Quantifying the Impact of Impact Investing*

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Abstract

We propose a quantitative framework for assessing the financial impact of any form of impact investing, including socially responsible investing (SRI), environmental, social, and governance (ESG) objectives, and other non-financial investment criteria. We derive conditions under which impact investing detracts from, improves on, or is neutral to the performance of traditional mean-variance optimal portfolios, which depends on whether the correlations between the impact factor and unobserved excess returns are negative, positive, or zero, respectively. Using Treynor-Black portfolios to maximize the risk-adjusted returns of impact portfolios, we propose a quantitative measure for the financial reward, or cost, of impact investing compared to passive index benchmarks. We illustrate our approach with applications to biotech venture philanthropy, divesting from "sin" stocks, investing in ESG, and "meme" stock rallies such as GameStop in 2021.

Keywords: Impact Investing; Environmental, Social, and Governance Investing; Socially Responsible Investing; Venture Philanthropy; Investments.

JEL Classification: C10, C20, G11, G12

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1 Introduction

Impact investing—broadly defined as investments that consider not only financial objectives but also other goals that support certain social priorities and agendas—has drawn an increasing amount of attention in recent years. This concept was first introduced through populist efforts to effect social change by encouraging institutional investors to divest from companies engaged in businesses viewed by critics as unethical, immoral, or otherwise objectionable, e.g., exploitation of child labor; tacit support of apartheid or religious persecution; or gambling, pornography, alcohol, tobacco, and firearms businesses (collectively known as "sin stocks"). Such "socially responsible" investing (SRI) initially involved imposing filters so that certain companies were excluded from investable universes, but its scope has expanded significantly to include environmental, social, and governance (ESG) criteria, and has been given different names such as "sustainable" and "green" investing. As of December 2021, 4,578 organizations representing over \$100 trillion in assets under management have become signatories to the United Nations Principles of Responsible Investment (UNPRI).

Conventional wisdom typically views impact investing as a standard portfolio selection problem with additional constraints related to the degree of social impact of the underlying securities, thereby implying a non-superior risk/reward profile compared to the unconstrained case. Given that the constrained portfolio contains a proper subset of securities of the unconstrained version, mathematical logic suggests that the constrained optimum is, at best, equal to the unconstrained optimum or, more likely, inferior.

However, the non-superiority of constrained optima relies on a key assumption that is almost never explicitly stated: the constraint is assumed to be statistically independent of the securities' returns. In some cases, such an assumption is warranted—imagine constructing a subset of securities with CUSIP identifiers that contain prime numbers. Clearly such a constraint has no relation to the returns of any security, hence imposing such a constraint can only reduce the risk-adjusted return of the optimized portfolio.

But what if the constraint is *not* independent of the returns? For example, consider the constraint "invest only in those companies for which their stock prices will appreciate by more than 10% over the next 12 months." Apart from the infeasibility of imposing such a condition, it should be obvious that this constraint would, in fact, increase the risk-adjusted return of the optimized portfolio. Therefore, the answer to the question of what is the impact of impact investing rests entirely on whether and how the impact criteria are related to the performance characteristics of the securities being considered.

In this paper, we develop a general framework to quantify the financial impact of impact investing. We formalize impact investing as the sorting and selection of an investment universe of N securities based on an *impact factor*, X_i, for security i, so that higher values of X_i correspond to greater impact, e.g., lower carbon emissions, greater sustainability, higher ESG score, etc. As a result, other things equal, impact investors are assumed to prefer securities with higher values of X_i. This impact factor defines a rank ordering for all securities in the universe from which an impact portfolio can be constructed, i.e., the top decile of ESG-ranked securities or the bottom decile of carbon-emissions-ranked securities. Therefore, the impact on investment performance is determined by the joint distribution of the vector $\mathbf{X} \equiv [\, X_1 \; X_2 \; \cdots \; X_N \,]^T$ of impact measures with the investment performance of individual securities.

To formalize this idea, we first propose a general linear multi-factor model for asset returns and define excess returns or "alpha" as non-zero intercepts that we model as mean-zero random variables. This framework allows for the possibility of superior investment performance for individual securities, but also includes the conventional case of equilibrium or no-arbitrage pricing if we set the variance of the alphas to zero. In fact, the implications from our model are broadly applicable to the equilibrium asset-pricing set-up with no alpha but there exist omitted factors of which investors are unaware. Such an agnostic approach to investment performance allows us to determine conditions under which impact investing does and does not change the risk/reward profile of a given investment product or strategy.

In particular, we derive—both in finite samples and asymptotically (as the number of securities increases without bound)—the distribution of individual alphas that have been ranked according to their impact factors $\mathbf{X}$. It is well known that ranked random variables—known as *order statistics*—have different distributions than their unranked versions, and a large body of literature has developed many results for the distributions of various types of order statistics. However, for our purposes, a more relevant strand of that literature focuses on *induced order statistics*, in which random variables are ranked not by their own values but by the values of other random variables, e.g., ranking the returns of a collection of mutual funds not by their returns but by the funds' market betas. We use properties of induced order statistics to derive the distribution of an impact portfolio's alphas ranked by an arbitrary impact score $\mathbf{X}$, allowing us to quantify the impact of impact investing.

Using this framework, we show that the expected alpha from the induced ordering is just a discounted version of the expected alpha from ordering securities based on alpha (i.e., via an all-knowing oracle which, in reality, is of course unattainable because alphas are unobservable). This simple but profound result highlights the mechanism through which an impact factor's excess return is influenced by the induced ordering of alpha—it achieves a fraction of the maximum possible alpha with perfect knowledge, where the fraction is simply the correlation between $\mathbf{X}$ and the individual securities' alphas.

Using this insight, we quantify the alphas of portfolios formed based on the impact factor, **X**—including common heuristics of creating portfolios from the top or bottom impact-factor quantiles—and then apply the Treynor and Black (1973) framework to derive the optimal weights when forming these portfolios to maximize Sharpe ratio. We show that such impact portfolios are associated with "super-efficient frontiers" as long as the impact factor, **X**, is positively correlated with the unobserved alphas of the individual securities. We also provide an equilibrium/no-arbitrage interpretation of our results in which excess returns arise from omitted factors that investors may not be aware of, but to which impact portfolio managers have access. In this case, the excess returns are simply "excess" with respect to factors that investors observe, and represent risk premia from specific impact factors.

The Treynor-Black portfolio allows us to construct a natural measure of the financial impact of impact investing: an impact factor has positive alpha when it is positively correlated with individual securities' unobserved alphas. On the other hand, an impact factor can impose a cost—also quantifiable in our framework—when it is negatively correlated with alphas and investors divest of the bottom-ranked securities (which have positive alphas on average due to the negative correlation with **X**). This provides a possible explanation for the inconsistent and sometimes contradictory empirical findings on the effects of adopting impacting investing. The correlation between the impact factor and alpha is affected by different measures of impact,[1] different market conditions,[2] and different asset-pricing models for alpha,[3] all of which can influence the final estimate of the benefit or cost of impact investing.

To illustrate the practical relevance of our results, we apply our framework to four specific impact-investing contexts. The first is biotech venture philanthropy, a particular form of impact investing in biomedicine where nonprofit and mission-driven organizations fund initiatives to advance their objectives and potentially achieve returns that can be reinvested toward their mission. We take the case study by Kim and Lo (2019) about the Cystic Fibrosis Foundation, a leading venture philanthropy organization dedicated to treating and, eventually, curing cystic fibrosis. This example shows that a significantly positive alpha can be achieved by advancing drug development for rare diseases, which illustrates the feasibility of "doing well by doing good" (Falck and Heblich, 2007; Eichholtz, Kok, and Quigley, 2010).

The second application involves measuring the cost of divesting from sin stocks, stocks of companies engaged in businesses considered by some to be socially undesirable but that are

[1] There is a substantial literature documenting the divergence of ESG ratings for the same firms (Dortfleitner, Halbritter, and Nguyen, 2015; Semenova and Hassel, 2015; Berg, Koelbel, and Rigobon, 2020).

[2] See, for example, the "luxury-good effect" of Bansal, Wu, and Yaron (2018).

[3] See, for example, Geczy, Stambaugh, and Levin (2021) and Madhavan, Sobczyk, and Ang (2021).

documented to have positive alphas (Fabozzi, Ma, and Oliphant, 2008; Hong and Kacperczyk, 2009; Statman and Glushkov, 2009; Fauver and McDonald IV, 2014). Calibrating to Hong and Kacperczyk (2009) as an example, we estimate the cost of divestment to be 1.7%–3.3% in forgone alpha per annum, depending on the specific selection criteria. However, when calibrating to Blitz and Fabozzi (2017) where the authors estimate alpha by controlling for two new Fama and French (2015) quality factors—profitability and investment—in addition to classic factors, we obtain a smaller but still non-trivial cost to institutional investors, 0.6%–1.3% per annum. This example illustrates the dependence of the magnitude of estimated alpha on the specific asset-pricing model used, a well-known issue with all performance attribution exercises.

Third, we apply our framework to several ESG empirical studies (Baker et al., 2018; Bansal, Wu, and Yaron, 2018; MSCI, 2021). Correlations between the specific ESG measures in these studies and the unobserved alphas of individual securities determine the final estimate of the benefit (or cost) of ESG investing. They range from -0.05% for bonds (Baker et al., 2018) to 2.65% for equities in certain market conditions (Bansal, Wu, and Yaron, 2018). This underscores the importance of asset class, impact measures, and specific market conditions in determining the alpha of impact investing.

Finally, we apply our framework to explain the January 2021 price spike in GameStop Corp. and other "meme" stocks such as AMC Entertainment Holdings and Blackberry, where a decentralized short squeeze that exploited the short positions of institutional investors caused their prices to increase sharply before crashing. Classifying such phenomena as impact investing may seem strange, but based on the narrative that emerged from the WallStreetBets social media group, there is little doubt that a significant source of trading volume was motivated by a desire to punish institutional shortsellers as well as to provide moral support for the companies under attack. Perhaps a separate category called "price-impact investing" would be more appropriate. Based on the GameStop experience and similar episodes with other meme stocks, it is clear that the very act of trading can produce positive alpha, at least in the short term. By applying an optimal order-execution model with a simple market-impact function (Bertsimas and Lo, 1998), we are able to quantify the financial impact of price-impact investing. Of course, manipulating the prices of publicly traded equities clearly violates both securities law and anti-trust regulation, and our analysis is not meant to condone or encourage such activities. However, measuring the magnitude of such investments and understanding its financial implications can better inform regulators and policymakers as to the scope and severity of this phenomena so they can devote the appropriate sources to addressing it.

2 Literature Review

There is a growing literature theorizing the impact of SRI, ESG, and other non-financial objectives on asset pricing. Heinkel, Kraus, and Zechner (2001) build an equilibrium model in which exclusionary ethical investing leads to lower stock prices for polluting firms. Fama and French's (2007) taste model shows that, if investors prefer to invest in socially responsible companies, the expected return on such companies will be lower. Pástor, Stambaugh, and Taylor (2021b) provide a model for ESG investing where investors' taste for green assets imply lower returns, and assets can be priced in a two-factor model that includes the ESG factor and the market portfolio. Moreover, they show that the ESG factor exists when there is a large dispersion in investors' ESG tastes.[4]

However, other attempts to incorporate ESG explicitly into an asset-pricing framework imply that impact investing may positively predict expected returns in certain situations. Pedersen, Fitzgibbons, and Pomorski (2021) show that when the market is populated by ESG-motivated, ESG-aware, and ESG-unaware investors, the optimal allocation satisfies four-fund separation and is characterized by an ESG-efficient frontier. In their framework, ESG may either yield benefits to expected returns because it provides information about firm fundamentals (as in our example above in which constraints contain information about returns), or incur costs because it affects investor preferences and constraints. Chen and Mussalli (2020) and Sorensen, Chen, and Mussalli (2021) outline a quantitative approach to expand traditional portfolio theory to incorporate sustainability considerations for practitioners.

While these studies share some of the same implications as our framework, we add to this literature in several novel ways. The equilibrium frameworks of Fama and French (2007), Pástor, Stambaugh, and Taylor (2021b), and Pedersen, Fitzgibbons, and Pomorski (2021) highlight that the expected return of ESG investing depends on the mix of investors and preferences in the market.[5] But impact investing is still an evolving concept and their expected returns are dynamic and context-dependent. It is possible that market prices are still adjusting to reach a new equilibrium that reflects these considerations (Cornell and Damodaran, 2020).[6] Our unified econometric framework provides an explicit method to

[4]Other theoretical work on sustainable investing includes Friedman and Heinle (2016), Luo and Balvers (2017), Albuquerque, Koskinen, and Zhang (2019), Zerbib (2020), and Goldstein et al. (2021).

[5]Pedersen, Fitzgibbons, and Pomorski (2021) show that high-ESG stocks deliver high expected returns when the market has many ESG-unaware investors, and low expected returns when the market has many ESG-motivated investors. Pástor, Stambaugh, and Taylor (2021b) show that dispersion in ESG preferences increases the size of ESG investments and lowers equilibrium ESG returns, and the authors also point out that disentangling alphas from ESG taste shifts is a major challenge for empirical work in this area.

[6]For example, Bebchuk, Cohen, and Wang (2013) document the disappearance of a return premium

quantify the excess returns of any form of impact investment—including, but not limited to, the equilibrium setting—during different stages of this adaptive process. These results are, in turn, consistent with the equilibrium-based models when the correlation between $\mathbf{X}$ and security returns reflects the particular market condition and shift in preferences over time. From the adaptive markets (Lo, 2004, 2017) perspective, this correlation could reinforce itself as the amount of assets under management for a given impact factor increases over time, and eventually stabilizes as the size of the new sector reaches a steady state.

Our framework also differs from existing models in that we allow for the possibility of non-zero alphas, or omitted factors in the equilibrium/no-arbitrage interpretation, which is particularly relevant for the highly adaptive and dynamic ESG investment industry. An important insight from Pedersen, Fitzgibbons, and Pomorski (2021) is that ESG's information about firm fundamentals can yield benefits to its returns, while screening constraints will incur costs to ESG investing.[7] Our model shows that, when securities have non-zero alphas that are otherwise inaccessible to investor, ESG investing can derive financial benefit from constraints too, because of the information about returns implicit in these constraints. This effect is formalized statistically by the correlation between the impact factor, $\mathbf{X}$, and returns. As a result, in addition to the ESG-efficient frontier of Pedersen, Fitzgibbons, and Pomorski (2021), we are able to explicitly construct the optimal super-efficient portfolio from any $\mathbf{X}$ and explicitly quantify its financial impact.

Theories of SRI and ESG investing are also accompanied by a vast empirical literature focused on measuring their returns across asset classes and regions, and how much of these returns can be explained by traditional asset pricing factors.[8] On the one hand, several studies suggest that investments with ESG considerations may sacrifice returns in markets including stocks (Alessandrini and Jondeau, 2020), bonds (Baker et al., 2018), and venture capital funds (Barber, Morse, and Yasuda, 2021). This is also consistent with the literature that documents positive excess returns for sin stocks.[9] Geczy, Stambaugh, and Levin (2021) show that the SRI cost to mutual funds is minimal compared to a CAPM-investor but may be substantial when investors allow for size, value, and momentum factors.

associated with highly rated corporate governance during an earlier period, and Bansal, Wu, and Yaron (2018) find a "luxury-good effect," both of which suggest that the measured performance of stocks as a function of their ESG ratings will depend on the sample period.

[7]In their framework, the standard mean-variance tangency portfolio has the highest Sharpe ratio among all portfolios, and restricting portfolios to have any ESG score other than that of the tangency portfolio must yield a lower Sharpe ratio (Pedersen, Fitzgibbons, and Pomorski, 2021, p. 573).

[8]See, for example, Galema, Plantinga, and Scholtens (2008), Renneboog, Ter Horst, and Zhang (2008), Blitz and Fabozzi (2017), and Madhavan, Sobczyk, and Ang (2021).

[9]See, for example, Fabozzi, Ma, and Oliphant (2008), Hong and Kacperczyk (2009), Statman and Glushkov (2009), and Fauver and McDonald IV (2014).

On the other hand, recent empirical evidence from both academic research (Bansal, Wu, and Yaron, 2018; Madhavan, Sobczyk, and Ang, 2021) and industry advocates (Shing, 2021; Xiong, 2021) suggests that impact investing and, in particular, ESG measures, is associated with higher expected returns, at least under certain market conditions. For example, Pástor, Stambaugh, and Taylor (2021a) show that the high returns for green assets in recent years reflect unexpectedly strong increases in environmental concerns. In a recent review of about 2,200 individual studies, Friede, Busch, and Bassen (2015) report that a large majority of them show a positive relationship between ESG criteria and corporate financial performance. This raises the possibility that impact investing need not always imply lower risk-adjusted returns.

Moreover, there is substantial divergence among impact measures—such as those for ESG—even when they purport to capture the same concepts (Dortfleitner, Halbritter, and Nguyen, 2015; Semenova and Hassel, 2015; Berg, Koelbel, and Rigobon, 2020; Gibson, Krueger, and Schmidt, 2021). In particular, Khan, Serafeim, and Yoon (2016) find that only firms with good ratings on *material* sustainability issues significantly outperform firms with poor ratings on these issues.

These inconsistencies in the impact investing literature raise the question of what the real financial impact of impact investing is, which is precisely the motivation for our current contribution. Our framework explains not only how to measure the financial impact of impact investing, but also explains why there is such a wide range of empirical estimates for the expected returns of SRI and ESG investing. It stems from the wide range of impact definitions, date ranges, asset classes, and asset-pricing models for alpha, each of which leads to a different specification that may have potentially different correlation between the impact factor and asset returns. Our framework provides a unified methodology to quantify the financial consequences of all forms of impact investing, including SRI and ESG.

Additional literature on SRI and ESG investing includes climate change and its impact on asset pricing (Giglio et al., 2021; Stroebel and Wurgler, 2021),[10] preference toward sustainable investments (Bauer, Ruof, and Smeets, 2021), market responses to companies' eco-friendly behavior (Klassen and McLaughlin, 1996; Flammer, 2013, 2021; Krüger, 2015), transmission channels between ESG information and company valuation (Dunn, Fitzgibbons, and Pomorski, 2018; Giese et al., 2019), the real social impact generated by green investors (Dyck et al., 2019; Chen, Dong, and Lin, 2020), and implications for bank loans (Goss and Roberts, 2011). The empirical evidence of a causal relation between the sustainability classification and capital inflows for U.S. mutual funds (Hartzmark and Sussman, 2019) further

[10] See also Hong, Karolyi, and Scheinkman (2020), Giglio, Kelly, and Stroebel (2021), and references therein.

highlights both the popularity and importance of impact investing today.

More generally, our framework is applicable to portfolios constructed on the basis of any characteristic, including both impact proxies such as ESG and SRI measures and traditional factors such as value, size, momentum, and other variables. As such, our work is related to several strands of the asset pricing and econometrics literature. This includes a large literature devoted to identifying asset pricing factors,[11] a vast econometrics literature focused on factor models,[12] and the literature on data-snooping biases and the high dimensionality of cross-sectional asset-pricing models.[13] In particular, we make use of the same statistical results on induced order statistics first applied to financial data by Lo and MacKinlay (1990), albeit in a very different context.

3 The Framework

We consider a universe of N securities with returns R_{it} that satisfy the following linear multi-factor model:

$$R_{it} - R_{ft} = \alpha_i + \beta_{i1}(\Lambda_{1t} - R_{ft}) + \cdots + \beta_{iK}(\Lambda_{Kt} - R_{ft}) + \epsilon_{it} \tag{1}$$

$$\text{such that } \mathrm{E}[\epsilon_{it}|\Lambda_{kt}] = 0 \ , \quad k = 1, \ldots, K \tag{2}$$

where Λ_{kt} is the k-th factor return, $k = 1, \ldots, K$, R_{ft} is the risk-free rate, α_i and β_{ik} are the excess return and factor betas, respectively, and ϵ_{it} is the idiosyncratic return component. Because we consider only a static model in this article, we omit the subscript t throughout for notational simplicity.

