downward) upon capital call. This comparison allows us to evaluate the impact of quantity risk. Results in Table 2 show that switching off quantity risk leads to a significant increase of the optimal PE commitment. Instead of a commitment of 5.2% at time τ_0, the investor optimally invests 8.4% of their total wealth in PE at the time of capital call. The rest of the policies are hardly affected: consumption increases only slightly (Figure 3, Panel B), and the stock-bond split is unaffected (Panel D).

Along with the increase in allocation, there is a corresponding welfare gain of 1.25% of total wealth (Table 2, Panel A). This amount is large: it corresponds to 15% of the amount committed to PE. Equivalently, the investor is willing to give up a return premium of 1.10% out of PE's expected return, forever.

The intuition for a high commitment-quantity risk premium is that the investor's welfare declines when their portfolio moves away from the optimal value. This is true both in the commitment period and in the holding period. Allowing the investor to choose their commitment at the time of a capital call means that public market movements in the commitment period no longer affect the investor's portfolio. The investor is always able to begin the holding period at the optimal allocation rather than under-allocated to PE as in the baseline economy (Section 4.2).

4.3.3 Interaction between Timing and Quantity Risks

We next examine the interaction between timing and quantity risks by solving the investor's problem in Economy 3: deterministic commitment period with the ability to adjust commitments at the time of the capital call. Is it more valuable to be able to adjust the PE allocation at call time if this time is random or deterministic?

A random call time makes it both possible to have a short and a long commitment period. In the former case, the PE allocation stays close to its optimal level, i.e., there is little quantity risk. By contrast, in the latter case the PE allocation can fluctuate away from its mean (as the value of the stock changes), inducing substantial quantity risk. In fact, this distinction turns out to be irrelevant.

In Economy 2, quantity risk is turned off but timing risk is on. We compare this economy to Economy 3 where they are both turned off. Both allocations and costs are similar: The welfare difference between these two economies is almost zero, with 0.02% welfare cost and 0.01% return premium (Table 2, Panel A). In Section 4.3.1, we showed that commitment-timing risk did not carry any cost. Combining these two results, we conclude that there is no interaction between timing risk and quantity risk: timing risk carries a negligible premium irrespective of whether quantity risk is present in the economy. Commitment-timing risk and commitment-quantity risk are separate.

4.3.4 Removing the Commitment Period

Next we remove the commitment delay (Economy 4), so that the investor's capital is invested as soon as it is committed. This allows us to make two useful comparisons. First, by comparing Economy 4 to the baseline, we can assess the overall effect of capital commitment on asset allocations and welfare. Second, by comparing Economies 2 and 4 (both with no commitment-quantity risk), we can isolate the impact of the commitment delay, i.e., the specific effect of not earning PE returns during the commitment period.

Interestingly, the optimal portfolio allocation changes between Economies 0 and 2, but not between Economies 2 and 4. This means that the commitment delay has no effect on the allocation once quantity risk has been removed. The investor increases their consumption slightly, in anticipation of the distributions that they will now receive earlier. The stock-bond split does not change. Thus, it is commitment-quantity risk, not the commitment delay, that generates the under-allocation to PE.

However, removing the commitment delay has a large effect on welfare. The investor is willing to pay 2.8% of their wealth to switch from Economy 0 to Economy 4. This is more than twice the amount they are willing to pay to switch from Economy 0 to Economy 2, i.e. to adjust their allocation at call time. Equivalently, the return premium associated with moving from Economy 0 to Economy 4 is 2.4%, also more than twice that of moving to Economy 2. These effects are due to the increase in active investment time: more time is spent with assets invested in PE, as opposed to waiting in the commitment period.

4.3.5 Distribution Timing Risk

We now draw a comparison between timing risk applied to commitment and applied to distribution. To do so, we solve Economy 5, which has commitment risk, but a deterministic holding period: the time between the capital call and distribution is fixed at $\frac{1}{\lambda_D}$.

We find that distribution-timing risk is meaningful, and substantially more important than commitment-timing risk, but less impactful than commitment-quantity risk. Table 2 shows that the optimal PE commitment increases from Economy 0 to Economy 5, from 5.2% to 6.8%. The investor also increases their consumption during the commitment period (Figure 3, Panel A), and the PE allocation at which the investor strategically defaults is higher than in the baseline economy. However, the stock-bond allocation remains similar to the one in the baseline economy.

Distribution timing risk is costly: The investor is willing to pay an initial welfare cost of 0.4% to remove this risk, or equivalently to accept a permanent decrease in the PE fund return of 0.4% as well. This result is in line with what is documented in Ang et al. (2014): The investor prefers certainty in the timing of distributions. However, both the welfare cost and the return premium are less than half those of commitment-quantity risk.[23]

The cost of distribution-timing risk is different from that of commitment-timing risk. During the commitment period, the investor is only exposed to public market volatility whereas during the holding period, it is exposed to both public and private market volatility. As a result, it is more volatile (Figure 2, Panel B) and the investor is more sensitive to the duration of the holding period than to the duration of the commitment period. Certainty on the distribution timing allows them to better handle their consumption stream and is thus preferred.

[23] The discrepancies between our values of welfare costs and the ones presented in Ang et al. (2014) have two sources. First, the Ang et al. (2014) model does not have a commitment period. Second, our parameters are chosen from a detailed calibration to private equity data described in Section 4.1.

5 Sensitivity Analyses

In this section, we modify the model parameters both to assess the robustness of the results we just presented – the under-allocation to PE, the large welfare cost and return premium of commitment-quantity risk – and to better understand the mechanism behind these results.

5.1 Default

The investor has the option to strategically default any time during the commitment period and at capital call. In the event of default, the investor is no longer obligated to provide the committed capital and they do not have to pay the fees associated with their commitment any more. The cost of strategic default is that the investor is being banned from PE, so their investment opportunity set reduces to the stock and bond. In this section, we analyze the value of the investor's option to default.

In our baseline economy, the investor defaults when their PE allocation reaches 20.8% of total wealth during the commitment period. If this threshold is reached, they do not wait for capital call. The default option is therefore exercised well before the pledge approaches the allocation to the liquid risky bond (44.5%). Despite the early default, it is a near zero-probability event (Section 4.2). Thus, commitment default is both very early and very low probability.

To understand why the investor does not wait for capital call if their allocation reaches 21% of wealth, we vary the level of fees that the investor must pay to maintain their commitment. Figure 4, Panel A shows the investor's value function during the commitment period as a function of allocation and fees. The convexity is created by the option to default,[24] and the point of default is at a lower allocation (ξ) when the level of fees is higher: the investor is less willing to maintain the commitment in the face of higher fees.

We next compare the point at which the investor would default while waiting for a capital call to the point of default at the moment of capital call (Figure 4, Panel B). When fees are near zero, the investor always waits for the call to default but when fees are high, the investor defaults early. In our calibration, fees of 2% are enough to induce the investor to default early.

We also test whether these results are specific to our calibration by varying parameters such that the PE allocation can become large during the commitment period. We examine

[24] Note that while H is convex in ξ, ξ is a composition of two state variables, and the value function as a whole is not convex in liquid wealth.

four cases: (i) the stock has low expected returns, (ii) the stock has high return volatility, (iii) the investor has a low risk aversion, and (iv) the commitment period is longer. In all scenarios, strategic default occurs early and remains a near zero-probability event as shown in Table 4.

Although default is rare, the penalty to default – loss of access to private equity – is important for our results. We can consider an economy in which the investor can default without penalty: they can simply re-pledge immediately to another fund. By continuously exercising this option, the investor is always able to have the optimal commitment when capital is called. Thus, the investor achieves the same outcome as in the economy in which we remove commitment-quantity risk! Therefore the default penalty has the same ex-ante welfare cost for the investor as commitment-quantity risk. We conclude that the option to default is important but rarely exercised, matching the discussion in Section 2.4.

5.2 Secondary Market

The secondary market offers the possibility to exit an allocation that has become too large. However, in the baseline economy, the investor only infrequently makes secondary market sales. This result is not sensitive to the secondary market haircut. Eliminating the haircut on the secondary market hardly changes this result. The investor's commitment increases only slightly, from 5.2% to 5.7% (Table 3, Panel A), which remains far from the optimal level upon capital call. Similarly, increasing the haircut to 40% only decreases the allocation to 5% (Panel B).

The reason for this infrequent use is that, as in practice (see Section 2.5), the secondary market forces the investor to sell their entire stake in a fund and wait until the next fund is raised and capital deployed, without earning any PE returns in the meantime.

The secondary market has a larger *indirect* effect: it raises the premiums associated with capital commitment, particularly the premiums associated with quantity risk and the commitment delay (Table 3, Panels A and B). The welfare cost of commitment-quantity risk increases from 1.2% to 1.4% as the haircut declines from 13.8% to 0%. The willingness to pay to eliminate the commitment period increases as well, with a welfare cost going from 2.8% to 3.25%. The return premiums similarly increase. Similarly, improving the liquidity of the secondary market increases welfare more when other liquidity frictions are removed (Table 3, Panel C). For example, the investor is willing to give up 0.07% of their wealth for a liquid secondary market in the baseline economy, but 0.25% when they can freely adjust their PE allocation at a capital call.

Putting the two results together, we conclude that the two types of liquidity – ease of commitment and ease of access to the secondary market – are complements, not substitutes. Increasing liquidity along the first dimension increases the willingness to pay to remove frictions along the second dimension. Moreover, the indirect effect of the secondary market on the welfare cost of capital commitment is larger than the direct welfare effect of the secondary market itself.

Thus, our model implies that the development of a PE secondary market increases the investor's desire to alleviate commitment-quantity risk, rather than satiate that desire. Intuitively, the investor is willing to invest more in PE if they have an easier way out, so this out becomes more valuable when commitment-quantity risk is absent.[25]

5.3 Calibrated Parameters

We examine the sensitivity of the optimal PE commitment to the model parameters, and benchmark the variations to the ones that would be observed in a fully liquid model. Results are shown in Figure 5.

Quite strikingly, for all the parameters except the correlation, PE allocations are less sensitive to parameter changes in our model than they are in a fully liquid model. When PE expected return is below 12%, the optimal commitment to PE is zero (Panel A). Above that threshold, the PE commitment increases linearly with expected returns: each additional percentage of expected return increases the allocation by 2.25%. However, if PE were a fully liquid asset, the increase would be nearly twice as fast: 4.2% for each additional 1% return.

Similarly for PE volatility, in the baseline model, varying PE volatility from 35% to 25% changes the optimal commitment from 4% to 17.9% (Panel B). In contrast, in the fully liquid model, the corresponding allocation changes from 11.7% to 38.3%. We obtain similar results when varying stock's expected return (Panel C), stock volatility (Panel D), and the investor's risk aversion level (Panel F). The sensitivity of the allocation to changes in the correlation is low, but it was also the case in the Merton economy (Panel E).[26]

These differences in the sensitivity of the optimal commitment in our model compared to the fully liquid case are consistent with the hypothesis that it is stochastic illiquidity that

[25] All these results are also robust to allowing the investor to reinvest in a new fund as soon as they sell their position on the secondary market, instead of waiting for the fund's end of life (non-tabulated).

[26] The correlation result is consistent with Ang et al. (2014), who examine a model of illiquidity without commitment, and the risk aversion result is similar to the result in Giommetti and Sorensen (2020). Results are similar with recursive preferences. The welfare costs of quantity risk are reported for different values of the intertemporal elasticity of substitution in Online Appendix C.

creates an issue for the investor. Changing parameters so as to make PE more attractive pulls up the optimal allocation. However, because the investor cannot freely enter or exit their investment, they are less willing to take larger positions because there is an increased chance that their portfolio moves significantly away from the optimum, making their consumption more volatile.

To illustrate, we consider the case of an increase in PE expected returns from $\nu = 14\%$ to $\nu = 18\%$, the optimal commitment goes from 5.2% of wealth to 14.4%, and the probability that consumption decreases by 5% (relative to liquid wealth) during the holding period, assuming it is initially at its optimum, increases from 1.9% to 11.3%. In contrast, in a fully liquid Merton economy, the optimal allocation increases from 14.2% to 30%, but consumption, relative to liquid wealth, is maintained at its optimum. Illiquidity means that smaller changes in PE allocation are associated with more consumption volatility because the investor can only consume out of liquid wealth.

Changing the intensities of capital calls and distributions has moderate effects on the investor's optimal commitment. Increasing λ_C makes the commitment period shorter on average and less variable. This effect reduces both commitment risk and commitment delay and results in a larger PE commitment. Panel G shows that the optimal PE commitment more than doubles when the average commitment period changes from five years ($\lambda_C = 0.2$) to one year ($\lambda_C = 1$): 3.1% to 7.6%. In the limit of λ_C going to infinity, the optimal PE commitment goes to 8.3%, which is the allocation in Economy 4, when the investment is made at fund inception (no commitment period).

The sensitivity of the optimal PE commitment to the intensity of capital distributions is hump-shaped (Panel H): it is slightly increasing for holding periods that are longer than 6.5 years (i.e., $\lambda_D = 0.15$), and decreasing for holding periods that are shorter than 6.5 years. This hump shape is the result of two opposite effects. With a low λ_D – a long holding period – the PE allocation is highly variable over time; hence the investor pulls down their commitment. As λ_D increases, this variation is lower (the holding period is shorter), hence PE commitment increases. With a high λ_D, the active investment time is short, so the investor is not willing to pledge a large amount because that amount will not be optimal in the holding period.

5.4 PE Under-Allocation, Welfare Costs and Return Premiums

Our first key finding that the significant under-allocation to PE due to commitment-quantity risk and the premiums associated with commitment-quantity risk, are robust and stable with respect to changes in underlying return and preference parameters (i.e., parameters other than λ_C and λ_D). Figure 5, Panels A to F, show the under-allocation – the difference between the optimal allocation in Economy 2 (choose commitment on call, dashed line) and the optimal commitment in the baseline economy (solid line) – as we vary these return and preference parameters. Similarly, Figure 6, Panels A to F, display the sensitivity of the return premium of commitment-quantity risk (dashed lines). We observe little change when varying these parameters.

This finding does not hold for the welfare cost of commitment-quantity risk (solid lines on Figure 6, Panels A to F): The investor is willing to pay more to eliminate quantity risk when PE expected returns increase, or when either PE volatility, correlation with the stock returns, or risk aversion decrease. The welfare cost is indeed a one-time payment at time 0 for the total risk that the investor will bear, at $t = 0$. As such, it is mechanically increasing in the optimal PE commitment as described in Section 3.6); see Figure 5. In contrast, the return premium represents a cost *per unit of PE allocation* and is much less sensitive to the optimal PE commitment.

The two intensity parameters, (λ_C, λ_D), have large effects, both on the magnitude of the PE under-allocation and on the return premium of commitment-quantity risk. These two parameters drive the expected length of the commitment period relative to the overall fund life: a larger λ_C or a smaller λ_D yields a shorter average commitment period relative to fund life. In this case, there is less commitment risk, hence both the PE under-allocation and the return premium associated with commitment-quantity risk decrease (Figure 6, Panels G and H). Changes in the welfare cost are smaller because they are dampened by increases in the optimal PE commitment.

6 Commitment Quantity Risk & Liquidity Cycles

In this section, we examine how time varying liquidity impacts our results. The key idea is that capital calls and distributions are correlated with returns. In principle, cycles can exacerbate commitment risk if PE funds demand liquidity exactly when it is most difficult to provide. However, cycles may also moderate commitment risk if PE expected returns are higher when capital is called.[27]

[27] Part of the motivation for such an analysis in based on anecdotal evidence such as what Leibowitz and Bova (2009) report in 2008: "The horrendous declines presented liquidity problems even for many portfolio managers who were long-term oriented, had modest payment schedules, and a seemingly ample percentage of liquid assets. This perfect liquidity storm, layered on top of a perfect asset storm, resulted from a toxic combination of: 1) a need to fulfill prior commitments to private equity, venture capital, real estate, and

6.1 Model and Calibration

In our baseline model, the Poisson processes triggering capital calls and distributions are independent of each other and independent of PE fund returns. However, our setup enables us to relate the intensity of capital calls and distributions to expected returns.

In this section, we assume that the economy can be in one of two states $s_t = \{L, H\}$. State L corresponds to periods of low liquidity and state H to high liquidity. The state of liquidity s_t follows a continuous time Markov process with a transition probability matrix between t and $t + dt$ given by

$$M = \begin{pmatrix} 1 - \chi^L \, dt & \chi^L \, dt \\ \chi^H \, dt & 1 - \chi^H \, dt \end{pmatrix}. \tag{21}$$

We identify years of low liquidity as the vintage years that experienced the lowest intensity of capital call. Specifically, we calculate the fraction of capital that is called each quarter for each fund. We remove an age fixed effect, as in Robinson and Sensoy (2016), and average the residual quarterly intensity across all funds with a given vintage year. We rank the vintage years by their associated average intensity of capital calls. The lowest 30% are the vintage years 1992, 1993, 1999, 2000, 2002, 2007, 2008 and 2009. In line with intuition, these years include the burst of the dot-com bubble and the financial crisis.

We calibrate the stock return and volatility in the low liquidity state as the average S&P 500 log return and volatility over the five years starting with each low-liquidity year. We find $\mu^{low} = 6.63\%$ and $\sigma^{low} = 18.59\%$.

