

A Free and Fair Economy: A Game of Justice and Inclusion*

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Abstract

In this study, we show that principles of market justice guarantee equilibrium existence and efficiency in a free economy. Chief among these principles is that your pay should not depend on your *name*, and a more productive agent should not earn less. We generalize our findings to economies with social justice and inclusion, implemented in progressive taxation and redistribution, and guarantee a basic income to unproductive agents. Our analysis uncovers a new class of strategic form games by incorporating normative principles into non-cooperative game theory. Illustrations include applications to exchange economies, surplus distribution in a firm, and contagion and self-enforcing lockdown in a networked economy.

Keywords: Market justice, Social justice, Inclusion, Ethics, Discrimination, Self-enforcing contracts, Fairness in non-cooperative games, Pure strategy Nash equilibrium, Efficiency.

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1 Introduction

“For Aristotle, justice means giving people what they deserve, giving each person his or her due.”

Sandel [2010, P. 187]

It is generally acknowledged that *justice* is the foundation of a stable, cohesive, and productive society. However, violations of fundamental principles that defined *justice*¹ are highly prevalent in real-life settings. For example, discriminations based on race, gender, culture and several other factors have been widely documented (see, for instance, Reimers [1983], Wright and Ermisch [1991], Sen [1992], Bertrand and Mullainathan [2004], Anderson and Ray [2010], Pongou and Serrano [2013], Goldin et al. [2017], Bapuji et al. [2020], Hyland et al. [2020], Card et al. [2020], Koffi and Wantchekon [Forthcoming], and Advani et al. [Forthcoming, 2021]). These realities raise the fundamental question of how basic principles of justice affect individual incentives, and whether such principles can guarantee the stability and efficiency of contracts among private agents in a free and competitive economy. That the literature has remained silent on this question is a bit surprising, given the long tradition of ethical and normative principles in economic theory and the relevance of these principles to the real world [Sen, 2009, Thomson, 2016]. The main goal of this paper is to address this problem. In our treatment of this question, we incorporate elementary principles of justice and ethics into non-cooperative game theory. In doing so, we uncover a new class of strategic form games with a wide range of applications to classical and more recent economic problems.

We precisely address the following questions:

A: How do *principles of justice* affect the stability of social interactions in a free economy?

B: Under which conditions do *principles of justice* lead to *equilibrium efficiency*?

To formalize these questions, we introduce a model of a *free and fair economy*, where

¹The Merriam-Webster dictionary defines *justice* as “the maintenance or administration of what is just especially by the impartial adjustment of conflicting claims or the assignment of merited rewards or punishments.”

agents freely (and non-cooperatively) choose their inputs, and the surplus resulting from these input choices is shared following four elementary principles of justice, which are:

1. **Anonymity:** Your pay should not depend on your *name*.²
2. **Local efficiency:** No portion of the surplus generated at any profile of input choices should be wasted.
3. **Unproductivity:** An unproductive agent should earn nothing.
4. **Marginality:** A more productive agent should not earn less.

It is generally agreed that these ideals form the core principles of *market* (or *meritocratic*) *justice*, and are of long tradition in economic theory. They have inspired writers like Rousseau [1762], Aristotle [1946], and authors like Rawls [1971], Shapley [1953], Young [1985], Roemer [1998], De Clippel and Serrano [2008], Sen [2009], Sandel [2010], Thomson [2016], and Posner and Weyl [2018], among several others. However, a number of empirical observations have suggested that the real world does not always conform to these elementary principles of justice. For instance, studies have shown that *anonymity* is violated in job hiring [Kraus et al., 2019, Bertrand and Mullainathan, 2004], in wages [Charles and Guryan, 2008, Lang and Manove, 2011], in scholarly publishing [Laband and Piette, 1994, Ellison, 2002, Heckman et al., 2017, Serrano, 2018, Akerlof, 2020, Card et al., 2020], in school admission [Francis and Tannuri-Pianto, 2012, Grbic et al., 2015], in sexual norm enforcement [Pongou and Serrano, 2013], in health care [Balsa and McGuire, 2001, Thornicroft et al., 2007], in household resource allocations [Sen, 1992, Anderson and Ray, 2010], in scholarly citations [Card et al., 2020, Koffi, 2021], and in organizations [Small and Pager, 2020, Koffi and Wantchekon, Forthcoming]. These studies generally show that discrimination based on name, race, gender, culture, religion, and academic affiliation is prevalent in these different contexts. Violations of basic principles of justice therefore raise

²Here, *name* designates any unproductive individual characteristic such as first and last names, skin color, gender, religious or political affiliation, cultural background, etcetera. Anonymity means that a person's pay should not depend on their identity; in other words, given my input choice and that of others, my pay should not vary depending on whether I am called "Emily/Greg" or "Lakisha/Jamal" [Bertrand and Mullainathan, 2004], or depending on whether my skin color is black, white or green, or depending on whether I am a man or a woman.

the fundamental question of how these principles affect individual incentives, the stability of social interactions, and economic efficiency.

We examine these questions through the lens of a model of a free and fair economy. This model is a list $\mathcal{E} = (N, \times_{j \in N} X_j, o, f, \phi, (u_j)_{j \in N})$, where N is a finite set of agents, X_j a finite set of actions (or inputs) available to agent j , $o = (o_j)_{j \in N}$ a reference profile of actions, f a production (or surplus) function (also called technology) that maps each action profile $x \in \times_{j \in N} X_j$ to a measurable output $f(x) \in \mathbb{R}$, ϕ an allocation scheme that distributes any realized surplus $f(x)$ to agents, and u_j the utility function of agent j . The reference point o can be interpreted as an unproduced endowment of goods (or resources) that can be either consumed as such, or may be used in the production process when production opportunities are specified. Agent j 's action set X_j can be interpreted broadly, as we do not impose any particular structure on it other than it being finite. It may be viewed as a capability set [Sen, 2009], or may represent the set of different occupations (or functions) available to agent j based on agent j 's skills, or the set of effort levels that agent j may supply in a production environment. The nature of the set of actions can also be different for each agent. For each input profile x , the allocation scheme ϕ distributes the generated surplus $f(x)$ following the aforementioned principles of *anonymity*, *local efficiency*, *unproductivity*, and *marginality*, and each agent j derives utility from her payoff $u_j(x) = \phi_j(f, x)$.³

To define an equilibrium concept that captures individuals' incentives in a free and fair economy, we first observe that any economy \mathcal{E} induces a corresponding strategic form game $G^{\mathcal{E}} = (N, \times_{j \in N} X_j, (u_j)_{j \in N})$.⁴ Then, a profile of actions $x^* \in \times_{j \in N} X_j$ is said to be an **equilibrium** in the free and fair economy \mathcal{E} if and only if it is a **pure strategy Nash equilibrium** of the game $G^{\mathcal{E}}$.

Theorem 1 shows that the four principles of market justice stated above guarantee the

³The formalization of these principles differ depending on the context. Ours is a generalization of the classical formalization of Shapley [1953] and Young [1985] to our economic environment. Indeed, we show that these four principles uniquely characterize a pay scheme that generalizes the classical Shapley value (Proposition 1). This pay scheme is a multivariate function defined at each input profile x . Also, $u_j(x)$ can be any increasing function of the payoff $\phi_j(f, x)$, and the functional form might be different for each agent.

⁴The class of free and fair economies therefore defines a large class of games that can be characterized as *fair*. Any strategic form game is either fair or unfair, and some unfair games are simply a monotonic transformation of fair games.

existence of an equilibrium. One implication of this finding is that fair rules guarantee the existence of self-enforcing contracts between private agents in a free economy. Moreover, from a purely theoretical viewpoint, the incorporation of normative principles into non-cooperative game theory has led us to identify an interesting class of strategic form games that always have a pure strategy Nash equilibrium in spite of the fact that each player has a *finite* action set.⁵

Although a pure strategy equilibrium always exists in any free and fair economy, this equilibrium may be Pareto-inefficient. We uncover a simple structural condition that guarantees equilibrium Pareto-efficiency. More precisely, Theorem 2 shows that if the technology is *strictly monotonic*, there exists a unique equilibrium, and this equilibrium is Pareto-efficient. One direct implication of Theorem 2 is that fair rules yield production stability and economic efficiency, given that action choices at the unique equilibrium are pure strategies. Quite interestingly, we find that when a monotonic economy fails to satisfy the principles of market justice, even if an equilibrium exists, it may be Pareto-inefficient.⁶ A clear implication of this finding is that in the class of monotonic economies, any allocation scheme that violates the principles of market justice is welfare-inferior to the unique scheme that respects these principles.

Next, we extend our analysis to economies with *social justice*. The principles of market justice imply that unproductive agents (for example, agents with severe disabilities) should earn nothing. In most societies, however, social security benefits ensure that a basic income is allocated to agents who, for certain reasons, cannot produce as much as they would like to (see, for example, David and Duggan [2006], and Hanna and Olken [2018]). To account for this reality, we extend our model to incorporate *social justice* or *inclusion*. Generally, social justice includes solidarity and moral principles that individuals have equal access to social rights and opportunities, and it requires consideration beyond talents and skills since some agents have natural limitations, not allowing them to be productive.

⁵As is well known, a pure strategy Nash equilibrium does not exist in a finite strategic form game in general [Nash, 1951]. A growing literature seeks to identify conditions under which a pure strategy Nash equilibrium exists in a finite game (see, for example, Rosenthal [1973], Monderer and Shapley [1996], Mallick [2011], Carmona and Podczeck [2020], and the references therein). But unlike our paper, this literature has not approached this problem from a normative perspective. We therefore view our analysis as a contribution.

⁶A clear example is the prisoner's dilemma game. Economies that are modeled by such games are monotonic, although their unique equilibrium is Pareto-inefficient.

Social justice is incorporated into our model in the form of progressive taxation and redistribution. At any production choice, a positive fraction $(1 - \alpha)$ of output is taxed and shared equally among all agents, and the remaining fraction (α) is allocated according to the principles of market justice. It is direct that this allocation scheme satisfies the principles of *anonymity* and *local efficiency*, but violates *unproductivity* and *marginality* when $\alpha \neq 1$. Income is redistributed from the high skilled and talented (or more productive agents) to the least well-off. However, the income rank of a free and fair economy (without social justice) is maintained, provided that the entire surplus is not taxed. We generalize each of our results. In particular, Theorem 3 shows that a pure strategy equilibrium always exists regardless of the tax rate. In line with Theorem 2, we also find in Corollary 1 that if the production technology is *strictly monotonic*, there exists a unique equilibrium, and this equilibrium is Pareto-efficient.⁷

We uncover additional results on the efficiency of economies with social justice. In particular, Theorem 4 states that there exists a tax rate threshold above which there exists a pure strategy Nash equilibrium that is Pareto-efficient, even if the economy is not monotonic. Moreover, Theorem 5 shows that one can always change the reference point of any *non-monotonic* free economy with social justice to guarantee the existence of an equilibrium that is Pareto-efficient. Theorem 5 implies that if a free economy is able to choose its reference point, then it can always do so to induce a Pareto-efficient outcome that is self-enforcing.

We develop various applications of our model to classical and more recent economic problems. In particular, we develop applications to exchange economies [Walras, 1954, Arrow and Debreu, 1954, Shapley and Shubik, 1977, Osborne and Rubinstein, 1994], surplus distribution in a firm, and self-enforcing lockdown in a networked economy with contagion. This variety of applications is possible because we impose no particular assumptions on the structure of action sets, and the action set of each agent may be of a different nature.

⁷Following Monderer and Shapley [1996, Theorem 2.8, p. 131], we can use Lemma 1 to demonstrate that any strategic form game derived from a free and fair economy or a free economy with social justice admits a *potential function*. Therefore, the class of our strategic form games constitutes a sub-domain of *potential games*. It follows that we can deduce the existence of a pure strategy Nash equilibrium in Theorems 1 and 3 from Monderer and Shapley [1996, Corollary 2.2, p. 128] for finite potential games. However, in our framework, we provide alternative proofs of existence, additional insights on Pareto-efficiency, and a variety of applications.

We start with applying our theory to a production environment where an owner of the firm (or team leader) uses bonuses as a device to incentivize costly labor supply from rational workers. Our analysis shows that in addition to guaranteeing equilibrium existence, the owner can also achieve production efficiency, provided that the costs of labor supply are not too high. Next, we provide an application to contagion in a networked economy in which rational agents freely form and sever bilateral relationships. Rationality is captured by the concept of pairwise-Nash equilibrium, which refines the Nash equilibrium. Using a contagion index [Pongou and Serrano, 2013], we show how the costs of a pandemic can induce self-enforcing lockdown. Finally, we recast the model of an exchange economy in our framework, and show that our equilibrium is generally different from the Walrasian equilibrium. This difference is in part explained by the fact that the Walrasian model assumes linear pricing, whereas our model is fully non-parametric.

Contributions to the closely related literature. In this study, we propose a model of a free and fair economy, defining a new class of non-cooperative games, and we apply it to a variety of economic environments. We prove that four elementary principles of distributive justice, of long tradition in economic theory, guarantee the existence of a pure strategy Nash equilibrium in finite games. In addition, we show that when an economy violates these principles, a pure strategy equilibrium may not exist. We extend this model to incorporate social justice and inclusion. In this more general model, we also prove several results on equilibrium existence and efficiency.

