The Fiscal Theory of the Price Level with a Bubble*

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Abstract

This paper incorporates a bubble term in the standard Fiscal Theory of the Price Level equation to explain why countries with persistently negative primary surpluses can have a positively valued currency and low inflation. It also provides two illustrative models with closed-form solutions in which the return on government bonds is below the economy's growth rate. The government can "mine" the bubble by perpetually rolling over its debt. The welfare-maximizing bubble mining rate is independent of government spending, and additional spending needs should be financed exclusively through taxes. Despite the bubble, the price level remains determined provided government policy credibly promises primary surpluses off-equilibrium.

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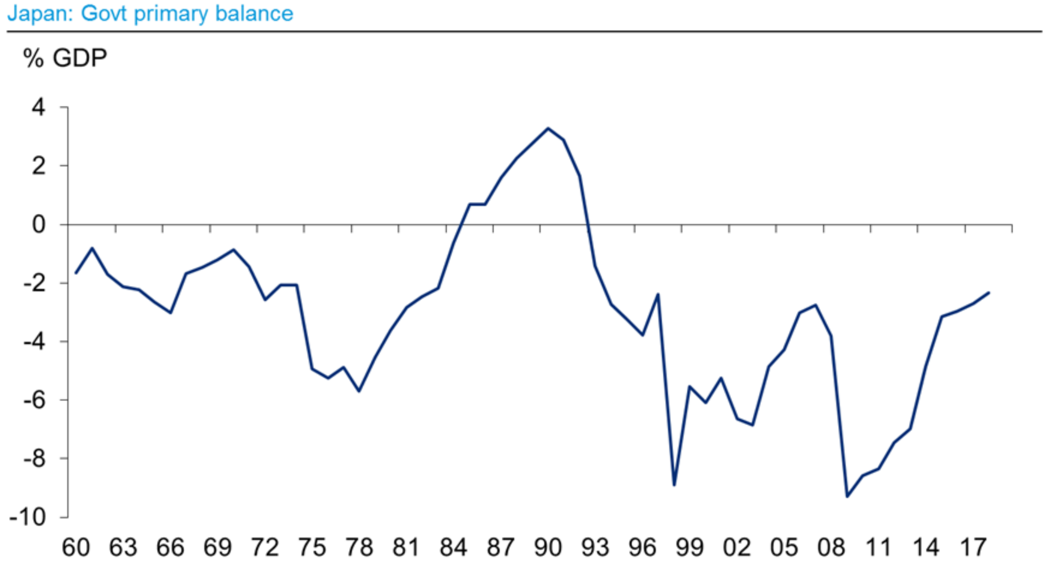

Figure 1: Japanese primary surplus 1960–2017

1 Introduction

Different monetary theories emphasize different roles of money and different equilibrium equations to determine the price level. The Fiscal Theory of the Price Level (FTPL) stresses the role of money as a store of value and argues that the real value of all outstanding government debt, i.e., the nominal debt level divided by the price level, is given by the discounted stream of future primary government surpluses. Primary surpluses are the difference between government revenue and expenditures excluding interest payments. Absent government default, an increase in expected future primary deficits leads to an increase in the price level, i.e., inflation, by devaluing outstanding debt.

Critics of the FTPL often point to Japan. Even though Japan has mostly run primary deficits since the 1960s (see Figure 1) and with no primary surpluses in sight, the price level has not risen much. Indeed, inflation levels are depressed even though the government and central bank leave no stone unturned to boost inflation closer to 2%.

In this paper, we revisit the key FTPL equation and argue that including the typically ignored bubble term reconciles the FTPL with Japan's experience. Indeed, we show that the transversality condition is often insufficient to rule out a bubble on the aggregate economy, refuting the usual justification to simply dismiss the bubble term. The bubble term cannot be ignored under either the monetary and fiscal dominance regime, a distinction of regimes the FTPL literature puts a lot of emphasis on.

A bubble term emerges whenever the real rate of return on government debt is persistently below the growth rate of the economy, i.e., whenever $r \leq g$. It is well known that this can be the case in overlapping generations models (Samuelson 1958), models of perpetual youth (Blanchard 1985), and incomplete market models with uninsurable idiosyncratic risk à la Bewley (1980). In this paper we spell out the details of the FTPL with a bubble in two simple illustrative models. The first is a version of the Blanchard (1985) model. The second is a simple Bewley-type model based on Brunnermeier and Sannikov (2016a,b) in which $r \leq g$ arises naturally due to precautionary savings demand when agents can invest in both physical capital and government bonds. In the perpetual youth model, government debt allows a transfer of resources from future to current generations. In the uninsurable idiosyncratic risk model, government debt takes on the role of a safe asset which allows citizens to partially insure their idiosyncratic risk.

By "printing" bonds, the government imposes an inflation tax that reduces the return on the bonds. Since government bonds are a bubble, the government in a sense "mines a bubble" to generate seigniorage revenue. The resulting seigniorage revenue can be used to finance government expenditures without ever having to raise extra taxes.

Despite this inflation tax logic, such bubble mining need not be inflationary. As faster bond issuance makes bonds less attractive to investors, investors shift portfolios from government bonds to private capital, stimulating investment and growth. Higher growth offsets the inflationary pressure. There is also an alternative way to "mine the bubble" not through bond issuance but by lowering the long-run nominal interest rate paid on bonds. This policy increases resources available to the government without the inflationary effect of bond issuance.

We also study optimal debt issuance policy. While possible, bubble mining is never optimal in the perpetual youth model that features a fixed growth rate. In contrast, in our second model with idiosyncratic risk, economic growth is endogenously determined. A positive rate of bubble mining with perpetually negative primary fiscal surpluses can be the optimal policy prescription, since bubble mining discourages bond holdings and boosts physical capital investments and thereby economic growth. Importantly, in both models the optimal debt issuance policy only corrects for pecuniary externalities but it never reacts to the size of or need for public expenditures. The important takeaway is that welfare-maximizing policy should rely on taxes, not bubble mining, as the marginal funding source for (additional) public expenditures.

Note that even though the possible bubble in the FTPL equation is crucial in our settings, e.g. if the stream of primary surpluses is always negative, the bubble size and hence the price level remain well defined. The size of the bubble is determined by wealth effects and goods market clearing. A larger bubble raises citizens' wealth and hence their demand for output. Output supply and goods market clearing pin down the bubble size.

The price level is uniquely determined if the fiscal authority backs the bubble to rule out equilibria that lead to hyperinflation. Such fiscal backing is only required off-equilibrium. Such backing also rules out bubbles on other assets than government debt. However, if the government does not have sufficient fiscal space for credible off-equilibrium backing, private bubbles can emerge.

Literature. Classic references for the FTPL are Leeper (1991), Sims (1994), and Woodford (1995). For more comprehensive treatments see Leeper and Leith (2016) and a recent book draft by Cochrane (2021). All of these references consider bubble-free environments. An exception is Bassetto and Cui (2018) who study the validity of the FTPL in low interest rate environments.[1] Our contribution differs to theirs in two ways. First, they focus exclusively on price determinacy and do not discuss the existence of a bubble and its implication for the government budget. Second, using a model that does feature the possibility of a bubble, a dynamically inefficient OLG setting, they conclude that the FTPL breaks down while we show in Section 7 how fiscal price level determination can succeed in the presence of a bubble.

Blanchard (2019) studies the cost of public debt when safe interest rates are low and concludes that public debt may have no fiscal cost. Brumm et al. (2021) presents four setting arguing that public debt expansion is not ideal policy to overcome the fundamental friction causes the low interest rate. Geerolf (2013) and Lian et al. (2020) document empirically when and for how long $r < g$.

Since circulation of a previous draft, some recent papers have taken up and extended the core insights from our paper. Like this paper, Reis (2021) emphasizes the bubble as a fiscal resource that has implications for debt sustainability, but his focus is on the interaction with other policies while we focus on FTPL aspects and optimal bubble mining. Brunnermeier et al. (2021) add aggregate risk to the example model presented in Section 3.2 to develop a safe asset theory of government debt. Kocherlakota (2021) studies a bubble on government debt caused by tail risk.

The FTPL equation in models with transaction benefits of money can always be written without the usual flow seigniorage term by discounting at the appropriate money rate, see e.g. Cochrane (2021, Section 6.4.6). If the convenience yield on money is sufficiently large, government debt appears to have a bubble in the resulting equation. This paper is not about the transaction benefits of money, but about bubbles that arise even under marginal utility discounting.

There is an extensive literature on rational bubbles. Survey papers include Miao (2014)

[1] Like Bassetto and Cui (2018), Farmer and Zabczyk (2020) also study the FTPL in an OLG model and conclude that the FTPL is unable to resolve equilibrium multiplicity. However, their result is based on indeterminancy in the underlying real model that is not directly related to either bubble multiplicity or indeterminacy of nominal valuations.

and Martin and Ventura (2018). More recently, Jiang et al. (2019) provide convincing empirical evidence that U.S. government debt has a bubble component.

2 The FTPL Equation with a Bubble

In this section, we derive the key equation of the fiscal theory of the price level in a generic partial equilibrium setting. We then discuss under which conditions this equation can possibly have a bubble term that has previously been ignored in the literature. To conclude, we mention the possible sources of seigniorage consistent with the equation. In the following sections, we elaborate more on these points in the context of two fully worked out examples in general equilibrium.