The other parameters are calibrated as described in Section 4.1, but keeping only PE funds that have a low-liquidity vintage year. We obtain $\nu^{low} = 11\%$, $\phi^{low} = 37.5\%$, $\lambda_C^{low} = 0.34$ (commitment period of 2.9 years) and $\lambda_D^{low} = 0.14$ (holding period of 7.1 years).

We set the parameters in the high liquidity state so that their state-weighted average match their value absent cycles. Finally, in line with Nadauld et al. (2019), we set the secondary market haircut to 28% in the low liquidity state and 9% in the high liquidity state. Parameter values are summarized in Table 5.

The low liquidity state is therefore characterized by lower expected returns, higher volatility and a higher correlation between public and private equity. In addition, intensities of capital call and distribution are lower than in the high liquidity state, but the intensity of capital call is less sensitive than the intensity of capital distributions. This result is in line

hedge funds, 2) reduced distributions from these asset classes..."

with the key insight of Robinson and Sensoy (2016): Net cash flows are procyclical because distributions are more procyclical than calls.

6.2 The Effect of Time-Varying Liquidity on Commitment Quantity Risk

Table 6 shows PE allocations, welfare costs and return premiums of commitment-quantity risk in an economy with no cycles and with the calibrated cycles from the previous section. The investor allocates slightly less to PE in the low state than in the high state (4.3% vs. 4.7%), and less to PE in both states than in the economy without cycles. As a result of this lower PE allocation, the welfare cost of commitment-quantity risk is also lower (in line with Section 3.6). However, the return premium is essentially unchanged from the economy without cycles.

To understand the results, we plot the value functions before and after a call and in both liquidity states in Figure 7, Panel A. Points A and B are low- and high-liquidity optimal commitments, while points C and D are welfare maximising allocations during the holding period. With time-varying liquidity, the optimal commitment changes from one state to the other, but so does the optimal allocation during the holding period. With our calibrated parameters, cycles act as a level shift in the value function: it is shifted up in both low and high liquidity states.[28] This level shift in the value function is due to PE becoming more desirable – higher expected returns, lower volatility, etc. – at the same time the stock becomes more desirable. The change in optimal PE commitment when switching state is modest, i.e., the investor would not re-allocate much if given the possibility.

To illustrate what happens when PE and the stock do not become more desirable together, we fix public market expected returns at $\mu = 0.08$ in both high and low states, and keep the other parameters as in our cycles calibration. With these parameters, the stock is relatively better in the low liquidity state (than in our cycles calibration), which is the state in which the investor needs liquid wealth as there are fewer calls but much fewer distributions. The value function is plotted in Figure 7, Panel B. The shift in the value function is now small but there is a change in its slope. The mismatch between stock returns and PE needs in liquid wealth (both high during the low liquidity state) makes the PE allocation lower in the low liquidity state (than with our cycles parameters) and higher in the high liquidity state. As a result, the investor would like to re-allocate at each change in the liquidity state.

[28]The reason why it is also higher in the low liquidity state than in the economy with no cycles is that the probability of switching to the high probability state pulls up welfare.

Table 6 shows that the resulting return premium is higher in *both* states.

We further test parameter sets where we switch one of public market returns (μ) or private market returns (ν) from the high state to the low state. Figure 7, Panel C and D and Table 6 report the results. In Panel C, the stock is even more desirable in the low liquidity state compared to PE than in Panel B. The gap between optimal PE commitments in both states widens and the risk premiums increase. In Panel D, PE is more desirable in the low liquidity state but not the stock. This increases the PE commitment compared to the high liquidity state and increases the risk premium. These results confirm that the relative difference between the desirability of public and private markets across states makes the return premium higher in *both* states.

Our setup being in continuous time, we cannot distinguish between realized and expected returns. Our cycles calibration was performed with realized returns and therefore returns are lower in the low liquidity state. In contrast, Haddad et al. (2017) suggest that expected returns to both public and private markets are higher in the low liquidity state. In Panel D, we therefore switch both public and private market returns from the high state to the low state. This makes both the stock and PE less desirable in the low liquidity state. In line with our previous results, Table 6 shows that the return premiums are almost the same as in the no-cycles economy. In this economy, expected returns are acting as an insurance. In the low (high) liquidity state, volatilities are high (low) but so are expected returns. As a result, welfare does not change much between the low and the high liquidity states. Moreover, there is no large relative difference between public and private markets, so PE allocations are basically unchanged.

To conclude, the impact of time-varying liquidity depends on the *relative* effect on private and public markets. It is commonly accepted that public and private markets are not completely segmented; our results show that the impact of cycles on commitment risk premiums crucially depends on the relative changes in expected returns.

7 Is Commitment Risk Diversifiable?

In this section, we study whether commitment-quantity risk is diversifiable. On the one hand, by spreading their allocation across several funds, the investor may smooth their capital inflows and outflows. It also allows the investor to sell only part of their PE exposure on the secondary markets. On the other hand, the investor risks a funding mismatch if earlier investments pay out late while later investments call early.

7.1 Two Funds

We solve the two-fund model described in Section 3.5 and show results in Figure 8 and Table 7. The optimal commitment allocates 4.2% to each fund (8.5% total), as opposed to a commitment of 5.2% when a single fund is available.

The first result is that access to a second fund is not particularly valuable. Diversification increases investor's welfare, but modestly: the investor is willing to give up 1.0% of their wealth, or accept a permanent reduction in PE returns of 0.9% in order to gain access to a second fund (Table 7, Panel C). For comparison, the investor would be willing to give up 1.25% of their wealth, or a return premium of 1.1% to alleviate commitment-quantity risk in the one-fund model. In other words, despite the gains from cash flow diversification, the investor prefers to control their quantity invested by adjusting their commitment than through access to a second fund.

Consistent with this result, the investor does not do much liquidity management – their allocation to a second fund does not much depend on their allocation to the first fund. If the investor has a large allocation, either a commitment or a holding, in fund 1, they will commit less to fund 2 (see Figure 8, Panel A). However, this effect is quite small: If the allocation to fund 1 is 8.8%, twice the optimal commitment to the first fund, the investor commits 3.8% to the second fund instead of 4.4%.[29]

In contrast with the one-fund case, a capital call does not always trigger a utility gain with multiple PE funds. Figure 8, Panel D, shows that when the commitment to fund 1 is called, the welfare change is positive if the second fund is in the commitment period (dashed

[29] This assumption is supported empirically by Robinson and Sensoy (2016): "Most variation in fund-level cash flows is purely idiosyncratic across funds of a given age at a given point in time (...) this suggests that liquidity shocks arising from the uncertain timing of calls and distributions can be significantly mitigated by holding a portfolio of investments diversified both across different funds of the same age and across funds of different ages. For example, for buyout funds the standard deviation of quarterly net cash flows averages 11.57% of committed capital, and this standard deviation shrinks to 4.54% in a portfolio of all buyout funds in the sample."

line) but negative if it is in the holding period (solid line). Indeed, if the second fund has not been called yet, there is no capital deployed in PE and liquid wealth is high (because of the previous funds' distributions). The capital call is hence good news. In contrast, if the second fund has already been called, the capital call of fund 1 occurs at a time of low liquid wealth. There is a *funding mismatch*, i.e., the capital call is bad news.

To evaluate whether diversification decreases commitment-quantity risk, we solve Economies 0, 2, and 4 with two PE funds; results are shown in Table 7 (Panel A). The optimal PE commitment increases compared to the one-fund case, but the investor still commits much less in Economy 0 than in Economy 2 (8.5% versus 11.4%). Hence, the under-allocation to PE remains strong.

Consistent with the increase in the investor's allocation, the welfare cost of commitment-quantity risk increases to 1.4% (versus 1.25% with one fund), but the return premium slightly decreases to 0.8% (versus 1.1% with one fund). We conclude that the investor only modestly benefits from liquidity diversification, even though we overstate its value by assuming purely idiosyncratic shocks.

To understand why the gains from diversification are small, we examine how the value function reacts to increases in PE allocation. In the economy with one fund, we have shown that the key issue for the investor is the possibility of a decline in the stock price, causing the PE allocation to be too large (denominator risk). This problem is equally present with multiple funds. In Figure 8, Panel B, we keep the allocation to fund 2 at its optimal level and vary the allocation to fund 1 (thin line). The penalty for having an allocation to fund 1 that becomes too large (moving right) is relatively small and comparable to the penalty in the one-fund model. However, if the public market value declines, the allocations to both funds increase together. To reflect this joint effect, we force ξ^2 to vary together with ξ_1 ($\xi^2 = \xi^1$). The value function becomes much more concave, with a larger penalty when the allocation to fund 1 increases. These results are similar when both funds are in their holding period (Panel C). In other words, *denominator risk* cannot be diversified.

7.2 Infinite Number of Funds

We now solve the infinite-fund problem described in Section 3.5. In this setup, each fund is infinitely small and the two key variables are the aggregate PE commitment and the aggregate PE investment as a function of liquid wealth: $\pi = \frac{X^\infty}{W+Y^\infty}$ and $\xi = \frac{Y^\infty}{W+Y^\infty}$. Figure 9 plots the former as a function of the latter. The optimal aggregate commitment decreases rapidly as the aggregate investment increases: it reaches 0 for $\xi \geq 21\%$. At the steady state, the fraction of liquid wealth invested in PE is 19.8% and only 2% is committed (thus uncalled). Therefore, the total PE allocation amounts to 21.9%, which is a large increase compared to the one- and two-fund case.

In addition, the investor sustains a level of investment that is much higher than the level of commitment. This is possible with an infinite number of funds because the timing of capital calls and distributions is no longer stochastic. There is therefore no longer a possible funding mismatch. As a result, the investor us willing to give up 5.4% of their wealth, or accept a reduction of 2.85% in PE returns to have access to an infinite number of funds instead of two funds (Table 7, Panel C).

However, we observe that the investor is still under-allocated to PE relative to what they would allocate without commitment risk (Table 7, Panel B). When given the possibility to choose-on-call, the PE allocation is 28.5%. As the change in allocation is large, the welfare premium associated with eliminating quantity risk is also large at 4.3% (versus 1.4% with two funds) and the return premium is only slightly smaller (0.7%) than with two funds (0.8%); see Panel C. The effects of diversification are therefore small because the denominator risk remains: undiversifiable public market movements still drive volatility in the PE allocation and thus excess volatility in consumption.

A final surprising result is that the secondary market remains mostly unused with an infinitely of funds. The investor can now sell any fraction of their aggregate investment on the secondary market. In contrast, with one fund, the investor has to sell all of the PE position on the secondary market or nothing. Even given the option to sell partial stakes, the investor rarely uses the secondary market and the reason is that they stop committing to new funds when the aggregate investment exceeds 21%. The invested amount then naturally decreases as capital is distributed (at rate $\lambda_D dt$). The larger λ_D, the less the investor needs the secondary market. Out model thus offers a rather unique insight regarding the secondary market: investors with few PE positions should be reluctant to use it because they need to liquidate a large chunk of their PE position if they do; and investors with many PE positions should be reluctant to use it because if they stop committing, the decrease in PE exposure is rapid.

8 Conclusion

This paper proposes an optimal dynamic portfolio allocation model that includes capital calls and distributions with uncertain timing. We calibrate this model to data on PE fund cash flows and show that ex-ante commitment has large effects on investors' portfolios and welfare.

One key finding is that investors want to change their capital allocations when capital is called – most often to increase the allocation. Hence, investors are under-allocated to PE, and willing to pay a large premium to adjust the quantity committed upon capital call. A direct implication of PE under-allocation is that the demand for changing the amount committed is asymmetric. Investors would rarely want to reduce their allocation to PE, but highly value the option to top-up their allocation when capital is called. For example, they would highly value the possibility to co-invest, which is consistent with the institutional details reviewed in Section 2.5. We could hence see recent key institutional developments as allowing some investors to alleviate commitment-quantity risk.

With one fund, the return premium of commitment-quantity risk amounts to 1.25% of the investors' initial wealth, or equivalently to a permanent loss of 1.1% of private equity returns. It is larger than the premium investors are willing to pay to eliminate other liquidity frictions: timing uncertainty and the limited tradability of PE investments. Furthermore, commitment risk premiums increase with secondary market liquidity, and increasing the number of funds does not allow the investor to diversify commitment risk, particularly when liquidity is time varying.

A natural question that arises is "Why do LPs not simply diversify the problem away?" Making multiple different capital commitments reduces the size of each commitment and call, removing the lumps. We find that this diversification with multiple funds is only weakly helpful. Investors care about the fraction of their wealth that they have committed, and public market movements change all the allocations' denominators at once, so the key risk – commitment-quantity risk – cannot be diversified. Worse, without perfect and full diversification, there is always a potential funding mismatch in which holdings in one fund impact the welfare value, through future capital calls, of commitments in other funds.

References

Ang, A., B. Chen, W. N. Goetzmann, and L. Phalippou (2018). Estimating private equity returns from limited partner cash flows. *Journal of Finance 73*, 1751–1783.

Ang, A., D. Papanikolaou, and M. M. Westerfield (2014). Portfolio choice with illiquid assets. *Management Science 60*(11), 2381–2617.

Arcot, S., Z. Fluck, J.-M. Gaspar, and U. Hege (2015). Fund managers under pressure: Rationale and determinants of secondary buyouts. *Journal of Finance 115*, 102–135.

Axelson, U., T. Jenkinson, P. Strömberg, and M. S. Weisbach (2013). Borrow cheap, buy high? the determinants of leverage and pricing in buyouts. *Journal of Finance 68*(6), 2223–2267.

Axelson, U., P. Strömberg, and M. S. Weisbach (2009). Why are buyouts levered? the financial structure of private equity funds. *Journal of Finance 64*(4), 1549–1582.

Banal-Estañol, A., F. Ippolito, and S. Vicente (2017). Default penalties in private equity partnerships. Working Paper.

Barber, B. M. and A. Yasuda (2017). Interim fund performance and fundraising in private equity. *Journal of Financial Economics 124*(1), 172–194.

Barrot, J.-N. (2017). Investor horizon and the life cycle of innovative firms: Evidence from venture capital. *Management Science 63*(9), 2773–3145.

Bollen, N. P. B. and B. A. Sensoy (2021). How much for a haircut? Illiquidity, secondary markets, and the value of private equity. *Financial Management*. Forthcoming.

Brown, G. W., O. Gredil, and S. N. Kaplan (2019). Do private equity funds manipulate reported returns? *Journal of Financial Economics 132*(2). 267-297.

Brown, G. W., R. S. Harris, W. Hu, T. Jenkinson, S. N. Kaplan, and D. T. Robinson (2020). Can Investors Time Their Exposure to Private Equity? Working paper.

Cavagnaro, D. R., B. A. Sensoy, Y. Wang, and M. S. Weisbach (2019). Measuring institutional investors' skill at making private equity investments. *Journal of Finance 74*(6), 3089–3134.

Cox, J. C. and C. Huang (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory 49*(1), 33 – 83.

Crain, N. G. and K. K. F. Law (2016). Spreading sunshine: Fair value accounting and information production. Working Paper.

Da Rin, M. and L. Phalippou (2017). The importance of size in private equity: Evidence from a survey of limited partners. *Journal of Financial Intermediation 31*, 64–76.

Dai, M., P. Li, H. Liu, and Y. Wang (2015). Portfolio Choice with Market Closure and Implications for Liquidity Premia. *Management Science 62*(2), 368–386.

Degeorge, F., J. Martin, and L. Phalippou (2016). On secondary buyouts. *Journal of Financial Economics 120*(1), 124–145.

Dimmock, S. G., N. Wang, and J. Yang (2019). The Endowment Model and Modern Portfolio Theory. Working paper.

Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica 60*(2), 353–394.

Dybvig, P. H. and C. Huang (1988). Nonnegative wealth, absence of arbitrage, and feasible consumption plans. *Review of Financial Studies 1*(4), 377–401.

Epstein, L. G. and S. E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica 57*(4), 937–69.

Epstein, L. G. and S. E. Zin (1991). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of Political Economy 99*(2), 263–86.

Fang, L., V. Ivashina, and J. Lerner (2015). The disintermediation of financial markets: Direct investing in private equity. *Journal of Financial Economics 116*(1), 160–178.

Gârleanu, N. (2009). Portfolio choice and pricing in illiquid markets. *Journal of Economic Theory 144*, 532–564.

Giommetti, N. and M. Sorensen (2020). Optimal Allocation to Private Equity. Working paper.

Haddad, V., E. Loualiche, and M. Plosser (2017). Buyout Activity: The Impact of Aggregate Discount Rates. *Journal of Finance 72*(3), 371–414.

Kahl, M., J. Liu, and F. Longstaff (2003). Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it? *Journal of Financial Economics 67*(3), 385–410.

Korteweg, A. (2019). Risk Adjustment in Private Equity Returns. *Annual Review of Financial Economics 1*(1), 131–152.

Korteweg, A. and M. Westerfield (2021). Asset allocation with private equity. Working Paper.

Kushner, H. J. and P. Dupuis (2001). *Numerical Methods for Stochastic Control Problems in Continuous Time* (Second ed.), Volume 24 of *Applications of Mathematics*. New York: Springer-Verlag.

Leibowitz, M. and A. Bova (2009). Portfolio Liquidity. *Morgan Stanley Research Portfolio Strategy*.

Lerner, J., J. Mao, A. Schoar, and N. R. Zhang (2021). Investing Outside the Box: Evidence from Alternative Vehicles in Private Capital. Working Paper.