Under suitable restrictions on the parameters $\{\alpha_i, \beta_{ik}\}$ and the definitions of the factor returns $\{\Lambda_k\}$, the linear multi-factor model (1) is consistent with a number of asset-pricing models such as the Sharpe-Lintner Capital Asset-Pricing Model (CAPM) (Sharpe, 1964; Lintner, 1965), Merton's Intertemporal CAPM (Merton, 1973), Ross's Arbitrage Pricing Theory (APT) (Ross, 1976), and the Fama-French multi-factor models (Fama and French, 1993, 2015). In particular, all of these asset-pricing models imply that $\alpha_i = 0$, and that returns are simply the sum of the risk-free rate plus all the risk premia multiplied by the

[11]See, for example, Chen, Roll, and Ross (1986), Fama and French (1993, 2015), Jegadeesh and Titman (1993), Jagannathan and Wang (1996), Lettau and Ludvigson (2001), Pástor and Stambaugh (2003), Yogo (2006), Adrian, Etula, and Muir (2014), Hou, Xue, and Zhang (2015), and He, Kelly, and Manela (2017).

[12]See, for example, Fama and MacBeth (1973), Ferson and Harvey (1991), Shanken (1992), Lewellen, Nagel, and Shanken (2010), Connor, Hagmann, and Linton (2012), Bai and Zhou (2015), Gagliardini, Ossola, and Scaillet (2016, 2019), Gu, Kelly, and Xiu (2020), and Raponi, Robotti, and Zaffaroni (2020).

[13]See, for example, Harvey, Liu, and Zhu (2016), Green, Hand, and Zhang (2017), Kozak, Nagel, and Santosh (2018), Feng, Giglio, and Xiu (2020), and Freyberger, Neuhierl, and Weber (2020).

asset's corresponding risk exposures or β_{ik}'s.

However, to measure the impact of impact investing, we take no position as to whether any particular asset-pricing model holds. Instead, we derive the implications of impact investing on the statistical properties of impact-portfolio returns without constraining excess returns to be zero. These properties can then be used to interpret impact from multiple perspectives.

3.1 The No-Impact Baseline Case

We begin by stating the near-trivial result that arbitrary portfolios formed according to criteria unrelated to the parameters of the return-generating processes $\{R_{it}\}$ are necessarily less than or equal to the investment performance of the mean-variance optimal portfolio.[14]

Proposition 1. *If asset returns satisfy (1)–(2) and $\alpha_1 = \cdots = \alpha_N = 0$, then any arbitrary subset $\mathcal{S} \subseteq \{1, \ldots, N\}$ formed independently of the joint distribution of returns, $\{R_{it}\}$, satisfies the following inequality:*

$$\max_{\left\{\omega_1,\ldots,\omega_N \,|\, \sum_{i=1}^N \omega_i = 1\right\}} \mathrm{E}[U(W)] \quad \geq \quad \max_{\left\{\omega_1^c,\ldots,\omega_N^c \,|\, \sum_{i\in\mathcal{S}} \omega_i^c = 1 \text{ and } \omega_i^c = 0 \text{ for } i \notin \mathcal{S}\right\}} \mathrm{E}[U(W^c)] \tag{3}$$

for any non-decreasing concave utility function $U(\cdot)$ where

$$W \;\equiv\; \sum_{i=1}^N \omega_i R_i \quad \text{and} \quad W^c \;\equiv\; \sum_{i\in\mathcal{S}} \omega_i^c R_i \;. \tag{4}$$

In addition, under certain fairly realistic conditions given in the Appendix, we show that the loss in utility by restricting to the subset $\mathcal{S}$ is generally small, as long as the number of securities excluded by $\mathcal{S}$ is small relative to the total number of securities, N.

This proposition confirms the common critique that skeptics often level against impact investing. If the constraint $\mathcal{S}$ has nothing to do with the characteristics of the underlying asset returns, $\{R_i\}$, then imposing such constraints can only reduce investment performance or, at best, achieve the unconstrained optimum. In particular, the independence of $\mathcal{S}$ and $\{R_i\}$ implies that the excess returns, $\{\alpha_i\}$, are indistinguishable from $\{\epsilon_i\}$, in which case we are essentially assuming zero excess returns, so there is no possibility of generating any excess performance.[15] In addition, although impact investing in this special case cannot improve

[14] Proofs of all propositions are provided in the Appendix.

[15] In this baseline portfolio selection problem, investors do not have information on the impact of individual securities, or $\mathbf{X}$ defined in the next section. In other words, investors maximize their unconditional mean-

returns, the under-performance is likely to be small (assuming no transactions costs or fees, of course).

However, suppose we allow for non-zero alphas that are unobserved to investors; in other words, the unconstrained optimization problem in (3) does not have the ability to find securities with positive alphas. If we relax the condition that $\mathcal{S}$ is independent of the joint distribution of $\{R_i\}$, then Proposition 1 clearly does not hold. For example, suppose:

$$\mathcal{S} \;\; = \;\; \{\, i \,:\, \alpha_i > 0 \;,\; i = 1, \ldots, N\} \;. \tag{5}$$

Clearly in this case, it is possible for the risk-adjusted returns of the $\mathcal{S}$-portfolio to beat those of the unconstrained portfolio, given that the subset contains all positive-alpha securities and the complement contains the reverse. This conclusion may seem counterintuitive because the constrained portfolio is, by definition, a feasible solution in the unconstrained case, so how can imposing the constraint ever improve performance? The answer lies in the fact that, in the unconstrained case, information about the $\{\alpha_i\}$ is not available—the constraint contains private information[16] that can dramatically improve performance. Therefore, the constrained solution is actually not feasible in the unconstrained case.

So the fundamental question of whether an impact investment has positive (or negative) financial impact reduces to the information content in the constraint, i.e., the relation between the constraint and the joint distribution of asset returns. No relation implies no information, hence no impact. But the presence of even the slightest amount of dependence between the constraint and returns implies the possibility of some degree of impact. We can quantify this degree by being explicit about the statistical relation between asset returns and the impact factor.

Of course, this counter-example assumes the existence of mispriced or positive-alpha securities (5), but an equally valid equilibrium/no-arbitrage interpretation is that the α_is are omitted factors from the investor's linear-factor benchmark. Either investors are unaware of these factors, or they do not have the ability to access them (e.g., exotic betas from private equity, distressed debt, event-driven opportunities, etc.). Under this interpretation, impact investing can be viewed as providing investors with alternative risk premia.

Our framework accommodates both interpretations—as we describe below—and offers a systematic and quantitative approach to measuring impact in either case.

variance utility, which corresponds to the "type-A (ESG-aware)" investors in Pedersen, Fitzgibbons, and Pomorski (2021).

[16] By private information, we mean that the alphas are assumed to be unobservable by investors. Without the constraint, $\mathcal{S}$, investors have no way to select securities with positive alphas. In this sense, the constraint is, in fact, a mechanism for alpha selection and therefore contains valuable information.

3.2 Impact Factors and Induced Order Statistics

To measure the effects of impact investing on investment performance, we assume that the excess return of the i-the security, α_i, is not observable, whereas the impact factor, X_i, for that security is. Contrary to the usual asset-pricing set-up in which the α_is are assumed to be fixed constants (and, in equilibrium or under no-arbitrage conditions, identically equal to 0), we assume that they are random variables.

Impact investors select a portfolio based on the impact factor, $\mathbf{X}$, and the excess return of their portfolio is determined by the corresponding vector of excess returns of the individual securities in that portfolio, $\boldsymbol{\alpha} \equiv [\,\alpha_1 \cdots \alpha_N\,]^T$. Specifically, suppose an investor ranks N securities according to $\mathbf{X}$. Let us re-order the bivariate vector $(X_i, \alpha_i)^T, i = 1, 2, \cdots, N$, according to the magnitudes of their first components:

$$\begin{pmatrix} X_{1:N} \\ \alpha_{[1:N]} \end{pmatrix}, \begin{pmatrix} X_{2:N} \\ \alpha_{[2:N]} \end{pmatrix}, \cdots, \begin{pmatrix} X_{N:N} \\ \alpha_{[N:N]} \end{pmatrix} \tag{6}$$

where $X_{1:N} < X_{2:N} < X_{N:N}$ and the notation $X_{i:N}$ denotes the i-th order statistic from a total of N random variables. The notation $\alpha_{[i:N]}$ represents the i-th *induced order statistic*,[17] where the order is *induced* by another variable $\mathbf{X}$.

3.3 Defining an Impact Portfolio

Impact investing essentially corresponds to the selection of securities based on the impact factor, $\mathbf{X}$. For example, an investor may choose to invest in the top n_0 securities ranked by $\mathbf{X}$, or form portfolios long the top decile and short the bottom decile. In general, we define an *impact portfolio* to be any portfolio $\mathcal{S}(\mathbf{X})$ formed as a function of the impact factor, $\mathbf{X}$. With portfolio weights $\{\omega_i,\ i \in \mathcal{S}\}$, the return of the impact portfolio is given by:

$$R_{\mathcal{S}} = \sum_{i \in \mathcal{S}} \omega_i R_i. \tag{7}$$

To characterize $R_{\mathcal{S}}$, we therefore need to quantify the distribution of the excess returns—or the induced order statistic, $\alpha_{[i:N]}$, given certain assumptions on the joint distribution of $(\mathbf{X}, \boldsymbol{\alpha})$.

[17] The term was coined by Bhattacharya (1974) to distinguish between random variables ranked by their own realized values versus random variables ranked by the realizations of related random variables. These indirectly ranked statistics are also referred to as *concomitants* of the order statistic, $X_{i:N}$ (David, 1973). Lo and MacKinlay (1990) applied these same statistical tools to quantify data-snooping biases in testing financial asset-pricing models.

Note that $\mathbf{X}$ can represent a variety of characteristics related to metrics for climate change, sustainable farming, tobacco usage, gambling, and any other SRI or ESG considerations. In fact, our framework applies more generally to any characteristics of a security including, for example, the traditional value, size, and momentum factors, as well as denizens of the "factor zoo" described in the recent literature (Harvey, Liu, and Zhu, 2016; Feng, Giglio, and Xiu, 2020; Hou, Xue, and Zhang, 2020). For the purposes of this study, we focus on the impact investing interpretation, but will discuss broader interpretations in Section 7.

4 Characterizing Excess Returns

To assess the impact of impact portfolios, we require the distribution of $\alpha_{[i:N]}$, which can be derived explicitly under the following assumption:

(A1) $(X_i, \alpha_i)^T, i = 1, 2, \cdots, N$, are independently and identically distributed (IID) bivariate normal random vectors with mean $(\mu_x, \mu_\alpha)^T$, variance $(\sigma_x^2, \sigma_\alpha^2)^T$, and correlation $\rho \in (-1, 1)$.

The assumption that α_i is random is somewhat unconventional, so a few clarifying remarks are in order. This assumption was first used in Lo and MacKinlay (1990) to represent cross-sectional estimation errors of intercepts from CAPM regressions. However, in our current context, we interpret the randomness in α_i as a measure of uncertainty as to the degree of mispricings of securities in our investment universe. This uncertainty can be interpreted from a Bayesian perspective as the degree of conviction that mispricings exist in the cross section. Under this interpretation, we will make the auxiliary assumption—without loss of much generality—that all α_is are mean 0 (μ_α=0). This corresponds to centering the Bayesian prior on zero average deviations from equilibrium or no-arbitrage pricing in our investment universe, a reasonable and more realistic first approximation that still allows for mispricings which, of course, motivates a significant portion of the asset management industry's products and services.[18] Moreover, we can calibrate the degree of mispricings in our model through σ_α^2—smaller values correspond to greater efficiency, and larger values correspond to lower efficiency and more active management opportunities.

However, our framework can also be interpreted from an equilibrium/no-arbitrage perspective, where non-zero α_is are due to the presence of omitted factors that investors are either unaware of or unable to access directly. Under this interpretation, we will see below

[18]In fact, Grossman and Stiglitz (1980) have argued that the presence of occasional mispricings is a pre-requisite for achieving informationally efficient markets, otherwise, no one has any incentive to gather information and incorporate it into market prices.

that the randomness in α_i is due to cross-sectional variability in security i's omitted-factor betas. In this case, however, it is possible for μ_α to be non-zero to reflect the risk premia of the omitted factors.

Regardless of the interpretation of α_i, the theory of induced order statistics allows us to completely characterize its statistical properties. We first present its finite-sample distribution, followed by asymptotic results when the number of securities, N, increases without bound.

4.1 Finite-Sample Distribution

We first observe that the mean and standard deviation of the impact factor, $\mathbf{X}$, do not actually matter for the distribution of $\alpha_{[i:N]}$'s, because it is only the relative order of X_i's that determines the order of $\alpha_{[i:N]}$'s. Therefore, we assume without loss of generality that $\mu_x = 0$ and $\sigma_x = 1$, so that $\mathbf{X}$ is a standard normal random vector. Then the following result characterizes the finite-sample distributions of the induced order statistics $\{\alpha_{[i:N]}\}$:

Proposition 2. *Under Assumption (A1), the expected value of the i-th induced order statistic* $\alpha_{[i:N]}, i = 1, 2, \cdots, N$ *is given by:*

$$\mu_i \equiv \mathrm{E}\left[\alpha_{[i:N]}\right] = \rho\sigma_\alpha \mathrm{E}\left[X_{i:N}\right]. \tag{8}$$

The variance of the i-th induced order statistic $\alpha_{[i:N]}, i = 1, 2, \cdots, N$ *is given by:*

$$\sigma_i^2 \equiv \mathrm{Var}\left(\alpha_{[i:N]}\right) = \sigma_\alpha^2\left(1 - \rho^2 + \rho^2 \mathrm{Var}\left(X_{i:N}\right)\right). \tag{9}$$

The covariance of the i-th and j-th induced order statistic, $\alpha_{[i:N]}$ *and* $\alpha_{[j:N]}$*, for* $i \neq j$ *is given by:*

$$\sigma_{ij} \equiv \mathrm{Cov}\left(\alpha_{[i:N]}, \alpha_{[j:N]}\right) = \sigma_\alpha^2 \rho^2 \mathrm{Cov}\left(X_{i:N}, X_{j:N}\right). \tag{10}$$

Proposition 2 gives us the first two moments of the induced order statistics, $\alpha_{[i:N]}$'s. We note that all three quantities in (8)–(10) depend on the distribution of the order statistics of standard normal random variables. In fact, the terms $\mathrm{E}\left[X_{i:N}\right]$ in (8), $\mathrm{Var}\left(X_{i:N}\right)$ in (9), and $\mathrm{Cov}\left(X_{i:N}, X_{j:N}\right)$ in (10) can be explicitly evaluated by numerical integration over the density function of $X_{i:N}$ (see David and Nagaraja (2004, Section 3.1), for example).

On the other hand, we can also explicitly evaluate the quantities in (8)–(10) based on the following approximation results:

Proposition 3. *Let* $p_i \equiv \frac{i}{N+1}$ *denote the relative position of the order i in the population of N securities. The expected value and variance of the i-th order statistic of the standard* *normal random variable, $X_{i:N}$, can be approximated up to order $(N+2)^{-2}$, when N increases without bound, by:*

$$\mathrm{E}\left[X_{i:N}\right] \approx \Phi^{-1}(p_i) + \frac{p_i(1-p_i)}{2(N+2)}Q_i'' + \frac{p_i(1-p_i)}{(N+2)^2}\left[\frac{1}{3}(1-2p_i)Q_i''' + \frac{1}{8}p_i(1-p_i)Q_i''''\right] \tag{11}$$

and

$$\mathrm{Var}\left(X_{i:N}\right) \approx \frac{p_i(1-p_i)}{N+2}Q_i'^2 + \frac{p_i(1-p_i)}{(N+2)^2}\left[2(1-2p_i)Q_i'Q_i'' + p_i(1-p_i)\left(Q_i'Q_i''' + \frac{1}{2}Q_i''^2\right)\right] \tag{12}$$

for $i = 1, 2, \cdots N$. And their covariances can be approximated up to order $(N+2)^{-2}$, when N increases without bound, by:

$$\begin{aligned}\mathrm{Cov}\left(X_{i:N}, X_{j:N}\right) \approx \frac{p_i(1-p_j)}{N+2}Q_i'Q_j' + \frac{p_i(1-p_j)}{(N+2)^2}\Big[&(1-2p_i)Q_i''Q_j' + (1-2p_j)Q_i'Q_j'' \\ &+ \frac{1}{2}p_i(1-p_i)Q_i'''Q_j' + \frac{1}{2}p_j(1-p_j)Q_i'Q_j''' + \frac{1}{2}p_i(1-p_j)Q_i''Q_j''\Big]\end{aligned} \tag{13}$$

for $1 \leq i < j \leq N$. Here Q_i', Q_i'', Q_i''', and Q_i'''' are the first four derivatives of $\Phi^{-1}(p_i)$:

$$Q_i' = \left(\Phi^{-1}(p_i)\right)' = \frac{1}{\phi\left(\Phi^{-1}(p_i)\right)} \tag{14}$$

$$Q_i'' = \left(\Phi^{-1}(p_i)\right)'' = \frac{\Phi^{-1}(p_i)}{\phi\left(\Phi^{-1}(p_i)\right)^2} \tag{15}$$

$$Q_i''' = \left(\Phi^{-1}(p_i)\right)''' = \frac{1 + 2\left(\Phi^{-1}(p_i)\right)^2}{\phi\left(\Phi^{-1}(p_i)\right)^3} \tag{16}$$

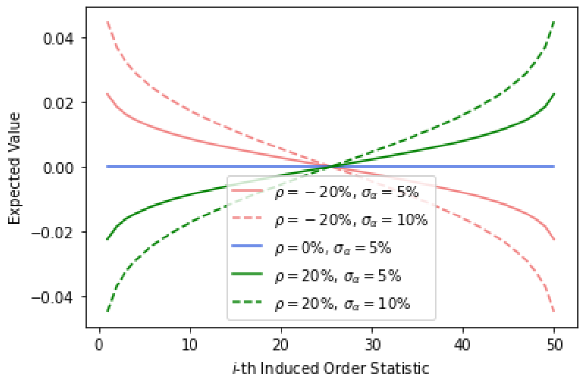
$$Q_i'''' = \left(\Phi^{-1}(p_i)\right)'''' = \frac{\Phi^{-1}(p_i)\left(7 + 6\left(\Phi^{-1}(p_i)\right)^2\right)}{\phi\left(\Phi^{-1}(p_i)\right)^4}. \tag{17}$$

Φ and ϕ are the cumulative distribution function (CDF) and density function of the standard normal distribution, respectively.

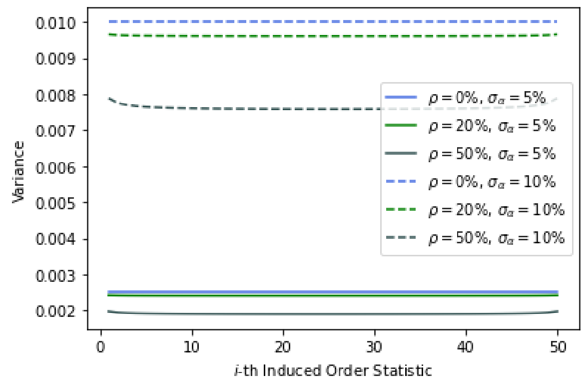
Although the approximations in Proposition 3 may seem daunting, their first-order terms are fairly intuitive. The first term in (11) is $\Phi^{-1}(p_i)$, which simply approximates $\mathrm{E}\left[X_{i:N}\right]$ by the inverse CDF applied to the relative rank, $p_i \equiv \frac{i}{N+1}$, of the i-th order statistic, which is a well-known first-order approximation by itself.[19]

[19] In fact, $\mathrm{E}\left[X_{i:N}\right] \approx \Phi^{-1}(p_i)$ is a reasonable first-order approximation. For example, David and Nagaraja (2004, Sections 4.5 and 4.6) give the following bound: $\Phi^{-1}\left(\frac{i-1}{N}\right) \leq \mathrm{E}\left[X_{i:N}\right] \leq \Phi^{-1}\left(\frac{i}{N}\right)$.