2.1 Revisiting the Derivation of the FTPL Equation

The derivation of the fiscal theory equation starts with the government flow budget constraint. In discrete time, this constraint is given by

$$\mathcal{B}_t + \mathcal{M}_t + \mathcal{P}_t T_t = (1 + i_{t-1})\,\mathcal{B}_{t-1} + \left(1 + i_{t-1}^m\right)\mathcal{M}_{t-1} + \mathcal{P}_t G_t,$$

where $\mathcal{B}_t$ is the nominal face value of outstanding government bonds, $\mathcal{M}_t$ is the nominal quantity of money in circulation, $\mathcal{P}_t$ is the price level, T_t are (real) taxes, G_t is (real) government spending, and i_t, i_t^m are the nominal interest rates paid on bonds and money, respectively. i_t^m can be smaller than i_t if money provides transaction services. If ξ_t is a real stochastic discount factor (SDF) process that prices government bonds, then $1 = \mathbb{E}_t\left[\xi_{t+1}/\xi_t \cdot \mathcal{P}_t/\mathcal{P}_{t+1}\,(1 + i_t)\right]$. Using this property, dividing the government budget constraint by $\mathcal{P}_t$ and rearranging yields

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}\,(1 + i_{t-1}) = T_t - G_t + \overbrace{\left(i_{t-1} - i_{t-1}^m\right)}^{\Delta i_{t-1} :=}\frac{\mathcal{M}_{t-1}}{\mathcal{P}_t} + \mathbb{E}_t\left[\frac{\xi_{t+1}}{\xi_t}\,(1 + i_t)\,\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_{t+1}}\right].$$

Iterating this forward until period T implies

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}\,(1 + i_{t-1}) = \mathbb{E}_t\left[\sum_{s=t}^{T}\frac{\xi_s}{\xi_t}\,(T_s - G_s)\right] + \mathbb{E}_t\left[\sum_{s=t}^{T}\frac{\xi_s}{\xi_t}\Delta i_{s-1}\frac{\mathcal{M}_{s-1}}{\mathcal{P}_s}\right] + \mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right].$$

Up to this point, we have merely rearranged and iterated the government budget constraint and assumed that there is some SDF process ξ_t that prices government bonds in equilibrium. To derive the fiscal theory equation, the literature now typically proceeds by invoking a private-sector transversality condition to eliminate the discounted terminal value of government debt when passing to the limit $T \to \infty$. In this paper, we focus on environments where the transversality condition does not eliminate the terminal value in the limit. When taking the limit $T \to \infty$, we therefore arrive at the more general equation[2]

$$\frac{\mathcal{B}_{t-1} + \mathcal{M}_{t-1}}{\mathcal{P}_t}(1 + i_{t-1}) = \underbrace{\mathbb{E}_t\left[\sum_{s=t}^{\infty} \frac{\xi_s}{\xi_t}(T_s - G_s)\right]}_{\text{PV of primary surpluses}} + \underbrace{\mathbb{E}_t\left[\sum_{s=t}^{\infty} \frac{\xi_s}{\xi_t} \Delta i_{s-1} \frac{\mathcal{M}_{s-1}}{\mathcal{P}_s}\right]}_{\text{PV of future transaction services}} + \underbrace{\lim_{T\to\infty} \mathbb{E}_t\left[\frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right]}_{\text{bubble}}.$$

Relative to the standard equation, this equation contains an additional bubble term. That term does not vanish in the limit if government debt has a bubble component. More generally, for any asset, we say that the asset has a bubble component if its market value exceeds its fundamental value. We define the fundamental value as the discounted present value of the asset's cash flows where cash flows are discounted using the SDF ξ generated by the marginal utility of the marginal holder of the asset.

From now on, we switch to continuous time in order to make our formal arguments more elegant. The continuous-time version of the last equation is given by[3]

$$\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = \mathbb{E}_t\left[\int_t^{\infty} \frac{\xi_s}{\xi_t}(T_s - G_s)\,ds\right] + \mathbb{E}_t\left[\int_t^{\infty} \frac{\xi_s}{\xi_t} \Delta i_s \frac{\mathcal{M}_s}{\mathcal{P}_s} ds\right] + \lim_{T\to\infty} \mathbb{E}_t\left[\frac{\xi_T}{\xi_t} \frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T}\right]. \quad (1)$$

This equation for the real value of government debt holds in any monetary model. While most conventional monetary models treat this equation as an intertemporal government budget constraint that holds on- and off-equilibrium, in the FTPL it is an equilibrium condition that determines the price level.

2.2 When Can a Bubble Exist?

Equation (1) differs from the standard fiscal theory only by the presence of an additional bubble term. When can this bubble term be nonzero? Well-known examples are bubbles in OLG (Samuelson 1958) and perpetual youth (Blanchard 1985) models. In Section 3.1, we analyze a simple version of the latter model. In Section 3.2, we present another example with incomplete idiosyncratic risk sharing. Here, we make some generic points that apply across models.

For tractability, let us focus on environments with a stationary debt-to-GDP ratio and no

[2] Mathematically, the sum of the three limits in the decomposition below may not be well-defined, even if the limit of the sum is. In this case, the right-hand side should be interpreted as the limit of the sum. For instance, this can happen if the bubble term is ∞, but the present value of surpluses is $-\infty$. While this may seem a pathological case, it can make sense economically because the bubble and surpluses are not separately tradeable, but necessarily bundled together to one asset: government debt. As long as the value of this asset is well-defined and finite, infinite subcomponents do not imply arbitrage opportunities or infinite utility.

[3] A formal derivation can be found in Appendix A.1.

risk. In this case, the real value of government debt is

$$\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} = \frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} e^{g(T-t)},$$

where g is the growth rate of the economy, and $\xi_T/\xi_t = e^{-r^f(T-t)}$ with r^f denoting the real risk-free rate. By substituting these expressions into equation (1), we see that the bubble term does not vanish in the limit if $r^f \leq g$. More generally, the correct risk-adjusted discount rate compensating for the real risk inherent in $\frac{\mathcal{B}_T+\mathcal{M}_T}{\mathcal{P}_T}$ must be used instead of the risk-free rate to determine whether a bubble is possible.[4]

For any agent with recursive isoelastic utility (which includes CRRA utility) that is marginal in the market for government debt, the risk-free rate is[5]

$$r^f = \rho + \psi^{-1}\mu^c - \frac{\gamma\left(1+\psi^{-1}\right)}{2}\|\sigma^c\|^2, \tag{2}$$

where $\rho > 0$ is the agent's time preference rate, γ is the relative risk aversion coefficient, ψ is the EIS, μ^c is the growth rate of agent-specific consumption, and σ^c is a vector or relative risk exposures of agent-specific consumption to Brownian risk factors.[6] $\|\cdot\|$ denotes the standard Euclidean norm. This equation is linked to the growth rate of the economy through individual consumption growth μ^c. For example, in a representative agent economy with a balanced growth path $\mu^c = g$.

Equation (2) suggests two reasons the growth rate may exceed the risk-free rate. First, expected consumption growth of individuals may be misaligned with aggregate growth, so that a higher growth rate g may not imply higher individual consumption growth μ^c. This is the case, for example, in OLG or perpetual youth models with population growth. We present a model of this type in Section 3.1. Second, large individual risk exposure (large $\|\sigma^c\|^2$) or risk aversion (large γ) may depress the risk-free rate through the last term in equation (2) and offset any positive effects of growth g on r^f through $\psi^{-1}\mu^c$.[7] We provide an example of this type in Section 3.2. Importantly, the main insights we derive from our two models do not depend on the precise channel through which a bubble can be sustained. These insights would equally apply to other environments in which a bubble term in equation (1) is possible.

[4] Bohn (1995) provides an example of a stochastic economy in which $r^f < g$ but no bubbles can exist.

[5] This equation assumes environments with non-stochastic investment opportunities.

[6] Here, we assume that all risk takes the form of Brownian risk. The intuition derived from the argument is unaltered if more general sources of consumption risk are permitted.

[7] In addition, bubble existence in this second case regularly requires idiosyncratic risk, i.e., the individual consumption growth volatility must differ from its aggregate counterpart. The reason is that in the presence of aggregate risk, the risk-free rate is not the correct comparison rate in the bubble existence condition. The correct rate adds a risk premium for aggregate risk on top of r^f that can offset the aggregate risk component in the last term of equation (2).

The possibility of $r^f \leq g$ is not merely a theoretical curiosity. Historically, real interest rates on government bonds of advanced economies have mostly been below the growth rate. Even Abel et al. (1989), who are often cited as providing evidence against the existence of rational bubbles, report that the safe interest rate r^f is smaller than g. With the more recent decline in r^f, as stressed by Blanchard (2019), the evidence for $r^f < g$ has become more clear-cut. See also Geerolf (2013) and Lian et al. (2020).

In a large class of canonical general equilibrium models, transversality conditions prevent the possibility of $r^f \leq g$ that leads to a bubble. This is regularly true for complete-market models with long-lived agents. However, many interesting incomplete market models do permit bubbles. The transversality condition has less bite in these models, because *individual* bond holdings matter for transversality, while it is the *aggregate* bond value that enters equation (1). Individual bond holdings can have different properties than the aggregate bond stock because agents find it optimal to trade the bonds in order to partially complete the market. For example, individual bond holdings may grow at a lower rate or have higher risk than the aggregate bond stock. Then, individual transversality conditions can hold, yet the bubble term in (1) cannot be ruled out. Our two illustrative models presented in Section 3 cover these two cases. In the context of these models, we discuss the transversality condition in Section 4

2.3 Three Forms of Seigniorage

Equation (1) suggests three forms of seigniorage, which here we define simply as government spending that is not backed by offsetting future taxes. The first takes the form of a dilution of private claims to future primary surpluses through surprise devaluations of existing government debt or money.[8] Under rational expectations, this cannot be a regular source of revenue for governments. For the U.S., Hilscher et al. (2014) assess the possibility of future surprise devaluation based on option-implied (risk-neutral) probabilities and conclude that this form of seigniorage is perceived to be a negligible source of revenue. The likelihood of a devaluation exceeding 5% of GDP is less than 1%.

A second form of seigniorage comes from exploiting the liquidity benefits (convenience yield) of "narrow" money ($\mathcal{M}$ in equation (1)). This form of seigniorage can only be extracted from the portion of government debt that takes the form of "narrow" money and provides liquidity services. It depends on the interest rate differential $\Delta i = i - i^m$ between illiquid and liquid government debt. It is small if either that differential is small or if the stock of "narrow" money is only a small part of total government debt.[9] This form of seigniorage is not an im-

[8]Without long-term debt as in equation (1) such dilution must work through a sudden surprise inflation (an unexpected upward jump in $\mathcal{P}_t$). In a more realistic setting with long-term debt, news of higher inflation going forward would have similar effects and work through bond prices instead of the general price level.