Our work contributes to several literatures. A closely related study to ours is the work by Pongou and Tondji [2018]. They examine the existence of a pure strategy equilibrium in a non-cooperative production game with Shapley payoffs, and they show that there exists a pure strategy Nash equilibrium when the production technology f is a non-decreasing function. Our setup is more general and it embeds the implications of social justice on economic stability. For instance, our results on the existence of a pure strategy Nash equilibrium in Theorems 1 and 3 require no monotonic property on the production function, f . Our study is also related to studies of group incentives in multi-agent problems under certainty. Holmstrom [1982] explores the effects of moral hazard in individual incentives and efficiency in organizations with and without uncertainty. Like Holmstrom [1982], we consider that in a free economy, any agent has the freedom to choose any action (or input) from their set of strategies, and the combination of actions from agents generates a mea-

surable output. However, unlike Holmstrom [1982], there is no uncertainty in the supply of inputs, and we assume that our allocation scheme follows basic principles of justice. It follows that our scope, analysis and applications are very different. Moreover, Holmstrom [1982] finds an impossibility result in his setup (see, Holmstrom [1982, Theorem 1, p. 326]), but our analysis implies that this result does not extend when we consider principles of market justice in a framework with finite action sets. Moreover, Theorem 2 shows that any free and fair economy which is strictly monotonic admits a unique equilibrium, and this equilibrium is optimal and Pareto-efficient. Our findings therefore underscore the role of justice in shaping individual incentives, stabilizing contracts among private agents, and enhancing welfare.

By incorporating normative principles into non-cooperative game theory, we have introduced a new class of finite strategic form games that always admit a Nash equilibrium in pure strategies. We view our study as contributing to the small but growing literature that seeks to uncover conditions under which a pure strategy Nash equilibrium exists in a non-cooperative game with simultaneous moves. Nash [1951] shows a very prolific result on the existence of equilibrium points in a finite non-cooperative games. Nash [1951] also shows that there always exists at least one pure strategy equilibrium in finite symmetric games. However, Nash [1951] was silent about the existence of pure strategy equilibrium in either finite or infinite non-symmetric strategic form games. Subsequent research has searched for sufficient and necessary conditions for the existence of pure strategy Nash equilibrium in different structure of strategic form games. Early contributions in this respect include, among others, Debreu [1952], Glicksberg [1952], Gale [1953], Schmeidler [1973], Mas-Colell [1984], Khan and Sun [1995], Athey [2001] in continuous games; Rosenthal [1973] in congestion games; Dasgupta and Maskin [1986a], Dasgupta and Maskin [1986b], Reny [1999], Carbonell-Nicolau [2011], Reny [2016], Nessah and Tian [2016] in discontinuous economic games; Monderer and Shapley [1996] in potential games; and Ziad [1999] in fixed-sum games. In these studies, scholars use different concepts of continuity, convexity and appropriate fixed point results along with some structures on utility functions to prove the existence of a pure strategy Nash equilibrium. Other contributions that guarantee the existence of equilibrium in pure strategies for finite games include, among others, Mallick [2011], Carmona and Podczeck [2020], and the references listed therein. We follow a different approach from this literature. Unlike our study, this literature has not approached

the issue of equilibrium existence in a non-cooperative game from a normative angle. We also apply our theory to different economic environments, including surplus distribution in a firm, exchange economies, and self-enforcing lockdown in networked economies facing contagion.

Finally, in addition to the previous point, our work can also be viewed as contributing to the Nash Program [Nash, 1953], which bridges non-cooperative and cooperative game theory. However, we significantly depart from the main approach taken in this literature so far. This approach has generally sought to define a non-cooperative game whose solution coincides with the outcomes of a cooperative solution concept; see Serrano [2021] for a recent survey on this literature. Our approach, on the contrary, follows the opposite direction. It asks if equilibrium can be found in a strategic form game in which payoffs obey natural axioms inspired by cooperative game theory.

The rest of this study is organized as follows. Section 2 introduces the model of a free and fair economy. Section 3 proves the existence of a pure strategy Nash equilibrium in a free and fair economy. Section 4 examines Pareto-efficiency in a free and fair economy. Section 5 extends our model to incorporate social justice and inclusion, and generalizes our results. Section 6 presents some applications of our analysis, and Section 7 concludes. Some proofs are collected in an appendix.

2 A free and fair economy: definition, existence and uniqueness

In this section, we introduce preliminary definitions and the key concepts of the study. We then show that there exists a unique economy that is free and fair.

2.1 A free economy

A free economy is an economy where agents freely choose their actions and derive utility from their pay. It is modeled as a list $\mathcal{E} = (N, \times_{j \in N} X_j, (o_j)_{j \in N}, f, \phi, (u_j)_{j \in N})$. $N = \{1, 2, \dots, n\}$ is a finite set of agents. Each agent j has a finite set of feasible actions X_j . We refer to an action profile $x = (x_j)_{j \in N}$ as an *outcome*, and denote the set $\times_{j \in N} X_j$ of outcomes by X . The *reference outcome* (also called *reference point*) is $o = (o_j)_{j \in N}$;

it can be interpreted as the *inaction* point, where agents do nothing or do not engage in any sort of transactions with other agents. A production (or surplus) function (also called technology) f transforms any choice x to a real number $f(x) \in \mathbb{R}$, with $f(o) = 0$.⁸ We denote by $P(X) = \{g : X \rightarrow \mathbb{R}, \text{ with } g(o) = 0\}$ the set of production functions on X . $\phi : P(X) \times X \rightarrow \mathbb{R}^n$ is a distribution scheme that assigns to each pair (f, x) a payoff vector $\phi(f, x)$. At each input profile x , each agent j derives utility $u_j(x) = \phi_j(f, x)$.⁹

2.2 A free and fair economy

A free and fair economy is a free economy $\mathcal{E} = (N, \times_{j \in N} X_j, (o_j)_{j \in N}, f, \phi, (u_j)_{j \in N})$ in which the surplus distribution scheme ϕ satisfies elementary principles of market justice. These principles, of long tradition in economic theory, are those of *anonymity*, *local efficiency*, *unproductivity*, and *marginality* stated in the Introduction. These principles are naturally interpreted, but their formalization varies depending on the context. A few preliminary definitions and notations will be needed for their formalization in our setting.

Definition 1. Let $x \in X$ a profile of actions. An outcome $x' \in X$ is a *sub-profile* of x if either $x' = x$ or $[x'_i \neq x_i \implies x'_i = o_i]$, for $i \in N$.

For each $x \in X$, we denote by $\Delta(x)$ the set of sub-profiles of x . Given a production function $f \in P(X)$, and an outcome $x \in X$, we define the function f^x as the restriction of f to $\Delta(x)$:

$$f^x : \Delta(x) \rightarrow \mathbb{R}, \text{ such that } f^x(y) = f(y), \text{ for each } y \in \Delta(x).$$

Definition 2. Let $i \in N$. We define the relation Δ_o^i on X by:

$$[x' \Delta_o^i x] \text{ if and only if } [x' \in \Delta(x) \text{ and } x'_i = o_i].$$

Let $x \in X$ be an outcome. We denote $\Delta_o^i(x) = \{x' \in X : x' \Delta_o^i x\}$, and by $N^x = \{i \in N : x_i \neq o_i\}$ the set of agents whose actions in x are different from their reference points. We also denote $|x| = |N^x|$ the cardinality of N^x .

⁸We normalize the surplus at the reference point to 0 for expositional purposes. It is possible that the surplus realized at o is not zero, and in this case, $f(x)$ should be interpreted as *net surplus* at x , that is, the realized surplus at x minus the realized surplus at o . We assume the reference o to be exogenously determined.

⁹As noted in the Introduction, $u_j(x)$ can be any increasing function of $\phi_j(f, x)$, where the functional form may be different for each agent.

Definition 3. Let $f \in P(X)$, $x \in X$, and $x' \in \Delta_o^i(x)$. The *marginal contribution* of agent i at a pair (x', x) is:

$$mc_i(f, x', x) = f(x'_{-i}, x_i) - f(x'),$$

where $(x'_{-i}, x_i) \in X$ is the outcome in which agent i chooses x_i , and every other agent j chooses x'_j .

Definition 4. Let $f \in P(X)$. Agent i is said to be *unproductive* if for each $x \in X$ and all $x' \in \Delta_o^i(x)$, $mc_i(f, x', x) = 0$.

A permutation π of N is a bijection of N into itself. We denote by \mathcal{S}_n the set of permutations of N . Let $x \in X$ be a profile of inputs, and let $\pi^x \in \mathcal{S}_n$ be a permutation of N whose restriction to $N \setminus N^x$ is the identity function, that is $\pi^x(i) = i$ for each $i \in N \setminus N^x$. Remark that π^x permutes only agents that are active in the profile x , and is therefore equivalent to a permutation $\pi^x : N^x \rightarrow N^x$ over N^x ; we denote by \mathcal{S}_n^x the set of such permutations. Let $x \in X$, $\pi^x \in \mathcal{S}_n^x$, and $y \in \Delta(x)$. We define the profile $\pi^x(y) = (\pi_j^x(y))_{j \in N}$, where

$$\pi_j^x(y) = \begin{cases} x_j & \text{if } y_k \neq o_k, j = \pi^x(k) \\ o_j & \text{if } y_k = o_k, j = \pi^x(k). \end{cases}$$

We now formalize the principles of market justice below.

Anonymity. An allocation ϕ satisfies x -Anonymity if for each $i \in N$ and $\pi^x \in \mathcal{S}_n^x$,

$$\phi_i(\pi^x f^x, x) = \phi_{\pi^x(i)}(f^x, x), \text{ where } \pi^x f^x(y) = f^x(\pi^x(y)), \text{ for } y \in \Delta(x).$$

The value ϕ satisfies Anonymity if ϕ satisfies x -Anonymity for all $x \in X$.

Local Efficiency. $\sum_{j \in N} \phi_j(f, x) = f(x)$ for any $f \in P(X)$ and $x \in X$.

Unproductivity. If agent i is unproductive, then $\phi_i(f, x) = 0$ for each $f \in P(X)$ and $x \in X$.

Marginality. Let $f, g \in P(X)$, and x an outcome. If

$$mc_i(f, x', x) \geq mc_i(g, x', x) \text{ for each } x' \in \Delta_o^i(x)$$

for an agent i , then $\phi_i(f, x) \geq \phi_i(g, x)$.

These axioms are interpreted naturally. *Anonymity* means that an agent's pay does not depend on their *name*. It states that every agent is treated the same way by the allocation rule: if two agents exchange their identities, their payoffs will remain unchanged. An important property that is implied by anonymity is *symmetry* (or *non-favoritism*), which means that equally productive agents should receive the same pay. *Local efficiency* simply requires that the surplus resulting from any input choice be fully shared among productive agents participating in the economy. *Unproductivity* means that an agent whose marginal contribution is zero at an input profile should get nothing at that profile. *Marginality* means that, if the adoption of a new technology increases the marginal contribution of an agent, that agent's pay should not be lower under this new technology relative to the old technology. In other words, more productive agents should not earn less compared to less productive agents. Throughout the paper, we abbreviate the four principles as **ALUM**.

Definition 5. A **free and fair economy** is a free economy (N, X, o, f, ϕ, u) such that the distribution scheme ϕ satisfies **ALUM**.

We have the following result.

Proposition 1. *There exists a unique distribution scheme, denoted \mathbf{Sh} , that satisfies **ALUM**. For any production function $f \in P(X)$, and any given outcome $x \in X$ and agent $i \in N$:*

$$\mathbf{Sh}_i(f, x) = \sum_{x' \in \Delta_o^i(x)} \frac{(|x'|)!(|x| - |x'| - 1)!}{(|x|)!} mc_i(f, x', x). \quad (1)$$

Proof of Proposition 1. See Appendix. □

Remark that for each agent i , the value $\mathbf{Sh}_i(f, x)$ is interpreted as agent i 's average contribution to output $f(x)$. It can be easily shown that the allocation rule \mathbf{Sh} generalizes the classical Shapley value [Shapley, 1953]. In fact, to obtain the classical Shapley value, one only has to assume that each agent's action set is the pair $\{0, 1\}$; the classical Shapley value is simply $\mathbf{Sh}_i(f, x)$ where $x = (1, 1, \dots, 1)$, which effectively corresponds to the assumption that the grand coalition is formed. Our setting generalizes the classical environment in three ways. First, it is not necessary to assume that all players have the same action set. Second, the action set of a player may have more than two elements. Third, the value can be computed for any input profile x , which effectively means that

$\mathbf{Sh}_i(f, x)$ as a multivariate function of x . Our model also generalizes that in Pongou and Tondji [2018] (when the environment is certain), Aguiar et al. [2018], and Aguiar et al. [2020]. Following these latter studies, we will call \mathbf{Sh} the *Shapley pay scheme*.

Below, we illustrate the notion of a free and fair economy, and provide an example of a free economy that is unfair.