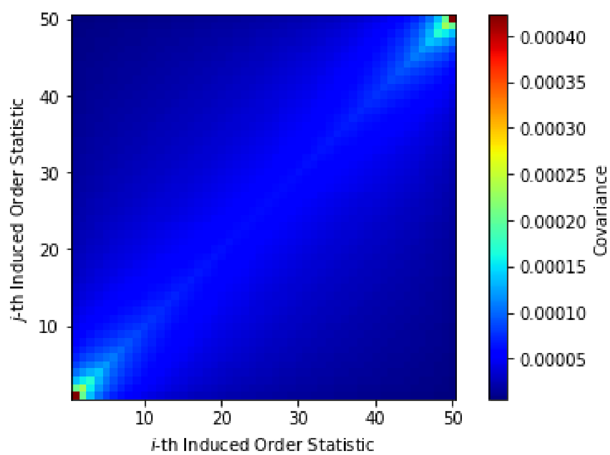
Figure 1 displays the mean, variance, and covariances of the induced order statistic, $\alpha_{[i:N]}$, for a collection of $N=50$ securities, as given in Proposition 2 using the approximations in Proposition 3. When the correlation, ρ, between $\boldsymbol{\alpha}$ and $\mathbf{X}$ is positive, the expected value of the induced order statistic increases as the order i increases (see Figure 1a). The dispersion of the mean is larger when the correlation, ρ, or the dispersion of the unknown $\boldsymbol{\alpha}$, σ_α, is larger.



(a) Expected Value



(b) Variance



(c) Covariances

Figure 1: Mean, variance, and covariances of the induced order statistic, $\alpha_{[i:N]}$. The total number of securities is set to 50 for illustrative purposes. In (c) we set $\rho = 20\%$ and $\sigma_\alpha = 5\%$.

In addition, Figure 1b shows that the variances, $\mathrm{Var}\left(X_{i:N}\right)$, stay relatively constant across the ordered securities i and are primarily determined by ρ and σ_α. In fact, we will see in Section 4.3 that as the number of securities increases without bound, the variance converges to a constant across all i.

Finally, the covariances, $\mathrm{Cov}\left(X_{i:N}, X_{j:N}\right)$, are very close to zero except when i and j are close to 0 or 50, the two extremes. We will also see in Section 4.3 that as N increases without bound, the covariances approach zero, implying that induced order statistics are mutually independent in the limit.

4.2 Comparison with Conventional Order Statistics

To develop further intuition for the effect of induced ordering, we compare the distributions of induced order statistics with their conventional order statistics counterparts, $\alpha_{i:N}$. Note that this comparison is merely meant to be an illustrative thought experiment; $\boldsymbol{\alpha}$ is unobservable by assumption, hence such rankings are not feasible in practice. Nonetheless, this provides a useful comparison to what can be achieved by ordering based on the impact factor, $\mathbf{X}$.

Proposition 4. *Under Assumption (A1), the first two moments of the induced order statistic, $\alpha_{[i:N]}$, are related to the order statistic, $\alpha_{i:N}$, by the following identities:*

$$\mu_i \equiv \mathrm{E}\left[\alpha_{[i:N]}\right] = \rho \mathrm{E}[\alpha_{i:N}] \tag{18}$$

$$\sigma_i^2 - \sigma_\alpha^2 \equiv \mathrm{Var}\left(\alpha_{[i:N]}\right) - \sigma_\alpha^2 = \rho^2\left[\mathrm{Var}\left(\alpha_{i:N}\right) - \sigma_\alpha^2\right] \tag{19}$$

$$\sigma_{ij} \equiv \mathrm{Cov}\left(\alpha_{[i:N]}, \alpha_{[j:N]}\right) = \rho^2 \mathrm{Cov}\left(\alpha_{i:N}, \alpha_{j:N}\right). \tag{20}$$

Proposition 4 tells us that the mean, variance, and covariances of the induced order statistics, $\alpha_{[i:N]}$, are essentially a discounted version of the corresponding moments of the conventional order statistics, $\alpha_{i:N}$. The discount factor, ρ, is precisely the correlation between $\mathbf{X}$ and $\boldsymbol{\alpha}$.

To visualize this effect, Figure 2 contains a comparison of the expected excess returns of the induced order statistic, $\alpha_{[i:N]}$, and the order statistic, $\alpha_{i:N}$, for a collection of $N = 50$ securities. As the correlation, ρ, increases to 1, the expected excess return approaches the hypothetical value of sorting based on $\boldsymbol{\alpha}$.

This result highlights the role that induced ordering plays in distinguishing securities with positive alpha from those with negative alpha. The correlation between the sorting variable (in our case, the impact factor) and the target variable (in our case, the unobserved $\boldsymbol{\alpha}$) determines how much of the mean, variance, and covariances from a hypothetical sorting based on $\boldsymbol{\alpha}$ can actually be achieved via the induced ordering of $\mathbf{X}$.

4.3 Asymptotic Distribution

As the number of securities, N, increases without bound, the limiting joint distribution of the induced order statistics, $\alpha_{[i:N]}$, has been derived by Yang (1977) and does not require the normality assumption (A1), hence we can rely on this asymptotic approximation for large samples.

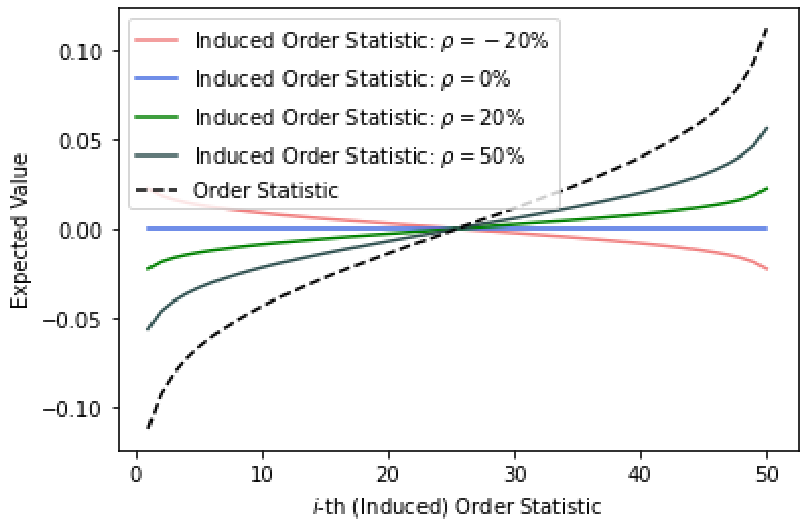


Figure 2: Comparison between the expected value of the induced order statistic, $\alpha_{[i:N]}$, and the order statistic, $\alpha_{i:N}$, with $N = 50$ and $\sigma_\alpha = 5\%$.

Proposition 5. *Assuming $(X_1, \alpha_1)^T, \cdots, (X_N, \alpha_N)^T$ are IID, for any sequence $1 < i_1 < \cdots < i_n < N$ such that, as $N \to \infty$, $i_k/N \to \xi_k \in (0, 1)$ for $k = 1, \cdots, n$, we have:*

$$\lim_{N\to\infty} \mathbb{P}\left(\alpha_{[i_1:N]} < a_1, \cdots, \alpha_{[i_n:N]} < a_n\right) = \prod_{k=1}^{n} \mathbb{P}\left(\alpha_k < a_k | F_x(X_k) = \xi_k\right), \tag{21}$$

where $F_x(\cdot)$ is the marginal CDF of X_i.

Proposition 5 implies that the induced order statistics at distinct quantiles are asymptotically independent, consistent with the finite sample observations in Proposition 3 and Figure 1. Also, because the conditional distribution of jointly normal random vectors is still normal, we can characterize the first two moments of the induced order statistics asymptotically via the following result.

Proposition 6. *Under Assumption (A1), as N increases without bound, the induced order statistics, $\alpha_{[i_k:N]}$ ($k = 1, \cdots, n$), converge in distribution to independent Gaussian random variables with mean $\mu(\xi_k)$ and variance $\sigma^2(\xi_k)$, where*

$$\mu(\xi_k) \equiv \rho(\sigma_\alpha/\sigma_x)\left[F_x^{-1}(\xi_k) - \mu_x\right] = \rho\sigma_\alpha\Phi^{-1}(\xi_k), \tag{22}$$

$$\sigma^2(\xi_k) \equiv \sigma_\alpha^2(1 - \rho^2). \tag{23}$$

Note that the mean and variance here are consistent with the finite-sample results in Proposition 2 when k/N converges to ξ_k. The mean depends on the order k (shown in Figure 3), and its shape is very similar to the finite-sample case (Figure 1a). On the other hand, the variance, $\sigma^2(\xi_k)$, is a constant across all quantiles.

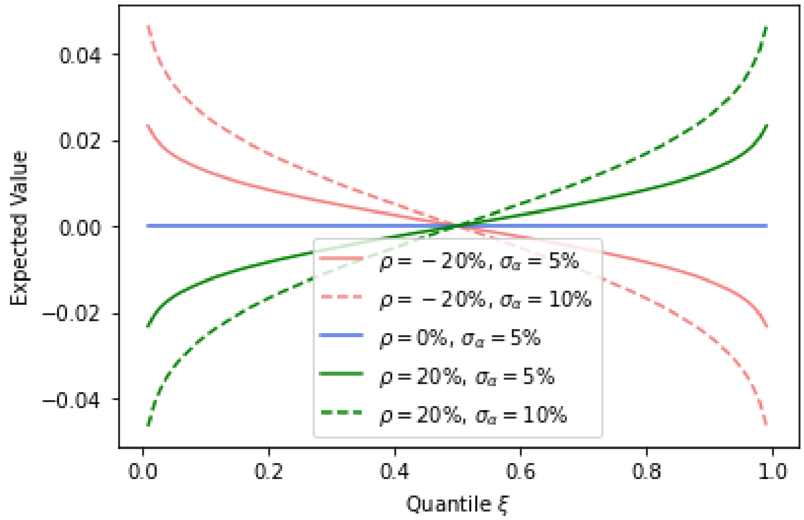


Figure 3: Asymptotic mean of the induced order statistic, $\alpha_{[i_k:N]}$, as $i_k/N \to \xi \in (0,1)$.

4.4 Interpreting Excess Return as Omitted Factors

Having completely characterized the stochastic properties of the excess returns $\boldsymbol{\alpha}$ of securities ranked according to an arbitrary impact factor $\mathbf{X}$, we now provide an explicit derivation of the equilibrium/no-arbitrage interpretation of $\boldsymbol{\alpha}$ as risk premia associated with omitted factors.

Let security returns follow the K-factor asset-pricing model as specified in (1)–(2), but now assume there are no mispricings. However, suppose that investors only account for the first factor Λ_1, without loss of generality, and are unaware of the remaing $K-1$ factors $\Lambda_2, \ldots, \Lambda_K$. We define:

$$\lambda_{ik} \equiv \beta_{ik}(\Lambda_k - R_f) \tag{24}$$

to be factor k's contribution to security i's return, for $i = 1, \ldots, N$ and $k = 2, \ldots, K$, and

$$\lambda_i \equiv \sum_{k=2}^{K} \lambda_{ik} \tag{25}$$

to be the total net contribution of all the omitted factors to security i's return. Given that investors are unaware of factors $2, \ldots, K$, the total excess expected returns for the securities in our universe appear to be alphas to such investors:

$$\alpha_i \quad \equiv \quad \mathrm{E}[\lambda_i] \quad = \quad \sum_{k=2}^{K} \beta_{ik}(\mathrm{E}[\Lambda_k] - R_f) \ . \tag{26}$$

To characterize the distribution of λ_i after ranking securities based on the impact factor $\mathbf{X}$, we make the following assumption:

(A2) $(X_i, \lambda_i)^T, i = 1, 2, \cdots, N$, are bivariate normal random vectors with their marginal distributions and paired correlations defined by:

$$\mu_x \equiv \mathrm{E}[X_i],\ \mu_\lambda \equiv \mathrm{E}[\lambda_i],\ \sigma_x^2 \equiv \mathrm{Var}(X_i),\ \sigma_\lambda^2 \equiv \mathrm{Var}(\lambda_i),\ \text{and}\ \rho_{x,\lambda} \equiv \mathrm{Corr}(X_i, \lambda_i) \tag{27}$$

for $i = 1, \ldots, N$. In addition, the correlation across different securities are defined by:

$$\rho_x \equiv \mathrm{Corr}(X_i, X_j),\ \rho_\lambda \equiv \mathrm{Corr}(\lambda_i, \lambda_j),\ \text{and}\ \tilde{\rho}_{x,\lambda} \equiv \mathrm{Corr}(X_i, \lambda_j) \tag{28}$$

for $i \neq j$.

Under this assumption, the cross-sectional randomness of λ_i can be interpreted as variations coming from both the factor values and the distribution of factor betas across companies in our universe. $(X_i, \lambda_i)^T$ can be correlated across securities, and their correlation structure is described by the four parameters $\rho_{x,\lambda}$, $\tilde{\rho}_{x,\lambda}$, ρ_x, and ρ_λ.

We can characterize the first two moments of $\lambda_{[i:N]}$. Recall that the notation $\lambda_{[i:N]}$ denotes the i-th induced order statistic where the order is *induced* by the impact factor $\mathbf{X}$. We again assume without loss of generality that $\mu_x = 0$ and $\sigma_x = 1$, so that $\mathbf{X}$ is a standard normal random vector. But we allow for a non-zero risk premium μ_λ.

Proposition 7. *Under Assumption (A2), define*

$$\rho_{adj} \equiv \frac{\rho_{x,\lambda} - \tilde{\rho}_{x,\lambda}}{1 - \rho_x} \tag{29}$$

to be an adjusted correlation. The expected value of the i-th induced order statistic $\lambda_{[i:N]}, i = 1, 2, \cdots, N$ is given by:

$$\mathrm{E}\left[\lambda_{[i:N]}\right] = \mu_\lambda + \rho_{adj}\sigma_\lambda \mathrm{E}\left[X_{i:N}\right]. \tag{30}$$

The variance of the i-th induced order statistic $\lambda_{[i:N]}, i = 1, 2, \cdots, N$ is given by:

$$\mathrm{Var}\left(\lambda_{[i:N]}\right) = \sigma_\lambda^2\left(1 - \rho_{adj}^2 + \rho_{adj}^2 \mathrm{Var}\left(X_{i:N}\right)\right). \tag{31}$$

The covariance of the i-th and j-th induced order statistic, $\lambda_{[i:N]}$ and $\lambda_{[j:N]}$, for $i \neq j$ is given by:

$$\mathrm{Cov}\left(\lambda_{[i:N]}, \lambda_{[j:N]}\right) = \sigma_\lambda^2 \rho_{adj}^2 \mathrm{Cov}\left(X_{i:N}, X_{j:N}\right) + \left(\rho_\lambda - \rho_x \rho_{adj}^2\right). \tag{32}$$

Proposition 7 characterizes the return from omitted factors for the i-th security induced by the impact factor $\mathbf{X}$. This result highlights an important implication when estimating the financial impact of impact investing. Given any definition of impact, $\mathbf{X}$, if the portfolio selected based on $\mathbf{X}$ produces a non-zero excess return, $\mathbf{X}$ must be correlated with some factors not previously accounted for in the asset-pricing framework. This may imply the existence of a new factor that corresponds to the very definition of $\mathbf{X}$, such as an "ESG factor" or a "carbon factor" (Bolton and Kacperczyk, 2021).

On the other hand, Proposition 7 also implies that, when forming a portfolio, if one uses a selection criteria that appears independent of return characteristics such as market betas and factor loadings, it may still be correlated with omitted factor risk premiums, in which case the selection criteria will produce nonzero excess returns. In other words, what appears to be an "impact factor" (a selection criteria $\mathbf{X}$ based on a particular concept) may just be correlations with other omitted factors that are, in fact, unrelated to the impact concept one intends to capture. Therefore, impact estimates may be inaccurate and misleading without first properly accounting for all known factors.

This observation is supported empirically by both Blitz and Fabozzi (2017) in the case of estimating excess returns for sin stocks, and Madhavan, Sobczyk, and Ang (2021) for ESG scores, both of which we discuss in more detail in Section 6.

5 Impact Portfolio Construction

Having quantified the distribution of the induced order statistics, $\alpha_{[i:N]}$, we can now construct portfolios based on the impact factor, $\mathbf{X}$, and characterize the statistical properties of their excess returns. We first quantify the performance of arbitrary impact portfolios, followed by a special case—equal-weighted portfolios—which is also related to how to estimate ρ and σ_α empirically. We then use the Treynor and Black (1973) framework to derive the optimal weights for each security, as well as the optimal way to combine an impact portfolio with any existing portfolio such as the passive market index. The latter result follows directly from our ability to completely characterize the statistical properties of individual alphas in our framework.

5.1 Properties of Arbitrary Impact Portfolios

Consider an arbitrary impact portfolio of n_0 securities with indexes in $\mathcal{S}$:

$$\mathcal{S} \equiv \{i_1, i_2, \cdots, i_{n_0}\} \tag{33}$$

which is obtained from a rank-ordering of securities from the investment universe according to the impact factor, $\mathbf{X}$. The excess return of the portfolio is then given by:

$$\tilde{\alpha} \equiv \sum_{i \in \mathcal{S}} \omega_i \alpha_{[i:N]} \tag{34}$$

where $\{\omega_i : i \in \mathcal{S}\}$ are arbitrary portfolio weights that sum to 1. Based on the distribution of the individual induced order statistics in Proposition 2, we have the following result for portfolio excess returns:

Proposition 8. *Under Assumption (A1), the expected excess return of a portfolio $\mathcal{S}$ defined in (33) is:*

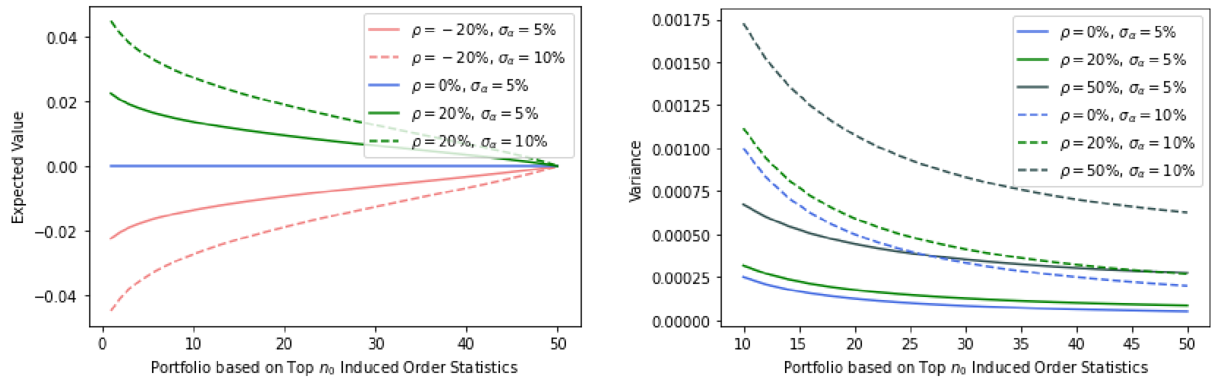
$$\mathrm{E}\left[\tilde{\alpha}\right] = \sum_{i \in \mathcal{S}} \omega_i \mu_i = \rho \sigma_\alpha \sum_{i \in \mathcal{S}} \omega_i \mathrm{E}\left[X_{i:N}\right], \tag{35}$$

and the variance is:

$$\begin{aligned} \mathrm{Var}\left(\tilde{\alpha}\right) =& \sum_{i \in \mathcal{S}} \omega_i^2 \sigma_i^2 + 2 \sum_{i<j \in \mathcal{S}} \omega_i \omega_j \sigma_{ij} \\ =& \sigma_\alpha^2 \left(1 - \rho^2 + \rho^2 \left(\sum_{i \in \mathcal{S}} \omega_i^2 \mathrm{Var}\left(X_{i:N}\right) + 2 \sum_{i<j \in \mathcal{S}} \omega_i \omega_j \mathrm{Cov}\left(X_{i:N}, X_{j:N}\right)\right)\right). \end{aligned} \tag{36}$$

Proposition 8 quantifies the distribution of excess returns for any portfolio constructed according to the impact factor, $\mathbf{X}$. This result implies that the full range of tools and results from modern portfolio theory can be applied here, including: the calculation of various performance metrics such as the Sharpe ratio (Sharpe, 1966), Sortino ratio (Sortino and Van Der Meer, 1991; Sortino and Price, 1994), and information ratios (Treynor and Black, 1973); performance attribution (Brinson and Fachler, 1985; Brinson, Hood, and Beebower, 1986; Brinson, Singer, and Beebower, 1991); and active portfolio management and enterprise risk management (Grinold and Kahn, 1999).