[9]In reality, one has to distinguish between reserves, whose quantity is nonnegligible, but which pay interest and

portant funding source for advanced economies. For example, in the U.S., Reis (2019) reports a flow revenue of approximately 0.36% of GDP and estimates a present value of $\approx$ 20% and, at most, 30% of GDP. Moreover, in the future the Δi term is likely to decline, because central banks pay interest on reserves and as money becomes more digitalized, its velocity rises.

Besides these standard forms of seigniorage, equation (1) suggests a third form of seigniorage that has remained unexplored in prior work and is the focus of this paper. The government can "mine" the bubble by using its outstanding government debt to run an ever-expanding Ponzi scheme: letting the stock of government debt grow generates a steady revenue flow that does not have to be paid for by future taxes as long as a bubble term is present in equation (1). Unlike a surprise dilution through inflation, dilution of the bubble value is feasible even if it is fully expected by the private sector. This form of seigniorage is arguably larger than the officially measured seigniorage from growing narrow money $\mathcal{M}$ because all revenue from growing $\mathcal{B} + \mathcal{M}$ is relevant for bubble mining.

3 Two Models with a Bubble

There are several model structures in which rational bubbles can exist and thus the bubble term in equation (1) does not necessarily disappear. We illustrate this in two simple examples. Each contains one of the two mechanisms discussed in Section 2.2 through which the risk-free rate (2) can fall below the growth rate.

In the first model, a bubble can be sustained by the difference between the individual and the aggregate consumption growth rates. That model is a very simple version of the perpetual youth model (Blanchard, 1985). Government debt may circulate as a bubble because it facilitates trade between the current and not yet born future generations.

In the second model, a bubble can be generated through incomplete idiosyncratic risk sharing. That model is a streamlined version of Brunnermeier and Sannikov (2016a) without banks.[10] Government debt may circulate as a bubble because bond trading allows agents to self-insure against idiosyncratic shocks.

For simplicity, we abstract in both models from the presence of additional "narrow" money that yields transaction benefits.[11]

have therefore a small Δi_t, and cash, which has a much larger Δi_t, but whose quantity is almost negligible relative to the overall stock of government debt.

[10]The model version without banks has previously been analyzed in Brunnermeier and Sannikov (2016b) and Di Tella (2020). These papers frame the model as a model of money. Here, we add fiscal policy and reinterpret money as bonds. The bond interpretation is also adopted in the safe asset framework of Brunnermeier et al. (2021).

[11]Other than adding an additional source of seigniorage, including transaction benefits into the analysis does not substantially alter our conclusions. For the FTPL with transaction benefits but no bubble, see, e.g., Sims (2019).

In this section, we set up both models and briefly outline their solution. We then discuss bubble existence, bubble mining policies, and price level determinations for both models simultaneously in the following Sections 4 to 7. A more formal derivation of the model solutions is presented in Appendix A.2.

3.1 Example I: Perpetual Youth

Environment. At each time t, there is a continuum of households indexed by $i \in [0, L_t]$, where L_t is the population mass at time t. We assume that population grows at a constant rate $g > 0$, $dL_t = gL_t dt$. For simplicity, we abstract from death, so that all households are infinitely lived.[12] Households $i \in [0, L_t]$ alive at time t have logarithmic preferences

$$V_t^i := \mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)} \log c_s^i ds\right]$$

with discount rate ρ.

Each household i born at a time $t_0 > 0$ is endowed at birth with one unit of human capital, $k_{t_0}^i = 1$. Human capital depreciates over time at a constant rate $\delta \geq 0$, $dk_t^i = -\delta k_t^i\, dt$, and k_t^i units of human capital produce an output flow of $ak_t^i dt$ ("labor income").

Denote by $K_t := \int_0^{L_t} k_t^i di$ the aggregate quantity of human capital in the economy. We choose initial conditions such that human capital per capita K_t/L_t is constant over time.[13] Then, K_t also grows at the constant rate g.

The key friction in the model is that agents are not able to trade with yet unborn generations. Instead, they can only enter financial contracts with other agents currently alive.

Besides households, there is a government that funds government spending, imposes taxes on labor income, and issues nominal bonds. The government has an exogenous need for real spending $\mathfrak{g}K_t dt$, where $\mathfrak{g}$ is a model parameter.[14] The government levies proportional labor income taxes (subsidies, if negative) τ_t on households. Outstanding government debt has a nominal face value of $\mathcal{B}_t$ and pays nominal interest i_t. $\mathcal{B}_t$ follows a continuous process $d\mathcal{B}_t = \mu_t^{\mathcal{B}} \mathcal{B}_t dt$, where the growth rate $\mu_t^{\mathcal{B}}$ is a policy choice of the government. In short, the government chooses the policy instruments τ_t, i_t, $\mu_t^{\mathcal{B}}$ as functions of histories of prices taking $\mathfrak{g}$ as given and

[12] At the expense of additional notation, one could easily assume a positive, but constant, death rate, provided there are annuity markets that insure against idiosyncratic death risk. See Blanchard (1985) for details. Other than simplifying notation, abstracting from death also stresses that an infinite lifespan does not preclude the existence of bubbles.

[13] Formally, the relevant condition is $K_0/L_0 = \frac{g}{g+\delta}$. This can be shown easily by computing the time derivative of K_t/L_t.

[14] Making spending proportional to total human capital K_t is equivalent to making spending proportional to population size L_t because K_t/L_t is constant.

subject to the nominal budget constraint[15,16]

$$\underbrace{\overbrace{\left(\mu_t^{\mathcal{B}} - i_t\right)}^{\text{bond growth in excess of payouts}}}_{=:\breve{\mu}_t^{\mathcal{B}}} \mathcal{B}_t \qquad + \qquad \overbrace{\mathcal{P}_t \underbrace{(\tau_t a - \mathfrak{g})}_{=:s_t} K_t}^{\text{(nominal) primary surpluses}} \qquad = 0, \tag{3}$$

where $\mathcal{P}_t$ denotes the price level as in Section 2.

The model is closed by the aggregate resource constraint

$$C_t + \mathfrak{g} K_t = a K_t, \tag{4}$$

where $C_t := \int_0^{L_t} c_t^i di$ is aggregate consumption.

Household Problem. Let b_t^i the (real) value of bond holdings of household i. Bond holdings satisfy the accumulation equation

$$db_t^i = \left(r_t^f b_t^i + (1 - \tau_t)\, a k_t^i - c_t^i\right) dt, \tag{5}$$

where, as in Section 2, r_t^f denotes the real risk-free rate, which equals the return on bonds in this model. The household chooses consumption $\{c_s^i\}_{s \geq t}$ to maximize utility V_t^i subject to the evolution of human capital and bond holdings (5) and subject to a standard no Ponzi condition.

In the appendix, we show that optimal consumption satisfies the familiar log-utility permanent-income consumption rule

$$c_t^i = \rho \left(b_t^i + q_t^K k_t^i\right), \tag{6}$$

where q_t^K is the shadow price of human capital. It is determined by the condition that the value of human capital must equal the present value of future after-tax labor income,

$$q_t^K k_t^i = \int_t^\infty e^{-\int_t^s r_{s'}^f ds'} (1 - \tau_s) a k_s^i ds. \tag{7}$$

Together with a standard transversality condition (to be discussed in Section 4), conditions (6) and (7) fully determine the optimal choices of each individual agent i.

[15] Letting policy depend on histories of endogenous price paths is common in the FTPL literature to discuss what happens off-equilibrium.

[16] At this point, we do not impose additional restrictions on government policy, including whether policy is characterized by monetary dominance or fiscal dominance, as the choice of the policy regime is irrelevant for most of our results. We do make more restrictive assumptions in Section 7 where we explain how to adjust the fiscal theory arguments for price level determination based on fiscal dominance if government debt has a bubble component.

Equilibrium. In this model, any equilibrium can be fully characterized by determining two "prices", the shadow value of human capital q_t^K and the nominal price level $\mathcal{P}_t$. Instead of working with the price level directly, it is more convenient to use the transformation $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$, which is the ratio of the real value of government debt to total human capital in the economy.[17] Total private net wealth consists of bond wealth, $q_t^B K_t$, and human capital wealth, $q_t^K K_t$.

The optimal consumption rule (6) implies that total wealth must be proportional to aggregate consumption,[18]

$$q_t^B + q_t^K = \frac{a - \mathfrak{g}}{\rho}.$$

To characterize the equilibrium, we therefore need to determine the share of total wealth that is due to bond wealth. We denote this share by ϑ_t:

$$\vartheta_t := \frac{q_t^B}{q_t^B + q_t^K}.$$

The equilibrium behavior of ϑ_t itself can be derived by combining the Fisher equation for the bond return r_t^f with the government budget constraint (3) and the human capital valuation equation (7). The former equation is in this model

$$r_t^f = i_t - \pi_t = \underbrace{i_t - \mu_t^{\mathcal{B}}}_{=-\breve{\mu}_t^{\mathcal{B}}} + g + \mu_t^{q,B}, \tag{8}$$

where inflation $\pi_t := \dot{\mathcal{P}}_t/\mathcal{P}_t$ depends on the nominal bond growth rate, $\mu_t^{\mathcal{B}}$, the growth rate of the economy, g, and the appreciation of the normalized real bond price (q_t^B), $\mu_t^{q,B} := \dot{q}_t^B/q_t^B$. Notice that equation (8) defines the growth in debt in excess of the amount used to pay nominal interest, $\breve{\mu}_t^{\mathcal{B}}$. This quantity is related to seigniorage.