Example 1. Consider a small economy $\mathcal{E} = (N, X, o, f, \phi, u)$, where $N = \{1, 2\}$, $X_1 = \{a_1, a_2\}$, $X_2 = \{b_1, b_2, b_3\}$, $o = (a_1, b_1)$, $X = X_1 \times X_2$, f is given by $f(a_1, b_1) = 0$, $f(a_1, b_2) = 5 = f(a_1, b_3)$, $f(a_2, b_1) = 2$, and $f(a_2, b_2) = 4 = f(a_2, b_3)$, and for each $x \in X$, $\phi(f, x) = u(f, x)$ is given in Table 1 below:

		Agent 2		
		b_1	b_2	b_3
Agent 1	a_1	(0, 0)	(0, 5)	(0, 5)
	a_2	(2, 0)	(0.5, 3.5)	(0.5, 3.5)

Table 1: A 2-agent free and fair economy

For each of the six payoff vectors presented in Table 1, the first component represents agent 1's payoff (for example, $u_1(f, (a_2, b_1)) = 2$) and the second component represents agent 2's payoff (for instance, $u_2(f, (a_2, b_1)) = 0$). We can check that for each $x \in X$, $u(f, x) = \phi(f, x) = \mathbf{Sh}(f, x)$. Therefore, \mathcal{E} is a free and fair economy.

		Agent 2		
		b_1	b_2	b_3
Agent 1	a_1	(0, 0)	(2, 3)	(3, 2)
	a_2	(1, 1)	(3, 1)	(2, 2)

Table 2: A 2-agent free and unfair economy

Now, we consider another economy \mathcal{E}' with the same characteristics as in \mathcal{E} except for the distribution scheme ϕ that is replaced by a new scheme ψ described in Table 2. In addition to the fact that $\psi \neq \mathbf{Sh}$, it is straightforward to show that the distribution ψ violates the *marginality* axiom. Therefore, \mathcal{E}' is not a free and fair economy.

One of our goals in this study is to answer the question of whether *principles of justice* (or fair principles) guarantee the existence of a pure strategy Nash equilibrium.

We can observe that in the free and fair economy described by Table 1, there are two pure strategy Nash equilibria, which are (a_2, b_2) and (a_2, b_3) . However, the modified economy \mathcal{E}' represented by Table 2 admits no equilibrium in pure strategies. In the next section, we will show that **ALUM** guarantee the existence of a pure strategy Nash equilibrium in a free economy, and when an economy violates these principles, a pure strategy Nash equilibrium may not exist.

3 Equilibrium existence in a free and fair economy

In a free and fair economy, agents make decisions that affect their payoff and the payoffs of other agents. One natural question that therefore arises is whether an *equilibrium* exists. In this section, we first show that a free economy can be modeled as a strategic form game and use the notion of pure strategy Nash equilibrium [Nash, 1951] to capture incentives and rationality. Our main result is that a strategic game derived from a free and fair economy always admits a pure strategy Nash equilibrium.

3.1 A free and fair economy as a strategic form game

A *strategic form game* is a 3-tuple (N, X, v) , where N is the set of players, $X = \times_{j \in N} X_j$ is the strategy space, and $v : X \rightarrow \mathbb{R}^n$ is the payoff function. For each $x \in X$, $v_i(x)$ is agent i 's payoff at strategy profile x , for each $i \in N$. A strategic form game is said to be finite if the set of agents N is finite, and for each agent i , the set X_i of actions is also finite.

A strategy profile $x^* \in X$ is a *pure strategy Nash equilibrium* in the game (N, X, v) if and only if for all $i \in N$, $v_i(x^*) \geq v_i(x_{-i}^*, y_i)$, for all $y_i \in X_i$, where (x_{-i}^*, y_i) is the strategy profile in which agent i chooses y_i and every other agent j chooses x_j^* .

A free economy $\mathcal{E} = (N, X, o, f, \phi, u)$ generates a strategic form game $G^{\mathcal{E}} = (N, X, u^{\mathcal{E}})$, where for each $x \in X$ and each $i \in N$, $u_i^{\mathcal{E}}(x) = u_i(f, x) = \phi_i(f, x)$. In the case \mathcal{E} is a free and fair economy, then for each outcome x , $\sum_{j \in N} u_j^{\mathcal{E}}(x) = f(x)$ since the distribution scheme ϕ satisfies *local efficiency*. For this reason, when \mathcal{E} is a free and fair economy, we may refer to the production function f as the *total utility function* of the strategic form game $G^{\mathcal{E}}$.

Definition 6. Let $\mathcal{E} = (N, X, o, f, \phi, u)$ be a free economy. A profile $x^* \in X$ is an *equilibrium* if and only if x^* is a pure strategy Nash equilibrium in the strategic form game $G^{\mathcal{E}}$.

3.2 Existence of an equilibrium

In this section, we state and prove the main result of this section.

Theorem 1. Any free and fair economy $\mathcal{E} = (N, X, o, f, \phi, u)$ admits an equilibrium.

The proof of Theorem 1 uses the concept of a *cycle of deviations* that we introduce below.

Definition 7. Let $G = (N, X, v)$ be a strategic form game and $L^k = (x^1, x^2, \dots, x^k)$ be a list of outcomes, where each $x^l \in X$ ($l = 1, \dots, k$) is a pure strategy. The k -tuple L^k is a *cycle of deviations* if there exist agents $j_1, \dots, j_k \in N$ such that

$$x^{l+1} = (x_{-j_l}^l, x_{j_l}^{l+1}) \text{ and } v_{j_l}(x^{l+1}) > v_{j_l}(x^l)$$

for each $l = 1, \dots, k$, and $x^{k+1} = x^1$.

Example 2. In the strategic form game represented in Table 3, consider the list $L^4 = (x^1, x^2, x^3, x^4)$, where $x^1 = (c, a)$, $x^2 = (d, a)$, $x^3 = (d, b)$, and $x^4 = (c, b)$.

		Agent 2	
		a	b
Agent 1	c	(0, 4)	(3, 0)
	d	(1, 0)	(0, 2)

Table 3: A 2-agent game that admits a cycle of deviations

L^4 forms a cycle of deviations. Indeed, agent 1 has an incentive to deviate from x^1 to x^2 . By doing so, agent 1 receives an excess payoff of 1. Similarly, agent 2 receives an excess payoff of 2 by deviating from x^2 to x^3 . Agent 1 receives an excess payoff of 3 by deviating from x^3 to x^4 ; and agent 2 receives an excess payoff of 4 by deviating from x^4 to x^1 . The sum of excess payoffs in the cycle L^4 is therefore equal to 10.

In the strategic form game in Table 4, the sum of excess payoffs in any cycle of outcomes equals 0. Therefore, the game does not admit a cycle of deviations. The profile $x^* = (a_2, b_3)$ is the only pure strategy Nash equilibrium of the game.

		Agent 2			
		b_1	b_2	b_3	b_4
Agent 1	a_1	(0, 0)	(0, 0)	(0, 12)	(0, 6)
	a_2	(13, 0)	$(\frac{13}{2}, -\frac{13}{2})$	$(\frac{3}{2}, \frac{1}{2})$	(4, -3)
	a_3	(3, 0)	(8, 5)	(-1, 8)	(-1, 2)

Table 4: A 2-agent game with Shapley payoffs

Note that the game in Table 4 is generated from a free and fair economy. From Definition 7, a sufficient condition for a finite strategic form game to admit a pure strategy Nash equilibrium is the absence of a cycle of deviations. The sum of excess payoffs in any cycle of deviations has to be strictly positive, as illustrated in Table 3 in Example 2. Such an example of a cycle of deviations can not be constructed in a strategic form game generated from a free and fair economy (see Table 4 in Example 2). We have the following result.

Lemma 1. *Let $\mathcal{E} = (N, X, o, f, \phi, u)$ be a free and fair economy, and $G^\mathcal{E} = (N, X, u^\mathcal{E})$ the strategic form game generated by \mathcal{E} . Then, the sum of excess payoffs in any cycle of deviations in $G^\mathcal{E}$ equals 0.*

Lemma 1 states that in the strategic form game $G^\mathcal{E}$ generated by a free and fair economy, the sum of excess payoffs in any cycle of deviations equals 0. In the proof of Lemma 1, we show that the sum of excess payoffs in any cycle of the game $G^\mathcal{E}$ is zero, which is equivalent to the game having a *potential function*; see Monderer and Shapley [1996, Theorem 2.8, p. 131].

Proof of Lemma 1. In this proof, we simply denote the payoff function $u^\mathcal{E}$ by u . Let $L^k = (x^1, x^2, \dots, x^k)$ be a cycle of deviations in the game $G^\mathcal{E}$, and let agents $j_1, \dots, j_k \in N$ be the associated sequences of defeaters. We denote by $S(L^k, u)$ the sum of excess payoffs in the cycle L^k :

$$S(L^k, u) = u_{j_k}(x^1) - u_{j_k}(x^k) + \sum_{l=1}^{k-1} [u_{j_l}(x^{l+1}) - u_{j_l}(x^l)].$$

We show that in the game $G^\mathcal{E}$

$$S(L^k, u) = 0.$$

For each agent $i \in N$, let \mathcal{R}_i be a total order on the set X_i such that $o_i \mathcal{R}_i x_i$ for all $x_i \in X_i$. For each outcome $x \in X$, define

$$f_x(T, y) = \begin{cases} |N^x| & \text{if } N^x \subseteq T \text{ and } x_i \mathcal{R}_i y_i \text{ for all } i \in N^x \\ 0 & \text{otherwise} \end{cases}$$

for all $T \subseteq N$ and $y \in X$.

We also define the following production function:

$$f_x(z) = f_x(N^z, z) \text{ for all } z \in X.$$

Note that the family $\{f_x, x \in X \setminus \{o\}\}$ forms a basis of the set of production functions on the the same set of players N , same set of outcomes X , and same reference outcome o . Therefore, there exists $(\alpha_x)_{x \in X \setminus \{o\}}$ such that

$$f(z) = \sum_{x \in X} \alpha_x f_x(z) \text{ for all } z \in X. \quad (2)$$

Furthermore, each f_x , $x \in X$, is the total utility function of a strategic form game with Shapley utilities $G^x = (N, X, v^x)$, where for each $i \in N$, v_i^x is given by

$$v_i^x(z) = \begin{cases} 1 & \text{if } i \in N^x, N^x \subseteq N^z, x_j \mathcal{R}_j z_j \text{ for all } j \in N^x \\ 0 & \text{otherwise.} \end{cases} \text{ for all } z \in X.$$

Step 1. We show that the sum of excess payoffs of the cycle L^k equals 0 in each strategic form game G^x . First observe that $v_i^x \equiv 0$ for all $i \notin N^x$, and $v_i^x \equiv v_j^x$ for all $i, j \in N^x$. This means that the sum of excess payoffs in any cycle of the game G^x , and in particular in the cycle L^k , equals the sum of excess payoffs of any $i \in N^x$, which is obviously 0.

Step 2. We show that $S(L^k, u) = 0$.

Using equation (2), $f = \sum_{x \in X} \alpha_x f_x$, we have that $u = \sum_{x \in X} \alpha_x v^x$. Given that $S(L^k, v^x) = 0$ for each outcome x , we can deduce that $S(L^k, u) = 0$. \square

Now, we derive the proof of Theorem 1.

Proof of Theorem 1. From Lemma 1, the game $G^\mathcal{E}$ admits no cycle of deviations. As $G^\mathcal{E}$ is finite, we conclude that $G^\mathcal{E}$ admits a pure strategy Nash equilibrium. \square

The principles of market justice that define a free and fair economy are only sufficient conditions for the existence of a pure strategy Nash equilibrium. However, an economy that violates **ALUM** may not have a pure strategy Nash equilibrium.

4 Equilibrium efficiency in a free and fair economy

In Section 3.2, Theorem 1 proves the existence of a pure strategy equilibrium in a free and fair economy. However, there is no guarantee that each equilibrium is Pareto-efficient. For instance, consider the strategic form game described in Table 4 in Example 2. The game admits a unique pure strategy Nash equilibrium $x^* = (a_2, b_3)$ with $\mathbf{Sh}(f, x^*) = (\frac{3}{2}, \frac{1}{2})$. However, the equilibrium x^* is Pareto-dominated by the strategy $x = (a_3, b_2)$ with $\mathbf{Sh}(f, x) = (8, 5)$. Below, we provide two conditions on the production function that address this issue. The first condition—*weak monotonicity*—guarantees the existence of a Pareto-efficient equilibrium in a free and fair economy, and the second condition—*strict monotonicity*—guarantees that there is a unique equilibrium and that this equilibrium is Pareto-efficient. Importantly, we also find that in a free economy that is not fair, these monotonicity conditions do not guarantee the existence of an equilibrium that is Pareto-efficient. Before presenting these results, we need some definitions.

Let $\mathcal{E} = (N, X, o, f, \phi, u)$ be a free economy, and for $i \in N$, we denote $X_{-i} = \prod_{j=1, j \neq i}^n X_j$.

Definition 8. An order R defined on X is *semi-complete* if for all $i \in N$ and $x_{-i} \in X_{-i}$, the restriction of R to A_i is complete, where $A_i = \{x_{-i}\} \times X_i$.

Definition 9. $f \in P(X)$ is:

1. *weakly monotonic* if there exists a semi-complete order R on X such that for any $x, y \in X$, if $x R y$, then $f(x) \leq f(y)$.
2. *strictly monotonic* if there exists a semi-complete order R on X such that for any $x, y \in X$, $[x R y \text{ and } x \neq y]$ implies $f(x) < f(y)$.

Definition 10. A free and fair economy $\mathcal{E} = (N, X, o, f, \phi, u)$ is weakly (resp. strictly) monotonic if f is weakly (resp. strictly) monotonic.

We have the following result.