To develop intuition for Proposition 8, consider a portfolio formed by selecting the top n_0 securities based on $\mathbf{X}$. For a market with $N = 50$ securities, Figure 4 displays the mean and variance of the excess return of portfolios formed in this way. As the number of securities in the portfolio, n_0, increases, the excess return decreases because more securities with weaker alphas are included. At the same time, the variance of the portfolio also decreases thanks to the diversification from more securities.

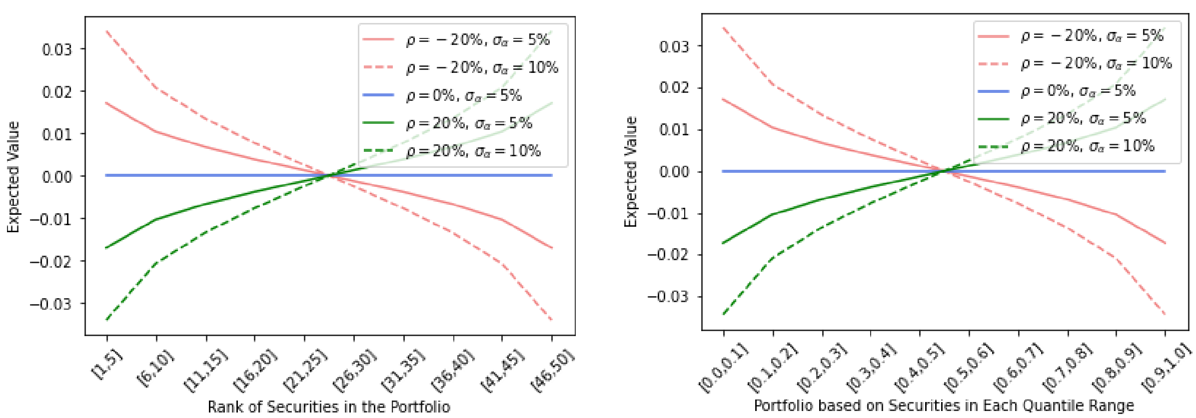


(a) Expected Value

(b) Variance

Figure 4: Distribution of portfolio excess return formed by the top n_0 securities ranked by the impact factor, $\mathbf{X}$. The number of total securities, N, is set to be 50.

Another typical way of forming portfolios is to sort all securities in the universe into 10 deciles based on $\mathbf{X}$. Figure 5a contains the expected excess returns of the 10 deciles, which has a similar shape to the expected excess returns of individual securities in Figure 1a.



(a) $N = 50$

(b) $N \to \infty$

Figure 5: Expected excess return for decile portfolios formed by ranking the impact factor, $\mathbf{X}$. In (a) the number of total securities, N, is set to be 50, and in (b) we show the case when N increases without bound.

Finally, we can also consider portfolios as N increases without bound. Suppose we divide the $[0, 1]$ interval into L segments each of length $1/L$, and pick M equally-spaced quantiles within each segment. Specifically, the l-th portfolio is formed by selecting the following quantiles:

$$\xi_{l,m} = \frac{l - 1 + \frac{m}{M+1}}{L}, \quad m = 1, 2, \cdots, M \tag{37}$$

for $l = 1, 2, \cdots, L$. Figure 5b shows the expected excess returns of this portfolio when $L = M = 10$, which, not surprisingly, has a similar shape to Figure 5a because the portfolio formed by (37) is the limit of the decile portfolio when N increases without bound.

5.2 Estimation of ρ and σ_α.

Two key parameters that characterize the distribution of induced order statistics in Propositions 2 and 6 are ρ, the correlation between unobserved $\boldsymbol{\alpha}$ and $\mathbf{X}$, and σ_α, the cross-sectional standard deviation of α_i. A special case of Proposition 8—equal-weighted portfolios—provides a way to estimate these parameters in practice. Consider an equal-weighted portfolio $\mathcal{S}$ defined in (33) with portfolio weights $\omega_i = 1/n_0$. In this case, Proposition 8 implies that the expected value and variance of portfolio alphas are given by:

$$\mathrm{E}\left[\tilde{\alpha}\right] = \frac{\rho\sigma_\alpha}{n_0} \sum_{i\in\mathcal{S}} \mathrm{E}\left[X_{i:N}\right], \tag{38}$$

$$\mathrm{Var}\left(\tilde{\alpha}\right) = \sigma_\alpha^2 \left(1 - \rho^2 + \frac{\rho^2}{n_0^2}\left(\sum_{i\in\mathcal{S}} \mathrm{Var}\left(X_{i:N}\right) + 2\sum_{i<j\in\mathcal{S}} \mathrm{Cov}\left(X_{i:N}, X_{j:N}\right)\right)\right). \tag{39}$$

Empirical studies usually report excess returns from equal-weighted portfolios formed by ranking some stock characteristics such as the P/E ratio, book-to-value, or ESG score. As a result, the expected value and variance of the impact-portfolio alpha in (38)–(39) lead to a natural estimator of these two parameters based on historical data.

In particular, suppose one empirically measures the portfolio alpha and its variance, which can be substituted into (38)–(39) to yield a system of two equations with respect to ρ and σ_α, where parameters such as the number of securities in the portfolio (n_0) and the total number of securities in the universe (N) can be easily obtained. This leads, in principle, to a solution for ρ and σ_α.

On the other hand, if the variance of the impact-portfolio alpha is difficult to estimate empirically, one can still use (38) to calibrate $\rho\sigma_\alpha$, from which ρ can be solved based on assumptions about the spread in cross-sectional $\boldsymbol{\alpha}$.

In addition, it is worth emphasizing that the estimation of ρ depends implicitly on the frequency of historical data used to estimate impact-portfolio excess returns, $\tilde{\alpha}$. In theory, if the two terms in (38), $\tilde{\alpha}$ and σ_α, both scale linearly as the frequency varies, the estimates of ρ should stay invariant with respect to weekly, monthly, or annual returns. However, they may lead to different empirical estimates in practice, and therefore, the correlation estimated from this procedure should be interpreted in the same frequency space as the return data used.

We apply these methods to four empirical examples in Section 6.

5.3 Treynor-Black Portfolios

A key advantage of our framework is the ability to characterize the alphas of arbitrary impact portfolios via induced order statistics. Given this representation, it is clear that equal-weighted portfolios are not optimal in terms of achieving the best risk-adjusted returns.

However, Treynor and Black (1973) provide a methodology that is designed to maximize a portfolio's Sharpe ratio, which can be directly applied in our case to derive optimal weights for securities selected by the impact factor. To apply the Treynor-Black framework, we rewrite the excess return of the i-th security, α_i, as its mean plus noise:

$$\alpha_i = \mu_i + \zeta_i \tag{40}$$

where $\{\zeta_i\}$ are independent random variables with zero means. We can then combine ζ_i with security i's idiosyncratic error, ϵ_i. Because ζ_i and ϵ_i are independent, the combined idiosyncratic variance for security i is simply $\sigma_i^2 + \sigma(\epsilon_i)^2$, where σ_i^2 is given in (9).

Given any number of securities selected by $\mathbf{X}$, we can form an optimal portfolio based on the Treynor-Black weights, which we summarize in the following result:

Proposition 9. *Under Assumption (A1), the Treynor-Black weight of security i is proportional to its expected alpha divided by its combined idiosyncratic variance:*

$$\omega_i \propto \frac{\mu_i}{\sigma_i^2 + \sigma(\epsilon_i)^2}. \tag{41}$$

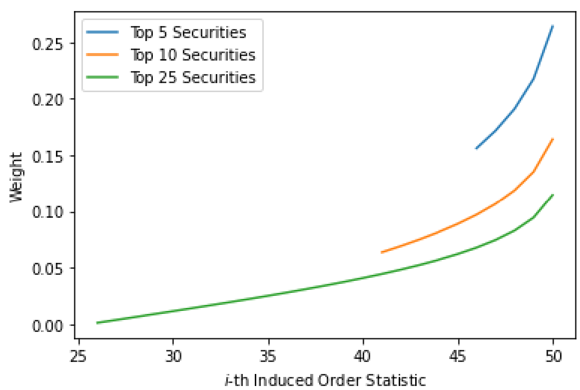
In addition, if the idiosyncratic volatility, $\sigma(\epsilon_i)$, is constant across securities i, as N increases without bound, the Treynor-Black weight of security i in (41) can be further simplified to:

$$\omega_i \propto \frac{\rho\sigma_\alpha\Phi^{-1}(\xi_i)}{\sigma_\alpha^2(1-\rho^2) + \sigma(\epsilon_i)^2} \propto \Phi^{-1}(\xi_i) \cdot \textit{Constant}. \tag{42}$$

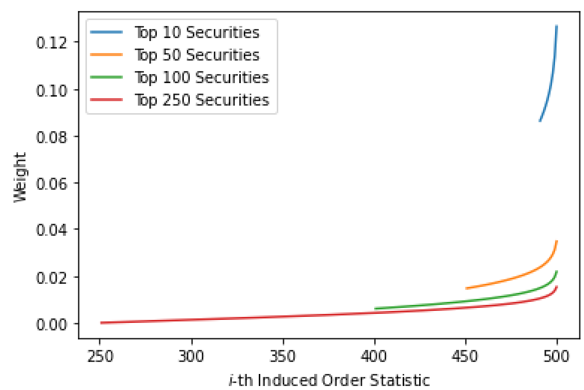
For further intuition behind (41), recall that the variance of the i-th induced order statistic, σ_i^2, is approximately a constant when N is large (see Figure 1b and Proposition 6). The expected excess return, $\mu_i = \rho\sigma_\alpha \mathrm{E}\left[X_{i:N}\right]$, varies with respect to i only through the last term $\mathrm{E}\left[X_{i:N}\right]$. As a result, if each security's idiosyncratic volatility is the same, the Treynor-Black weights of security i in (41) depend only on their relative ranking in the universe of N securities.

Proposition 9 gives an explicit formula for the Treynor-Black weights that optimize the risk-adjusted returns of the impact portfolio, which can easily be implemented in practice. In addition, (42) highlights an important link between the Treynor-Black weights, ω_i, and $\Phi^{-1}(\xi_i)$, the key term that determines the distribution of the i-th induced order statistic, in the special case when idiosyncratic volatility is a constant.

For an illustrative example, consider a portfolio formed by the top n_0 securities ranked by $\mathbf{X}$, and let n_0 vary from 1 to 250. We assume for a moment that the idiosyncratic volatility is 15% for all securities. Figure 6 depicts the weights of this portfolio. As expected, securities that rank higher have higher weight. Based on Proposition 9, the weights in Figure 6 are determined only by the relative rank of the i-th security in the universe of N securities. In other words, changing the correlation, ρ, between $\boldsymbol{\alpha}$ and $\mathbf{X}$ does not affect these weights.



(a) $N=50$



(b) $N=500$

Figure 6: Treynor-Black weights of the securities in the impact portfolio formed by top-ranking securities based on the impact factor, $\mathbf{X}$, with (a) $N=50$; and (b) $N=500$.

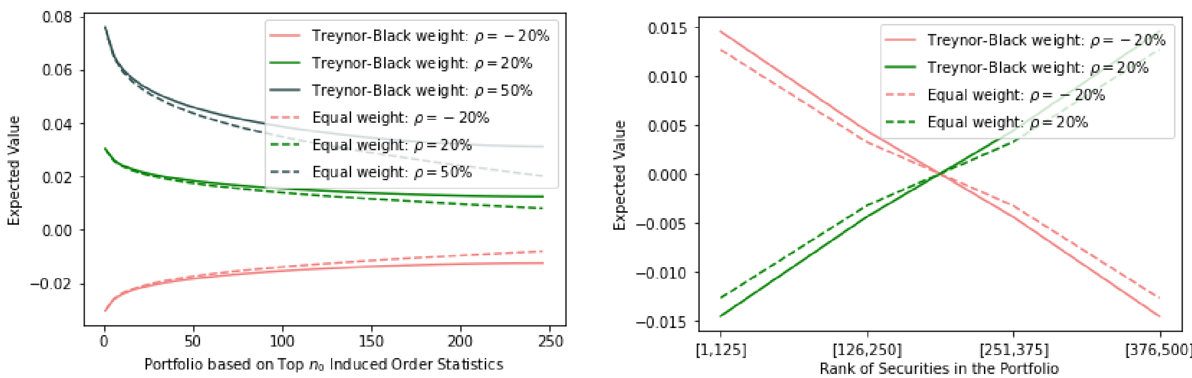
The portfolio selected by ranking $\mathbf{X}$ and applying the Treynor-Black weights in (41) is one specific example of an *impact portfolio* we defined in Section 3.3. Treynor and Black (1973) call this the "active management" portfolio, and its return characteristics are given by:

$$\alpha_A = \sum_{k=1}^{n} \omega_k \mu_k, \tag{43}$$

$$\beta_A = \sum_{k=1}^{n} \omega_k \beta_k, \tag{44}$$

$$\sigma(\epsilon_A)^2 = \sum_{k=1}^{n} \omega_k^2 \left(\sigma_k^2 + \sigma(\epsilon_k)^2\right). \tag{45}$$

Figure 7 contains the expected excess return, α_A, for two examples of the impact portfolio in a collection of $N = 500$ securities. Figure 7a depicts portfolios formed by selecting the top n_0 securities ranked by $\mathbf{X}$. The expected value decreases as n_0 increases and more securities are included. Figure 7b depicts portfolios formed by dividing all securities into four quantiles based on the ordering of $\mathbf{X}$. In both cases, Treynor-Black portfolios (solid line) achieve higher expected excess returns than the equal-weighted portfolios (dashed line).



(a) Top n_0 Portfolio

(b) Quantile Portfolio

Figure 7: Expected excess return of the impact portfolio formed based on Treynor-Black weights, with $N = 500$ and $\sigma_\alpha = 5\%$. The expected excess return of the corresponding equal-weighted portfolios are shown in dashed lines for comparison. (a) shows the case where the top-ranking securities are selected. (b) shows the case where all securities are divided into four segments based on ranking.

5.4 Combining Impact and Passive Portfolios

Once the relative weights of the securities within an impact portfolio are determined, one can combine the portfolio with any other portfolio. For example, we may form an impact portfolio by ranking a company's impact on global warming, which can be combined with other characteristics such as sustainable farming, tobacco usage, and gaming, to form an overall “ESG” portfolio. We can also add the impact portfolio to the suite of portfolios mimicking more traditional asset pricing factors such as value, size, and momentum.

However, perhaps the most natural application is to consider combining the impact portfolio with a passive index fund such as the market portfolio. Let ω_A denote the weight of the impact portfolio, and $1-\omega_A$ the weight of a passive portfolio. To maximize the Sharpe ratio of the combined portfolio, the relative weight is determined by the impact portfolio's excess return and idiosyncratic volatility:

$$\omega_A = \frac{\frac{\alpha_A}{\sigma(\epsilon_A)^2}}{\frac{\mathrm{E}[R_m]-R_f}{\sigma_m^2}} \tag{46}$$

where $\mathrm{E}[R_m]$ and σ_m^2 are the expected return and variance of the passive portfolio, respectively.

We illustrate the impact portfolio's alpha and its corresponding weight, ω_A, using a numerical example. Suppose the passive portfolio has an annualized risk premium of $\mathrm{E}[R_m]-R_f = 6\%$ and volatility of $\sigma_m = 15\%$. The idiosyncratic volatility is a constant $\sigma(\epsilon_i) = 15\%$ for all securities.[20] Consider again a collection of $N = 500$ securities. We divide them into 10 decile portfolios ranked by the impact factor, $\mathbf{X}$. For several different values of ρ and σ_α, Table 1 reports the weight, ω_A, and expected excess return of the impact portfolio, where we present the top and bottom two decile portfolios.

We first consider the case in which the cross-sectional standard deviation $\sigma_\alpha = 1\%$. In other words, most securities' excess returns are within $[-2\sigma_\alpha, 2\sigma_\alpha] = [-2\%, 2\%]$. This is a very conservative assumption for U.S. equities, but even with such a modest range of $\boldsymbol{\alpha}$, the optimal portfolio contains significant weight from the impact portfolio. For example, ω_A for the top decile is 0.66 when the correlation $\rho = 30\%$, and 0.22 when $\rho = 10\%$. Observe that ρ^2 is simply the R^2 of the cross-sectional regression of $\boldsymbol{\alpha}$ on $\mathbf{X}$, so a 30% (10%) correlation implies that only 9% (1%) of the variation in $\boldsymbol{\alpha}$ is explained by $\mathbf{X}$, which is a fairly plausible assumption for a typical impact factor.

When the cross-sectional standard deviation, σ_α, is doubled to 2%, the optimal weight, ω_A, becomes larger. In addition, the expected excess return α_As are also substantial, even with a mild correlation, ρ.

When $\sigma_\alpha = 5\%$, the impact portfolio has a weight of 3.23 for the top decile when $\rho = 30\%$. This implies a highly leveraged portfolio in which more than 200% of the passive portfolio is

[20] This is an innocuous assumption and we show later via simulation that cross-sectional heterogeneity in idiosyncratic volatilities does not affect our conclusions.

Table 1: Expected excess returns for the impact portfolios and their corresponding Treynor-Black weights when combined with a passive portfolio. We set $N = 500$ and assume that the passive portfolio has an annualized expected excess return of $\mathrm{E}[R_m] - R_f = 6\%$ and volatility of $\sigma_m = 15\%$. The idiosyncratic volatility is a constant $\sigma(\epsilon_i) = 15\%$ for all securities.

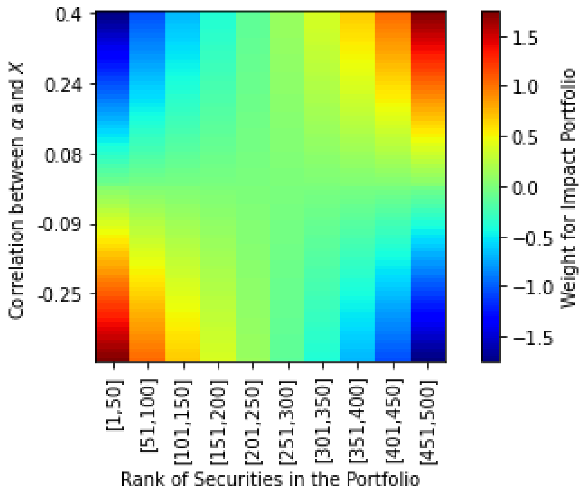
Correlation ρ	Weight ω_A				Expected Excess Return α_A			
	Bottom	2^{nd}	9^{th}	Top	Bottom	2^{nd}	9^{th}	Top
				$\sigma_\alpha = 1\%$				
30% ($R^2 = 9\%$)	-0.66	-0.39	0.39	0.66	-0.6%	-0.3%	0.3%	0.6%
10% ($R^2 = 1\%$)	-0.22	-0.13	0.13	0.22	-0.2%	-0.1%	0.1%	0.2%
-10% ($R^2 = 1\%$)	0.22	0.13	-0.13	-0.22	0.2%	0.1%	-0.1%	-0.2%
-30% ($R^2 = 9\%$)	0.66	0.39	-0.39	-0.66	0.6%	0.3%	-0.3%	-0.6%
				$\sigma_\alpha = 2\%$				
30% ($R^2 = 9\%$)	-1.31	-0.78	0.78	1.31	-1.1%	-0.6%	0.6%	1.1%
10% ($R^2 = 1\%$)	-0.44	-0.26	0.26	0.44	-0.4%	-0.2%	0.2%	0.4%
-10% ($R^2 = 1\%$)	0.44	0.26	-0.26	-0.44	0.4%	0.2%	-0.2%	-0.4%
-30% ($R^2 = 9\%$)	1.31	0.78	-0.78	-1.31	1.1%	0.6%	-0.6%	-1.1%
				$\sigma_\alpha = 5\%$				
30% ($R^2 = 9\%$)	-3.23	-1.93	1.93	3.23	-2.8%	-1.6%	1.6%	2.8%
10% ($R^2 = 1\%$)	-1.08	-0.64	0.64	1.08	-0.9%	-0.5%	0.5%	0.9%
-10% ($R^2 = 1\%$)	1.08	0.64	-0.64	-1.08	0.9%	0.5%	-0.5%	-0.9%
-30% ($R^2 = 9\%$)	3.23	1.93	-1.93	-3.23	2.8%	1.6%	-1.6%	-2.8%

shorted. The corresponding gain in expected excess return is 2.8% when $\rho = 30\%$ and 0.9% when $\rho = 10\%$.