We combine these equilibrium conditions in the appendix and show that ϑ_t is characterized by the equation

$$\vartheta_t = \int_t^\infty e^{-\rho s} \left((1 - \vartheta_s) \left(\delta + g \right) - \breve{\mu}_s^{\mathcal{B}} \right) \vartheta_s ds. \tag{9}$$

This equation relates the current bond wealth share ϑ_t positively to the future flow $(1 - \vartheta_s)(\delta + g)$. Note that $g + \delta$ is the gap between aggregate and individual human capital growth. Bonds trading is more beneficial as this gap increases. The equation also relates ϑ_t negatively to the expected future path of $\breve{\mu}_t^{\mathcal{B}}$, which measures the dilution of existing bond holders through the issuance of new bonds.

[17] It is more convenient to work with this normalized version of the inverse price level $1/\mathcal{P}_t$ because the latter depends on the scale of the economy and the nominal quantity of outstanding bonds in equilibrium, whereas q_t^B does not.

[18] The following equation also makes use of the aggregate resource constraint (4).

Steady-State Equilibria. We now focus on government policies that hold $\breve{\mu}^{\mathcal{B}}$ and τ constant over time and consider steady-state equilibria with constant q^B and q^K – and thus constant ϑ. All such equilibria must solve equation (9) with constant ϑ. One can show that there is at most one steady-state equilibrium in which bonds have positive value. It is given by

$$\vartheta = 1 - \frac{\rho + \breve{\mu}^{\mathcal{B}}}{\delta + g}.$$

That formula describes a valid equilibrium if both the value of human capital and the value of bonds are positive. This is the case if and only if

$$\delta + g > \rho + \breve{\mu}^{\mathcal{B}}.$$

We make this parameter assumption from now on and focus exclusively on this "monetary steady state" with a positive value of government bonds. We also remark that there is always a second steady-state equilibrium, in which bonds have no value, $q^B = \vartheta = 0$,[19] and there are many additional non-stationary equilibria. However, we show in Section 7 that a simple off-equilibrium modification to the fiscal policy rule can select the monetary steady state as the unique equilibrium.

3.2 Example II: Uninsurable Idiosyncratic Risk

Environment. There is a continuum of households indexed by $i \in [0, 1]$. All households have identical logarithmic preferences

$$V_0^i := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right]$$

with discount rate ρ.

Each agent operates one firm that produces an output flow $ak_t^i dt$, where k_t^i is the (physical) capital input chosen by the firm. Absent market transactions of capital, capital of firm i evolves according to

$$\frac{dk_t^i}{k_t^i} = \left(\Phi\left(\iota_t^i\right) - \delta\right) dt + \tilde{\sigma} d\tilde{Z}_t^i,$$

where $\iota_t^i k_t^i dt$ are physical investment expenditures of firm i (in output goods), Φ is a concave function that captures adjustment costs in capital accumulation, δ is the depreciation rate, and $\tilde{Z}^i$ is an agent-specific Brownian motion that is i.i.d. across agents i. $\tilde{Z}^i$ introduces firm-specific

[19] In that equilibrium, the price level is infinite, $\mathcal{P} = \infty$, and the government does not raise any primary surpluses, $\tau a = \mathfrak{g}$.

idiosyncratic risk. To obtain simple closed-form expressions, we choose the functional form $\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.

The key friction in the model is that agents are not able to share idiosyncratic risk. While they are allowed to trade physical capital and risk-free assets, they cannot write financial contracts contingent on individual $\tilde{Z}^i$ histories. As a consequence, all agents have to bear the idiosyncratic risk inherent in their physical capital holdings.

As in Section 3.1, there is also a government that funds government spending, imposes taxes on firms, and issues nominal government bonds. With $K_t := \int k_t^i di$ denoting the aggregate capital stock, the government flow budget constraint is precisely as in the perpetual youth model (equation (3)).

The aggregate resource constraint in this model is

$$C_t + \mathfrak{g}K_t + I_t = aK_t, \tag{10}$$

where, compared to Section 3.1 (equation (4)), here output can also be used for physical investment, $I_t := \int \iota_t^i k_t^i di$.

Price Processes and Returns. We use notation in complete analogy to Section 3.1. Let q_t^K be the market price of a single unit of physical capital and let $q_t^B := \frac{\mathcal{B}_t/\mathcal{P}_t}{K_t}$ be the ratio of the real value of government debt to total capital in the economy. As before, we define $\vartheta_t := q_t^B/(q_t^B + q_t^K)$ as the share of total wealth that is due to bond wealth. We denote by $\mu_t^{q,B} := \dot{q}_t^B/q_t^B$ and $\mu_t^{q,K} := \dot{q}_t^K/q_t^K$ the instantaneous growth rates of q_t^B and q_t^K, respectively.

Households can trade two assets in positive net supply (if $q_t^B \neq 0$), bonds and capital. Assume that in equilibrium $\iota_t = \iota_t^i$ for all i (to be verified below) so that aggregate capital grows deterministically at rate $g_t = \Phi(\iota_t) - \delta$. Then, the return on bonds is

$$dr_t^{\mathcal{B}} = \left(-\check{\mu}_t^{\mathcal{B}} + \Phi(\iota_t) - \delta + \mu_t^{q,B}\right) dt \tag{11}$$

in full analogy to equation (8). The return on agent i's capital is

$$dr_t^{K,i}\left(\iota_t^i\right) = \left(\frac{(1-\tau_t)\,a - \iota_t^i}{q_t^K} + \Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}\right) dt + \tilde{\sigma} d\tilde{Z}_t^i.$$

The expected capital return consists of the after-tax dividend yield, $\frac{(1-\tau_t)a-\iota_t^i}{q_t^K}$, and the capital gains rate, $\Phi\left(\iota_t^i\right) - \delta + \mu_t^{q,K}$. Capital returns are risky due to the presence of idiosyncratic risk $\tilde{\sigma} d\tilde{Z}_t^i$.

Household Problem and Equilibrium. The household problem is analogous to the one presented in Section 3.1 with the exception that agents now also choose the capital investment rate ι_t^i and the share of wealth θ_t^i invested in bonds as opposed to capital. We relegate the details of this problem to the appendix. The first-order conditions for the three choices are

$$q_t^K = \frac{1}{\Phi'\left(\iota_t^i\right)}, \qquad \text{Tobin's } q$$

$$c_t^i = \rho n_t^i, \qquad \text{permanent income consumption}$$

$$\frac{a - \mathfrak{g} - \iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \breve{\mu}_t^{\mathcal{B}}}{1 - \vartheta_t} = \left(1 - \theta_t^i\right)\tilde{\sigma}^2, \qquad \text{Merton portfolio}$$

where n_t^i denotes the net worth of agent i, which consists of both capital and bond holdings, and $\mu_t^\vartheta := \dot{\vartheta}_t/\vartheta_t$ is the growth rate of the bond wealth share ϑ_t. Relative to Section 3.1, only the second condition is identical. The first condition is entirely new. It captures the optimal physical investment choice.[20] The third condition (portfolio choice) replaces condition (7), the valuation equation for nontraded human capital in Section 3.1. It equates the excess return on capital with the required risk premium $\left(1 - \theta_t^i\right)\tilde{\sigma}^2$ for bearing idiosyncratic risk.[21]

As in Section 3.1, the optimal consumption rule implies that total wealth is proportional to total consumption. The difference here is that total consumption is no longer a fixed proportion of aggregate capital but depends on the endogenous reinvestment choice ι_t. Nevertheless, one can show that asset prices q_t^B and q_t^K and the investment rate ι_t are still simple functions of ϑ_t,

$$\iota_t = \frac{(1-\vartheta_t)(a-\mathfrak{g}) - \rho}{1 - \vartheta_t + \phi\rho},$$

$$q_t^B = \vartheta_t \frac{1 + \phi(a - \mathfrak{g})}{1 - \vartheta_t + \phi\rho},$$

$$q_t^K = (1-\vartheta_t)\frac{1 + \phi(a - \mathfrak{g})}{1 - \vartheta_t + \phi\rho},$$

so that, again, the equilibrium is determined uniquely up to the dynamics of the bond wealth share ϑ_t. Similar to Section 3.1, ϑ_t must solve a valuation equation

$$\vartheta_t = \int_t^\infty e^{-\rho s}\left((1-\vartheta_s)^2\tilde{\sigma}^2 - \breve{\mu}_s^{\mathcal{B}}\right)\vartheta_s ds \tag{12}$$

that relates the current bond wealth share ϑ_t positively to expected future "services" $(1-\vartheta_s)^2\tilde{\sigma}^2$ and negatively to future bond dilution $\breve{\mu}_t^{\mathcal{B}}$. Equation (12) is structurally identical to equation (9)

[20] In particular, because all agents face the same capital price q_t^K, they all choose the same investment rate ι_t^i, verifying our previous assumption.

[21] Government spending $\mathfrak{g}$ enters that third equation because we have used the government budget constraint (3) to substitute out taxes.

in the perpetual youth model. The only difference is that services are now derived from self-insurance against idiosyncratic risk and thus related to the magnitude of risk $\tilde{\sigma}$ the agents are exposed to.

Steady-State Equilibria. We consider again steady-state equilibria with constant policy choices $\breve{\mu}^{\mathcal{B}}$ and τ and constant (scaled) asset prices q^B and q^K. As in Section 3.1, there is at most one steady state equilibrium in which bonds have positive value.[22] In that "monetary steady state", if it exists, the bond wealth share is

$$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$$

and the remaining equilibrium quantities are given by

$$\iota = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(a - \mathfrak{g}) - \rho\tilde{\sigma}}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \qquad q^B = \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}, \qquad q^K = \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(1 + \phi\,(a - \mathfrak{g}))}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\rho\tilde{\sigma}}.$$

These formulas describe a valid equilibrium only if idiosyncratic risk is sufficiently large,

$$\tilde{\sigma}^2 > \rho + \breve{\mu}^{\mathcal{B}}.$$

In what follows, we always make this assumption.