Lerner, J. and A. Schoar (2004). The illiquidity puzzle: theory and evidence from private equity. *Journal of Financial Economics 72*(1), 3–40.

Longstaff, F. A. (2001). Optimal Portfolio Choice and the Valuation of Illiquid Securities. *Review of Financial Studies 14*(2), 407–431.

Longstaff, F. A. (2009). Portfolio Claustrophobia: Asset Pricing in Markets with Illiquid Assets. *American Economic Review 99*(4), 1119–1144.

Metrick, A. and A. Yasuda (2010). The Economics of Private Equity Funds. *Review of Financial Studies 23*(6), 2303–2341.

Nadauld, T. D., B. A. Sensoy, K. Vorkink, and M. S.Weisbach (2019). The liquidity cost of private equity investments: Evidence from secondary market transactions. *Journal of Financial Economics 132*(3), 158–181.

Phalippou, L. (2021). *Private Equity Laid Bare*. Independent.

Robinson, D. T. and B. A. Sensoy (2016). Cyclicality, performance measurement, and cash flow liquidity in private equity. *Journal of Financial Economics 122*(3), 521–543.

Sorensen, M., N. Wang, and J. Yang (2014). Valuing Private Equity. *Review of Financial Studies 27*(7), 1977–2021.

Table 1: Calibrated Parameters

This table displays the values of parameters obtained from the calibration described in Section 4.1.

Panel A: Calibrated parameters in the model with a finite number of funds

Parameter	Symbol	Parameter value
Risk-free rate	r	0.03
PE expected returns	ν	0.14
Stock expected returns	μ	0.08
PE fund volatility	ψ	0.335
Stock volatility	σ	0.150
Correlation stock & PE	ρ_L	0.66
Correlation between PE funds	ρ_{PE}	0.68
Intensity of capital call	λ_C	0.344
Intensity of capital distribution	λ_D	0.174
Secondary market haircut	α	13.8%
Investor's time discounting	δ	0.05
Investor's risk aversion	γ	4
Fee on commitment	f	2%

Panel B: Calibrated parameters in the model with an infinite number of funds

Parameter	Symbol	Parameter value
Correlation between stock & PE portfolios	ρ_L^∞	0.80
PE portfolio volatility	ψ^∞	0.276

Table 2: Optimal PE Allocation & Liquidity Frictions

The baseline economy (E0) is our central model, i.e. an economy in which investors need to commit ex-ante on the amount that will be called, wait for capital to be called and receive capital calls and distributions at random times. These frictions are removed one at a time in Economies 1 through 4, as described in Section 3.4. Economy 5 contains commitment risk, but the distribution time is deterministic. The PE allocation refers to the optimal PE commitment made at time τ_0 in Economies 0, 1 and 5 and to the optimal investment made at time τ_C in Economies 2, 3 and 4. As defined in Section 3.6, the welfare cost is the amount investors are willing to pay to switch from E0 to a given economy, i.e. the willingness to pay to remove a given friction. The return premium is the additional return that PE should deliver (ν) in Economy 0 for the economy under consideration to be equivalent to E0, i.e. the return premium associated with a given friction. Results are shown with a secondary market haircut of 13.8%.

		PE allocation	Welfare cost	Return premium
E0	Baseline (All frictions)	5.23%		
E1	Deterministic call time	5.18%	0.01%	0.01%
E2	Choose quantity on call	8.32%	1.25%	1.10%
E3	E1 $\cap$ E2	8.32%	1.27%	1.11%
E4	No commitment period	8.32%	2.79%	2.41%
E5	Deterministic payout time	6.79%	0.43%	0.38%

Table 3: Secondary Market

This table reports the marginal impact of changing the secondary market haircut on investor welfare, in a given economy. The welfare cost of the default haircut, i.e., the amount that the investor is willing to pay to change this haircut from 13.8% to 0% (resp., from 13.8% to 40%) is displayed in the second (resp., third) column. Each row references an economy with a different liquidity friction.

Panel A: Secondary market haircut of 0% (liquid market)

		PE allocation	Welfare cost	Return premium
E0	Baseline (All frictions)	5.67%		
E1	Deterministic call time	5.61%	0.01%	0.01%
E2	Choose quantity on call	9.62%	1.43%	1.33%
E3	E1 ∩ E2	9.62%	1.44%	1.35%
E4	No commitment period	9.71%	3.25%	2.97%
E5	Deterministic payout time	6.79%	0.36%	0.34%

Panel B: Secondary market haircut of 40% (illiquid market)

		PE allocation	Welfare cost	Return premium
E0	Baseline (all frictions)	4.98%		
E1	Deterministic call time	4.93%	0.01%	0.01%
E2	Choose quantity on call	7.80%	1.17%	0.99%
E3	E1 ∩ E2	7.80%	1.19%	1.01%
E4	No commitment period	7.73%	2.61%	2.18%
E5	Deterministic payout time	6.79%	0.47%	0.40%

Panel C: Welfare cost of changing the haircut in each economy

		Calibration (h=13.8%)	Liquid (h=0%)	Illiquid (h=40%)
E0	Baseline	0%	0.07%	-0.04%
E1	Deterministic call time	0%	0.07%	-0.04%
E2	Choose quantity on call	0%	0.25%	-0.13%
E3	E1 ∩ E2	0%	0.25%	-0.13%
E4	No commitment period	0%	0.54%	-0.23%
E5	Deterministic payout time	0%	0.00%	0.00%

Table 4: Sensitivity of the probability of strategic default to changes in parameters

This table illustrates the marginal impact of changing one parameter on the investor's point of strategic default. All other parameters are as in our calibration (see Table 1). The optimal PE commitment and bond allocation, and the probability of strategic default in Economy 0 (baseline economy) are reported in columns 2 to 4. The last column displays the welfare cost of commitment-quantity risk, obtained by comparing Economy 2 (the investor can update their commitment upon capital call) to the baseline economy.

	PE comm.	Bond alloc.	P(default)	Welfare cost
Baseline (E0)	5.23%	45%	$\approx$ 0%	1.10%
Low stock expected returns (μ=6%)	12.00%	67%	$\approx$ 0%	2.78%
High stock volatility (σ=0.20)	9.63%	69%	$\approx$ 0%	2.27%
Low risk aversion (γ=2)	10.65%	0%	$\approx$ 0%	0.75%
Long commitment period (λ_C=0.2)	3.27%	44%	$\approx$ 0%	1.41%

Table 5: Calibrated Parameters with Cycles

This table displays the values of parameters obtained in our extension of the model with liquidity cycles, following the calibration described in Section 6.1. The values are given in the low and high liquidity states.

Parameter	Symbol	Low liquidity	High liquidity
Probability to switch from state H to L	χ^H	-	0.143
Probability to switch from state L to H	χ^L	0.333	-
Stock expected returns	μ	0.066	0.086
Stock volatility	σ	0.186	0.135
PE expected returns	ν	0.110	0.153
PE fund volatility	ψ	0.380	0.316
Correlation between stocks & PE	ρ_L	0.685	0.649
Intensity of capital call	λ_C	0.340	0.346
Intensity of capital distribution	λ_D	0.136	0.190
Secondary market haircut	α	28%	9%

Table 6: Optimal PE Allocation & Liquidity Frictions with Cycles

This table reports the impact of changing the return parameters of the stock and PE on the cost of commitment quantity risk. The welfare cost is the amount investors are willing to pay to switch from E0 to E2, i.e. the willingness to pay to remove commitment quantity risk. The return premium is the additional return that PE should deliver (ν) in Economy 0 for E2 to be equivalent to E0, i.e. the return premium associated with commitment quantity risk.

					Opt. comm.		Welfare cost		Ret. premium	
	μ^L	μ^H	ν^L	ν^H	Low	High	Low	High	Low	High
no cycles	8%		14%		5.23%		1.25%		1.10%	
calibrated	6.6%	8.6%	11.0%	15.3%	4.33%	4.74%	1.07%	1.11%	1.09%	1.11%
μ fixed	8.0%	8.0%	11.0%	15.3%	3.96%	5.13%	1.15%	1.24%	1.14%	1.16%
μ switched	8.6%	7.7%	11.0%	15.3%	3.92%	5.50%	1.23%	1.35%	1.18%	1.20%
ν switched	6.6%	8.6%	15.3%	13.4%	6.27%	4.83%	1.31%	1.21%	1.15%	1.14%
both switched	8.6%	7.7%	15.3%	13.4%	5.34%	5.13%	1.24%	1.24%	1.09%	1.10%

Table 7: Multiple Funds

This table reports the benefits of increasing the number of funds in the investor's opportunity set. Panel A reports the total optimal PE commitment in Economy 0 (baseline economy; first line), Economy 2 (the investor can update their commitment upon capital call; second line) and Economy 4 (no commitment period; third line), when the investor has access to two private equity funds. The last two columns display the welfare costs and return premiums of Economies 2 and 4 compared to the baseline economy. Panel B reports the optimal PE aggregate commitment and investment, in the steady state, in Economies 0 and 2, as well. The steady state is defined as the state in which the expected change in invested capital is zero: $E[dX_t^\infty] = 0$, where X_t^∞ follows the dynamics given in equation (12). The last two columns display the welfare cost and return premium of commitment-quantity risk. Panel C reports the welfare costs and return premiums of having access to one fund instead of two, and two funds instead of an infinity.

Panel A: Two-fund allocations and costs of commitment-quantity risk

		PE allocation	Welfare cost	Return premium
E0	Baseline economy	8.48%		
E2	Choose quantity on call	11.44%	1.37%	0.79%
E4	No commitment period	11.34%	3.39%	3.82%

Panel B: Infinite-fund allocations and costs of commitment-quantity risk

		PE comm.	PE inv.	Welfare cost	Return premium
E0	Baseline economy	2.24%	19.62%		
E2	Choose quantity on call		28.50%	4.32%	0.74%

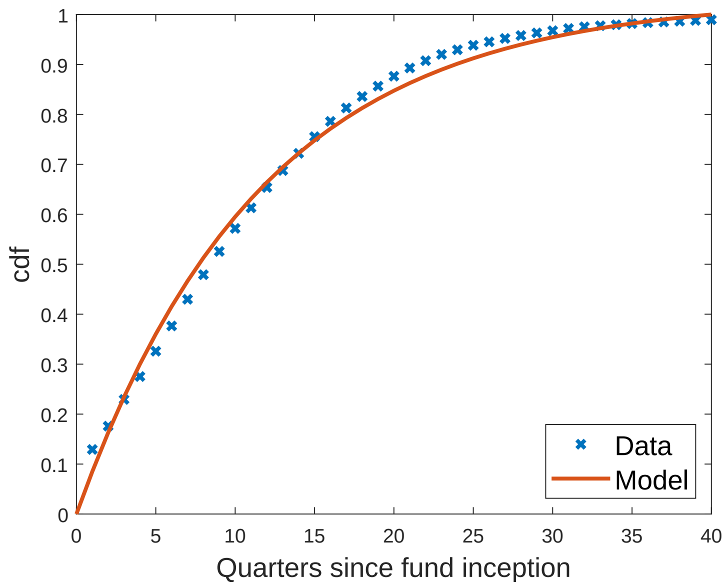
Panel C: Benefits of diversification

		Welfare cost	Return premium
E0	Increasing from 1 fund to 2	0.98%	0.86%
E0	Increasing from 2 funds to an infinity	5.48%	3.02%

Figure 1: Model Validation

This figure illustrates the output of our calibration procedure, described in Section 4.1. In Panel A, the blue marks represent the empirical fraction of called capital after n quarters since capital commitment, for n between 1 and 40. The red curve represents the model-implied fraction of capital calls for the our calibrated λ_C of 0.344. Panel B displays the empirical cumulative distribution function (cdf) of PMEs in our data sample (blue marks), and the model-implied cdf (red line). We consider the first payment (fee) as capital commitment date.

Panel A: Fraction of called capital



Panel B: Distribution of PME

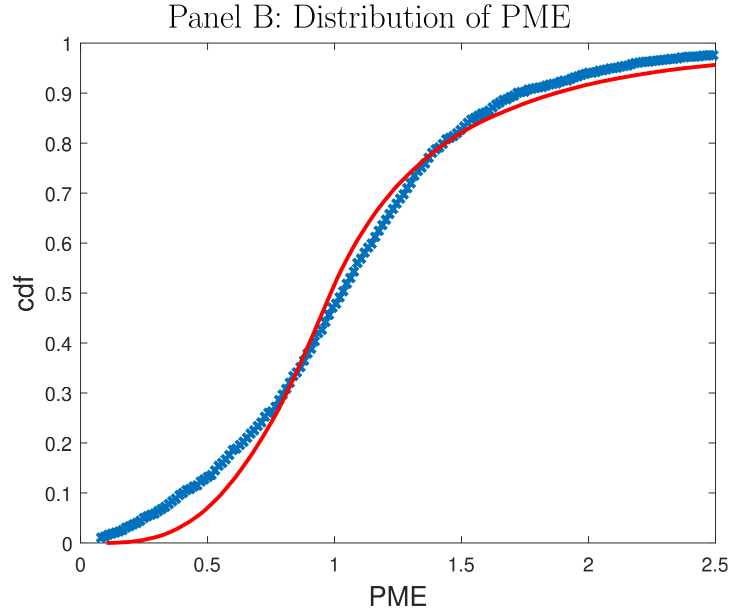


Figure 2: Optimal Allocation and Policies in the Baseline Economy

Panel A represents the value function of the investor during the commitment period (dashed line) and the holding period (plain line) after adjusting for the jump in liquid wealth. Default is represented as a circle, sale on the secondary market as a square. Panel B displays the distribution of the PE allocation during the commitment period (dashed line) and holding period (plain line). Panel C displays the optimal consumption of the investor given their PE allocation. Panel D displays the optimal stock allocation. The vertical dotted line represents the optimal PE commitment ξ^*.

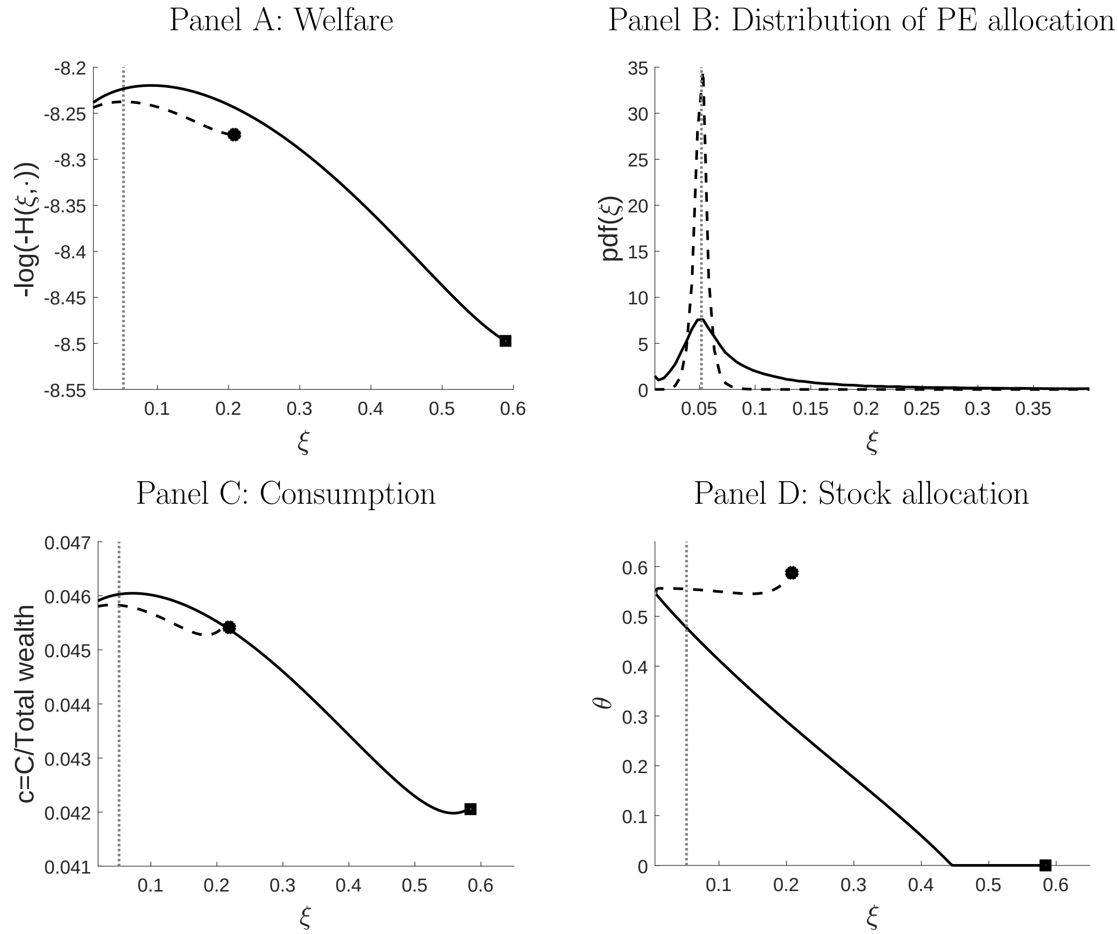


Figure 3: Consumption and Stock Allocation in Economies E0 to E5

This figure represents the optimal consumption rate and stock allocation before capital call (Panels A and C) and after capital call (Panel B), in the different economies. In Economies 2 to 4, the amount of capital invested in private equity is chosen at capital call therefore the relation between consumption (resp. stock allocation) and PE allocation before capital call is not displayed. After capital call, stock allocations overlap in economies E0 to E5.