Theorem 2. *A weakly monotonic free and fair economy $\mathcal{E} = (N, X, o, f, \phi, u)$ admits an equilibrium that is Pareto-efficient. If \mathcal{E} is strictly monotonic, then, the equilibrium is unique and Pareto-efficient.*

Proof of Theorem 2. The result in Theorem 2 follows from the fact that each agent i 's payoff $\mathbf{Sh}_i(f, x)$ at x depends only on the marginal contributions $\{f(y_{-i}, x_i) - f(y), y \in \Delta_o^i(x)\}$ of that agent at x . Since f is weakly monotonic, the underlying semi-complete relation, say R , satisfies the following condition: there exists $\bar{x} \in X$ such that f reaches its maximum at \bar{x} , and for all $i \in N$ and $x_{-i} \in X_{-i}$, we have $x R (x_{-i}, \bar{x}_i)$. Therefore, each marginal contribution of agent i at a given outcome x is less than or equal to his or her corresponding marginal contribution at the outcome (x_{-i}, \bar{x}_i) . Given that the Shapley distribution scheme, $\mathbf{Sh}(f, \cdot)$, is increasing in marginal contributions, agent i 's choice \bar{x}_i is a weakly dominant strategy of agent i in the game $G^\mathcal{E}$. Therefore, \bar{x} is a Nash equilibrium. The profile \bar{x} is also Pareto-efficient as it maximizes f . If f is strictly monotonic, then each \bar{x}_i is strictly dominant and \bar{x} is the unique Nash equilibrium of the game $G^\mathcal{E}$. \square

Theorem 2 ensures the uniqueness and Pareto-efficiency of the equilibrium in a strictly monotonic free and fair economy. The strategic form game described in Table 4 admits the profile $x^* = (a_2, b_3)$ as the only pure strategy Nash equilibrium. However, x^* is Pareto-dominated by the profile $x = (a_3, b_2)$, which is not an equilibrium. Such a result can not arise in a strictly monotonic free and fair economy. In addition to providing a condition that guarantees the existence of a Pareto-efficient equilibrium, Theorem 2 also provides a condition that rules out multiplicity of equilibria in the domain of free and fair economies.

In Theorem 2, we show that each weakly monotonic free and fair economy admits an equilibrium that is Pareto-efficient. Consider the strategic form game described in Table 5 below. The latter is derived from a free and fair economy with the profile $o = (c, a)$ as the reference point. The economy admits two equilibria, namely, outcomes (c, a) and (d, b) . The profile (d, b) is Pareto-efficient and it dominates the outcome (c, a) .

		Agent 2	
		a	b
Agent 1	c	$(0, 0)$	$(0, 0)$
	d	$(0, 0)$	$(1, 1)$

Table 5: A 2-agent free and fair economy with a Pareto-dominated equilibrium

We relate the existence of an equilibrium that is Pareto-dominated in the free and fair economy described in Table 5 to the fact that the production function is weakly

		Agent 2	
		b_1	b_2
Agent 1	a_1	(0, 0)	(2, -1)
	a_2	(2, 0)	(1, 2)

Table 6: A 2-agent strictly monotonic free and unfair economy

monotonic. However, it is essential to emphasize that the existence of an equilibrium is due to the fact that the economy is fair and not to the monotonicity property of the technology. For instance, consider a free economy \mathcal{E}^f , where agents 1 and 2 have strategies, $X_1 = \{a_1, a_2\}$, and $X_2 = \{b_1, b_2\}$, and the production function f is given by: $f(a_1, b_1) = 0$, $f(a_1, b_2) = 1$, $f(a_2, b_1) = 2$, and $f(a_2, b_2) = 3$. Agents' payoffs are described in Table 6. The environment \mathcal{E}^f describes a strictly monotonic economy, but it is unfair. Similarly, by replacing the production function f by another function g defined by: $g(a_1, b_1) = 0$, $g(a_1, b_2) = g(a_2, b_1) = 1$, and $g(a_2, b_2) = 3$, we obtain a weakly free monotonic and unfair economy \mathcal{E}^g with agents' payoffs described in Table 7.

		Agent 2	
		b_1	b_2
Agent 1	a_1	(0, 0)	(2, -1)
	a_2	(2, -1)	(1, 2)

Table 7: A 2-agent weakly monotonic free and unfair economy

		Agent 2	
		Cooperate	Defect
Agent 1	Cooperate	(0, 0)	(-2, 1)
	Defect	(1, -2)	(-1, -1)

Table 8: A prisoner's dilemma game

Note also that neither strategic form game $G^{\mathcal{E}^f}$ described in Table 6, nor $G^{\mathcal{E}^g}$ described in Table 7 admit a pure strategy Nash equilibrium. This shows that the monotonicity conditions do not guarantee the existence of a pure strategy Nash in a free economy that

is unfair; and even when an equilibrium exists in such an economy, it may be Pareto-inefficient. This latter situation occurs, for example, in the prisoner's dilemma game. An economy that is represented by a prisoner's dilemma game is monotonic, but its unique equilibrium is Pareto-inefficient (see, for instance, the game described in Table 8; the unique pure strategy Nash equilibrium (Defect, Defect) is Pareto-inefficient).

5 A free economy with social justice and inclusion

Our conception of a free economy with social justice embodies both the ideals of market justice and social inclusion. Members of a society do not generally have the same abilities. Consequently, distribution schemes that are based on market justice alone will penalize individuals with less opportunities or those who are unable to develop a positive productivity to the economy.

One of the goals of social justice is to remedy this social disadvantage that results mainly from arbitrary factors in the sense of moral thought. Social justice requires caring for the least well-off and those who have natural limitations not allowing them to achieve as much as they would like to. This requirement goes beyond the considerations of a free and fair economy in which agents have equal access to civic rights, wealth, opportunities, and privileges. The ideal of social justice could be implemented in a fair society through specific redistribution rules, and that is the main message that we intend to provide in this section.

Market justice as defined in the previous sections requires that the collective outcome must be distributed based on individual marginal contributions. Thus, a citizen who is not able to contribute a positive value to the economy shouldn't receive a positive payoff.

Social justice differs to market justice in the sense that everyone should receive a basic worth for living. This principle is consistent with the results found by De Clippel and Rozen [2013] in a recent experimental study in which neutral agents (called "Decision Makers") are called upon to distribute collective rewards among other agents (called "Recipients"). They show that even if collective rewards depend on complementarity and substitutability between recipients, some decision markers still allocate positive rewards to those who bring nothing to the economy. Moreover, a linear convex combination of the Shapley value [Shapley, 1953] and the *equal split* scheme arises as a one-parameter allocation estimate of

data. This convex allocation is also known as an *egalitarian Shapley value* [Joosten, 1996]. Intuitively, this pay scheme can be viewed as implementing a progressive redistribution policy where a positive amount of the total surplus in an economy is taxed and redistributed equally among all the agents. We use this distribution scheme to showcase our purpose. We will see that some properties of an economy that embeds the idea of social justice depends on the tax rate. Below, we define the *equal-split* and an *egalitarian Shapley value* schemes.

Definition 11. Let $\mathcal{E} = (N, X, o, f, \phi, u)$ be a free economy.

1. ϕ is the *equal split* distribution scheme, if

$$\phi_i(f, x) = \frac{f(x)}{n}, \text{ for all } f \in P(X), x \in X, \text{ and } i \in N.$$

2. ϕ is an *egalitarian Shapley value* if there exists $\alpha \in [0, 1]$ such that for all $f \in P(X)$, and $i \in N$,

$$\phi_i(f, x) = \alpha \cdot \mathbf{Sh}_i(f, x) + (1 - \alpha) \cdot \frac{f(x)}{n}, \text{ for all } x \in X.$$

We denote by \mathbf{ES}^α the egalitarian Shapley value associated to a given $\alpha \in [0, 1]$. Consider a simple case in which $N = \{1, 2\}$, $X_1 = X_2 = \{0, 1\}$, and two production functions f and g defined as follows: $f(0, 0) = g(0, 0) = 0$, $f(1, 0) = g(1, 0)$, $f(0, 1) \neq g(0, 1)$, $f(1, 1) \neq g(1, 1)$, and $f(1, 1) - f(0, 1) = g(1, 1) - g(0, 1)$. We can show that $mc_1(f, x', x) = mc_1(g, x', x)$ for any $x' \in \Delta_o^i(x)$. However, for any $\alpha \in [0, 1]$, it holds that $\mathbf{ES}_1^\alpha(f, (0, 1)) \neq \mathbf{ES}_1^\alpha(g, (0, 1))$ and $\mathbf{ES}_1^\alpha(f, (1, 1)) \neq \mathbf{ES}_1^\alpha(g, (1, 1))$. The latter shows that in addition of violating unproductivity, the mixing equal split and Shapley value also violates marginality. It is direct that \mathbf{ES}^α satisfies the principles of anonymity and local efficiency.¹⁰ The allocation scheme \mathbf{ES}^α has a very natural interpretation. The parameter α makes the reconciliation between marginalism and egalitarianism. Given an outcome x , the technology f produces the output $f(x)$. A share (α) of the latter is shared among agents according to their marginal contributions, while the remaining $(1 - \alpha)$ is shared equally among the entire population; the fraction $1 - \alpha$ is the tax

¹⁰Several axiomatizations of the Shapley value, the equal division scheme, and the α -egalitarian Shapley value appear in the literature. For a brief survey on these studies, we refer to the recent work of Choudhury et al. [2021] and the references therein.

rate. Immediately, those who are more talented will still receive more under a given egalitarian Shapley value scheme. Indeed, consider a production function $f \in P(X)$, and two agents i and j such that $\mathbf{Sh}_i(f, x) \geq \mathbf{Sh}_j(f, x)$ for $x \in X$. It is direct that for any $\alpha \in [0, 1]$, $\mathbf{ES}_i^\alpha(f, x) \geq \mathbf{ES}_j^\alpha(f, x)$, since $\alpha \geq 0$. Additionally, those who do not have the opportunity to contribute to their optimum scale will still be rewarded. We have the following definition.

Definition 12. $\mathcal{E} = (N, X, o, f, \phi, u)$ is a *free economy with social justice* if there exists $\alpha \in [0, 1[$ such that $\phi = \mathbf{ES}^\alpha$. We call $\mathcal{E}^\alpha = (N, X, o, f, \mathbf{ES}^\alpha, u)$ an α -free economy with social justice.

In Section 5.1, we analyze equilibrium existence and Pareto-efficiency in free economies with social justice. Our methodology is similar to the one followed in Sections 3 and 4. In Section 5.2, we prove that an economy can always choose its reference point to induce equilibrium efficiency, even when the economy is not monotonic.

5.1 Equilibrium existence and efficiency in a free economy with social justice

In what follows, we study the existence of equilibrium in an α -free economy with social justice. As defined in Section 3.1, a free economy with social justice admits an equilibrium if the strategic form game derived from that economy possesses a pure strategy Nash equilibrium. A meritocratic planner will choose a higher α when allocating resources since talents and merits have more value in such a society. An egalitarian planner will put a higher weight on equal distribution. It follows that a choice of α reveals a trade-off between market justice (or marginalism) and egalitarianism. The good news is that there exists a self-enforcing social contract irrespective of the size of α . We have the result hereunder.

Theorem 3. *Any α -free economy with social justice $\mathcal{E}^\alpha = (N, X, o, f, \mathbf{ES}^\alpha, u)$ admits an equilibrium.*

Proof of Theorem 3. Consider $\alpha \in [0, 1]$ such that $\phi = \mathbf{ES}^\alpha$. In the proof of Theorem 1, we show that the sum of excess payoffs in any cycle of deviations from any strategic form game derived from a fair economy equals 0. The same result holds for any strategic form

game derived from an α -free economy with social justice, since an egalitarian Shapley value is a linear combination of the Shapley value and equal division. Thus, we conclude the proof.¹¹ \square

We also provide a condition under which a free economy with social justice has an equilibrium which is Pareto-efficient. The following result is deduced from Theorem 2.

Corollary 1. *A weakly monotonic α -free economy with social justice \mathcal{E}^α admits an equilibrium that is Pareto-efficient. If f is strictly monotonic, then, the equilibrium is unique and Pareto-efficient.*

The proof of Corollary 1 is similar to that of Theorem 2. Next, we provide an additional result about Pareto-efficiency of equilibria in a free economy with social justice. Before stating Theorem 4, we introduce the following definition.

Definition 13. Let $\mathcal{E}^\alpha = (N, X, o, f, \mathbf{ES}^\alpha, u)$ be an α -free economy with social justice. An *optimal outcome* is any outcome $x \in \arg \max_{y \in X} f(y)$ at which f is maximized.