4 Transversality Condition and Existence of a Bubble

In both models, government debt can have value even in the absence of primary surpluses ($\breve{\mu}^{\mathcal{B}} \geq 0$) because it has a bubble component. In the perpetual youth model, it provides a store of value which allows agents to exchange some of their present labor income for a claim to the labor income of future generations. In the idiosyncratic risk model, bonds are the only store of value that is free of idiosyncratic risk and thus allow agents to self-insure against their risk exposures. In this section, we discuss why the private-sector transversality condition may not rule out the existence of a bubble despite the infinite lifespan of all agents. The key insight is that a bubble can exist because agents do not buy and hold government bonds, but optimally trade them. Such trading makes their individual bond portfolios look very different from the aggregate bond stock.

[22] There are again a non-monetary steady state with $q^B = 0$ and many nonstationary equilibria. As for the perpetual youth model, all these additional equilibria can be ruled out by a modification of the fiscal policy rule (see Section 7).

For each individual agent, a transversality condition for bond holdings,

$$\lim_{T\to\infty} \mathbb{E}\left[\xi_T^i b_T^i\right] = 0,$$

is necessary for an optimal choice. Here, as in Section 2, $\xi_t^i := e^{-\rho(t-t_0)}\frac{1}{c_t^i}$ denotes the SDF process of agent i.[23] The transversality condition appears to suggest that it should not be possible to have a nonzero bubble term in the FTPL equation (1). However, this argument overlooks that individual bond wealth b_T^i that enters the transversality condition differs from the aggregate value of bonds $\mathcal{B}_T/\mathcal{P}_T$ that enters the FTPL equation.

In the perpetual youth model, this is the case because agents' optimal savings decision leads them to eventually decumulate their bond holdings to support additional consumption. Bond decumulation at the individual level is sustainable without a reduction in the aggregate value of bonds because agents can pass bonds on to newly born generations.

In the idiosyncratic risk model, the aggregate bond stock $B_T/\mathcal{P}_T$ evolves deterministically, yet individual bond wealth b_T^i is optimally chosen to be stochastic because agents constantly rebalance their portfolios in response to idiosyncratic shocks. Agents thus discount b_t^i at a risk-adjusted rate that takes into account their idiosyncratic risk. As idiosyncratic risk cancels out in the aggregate, when valuing a fixed fraction of the outstanding bond stock, as in the FTPL equation (1), the relevant discount rate from the perspective of all agents is instead the risk-free rate.

Formally, we have in both models $c_t^i = \rho(b_t^i + q^K k_t^i)$ and $q^K k_t^i \geq 0$, so that

$$\mathbb{E}\left[\xi_T^i b_T^i\right] = e^{-\rho(T-t_0)}\frac{1}{\rho}\mathbb{E}\left[\frac{b_T^i}{b_T^i + q^K k_T^i}\right] \leq \frac{1}{\rho}e^{-\rho(T-t_0)} \to 0 \qquad (T\to\infty)$$

and thus the individual transversality condition is clearly satisfied in the equilibria determined in Sections 3.1 and 3.2. Yet, when determining agent i's time t_0 valuation of the entire government bond stock at time T, we obtain (up to a scaling constant)

$$\mathbb{E}\left[\xi_T^i \int b_T^j dj\right] = \mathbb{E}\left[\xi_T^i q^B K_T\right] = e^{-r^f(T-t_0)} q^B K_T = e^{\left(g-r^f\right)(T-t_0)} q^B K_{t_0}$$

and the latter expression does not converge to zero, if $r^f \leq g$.

The difference in the two equations is the presence of the dj-integral. In the perpetual youth model, that integral runs over an expanding interval $[0, L_t]$ of agents, so that aggregate bond wealth can grow at a rate g, even though each individual integrand b_T^j grows (asymptotically)

[23] t_0 denotes the birth time of agent i in the perpetual youth model and is simply set to zero in the idiosyncratic risk model.

only at the lower rate $g - \check{\mu}^{\mathcal{B}} - \rho$. When the risk-free rate is in between the two growth rates, a bubble can exist, yet individual transversality conditions are satisfied.

In the idiosyncratic risk model, the integral is over a fixed interval $[0,1]$, so the (expected) growth rates of aggregate bond and individual bond wealth must be identical. Instead, the integral averages out idiosyncratic shocks and changes the risk characteristics relative to the individual integrands. All individual integrands b^i_T have idiosyncratic risk that is negatively correlated with agent i's SDF ξ^i_t. The effective discount rate in the individual transversality condition therefore contains a covariance term (risk premium) that raises the discount rate above r^f. For the total bond stock, idiosyncratic risk averages out and discounting happens at the risk-free rate.

Nothing in the model prevents the bubble existence condition $r^f \leq g$. Indeed, the growth rate of the economy equals the (human or physical) capital growth rate g and the risk-free rate equals the return on bonds, by equations (8) or (11), respectively

$$r^f = g - \check{\mu}^{\mathcal{B}}. \tag{13}$$

Consequently, $r^f \leq g$, if and only if $\check{\mu}^{\mathcal{B}} \geq 0$. A nonnegative value of $\check{\mu}^{\mathcal{B}}$ is consistent with the existence condition of a monetary equilibrium if the specific reason that generates bond savings demand in the models is sufficiently strong relative to the time preference rate. In the perpetual youth model, this is the case if $\delta + g \geq \rho$, i.e. if population growth (g) and/or the decay rate in individual life-cycle labor income (δ) are sufficiently large. In the idiosyncratic risk model, this is the case if $\tilde{\sigma}^2 \geq \rho$, i.e. if idiosyncratic risk is sufficiently large.

5 Mining the Bubble

In this section, we show how the government can mine a bubble, i.e. finance government expenditures without ever raising taxes for them. We also discuss under which circumstances bubble mining is inflationary.

Primary surplus is defined as $T_t - G_t = \tau a K_t - \mathfrak{g} K_t = s K_t$. Due to our assumptions on fiscal policy, it grows at the same rate as K_t. From the government budget constraint (3), $s = -\check{\mu}^{\mathcal{B}} q^B$. Hence, in our two models, the fiscal theory equation (1) reduces to[24]

$$q^B K_0 = \lim_{T\to\infty} \left(\underbrace{\int_0^T e^{-(r^f-g)t} s K_0 dt}_{=:PVS_{0,T}} + e^{-(r^f-g)T} q^B K_0 \right).$$

[24]Since the models do not include "narrow" money, there is no Δi term.

Provided $q^B > 0$, equation (13) implies precisely three cases:[25]

1. $s > 0, \breve{\mu}^{\mathcal{B}} < 0$: then $r^f > g$, $PVS_{0,\infty} > 0$ and a bubble cannot exist. This is the "conventional" situation considered in the literature.

2. $s = \breve{\mu}^{\mathcal{B}} = 0$: then $r^f = g$, $PVS_{0,\infty} = 0$ and there is a finite positive bubble whose value exactly equals $q^B K_0$ and grows over time at the growth rate/risk-free rate.

3. $s < 0, \breve{\mu}^{\mathcal{B}} > 0$: then $r^f < g$ and thus the integral $PVS_{0,T}$ converges to $-\infty$ as $T \to \infty$. Yet, q^B is still positive, which is only possible if there is an offsetting infinite positive bubble. These infinite values do not violate any no-arbitrage condition and are also not otherwise economically problematic, since the bubble cannot be traded separately from the claim on surpluses. Both are necessarily bundled in the form of government bonds. As long as $\frac{\mathcal{B}_t}{\mathcal{P}_t} = q^B K_t$ is determined and finite in equilibrium, the model remains economically and mathematically sensible despite the infinite values in the decomposition of the value of government bonds.[26]

In all three cases, the (possible) presence of a bubble represents a fiscal resource that grants the government some extra leeway. Clearly in case 3, the government can run a perpetual deficit, "mine the bubble" and never has to raise taxes to fully fund all government expenditures. In case 2, the existence of the bubble is beneficial, because the value of government debt is positive – allowing agents to transfer resources across generations or self-insure against risk – despite the fact that the present value of primary surpluses is zero. Even in case 1, government debt is more sustainable since an unexpected drop of primary surpluses to zero results in a bubble instead of a total collapse of the value of debt.

Is bubble mining inflationary? Not necessarily. Among steady-state policies the answer depends on how the government mines the bubble, by issuing more debt or paying less interest, and on the impact of policy on economic growth.

Specifically, by the Fisher equation, inflation in our models is

$$\pi = i - r^f = i + \breve{\mu}^{\mathcal{B}} - g.$$

For a given nominal interest rate i, there is a direct inflationary effect from an increase in bubble mining $\breve{\mu}^{\mathcal{B}}$. Higher bubble mining at a given interest rate requires the government to grow its

[25] The apparent dichotomy, a positive bubble value and nonnegative surpluses or positive surpluses and no bubble, is due to the steady-state nature of our analysis. In a more general model, a positive present value of surpluses and a bubble can coexist.

[26] However, surpluses s cannot become arbitrarily small, because q^B is decreasing in $\breve{\mu}^{\mathcal{B}}$ and reaches zero at the finite value $\breve{\mu}^{\mathcal{B}} = \delta + g - \rho$ (or $\breve{\mu}^{\mathcal{B}} = \tilde{\sigma}^2 - \rho$ in the idiosyncratic risk model); there is a Laffer curve for bubble mining (see Brunnermeier et al. (2021) for a more detailed discussion).

debt at a larger rate $\mu^{\mathcal{B}}$. When the growth rate is exogenously given, as in the perpetual youth example, then this is the only effect. But in general there could be an additional indirect effect that operates through the growth rate g. This is the case in the idiosyncratic risk model: bubble mining decreases the attractiveness of bonds, making the agents want to hold more physical capital, which stimulates real investment and increases the steady-state growth rate g.[27] This latter effect tends to be deflationary.