Economies are summarized below:

Baseline economy E0	Model described in Section 3	All risks on
Economy E1	Deterministic call time	Commitment-timing risk off
Economy E2	Choose quantity when called	Commitment-quantity risk off
Economy E3	Choose quantity when called + deterministic call time	Commitment risk off
Economy E4	No commitment delay	Commitment risk off
Economy E5	Deterministic payout delay	Distribution-timing risk off

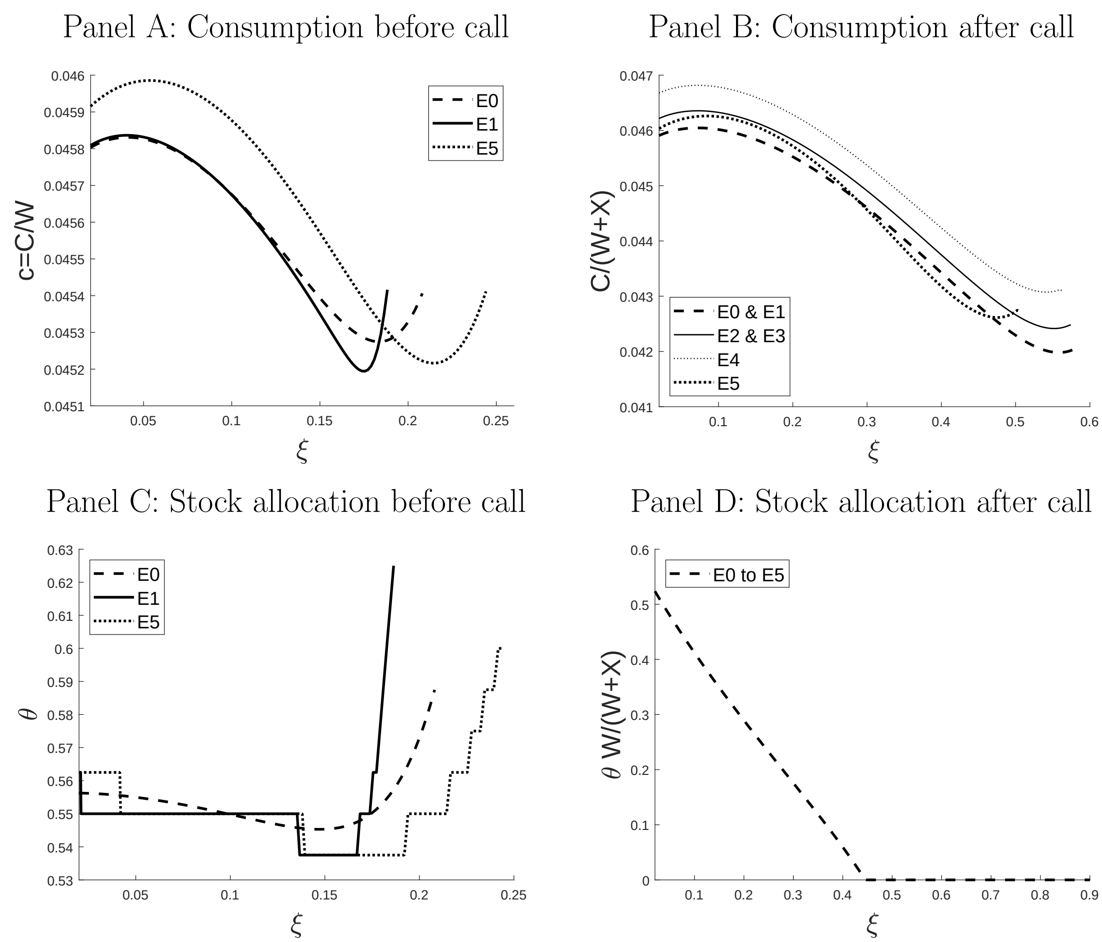
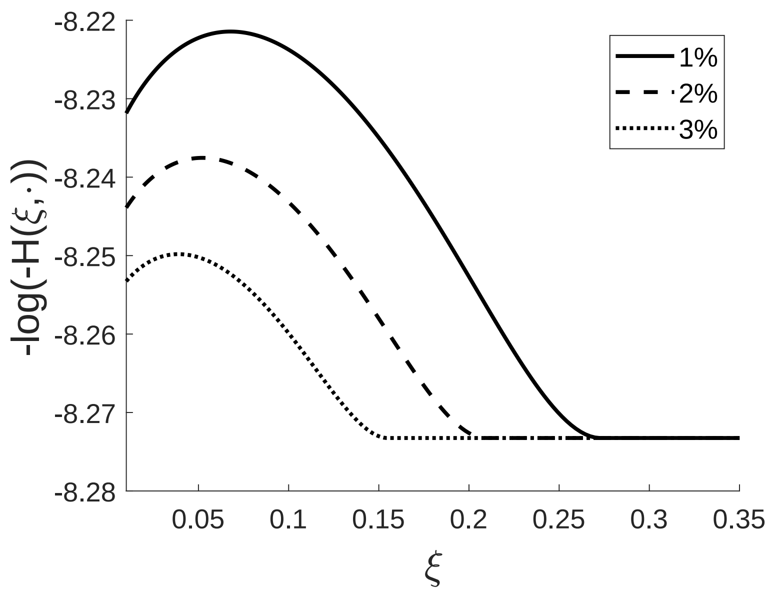


Figure 4: Strategic Default in the Baseline Economy

Panel A displays the value function of the investor during the commitment period, for three levels of fees: 1%, 2% and 3% of the committed amount. The investor strategically defaults when their value function becomes lower than the value function in the Merton problem (no access to PE). Panel B displays the threshold allocation at which the investor strategically default during the commitment period, and at capital call, as a function of the fee level.

Panel A: Value function as a function of the fee



Panel B: Strategic default allocation as a function of the fee

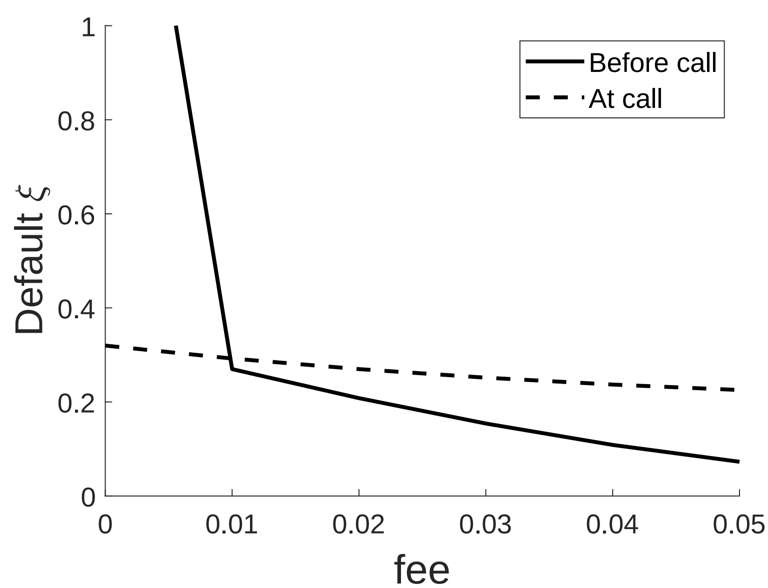
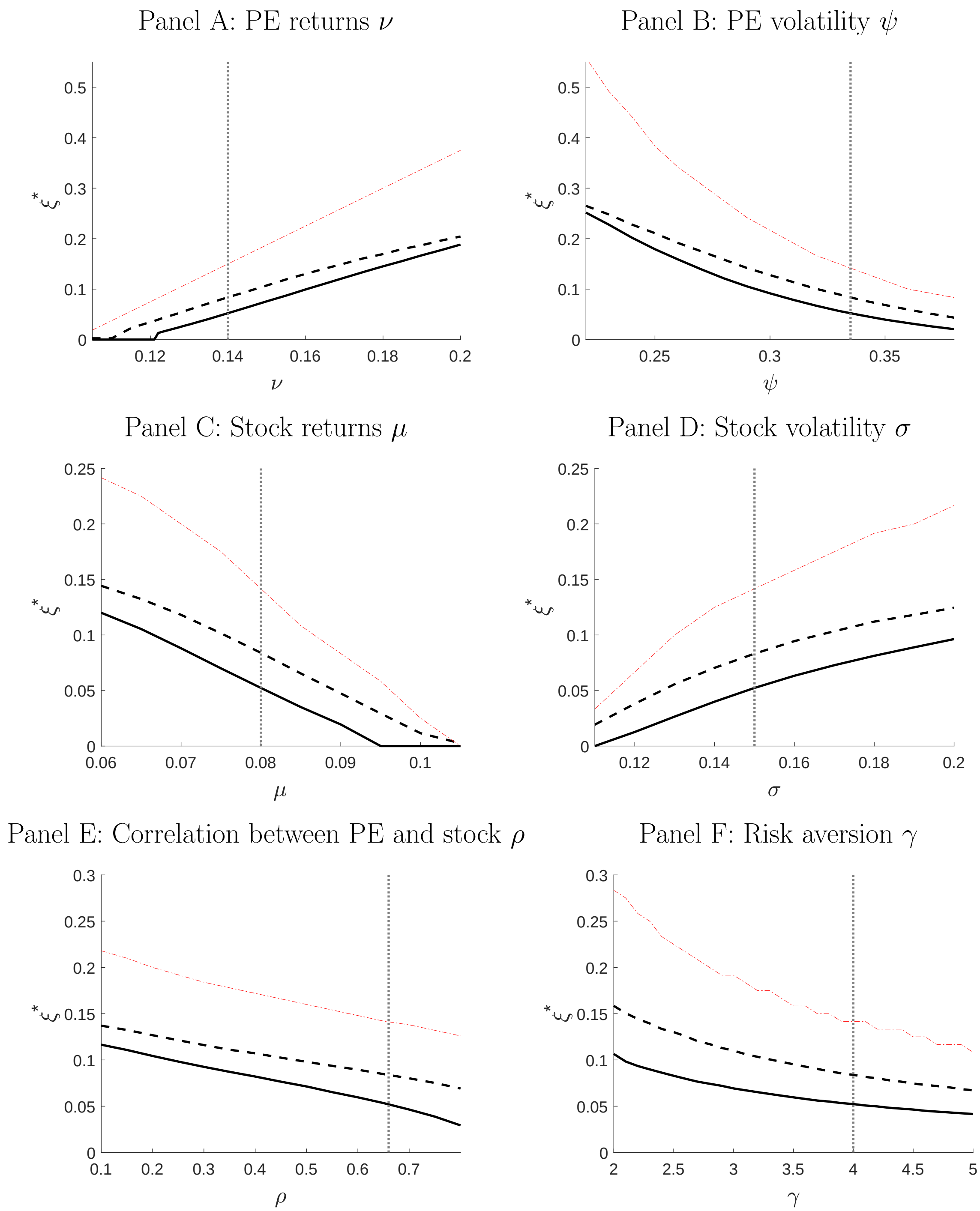
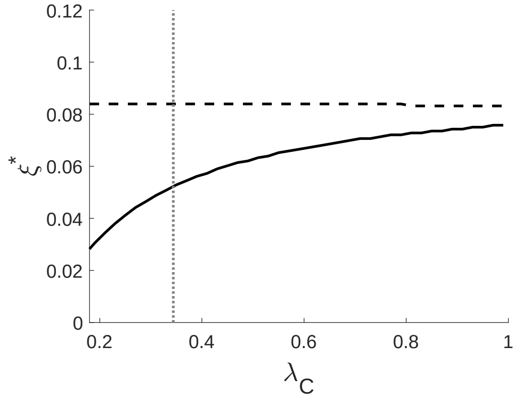


Figure 5: Sensitivity of Optimal PE Allocation to Model Parameters

This figure shows the optimal PE commitment ξ^* as we vary model parameters. Each Panel A to H varies one parameter around the standard calibration value (dotted vertical line), keeping the others fixed at their values in our standard calibration (Table 1). The solid line represents PE commitment in the baseline economy (E0); the dashed line is for the economy without commitment-quantity risk (choose quantity on call, E2); the dot-dash red line is for an economy in which PE is fully liquid (the Merton two-risky-asset model).



Panel G: Intensity of calls λ_C



Panel H: Intensity of distributions λ_D

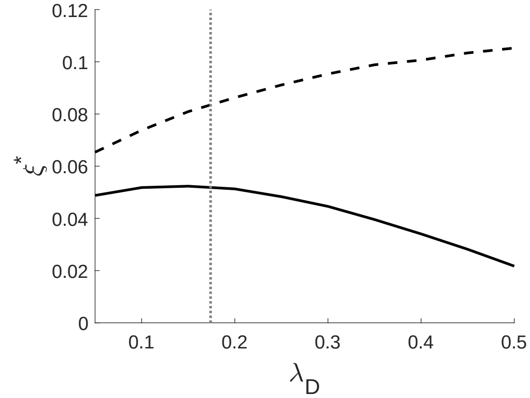
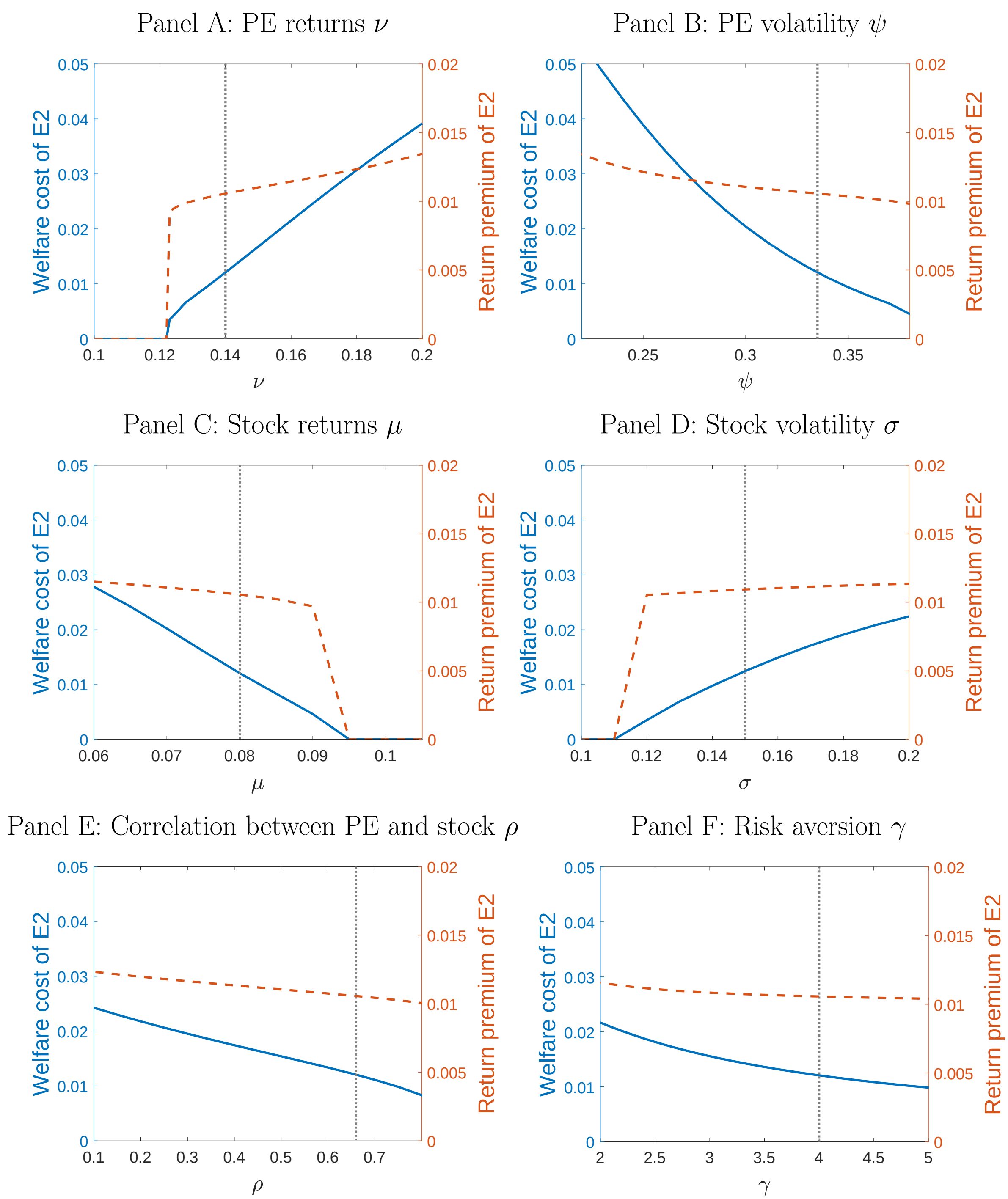
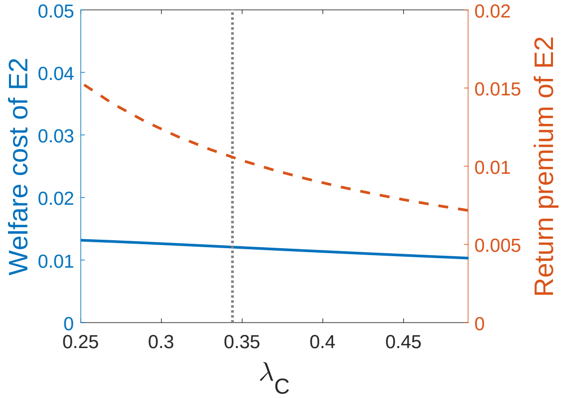


Figure 6: Sensitivity of Return Premium and Welfare Cost of Commitment-Quantity Risk to Model Parameters

This figure represents the return premium and the welfare cost of commitment-quantity risk, as functions of the risk-return parameters of the model. We only vary one parameter while keeping the other parameters constant. The fixed parameters are those of our standard calibration (Table 1). The vertical dotted lines represent the value used in the standard calibration for the parameter that we vary.