Theorem 4. *There exists $\alpha_0 \in (0, 1)$ such that for all $\alpha \in [0, \alpha_0]$, the α -free economy with social justice $\mathcal{E}^\alpha = (N, X, o, f, \mathbf{ES}^\alpha, u)$ admits an equilibrium that is Pareto-efficient.*

Proof of Theorem 4. Assume that α is sufficiently small. If f admits a unique optimal outcome x , then x is a pure strategy Nash equilibrium of the game generated by any α -free economy with social justice \mathcal{E}^α . In the case f admits two or more optimal outcomes, then, for strictly positive but sufficiently small α , no agent has any incentive to deviate from an optimal outcome to a non-optimal outcome. As games generated by α -free economies with social justice do not admit cycles of deviations, it is not possible to construct any cycle of deviations within the set of optimal outcomes. It follows that at least one optimal outcome is a pure strategy Nash equilibrium. The latter profile is also Pareto-efficient as it maximizes the sum of agents' payoffs. \square

Example 3 (Taxation and Social Justice). Consider a small economy involving three agents, $N = \{1, 2, 3\}$, who live in three different states or regions in a given country. One can assume that each agent is the “typical” representative of each state. Agents face

¹¹Alternatively, one can think of the strategic form game generated by an α -free economy as a convex sum of two *potential games*: the game $G^\mathcal{E}$, and the game generated by the equal division.

different occupational choices. Agent 1 can decide to stay unemployed (strategy “a”), work in a middle class job (strategy “b”) that provides an annual salary of \$188,000, or accumulate experience to land a higher skilled job (strategy “c”) that pays an annual salary of \$200,000. Agent 2 can only choose between strategies “a” and “b”. For many reasons including health concerns, natural disasters such as hurricane, pandemics or wildfire, or civil war violence, agent 3 does not have the opportunities available to other agents; he or she can not work, and is therefore considered as unemployed. The government uses marginal tax rates to determine the amount of income tax that each agent must pay to the tax collector. The aggregate annual fiscal revenue function f for the economy depends on agents’ strategies and it is described as follows: $f(a, a, a) = 0$, $f(a, b, a) = \$41,175.5$, $f(b, a, a) = \$41,175.5$, $f(b, b, a) = \$82,351$, $f(c, a, a) = \$45,015.5$, and $f(c, b, a) = \$86,191$. Numerous countries over the world use marginal tax brackets to collect income taxes (see, for example, a report by Bunn et al. [2019] for the Organisation for Economic Co-operation and the Development (OECD) and European Union (EU) countries). The function f is a simplified version of such fiscal revenue rules. With the tax revenue collected, the government provides public goods. However, the type of public investment received by an agent’s state depends on the agent’s marginal contribution to the aggregate annual fiscal revenue. Using the Shapley scheme $\phi = \mathbf{Sh}$ in the distribution of public investments yields the outcome $x^* = (c, b, a)$ as the unique pure strategy Nash equilibrium in this free and fair economy. At this equilibrium, the state of agent 1 receives a public good that is worth \$45,015.5, agent 2’s state receives a public investment of \$41.175.5, and agent 3’s state receives nothing. However, if the egalitarian Shapley scheme $\phi = \mathbf{ES}^{4/5}$ is used instead to redistribute the fiscal revenue, then $x^* = (c, b, a)$ is still the unique pure strategy Nash equilibrium in the free economy with social justice. In that case, the outcome x^* is still Pareto-efficient and the ranking of the size of investment across states does not change. Agent 3’s state receives a public investment of \$5,746, agent 2’s state receives \$38,686.5, and agent 1’s state receives \$41,758.5. Although the allocation $\mathbf{ES}^{4/5}(f, x^*) = (\$41,758.5, \$38,686.5, \$5,746)$ might not be the “best” decision for some people living in that society, it is a significant improvement (at least for agent 3’s state) from the market allocation $\mathbf{Sh}(f, x^*) = (\$45,015.5, \$41,175.5, 0)$.

Using Theorem 4, we deduce the following corollary.

Corollary 2. *Let $\mathcal{E}^\alpha = (N, X, o, f, \mathbf{ES}^\alpha, u)$ be an α -free economy with social justice.*

Assume that f only takes non-negative values. Then, each agent receives a non-negative payoff at any equilibrium.

The intuition behind Corollary 2 is straightforward. Assuming that at a given outcome $x \in X$, $f(x)$ is non-negative, then for all $i \in N$, agent i 's payoff is non-negative if instead of choosing x_i , the agent chooses the reference point o_i .

5.2 Choosing a reference point to achieve equilibrium efficiency

So far, we have assumed that the reference point o is exogenously determined and that in a free economy, the surplus function f is such that $f(o_1, o_2, \dots, o_n) = 0$. As noted earlier, this latter point is just a simplifying normalization. We have also shown that in a free and fair economy, all the equilibria may be Pareto-inefficient, especially in the absence of monotonicity. Similarly, in a free economy with social justice, if the tax rate $(1 - \alpha)$ is too small, a Pareto-efficient equilibrium may not exist either. This section shows that we can achieve equilibrium efficiency simply by changing the reference point of any free and fair economy or any free economy with social justice.

Without loss of generality, we assume that $f(o)$ is strictly positive and modify the Shapley distribution scheme such that for $i \in N$, and $x \in X$, agent i 's payoff at (f, x) , denoted $\overline{\mathbf{Sh}}(f, x)$, is given by $\overline{\mathbf{Sh}}(f, x) = \mathbf{Sh}_i(f - f(o), x) + \frac{f(o)}{n}$. Let us denote $\overline{P}(X) = \{g : X \rightarrow \mathbb{R}, \text{ with } g(o) > 0\}$. Our next result says that any optimal outcome can be achieved via an equilibrium profile in any α -free economy with social justice endowed with the distribution scheme $\overline{\mathbf{ES}}^\alpha$, where $\overline{\mathbf{ES}}^\alpha(f, x) = \alpha \cdot \overline{\mathbf{Sh}}_i(f, x) + (1 - \alpha) \cdot \frac{f(x)}{n}$, for all $x \in X$ and $f \in \overline{P}(X)$.

Theorem 5. *For all free economy $\mathcal{E}^\alpha(o) = (N, X, o, f, \overline{\mathbf{ES}}^\alpha, u)$, there exists another reference outcome o' such that the α -free economy $\mathcal{E}^\alpha(o') = (N, X, o', f, \overline{\mathbf{ES}}^\alpha, u)$ admits an optimal equilibrium x^* .*

Proof of Theorem 5. Assume $\alpha = 1$. Let o' be a profile of inputs such that $f(o') = \max_{x \in X} f(x)$. No agent has any strict incentive to deviate from o' . Indeed if agent i deviates and chooses x_i , then agent i is the only active agent at the new outcome (o'_{-i}, x_i) . As each inactive agent receives $\frac{f(o')}{n}$ at (o'_{-i}, x_i) , and $f(o')$ maximizes the production, it follows from the local efficiency axiom of the Shapley distribution scheme that the deviation x_i is not strictly profitable. A similar argument holds for any other $\alpha \in [0, 1)$. Indeed, at the

profile (o'_{-i}, x_i) , agent i receives $\alpha \left(f(o'_{-i}, x_i) - f(o') + \frac{f(o')}{n} \right) + (1 - \alpha) \frac{f(o'_{-i}, x_i)}{n}$, which is less than $\frac{f(o')}{n}$. \square

Remark that this result holds for any value of α , including for $\alpha = 1$, which corresponds to a situation where the tax rate is zero. In that case, the entire surplus of the economy is distributed following the Shapley pay scheme. The analysis implies that if an economy can choose its reference point, it can always do so to lead to equilibrium efficiency.

6 Some applications

There a wide variety of applications of our theory. In this section, we provide applications to the distribution of surplus in a firm, exchange economies, and self-enforcing lockdown in a networked economy facing a pandemic.

6.1 Teamwork: surplus distribution in a firm

In this first application, we use our theory to show how bonuses can be distributed among workers in a way that incentivizes them to work efficiently.

Consider a firm which consists of a finite set of workers $N = \{1, 2, \dots, n\}$. Each worker $i \in N$ privately and freely chooses an effort level $x_i^j \in X_i$, and bears a corresponding non-negative cost $c_i^j = c(x_i^j)$, where $c(\cdot)$ denotes the cost function. The cost of labor supply includes any private resources or extra working time that worker i puts into the project (for example, transportation costs, time, etcetera). Workers' labor supply choices are made simultaneously and independently. The owner of the firm (or the team leader) knows the cost associated to each effort level. At each effort profile $x = (x_1, \dots, x_n)$, a corresponding monetary output $F(x)$ is produced. A fraction of the monetary output, $f = \gamma \cdot F$, with $\gamma \in (0, 1)$, is redistributed to workers in terms of bonuses.

The existence of a pure strategy Nash equilibrium in this teamwork game follows from Lemma 1. To see this, observe that the payoff function of a worker can be decomposed in two parts: the bonus that is determined by the Shapley payoff and the cost function. Lemma 1 shows that the sum of excess payoffs in any cycle of deviations equals 0 in any free and fair economy (or any strategic game with Shapley payoffs). The reader can check that the sum of excess costs in any cycle of strategy profiles in the game is zero.

The latter implies that the sum of excess payoffs in any cycle of strategy profiles of the teamwork game is zero. Therefore, the teamwork game admits no cycle of deviations. As the game is finite, we conclude that it admits at least a Nash equilibrium profile in pure strategies. (Recall that the total output of the firm, F , and the total bonus, f , are perfectly correlated.) We should point out that a pure strategy Nash equilibrium always exists in the teamwork game, even if costs are high. In the latter case, some workers, if not all, might find it optimal to remain inactive at the equilibrium. In such a situation, the owner might want to raise the total bonus to be redistributed to workers.

Illustration. We now provide a numerical example with two workers called Bettina and Diana. Bettina has four possible effort levels: b_1 , b_2 , b_3 and b_4 ; and Diana has four possible effort levels as well: d_1 , d_2 , d_3 and d_4 . The cost functions of the two workers are given by: $c(b_1) = c(d_1) = 0$, $c(b_2) = c(b_3) = c(d_2) = c(d_3) = 4$, $c(b_4) = 3$, and $c(d_4) = 5$. The fraction f of the output to be redistributed as bonus is described in Table 9. The number $f(b, d)$ is the bonus to be distributed at the profile of efforts (b, d) ; for instance, $f(b_1, d_1) = 0$.

		Diana			
		d_1	d_2	d_3	d_4
Bettina	b_1	0	5	1	13
	b_2	2	8	10	2
	b_3	5	13	1	13
	b_4	3	9	13	2

Table 9: Total bonus function in a teamwork game

		Diana			
		d_1	d_2	d_3	d_4
Bettina	b_1	(0, 0)	(0, 5)	(0, 1)	(0, 13)
	b_2	(2, 0)	$(\frac{5}{2}, \frac{11}{2})$	$(\frac{11}{2}, \frac{9}{2})$	$(-\frac{9}{2}, \frac{13}{2})$
	b_3	(5, 0)	$(\frac{13}{2}, \frac{13}{2})$	$(\frac{5}{2}, -\frac{3}{2})$	$(\frac{5}{2}, \frac{21}{2})$
	b_4	(3, 0)	$(\frac{7}{2}, \frac{11}{2})$	$(\frac{15}{2}, \frac{11}{2})$	(-4, 6)

Table 10: Shapley payoffs: redistribution of total bonus in a teamwork game

		Diana			
		d_1	d_2	d_3	d_4
Bettina	b_1	$(0, 0)$	$(0, 1)$	$(0, -3)$	$(0, 8)$
	b_2	$(-2, 0)$	$(-\frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(-\frac{17}{2}, \frac{3}{2})$
	b_3	$(1, 0)$	$(\frac{5}{2}, \frac{5}{2})$	$(-\frac{3}{2}, -\frac{11}{2})$	$(-\frac{3}{2}, \frac{11}{2})$
	b_4	$(0, 0)$	$(\frac{1}{2}, \frac{3}{2})$	$(\frac{9}{2}, \frac{3}{2})$	$(-7, 1)$

Table 11: Bettina and Diana's net payoffs in a teamwork game

The corresponding Shapley payoffs are described in Table 10 and the net payoffs of Bettina and Diana in the teamwork game are described in Table 11. The profile (b_4, d_3) is a pure strategy Nash equilibrium. Therefore, the owner of the firm can implement the profile (b_4, d_3) without any need of monitoring the actions of Bettina and Diana, as (b_4, d_3) is self-enforcing. The owner can also implement the profile (b_1, d_4) . Note that the set of equilibrium effort profiles depend on the cost functions, and that no worker receives a non positive bonus at the equilibrium. The reason is that each worker i always has the option to remain inactive, which is equivalent to Bettina choosing b_1 or Diana choosing d_1 in this illustration. The two equilibria in this teamwork game are Pareto-efficient.

6.2 Contagion and self-enforcing lockdown in a networked economy

In this section, we provide an application of a free and fair economy to contagion and self-enforcing lockdown in a networked economy. We show how the costs of a pandemic from a virus outbreak can affect agents' decisions to form and sever bilateral relationships in the economy. Specifically, we illustrate this application by using the contagion potential of a network [Pongou, 2010, Pongou and Serrano, 2013, 2016, Pongou and Tondji, 2018].

Consider an economy \mathcal{M} involving agents who freely form and sever bilateral links according to their preferences. Agents' choices lead to a network, defined as a set of bilateral links. Assume that rational behavior is captured by a certain equilibrium notion (for example, Nash equilibrium, pairwise-Nash equilibrium, etcetera). Such an economy may have multiple equilibria. Denote by $\mathcal{E}(\mathcal{M})$ the set of its equilibria. Our main goal is to assess agent's decisions in response to the spread of a random infection (for example,

COVID-19) that might hit the economy. As the pandemic evolves in the economy, would some agents decide to sever existing links and self-isolate themselves? How does network structure depend on the infection cost?