When the growth rate is endogenous, an increase in $\breve{\mu}^{\mathcal{B}}$ may therefore in principle lower the $\breve{\mu}^{\mathcal{B}} - g$ term and thus inflation. However, this is unlikely to be the case for any realistic calibration of our idiosyncratic risk model: the effect on growth g is largest without capital adjustment costs ($\phi = 0$) and then

$$\frac{dg}{d\breve{\mu}^{\mathcal{B}}} = \frac{d\iota}{d\breve{\mu}^{\mathcal{B}}} = \frac{1}{2}\frac{\rho}{\rho + \breve{\mu}^{\mathcal{B}}}\frac{1}{1-\vartheta}.$$

For $\breve{\mu}^{\mathcal{B}} \geq 0$, this derivative can only be larger than 1 if $\vartheta > 1/2$, that is if the majority of private wealth is bond wealth. Despite the recent rise in the levels of public debt throughout advanced economies, this condition is unlikely to be satisfied in the foreseeable future. The most plausible situation is therefore the one in which the direct effect dominates the indirect growth effect. Thus, for a fixed nominal interest rate i, an increase in bubble mining is inflationary.

The government can also offset the inflationary effect of bubble mining further by lowering the interest rate i. This is possible whenever there is no binding lower bound on nominal interest rates. If i fully offsets the rise in $\breve{\mu}^{\mathcal{B}}$, so $i + \breve{\mu}^{\mathcal{B}} = \mu^{\mathcal{B}}$ is unaffected, then only the indirect deflationary effect due to higher growth remains. By using the policy tools of debt growth and interest rate in the right proportion, the government can increase bubble mining in an inflation-neutral way.

Note, however, that the previous discussion solely centers on the steady-state inflation rate as a result of a steady-state level of bubble mining $\breve{\mu}^{\mathcal{B}}$. If the government was to announce more aggressive bubble mining going forward, government debt would become less attractive as the debt is diluted at a faster rate and hence the real value of the debt would have to fall. This is brought about in equilibrium by an inflationary upward jump in the price level.[28]

[27] This presumes that revenues from bubble mining are used to lower the output tax rate τ and government consumption $\mathfrak{g}K_t$ is kept constant. If instead bubble mining revenues are used to increase $\mathfrak{g}$, then the effect on the economic growth rate g is ambiguous as a larger $\mathfrak{g}$ tends to decrease both private consumption and private investment.

[28] If we were to add price stickiness to the model, this initial price level jump would translate into a transition period of larger inflation.

6 Optimal Bubble Mining

Even if bubble mining is possible, is it ever socially optimal for the government to engage in mining? In this section, we characterize the optimal policy and draw two key conclusions:

First, a bubble in settings with frictions facilitates trade along certain dimensions – between generations or in response to idiosyncratic shocks – and mining the bubble inhibits these beneficial trades. Optimal policy therefore only calls for a positive rate of bubble mining, $\breve{\mu}^{\mathcal{B}} > 0$, if pecuniary externalities generate an equilibrium bubble that is "too large". Such a situation is never possible in the perpetual youth example, but can arise in the idiosyncratic risk example because the bubble crowds out real investment ι_t. The optimal policy there balances a trade-off between growth and risk sharing and may call for a positive rate of bubble mining, $\breve{\mu}^{\mathcal{B}} > 0$, if idiosyncratic risk is sufficiently large.

Second, the optimal degree of bubble mining is independent of the government spending need $\mathfrak{g}$. This implies that under the optimal policy, any additional government spending is optimally funded by raising taxes, not by bubble mining. This conclusion holds in both models.

We limit our formal analysis in this section to the idiosyncratic risk model, as its welfare implications are richer and less well-known. A brief welfare analysis in the perpetual youth example can be found in Appendix A.4.

Formally, expected utility of an agent with initial wealth share $\eta_0^i := n_0^i / \left((q_0^B + q_0^K) K_0 \right)$ is[29]

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right] = \frac{\log \eta_0^i + \log K_0}{\rho}$$
$$+ \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\underbrace{\log\left(\frac{\rho\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right)}_{=\log(a-\mathfrak{g}-\iota_t)} + \underbrace{\frac{1}{\phi\rho}\log\left(\frac{\left(1-\vartheta_t\right)\left(1+\phi\left(a-\mathfrak{g}\right)\right)}{1-\vartheta_t+\phi\rho}\right) - \frac{\delta}{\rho}}_{=\frac{(\Phi(\iota_t)-\delta)}{\rho}} - \underbrace{\frac{\left(1-\vartheta_t\right)^2\tilde{\sigma}^2}{2\rho}}_{=\frac{(1-\vartheta_t)^2\tilde{\sigma}^2}{2\rho}} \right) dt\right]. \tag{14}$$

For arbitrary Pareto weights, a social planner would like to manipulate ϑ_t period by period to maximize the integrand in the second line.[30] The first term in the integrand is utility from consumption $a - \mathfrak{g} - \iota_t$, which is increasing in ϑ_t because a higher ϑ_t depresses investment and leaves more resources for consumption. The second term is proportional to the endogenous component $\Phi(\iota_t)$ of the growth rate, which is decreasing in ϑ_t. The last term represents the reduction of utility due to idiosyncratic risk. Higher ϑ_t reduces residual consumption risk

[29] We provide a derivation of this equation in Appendix A.3.

[30] This is the case because the second line is the same for all agents i, whereas the first line depends only on initial conditions that cannot be affected by bubble mining policy.

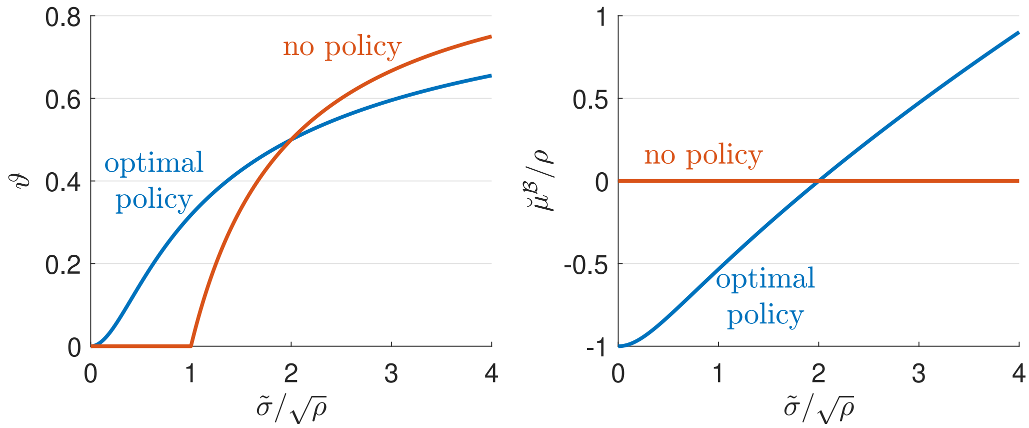

Figure 2: Optimal policy versus no policy ($\breve{\mu}^{\mathcal{B}} = 0$) in the idiosyncratic risk model for $\phi = 0$ as a function of $\tilde{\sigma}/\sqrt{\rho}$. The left panel depicts the nominal wealth share ϑ, the right panel the associated bubble mining policy $\breve{\mu}^{\mathcal{B}}$ normalized by the time-preference rate ρ.

$(1 - \vartheta_t)\,\tilde{\sigma}$ and thereby increases this term.[31]

While ϑ_t is not a policy instrument, the government can effectively choose ϑ_t directly by adjusting $\breve{\mu}_t^{\mathcal{B}}$. We show in the appendix that there is a unique optimal solution ϑ^{opt} for ϑ_t, which is time-invariant, depends only on ρ, $\tilde{\sigma}$, and ϕ, and is strictly increasing in idiosyncratic risk $\tilde{\sigma}$. Figure 2 depicts this optimal nominal wealth share ϑ and the bubble mining rate $\breve{\mu}^{\mathcal{B}}$ required to implement it as function of idiosyncratic risk.[32] It also compares the optimal policy to the competitive equilibrium without policy intervention ($\breve{\mu}^{\mathcal{B}} = 0$). Relative to that benchmark, optimal policy backs the value of government debt by primary surpluses (negative $\breve{\mu}^{\mathcal{B}}$) if risk is low. In these cases, the bubble created by market forces is too small (for $\tilde{\sigma} > \sqrt{\rho}$) or even absent (for $\tilde{\sigma} \leq \sqrt{\rho}$) and risk-sharing is suboptimal. If risk is high, market forces generate a bubble that is too large. Optimal policy then runs deficits (positive $\breve{\mu}^{\mathcal{B}}$) and funds government expenditures out of the bubble to encourage higher real investment and growth.

Market forces may fail to generate a bubble that achieves the optimal trade-off between growth and risk sharing. Inefficiencies are possible due to pecuniary externalities with respect to agents' portfolio choices because agents take returns as given when making these choices, yet their collective choice affects the risk-free rate and risk-premium on capital.[33] On one hand, a greater portfolio allocation to bonds discourages real investment ι in the economy, which in

[31] Representing the objective in this way highlights similarities to the classic analysis of the optimal quantity of debt by Aiyagari and McGrattan (1998). In their framework, a larger value of government debt increases liquidity by effectively relaxing borrowing constraints, but reduces the quantity of capital. Here, a larger debt wealth share directly improves risk sharing (even in the absence of borrowing constraints) but reduces the growth rate of capital and output.

[32] The figure assumes no capital adjustment cost, $\phi = 0$. It looks qualitatively identical for $\phi \in (0, \infty)$.

[33] These pecuniary externalities have been previously identified by Brunnermeier and Sannikov (2016b) and Di Tella (2020) in closely related frameworks.

turn affects the real return on all assets through the growth term in the risk-free rate. This force tends to generate too much bond demand, a too high ϑ and thus under-investment in capital. On the other hand, a greater allocation to bonds increases the total value of bonds and thus reduces the residual (proportional) idiosyncratic consumption risk $(1-\vartheta)\tilde{\sigma}$ that each agent has to bear.[34] This in turn affects asset returns through the precautionary motive in the risk-free rate and through the risk premium on capital. This second force tends to generate too little bond demand, a too low ϑ and thus over-investment in capital.