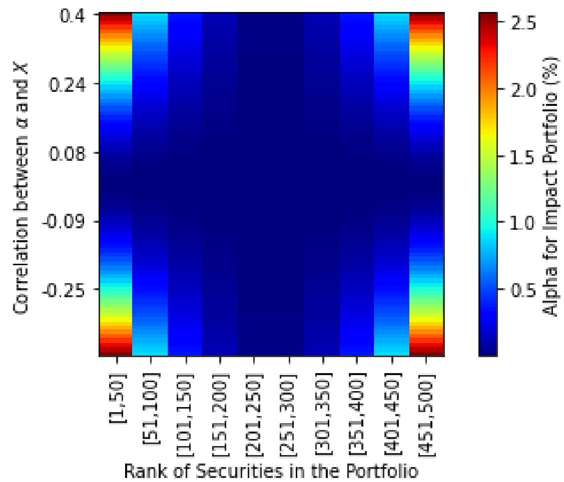
More generally, Figure 8 displays two metrics for the combined portfolio that consists of the impact and passive portfolios, with two different levels of σ_α. In Figures 8a and 8b we consider $\sigma_\alpha = 2\%$. In other words, most of the securities have an alpha within $[-4\%, 4\%]$. The weights of the active portfolio range from -1.5 to 1.5, depending on the correlation between $\boldsymbol{\alpha}$ and $\mathbf{X}$ (Figure 8a). The expected excess return of the combined portfolio ranges from 0% to over 2.5% (Figure 8b).

In Figures 8c and 8d we consider $\sigma_\alpha = 5\%$. In other words, most of the securities have an alpha within $[-10\%, 10\%]$. This is not unimaginable in some highly volatile sectors such as biotech. The weights of the active portfolio can be as high as two, indicating a leveraged impact portfolio and a short position in the passive market portfolio (Figure 8c). In this case, the expected excess return of the impact portfolio can yield up to 14% (Figure 8d)!

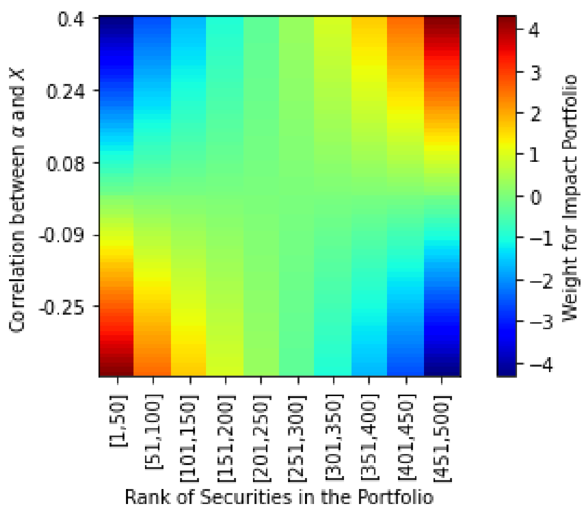
Finally, we show how our impact portfolio improves the efficient frontier and the Capital Market Line to achieve a "super-efficient frontier" under the assumption that $\boldsymbol{\alpha}$ are mispricings. Under the alternate omitted-factor interpretation, the "super-efficiency" of the new frontier may be viewed as the result of additional risk premia not accessible to investors



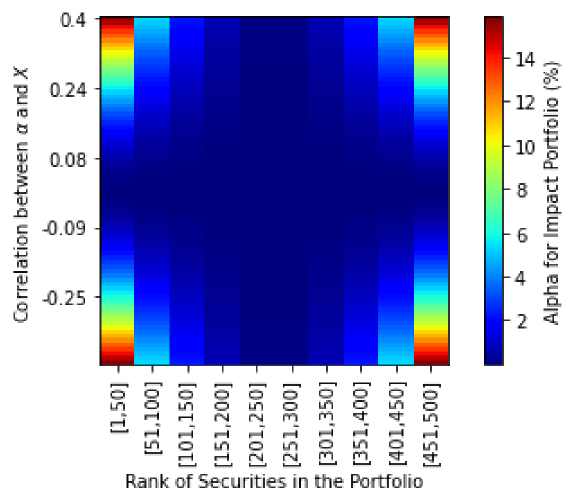
(a) $\sigma_\alpha = 2\%$: Weights



(b) $\sigma_\alpha = 2\%$: Excess Return



(c) $\sigma_\alpha = 5\%$: Weights



(d) $\sigma_\alpha = 5\%$: Excess Return

Figure 8: Performance metrics for the combined portfolio that consists of the impact portfolio with $N = 500$ and passive market portfolio with an annualized risk premium of $\mathrm{E}[R_m] - R_f = 6\%$ and volatility of $\sigma_m = 15\%$. The idiosyncratic volatility is a constant $\sigma(\epsilon_i) = 15\%$ for all securities. (a) and (b) show the Treynor-Black weight for the impact portfolio and the overall expected excess return, respectively, for $\sigma_\alpha = 2\%$. (c) and (d) show the same metrics for $\sigma_\alpha = 5\%$.

except through impact portfolio managers.

Proposition 10. *Under Assumption (A1), the return of the final portfolio, P, that consists of the impact portfolio with Treynor-Black weights and the passive market portfolio is:*

$$R_P - R_f = \omega_A R_A + (1-\omega_A)R_m - R_f = \omega_A \alpha_A + (R_m - R_f)(\beta_A \omega_A + (1-\omega_A)) + \omega_A \epsilon_A, \quad (47)$$

where R_m is the return of the passive portfolio. The expected value and variance of R_P are given by:

$$\mathrm{E}[R_P] - R_f = \omega_A \alpha_A + (\mathrm{E}[R_m] - R_f)(\beta_A \omega_A + (1-\omega_A)), \quad (48)$$

$$\mathrm{Var}[R_P] = \mathrm{Var}[R_m](\beta_A \omega_A + (1-\omega_A))^2 + \omega_A^2 \sigma(\epsilon_A)^2. \quad (49)$$

This forms a super-efficient frontier in comparison to the Capital Market Line associated with the passive portfolio.

Figure 9 displays the passive portfolio as well as several combinations with impact portfolios in relation to the efficient frontier. We continue to assume that the passive portfolio has an annualized risk premium of $\mathrm{E}[R_m] - R_f = 6\%$ and volatility of $\sigma_m = 15\%$. In Figure 9a, the idiosyncratic volatility is assumed to be a constant $\sigma(\epsilon_i) = 15\%$ for all securities. As the correlation, ρ, and variance in alpha, σ_α^2, increase, the impact portfolios (defined as the top half of the securities ranked by $\mathbf{X}$) are able to improve the original Capital Market Line, leading to super-efficient frontiers.

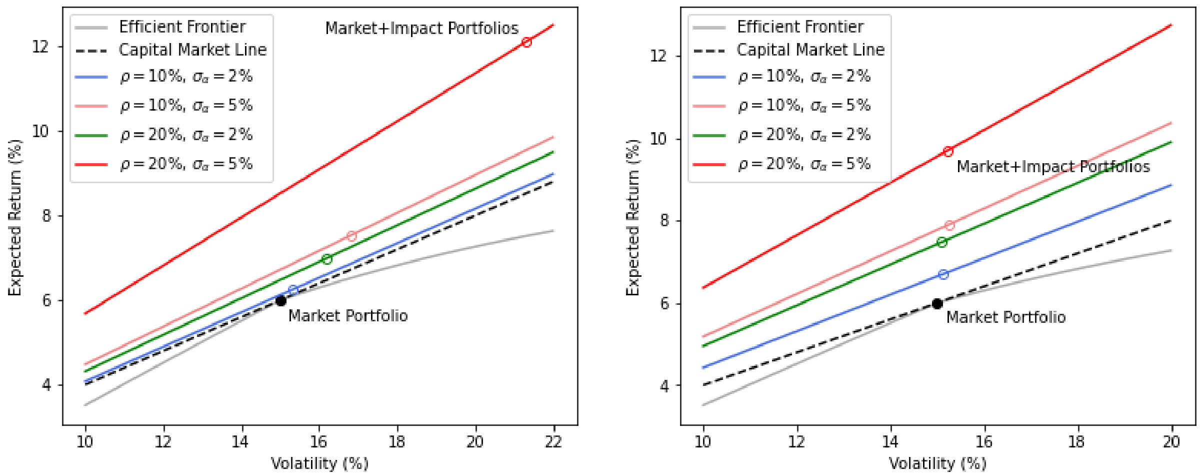
The results in this section have so far relied on the unrealistic assumption that the idiosyncratic volatility, $\sigma(\epsilon_i)$, is cross-sectionally constant. To check the robustness of our results, we simulate a collection of securities where the i-th security's idiosyncratic volatility follows a lognormal distribution:

$$\log(\sigma(\epsilon_i)) \sim \mathrm{Normal}(\mu_\epsilon, \sigma_\epsilon). \quad (50)$$

Calibrating this process to empirically plausible values in the literature (e.g., Kuntz (2020)), we perform simulations for $\log(\mu_\epsilon) = 15\%$ and $\sigma_\epsilon = 1$.

Figure 9b confirms that even with such cross-sectional heterogeneity, the Capital Market Line is still improved, leading to super-efficient frontiers. Compared with the simpler case in Figure 9a, the magnitude of the improvements in alpha for the combined portfolios is bigger.

The two examples in Figure 9 are based on impact portfolios formed with the top half of the securities ranked according to $\mathbf{X}$. More generally, each investor can decide on the most suitable subset of securities depending on their desired level of impact. In the special case



(a) Constant $\sigma(\epsilon_i)$ (b) Simulated $\sigma(\epsilon_i)$

Figure 9: Super-efficient frontiers from the combined portfolio that consists of the impact portfolio with $N = 500$ and passive market portfolio with an annualized risk premium of $\mathrm{E}[R_m] - R_f = 6\%$ and volatility of $\sigma_m = 15\%$. In (a) the idiosyncratic volatility is a constant $\sigma(\epsilon_i) = 15\%$ for all securities. In (b) we simulated idiosyncratic volatility based on (50) and apply a maximum leverage ratio of 3:1.

of an ESG metric, this process yields the "ESG-efficient frontier" of Pedersen, Fitzgibbons, and Pomorski (2021), which provides the highest attainable Sharpe ratio for each ESG level.

Our results add to Pedersen, Fitzgibbons, and Pomorski (2021) in two ways. First, we provide the explicit construction of the optimal impact portfolio, thanks to our ability to characterize impact-ranked returns for individual securities in Proposition 2-9, which yields an explicit measure of the degree to which $\mathbf{X}$ improves or worsens the efficient frontier. Second, in their framework, the standard mean-variance tangency portfolio has the highest Sharpe ratio among all portfolios, and restricting portfolios to have any ESG score other than that of the tangency portfolio must yield a lower Sharpe ratio (Pedersen, Fitzgibbons, and Pomorski, 2021, p. 573). Our framework shows that, when there are non-zero $\boldsymbol{\alpha}$ that are otherwise inaccessible to investor (either interpreted as mispricings or omitted factors), simply using the impact factor, $\mathbf{X}$, to form portfolios based on ranking or subsetting can also improve the efficient frontier, because of the information implicit in the selection criteria.

6 Applications to Four Impact Investments

In this section, we apply our framework to four particular examples of impact investing: biotech venture philanthropy, divesting from sin stocks, ESG investing, and the GameStop short squeeze during January 2021.

6.1 Venture Philanthropy: The Cystic Fibrosis Foundation

The concept of venture philanthropy (VP) was introduced by Letts, Ryan, and Grossman (1997), who suggested that nonprofit organizations could learn useful practices from venture capitalists, including due diligence, risk management, performance measurement, relationship management, investment duration and size management, and exit strategies. This approach has received a great deal of attention both within and outside the field (Grossman, Appleby, and Reimers, 2013), and has now been applied to education (Scott, 2009), community redevelopment (Van Slyke and Newman, 2006), and medical R&D (Scaife, 2008; Salzman, 2016), among other fields. In particular, recent biomedical advances have created significant opportunities for a new generation of therapeutics (Sharp and Hockfield, 2017). However, early-stage R&D efforts often face a dearth of funding, given the high risk of failure and significant funding requirements. This has been particularly true for rare disease drug development, where market sizes are often too small to attract much attention and funding (Kim and Lo, 2019).

We consider the example of the Cystic Fibrosis (CF) Foundation—profiled in the case study by Kim and Lo (2019)—and conclude that VP in biomedicine can produce significant positive excess returns. This example illustrates the possibility of an impact investment that is positively correlated with $\boldsymbol{\alpha}$, or an omitted factor that patient advocacy groups can more easily exploit than typical investors.

The CF Foundation is the world's leading philanthropic organization for CF, a rare genetic disease that currently affects more than thirty thousand Americans. Over a period of 12 years, the CF Foundation invested \$150 million to fund CF drug development efforts at Vertex Pharmaceuticals, a Boston-based biotechnology firm. This work led to the identification and development of Kalydeco, the first FDA-approved treatment to address the underlying causes of CF. The Foundation's investment entitled them to receive royalties calculated as a percentage of future sales of successful CF drugs. In 2014, their rights to Vertex royalties were sold to an outside investment firm, New York City-based Royalty Pharma, for \$3.3 billion in cash.

From the financial perspective, a \$3.3 billion return from a \$150 million investment is the dream scenario for any investor, but it could seem like just one individual success story. If we consider CF Foundation's entire portfolio of VP efforts, they allocated a medical and research budget of \$87 million across more than 500 awards in 2012, and over \$160 million across more than 1,100 awards in 2016 (Kim and Lo, 2019). Apparently, from the portfolio perspective, the \$3.3 billion return is still very attractive after factoring in CF Foundation's investments in other projects, even assuming everything else did not produce any financial reward.[21]

If we assume for simplicity that the \$150 million investment was made upfront and the \$3.3 billion sale occurred 12 years later, this implies a compound annual return of 29.4% over this period. To estimate the realized α of this investment, we require an estimate of the cost of capital of Vertex during the 12-year investment period from 2002 to 2013, prior to the 2014 royalty sale. Figure 10 displays the 250-day rolling-window daily estimated beta of Vertex from 17 July 1992 to 30 December 2020, and the average value between 2 January 2001 and 31 December 2013 was 1.42. The average 5-year constant-maturity Treasury yield from January 2001 to December 2013 was 2.8%,[22] and the annualized compound return of the CRSP value-weighted returns index with dividends during this period was 5.4%, hence a simple CAPM estimate of the cost of capital is $1.42 \times (5.4\% - 2.8\%) + 2.82\% = 6.5\%$.

Of course, this crude estimate does not account for the illiquid nature of biomedical assets and the financing risks that their multi-year investment horizons pose. A cost of capital of 20% for privately held biotech investments is a commonly used industry benchmark. Therefore, a plausible range for the α of the CF Foundation's investment in Vertex is 9.4% (using a 20% cost of capital) to 22.9% (using a 6.5% cost of capital).

Using this estimated range for the CF Foundation's α, and making a few additional assumptions about auxiliary parameters, we can reverse-engineer the implied correlation, ρ, that is consistent with this performance range, which is $[35\%, 86\%]$.[23] Our highly stylized calculations are not meant to yield a rigorous estimate of the true alpha associated with drug development for rare diseases, and the plausible range of the true alpha is likely larger, potentially including zero. But more systematic empirical analyses of the biopharma industry show that pharmaceutical companies have become increasingly profitable, with risk-adjusted returns outperforming the aggregate stock market in recent years (Thakor et al., 2017; Lo and Thakor, 2019). The example of the CF Foundation provides additional intuition for how impact and investment performance need not be a zero-sum game in the presence of sufficient correlation between impact and performance.

However, there is a deeper message in this striking example, which is that, in certain

[21] In fact, since 2014, the CF Foundation has sold additional royalty interests, bringing their total investment returns to over \$4 billion since inception. However, for our purposes, we focus only on the single sale to Royalty Pharma for simplicity since it occurred at a single point in time.

[22] See https://fred.stlouisfed.org/series/GS5/.

[23] We assume that the cross-sectional standard deviation of $\boldsymbol{\alpha}$ is $\sigma_\alpha = 10\%$, and the CF Foundation's investment in Vertex ranks at the top 1% of $N = 10,000$ securities based on a "rare disease impact investing" factor. If we assume, instead, that $\sigma_\alpha = 20\%$, the implied correlation range is $[18\%, 43\%]$.

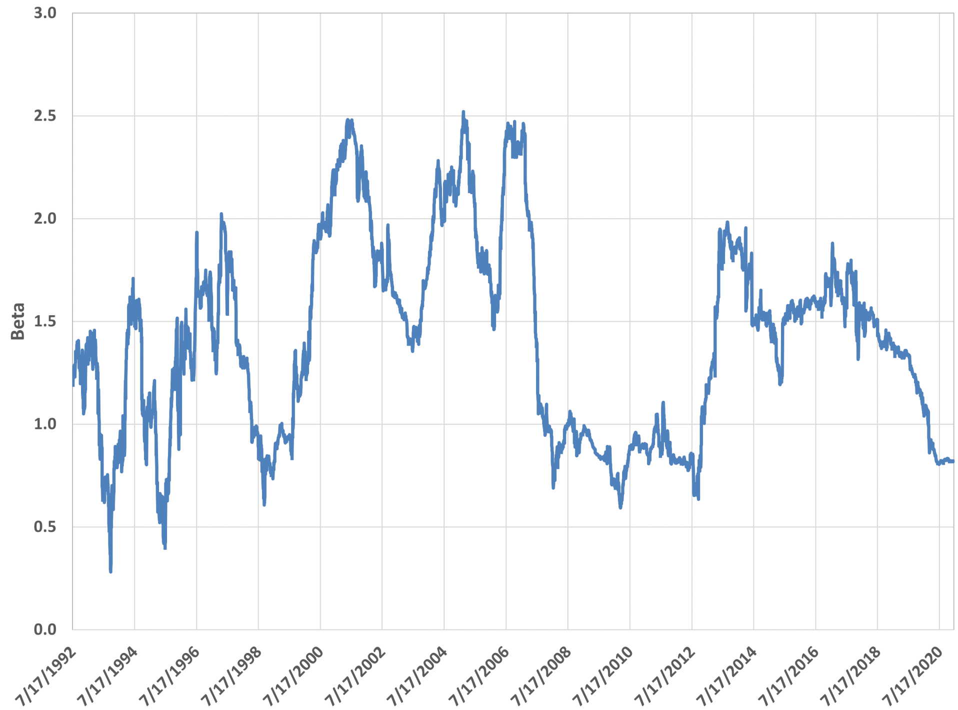


Figure 10: 250-day rolling-window estimated daily beta coefficients for Vertex Pharmaceuticals from 17 July 1992 to 30 December 2020.

cases, impact is a *pre-requisite* for performance. The CF Foundation's main objective—helping to create a disease-modifying drug for CF—was, in fact, the primary source of Vertex's outsized investment performance. The fact that the Foundation focused on this one long-term goal—to the exclusion of shorter-term financial metrics and milestones—and was willing to continue investing in Vertex over multiple years despite business cycle fluctuations (including the 2008 Financial Crisis) contributed significantly to its success (both in impact and in financial returns). Indeed, many traditional venture capitalists have shied away from investing in projects with such high risks and long-term capital commitments. In other words, in this case, correlation may actually be *causation*; impact can sometimes be responsible for financial success.