Panel G: Intensity of calls λ_C



Panel H: Intensity of distributions λ_D

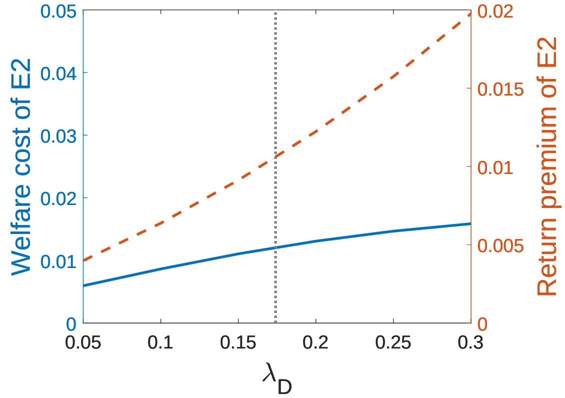
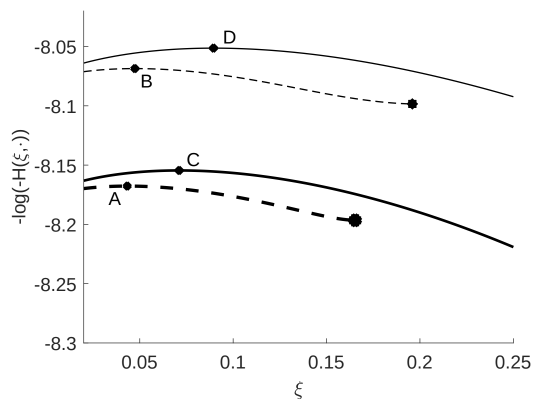


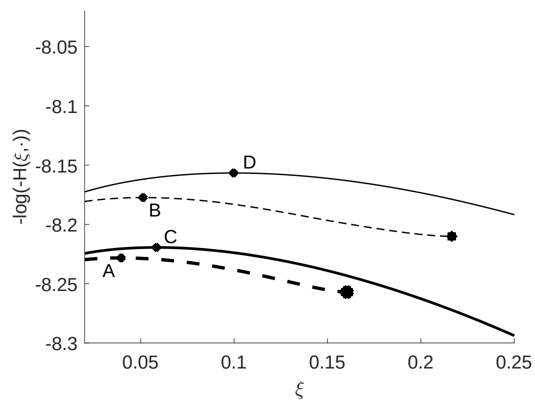
Figure 7: Welfare in the baseline economy with cycles

This figure represents the investor's value function in the low liquidity state during the commitment period (thick dashed line, point A marks the optimal PE commitment) and the holding period (thick plain line, point C marks the optimal PE allocation), and in the high liquidity state (thin dashed line and thin plain line, B and D mark the optimal commitment and allocation). Panel A uses the calibrated cycles parameters (Table 5). In the other panels, we keep all calibrated parameters except those indicated. In Panel B, μ is set to its value in the economy with no cycles ($\mu = 8\%$ in both liquidity states). In Panel C, μ and ν are high (low) in the low (high) liquidity state: $\mu = 8.6\%$ and $\nu = 15.3\%$ ($\mu = 7.7\%$ and $\nu = 13.4\%$). In Panel D, μ is high (low) in the low (high) liquidity state: $\mu = 8.6\%$ ($\mu = 7.7\%$). In Panel E, ν is high (low) in the low (high) liquidity state: $\nu = 15.3\%$ ($\nu = 13.4\%$).

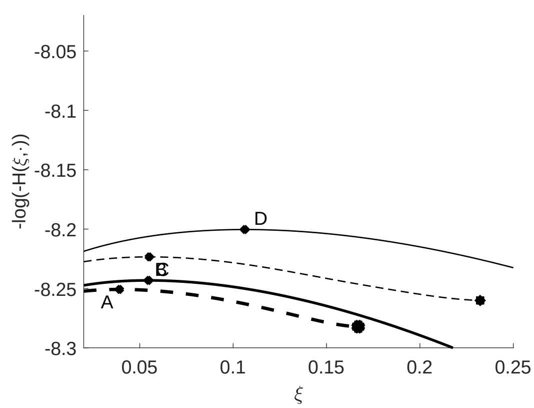
Panel A: Cycles calibration



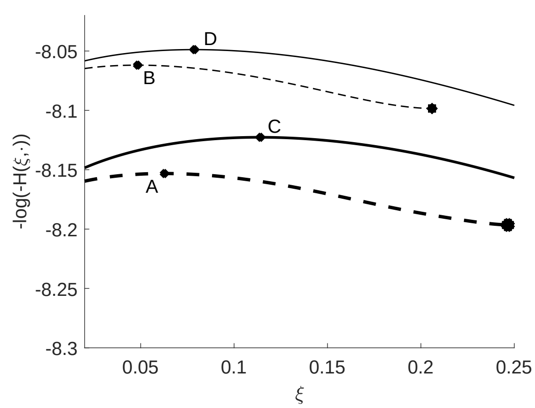
Panel B: $\mu = 0.08$ in both states



Panel C: μ high in low liquidity state



Panel D: ν high in low liquidity state



Panel E: μ and ν high in low liquidity state

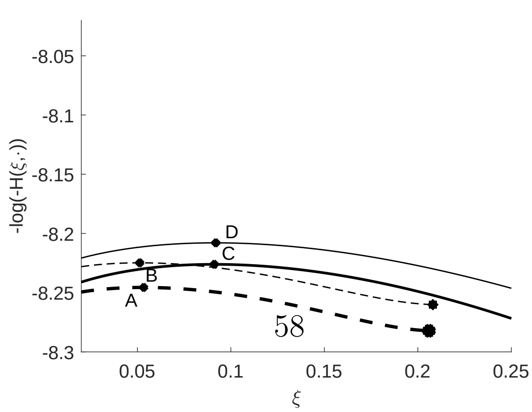


Figure 8: PE Allocation and Welfare in the Two-Fund Model

Panel A represents the optimal commitment of the investor to the second fund, ξ^{2*}, as a function of their ongoing allocation to the first fund ξ^1, if the first fund is (i) in the commitment period (dashed line) and (ii) in the holding period (solid line). The circle indicates strategic default in fund 1. The square indicates fund 1 being sold on the secondary market. The dotted vertical line marks the optimal allocation in the first fund, ξ^{1*}. Panels B and C display the value function of the investor as a function of the allocation in fund 1, (B) when both funds are in the commitment period, (C) when both funds are in the holding period. Thin lines correspond to the case in which the allocation in fund 2 is optimal (B) at inception of the fund and (C) at capital call. Thick lines correspond to the case in which the same fraction of wealth is allocated to both funds. Panel D represents the utility gain when fund 1 calls, if fund 2 is in its commitment period (dashed line) and if it is in its holding period (solid line). The utility gain is defined as the difference in the log value function after adjusting for the jump in liquid wealth at capital call.

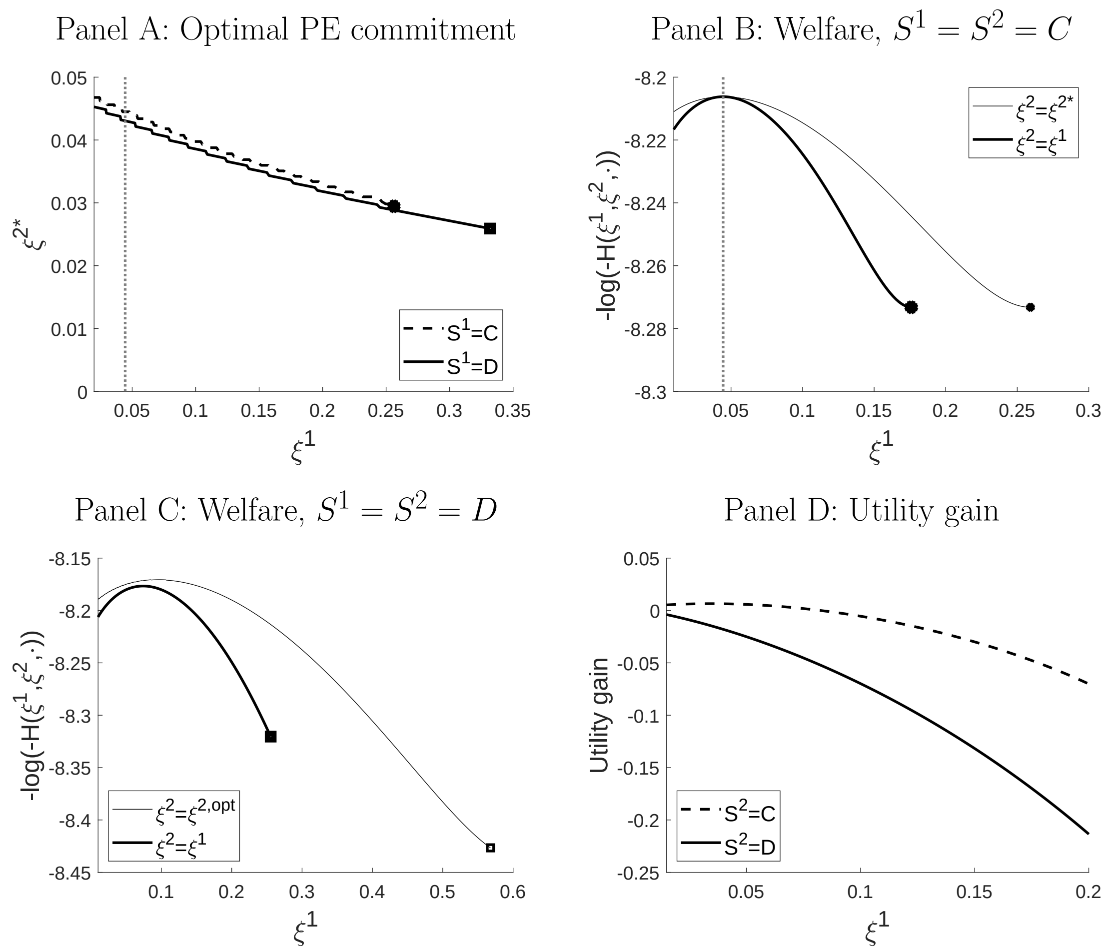


Figure 9: PE Allocation in the Infinite-Fund Case

This figure represents the optimal aggregate commitment of the investor, π^*, as a function of the capital that is already invested, ξ, in the infinite-fund model. The diagonal line represents the committed capital as a function as the invested capital in the steady state, see equation (17). The steady state is defined as the state in which the expected change in invested capital is zero: $E[dY_t^\infty] = 0$, where Y_t^∞ follows the dynamics given in equation (12). The dotted vertical line marks the optimal investment in the steady state.

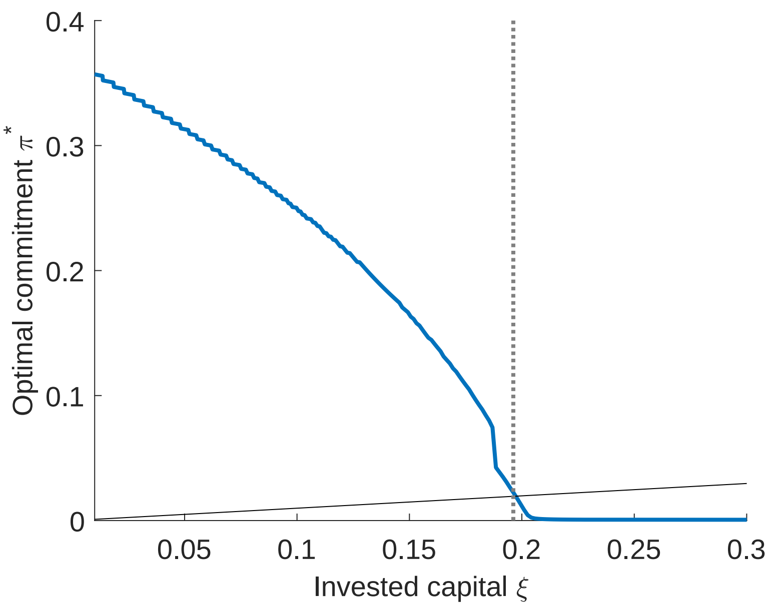
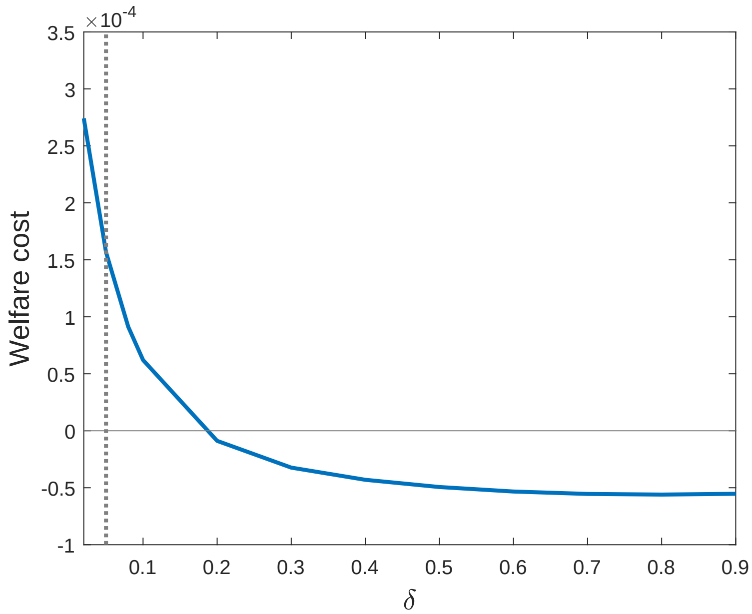


Figure 10: Welfare cost of timing risk

This figure represents the welfare cost of commitment-timing risk as a function of the subjective discount factor δ. We only vary δ while keeping the other parameters constant and as given in our standard calibration (Table 1). In the standard calibration we use $\delta = 0.05$ (vertical dotted line). The horizontal line is at zero.



ONLINE APPENDIX

A Model solutions

A.1 Solution in Economy 0 - 1 fund

The investor solves problem (6). The function F solves the following HJB equation:

$$\begin{aligned}\delta F = \max_{\{\theta, c\}} \Big\{ & U(cW) + \left(r + (\mu - r)\theta - c - \mathbb{1}_{S=C} f \frac{X}{W}\right) W F_W + \frac{1}{2}\theta^2\sigma^2 W^2 F_{WW} \\ & + \nu X F_X \mathbb{1}_{S=D} + \frac{1}{2}\psi^2 X^2 F_{XX}\mathbb{1}_{S=D} + \theta\sigma\psi\rho_L W X F_{WX}\mathbb{1}_{S=D} \\ & + \lambda_C \left(F^{+C} - F\right)\mathbb{1}_{S=C} + \lambda_D \left(F^{+D} - F\right)\mathbb{1}_{S=D}\Big\}, \end{aligned} \tag{22}$$

where $F^{+C}(W, X, S = C) \equiv F(W - X, X, S = D)$ and $F^{+D}(W, X, S = D) = \max_{X'} F(W + X, X', S = C)$. The value function F can be written as (9). We give below the reduced HJB equation solved by H.