To illustrate these concepts and answer the above questions, we consider an economy involving a finite set of agents $N = \{1, \dots, n\}$. All agents simultaneously announce the direct links they wish to form. For every agent i , the set of strategies is an n -tuple of 0 and 1, $X_i = \{0, 1\}^n$. Let $x_i = (x_{i1}, \dots, x_{ii-1}, 1, x_{ii+1}, \dots, x_{in})$ be an element in X_i . Let x_{ij} denote the j th coordinate of x_i . Then, $x_{ij} = 1$ if and only if i chooses a direct link with j ($j \neq i$), or $j = i$ (and thus $x_{ij} = 0$, otherwise). We assume that the formation of a link requires mutual consent, that is, a link ij is formed in a network if and only if $x_{ij}x_{ji} = 1$. We denote $X = \times_{j \in N} X_j$. An outcome $x \in X$ yields a unique network $g(x)$. However, a network can be formed from multiple outcomes. We denote $o = (0, \dots, 0)$ the reference outcome, and $g(o)$ the empty network. It follows that the networked economy \mathcal{M} can be represented by a free economy (N, X, o, f, ϕ, u) , where f is the production function and $u = \phi$ the payoff function (see below).

Assume that rationality is captured by the notion of pairwise-Nash equilibrium as defined by, among others, Calvó-Armengol [2004], Goyal and Joshi [2006], and Bloch and Jackson [2007]. The concept of pairwise-Nash equilibrium refines Nash equilibrium building upon the pairwise stability concept in Jackson and Wolinsky [1996]. Pairwise-equilibrium networks are such that no agent gains by reshaping the current configuration of links, neither by adding a new link nor by severing any subset of the existing links. Let g be a network and $ij \in g$ a link. We let $g + ij$ denote the network found by adding the link ij to g , and $g - ij$ denote the network obtained by deleting the link ij from g . Formally, g is a pairwise-Nash equilibrium network if and only if there exists a Nash equilibrium outcome x^* that supports g , that is $g = g(x^*)$, and for all $ij \notin g$, $\phi_i(f, g + ij) > \phi_i(f, g)$ implies $\phi_j(f, g + ij) < \phi_j(f, g)$.

The contagion function is the contagion potential of a network [Pongou, 2010, Pongou and Serrano, 2013, 2016, Pongou and Tondji, 2018]. To define this function, we consider a network g that has k components, where a component is a maximal set of agents who are directly or indirectly connected in g ; and n_j the number of individuals in the j^{th} component ($1 \leq j \leq k$). Pongou [2010] shows that if a random agent is infected with a virus, and if that agent infects his or her partners who also infect their other partners and

so on, the fraction of infected agents is given by the contagion potential of g , which is:

$$\mathcal{P}(g) = \frac{1}{n^2} \sum_{j=1}^k n_j^2.$$

However, in a network g , each agent is exogenously infected with probability $\frac{1}{n}$, and given that agents are not responsible for exogenous infections, the part of contagion for which agents are collectively responsible in g is:

$$\tilde{c}(g) = \mathcal{P}(g) - \frac{1}{n}.$$

We assume that the infection by a communicable virus leads to a disease outbreak in the economy. Measures that are implemented to fight the pandemic bring economic costs to society. To assess those costs, we assume that the collective contagion function \tilde{c} generates a pandemic cost function \mathcal{C} so that, for each network g :

$$\mathcal{C}(g) = F(\tilde{c}(g)), \text{ } F \text{ being a well-defined function.}$$

The pandemic and network formation affect economic activities. The formation of a network g brings an economic value $v(g) \in \mathbb{R}$ to the economy. Given the cost function \mathcal{C} , the economic surplus of a network g is:

$$f(g) = v(g) - \mathcal{C}(g).$$

Our main goal is to examine each agent's behavior in forming or severing bilateral links as the pandemic spreads in the economy. Let g be a network and S be a set of agents. We denote by g^S the restriction of the network g to S . This restriction is obtained by severing all the links involving agents in $N \setminus S$. Also, let i be an agent. We denote by $g^S + i$ the network $g^{S \cup \{i\}}$ obtained from g^S by connecting i to all the agents in S to whom i is connected in the network g . The structure of the networked economy provides a natural setting for the use of the Shapley distribution scheme. In a competitive environment where marginal contributions are the only inputs that matter in the economy, we can expect that an agent who adds no value to any network configuration receives no payoff, and a more productive agent in a network structure receives a payoff that is greater relative to that of less productive agents. Assuming that the output from individual contributions are entirely shared among agents, it becomes natural to consider that agent i 's payoff in a

network g is given by the Shapley distribution scheme (1):

$$\phi_i(f, g) \equiv \mathbf{Sh}_i(f, g) = \sum_{S \subseteq N, i \notin S} \frac{s!(n-s-1)!}{n!} \{f(g^S + i) - f(g^S)\}, \quad s = |S|.$$

The networked economy $\mathcal{M} = (N, X, o, f, \mathbf{Sh}, u)$ describes a free and fair economy. We have the following result.

Proposition 2. *Pairwise-Nash equilibrium networks always exist: $\mathcal{E}(\mathcal{M}) \neq \emptyset$.*

Proposition 2 partly follows from Theorem 1, but is stronger because the notion of pairwise-Nash equilibrium refines the Nash equilibrium. The proof is left to the reader. We illustrate it below.

Illustration. Let $N = \{1, 2, 3\}$. Assume the set of an agent i 's direct links in a network g is $L_i(g) = \{jk \in g : j = i \text{ or } k = i, \text{ and } j \neq k\}$, of size $l_i(g)$. The size of g is $l(g) = \sum_{i \in N} l_i(g)/2$. Note that $l(g) = 0$ if and only if g is the empty network. For illustration, we assume that for each network g :

$$\begin{aligned} v(g) &= [l(g)]^{1/2} \\ \mathcal{C}(g) &= \lambda \tilde{c}(g) = \lambda [\mathcal{P}(g) - \frac{1}{n}], \quad \lambda > 0 \\ f(g) &= [l(g)]^{1/2} - \lambda [\mathcal{P}(g) - \frac{1}{n}], \quad \lambda > 0. \end{aligned}$$

We can rewrite f as follows (note that $\mathcal{P}(\emptyset) = \frac{1}{n}$):

$$f(g) = \begin{cases} 0 & \text{if } l(g) = 0 \\ 1 - \frac{2\lambda}{9} & \text{if } l(g) = 1 \\ \sqrt{2} - \frac{2\lambda}{3} & \text{if } l(g) = 2 \\ \sqrt{3} - \frac{2\lambda}{3} & \text{if } l(g) = 3 \end{cases}$$

Given that there is only three agents, we can fully represent the set of networks in \mathcal{M} . The agents are labeled as described in Figure 1. In Figure 2, we display the different network configurations in \mathcal{M} . In each network, the payoff of each agent is given next to the corresponding node. The pairwise stability concept facilitates the search of equilibrium networks.

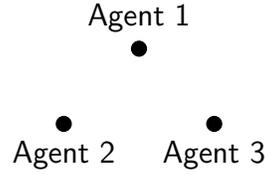


Figure 1: Disposition of agents in a network

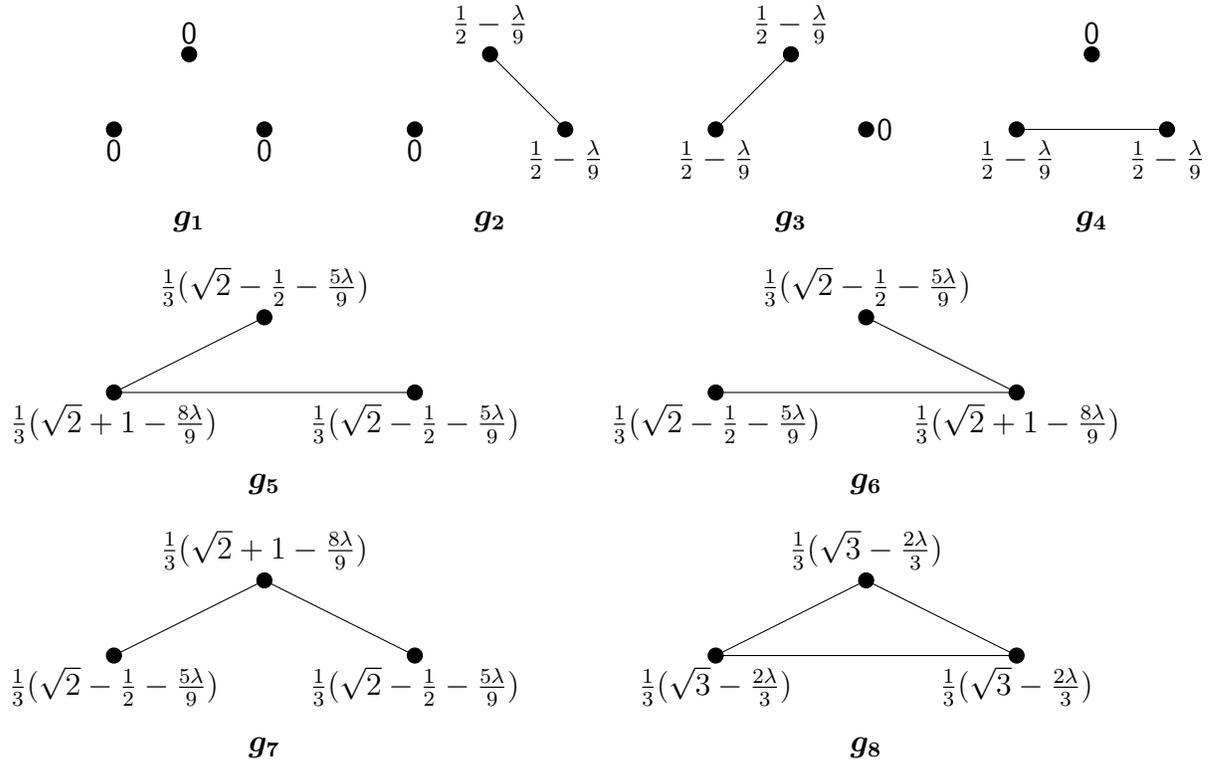


Figure 2: Possible network formation in \mathcal{M}

We have the following result. We denote by g^N the complete network.

Proposition 3. *Let g be a network. If:*

1. $\lambda < 1.8\sqrt{2} - 0.9$, then $\mathcal{E}(\mathcal{M}) = \{g^N\}$.
2. $1.8\sqrt{2} - 0.9 < \lambda < \frac{3\sqrt{3}}{2}$, then $g \in \mathcal{E}(\mathcal{M})$ if and only if $l(g) \in \{1, 3\}$.
3. $\frac{3\sqrt{3}}{2} < \lambda < 4.5$, then $g \in \mathcal{E}(\mathcal{M})$ if and only if $l(g) = 1$.
4. $\lambda > 4.5$, then $\mathcal{E}(\mathcal{M}) = \{g(o)\}$.

The proof of Proposition 3 is straightforward and left to the reader. Clearly, Proposition 3 shows that pandemic costs affect agents' decisions in the networked economy. The parameter λ summarizes the negative effects of the contagion in the economy. When there is no disease outbreak, or the pandemic costs are very low (lower values of λ), each agent gains by keeping bilateral relationships with others. In that situation, the complete network is likely to sustain as the equilibrium social structure in the economy. No agent has an incentive to self-isolate. However, as the pandemic costs rise, agents respond by severing some bilateral connections. For intermediate values of λ ($\frac{3\sqrt{3}}{2} < \lambda < 4.5$), only networks with one link will be sustained in the equilibrium. This means that some agents find it rational to partially or fully self-isolate in order to reduce the spread of the virus. In the extreme case where the contagion costs are very high ($\lambda > 4.5$), a complete lockdown arises, and the empty network is the only equilibrium.

Interestingly, the value of λ depends on the nature of the virus. Viruses induce different severity levels. For example, COVID-19 and the flu virus have different values, inducing different network configurations in equilibrium. The different network configurations in Figure 2 can therefore be interpreted as the networks that will arise in different scenarios regarding the nature of the virus.

6.3 Exchange economies

In this section, we apply our theory to pure exchange economies (Section 6.3.1) and markets with transferable payoff (Section 6.3.2).

6.3.1 Pure exchange economies

There are no production opportunities in a pure exchange economy (or, simply, an exchange economy), and agents trade initial stocks, or endowments, of goods (or commodities) that they possess according to a specific rule and attempt to maximize their preferences or utilities. Generally, an exchange economy consists of a list $\Omega = (N, l, (w_i), (u_i))$, where:

- (a) N is a finite set of agents ($|N| = n < \infty$);
- (b) l is a positive integer (the number of goods or commodities);

- (c) the vector w_i is agent i 's endowment vector ($w_i \in X_i \subseteq \mathbb{R}_+^l$), with \mathbb{R}_+ being the set of non-negative real numbers, and X_i the agent i 's consumption set; and
- (d) $u_i : X_i \rightarrow \mathbb{R}$ is agent i 's utility function.

The amount of good k that agent i demands in the market is denoted x_{ik} , so that agent i 's consumption bundle is denoted $x_i = (x_{i1}, x_{i2}, \dots, x_{il}) \in X_i$. An *allocation* is a distribution of the total endowment among agents: that is, an outcome $x = (x_j)_{j \in N}$, with $x_j \in X_j$ for all $j \in N$ and $\sum_{j \in N} x_j \leq \sum_{j \in N} w_j$. A *competitive equilibrium* of an exchange economy is a pair (p^*, z^*) consisting of a vector $p^* \in \mathbb{R}_+^l$, with $p^* \neq 0$ (the price vector), and an allocation $x^* = (x_j^*)_{j \in N}$ such that, for each agent i , we have:

$$p^* x_i^* \leq p^* w_i, \text{ and } u_i(x_i^*) \geq u_i(x_i) \text{ for which } p^* x_i \leq p^* w_i, x_i \in X_i.$$

We say that $x^* = (x_j^*)_{j \in N}$ is a *competitive allocation*.