More generally, most early-stage drug development programs have low probabilities of success, long time horizons, and large capital requirements (Fagnan et al., 2013), making them less attractive investments than alternatives in other industries like software, social media, telecommunications, etc. In recent years, new tools have emerged to quantify and diversify the risk in these investments (Fagnan et al., 2013; Thakor et al., 2017). Our impact framework provides a systematic approach for constructing impact portfolios and measuring their financial performance, and properly measuring and managing the risk of these investments is the first step towards encouraging more capital to be allocated to accelerate drug development and build greater social value.

6.2 Divesting Sin Stocks

Another particular type of impact investing is avoiding or divesting sin stocks—stocks from companies involved in or associated with activities considered unethical or immoral. Although there may be a degree of subjectivity involved in determining what is considered sinful, common examples include companies involved in producing, distributing, or otherwise supporting alcohol, tobacco, gambling, sex-related industries, and firearms. It has been found that sin stocks are less held by norm-constrained institutions such as pension plans as compared to mutual or hedge funds, and receive less coverage from analysts. As a result, sin stocks seem to yield higher expected returns (Fabozzi, Ma, and Oliphant, 2008; Hong and Kacperczyk, 2009; Statman and Glushkov, 2009), an observation also shared by Fauver and McDonald IV (2014) on international stocks.

This empirical fact implies a negative correlation between a stock's excess return and an "anti-sin stock" factor. In terms of the super-efficient frontier shown in Figure 9, divesting from sin stocks likely yields a negative return and a lower efficient frontier. This leads to a natural definition of the cost to this specific impact factor.

We use Hong and Kacperczyk (2009) to calibrate our model and focus on tobacco, alcohol, and gambling as proxies for sin stocks. The authors report a monthly excess return of 0.26% for an equal-weighted portfolio long sin stocks and short their comparables, by running a time series regression controlling for market, size, value, and momentum factors, using equity data in United States from 1965 to 2006.[24] This estimate can be used to calculate the implied correlation between $\boldsymbol{\alpha}$ and $\mathbf{X}$ in our model, using results from Propositions 2 and 6 (see also discussions in Section 5.2).

Panel A of Table 2 summarizes these calibration results.[25] The implied correlation is 27% ($R^2 = 7.2\%$), assuming a standard deviation of cross-sectional alpha of $\sigma_\alpha = 5\%$.[26] This leads to a measure of the cost of avoiding sin stocks. If we form an impact portfolio based on the top half of all securities based on the anti-sin factor, it leaves an excess return of

[24]Hong and Kacperczyk (2009) use monthly returns. The authors also run a cross-sectional regression controlling firm characteristics and get similar excess returns.

[25]Hong and Kacperczyk (2009) report 193 sin stocks in their selection, and Blitz and Fabozzi (2017) report that sin stocks are about 2.5% of the universe. We calibrate to these parameters when determining the quantiles of the induced order statistics in, for example, (22).

[26]Note that different assumptions about σ_α lead to different estimates of ρ, but not the final estimates of the cost of avoiding sin stocks because the expected alpha in (22), for example, is invariant of the product of the two ($\rho\sigma_\alpha$).

1.7% per annum on the table. If we form a Treynor-Black portfolio based on the omitted sin stocks and the passive market portfolio, we could have achieved a leveraged alpha of 14.4% per annum with a (leveraged) weight of 8.58 for the sin stocks portfolio. On the other hand, if we form an impact portfolio by leaving out only the top decile (or top 2%) of the most sinful stocks, the opportunity cost is 2.5% (3.3%).

Table 2: Estimated cost in excess return per annum for avoiding sin stocks, calibrated to prior empirical studies. Here we assume that the passive portfolio has an annualized risk premium of $\mathrm{E}[R_m] - R_f = 6\%$ and volatility of $\sigma_m = 15\%$.

Impact Portfolio	Weight of Impact Portfolio ω_A	Expected Excess Return: Impact Portfolio α_A	Expected Excess Return: Combined with Passive Portfolio $\omega_A \alpha_A$
Panel A: Hong and Kacperczyk (2009)			
Implied correlation $\rho = 27\%$ ($R^2 = 7.2\%$) assuming $\sigma_\alpha = 5\%$.			
Top Half	8.58	1.7%	14.4%
Top Decile	3.78	2.5%	9.35%
Top 2%	1.04	3.3%	3.4%
Panel B: Blitz and Fabozzi (2017)			
Implied correlation $\rho = 10\%$ ($R^2 = 1.1\%$) assuming $\sigma_\alpha = 5\%$.			
Top Half	3.30	0.6%	2.1%
Top Decile	1.45	1.0%	1.4%
Top 2%	0.40	1.3%	0.5%

In fact, a few studies have tried to understand why sin stocks appear to show positive excess returns.[27] In particular, Blitz and Fabozzi (2017) show that sin stocks indeed exhibit a significantly positive CAPM alpha, but this alpha disappears completely when controlling not only for classic factors such as size, value, and momentum, but also for exposures to the two new Fama and French (2015) quality factors—profitability and investment. We also summarize the implied correlation and cost to avoid sin stocks based on Blitz and Fabozzi (2017) in Panel B of Table 2.[28] Both the correlation and sin-stock excess returns decrease sharply based on this study.

This example highlights the fact that the measurement of excess returns of impact investing depends on the specific asset-pricing model used to estimate alpha. Our framework can be applied to any number of factors as specified in (1)–(2). Indeed, a factor may yield positive correlation with alpha under one asset-pricing model (implying a positive excess

[27] For example, Kim and Venkatachalam (2011) find that sin stocks' financial report quality is superior to other comparable stocks.

[28] Blitz and Fabozzi (2017) use monthly returns. For U.S. data in 1963–2016, the authors report a non-significant monthly excess return of 0.10%. This number becomes negative when restricting to data after 1990.

return), and may disappear or change sign after controlling for additional factors.

6.3 ESG Investing

More generally, SRI and ESG-aware investing have both drawn an increasing amount of attention in recent years. Our model provides a systematic framework to measure the financial impact of SRI and ESG—positive or negative—and construct optimal portfolios based on the correlation between the impact characteristic and excess returns.

Compared to sin stocks, the empirical evidence on ESG's excess returns is mixed. On the one hand, several studies find that portfolios or funds with high environmental scores tend to outperform otherwise comparable investments (Bansal, Wu, and Yaron, 2018; Madhavan, Sobczyk, and Ang, 2021; Shing, 2021). On the other hand, others argue that the evidence that markets reward companies for being "good" is weak to non-existent (Alessandrini and Jondeau, 2020; Cornell and Damodaran, 2020), which is supported by recent evidence that green bonds—bonds whose proceeds are used for environmentally sensitive purposes—are indeed priced at a premium, implying a lower yield compared to otherwise equivalent bonds (Baker et al., 2018).[29]

As with the sin stocks in Section 6.2, we also calibrate our model with respect to several studies in Table 3. Panel A uses the MSCI World ESG Leaders index, which yields an excess return of 0.07% per annum compared to the MSCI World index over the past 10 years.[30] This implies a correlation of 1.6% ($R^2 = 0.0\%$) between stock alpha and the MSCI ESG scores based on our model, which leads to an excess return of 0.10% (0.20%) per annum for the impact portfolio formed by the top half (top 2%) of the MSCI ESG stocks. This is consistent with opinions from industry advocates of ESG, although the magnitude of excess returns here is quite small. For example, Edmund Shing, the Global Chief Investment Officer of BNP Paribas Wealth Management, wrote in Shing (2021): "Responsible investing not an EITHER/OR choice, but an AND...you can choose a sustainable/responsible investment strategy and outperform non-sustainable benchmarks."

In contrast, Baker et al. (2018) study the U.S. bond market over the period 2010–2016 and report a yield difference of 6 basis points at issuance for green bonds below other ordinary bonds.[31] This corresponds to a plausible and economically meaningful 0.6% difference in value on a bond with a 10-year duration. Panel B of Table 3 shows the implied correlation

[29]For additional discussion on this topic, see Renneboog, Ter Horst, and Zhang (2008), de Franco (2020), and Pedersen, Fitzgibbons, and Pomorski (2021).

[30]See MSCI (2021) which reports that 724 stocks are included in the MSCI World ESG Leaders index compared to 1,559 for the MSCI World index.

[31]2,083 green U.S. municipal bonds are used in the sample, compared to 643,299 ordinary bonds.

Table 3: Estimated ESG excess returns per annum, calibrated to prior empirical studies. Here we assume that the passive portfolio has an annualized risk premium of $\mathrm{E}[R_m] - R_f = 6\%$ and volatility of $\sigma_m = 15\%$.

Impact Portfolio	Weight of Impact Portfolio	Expected Excess Return	
		Impact Portfolio	Combined with Passive Portfolio
	ω_A	α_A	$\omega_A \alpha_A$
Panel A: MSCI (2021)			
Implied correlation $\rho = 1.6\%$ ($R^2 = 0.0\%$) assuming $\sigma_\alpha = 5\%$.			
Top Half	0.11	0.10%	0.01%
Top Decile	0.05	0.15%	0.00%
Top 2%	0.01	0.20%	0.00%
Panel B: Baker et al. (2018)			
Implied correlation $\rho = -2.0\%$ ($R^2 = 0.04\%$) assuming $\sigma_\alpha = 1\%$.			
Top Half	-1.06	-0.02%	-0.03%
Top Decile	-0.47	-0.04%	-0.02%
Top 2%	-0.13	-0.05%	-0.00%
Panel C: Bansal, Wu, and Yaron (2018) ("good times")			
Implied correlation $\rho = 22\%$ ($R^2 = 4.7\%$) assuming $\sigma_\alpha = 5\%$.			
Top Half	2.69	1.35%	3.64%
Top Decile	1.19	1.99%	2.37%
Top 2%	0.33	2.65%	0.88%
Panel D: Bansal, Wu, and Yaron (2018) ("bad times")			
Implied correlation $\rho = -0.2\%$ ($R^2 = 0.0\%$) assuming $\sigma_\alpha = 5\%$.			
Top Half	-0.02	-0.01%	-0.00%
Top Decile	-0.01	-0.02%	-0.00%
Top 2%	-0.00	-0.02%	-0.00%

of -2%.[32] This result points to a negative ESG alpha in the bond market.

In addition, using stock data from S&P 500 and Russell 3000 in 1993–2013, Bansal, Wu, and Yaron (2018) document a "luxury-good effect" for ESG.[33] Stocks with higher ESG ratings significantly outperform lower-ranked ones during good economic times, but not during bad economic times, resembling the demand for luxury goods.

We report the implied correlation between stock alpha and the ESG factor based on their estimates for good and bad economic times, in Panels C and D of Table 3, respectively. During good economic times defined by the cyclically-adjusted real P/E (CAPE) ratios from Shiller (2005), Bansal, Wu, and Yaron (2018) report a monthly Fama-French four-factor alpha of 0.315% for the top-bottom ESG portfolio. This implies a 22% correlation between stock alpha and the ESG factor, and sizable positive excess returns for the impact portfolios. However, during bad economic times, the monthly Fama-French four-factor alpha in Bansal, Wu, and Yaron (2018) becomes -0.0026%, rendering all of our estimates of correlation and ESG alpha to be essentially zero.

The three studies we highlight in Table 3 underscore the difficulty in measuring consistent excess returns of ESG, which depend on many factors including the asset class, region, and time period. In addition, the specific choice of asset-pricing model also affects the empirical estimates of ESG alpha. For example, Madhavan, Sobczyk, and Ang (2021) show that the security selection alpha by U.S. equity mutual fund managers is related to ESG scores, but only through the component correlated with existing style factors such as value, quality, and momentum. In contrast, no significant relationship was found with the idiosyncratic ESG components not related to style factors. In the context of mutual funds, Geczy, Stambaugh, and Levin (2021) show that the SRI cost depends on the investor's views about asset-pricing models and manager skills. In particular, the SRI cost is minimal compared to a CAPM-investor but may be substantial when investors allow for size, value, and momentum factors, as well as managerial skill.

6.4 The GameStop Phenomenon

In January 2021, the share price of Gamestop Corp. (GME)—a struggling videogame retailer that had recently announced a 30% decline in 2020Q3 net sales, due in part to an 11% reduction in their store base—went from \$17.25 on January 5 to an all-time high of \$347.51 on January 27. Although few investment professionals would consider GME an "impact

[32] We assume the standard deviation of cross-sectional alpha is $\sigma_\alpha = 1\%$ because of smaller magnitudes for bond returns. Similar to our results for sin stocks, different assumptions about σ_α lead to different estimates of ρ, but not the final estimates of the ESG alpha.

[33] Bansal, Wu, and Yaron (2018) use monthly returns.

investment," it is difficult to categorize it as anything else given the apparent origin of its meteoric price spike.

The key turning point for GME seemed to be growing interest among retail investors affiliated with the Reddit forum "r/WallStreetBets."[34] While it is difficult to determine the exact cause and motivation behind the early initiators,[35] the GME price spike is unlikely to have been driven by changes in the fundamentals of the company,[36] but rather caused by a combination of a grass-roots "David vs. Goliath" conflict between retail investors and hedge-fund shortsellers, and trend followers taking advantage of this dynamic. Other stocks that seemed to be involved in this movement included AMC Entertainment Holdings (AMC) and Blackberry (BB), both of which were facing shortselling pressure from institutional investors in late 2020 and into January 2021. These events attracted substantial media attention due to the populist narrative that was playing out on social media at the time, as well as the extraordinary price gyrations and wealth transfers involved. As shown in Figure 11, if an investor bought $1 of GME at the beginning of December 2020, she would have gained over $20 at the end of January 2021, before crashing back to under $3 the next month.

In this sense, WallStreetBets participants can be viewed as impact investors. And by most accounts, they have been highly successful in achieving the impact they desired, i.e., punishing the shortsellers and pushing up the price of an underdog company bullied by elite institutional investors. However, to distinguish this type of activity from traditional impact investing, we shall call the GME phenomenon "price-impact investing".

In the case of GME, it is almost obvious in retrospect that the very act of investing can produce a positive α, at least for a short period of time.[37] However, the same strategy may not work as well for other stocks. In general, all stocks can be affected by such price-impact investors in theory, but the degree to which each of them is susceptible depends on a number of factors, including its market capitalization, liquidity, price dynamics, main shareholders, amount of short interest, sentiment and attention from the general public, and so on. Moreover, manipulating the prices of publicly traded equities clearly violates both securities law and anti-trust regulation,[38] hence there are significant ethical and legal

[34] A literature quantifying the effect of Reddit activities on the GME mania has already emerged. See Long, Lucey, and Yarovaya (2021), Lyócsa, Baumöhl, and Vŷrost (2021), and Umar et al. (2021) for example.

[35] For example, Hasso et al. (2021) profile retail investors participating in the frenzy using individual brokerage data, and find that GME investors had a history of investing in lottery-like stocks prior to investing in GME. This implies that individual investors may not purely engage in a digital protest against Wall Street.

[36] In fact, GME's revenue has been declining year over year since 2017 and its earnings-per-share has been negative since 2018.

[37] What constitutes a "short" period of time is clearly subjective and context dependent—as of August 13, 2021, GME's close price was $157.41, still nearly 10 times higher than what it was at the start of 2021.

[38] See the Securities Exchange Act of 1934, Section 9, and the Sherman Act and the Commodity Exchange Act.

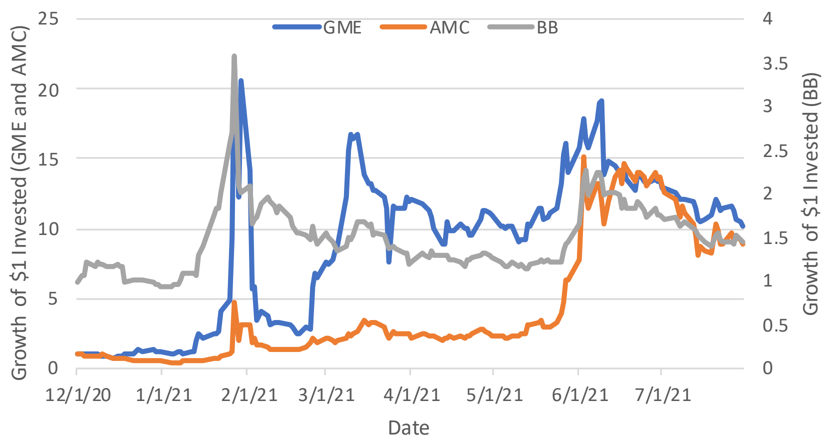


Figure 11: Growth of one dollar invested in GameStop (GME), AMC Entertainment Holdings (AMC), and Blackberry (BB).

ramifications of this type of impact investing that have yet to be fully explored. Nevertheless, our impact framework provides a means to measure the magnitude of such investments, which could be an important component of policy debates on whether and how to regulate this type of activity. To that end, we can apply well-known market-microstructure models such as Bertsimas and Lo (1998) to first quantify the relation between short-term trading programs and market price reactions.

A Simple Execution Model. Following Bertsimas and Lo's (1998) framework and notation, we assume that an investor seeks to purchase a total of $\bar{S}$ shares of a particular security over a fixed time interval, $[0, T]$. The investor decides how to divide $\bar{S}$ into smaller purchases distributed throughout the interval so as to maximize the final price-impact of the security.[39] The answer depends, of course, on the degree to which a single purchase affects the market price, i.e., the "price impact" and the dynamics of future market prices. Given a particular price-impact function and a specification for the price dynamics, an optimal trading strategy that maximizes the price impact of acquiring $\bar{S}$ in $[0, T]$ may be obtained.

Specifically, denote by S_t the number of shares acquired in period t at price P_t, where $t = 1, 2, \cdots, T$. Then the investor's objective of maximizing final price impact is given by:

$$\max_{\{S_t\}} \mathrm{E}[P_T] \tag{51}$$

[39]Note that this is *not* the objective function considered by Bertsimas and Lo (1998)—the problem they pose is how to divide $\bar{S}$ so as to maximize *cumulative profits*, which they solve via stochastic dynamic programming.

subject to the constraint that the desired number of shares are acquired:

$$\sum_{t=1}^{T} S_t = \bar{S}. \tag{52}$$

We assume that the security price follows the bivariate stochastic process:

$$\begin{aligned} P_t &= P_{t-1} + \theta S_t^z + \gamma F_t + \epsilon_t, \quad \theta > 0,\, z \in (0, 1] \\ F_t &= \delta F_{t-1} + \eta_t, \quad \delta \in (-1, 1) \end{aligned} \tag{53}$$

where ϵ_t and η_t are independent white noise processes with mean 0 and variance σ_ϵ^2 and σ_η^2, respectively. The parameter θ specifies the magnitude of the price impact, which is assumed to follow a power law in S_t, where the parameter z specifies the "price sensitivity" of the security or, equivalently, the security's degree of illiquidity. The latter interpretation is motivated by Kyle's (1985) market microstructure model in which liquidity is measured by a loglinear-regression estimate of the log-volume required to move the price by one dollar. Sometimes referred to as "Kyle's lambda," this measure is an inverse proxy of liquidity, with higher values of lambda implying lower liquidity and lower market depth.[40]

The presence of F_t in the law of motion for P_t captures the potential impact of market conditions or private information about the security. For example, F_t might represent new business opportunities created by the company. However, F_t can also represent the impact of popular sentiment, as in the case of GME, as well as any of the other factors mentioned above. In either case, the impact of F_t on trading profits and the time series properties of F_t both have important implications for the feasibility and profitability of price-impact investing. With these price dynamics, the following result completely characterizes the optimal price-impact strategy and its corresponding expected profit:

Proposition 11. *Under the price dynamics specified by (53), the strategy that maximizes the total price impact, (51), is given by:*

$$S_1 = S_2 = \cdots = S_T = \frac{\bar{S}}{T}, \tag{54}$$

[40]See also Lillo, Farmer, and Mantegna (2003) and Almgren et al. (2005) for more detailed explorations of the power law of price impact in equity markets. When $z = 1$, this reduces to the "linear price impact with information" specification from Bertsimas and Lo (1998).

and its corresponding expected profit is given by:

$$V^* = \left(\frac{\theta \bar{S}^z (T-1)}{2T^z} + \frac{\gamma \delta F_1 \left(1 - T\delta^{T-1} + (T-1)\delta^T\right)}{(1-\delta)^2 T} \right) \bar{S}. \tag{55}$$

In fact, when $z=1$ and price impact is linear in trading quantities,[41] it does not matter how trades are allocated because the total impact from T trades is always equal to the impact of one single trade of size $\bar{S}$. However, when the price impact is a concave function in general ($0<z<1$), the optimal strategy is to simply divide the total order $\bar{S}$ into T equal "waves," and trade them at regular intervals, as specified in (54).