Proposition 2 (Economy 0, 1 fund) *The investor's value function can be written as in (9), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(\xi, S = C)$, which exists. During the commitment period, $H(\xi, S = C)$ is characterized by*

$$\begin{aligned} 0 = \max_{c,\theta} \Big[& \frac{c^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S = C) H + A_1(\xi, c, \theta, S = C) H_\xi \\ & + A_2(\xi, c, \theta, S = C) H_{\xi\xi} + \lambda_C (H^{+C} - H) \Big] \end{aligned} \tag{23}$$

where

$$A_0(\xi, c, \theta, S = C) = (1-\gamma)(r + \theta(\mu - r) - c - f\xi) + \frac{\gamma}{2}(\gamma - 1)\sigma^2\theta^2 \tag{24}$$

$$A_1(\xi, c, \theta, S = C) = \left[-(r + \theta(\mu - r) - c - f\xi) + \sigma^2\theta^2\gamma\right]\xi \tag{25}$$

$$A_2(\xi, c, \theta, S = C) = \frac{1}{2}\sigma^2\theta^2\xi^2 \tag{26}$$

$$H^{+C}(\xi) = H(\xi, S = D). \tag{27}$$

The investor strategically defaults at time $t \in (\tau_0, \tau_C]$ if $\xi_t \geq \xi_{def}$, where ξ_{def} is defined by

$H(\xi_{def}, S=C) = H^{Merton}$.[30] *The smooth pasting and super contact conditions are given by:*

$$\lim_{\xi \to \xi_{def}} H_\xi(\xi, S=C) = 0 \tag{28}$$

$$\lim_{\xi \to \xi_{def}} H_{\xi\xi}(\xi, S=C) = 0. \tag{29}$$

During the holding period, $H(\xi, S=D)$ is characterized by

$$\begin{aligned} 0 = \max_{c,\theta} \Big[& \frac{[c(1-\xi)]^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S=D)H + A_1(\xi, c, \theta, S=D)H_\xi \\ & + A_2(\xi, c, \theta, S=D)H_{\xi\xi} + \lambda_D(H^{+D} - H)\Big] \end{aligned} \tag{30}$$

where

$$\begin{aligned} A_0(\xi, c, \theta, S=D) &= (1-\xi)(1-\gamma)(r + \theta(\mu - r) - c) + \xi(1-\gamma)\nu \\ &\quad + \frac{\gamma}{2}(\gamma - 1)\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) \end{aligned} \tag{31}$$

$$\begin{aligned} A_1(\xi, c, \theta, S=D) &= -\xi(1-\xi)(r + \theta(\mu - r) - c) + \xi(1-\xi)\nu \\ &\quad + \gamma\left(-\psi^2\xi^2(1-\xi) + \sigma^2\theta^2(1-\xi)^2\xi - \xi(1-\xi)(1-2\xi)\rho_L\psi\sigma\theta\right) \end{aligned} \tag{32}$$

$$A_2(\xi, c, \theta, S=D) = \frac{1}{2}\xi^2(1-\xi)^2\left(\psi^2 - 2\rho_L\sigma\theta\psi + \sigma^2\theta^2\right) \tag{33}$$

$$H^{+D}(\xi) = \max_{\xi'} H(\xi', S=C). \tag{34}$$

The investor strategically sells their PE stakes on the secondary market at $t \in [\tau_C, \tau_D)$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\xi_{sec\_mkt}, S=D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\xi', S=C)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\xi, S=D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(\xi', S=C) \tag{35}$$

$$\begin{aligned} \lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\xi, S=D) = & (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}}\left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma\right] \\ & \max_{\xi'} H(\xi', S=C). \end{aligned} \tag{36}$$

Proposition 1 is a corollary to Proposition 2. The propositions in this Online Appendix are all straightforward applications of the HJB equation, combined with the usual verification arguments. As such, the proofs are omitted.

[30] We define H^{Merton} from the standard Merton problem with access to the liquid stock and bond but without the private equity sector. In line with Section 3.3, we denote the value function of a Merton investor $F^{Merton} = W^{1-\gamma}H^{Merton}$.

A.2 Solution in Economy 1 - 1 fund

In Economy 1, the time of capital call τ_C is known and equal to $\tau_0 + \frac{1}{\lambda_C}$. The commitment period has duration fixed at $\frac{1}{\lambda_C}$ and we use $t \in [0, 1/\lambda_C]$ during the commitment period. In an abuse of notation, we will write the agent's value function as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, with $S \in \{C, D\}$. However, H is not a function of t during the holding period because the duration is controlled by a Poisson process.

The following proposition characterizes the solution to the investor's problem:

Proposition 3 (Economy 1, 1 fund) *The investor's value function can be written as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, where $H(t, \xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(0, \xi, S = C)$, which exists. Between private equity commitments and capital calls, $H(t, \xi, S = C)$ is characterized by*

$$0 = \max_{c,\theta} \Big[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + H_t + A_0(\xi, c, \theta, S = C)H + A_1(\xi, c, \theta, S = C)H_\xi + A_2(\xi, c, \theta, S = C)H_{\xi\xi} \Big] \tag{37}$$

with A_0, A_1 and A_2 given by equations (24)-(26).

The investor strategically defaults at time $t \in (0, \tau_C]$ if $\xi_t \geq \xi_{def}$, where ξ_{def} is defined by $H(t, \xi_{def}, S = C) = H^{Merton}$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{def}} H_\xi(t, \xi, S = C) = 0 \tag{38}$$

$$\lim_{\xi \to \xi_{def}} H_{\xi\xi}(t, \xi, S = C) = 0. \tag{39}$$

.

At capital call,

$$H(1/\lambda_C, \xi, S = C) = H(\cdot, \xi, S = D). \tag{40}$$

Between capital calls and distributions, H is characterized by

$$0 = \max_{c,\theta} \Big[\frac{[c(1-\xi)]^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S = D)H + A_1(\xi, c, \theta, S = D)H_\xi + A_2(\xi, c, \theta, S = D)H_{\xi\xi} + \lambda_D(H^{+D} - H) \Big] \tag{41}$$

where A_0, A_1 and A_2 are given by equations (31)-(33). H^{+D} is given by

$$H^{+D}(\cdot, \xi) = \max_{\xi'} H(0, \xi', S = C). \tag{42}$$

The investor strategically sells their PE stakes on the secondary market in $S = D$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\cdot, \xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(0, \xi', S = C)$.

The smooth pasting and super contact conditions are given for $S = D$ by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\cdot, \xi, S = D) = \quad (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(0, \xi', S = C) \tag{43}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\cdot, \xi, S = D) = \quad (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma\right] \max_{\xi'} H(0, \xi', S = C). \tag{44}$$

A.3 Solution in Economy 2 - 1 fund

In Economy 2, the investor can update their commitment to PE at time of capital call τ_C. This implies that the agent commits $\xi^* = 0$ to avoid fees and increases the commitment fully at the time of the capital call. We abuse notation by writing H as a function of ξ in $S = C$ even though ξ does not vary. The following proposition characterizes the solution to the investor's problem:

Proposition 4 (Economy 2, 1 fund) *The investor's value function can be written as in (9), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* = 0$. Upon capital call, they select $\xi^* \equiv \arg\max_\xi H(\xi, S = D)$, which exists. During the commitment period, $H(\xi = 0, S = C)$ is given by*

$$0 = \max_{c,\theta}\left[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi = 0, c, \theta, S = C)H + \lambda_C(H^{+C} - H)\right] \tag{45}$$

where $A_0(\xi = 0, c, \theta, S = C)$ is characterized by equation (24) and

$$H^{+C} = \max_{\xi'} H(\xi', S = D). \tag{46}$$

During the holding period, $H(\xi, S = D)$ is characterized by equations (30)-(33). Equation (34) becomes

$$H^{+D} = H(\xi = 0, S = C). \tag{47}$$

The investor strategically sells their PE stakes on the secondary market in $S = D$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H(0, S = C)$. The *smooth pasting and super contact conditions are given by*

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} H(0, S = C) \tag{48}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma\right] H(0, S = C). \tag{49}$$

A.4 Solution in Economy 3 - 1 fund

In Economy 3, the time of capital call τ_C is known and equal to $\tau_0 + \frac{1}{\lambda_C}$. As in the previous two economies, we abuse notation in two ways. The commitment period has duration fixed at $\frac{1}{\lambda_C}$ and we use $t \in [0, 1/\lambda_C]$ during the commitment period. We write the agent's value function as $F(t, W, X, S) = TW^{1-\gamma} H(t, \xi, S)$, with $S \in \{C, D\}$. However, H is not a function of t during the holding period because the duration is controlled by a Poisson process. In addition, the investor can update their commitment to PE at time of capital call τ_C. This implies that the agent commits $\xi^* = 0$ to avoid fees and increases the commitment fully at the time of the capital call. We write H as a function of ξ in $S = C$ even though ξ does not vary.

The following proposition characterizes the solution to the investor's problem:

Proposition 5 (Economy 3, 1 fund) *The investor's value function can be written as* $F(t, W, X, S) = TW^{1-\gamma} H(t, \xi, S)$*, where* $H(t, \xi, S)$ *exists and is finite, continuous, and concave for* $\xi \in [0, 1)$*. Whenever the investor can commit capital, they select* $\xi^* = 0$*. Between private equity commitments and capital calls,* $H(t, \xi = 0, S = C)$ *is characterized by*

$$0 = \max_{c,\theta} \left[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + H_t + A_0(c, \theta, S = C) H\right] \tag{50}$$

with

$$A_0(\xi, c, \theta, S = C) = (1-\gamma)(r + \theta(\mu - r) - c) + \frac{\gamma}{2}(\gamma - 1)\sigma^2\theta^2 \tag{51}$$

Upon capital call, the investor selects $\xi^* \equiv \arg\max_\xi H(\cdot, \xi, S = D)$*, which exists.*

Between capital calls and distributions, H *is characterized by equation* (41) *with* A_0*,* A_1 *and* A_2 *given by equations* (31)-(33)*. Equation* (34) *becomes*

$$H^{+D}(\cdot, \xi) = H(0, \xi = 0, S = C). \tag{52}$$

The investor strategically sells their PE stakes on the secondary market in $S = D$ *if* $\xi_t = \xi_{sec\_mkt}$*, where* ξ_{sec_mkt} *solves* $H(\cdot, \xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H(0, \xi = 0, S = C)$*.*

The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\cdot, \xi, S = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} H(0, \xi = 0, S = C) \tag{53}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\cdot, \xi, S = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma\right] H(0, \xi = 0, S = C). \tag{54}$$

A.5 Solution in Economy 4 - 1 fund

In Economy 4, the investor immediately invests in PE, there is no commitment period. $S = C$ does not exist, and we have only $S = D$. The following proposition characterizes the solution to the investor's problem:

Proposition 6 (Economy 4, 1 fund) *The investor's value function can be written as in (9), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(\xi, S = D)$, which exists. $H(\xi, S = D)$ is characterized by equations (30)-(33). Equation (34) becomes*

$$H^{+D}(\xi) = \max_{\xi'} H(\xi', S = D). \tag{55}$$

The investor strategically sells their PE stakes on the secondary market at t if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\xi', S = D)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\xi, S = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(\xi', S = D) \tag{56}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\xi, S = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma\right] \max_{\xi'} H(\xi', S = D). \tag{57}$$

A.6 Solution in Economy 5 - 1 fund

In Economy 5,the payout time τ_D is deterministic and happens at time $\tau_C + \frac{1}{\lambda_D}$. The holding period has duration fixed at $\frac{1}{\lambda_D}$ and we use $t \in [0, 1/\lambda_D]$ during the holding period. We abuse notation by writing the agent's value function as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, with $S \in \{C, D\}$. However, H is not a function of t during the commitment period because the duration is controlled by a Poisson process.

The following proposition characterizes the solution to the investor's problem:

Proposition 7 (Economy 5, 1 fund) *The investor's value function can be written as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, where $H(t, \xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select*

$\xi^* \equiv \arg\max_\xi H(\cdot, \xi, S = C)$, *which exists. Between private equity commitments and capital calls, H is characterized by*

$$0 = \max_{c,\theta} \Big[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S = C) H + A_1(\xi, c, \theta, S = C) H_\xi \\ + A_2(\xi, c, \theta, S = C) H_{\xi\xi} + \lambda_C (H^{+C} - H) \Big] \tag{58}$$

where A_0, A_1 and A_2 are given by equations (24)-(26). H^{+C} is given by

$$H^{+C}(\cdot, \xi) = H(0, \xi, S = D). \tag{59}$$

The investor strategically defaults in $S = C$ if $\xi_t \geq \xi_{def}$, where ξ_{def} is defined by $H(\cdot, \xi_{def}, S = C) = H^{Merton}$. The smooth pasting and super contact conditions are given by equations (28)-(29).

Between capital calls and distributions, $H(t, \xi, S = D)$ is characterized by

$$0 = \max_{c,\theta} \left[\frac{[c(1-\xi)]^{1-\gamma}}{1-\gamma} - \delta H + H_t + A_0(\xi, c, \theta, S = D) H + A_1(\xi, c, \theta, S = D) H_\xi \right. \\ \left. + A_2(\xi, c, \theta, S = D) H_{\xi\xi} \right] \tag{60}$$

where A_0, A_1 and A_2 are given by equations (31)-(33).

The investor strategically sells their PE stakes on the secondary market at $t \in [0, 1/\lambda_D)$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(t, \xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\cdot, \xi', S = C)$. The smooth pasting and super contact conditions are given by equations (35)-(36).

A.7 Solution in Economy 0 - 2 funds

In Economy 0 of the 2-fund problem, the Hamilton-Jacobi-Bellman (HJB) equation is as follows:

$$\begin{aligned} \delta F = \max_{\{\theta, c\}} \Big\{ & U(c) + \left(r + (\mu - r)\theta - c - \mathbb{1}_{S^1=C} f \frac{X^1}{W} - \mathbb{1}_{S^2=C} f \frac{X^2}{W} \right) W F_W \\ & + \frac{1}{2}\theta^2\sigma^2 W^2 F_{WW} + \nu X^1 F_{X^1} + \nu X^2 F_{X^2} + \frac{1}{2}\psi^2 (X^1)^2 F_{X^1X^1} + \frac{1}{2}\psi^2 (X^2)^2 F_{X^2X^2} \\ & + \theta\sigma\psi\rho_L W (X^1 F_{WX^1} + X^2 F_{WX^2}) + \psi^2 \rho_{PE} X^1 X^2 F_{X^1X^2} \\ & + \lambda_C \left[\left(F^{1+C} - F\right) \mathbb{1}_{S^1=C} + \left(F^{2+C} - F\right) \mathbb{1}_{S^2=C} \right] \\ & + \lambda_D \left[\left(F^{1+D} - F\right) \mathbb{1}_{S^1=D} + \left(F^{2+D} - F\right) \mathbb{1}_{S^2=D} \right] \Big\} \end{aligned} \tag{61}$$

with

$$F^{1+C}(W,X^1,X^2,S^1=C,S^2)=F(W-X^1,X^1,X^2,S^1=D,S^2),$$
$$F^{2+C}(W,X^1,X^2,S^1,S^2=C)=F(W-X^2,X^1,X^2,S^1,S^2=D),$$
$$F^{1+D}(W,X^1,X^2,S^1=D,S^2)=\max_{X^1}F(W+X^1,X^1,X^2,S^1=C,S^2),$$
$$F^{2+D}(W,X^1,X^2,S^1,S^2=D)=\max_{X^2}F(W+X^2,X^1,X^2,S^1,S^2=C).$$

Similarly to the one-fund case, the value function can be decomposed as follows:

$$F(W,X^1,X^2,S^1,S^2)=TW^{1-\gamma}H\left(\xi^1,\xi^2,S^1,S^2\right), \tag{62}$$

with $TW\equiv W+X^1\mathbb{1}_{S^1=D}+X^2\mathbb{1}_{S^2=D}$. We give below the reduced HJB equation solved by H.

Proposition 8 (Economy 0, 2 funds) *The investor's value function can be written as in (62), where $H(\xi^1,\xi^2,S^1,S^2)$ exists and is finite, continuous, and concave for $(\xi^1,\xi^2)\in[0,1)^2$. At time 0, the investor selects $(\xi^1,\xi^2)^*\equiv\arg\max_{\xi^1,\xi^2}H(\xi^1,\xi^2,S^1=C,S^2=C)$, which exists. Whenever the investor can commit capital to fund $i\in\{1,2\}$, she selects $\xi^{i*}\equiv\arg\max_{\xi^i}H(\xi^i,\xi^{\neq i},S^i=C,S^{\neq i})$, which exists.*

When both funds are in their commitment periods, $H(\xi^1,\xi^2,S^1=C,S^2=C)$ is characterized by

$$\begin{aligned}0=\max_{c,\theta}\Big[&\frac{c^{1-\gamma}}{1-\gamma}-\delta H+A_0(c,\theta,S^1=C,S^2=C)H+A_1(\xi^1,\xi^2,c,\theta,S^1=C,S^2=C)H_{\xi^1}\\&+A_2(\xi^1,c,\theta,S^1=C,S^2=C)H_{\xi^1\xi^1}+A_3(\xi^1,\xi^2,c,\theta,S^1=C,S^2=C)H_{\xi^2}\\&+A_4(\xi^2,c,\theta,S^1=C,S^2=C)H_{\xi^2\xi^2}+A_5(\xi^2,c,\theta,S^1=C,S^2=C)H_{\xi^1\xi^2}\\&+\lambda_C\left(H^{1+C}+H^{2+C}-2H\right)\Big]\end{aligned} \tag{63}$$

where

$$\begin{aligned}A_0(\xi^1,\xi^2,c,\theta,S^1=C,S^2=C)=&\ (1-\gamma)(r+\theta(\mu-r)-c-f\xi^1-f\xi^2)+\frac{\gamma}{2}(\gamma-1)\sigma^2\theta^2\\A_1(\xi^1,\xi^2,c,\theta,S^1=C,S^2=C)=&\ \left[-(r+\theta(\mu-r)-c-f\xi^1-f\xi^2)+\sigma^2\theta^2\gamma\right]\xi^1\\A_2(\xi^1,c,\theta,S^1=C,S^2=C)=&\ \frac{1}{2}\sigma^2\theta^2(\xi^1)^2\\A_3(\xi^1,\xi^2,c,\theta,S^1=C,S^2=C)=&\ \left[-(r+\theta(\mu-r)-c-f\xi^1-f\xi^2)+\sigma^2\theta^2\gamma\right]\xi^2\\A_4(\xi^2,c,\theta,S^1=C,S^2=C)=&\ \frac{1}{2}\sigma^2\theta^2(\xi^2)^2\\A_5(\xi^1,\xi^2,c,\theta,S^1=C,S^2=C)=&\ \sigma^2\theta^2\xi^1\xi^2\\H^{1+C}(\xi^1,\xi^2,S^1=C,S^2=C)=&\ H(\xi^1,\xi^2,S^1=D,S^2=C)\\H^{2+C}(\xi^1,\xi^2,S^1=C,S^2=C)=&\ H(\xi^1,\xi^2,S^1=C,S^2=D).\end{aligned}$$