In an exchange economy, we can assimilate an agent's consumption bundle to that agent's action in the market. In that respect, we can formulate an exchange economy under mild assumptions as a free and fair economy. Consider an exchange economy $\Omega = (N, l, (w_i), (u_i))$ in which the number of goods is finite ($l < \infty$), and each agent i 's consumption set X_i is finite ($|X_i| < \infty$). For instance, one can assume that agents can only purchase or sell indivisible units of goods in the market. We can model Ω as a free and fair economy $\mathcal{E}^\Omega = (N, X = \times_{j \in N} X_j, o, F, \mathbf{Sh}, \bar{u})$ where:

- (i) each agent i 's action $x_i \in X_i$;
- (ii) the reference outcome o is the vector of endowments w ;
- (iii) $F : X \rightarrow \mathbb{R}$ is the net aggregate utility function, i.e., for $x = (x_j)_{j \in N} \in X$,

$$F(x) = \sum_{j \in N} [u_j(x_j) - u_j(w_j)], \text{ with } F(w) = 0; \text{ and}$$

- (iv) the Shapley allocation scheme $\mathbf{Sh} = \bar{u}$ distributes the net aggregate utility $F(x)$ between agents at each profile $x \in X$: $\bar{u}_i(x) = \mathbf{Sh}_i(F, x)$ for each $i \in N$.

Only allocations in the free and fair economy can be selected in the equilibrium. This means that an outcome $x = (x_j)_{j \in N} \in X$ is an equilibrium in the free and fair economy if

$$(1) \sum_{j \in N} x_j \leq \sum_{j \in N} w_j, \text{ and}$$

(2) x is a pure strategy Nash equilibrium of the strategic game (N, X, \mathbf{Sh}) .

Our model differs from the exchange economy in at least two important respects. First, the incentive mechanism is different. Second, the equilibrium prediction from free exchanges between agents in both economies is different in general. A competitive equilibrium exists in an exchange economy when some assumptions exist on agents' utilities and endowments. For instance, when utilities are continuous, strictly increasing, and quasi-concave and each agent initially owns a positive amount of each good in the market, a competitive equilibrium exists, and many equilibria might arise. However, under such assumptions on agents' utilities, the net aggregate utility function F is strictly increasing, and thanks to Theorem 2, the free and fair economy admits a unique equilibrium. Additionally, it is not necessary to impose any assumptions on utilities and endowments to guarantee the existence of an equilibrium in a free and fair economy. We illustrate these points in the following examples.

Example 4. Consider an exchange economy with two goods (1 and 2) and two agents (A and B) in which agent A initially owns a positive amount of good 1, $w_A = (1, 0)$, while agent B owns a positive amount of both goods, $w_B = (2, 1)$. We assume that agent A's consumption set is $X_A = \{(1, 0), (0, 0)\}$ and utility is $u_A(x_A) = u_A(x_{A1}, x_{A2}) = x_{A1} + x_{A2}$. Agent B's consumption set is $X_B = \{(2, 1), (1, 1), (0, 1), (2, 0), (1, 0), (0, 0)\}$ and utility is $u_B(x_B) = u_B(x_{B1}, x_{B2}) = \min\{x_{B1}, x_{B2}\}$. An allocation $x = (x_A, x_B) \in X_A \times X_B$ is such that $x_{A1} + x_{B1} \leq 3$ and $x_{A2} + x_{B2} \leq 1$. We can show that there is no competitive equilibrium in this exchange economy (one reason is the fact that agent A owns zero units of good 2), while the free and fair economy admits two equilibria $x_1^{Sh} = (w_A, w_B)$ and $x_2^{Sh} = (w_A, (1, 1))$. Each equilibrium maximizes the net aggregate utility, $F(x_1^{Sh}) = F(x_2^{Sh}) = 0$, with $\mathbf{Sh}_A(F, x_1^{Sh}) = \mathbf{Sh}_B(F, x_1^{Sh}) = 0$, and $\mathbf{Sh}_A(F, x_2^{Sh}) = \mathbf{Sh}_B(F, x_2^{Sh}) = 0$. This example shows that a free and fair exchange economy has an equilibrium while a competitive equilibrium does not exist. The next example will show that the equilibrium of a free and fair exchange economy can coincide with the competitive equilibrium.

Example 5. Consider a Shapley-Shubik economy [Shapley and Shubik, 1977] in which there are two agents and two goods. Agent A is endowed with 2 units of good 1, $w_A =$

$(2, 0)$, and agent B is endowed with 2 units of good 2, $w_B = (0, 2)$. We assume that agent A's consumption set is

$$X_A = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\},$$

and his or her utility function is $u_A(x_{A1}, x_{A2}) = x_{A1} + 3x_{A2} - \frac{1}{2}(x_{A2})^2$; agent B's consumption set is

$$X_B = \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2)\},$$

and his or her utility function is $u_B(z_{B1}, x_{B2}) = x_{B2} + 3x_{B1} - \frac{1}{2}(x_{B1})^2$. Assume that good 1 is the numeraire ($p_1 = 1$), and let $p = p_2$ and $X = X_A \times X_B$. It is straightforward to note that not all pairs of actions in X are feasible in the economy. We can show that the pair $E^* = (p^*, x^*)$, where $p^* = 1$, and $x^* = (x_A^* = (0, 2), x_B^* = (2, 0))$, is the unique competitive equilibrium of the market. At the equilibrium allocation (p^*, x^*) , agents exchange endowments, and that transaction results in utilities: $u_A(x_1^*) = u_B(x_2^*) = 4$. Similarly, strategic interactions among agents in the free and fair market yield the same outcome x^* . To show that result, we use an approach that allows us to simplify calculations in the free and fair economy.

Let us denote by \bar{X} the subset of allocations ($\bar{X} \subset X$), and consider the following decisions: a "keep the full endowment", b "sell 1 unit of good", and c "sell the full endowment." Consider $\bar{X}_A = \bar{X}_B = \{a, b, c\}$ as each agent's set of decisions. Each vector of decisions in $\bar{X}_A \times \bar{X}_B$ yields a unique outcome $(x_A, x_B) \in \bar{X}$. Precisely, the vector (a, a) entails the unique profile $x = (w_A, w_B) = ((2, 0), (0, 2))$; (a, b) corresponds to $x = ((2, 1), (0, 1))$; (a, c) corresponds to $x = ((2, 2), (0, 0))$; (b, a) corresponds to $x = ((1, 0), (1, 2))$; (b, b) corresponds to $x = ((1, 1), (1, 1))$; (b, c) corresponds to $x = ((1, 2), (1, 0))$; (c, a) corresponds to $x = ((0, 0), (2, 2))$; (c, b) corresponds to $x = ((0, 1), (2, 1))$; and (c, c) corresponds to $x = ((0, 2), (2, 0))$. The net aggregate utility function F is defined as: $F(x) = F(x_A, x_B) = u_A(x_A) + u_B(x_B) - 4$. Using the strategy profile (a, a) as the reference point, Table 12 describes agents' utilities in the free and fair economy. For each agent, decision c strictly dominates decisions a and b . It follows that the vector (c, c) which corresponds to the outcome $x^{Sh} = ((0, 2), (2, 0)) = x^*$ is the unique equilibrium in the free and fair economy. In this case, the equilibrium coincides with the competitive allocation.

		Agent B		
		a	b	c
Agent A	a	(0, 0)	(0, 1.5)	(0, 2)
	b	(1.5, 0)	(1.5, 1.5)	(1.5, 2)
	c	(2, 0)	(2, 1.5)	(2, 2)

Table 12: Utilities in the free and fair economy

6.3.2 Markets with transferable payoff

A market with transferable payoff is a variant of a pure exchange economy in which each agent in the economy is endowed with a bundle of goods that can be used as inputs in a production system that the agent operates. All production systems transform inputs into the same kind of output (i.e., money), and this output can be transferred between the agents. In a market, the payoff can be directly transferred between agents, while in a pure exchange economy only goods can be directly transferred. Following Osborne and Rubinstein [1994], a market with transferable payoff consists of a list $\Pi = (N, l, (w_i), (f_i), (u_i))$, where:

- (a) N is a finite set of agents ($|N| = n < \infty$);
- (b) l is a positive integer (the number of input goods);
- (c) the vector w_i is agent i 's endowment vector ($w_i \in X_i \subseteq \mathbb{R}_+^l$), with X_i being the agent i 's input set;
- (d) $f_i : X_i \rightarrow \mathbb{R}$ is agent i 's continuous, non-decreasing, and concave production function; and
- (e) u_i is agent i 's utility function: $u_i(f_i, p, x_i) = f_i(x_i) - p(x_i - w_i)$, with $p \in \mathbb{R}_+^l$ (the vector of positive input prices), and $x_i \in X_i$.

In the market, an *input vector* is a member of X_i , and a profile $(x_j)_{j \in N}$ of input vectors for which $\sum_{j \in N} x_j \leq \sum_{j \in N} w_j$ is an *allocation*. We denote $w = (w_j)_{j \in N}$. Agents can exchange inputs at fixed prices $p \in \mathbb{R}_+^l$, which are expressed in terms of units of output. At the end of the trade, if agent i holds the bundle x_i , then his or her net expenditure, in units

of output, is $p(x_i - w_i)$. Agent i can produce $f_i(x_i)$ units of output, so that his or her net utility is $u_i(f_i, p, x_i)$. A price vector $p^* \in \mathbb{R}_+^l$ generates a *competitive equilibrium* if, when agent i chooses his or her trade to maximize his or her utility, the resulting profile $(x_i^*)_{i \in N}$ of input vectors is an allocation. Formally, a *competitive equilibrium* of a market is a pair $(p^*, (x_i^*)_{i \in N})$ consisting of a vector $p^* \in \mathbb{R}_+^l$ and an allocation $(x_i^*)_{i \in N}$ such that, for each agent i , the vector x_i^* maximizes his or her utility $u_i(f_i, p^*, x_i)$, for each $x_i \in X_i$. The list $(N, l, w, (f_i), (u_i))$ defines a competitive market with transferable payoff.

In a market with transferable payoff, we can view an agent's input vector as an agent's action in the market. Therefore, as in section 6.3.1, we can write a market with transferable payoff under mild assumptions as a free and fair economy. Consider a market with transferable payoff $\Pi = (N, l, w, (f_i), (u_i))$ in which the number of input goods is finite ($l < \infty$), and each agent i 's input set X_i is finite ($|X_i| < \infty$). As in Section 6.3.1, we can model Π as a free and fair market $\mathcal{E}^\Pi = (N, X = \times_{j \in N} X_j, o, F, \mathbf{Sh}, \bar{u})$, with the difference that for $x = (x_j)_{j \in N} \in X$,

$$F(x) = \sum_{j \in N} [f_j(x_j) - f_j(w_j)].$$

As in the analysis in section 6.3.1 below, we provide examples that show similarities (Example 6) and differences (Example 7) between the predictions of free and fair markets and markets with transferable payoff.

Example 6. We consider a single-input market with transferable payoff in which there are two homogeneous agents who have the same production, $w_1 = w_2 = 1$, $f_i(x_i) = \sqrt{x_i}$, $i \in \{1, 2\}$, and $X_1 = X_2 = \{0, 1, 2\}$. The pair $E^* = (p^* = \frac{1}{2}, x^* = (w_1, w_2))$ is the unique competitive equilibrium of the market, and $u_1(p^*, x_1^*) = u_2(p^*, x_2^*) = 1$. Similarly, strategic interactions among agents in the free and fair market yield the same outcome x^* .

Example 7. As mentioned in Section 6.3.1, generally, the equilibrium predictions of a free and fair economy and a market with transferable payoff do not coincide. To showcase this point, we consider a market in which agents' production functions are not concave. Consider a single-input market with transferable payoff in which there are two heterogeneous agents in production: $w_1 = 1$, $X_1 = \{0, 1, 2, 3\}$, and $f_1(x_1) = \frac{1}{2}x_1^2$; and $w_2 = 2$, $X_2 = \{0, 1, 2, 3\}$, and $f_2(x_2) = x_2^2$. In the competitive market, the utility functions are

convex and given by: $u_1(p, x_1) = \frac{1}{2}x_1^2 - p(x_1 - 1)$ and $u_2(p, x_2) = z_2^2 - p(x_2 - 2)$. There is no exchange in this market, while strategic interactions among agents in the free and fair market yield a different outcome: $x^{Sh} = (0, 3)$.

In Section 6.3, we write an exchange economy as a free and fair economy under mild conditions. The axioms that characterize the Shapley allocation scheme reflect some ethical and fair considerations. Consider a domain \mathcal{D} of economies in which the set of competitive allocations is non-empty. Given the examples that we provide in Sections 6.3.1 and 6.3.2, one might be interested in formally examining the relationship between competitive allocations and equilibria as defined in a free and fair economy in the domain \mathcal{D} . Such a study contributes to the literature characterizing Walrasian allocations in terms of social choice axioms. Pioneer works in this literature include, Dubey et al. [1980], Gevers [1986], Thomson [1988], and Nagahisa [1994], among others. Though this exercise is beyond the scope of our study, it is a possible avenue for future research.