The expression for V^* in (55) shows that the expected profit of price-impact investing depends on two factors: the market impact as parameterized by θ and z, and influences from other factors (sentiment, liquidity, private information, etc.) as parameterized by γ and the AR(1) coefficient governing these other factors (δ).

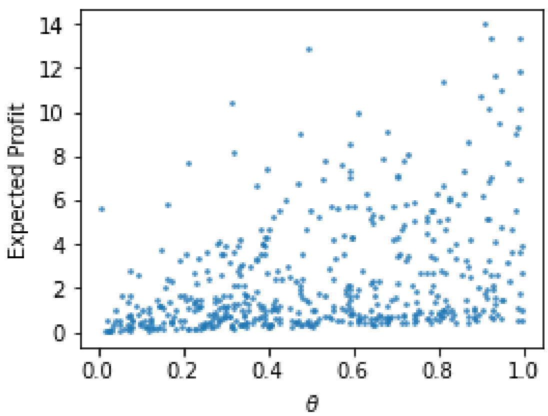
To illustrate the effect of these parameters on trading profit V^*, we simulate a universe of $N=500$ securities where the parameters, θ, z, γ, and δ, are generated by four independent uniform distributions on $[0,1]$. In the following analysis, we assume that the first realization of $X_1=1$, without loss of generality.

In Figure 12a, we first show the relationship of the expected profit V^* with respect to market impact (θ). As θ increases, expected profit increases as well. This is quite intuitive because the stronger the market impact, the easier it is for short squeezers to induce price momentum and generate profits. If we consider a collection of securities each with a different θ, the correlation between their market-impact coefficients and expected profit is 37%, implying that sorting $\boldsymbol{\alpha}$ based on θ will generate positive excess returns in the context of our impact-investing framework.

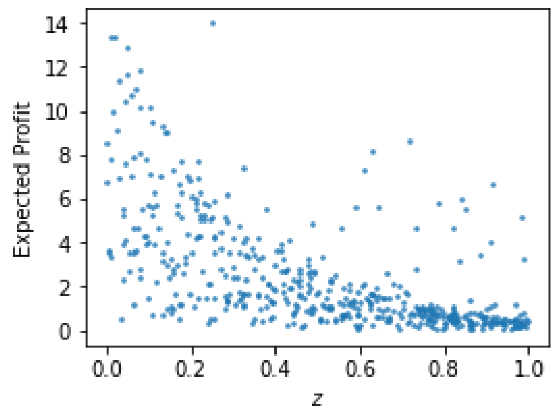
Figure 12b displays the relation between expected profit V^* and sensitivity z. As power increases from 0 to 1, the expected profit decreases. This is because lower values of z correspond to more concave price-impact functions, for which each small trading segment has larger price impact. The correlation between z and expected profit is -63%. In other words, one can achieve positive excess returns by selecting securities based on the reverse ordering of sensitivity z.

Figure 12c displays the relation between expected profit V^* and influences from other factors (γ), which has a weak positive correlation of 9%. Finally, Figure 12d displays the relation between AR coefficient (δ) and expected profit. The expected profit is larger when δ is larger. This is because we have assumed the first realization of F_t is positive, and

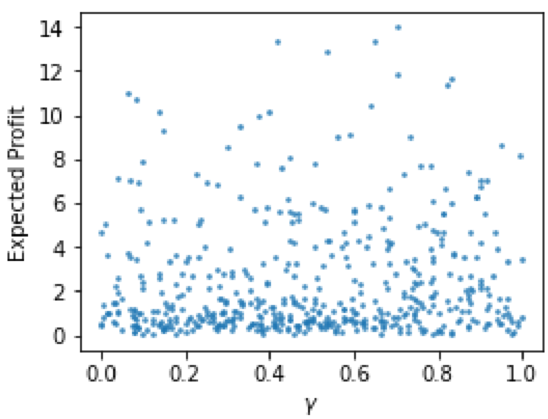
[41] See also Bertsimas and Lo (1998).



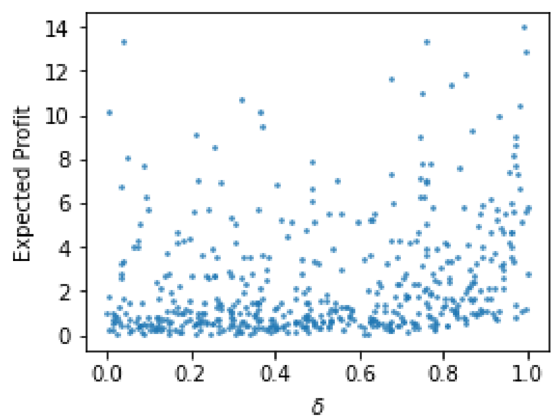
(a) Price impact (θ)



(b) Market sensitivity (z)



(c) Influences from other factors (γ)



(d) AR coefficient (δ)

Figure 12: The expected profit, V^*, of price-impact investing as a function of four parameters in (53), for a market with $N=500$ securities with simulated parameters. Here we set $\theta=1$, $z=1$, $\gamma=1$, $\delta=10\%$, $\bar{S}=1$, $F_1=1$, and $T=30$ by default, and vary each parameter accordingly.

higher autocorrelations imply stronger momentum. Indeed, the correlation between the AR coefficient, δ, and expected profit is 26% in this simulated market.

Implied Excess Returns. We summarize the results from Figure 12 in Table 4, and provide their implied α when applied to a collection of 500 securities simultaneously, each with different price dynamics as specified in (53). Panels A, B, and C show the expected excess returns if investors apply θ, z, γ, and δ, respectively, to rank securities, where the correlations with trading profits are obtained from our simple execution model. The expected α can be very high with a leveraged portfolio, driven by the high correlation between stock $\boldsymbol{\alpha}$ and the price-impact investing factor in certain cases.

Table 4: Estimated excess returns per annum for the price-impact investing factor, based on the optimal strategy's profit in (55) and its implied correlations with respect to various characteristics of individual securities. Here we assume $\sigma_\alpha = 5\%$, and the passive portfolio has an annualized risk premium of $\mathrm{E}[R_m] - R_f = 6\%$ and volatility of $\sigma_m = 15\%$.

Impact Portfolio	Weight of Impact Portfolio	Expected Excess Return	
		Impact Portfolio	Combined with Passive Portfolio
	ω_A	α_A	$\omega_A \alpha_A$
Panel A: Ranking based on price impact (θ)			
Model-implied correlation with alpha: $\rho = 37\%$.			
Top Half	9.19	2.3%	21.4%
Top Decile	4.09	3.4%	14.0%
Top 2%	1.20	4.5%	5.4%
Panel B: Ranking based on market sensitivity (z); reverse order			
Model-implied correlation with alpha: $\rho = 63\%$.			
Top Half	15.58	3.9%	61.5%
Top Decile	6.95	5.8%	40.3%
Top 2%	2.04	7.6%	15.5%
Panel C: Ranking based on other factors (γ)			
Model-implied correlation with alpha: $\rho = 9\%$.			
Top Half	2.20	0.6%	1.2%
Top Decile	0.98	0.8%	0.8%
Top 2%	0.29	1.1%	0.3%
Panel D: Ranking based on AR coefficient for other factors (δ)			
Model-implied correlation with alpha: $\rho = 26\%$.			
Top Half	6.43	1.6%	10.5%
Top Decile	2.87	2.4%	6.9%
Top 2%	0.84	3.2%	2.7%

In practice, it is difficult to accurately calibrate the relevant parameters for each stock, hence the expected profit of engaging in GME-like price-impact investing is correspondingly difficult to estimate. However, this example highlights the fundamental determinants for a price-impact investor's α: the correlation between each stock's trading profit and stock characteristics, e.g., market capitalization, liquidity, specific forms of market impact, attention from the general public, main shareholders, short interest, or anything correlated with stock returns. Higher correlations lead to higher alpha when following that particular characteristic to select target stock. In fact, stocks like GME, AMC Entertainment Holdings and Blackberry were the perfect targets for the short squeezes that occurred at the end of 2020 to early 2021 because of their highly publicized amounts of short interest from hedge funds, and high customer concentration in the young people, both of which are arguably correlated features with short-squeeze profits.

7 Conclusion

In this article, we propose a new framework to quantify the financial value-added/subtracted of impact investing. Using the theory of induced order statistics, we show that the correlation between the impact factor, $\mathbf{X}$, and the excess returns of individual securities determines the excess return of the impact portfolio. The impact factor provides a ranking and selection mechanism for portfolio construction, and its correlation with $\boldsymbol{\alpha}$ provides additional information that can be used to achieve better risk-adjusted returns as well as impact.

The ability to quantify the distribution of alphas for impact-sorted securities allows us to form Treynor-Black portfolios to exploit the alphas optimally. This is particularly relevant for the investment management industry as it strives to bridge the gap between traditional investment products and the growing demand for impact investments. Regardless of the nature of the desired impact—whether it is biomedical innovation, promoting ESG, avoiding socially unsavory businesses, or attempting to achieve certain price objectives—our framework can be used to construct the most efficient way of investing in impact portfolios. And by comparing the properties of impact portfolios on the Black-Treynor super-efficient-frontier with those of non-impact investments, we have a concrete metric of the reward (or cost) of impact investing, as demonstrated in the four examples above.

This provides all stakeholders with a practical toolkit for constructing impact portfolios, and asset owners with a framework for assessing whether asset managers have satisfied their fiduciary duties by engaging in specific types of impact investing. Our model compares traditional investments without impact considerations, i.e., investments unconditioned on any impact information $\mathbf{X}$, with impact-aware investments, because this is the primary consideration for fiduciaries deciding whether or not to invest in ESG and other impact-related portfolios. On the other hand, one can also separate the information component from the impact factor **X** from investor preferences for portfolios with more impact, as Pedersen, Fitzgibbons, and Pomorski (2021) demonstrated in their framework.[42]

In fact, an investment's alpha can itself be influenced by its impact, as demonstrated in our example of venture philanthropy. If the Cystic Fibrosis Foundation were not able to achieve the impact to develop effective drugs for cystic fibrosis, it is unlikely that they could have generated any meaningful return. In this sense, there is an endogenous and likely a *causal* relationship between **X** and alpha. Another example of realizing alpha by achieving impact is activist investing, for which it has been empirically documented that activists may help their portfolio companies improve production efficiency (Brav, Jiang, and Kim, 2015), long-term fundamentals (Bebchuk, Brav, and Jiang, 2015), and stock performance (Dimson, Karakaş, and Li, 2015).

In practice, the correlations between excess returns and the impact factor **X** are not static. As new concepts emerge and grow, investors move into them progressively. For example, the idea that portfolio managers should include company-specific carbon risk exposures in their investment process was greeted with mainly skepticism in a not-so-distant 2010 conference (Andersson, Bolton, and Samama, 2016), in contrast to today's 4,578 UNPRI signatories,[43] of which 74% are asset managers, 15% are asset owners, and 11% are service providers. In this adaptive process (Lo, 2004, 2017), the correlation reinforces itself as the amount of assets under management and the number of products that are attempting to take advantage of a given **X** increase over time, and eventually stabilizes as the size of the new sector reaches a steady state. In this respect, the GameStop example of Section 6.4 offers an alternate explanation of the presence of climate-related risk premia as documented by Bolton and Kacperczyk (2021). Even in the absence of any real relationship between a company's carbon emissions and its business prospects, if enough investors care about its carbon footprint because of general environmental awareness, this factor can have an impact on the company's returns, thereby inducing a risk premium.

More broadly, our framework is relevant not just to impact proxies such as SRI and ESG metrics, but applies to any characteristics that may be correlated with excess returns. This includes traditional factors such as value, quality, size, and momentum, as well as hundreds of new factors and anomalies in the "Factor Zoo" discussed in the recent literature (Harvey, Liu, and Zhu, 2016; Feng, Giglio, and Xiu, 2020; Hou, Xue, and Zhang, 2020). From this

[42] Pedersen, Fitzgibbons, and Pomorski (2021) consider three types of investors termed Type-U (ESG-unaware), Type-A (ESG-aware) and Type-M (ESG-motivated). As a result, they derive an ESG-efficient frontier which shows the highest attainable Sharpe ratio for each ESG level.

[43] `https://www.unpri.org/signatories/signatory-resources/signatory-directory` (accessed 7 December 2021).

perspective, our model has defined a measure for the alpha of any factor, providing a unified framework for SRI, ESG, and beyond.

Our framework may also help inform regulators and policymakers on the most appropriate tools to encourage investments with more socially-aware goals. Not all types of impact investing are created equal. When these investments create positive excess returns, one must understand what drives the initial under-valuation in the first place, and what risks are preventing investors from participating in these opportunities. In the case of venture philanthropy in biomedical research and development, for example, it is crucial to develop new tools to mitigate risks from low probabilities of success, long time horizons, and large capital requirements (Fagnan et al., 2013; Thakor et al., 2017).

On the other hand, when impact investing incurs a cost to investors, at the very least, it suggests the need for more explicit investor disclosures. It may also justify certain incentives and industrial policies,[44] such as tax benefits and R&D grants to encourage the growth of these socially beneficial firms and organizations. One case in point is the area of green energy where, for example, Baker et al. (2018) document a lower yield for green bonds compared to otherwise equivalent bonds. Governments around the world are designing policies to help grow industries such as clean energy and electric vehicles. Even if they incur a cost in the short to medium term, as a society we need to invest in them if we value greater sustainability.

Indeed, our analysis underscores the fact that finance need not be a zero-sum game. While impact investing does imply sacrificing excess returns in certain situations, in other situations it is, in fact, possible to achieve impact and attractive financial returns at the same time. We hope to apply our framework more broadly so as to identify more opportunities for doing well by doing good.

[44] See Lin and Chang (2009) for example.

A Appendix

In this Appendix, we provide proofs for all the propositions.

A.1 Proof of Proposition 1

The constraints on the right-hand side optimization problem of (3) is a subset of the left-hand side optimization problem. Therefore the inequality follows.

To give a bound on the utility loss between the unconstrained portfolio W and the constrained portfolio W^c, we consider an intermediate portfolio W^{c1} that is also constrained to the subset $\mathcal{S}$, but with equal factor loadings as the unconstrained portfolio W. In other words, the portfolio weights for W^{c1} satisfy the following conditions:

$$\omega_i^{c1} = 0 \text{ for } i \notin \mathcal{S} \tag{A.1}$$

$$\sum_{i \in \mathcal{S}} \omega_i^{c1} = 1 \tag{A.2}$$

$$\sum_{i=1}^{N} \omega_i \beta_{ik} = \sum_{i \in \mathcal{S}} \omega_i^{c1} \beta_{ik} \text{ for } k = 1, \ldots, K. \tag{A.3}$$

Because W^c maximizes the utility in (3),

$$\begin{aligned}
\mathrm{E}[U(W^c)] &= \mathrm{E}\left[U\left(R_f + \sum_{k=1}^{K}\sum_{i\in\mathcal{S}} \omega_i^c \beta_{ik}\left(\Lambda_k - R_f\right) + \sum_{i\in\mathcal{S}} \omega_i^c \epsilon_i\right)\right] \\
&\geq \mathrm{E}\left[U\left(R_f + \sum_{k=1}^{K}\sum_{i\in\mathcal{S}} \omega_i^{c1} \beta_{ik}\left(\Lambda_k - R_f\right) + \sum_{i\in\mathcal{S}} \omega_i^{c1} \epsilon_i\right)\right] \\
&= \mathrm{E}\left[U\left(R_f + \sum_{k=1}^{K}\sum_{i=1}^{N} \omega_i \beta_{ik}\left(\Lambda_k - R_f\right) + \sum_{i\in\mathcal{S}} \omega_i^{c1} \epsilon_i\right)\right] = \mathrm{E}[U(W^{c1})].
\end{aligned} \tag{A.4}$$

Now we consider the utility of the following two portfolios,

$$\begin{aligned}
\mathrm{E}[U(W)] &= \mathrm{E}\left[U\left(R_f + \sum_{k=1}^{K}\sum_{i=1}^{N} \omega_i \beta_{ik}\left(\Lambda_k - R_f\right) + \sum_{i=1}^{N} \omega_i \epsilon_i\right)\right], \\
\mathrm{E}[U(W^{c1})] &= \mathrm{E}\left[U\left(R_f + \sum_{k=1}^{K}\sum_{i=1}^{N} \omega_i \beta_{ik}\left(\Lambda_k - R_f\right) + \sum_{i\in\mathcal{S}} \omega_i^{c1} \epsilon_i\right)\right].
\end{aligned} \tag{A.5}$$

Note that they only differ in the last term in the parenthesis, the idiosyncratic volatilities. Denote $A \equiv R_f + \sum_{k=1}^{K}\sum_{i=1}^{N} \omega_i \beta_{ik}\left(\Lambda_k - R_f\right)$ and $B \equiv \sum_{i=1}^{N} \omega_i \epsilon_i$ (or $\sum_{i\in\mathcal{S}} \omega_i^{c1} \epsilon_i$). For any well-behaved utility function U, because $\mathrm{E}[B]=0$, we have:

$$\mathrm{E}[U(A+B)] \approx \mathrm{E}\left[U(A)+U'(A)B+\frac{1}{2}U''(A)B^2\right] = \mathrm{E}[U(A)]+\frac{1}{2}\mathrm{E}[U''(A)]\mathrm{Var}[B^2] \quad \text{(A.6)}$$

by second-order Taylor expansion around $B=0$. Since $\mathrm{E}[U(W)]$ and $\mathrm{E}[U(W^{c1})]$ differs only through the idiosyncratic volatility term, B, we have:

$$\begin{aligned}\mathrm{E}[U(W)]-\mathrm{E}[U(W^{c})] &\leq \mathrm{E}[U(W)]-\mathrm{E}[U(W^{c1})] \\ &\approx \frac{1}{2}\mathrm{E}[U''(A)]\left(\mathrm{Var}\left(\sum_{i=1}^{N}\omega_i\epsilon_i\right)-\mathrm{Var}\left(\sum_{i\in\mathcal{S}}\omega_i^{c1}\epsilon_i\right)\right).\end{aligned} \quad \text{(A.7)}$$

When the number of securities, N, is large, suppose further that:

$$\omega_i \approx \frac{1}{N},\ \omega_i^{c1} \approx \frac{1}{N-n},\ \sigma(\epsilon_i)\approx\sigma_\epsilon, \quad \text{(A.8)}$$

where n is the number of securities excluded in $\mathcal{S}$, and σ_ϵ is the common idiosyncratic volatility for all securities. We have:

$$\mathrm{E}[U(W)]-\mathrm{E}[U(W^{c})] \leq \frac{1}{2}\mathrm{E}[U''(A)]\left(\frac{\sigma_\epsilon^2}{N}-\frac{\sigma_\epsilon^2}{N-n}\right) = -\frac{1}{2}\mathrm{E}[U''(A)]\sigma_\epsilon^2\left(\frac{n}{N(N-n)}\right). \quad \text{(A.9)}$$

When the number of securities excluded in $\mathcal{S}$, n, is small relative to the total number of securities, N, the utility loss (A.9) is also small.