The investor strategically defaults at time t if $\xi^1_t \geq \xi^1_{def}(\xi^2_t)$ or $\xi^2_t \geq \xi^2_{def}(\xi^1_t)$, where $\xi^1_{def}(\xi^2)$ solves $H(\xi^1_{def}, \xi^2, S^1 = C, S^2 = C) = H^{Merton}$ and $\xi^2_{def}(\xi^1)$ solves $H(\xi^1, \xi^2_{def}, S^1 = C, S^2 = C) = H^{Merton}$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi^1 \to \xi^1_{def}} H_{\xi^1}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0 \tag{64}$$

$$\lim_{\xi^2 \to \xi^2_{def}} H_{\xi^2}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0 \tag{65}$$

$$\lim_{\xi^1 \to \xi^1_{def}} H_{\xi^1\xi^1}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0. \tag{66}$$

$$\lim_{\xi^2 \to \xi^2_{def}} H_{\xi^2\xi^2}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0. \tag{67}$$

When one fund is in its commitment period and the other is in its holding period (w.l.o.g., assume that $S^1 = C$ and $S^2 = D$), $H(\xi^1, \xi^2, S^1 = C, S^2 = D)$ is characterized by

$$\begin{aligned} 0 = \max_{c,\theta} \Big[& \frac{[c(1-\xi^2)]^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) H \\ & + A_1(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) H_{\xi^1} + A_2(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) H_{\xi^1\xi^1} \\ & + A_3(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) H_{\xi^2} + A_4(\xi^2, c, \theta, S^1 = C, S^2 = D) H_{\xi^2\xi^2} \\ & + A_5(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) H_{\xi^1\xi^2} + \lambda_C \left(H^{1+C} - H\right) + \lambda_D \left(H^{2+D} - H\right) \Big] \end{aligned} \tag{68}$$

where

$$\begin{aligned} & A_0(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) = \ (1-\xi^2)(1-\gamma)(r + \theta(\mu - r) - c - f\xi^1) + \xi^2(1-\gamma)\nu \\ & \quad + \frac{\gamma}{2}(\gamma - 1)[(\xi^2)^2\psi^2 + \sigma^2\theta^2(1-\xi^2)^2 + 2\xi^2(1-\xi^2)\rho_L\psi\sigma\theta] \\ & A_1(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) = -\xi^1(1-\xi^2)(r + \theta(\mu - r) - c - f\xi^1) - \nu\xi^1\xi^2 \\ & \quad + \gamma\xi^1 \left[\sigma^2\theta^2(1-\xi^2)^2 + \xi^2(1-\xi^2)2\rho_L\psi\sigma\theta + \psi^2(\xi_2)^2\right] \\ & A_2(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) = \ \frac{1}{2}(\xi^1)^2 \left[2\rho_L\sigma\theta\psi(\xi^2)^2(1-\xi^2)^2 + \sigma^2\theta^2(1-\xi^2)^2 + \psi^2(\xi^2)^2\right] \\ & A_3(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) = \ [-(r + \theta(\mu - r) - c - f\xi^1) + \sigma^2\theta^2\gamma(1-\xi^2) - \psi\gamma\xi^2 \\ & \quad - \theta\sigma\psi\rho_L\gamma(1-2\xi^2) + \nu\xi^2] + \xi^2(1-\xi^2) \\ & A_4(\xi^2, c, \theta, S^1 = C, S^2 = D) = \ \frac{1}{2}(\xi^2)^2(1-\xi^2)2[\sigma^2\theta^2 + \psi^2 - 2\theta\sigma\psi\rho_L] \\ & A_5(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = D) = \ \xi^1\xi^2(1-\xi^2)[\theta^2\sigma^2(1-\xi^2) - \psi^2(\xi^2) - \theta\sigma\psi\rho_L(1-2\xi^2)] \\ & H^{1+C}(\xi^1, \xi^2, S^1 = C, S^2 = D) = H(\xi^1, \xi^2, S^1 = D, S^2 = D) \\ & H^{2+D}(\xi^1, \xi^2, S^1 = C, S^2 = D) = \max_{\xi^2} H(\xi^1, \xi^2, S^1 = C, S^2 = C). \end{aligned}$$

The investor strategically defaults at time t if $\xi^1_t \geq \xi^1_{def}(\xi^2_t)$, where $\xi^1_{def}(\xi^2)$ solves $H(\xi^1_{def}, \xi^2, S^1 = C, S^2 = C) = H^{Merton}$. The smooth pasting and super contact conditions are given by equa-

tions (64) *and* (66).

The investor strategically sells their PE stakes on the secondary market at t *if* $\xi_t^2 = \xi^2_{sec\_mkt}$, *where* $\xi^2_{sec\_mkt}$ *solves* $H(\xi^1, \xi^2_{sec\_mkt}, S^1 = C, S^2 = D) = \left(\frac{1+\alpha\xi^2_{sec\_mkt}}{1+\xi^2_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\xi^1, \xi', S^1 = C, S^2 = C)$. *The smooth pasting and super contact conditions are given by*

$$\lim_{\xi^2 \to \xi^2_{sec\_mkt}} H_{\xi^2}(\xi^1, \xi^2, S^1 = C, S^2 = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^2_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(\xi^1, \xi', S^1 = C, S^2 = C) \tag{69}$$

$$\lim_{\xi^2 \to \xi^2_{sec\_mkt}} H_{\xi^2\xi^2}(\xi^1, \xi^2, S^1 = C, S^2 = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^2_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi^2_{sec\_mkt})}{1+\alpha\xi^2_{sec\_mkt}} - 2 + \gamma\right] \max_{\xi'} H(\xi^1, \xi', S^1 = C, S^2 = C). \tag{70}$$

Finally, when both funds are in their holding period, $H(\xi^1, \xi^2, S^1 = D, S^2 = D)$ *is characterized by*

$$\begin{aligned} 0 = \max_{c,\theta}\Big[& \frac{[c(1-\xi^1-\xi^2)]^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi^1, \xi^2, c, \theta, S^1 = D, S^2 = D)H \\ & + A_1(\xi^1, \xi^2, c, \theta, S^1 = D, S^2 = D)H_{\xi^1} + A_2(\xi^1, \xi^2, c, \theta, S^1 = D, S^2 = D)H_{\xi^1\xi^1} \\ & + A_3(\xi^1, \xi^2, c, \theta, S^1 = D, S^2 = D)H_{\xi^2} + A_4(\xi^1, \xi^2, c, \theta, S^1 = D, S^2 = D)H_{\xi^2\xi^2} \\ & + A_5(\xi^1, \xi^2, c, \theta, S^1 = D, S^2 = D)H_{\xi^1\xi^2} + \lambda_D\left(H^{1+D} + H^{2+D} - 2H\right)\Big] \end{aligned} \tag{71}$$

where

$$A_0(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D) = (1-\xi^1-\xi^2)(1-\gamma)(r+\theta(\mu-r)-c)+(\xi^1+\xi^2)(1-\gamma)\nu$$
$$+\frac{\gamma}{2}(\gamma-1)[((\xi^1)^2+(\xi^2)^2)\psi^2+\sigma^2\theta^2(1-\xi^1-\xi^2)^2+2\xi^1\xi^2\rho_{PE}\psi^2+2(1-\xi^1-\xi^2)\rho_L\psi\sigma\theta(\xi^1+\xi^2)]$$
$$A_1(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D) = -\xi^1(1-\xi^1-\xi^2)(r+\theta(\mu-r)-c)+\nu\xi^1(1-\xi^1-\xi^2)$$
$$+\gamma\xi^1\sigma^2\theta^2(1-\xi^1-\xi^2)^2+\psi^2(\xi^1)^2(1-\gamma)(1-\xi^1)+\psi^2(\xi^2)^2\gamma\xi^1-\psi^2\rho_{PE}\xi^1\xi^2\gamma(1-2\xi^1)$$
$$+\rho_L\psi\sigma\theta(1-\xi^1-\xi^2)\xi^1[(2-\gamma)(1-\xi^1)+\gamma\xi^2]$$
$$A_2(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D) = \frac{1}{2}(\xi^1)^2\left[2\rho_L\sigma\theta\psi(1-\xi^1+\xi^2)(1-\xi^1-\xi^2)+\sigma^2\theta^2(1-\xi^1-\xi^2)^2\right.$$
$$\left.+\psi^2[(\xi^2)^2-(1-\xi^1)^2-2\rho_{PE}\xi^2(1-\xi^1)]\right]$$
$$A_3(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D) = \left[-(r+\theta(\mu-r)-c)+\sigma^2\theta^2\gamma\xi^2(1-\xi^1-\xi^2)-\theta\sigma\psi\rho_L\gamma(1-\xi^1-\xi^2)+\nu\right]$$
$$A_4(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D) = \frac{1}{2}(\xi^2)^2(1-\xi^1-\xi^2)^2\sigma^2\theta^2+\frac{1}{2}(\xi^2)^2\psi^2[(\xi^1)^2+(1-\xi^2)^2$$
$$-2\rho_{PE}\xi^1(1-\xi^2)]-\theta\sigma\psi\rho_L(1-\xi^1-\xi^2)^2(\xi^2)^2$$
$$A_5(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D) = \xi^1\xi^2(1-\xi^1-\xi^2)[\theta^2\sigma^2(1-\xi^1-\xi^2)-\psi^2(\xi^1)(1-\xi^2)$$
$$+\theta\sigma\psi\rho_L2\xi^2]+\psi^2\rho_{PE}\xi^1\xi^2[(1-\xi^1)(1-\xi^2)+\xi^1\xi^2]$$
$$H^{1+D}(\xi^1,\xi^2,S^1=D,S^2=D)=\max_{\xi^1}H(\xi^1,\xi^2,S^1=C,S^2=D)$$
$$H^{2+D}(\xi^1,\xi^2,S^1=D,S^2=D)=\max_{\xi^2}H(\xi^1,\xi^2,S^1=D,S^2=C).$$

The investor strategically sells their stakes in fund 1 on the secondary market at t if $\xi^1_t = \xi^1_{sec\_mkt}$, where $\xi^1_{sec\_mkt}$ solves $H(\xi^1_{sec\_mkt},\xi^2,S^1=C,S^2=D)=\left(\frac{1+\alpha\xi^1_{sec\_mkt}+\xi^2}{1+\xi^1_{sec\_mkt}+\xi^2}\right)^{1-\gamma}\max_{\xi'}H(\xi',\xi^2,S^1=C,S^2=D)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi^1\to\xi^1_{sec\_mkt}} H_{\xi^1}(\xi^1,\xi^2,S^1=D,S^2=D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^1_{sec\_mkt}+\xi^2)^{-\gamma}}{(1+\xi^1_{sec\_mkt}+\xi^2)^{2-\gamma}}\max_{\xi'}H(\xi',\xi^2,S^1=C,S^2=D) \quad (72)$$

$$\lim_{\xi^1\to\xi^1_{sec\_mkt}} H_{\xi^1\xi^1}(\xi^1,\xi^2,S^1=D,S^2=D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^1_{sec\_mkt}+\xi^2)^{-\gamma}}{(1+\xi^1_{sec\_mkt}+\xi^2)^{3-3\gamma}}\left[\frac{-\gamma\alpha(1+\xi^1_{sec\_mkt}+\xi^2)}{1+\alpha\xi^1_{sec\_mkt}+\xi^2}-2+\gamma\right]\max_{\xi'}H(\xi',\xi^2,S^1=C,S^2=D). \quad (73)$$

The investor strategically sells their stakes in fund 2 on the secondary market at t if $\xi^2_t = \xi^2_{sec\_mkt}$, where $\xi^2_{sec\_mkt}$ solves $H(\xi^1,\xi^2_{sec\_mkt},S^1=C,S^2=D)=\left(\frac{1+\xi^1+\alpha\xi^2_{sec\_mkt}}{1+\xi^1+\xi^2_{sec\_mkt}}\right)^{1-\gamma}\max_{\xi'}H(\xi^1,\xi',S^1=D,S^2=C)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi^2 \to \xi^2_{sec\_mkt}} H_{\xi^2}(\xi^1, \xi^2, S^1 = D, S^2 = D) =$$

$$(1-\gamma)(\alpha-1)\frac{(1+\xi^1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^1+\xi^2_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(\xi^1, \xi', S^1 = D, S^2 = C) \quad (74)$$

$$\lim_{\xi^2 \to \xi^2_{sec\_mkt}} H_{\xi^2\xi^2}(\xi^1, \xi^2, S^1 = D, S^2 = D) =$$

$$(1-\gamma)(\alpha-1)\frac{(1+\xi^1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^1+\xi^2_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi^1+\xi^2_{sec\_mkt})}{1+\xi^1+\alpha\xi^2_{sec\_mkt}} - 2 + \gamma\right]$$

$$\max_{\xi'} H(\xi^1, \xi', S^1 = D, S^2 = C). \quad (75)$$

In Economies 2 and 4, the solution to the two-fund problem is derived as in the one-fund problem, see Sections A.3 and A.5.

A.8 Solution in Economy 0 - Infinity of funds

The investor solves problem (6). The function F can be written as (16). We give below the reduced HJB equation solved by H^∞.

We give below the reduced HJB equation solved by H^∞.

Proposition 9 (Economy 0, infinity of funds) *The investor's value function can be written as in (16), where $H^\infty(\pi,\xi)$ exists and is finite, continuous, and concave for $\xi \in [0,1)$ and $\pi \in [0, 1-\xi)$.*

The investor chooses dJ_t such that $\frac{dJ_t}{W_t+Y_t^\infty} = \max(0, \pi^*(\xi_t) - \pi_t)$, where $\pi^*(\xi)$ is characterized by the value matching and super-contact conditions $H^\infty_\pi(\pi^*,\xi) = H^\infty_{\pi\pi}(\pi^*,\xi) = 0$.

On $\pi \in [\pi^*(\xi), 1-\xi)$ and $\xi \in [0,1)$, $H^\infty(\pi,\xi)$ is characterized by

$$0 = \max_{c,\theta}\left[\frac{1}{1-\gamma}c^{1-\gamma}(1-\xi)^{1-\gamma} - \delta H + A(\pi,\xi,c,\theta)H + B(\pi,\xi,c,\theta)H_\xi \right. \quad (76)$$

$$\left. + C(\pi,\xi,c,\theta)\frac{1}{2}H_{\xi\xi} + D(\pi,\xi,c,\theta)H_\pi + E(\pi,\xi,c,\theta)\frac{1}{2}H_{\pi\pi} + F(\pi,\xi,c,\theta)H_{\xi\pi}\right]$$

where

$$
\begin{aligned}
A(\pi,\xi,c,\theta) =& (1-\gamma)\left((1-\xi)(r+\theta(\mu-r)-c)+\xi\lambda_D-\pi\lambda_C-\pi f\right)+(1-\gamma)\left(\xi\nu+\pi\lambda_C+\pi f-\xi\lambda_D\right)\\
&+\frac{\gamma}{2}(\gamma-1)\left(\xi^2\psi^2+\sigma^2\theta^2(1-\xi)^2+2\xi(1-\xi)\rho_L\psi\sigma\theta\right)\\
B(\pi,\xi,c,\theta) =& -\xi(1-\xi)(r+\theta(\mu-r)-c)+(1-\xi)\left(\xi\nu+\pi\lambda_C+\pi f-\xi\lambda_D\right)\\
&+\gamma\left(-\psi^2\xi^2(1-\xi)+\sigma^2\theta^2(1-\xi)^2\xi-\xi(1-\xi)(1-2\xi)\rho_L\psi\sigma\theta\right)\\
C(\pi,\xi,c,\theta) =& \xi^2(1-\xi)^2\left(\psi^2-2\rho_L\sigma\theta\psi+\sigma^2\theta^2\right)\\
D(\pi,\xi,c,\theta) =& -\pi\left((1-\xi)(r+\theta(\mu-r)-c)-\xi\lambda_D+\pi\lambda_C+\pi f\right)-\pi\left(\lambda_C+\pi\lambda_C+\pi f-\xi\lambda_D+\xi\nu\right)\\
&+\frac{\gamma}{2}\left(\xi^2\psi^2+\sigma^2\theta^2(1-\xi)^2+2\xi(1-\xi)\rho_L\psi\sigma\theta\right)\\
E(\pi,\xi,c,\theta) =& \pi^2\xi^2\psi^2+2\xi\pi^2(1-\xi)\rho_L\sigma\theta\psi+\pi^2(1-\xi)^2\sigma^2\theta^2\\
F(\pi,\xi,c,\theta) =& -\xi(1-\xi)\pi\left(\xi\psi^2+(1-2\xi)\rho_L\sigma\theta\psi-(1-\xi)\sigma^2\theta^2\right).
\end{aligned}
$$

The investor strategically defaults at time t if $\xi_t \geq \xi_{def}(\pi_t)$, where $\xi_{def}(\pi)$ solves $H^\infty(\pi,\xi_{def}) = H^{Merton}$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi\to\xi_{def}} H^\infty_\xi(\pi,\xi) = 0 \tag{77}$$

$$\lim_{\xi\to\xi_{def}} H^\infty_{\xi\xi}(\pi,\xi) = 0. \tag{78}$$

The investor strategically sells stakes on the secondary market at t if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H^\infty(\pi,\xi_{sec\_mkt}) = \max_{\xi'\leq\xi_{sec\_mkt}}\left(\frac{1+\alpha(\xi_{sec\_mkt}-\xi')+\xi'}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H^\infty(\pi,\xi')$. Let us denote by $\xi'=\xi_m$ the value of ξ' at which the maximum is reached. The smooth pasting and super contact conditions are given by

$$\lim_{\xi\to\xi_{sec\_mkt}} H^\infty_\xi(\pi,\xi) = (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}}(1+\xi_m)H(\pi,\xi_m) \tag{79}$$

$$
\begin{aligned}
\lim_{\xi\to\xi_{sec\_mkt}} H_{\xi\xi}(\pi,\xi) =& (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}}\\
&\left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m}-2+\gamma\right]H(\pi,\xi_m).
\end{aligned} \tag{80}
$$

A.9 Solution in Economy 2 - Infinity of funds

The following proposition characterizes the solution to the investor's problem:

Proposition 10 (Economy 2, infinity of funds) *The investor's value function can be written as $F^\infty(W,X^\infty,Y^\infty) = (W+Y^\infty)^{1-\gamma}H^\infty(\xi)$, where $H^\infty(\xi)$ exists and is finite, continuous, and concave for $\xi\in[0,1)$.*

The investor chooses dJ_t *such that* $\frac{dJ_t}{W_t+Y_t^{\infty}} = \max(0, \xi_t^* - \xi_t)$*, where* ξ^* *is characterized by the value matching and super-contact conditions* $H_\xi^{\infty}(\xi^*) = H_{\xi\xi}^{\infty}(\xi^*) = 0$.