It is instructive, however, that the optimal value ϑ^{opt} for ϑ is independent of the government spending need $\mathfrak{g}$. Because $\mathfrak{g}$ does not appear in equation (12) either, then also the optimal degree of bubble mining $\breve{\mu}^{\mathcal{B}}$ to implement $\vartheta = \vartheta^{\text{opt}}$ must be independent of $\mathfrak{g}$. While the government could increase $\breve{\mu}^{\mathcal{B}}$ in response to an (unanticipated) increase in $\mathfrak{g}$ in order to fund the additional spending, this is never optimal.[35] The optimal policy should rely on taxes as the marginal funding source for additional government spending.

The reason for this result is that when government spending $\mathfrak{g}$ increases, the government must transfer a larger fraction of current output away from the private sector to itself. Taxing current output is the most direct way of doing so and does not distort the portfolio choice between capital and bonds.[36] In contrast, funding additional spending by increasing primary deficits and bubble mining dilutes the bubble at a faster rate and thereby distorts agents' portfolio choice in favor of larger capital holdings. Because the pecuniary externalities just discussed do not depend on either the level of government spending $\mathfrak{g}K_t$ or total output left for private uses $(a-\mathfrak{g})K_t$, the optimal portfolio distortion induced by $\breve{\mu}^{\mathcal{B}}$ is also independent of these quantities.[37]

In our models, government spending $\mathfrak{g}K_t$ is exogenously given and does not provide any utility to agents. We remark, that all the results in this section would equally apply if agents derived additively separable utility from public spending and also $\mathfrak{g}_t$ was a policy choice.

[34] Specifically, higher bond prices benefit the bond-selling agents: those who suffered idiosyncratic losses and who have higher marginal utilities, on average.

[35] $\mathfrak{g}$ is here a model parameter and thus any change in $\mathfrak{g}$ must by construction be an unanticipated change in government spending. However, this fact is irrelevant for our conclusion. We would obtain the same result if we were to add government spending shocks to the model (including transitory changes that mean-revert in the long run).

[36] Similarly, in the perpetual youth example, it does not distort the inter-generational resource transfer, which is ultimately about how *future* output left for private uses is split across individuals.

[37] The size of the pecuniary externalities does depends on the aggregate consumption-wealth ratio which equals the time preference rate ρ in our model with log utility. Admittedly, this is a somewhat knife-edge case that only holds for unit EIS. For general EIS, the aggregate consumption-wealth ratio depends on the growth rate of the economy, which in turn is increasing in output left for private uses $a-\mathfrak{g}$ per unit of capital. Nevertheless, our result represents an important benchmark case and the broader point that optimal bubble mining only adjusts to correct pecuniary externalities remains valid also for EIS $\neq 1$.

7 Price Level Determination, Uniqueness, and Off-Equilibrium Policy

The key equation (1) of the fiscal theory of the price level without a bubble term can be solved for the price level as a function of the present value of primary surpluses and the outstanding quantity of nominal government debt. For a given real allocation and initial quantity of debt, this equation alone therefore pins down the price level.[38] In the fiscal theory with a bubble, this is no longer true because the size of the bubble is not determined by the present value identity itself. Instead, goods market clearing determines the price level. A larger real value of bonds, holding taxes constant, means bonds represent more net wealth for the private sector, which increases consumption demand through a wealth effect. The equilibrium price level is the price level at which consumption demand equals consumption supply.[39] The fiscal theory equation itself determines the size of the bubble as the residual value of government debt that is not explained by the present value of primary surpluses.

As a consequence, the presence of a bubble makes price level determination based on fiscal dominance more challenging because it eliminates the simple one-to-one relationship between the present value of primary surpluses and the price level. Even making primary surpluses completely exogenous may not be sufficient to determine the price level uniquely if government policy fails to pin down the value of the bubble. The same path of surpluses can be consistent with multiple paths for the bubble value and thus with multiple initial price levels.

The steady state equilibria derived above are consistent with a policy that fixes $\breve{\mu}^{\mathcal{B}}$ at a constant level and adjusts taxes τ such that the government budget constraint (3) holds after any price history.[40] While simple, such a policy is clearly inadequate to determine the price level. This is evident from the existence of a nonmonetary equilibrium in which nominal government bonds are worthless ($q^{\mathcal{B}} = 0$). However, government policy can easily be modified off-equilibrium to select the monetary steady state as the unique equilibrium.[41]

[38] This conclusion does not rely on the assumption of fiscal dominance. Assuming fiscal dominance just ensures that surpluses do not react too strongly to the price level to make the "given real allocation" the only possible equilibrium allocation.

[39] The same mechanism is present in the fiscal theory without a bubble, but it may not be as clearly visible because one can mechanically solve the model by reading off the price level from the fiscal theory equation.

[40] We have opted not to choose the opposite specification where τ is constant and $\breve{\mu}^{\mathcal{B}}$ adjusts to make the government budget constraint hold because this is only a valid policy specification if $\tau \geq \mathfrak{g}/a$. For $\tau < \mathfrak{g}/a$, there are histories of prices in which no value of $\breve{\mu}^{\mathcal{B}}$ is consistent with equation (3) (e.g. $\mathcal{P} = \infty$, i.e., the moneyless equilibrium).

[41] Here, we focus on equilibria that are deterministic and feature absolutely continuous price paths. With additional technical arguments, one can also rule out non-time-continuous equilibria and equilibria driven by sunspot noise.

To see this, recall that along any equilibrium path, ϑ_t must satisfy equation (9) in example model I and equation (12) in example model II. By taking the time derivative, we can write both equations as an ordinary differential equation (ODE) of the form

$$\dot{\vartheta}_t = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - f\left(1 - \vartheta_t\right)\right)\vartheta_t \tag{15}$$

where f is a strictly increasing function (for positive arguments) given by

$$f(x) = \begin{cases} x\left(\delta + g\right), & \text{perpetual youth model} \\ x^2\tilde{\sigma}^2, & \text{idiosyncratic risk model} \end{cases}.$$

In other words, ODE (15) must hold along any equilibrium path. The converse is also true, provided ϑ is contained in $[0, 1]$:

Lemma 1. *An absolutely continuous function $[0, \infty) \to \mathbb{R}, t \mapsto \vartheta_t$ corresponds to a model equilibrium, if and only if it satisfies equation (15) and $0 \leq \vartheta_t \leq 1$ for all t.*

We have already noted that equation (15) is necessary for an equilibrium. The proof of sufficiency requires a number of technical verification arguments that we relegate to Appendix A.5.

With constant $\breve{\mu}^{\mathcal{B}}$, there is a continuum of solution paths for ϑ consistent with the requirement in Lemma 1, which can be indexed by the initial value $\vartheta_0 \in [0, \vartheta^*]$, where ϑ^* denotes the monetary steady-state level of ϑ. For any initial value but the right endpoint, ϑ asymptotically converges to 0.[42] Conversely, if for some reason agents expected that the equilibrium value of ϑ could never fall below a positive threshold $\underline{\vartheta} > 0$, then all equilibria but the monetary steady state $\vartheta_0 = \vartheta^*$ could be ruled out.

These considerations suggest a simple off-equilibrium modification of the fiscal policy rule to achieve equilibrium uniqueness: fix an arbitrary threshold $0 < \underline{\vartheta} \leq \vartheta^*$ and, whenever ϑ falls below $\underline{\vartheta}$, switch from a constant debt growth rule (constant $\breve{\mu}^{\mathcal{B}}$) to a positive surplus rule with a constant output tax rate $\bar{\tau} > \mathfrak{g}/a$ for as long as $\vartheta < \underline{\vartheta}$.[43] For this to work even when there are positive surpluses in steady state (the case $\breve{\mu}^{\mathcal{B}} < 0$), we assume in addition $\bar{\tau} \geq \tau$, where τ is the equilibrium tax rate in the steady state.[44]

From the government budget constraint (3), it follows that under this modified fiscal policy

[42] This is implied by the fact that the right-hand side of equation (15) is negative for all $\vartheta_t \in (0, \vartheta^*)$.

[43] The proposed modification is reminiscent of Obstfeld and Rogoff (1983), who show how an off-equilibrium commodity backing of money can rule out hyperinflationary equilibria in models of money as a medium of exchange.

[44] If $\breve{\mu}^{\mathcal{B}} \geq 0$ and thus $\tau \leq \mathfrak{g}/a$ (nonpositive surpluses), this condition is clearly redundant. Otherwise, it is required to avoid equilibria in which the expectation of lower, yet still positive, primary surpluses becomes self-fulfilling.

ODE (15) becomes

$$\dot{\vartheta}_t = \begin{cases} \left(\rho - f\left(1-\vartheta_t\right) + \breve{\mu}^{\mathcal{B}}\right)\vartheta_t, & \vartheta_t \geq \underline{\vartheta} \\ \left(\rho - f\left(1-\vartheta_t\right)\right)\vartheta_t - \left(\bar{\tau}a - \mathfrak{g}\right) h\left(1-\vartheta_t\right), & \vartheta_t < \underline{\vartheta} \end{cases},$$

where h is a (weakly) increasing function given by

$$h(x) = \begin{cases} \frac{\rho}{a-\mathfrak{g}}, & \text{perpetual youth model} \\ \frac{x+\phi\rho}{1+\phi(a-\mathfrak{g})}, & \text{idiosyncratic risk model} \end{cases}.$$

In particular, the values of h (for positive arguments) are always positive and bounded away from zero.

It is then easy to see that this modified ODE has a strictly negative left-hand side on the interval $[0, \vartheta^*)$ including the left endpoint, and therefore all solutions that start inside this interval turn negative in finite time. By Lemma 1, the only possible equilibrium path for ϑ is therefore the steady-state equilibrium $\vartheta = \vartheta^*$.