Finally, we observe that the assumptions in (A.8) are non-critical for our main conclusions here, and can be relaxed at the expense of simplicity of the mathematical derivation.

A.2 Proof of Proposition 2

Because $\mathbf{X}$ and $\boldsymbol{\alpha}$ are jointly normal, we can express α_i with the following linear relationship:

$$\alpha_i = \mu_\alpha + \rho\frac{\sigma_\alpha}{\sigma_x}(X_i-\mu_x)+e_i, \quad \text{(A.10)}$$

where e_i are normal random variables with $\mathrm{E}[e_i]=0$ and $\mathrm{Var}(e_i)=\sigma_\alpha^2(1-\rho^2)$, and the X_i and the e_i are mutually independent. Ordering securities based on X_i, we have:

$$\alpha_{[i:N]} = \mu_\alpha + \rho\frac{\sigma_\alpha}{\sigma_x}(X_{i:N}-\mu_x)+e_{[i]}, \quad \text{(A.11)}$$

where $e_{[i]}$ denotes the particular e_i associated with $X_{i:N}$. Note that $X_{i:N}$ on the right-hand side are the usual order statistics, while $\alpha_{[i:N]}$ on the left-hand side are the *induced* order statistics. Because X_i and e_i are independent, the set of $X_{i:N}$ and the set of $e_{[i]}$ are also independent. Therefore, we can calculate the first two moments of $\boldsymbol{\alpha}$ based on the relationship in (A.11):

$$\mathrm{E}\left[\alpha_{[i:N]}\right] = \mu_\alpha + \rho\frac{\sigma_\alpha}{\sigma_x}(\mathrm{E}[X_{i:N}] - \mu_x) + e_{[i]} = \mu_\alpha + \rho\sigma_\alpha\mathrm{E}[X_{i:N}], \tag{A.12}$$

$$\mathrm{Var}\left(\alpha_{[i:N]}\right) = \rho^2\frac{\sigma_\alpha^2}{\sigma_x^2}\mathrm{Var}\left(X_{i:N}\right) + \sigma_\alpha^2(1-\rho^2) = \sigma_\alpha^2\left(1-\rho^2+\rho^2\mathrm{Var}\left(X_{i:N}\right)\right), \tag{A.13}$$

$$\mathrm{Cov}\left(\alpha_{[i:N]}, \alpha_{[j:N]}\right) = \mathrm{Cov}\left(\rho\frac{\sigma_\alpha}{\sigma_x}X_{i:N}, \rho\frac{\sigma_\alpha}{\sigma_x}X_{j:N}\right) = \sigma_\alpha^2\rho^2\mathrm{Cov}\left(X_{i:N}, X_{j:N}\right). \tag{A.14}$$

See also David and Nagaraja (2004, Section 6.8).

A.3 Proof of Proposition 3

We first observe that $U_{i:N} \equiv \Phi(X_{i:N})$ maps the i-th normal order statistics to the i-th order statistics from a uniform distribution on $[0,1]$, where Φ is the cumulative distribution function of standard normal random variables. We define $Q \equiv \Phi^{-1}$ and write $X_{i:N} = Q(U_{i:N})$. We then expand $Q(U_{i:N})$ in a Taylor series around the expected value of $Q(U_{i:N})$:

$$\mathrm{E}[Q(U_{i:N})] = \frac{i}{n+1} = p_i, \tag{A.15}$$

which gives:

$$X_{i:N} = Q(U_{i:N}) = Q(p_i) + (U_{i:N} - p_i)\,Q'(p_i) + \frac{1}{2}\left(U_{i:N} - p_i\right)^2 Q''(p_i) + \frac{1}{6}\left(U_{i:N} - p_i\right)^3 Q'''(p_i) + \cdots. \tag{A.16}$$

Substituting (A.16) into the definition of $\mathrm{E}\left[X_{i:N}\right]$, $\mathrm{Var}\left(X_{i:N}\right)$, and $\mathrm{Cov}\left(X_{i:N}, X_{j:N}\right)$, and rearranging the terms lead to (11)-(13) in Proposition 3. See also David and Nagaraja (2004, Section 4.6).

In particular, for standard normal random variables we have $Q'(p_i) = 1/\phi(Q)$ where ϕ is the density function for standard normal random variables. Therefore we can calculate:

$$Q''(p_i) = \frac{d\left(1/\phi(Q)\right)}{d\Phi(Q)} = \frac{d\left(1/\phi(Q)\right)}{dQ}\frac{dQ}{d\Phi(Q)} = \frac{Q}{\phi^2(Q)}, \tag{A.17}$$

$$Q'''(p_i) = \frac{1+2Q^2}{\phi^3(Q)}, \tag{A.18}$$

$$Q''''(p_i) = \frac{Q(7+6Q^2)}{\phi^4(Q)}, \tag{A.19}$$

which completes the proof for (14)-(17).

A.4 Proof of Proposition 4

Because $\mathbf{X}$ and $\boldsymbol{\alpha}$ are both normally distributed, we observe that $\frac{X_i-\mu_x}{\sigma_x}$ and $\frac{\alpha_i-\mu_\alpha}{\sigma_\alpha}$ both follow the standard normal distribution. Therefore,

$$\mathrm{E}\left[\frac{X_{i:N}-\mu_x}{\sigma_x}\right] = \mathrm{E}\left[\frac{\alpha_{i:N}-\mu_\alpha}{\sigma_\alpha}\right], \tag{A.20}$$

$$\mathrm{Var}\left(\frac{X_{i:N}-\mu_x}{\sigma_x}\right) = \mathrm{Var}\left(\frac{\alpha_{i:N}-\mu_\alpha}{\sigma_\alpha}\right), \tag{A.21}$$

$$\mathrm{Cov}\left(\frac{X_{i:N}-\mu_x}{\sigma_x}, \frac{X_{j:N}-\mu_x}{\sigma_x}\right) = \mathrm{Var}\left(\frac{\alpha_{i:N}-\mu_\alpha}{\sigma_\alpha}, \frac{\alpha_{j:N}-\mu_\alpha}{\sigma_\alpha}\right). \tag{A.22}$$

We have assumed, without loss of generality, that $\mu_\alpha = \mu_x = 0$ and $\sigma_x = 1$, which leads to:

$$\mathrm{E}[X_{i:N}] = \frac{\mathrm{E}[\alpha_{i:N}]}{\sigma_\alpha}, \tag{A.23}$$

$$\mathrm{Var}\left(X_{i:N}\right) = \frac{\mathrm{Var}\left(\alpha_{i:N}\right)}{\sigma_\alpha^2}, \tag{A.24}$$

$$\mathrm{Cov}\left(X_{i:N}, X_{j:N}\right) = \frac{\mathrm{Cov}\left(\alpha_{i:N}, \alpha_{j:N}\right)}{\sigma_\alpha^2}. \tag{A.25}$$

This together with (8)-(10) gives:

$$\mu_i = \mathrm{E}\left[\alpha_{[i:N]}\right] = \rho\sigma_\alpha \mathrm{E}\left[X_{i:N}\right] = \rho \mathrm{E}[\alpha_{i:N}]. \tag{A.26}$$

$$\sigma_i^2 - \sigma_\alpha^2 = \sigma_\alpha^2\rho^2\left[\mathrm{Var}\left(X_{i:N}\right) - 1\right] = \rho^2\left[\mathrm{Var}\left(\alpha_{i:N}\right) - \sigma_\alpha^2\right], \tag{A.27}$$

$$\sigma_{ij} \equiv \mathrm{Cov}\left(\alpha_{[i:N]}, \alpha_{[j:N]}\right) = \sigma_\alpha^2\rho^2 \mathrm{Cov}\left(X_{i:N}, X_{j:N}\right) = \rho^2 \mathrm{Cov}\left(\alpha_{i:N}, \alpha_{j:N}\right). \tag{A.28}$$

A.5 Proof of Proposition 5

This proposition follows from Yang (1977). See also Lo and MacKinlay (1990) for an application in a different context.

A.6 Proof of Proposition 6

This follows from Proposition 5 by observing that $\Phi(\xi_k) = F_x(\xi_k\sigma_x + \mu_x)$. Alternatively, this result can be proved by taking the limit as $N \to \infty$ based on the finite-sample results in Proposition 2-3.

A.7 Proof of Proposition 7

For simplicity, we define $\boldsymbol{\lambda} \equiv [\,\lambda_1 \,\cdots\, \lambda_N\,]^T$, and observe that $\mathbf{X}$ and $\boldsymbol{\lambda}$ can be rewritten as:

$$\begin{aligned} \mathbf{X} &= \mu_x \mathbf{1} + \mathbf{C}_x \mathbf{N}_x \\ \boldsymbol{\lambda} &= \mu_\lambda \mathbf{1} + \mathbf{C}_\lambda \mathbf{N}_\lambda \end{aligned} \tag{A.29}$$

where $\mathbf{1} \equiv [1 \cdots 1]^T$ is a column vector of ones with size N, $\mathbf{N}_x$ and $\mathbf{N}_y$ are both N-dimensional standard normal random vectors with $\mathrm{Cov}\left(\mathbf{N}_x, \mathbf{N}_y\right) = \boldsymbol{\Sigma}$, and $\mathbf{C}_x$ and $\mathbf{C}_y$ are both $N \times N$ deterministic matrices. The specification in (A.29) completely characterizes the joint distribution of $\mathbf{X}$ and $\boldsymbol{\lambda}$. In light of the parameterization in Assumption (A2), we have (see also Wu (2021)):

$$\begin{aligned} \mathbf{C}_x &= \sqrt{1-\rho_x}\sigma_x \mathbf{I} + \left(\sqrt{1+(N-1)\rho_x} - \sqrt{1-\rho_x}\right)\sigma_x \mathbf{L} \\ \mathbf{C}_\lambda &= \sqrt{1-\rho_\lambda}\sigma_\lambda \mathbf{I} + \left(\sqrt{1+(N-1)\rho_\lambda} - \sqrt{1-\rho_\lambda}\right)\sigma_\lambda \mathbf{L} \\ \boldsymbol{\Sigma} &= \frac{\rho_{x\lambda} - \tilde{\rho}_{x\lambda}}{\sqrt{(1-\rho_x)(1-\rho_\lambda)}}\mathbf{I} + \left(\frac{\rho_{x\lambda} + (n-1)\tilde{\rho}_{x\lambda}}{\sqrt{(1+(n-1)\rho_x)(1+(n-1)\rho_\lambda)}} - \frac{\rho_{x\lambda} - \tilde{\rho}_{x\lambda}}{\sqrt{(1-\rho_x)(1-\rho_\lambda)}}\right)\mathbf{L} \end{aligned} \tag{A.30}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{L} \equiv \frac{\mathbf{1}\cdot\mathbf{1}^T}{N}$ is a matrix whose elements are all $1/N$.

We now define a projection matrix:

$$\begin{aligned} \mathbf{P} &\equiv \left(\mathbf{C}_\lambda \boldsymbol{\Sigma}^T \mathbf{C}_x^T\right)\left(\mathbf{C}_x \mathbf{C}_x^T\right)^{-1} \\ &= \underbrace{\rho_{adj}\frac{\sigma_\lambda}{\sigma_x}}_{a}\mathbf{I} + \underbrace{\left(\frac{\rho_{x\lambda} + (n-1)\tilde{\rho}_{x\lambda}}{1+(n-1)\rho_x} - \rho_{adj}\right)\frac{\sigma_\lambda}{\sigma_x}}_{b}\mathbf{L} \\ &= a\mathbf{I} + b\mathbf{L} \end{aligned} \tag{A.31}$$

and it is easy to show that:

$$\boldsymbol{\lambda} - \mathbf{P}\mathbf{X} \perp \mathbf{X}. \tag{A.32}$$

Therefore, when assuming $\mu_x = 0$ and $\sigma_x = 1$, we have:

$$\begin{aligned} \mathrm{E}\left[\lambda_{[i:N]}\right] &= \mathrm{E}\left[(\boldsymbol{\lambda} - \mathbf{P}\mathbf{X})_{[i:N]}\right] + \mathrm{E}\left[(\mathbf{P}\mathbf{X})_{[i:N]}\right] \\ &= \mu_\lambda - (a+b)\mu_x + a\mathrm{E}\left[X_{i:N}\right] + b\mu_x \\ &= \mu_\lambda + \rho_{adj}\sigma_\lambda \mathrm{E}\left[X_{i:N}\right], \end{aligned} \tag{A.33}$$

which proves (30). The variances and covariances in (31)-(32) can be proven similarly following the same decomposition in (A.33). See also Lee and Viana (1999) and Wu (2021).

A.8 Proof of Proposition 8

The expected excess return follows directly from the distribution of alphas for single securities in Proposition 2. The variance also follows by rearranging terms:

$$\begin{aligned}
\mathrm{Var}\left(\tilde{\alpha}\right) =& \sum_{i\in P}\omega_i^2\sigma_i^2 + 2\sum_{i<j\in P}\omega_i\omega_j\sigma_{ij} \\
=&\sigma_\alpha^2\left(1-\rho^2+\rho^2\sum_{i\in P}\omega_i^2\mathrm{Var}\left(X_{i:N}\right)+2\rho^2\sum_{i<j\in P}\omega_i\omega_j\mathrm{Cov}\left(X_{i:N},X_{j:N}\right)\right) \\
=&\sigma_\alpha^2\left(1-\rho^2+\rho^2\left(\sum_{i\in P}\omega_i^2\mathrm{Var}\left(X_{i:N}\right)+2\sum_{i<j\in P}\omega_i\omega_j\mathrm{Cov}\left(X_{i:N},X_{j:N}\right)\right)\right).
\end{aligned} \tag{A.34}$$

A.9 Proof of Proposition 9

Because of the decomposition in (40), and the fact that ζ_i are independent of ϵ_i, the combined idiosyncratic variance for security i is simply $\sigma_i^2+\sigma(\epsilon_i)^2$, where σ_i^2 is the variance of the i-th induced order statistic given in (9), and $\sigma(\epsilon_i)^2$ is the original idiosyncratic variance for security i given in (1). The classical result of Treynor and Black (1973) maintains that to maximize the Sharpe ratio of the portfolio, security weights should be proportional to the expected excess returns divided by the idiosyncratic variance, which proves (41).

(42) follows from plugging in results from Proposition 6 into (41).

A.10 Proof of Proposition 10

By definitions in (43)-(44), the return of the impact portfolio in excess of the risk-free rate can be written as:

$$R_A - R_f = \alpha_A + \beta_A(R_m - R_f) + \epsilon_A. \tag{A.35}$$

When combining with the passive market portfolio, the weight of the impact portfolio, ω_A, is given in (46). Therefore, the return of the combined portfolio, in excess of the risk-free rate, is

$$\begin{aligned}
R_P - R_f &= \omega_A(R_A-R_f)+(1-\omega_A)(R_m-R_f) \\
&= \omega_A\left(\alpha_A+\beta_A(R_m-R_f)+\epsilon_A\right)+(1-\omega_A)(R_m-R_f) \\
&= \omega_A\alpha_A+(R_m-R_f)(\beta_A\omega_A+(1-\omega_A))+\omega_A\epsilon_A,
\end{aligned} \tag{A.36}$$

which completes the proof of (47). (48) and (49) follow directly from simple calculations of the expected value and variance of R_P based on (A.36).

A.11 Proof of Proposition 11

Based on the price process, (53), the investor's objective, (51), can be written as:

$$\begin{aligned}
\mathrm{E}[P_T] &= \mathrm{E}[P_{T-1} + \theta S_T^z + \gamma F_T + \epsilon_T] \\
&= \mathrm{E}[P_{T-2} + \theta S_{T-1}^z + \gamma F_{T-1} + \epsilon_{T-1} + \theta S_T^z + \gamma F_T + \epsilon_T] \\
&= P_0 + \theta(S_1^z + \cdots + S_T^z) + \gamma(F_1 + \delta F_1 + \cdots + \delta^{T-1} F_1] \\
&= P_0 + \theta(S_1^z + \cdots + S_T^z) + \frac{\gamma(1-\delta^T)F_1}{1-\delta}.
\end{aligned} \tag{A.37}$$

Maximizing $\mathrm{E}[P_T]$ over $S_1, S_2, \ldots, S_T$ is the same as maximizing the middle term in (A.37):

$$(S_1^z + \cdots + S_T^z). \tag{A.38}$$

When $z = 1$, it does not matter how trades are allocated because (A.38) is always equal to $\bar{S}$. When $0 < z < 1$, (A.38) is a concave function with respect to $S_1, S_2, \ldots, S_T$, and is maximized when $S_1 = S_2 = \cdots = S_T = \bar{S}/T$, which completes the proof of the optimal strategy, (54).

The optimal profit is simply the total value of the position subtracted by the average execution cost:

$$V^* = \mathrm{E}[P_T] \cdot \bar{S} - \mathrm{E}\left[\sum_{t=1}^{T} P_t S_t\right] = \left(\mathrm{E}[P_T] - \frac{1}{T}\sum_{t=1}^{T} \mathrm{E}[P_t]\right)\bar{S}. \tag{A.39}$$

Based on a similar derivation to (A.37), it is easy to show that

$$\mathrm{E}[P_t] = P_0 + \theta(S_1^z + \cdots + S_t^z) + \frac{\gamma(1-\delta^t)F_1}{1-\delta} = P_0 + \theta t \frac{\bar{S}^z}{T^z} + \frac{\gamma(1-\delta^t)F_1}{1-\delta}, \tag{A.40}$$

for $t = 1, 2, \ldots, T$. Substituting (A.40) into (A.39), we have

$$
\begin{aligned}
V^* &= \left(\mathrm{E}[P_T] - \frac{1}{T} \sum_{t=1}^{T} \mathrm{E}[P_t] \right) \bar{S} \\
&= \left(P_0 + \theta T \frac{\bar{S}^z}{T^z} + \frac{\gamma(1-\delta^T)F_1}{1-\delta} - \frac{1}{T} \sum_{t=1}^{T} \left(P_0 + \theta t \frac{\bar{S}^z}{T^z} + \frac{\gamma(1-\delta^t)F_1}{1-\delta} \right) \right) \bar{S} \\
&= \left(\theta T \frac{\bar{S}^z}{T^z} + \frac{\gamma(1-\delta^T)F_1}{1-\delta} - \frac{1}{T} \sum_{t=1}^{T} \left(\theta t \frac{\bar{S}^z}{T^z} + \frac{\gamma(1-\delta^t)F_1}{1-\delta} \right) \right) \bar{S} \\
&= \left(\theta \frac{\bar{S}^z}{T^z} \left(T - \frac{1}{T} \sum_{t=1}^{T} t \right) + \frac{\gamma F_1}{1-\delta} \left((1-\delta^T) - \frac{1}{T} \sum_{t=1}^{T} (1-\delta^t) \right) \right) \bar{S} \\
&= \left(\theta \frac{\bar{S}^z}{T^z} \left(T - \frac{1+T}{2} \right) + \frac{\gamma F_1}{1-\delta} \left(\frac{1}{T} \sum_{t=1}^{T} \delta^t - \delta^T \right) \right) \bar{S} \\
&= \left(\theta \frac{\bar{S}^z}{T^z} \left(\frac{T-1}{2} \right) + \frac{\gamma \delta F_1}{1-\delta} \left(\frac{1-\delta^T}{T(1-\delta)} - \delta^{T-1} \right) \right) \bar{S} \\
&= \left(\frac{\theta \bar{S}^z (T-1)}{2T^z} + \frac{\gamma \delta F_1 \left(1 - T\delta^{T-1} + (T-1)\delta^T \right)}{(1-\delta)^2 T} \right) \bar{S}
\end{aligned}
\tag{A.41}
$$

which completes the proof of (55).

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