On $\xi \in [0,1)$*,* $H^{\infty}(\xi)$ *is characterized by*

$$0 = \max_{c,\theta}\left[\frac{1}{1-\gamma}c^{1-\gamma}(1-\xi)^{1-\gamma} - \delta H + A(\xi,c,\theta)H + B(\xi,c,\theta)H_\xi + C(\xi,c,\theta)\frac{1}{2}H_{\xi\xi}\right] \tag{81}$$

where

$$\begin{aligned} A(\xi,c,\theta) =& (1-\gamma)\left((1-\xi)(r+\theta(\mu-r)-c)+\xi\lambda_D\right) + (1-\gamma)\left(\xi\nu - \xi\lambda_D\right) \\ &+ \frac{\gamma}{2}(\gamma-1)\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) \\ B(\xi,c,\theta) =& -\xi(1-\xi)(r+\theta(\mu-r)-c) + (1-\xi)\left(\xi\nu-\xi\lambda_D\right) \\ &+ \gamma\left(-\psi^2\xi^2(1-\xi) + \sigma^2\theta^2(1-\xi)^2\xi - \xi(1-\xi)(1-2\xi)\rho_L\psi\sigma\theta\right) \\ C(\xi,c,\theta) =& \xi^2(1-\xi)^2\left(\psi^2 - 2\rho_L\sigma\theta\psi + \sigma^2\theta^2\right). \end{aligned}$$

The investor strategically sells stakes on the secondary market at t if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H^{\infty}(\xi_{sec\_mkt}) = \max_{\xi' \le \xi_{sec\_mkt}} \left(\frac{1+\alpha(\xi_{sec\_mkt}-\xi')+\xi'}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H^{\infty}(\xi')$. Let us denote by $\xi' = \xi_m$ the value of ξ' at which the maximum is reached. The smooth pasting and super contact conditions are given by

$$\lim_{\xi\to\xi_{sec\_mkt}} H_\xi^{\infty}(\xi) = (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}}(1+\xi_m)H(\xi_m) \tag{82}$$

$$\begin{aligned} \lim_{\xi\to\xi_{sec\_mkt}} H_{\xi\xi}(\xi) =& (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \\ &\left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m} - 2 + \gamma\right]H(\xi_m). \end{aligned} \tag{83}$$

B Numerical Methods

After changing variables: $\xi = e^Z$ when $S = C$ and $\xi = \frac{1}{1+e^{-Z}}$ when $S = D$, the value function (9) can be rewritten as

$$F(W,X,S) = W^{1-\gamma}G(Z,S), \tag{84}$$

where $Z = \log(X/W)$, and G solves the following transformed reduced HJB equation:

$$0 = \max_{c,\theta}\left\{\frac{c^{1-\gamma}}{1-\gamma} + B_0(c,\theta,Z)G + B_1(c,\theta,Z)G_Z + B_2(c,\theta)G_{ZZ} + \lambda_S\left(G^{+S} - G\right)\right\}, \tag{85}$$

where $\lambda_S = \lambda_C$ (resp. $\lambda_S = \lambda_D$) when $S = C$ (resp. $S = D$). The functions B_0, B_1 and B_2 and G^{+S} are defined as follows:

$$\begin{cases} B_0(c,\theta,Z) = -\delta + (1-\gamma)\left(r + (\mu - r)\theta - c - fe^Z\right) - \frac{1}{2}\gamma(1-\gamma)\theta^2\sigma^2 \\ B_1(c,\theta,Z) = -\left(r + (\mu - r)\theta - c - fe^Z\right) + \theta^2\sigma^2(\gamma - \frac{1}{2}) + \nu - \frac{1}{2}\psi^2 + \theta\sigma\psi\rho_L(1-\gamma) \\ B_2(c,\theta) = \frac{1}{2}\theta^2\sigma^2 + \frac{1}{2}\psi^2 - \theta\sigma\psi\rho_L \\ G^{+S} = G(Z, S = D) \text{ if } S = C \text{ and } G^{+S} = G(Z, S = C) \text{ if } S = D. \end{cases} \tag{86}$$

When $S = C$, the parameters ρ_L, ψ and ν are set equal to zero.

We use the Markov Chain Approximation (MCA) of Kushner and Dupuis (2001) to evaluate equations (85), for $S = C$ and $S = D$. The method used applies an MCA to continuous-time continuous state stochastic control problems, by renormalizing finite differences forms as proper Markov chain transition probabilities. These transition probabilities arise when deriving finite difference versions of the HJB equation. In our case, the ODEs (PDEs with more than one fund) are linked: $G(Z, S = C)$ depends on $G(Z, S = D)$ and vice versa.

We present below the main steps of the algorithm, for one private equity fund and with power utility.

We define a grid z_i of values taken by Z, ranging from -10 to 5 with intervals of $h = \frac{1}{100}$. For each state S, we discretize the value function G on the grid of z_i: $g_i = G(z_i)$. g_i satisfies an equation of the form:

$$0 = \max_{c,\theta}\left[\tilde{U}(z_i,c,\theta) + B_0(z_i,c,\theta)g_i + B_1^+(z_i,c,\theta)\frac{g_{i+1} - g_i}{h} + B_1^-(z_i,c,\theta)\frac{g_i - g_{i-1}}{h} + \right.$$
$$\left. B_2(z_i,c,\theta)\frac{g_{i+1} + g_{i-1} - 2g_i}{h^2}\right],$$

where $\tilde{U}(z_i,c,\theta)$ includes the utility term as well as the jump term, $B_1^+(z_i,c,\theta)$ (resp. $B_1^-(z_i,c,\theta)$ is the positive (resp. negative) part of $B_1(z_i,c,\theta)$.

The optimal allocation and consumption policy, as functions of f_i, are obtained from the first order condition.

We rearrange the terms to define transition probabilities and the time step of a Markov chain:

$$\begin{cases} p_u(z_i,c,\theta) = \dfrac{\frac{1}{h}B_1^+(z_i,c,\theta) + \frac{1}{h^2}B_2(z_i,c,\theta)}{-B_0(z_i,c,\theta) + \frac{1}{h}B_1^+(z_i,c,\theta) - \frac{1}{h}B_1^-(z_i,c,\theta) + \frac{2}{h^2}B_2(z_i,c,\theta)} \\ p_d(z_i,c,\theta) = \dfrac{-\frac{1}{h}B_1^-(z_i,c,\theta) + \frac{1}{h^2}B_2(z_i,c,\theta)}{-B_0(z_i,c,\theta) + \frac{1}{h}B_1^+(z_i,c,\theta) - \frac{1}{h}B_1^-(z_i,c,\theta) + \frac{2}{h^2}B_2(z_i,c,\theta)} \\ \delta(z_i,c,\theta) = \dfrac{1}{-B_0(z_i,c,\theta) + \frac{1}{h}B_1^+(z_i,c,\theta) - \frac{1}{h}B_1^-(z_i,c,\theta) + \frac{2}{h^2}B_2(z_i,c,\theta)} \end{cases} \tag{87}$$

Conditional on the allocation and consumption policy, the value function g_i is then given by:

$$g_i = p_u(z_i, c, \theta) g_{i+1} + p_d(z_i, c, \theta) g_{i-1} + \tilde{U}(z_i, c, \theta)\delta(z_i, c, \theta) \tag{88}$$

We observe that g_i depends on $\tilde{U}(z_i, c, \theta)$, which itself depends on the jump term $G^{+S} - G$ in (85), hence on g_i.

The algorithm then consists of two steps: i) the estimation of the value function g_i, for the two states, conditional on the liquid asset portfolio policy, the consumption policy, and the previous round's estimate of g_i used in $\tilde{U}(z_i, c, \theta)$ and ii) the estimation of the optimal allocation and consumption policy given the value function. These two steps are iterated until convergence. We deviate from strictly alternating between (i) and (ii) by running (i) multiple times in succession with updated values for g_i in order to speed up convergence. Denote the value function at iteration k by g_i^k. We stop the procedure when the sum of the absolute value of all innovations is below 10^{-6}: $\sum_i \left| g_i^{k+1} - g_i^k \right| < 10^{-6}$.

To include strategic default and the secondary market, g is set in step (i) equal to the maximum of its derived value and the value the agent would obtain by defaulting in state $S = C$, or selling their stake on the secondary market in state $S = D$. Thus, the smooth pasting and super-contact conditions emerge from optimality and are checked; they are not imposed directly.

For two private equity funds, we use grid intervals for z_i of $\frac{1}{50}$ and a total tolerance of 10^{-1}. Because $N = 2$ require two state variables, we are solving linked-PDEs on $751 \times 751 = 564,001$ points, rather than linked-ODEs on 1501 points. Recall that our tolerances of 10^{-6} and 10^{-1} are the sum of absolute deviations, not the average of absolute deviations.

For probability distributions we use Monte Carlo methods: we use $dt = \frac{1}{100}$ ($dt = \frac{1}{50}$ for $N = 2, \infty$) and simulate wealth shocks using $dZ_t = \epsilon \sim N(0, \sqrt{dt})$. We create a single times series lasting for $1,000,000$ years, taking the evolution of wealth from the budget equations and optimal allocations and consumption from the HJB equation.

Our standard parameters are shown in Table 1.

C Sensitivity of Results to Preferences

A drawback of power utility is that it ties the degree of consumption smoothing of the investor across states (governed by the risk aversion) to the degree of smoothing over time (governed by the elasticity of intertemporal substitution, a.k.a. EIS). In practice, the reciprocal relation that is imposed by power utility on the risk aversion and EIS coefficients does not necessarily hold. To disentangle the two effects, recursive preferences were introduced by Epstein and Zin (1989) and Epstein and Zin (1991) in a discrete-time setup, and extended by Duffie and Epstein (1992) in continuous-time. We consider a variation of our baseline problem using recursive utility. The investor's value function is given by:

$$F^{EZ}(W_t, X_t, S_t) = \max_{\{\theta, X, c\}} \mathrm{E}_t \left[\int_t^\infty f\left(C_u, F^{EZ}(W_u, X_u, S_u)\right) du \right], \tag{89}$$

where the time aggregator f is defined as in Duffie and Epstein (1992):

$$f(C, J) = \frac{\delta}{1-\zeta}\left(\frac{C^{1-\zeta}}{((1-\gamma)J)^{\frac{\gamma-\zeta}{1-\gamma}}} - (1-\gamma)J\right). \tag{90}$$

The maximization is subject to the budget constraints (3) and (5). ζ denotes the inverse of the elasticity of intertemporal substitution. When $\zeta = \gamma$, the problem reduces to our baseline problem with CRRA preferences.

If the investor has Epstein-Zin preferences and solves Problem (89), the function H^{EZ} solves analogous ODEs to those describer in Section A.1.

Proposition 11 (Baseline Epstein-Zin, one fund) *The investor's value function can be written as in (9). Between private equity commitments and capital calls, $H^{EZ}(\xi, S = C)$ is characterized by*

$$\begin{aligned} 0 = \max_{c,\theta} \Bigg[& \frac{\delta}{1-\zeta}\left(\frac{c^{1-\zeta}}{((1-\gamma)H^{EZ})^{\frac{\gamma-\zeta}{1-\gamma}}} - (1-\gamma)H^{EZ}\right) + A_0(c, \theta, S = C)H^{EZ} \\ & + A_1(\xi, c, \theta, S = C)H^{EZ}_{\xi} + A_2(\xi, c, \theta, S = C)H^{EZ}_{\xi\xi} + \lambda_C\left(H^{EZ,+C} - H^{EZ}\right)\Bigg] \end{aligned} \tag{91}$$

Between capital calls and distributions, $H(\xi, S = D)$ is characterized by

$$\begin{aligned} 0 = \max_{c,\theta} \Bigg[& \frac{\delta}{1-\zeta}\left(\frac{[c(1-\zeta)]^{1-\zeta}}{((1-\gamma)H^{EZ})^{\frac{\gamma-\zeta}{1-\gamma}}} - (1-\gamma)H^{EZ}\right) + A_0(\xi, c, \theta, S = D)H^{EZ} \\ & + A_1(\xi, c, \theta, S = D)H^{EZ}_{\xi} + A_2(\xi, c, \theta, S = D)\frac{1}{2}H^{EZ}_{\xi\xi} + \lambda_D(\max_{\xi} H^{EZ,+D} - H^{EZ})\Bigg] \end{aligned} \tag{92}$$

The functions A_0, A_1 and A_2 are defined as in Online Appendix A.1.

Figure A1 shows the investor's consumption, as a function of the PE allocation and for two values of the EIS: 2 and 4 (reduces to power utility). The thin dashed (respectively, plain) line represents the consumption policy of an investor with Epstein-Zin utility during the commitment period (resp. holding period), to be compared to the wide dashed (resp. plain) line which represents the consumption policy of an investor with power utility. As a lower EIS captures a weaker desire to smooth consumption over time, consumption is higher when using Epstein-Zin utility with $\zeta = 2$ than when using power utility ($\zeta = \gamma = 4$), for all PE allocations. Interestingly, the choice of the utility function does not impact the allocation in liquid asset, similarly to Ang et al. (2014). It only affects consumption, and hence the allocation in the risk-free asset.

Figure A1: Consumption under Recursive Preferences

This figure represents the optimal consumption of an investor as a function of their PE allocation. The consumption of an investor with power utility is the thicker dashed line during the commitment period and the thicker plain line during the holding period. The consumption of an investor with Epstein-Zin utility and EIS $= 6$ is the thinner upper dashed line during the commitment period and the thinner upper plain line during the holding period. With EIS=2, it is, respectively the thinner lower dashed and plain lines. Default is represented as a circle, sale on the secondary market as a square.

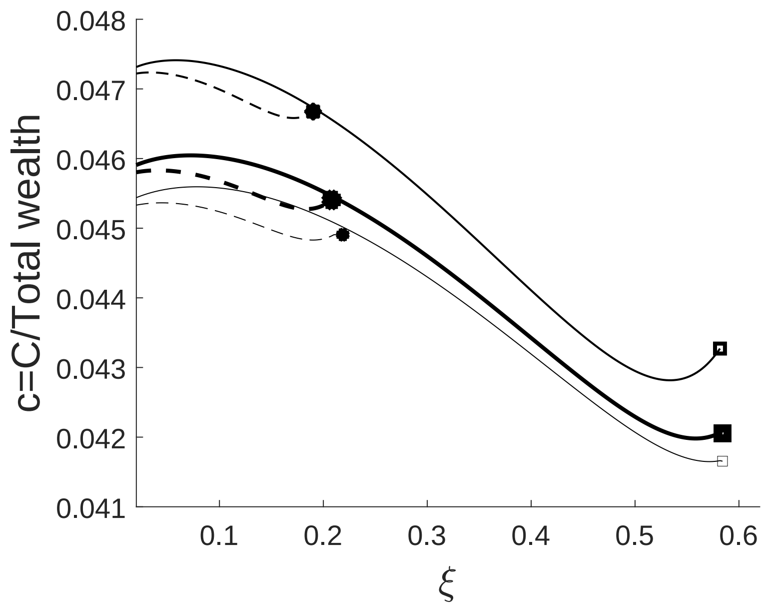


Table A1 reports the optimal PE commitments as well as the welfare costs and return premiums of commitment-quantity risk for investors with $\zeta = 2, 4$ and 6. $\gamma = 4$ throughout. All our findings still hold with Epstein-Zin utility.

Table A1: PE allocations, welfare costs and premiums with recursive utility

This table displays the results of the calibration of our one-fund model described in Section 3 for an investor with Epstein-Zin utility and an EIS varying from 2 (first line) to 6 (last line). The optimal commitment to PE is reported in column 2. The welfare cost and the return premium of commitment-quantity risk (Economy 2 relative to Economy 0) are in columns 3 and 4. Parameters are set as listed in Table 1. The case of EIS=4 corresponds to an investor with power utility and reproduces the results presented in Sections 4.2.

	PE allocation	Welfare cost	Return premium
ζ=2	5.13%	1.17%	1.06%
ζ=4	5.23%	1.21%	1.05%
ζ=6	5.23%	1.22%	1.03%