7 Conclusion

In this study, we examine how elementary principles of justice and ethics, of long tradition in economic theory, affect individual incentives in a competitive environment and determine the existence and efficiency of self-enforcing social contracts. To formalize this problem, we introduce a model of a *free and fair economy*, in which each agent freely and non-cooperatively chooses their input from a finite set, and the surplus generated by these choices is distributed following four ideals of market justice, which are anonymity, local efficiency, unproductivity, and marginality. We show that these ideals guarantee the existence of a pure strategy Nash equilibrium. However, an equilibrium need not be unique or Pareto-efficient. We uncover an intuitive condition—*strict technological monotonicity*—, which guarantees equilibrium uniqueness and efficiency. Interestingly, this condition does not guarantee equilibrium efficiency (or even existence) when ideals of market justice are violated in an economy. These ideals therefore lead to positive incentives, given their desirable equilibrium and efficiency properties.

We extend our analysis to incorporate social justice and inclusion, implemented in the form of progressive taxation and redistribution and guaranteeing a basic income to unproductive agents. In this more general setting, we generalize all of our findings. Additionally,

we examine how the tax policy affects efficiency, showing that there is a tax rate threshold above which a pure strategy Nash equilibrium that is Pareto-efficient always exists in the economy, even in the absence of technological monotonicity. Moreover, we show that if a free economy is able to choose its reference point, it can always do so to induce an efficient outcome that is self-enforcing, even if this economy is not monotonic.

By incorporating normative principles into non-cooperative game theory, we define a new class of finite strategic form games that always admit a pure strategy Nash equilibrium. We develop applications to some classical and recent economic problems, including the allocation of goods in an exchange economy, surplus distribution in a firm, and self-enforcing lockdown in a networked economy facing contagion. This variety of applications is possible because we impose no particular assumptions on the structure of agents' action sets, and our setting is fully non-parametric.

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Appendix

Proof of Proposition 1.

Sufficiency. We show that the allocation scheme \mathbf{Sh} satisfies **ALUM**.

Anonymity. Let $f \in P(X)$, $x \in X$, $\pi^x \in \mathcal{S}_n^x$, and i be an agent. We show that $\mathbf{Sh}_i(\pi^x f^x, x) = \mathbf{Sh}_{\pi^x(i)}(f^x, x)$.

1. If $i \notin N^x$, then $x_i = o_i$, and $\pi^x(i) = i$.

$$\begin{aligned} \mathbf{Sh}_i(f^x, x) &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{f^x(a + x_i e_i) - f^x(a)\} \\ &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{f^x(a) - f^x(a)\} \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{Sh}_i(\pi^x f^x, x) &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{\pi^x f^x(a + x_i e_i) - \pi^x f^x(a)\} \\ &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{f^x(\pi^x(a + x_i e_i)) - f^x(\pi^x(a))\}. \end{aligned}$$

For $a \in \Delta_0^i(x)$ and $x_i = o_i$, we have $\pi^x(a + x_i e_i) = \pi^x(a)$, and $\mathbf{Sh}_i(\pi^x f^x, x) = 0$. Therefore, for each $i \notin N^x$, we can conclude that $\mathbf{Sh}_i(\pi^x f^x, x) = \mathbf{Sh}_{\pi^x(i)}(f^x, x)$.

2. If $i \in N^x$, then $x_i \neq o_i$. Assume that $\pi^x(i) = j$. Then, $j \in N^x$ and $x_j \neq o_j$.

$$\begin{aligned} \mathbf{Sh}_j(f^x, x) &= \sum_{a \in \Delta_0^j(x)} \varphi(a, x) \{f^x(a + x_j e_j) - f^x(a)\} \\ &= \sum_{a \in \Delta_0^j(x)} \varphi(a, x) \{f(a + x_j e_j) - f(a)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{Sh}_i(\pi^x f^x, x) &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{\pi^x f^x(a + x_i e_i) - \pi^x f^x(a)\} \\ &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{f^x(\pi^x(a + x_i e_i)) - f^x(\pi^x(a))\} \\ &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{f(\pi^x(a + x_i e_i)) - f(\pi^x(a))\} \end{aligned}$$

$a \in \Delta_0^i(x)$ implies $a = (a_1, \dots, \underbrace{o_i}_{i\text{th component}}, \dots, a_n)$. The vector $\pi^x(a) = (\pi_1^x(a), \dots, \underbrace{\pi_j^x(a)}_{j\text{th component}}, \dots, \pi_n^x(a))$.

Given that $j = \pi^x(i)$ and $a_i = o_i$, it follows that $\pi_j^x(a) = o_j$ and $\pi^x(a) \in \Delta_0^j(x)$.

We also have $a + x_i e_i = (a_1, \dots, \underbrace{x_i}_{i\text{th component}}, \dots, a_n)$. Given that $j = \pi^x(i)$ and

$(a + x_i e_i)_i = x_i \neq o_i$, it follows that $\pi_j^x(a + x_i e_i) = x_j$. Note that we can write $\pi^x(a + x_i e_i) = \pi^x(a) + x_j e_j$. Therefore,

$$\begin{aligned} \mathbf{Sh}_i(\pi^x f^x, x) &= \sum_{a \in \Delta_0^i(x)} \varphi(a, x) \{f(\pi^x(a) + x_j e_j) - f(\pi^x(a))\} \\ &= \sum_{b \in \Delta_0^j(x)} \varphi(b, x) \{f(b + x_j e_j) - f(b)\}, \text{ where } b = \pi^x(a) \\ &= \mathbf{Sh}_j(f^x, x). \end{aligned}$$

It follows that the allocation \mathbf{Sh} satisfies x -Anonymity for each $x \in X$. Hence, \mathbf{Sh} satisfies Anonymity.

Local Efficiency. For any $f \in P(X)$ and $x \in X$, it is immediate that $\sum_{i \in N} \mathbf{Sh}_i(f, x) = f(x)$.

Unproductivity. If agent i is unproductive, then for any $f \in P(X)$ and $x \in X$, it is immediate that $\mathbf{Sh}_i(f, x) = 0$, since $mc(i, f, a, x) = 0$ for each $a \in \Delta_0^i(x)$.

Marginality. Let $f, g \in P(X)$ such that $mc(i, f, x', x) \geq mc(i, g, x', x)$ for all $i \in N$, $x \in X$ and $x' \in \Delta_0^i(x)$. By the definition of the value \mathbf{Sh} , it is immediate that $\mathbf{Sh}_i(f, x) \geq \mathbf{Sh}_i(g, x)$.

Necessity. In this part of the proof, we prove the uniqueness of the Shapley value. Consider another allocation procedure ϕ which satisfies **ALUM**.

Define the following production function $f_x \in P(X)$ for each $x \in X$ by:

$$f_x(y) = \begin{cases} 1 & \text{if } x \in \Delta(y) \\ 0 & \text{if } x \notin \Delta(y) \end{cases}$$

where $x \in \Delta(y)$ if and only if $[x_i \neq y_i \Rightarrow x_i = o_i]$.

Lemma 2 (Pongou and Tondji [2018]). *Any production function is a linear combination of the production functions f_x :*

$$f = \sum_{x \in X} c_x(f) f_x, \text{ where } c_x(f) = \sum_{x' \in \Delta(x)} (-1)^{|x|-|x'|} f(x').$$

Let $f \in P(X)$. Define the index I of the production function f to be the number of non-zero terms in some expression for f in (2). The theorem is proved by induction on I .

- a) If $I = 0$, then $f \equiv 0$. Let $x \in X$ and $i \in N$. Then, $mc(i, f, a, x) = 0$ for all $a \in X$ such that $a \in \Delta_0^i(x)$. Therefore, by Unproductivity, $\mathbf{Sh}_i(f, x) = \phi_i(f, x) = 0$.
- b) If $I = 1$, then $f = c_x(f) f_x$ for some $x \in X$. Consider $N^x = \{l \in N : x_l \neq o_l\}$.

Step 1. Let $i \notin N_x$, i.e., $x_i = o_i$.

For any $a \in X$ such that $a \in \Delta_0^i(x)$, we have $f(a + x_i e_i) - f(a) = 0$, i.e., $mc(i, f, a, x) = 0$. It follows that $\mathbf{Sh}_i(f, x) = 0$. Let $y \in X$ with $y \neq x$. Then, $x \in \Delta(y)$ or $x \notin \Delta(y)$.

- If $x \in \Delta(y)$, then $x_l = y_l$ for each $l \in N^x$. If $y_i = o_i$, then $\phi_i(f, y) = 0 = \mathbf{Sh}_i(f, y)$. Assume $y_i \neq o_i$. Then, for any $a \in \Delta_0^i(y)$, we have $mc(i, f, a, y) = f(a + y_i e_i) - f(a)$. If $x \in \Delta(a)$, we also have $x \in \Delta(a + y_i e_i)$ because $x_i = o_i$ and $y_i \neq o_i$. Similarly if $x \notin \Delta(a)$, then $x \notin \Delta(a + y_i e_i)$. Therefore, $mc(i, f, a, y) = 0$ for each $a \in \Delta_0^i(y)$, and $\mathbf{Sh}_i(f, y) = 0$.

- $x \notin \Delta(y)$, then $f(y) = 0$. If $y_i = o_i$, then $\mathbf{Sh}_i(f, y) = 0$. Assume $y_i \neq o_i$. Then, for any $a \in \Delta_0^i(y)$, we have $mc(i, f, a, y) = f(a + y_i e_i) - f(a)$. If $x \in \Delta(a)$, then for each $l \in N^x$, $x_l = a_l \neq o_l$. Or $a \in \Delta(y)$ implies that for each $l \in N^x$, we will have $a_l = y_l$, because $a_l \neq o_l$. Therefore, for each $l \in N^x$, $a_l = y_l = x_l$, and given that $y_i \neq o_i$ and $x_i = o_i$, we have $x \in \Delta(y)$, a contradiction. In fact $x \in \Delta(a)$ if and only if $x \in \Delta(a + y_i e_i)$. Thus, $mc(i, f, a, y) = 0$ for each $a \in \Delta_0^i(y)$, and $\mathbf{Sh}_i(f, y) = 0$.

Given that agent i is unproductive, it follows that $\phi_i(f, y) = \mathbf{Sh}_i(f, y) = 0$ for each $y \in X$.

Step 2. Let $i, j \in N$ such that $i, j \in N^x$ and $y \in X$. Let $\pi^y = (ij)$ a permutation. Given that ϕ satisfies Anonymity, it follows that ϕ satisfies y -Anonymity, and $\phi_i(\pi^y f^y, y) = \phi_j(f^y, y)$. For each $z \in \Delta(y)$, we have $\pi^y f^y(z) = f^y(z)$. Thus, $\pi^y f^y = f^y$, and $\phi_i(f^y, y) = \phi_j(f^y, y)$. By Local efficiency, $\sum_{k \in N^x} \phi_k(f^y, y) = f^y(y) = f(y)$. Therefore, $\sum_{k \in N^x} \phi_k(f^y, y) = |N^x| \phi_k(f^y, y)$, and for each $k \in N^x$, $\phi_k(f^y, y) = \frac{f^y(y)}{|N^x|} = \frac{f(y)}{|N^x|}$. If $x \in \Delta(y)$, then $f(y) = c_x(f)$. Otherwise, $f(y) = 0$, and for each $k \in N^x$, $\phi_k(f, y) = \phi_k(f^y, y) = \mathbf{Sh}_k(f, y)$.

- c) Assume now that ϕ is the value \mathbf{Sh} whenever the index of f is at most I and let f have index $I + 1$, with:

$$f = \sum_{k=1}^{I+1} c_{x^k}(f) f_{x^k}, \text{ all } c_{x^k} \neq 0, \text{ and } x^k \in X.$$

For $k \in \{1, 2, \dots, I + 1\}$, consider:

$$N^{x^k} = \{l \in N : x_l^k \neq o_k\}, \bar{N} = \bigcap_{k=1}^{I+1} N^{x^k}, \text{ and assume } i \notin \bar{N}.$$

Define the following production function:

$$g = \sum_{k: i \in N^{x^k}} c_{x^k}(f) f_{x^k}.$$

The index of g is at most I . Let $x, a \in X$ such that $a \in \Delta_0^i(x)$. Then $f(a + x_i e_i) - f(a) = g(a + x_i e_i) - g(a)$. Consequently, using Marginality, $\phi_i(f, x) = \phi_i(g, x)$.

By induction, we have:

$$\phi_i(f, x) = \sum_{k:i \in N^{x^k}} \frac{c_{x^k}(f) f_{x^k}(x)}{|x^k|} = \mathbf{Sh}_i(f, x), \text{ for } x \in X.$$

It remains to show that for each $x \in X$, $\phi_i(f, x) = \mathbf{Sh}_i(f, x)$ when $i \in \bar{N}$. Let $x \in X$. By Anonymity, $\phi_i(f, x)$ is a constant φ for all members of \bar{N} ; likewise the value $\mathbf{Sh}_i(f, x)$ is some constant φ' for all members of \bar{N} (with $|\bar{N}| > 0$). By Local efficiency,

$$|\bar{N}| \phi_i(f, x) = |\bar{N}| \varphi = f(x),$$

so that,

$$\varphi = \frac{f(x)}{|\bar{N}|}.$$

Similarly,

$$|\bar{N}| \mathbf{Sh}_i(f, x) = |\bar{N}| \varphi' = f(x),$$

so that,

$$\varphi' = \frac{f(x)}{|\bar{N}|}.$$

It follows that $\varphi = \varphi'$, and concludes the proof.

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