The above rule modifies fiscal policy only off-equilibrium, whereas along the equilibrium path of the remaining unique equilibrium, the government is free to choose any debt growth rate net of interest payments $\breve{\mu}^{\mathcal{B}}$. This raises questions about the credibility and fiscal capacity to promise off-equilibrium surpluses. These issues are beyond the scope of the present paper.

Bubbles on Other Assets. Our analysis so far has presumed that a bubble is either on government debt or there is no bubble at all. However, a bubble could also be on other assets than government debt. Specifically, any intrinsically worthless asset in limited supply could have a bubble component.[45] Furthermore, the aggregate bubble could be split between the government bond and these other bubbly assets. Can a similar policy as just discussed also rule out these other equilibria?

The answer is yes if the taxation threshold $\underline{\vartheta}$ is tight, i.e. $\underline{\vartheta} = \vartheta^*$. To see this, we extend our models by adding an intrinsically worthless additional asset that can potentially also have a bubble component.[46] For simplicity, we assume that this asset is in constant supply. Denote by $\hat{q}_t^B K_t$ the value of the portfolio consisting of all government bonds and and the other potential bubble asset and define $\hat{\vartheta}_t := \hat{q}_t^B / \left(\hat{q}_t^B + q_t^K\right)$. With this notation, clearly $q_t^B \leq \hat{q}_t^B$ and $\vartheta_t \leq \hat{\vartheta}_t$.[47]

[45] We limit the discussion to bubbles on intrinsically worthless assets in fixed supply, but our economic points also apply to bubbles on other assets.

[46] It is without loss of generality to consider only a single additional asset as we could always combine the portfolio of all potential bubble assets in the economy into a single asset.

[47] As before, $\vartheta_t = q_t^B / \left(\hat{q}_t^B + q_t^K\right)$ denotes the fraction of total net wealth that is due to government debt.

We can solve these augmented models as before[48] if we replace the return on bonds $dr_t^{\mathcal{B}}$ with the return on the portfolio of all government bonds and the other bubble asset, which is given by

$$d\hat{r}_t^{\mathcal{B}} = \frac{q_t^B}{\hat{q}_t^B} dr_t^{\mathcal{B}} + \frac{\hat{q}_t^B - q_t^B}{\hat{q}_t^B} \frac{d\left(\left(\hat{q}_t^B - q_t^B\right) K_t\right)}{\left(\hat{q}_t^B - q_t^B\right) K_t} = \left(g_t - \frac{q_t^B}{\hat{q}_t^B} \breve{\mu}_t^{\mathcal{B}} + \mu_t^{\hat{q},B}\right) dt.$$

Following the same solution procedure as in Section 3, we obtain a solution ODE for $\hat{\vartheta}_t$ in analogy to equation (15):

$$\dot{\hat{\vartheta}}_t = \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right) \hat{\vartheta}_t + \breve{\mu}_t^{\mathcal{B}} \vartheta_t. \tag{16}$$

In addition, whenever both government bonds and the other bubble asset have positive value, then a no arbitrage condition between the two must hold, which implies a second ODE

$$\dot{\vartheta}_t = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right) \vartheta_t. \tag{17}$$

In both equations, the f function is the same as in equation (15) above.

In analogy to Lemma 1, time paths $t \mapsto (\vartheta_t, \hat{\vartheta}_t)$ correspond to a valid equilibrium if and only if they solve ODEs (16) and (17) and satisfy $0 \leq \vartheta_t \leq \hat{\vartheta}_t \leq 1$ for all t. Assuming the same threshold policy specification as before, the structure of equation (17) is sufficiently similar to that of equation (15) that we can rule out any solution paths for ϑ that ever fall below the threshold $\underline{\vartheta}$ by the same argument as previously: if ϑ ever fell below $\underline{\vartheta}$, then this low valuation of bonds would only be consistent with the positive surplus policy under $\vartheta < \underline{\vartheta}$ if agents expected ϑ to fall even further at a speed that would imply $\vartheta_T = 0$ at some finite time T. But because positive surpluses continue beyond time T, $\dot{\vartheta}_T < 0$ also at time T, so that the mathematical solution ϑ to ODE (17) must eventually turn negative and thus cannot correspond to an equilibrium solution of the model.

This argument shows that all equilibrium solution paths must satisfy $\vartheta_t \geq \underline{\vartheta}$ at all times. But if $\underline{\vartheta} < \vartheta^*$ and $\breve{\mu}^{\mathcal{B}} \geq 0$, one can show that there are multiple solution pairs $(\vartheta, \hat{\vartheta})$ to equations (16) and (17) that satisfy $0 \leq \vartheta_t \leq \hat{\vartheta}_t \leq 1$. Consequently, in the presence of other potential bubble assets, the above policy only selects the stationary equilibrium $\vartheta_t = \hat{\vartheta}_t = \vartheta^*$ if in addition $\underline{\vartheta} = \vartheta^*$.[49] In total, we have the following proposition. Here, we include an additional limit result whose proof requires a refinement of the previous arguments.

Proposition 1. *Under the threshold policy discussed in this section, all equilibria have the property that*

[48] We only sketch the solution procedure here and provide more details in Appendix A.6.

[49] It is easy to show that if $\vartheta_t = \vartheta^*$ then there is no "space" for an additional bubble on a different asset and thus $\hat{\vartheta}_t = \vartheta_t$ holds automatically. See Appendix A.7 for further details.

$\vartheta_t \geq \underline{\vartheta}$ for all t. In the limit as $t \to \infty$, $\hat{\vartheta}_t \to \vartheta^*$ and, if $\breve{\mu}^{\mathcal{B}} \neq 0$, also $\vartheta_t \to \vartheta^*$.[50] *If no bubbles on other assets are permitted or if $\underline{\vartheta} = \vartheta^*$, the equilibrium is unique and satisfies $\vartheta_t = \hat{\vartheta}_t = \vartheta^*$ for all t.*

The core content of the previous proposition follows directly from the arguments in this section. We provide additional details on the proof in Appendix A.7.

We conclude this section by pointing out how our results differ from the analysis of Bassetto and Cui (2018) in the context of a dynamically inefficient OLG model. They study constant tax policies that are not contingent on the price level and conclude that "the FTPL breaks down in [their] OLG economy" (p. 13). While seemingly the opposite of our conclusion, their result is fully consistent with the discussion in this paper, which highlights that contingent policy is required to obtain uniqueness of the price level. In order to back the bubble in equilibrium, the government must commit to off-equilibrium taxation that replaces the value of the bubble by a present value of primary surpluses. Constant tax policies are insufficient for this purpose.[51]

8 Conclusion

This paper integrates the typically ignored bubble term in the FTPL equation, which is necessary to explain low inflation in countries with persistently negative primary surpluses. Brunnermeier et al. (2021) expand on the idiosyncratic risk example provided in this paper to study the role of government debt as a safe asset in a generalized setting with time-varying idiosyncratic risk. During recessions idiosyncratic risk expands and with it the value of the service flow derived from re-trading the safe asset. Consequently, the safe asset appreciates during recessions, i.e. it has a negative CAPM-β. Also, the analysis in this paper can be easily extended to include a transaction role of money. In this case the interest rate on "narrow money" is further depressed by $\Delta i = i - i^M$, making it more likely to be below the growth rate of economy. This would also constitute another source of seigniorage besides the "bubble mining" emphasized in this paper.

[50] The condition $\breve{\mu}^{\mathcal{B}} \neq 0$ is required because $\breve{\mu}^{\mathcal{B}} = 0$ implies the special case with a zero surplus policy. Then, there are equilibria in which the aggregate bubble is split between government bonds and the other asset according to some fixed proportion. If $\underline{\vartheta} < \vartheta^*$, there is "space" for such an equilibrium in which the proportion of the bubble captured by the government bond is less than 1 and thus ϑ is permanently smaller than ϑ^*.

[51] In addition, even positive primary surpluses are associated with equilibrium multiplicity in the OLG economy of Bassetto and Cui (2018), whereas in our model the constant tax policy with $\bar{\tau} > \mathfrak{g}/a$ selects a unique equilibrium. This is due to a difference in assumptions: while we assume here that the government does not lend to the private sector, they allow for such lending, thereby effectively validating additional equilibrium paths that lead to a steady state with a negative value of debt. Because a government can simply refuse to lend to the private sector, this type of multiplicity is easily avoided by modifying government policy.

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A Appendix

A.1 Derivation of the Continuous-time Fiscal Theory Equation (Equation (1))

As in the discrete-time case, the derivation of the fiscal theory equation starts with the government flow budget constraint, which here is

$$\left(\mu_t^{\mathcal{B}}\mathcal{B}_t + \mu_t^{\mathcal{M}}\mathcal{M}_t + \mathcal{P}_t T_t\right) dt = \left(i_t\mathcal{B}_t + i_t^m\mathcal{M}_t + \mathcal{P}_t G_t\right) dt,$$

where $\mathcal{B}_t$, $\mathcal{M}_t$, T_t, G_t, i_t and i_t^m have the same meaning as in the main text and $\mu_t^{\mathcal{B}}$, $\mu_t^{\mathcal{M}}$ are the growth rates of nominal bonds and money, respectively.[52] Multiplying the budget constraint by the nominal SDF $\xi_t/\mathcal{P}_t$ and rearranging yields

$$\left(\left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t + \left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\right) dt = -\xi_t\left((T_t - G_t) + \Delta i_t\frac{\mathcal{M}_t}{\mathcal{P}_t}\right) dt. \tag{18}$$

Next, Ito's product rule implies

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\right) = \left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t dt + \frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right),$$
$$d\left(\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\right) = \left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t dt + \frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right).$$

Solving these last two equations for $\left(\mu_t^{\mathcal{B}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{B}_t dt$ and $\left(\mu_t^{\mathcal{M}} - i_t\right)\frac{\xi_t}{\mathcal{P}_t}\mathcal{M}_t dt$, respectively, and substituting the results into equation (18) yields (after rearranging)

$$d\left(\frac{\xi_t}{\mathcal{P}_t}\left(\mathcal{B}_t + \mathcal{M}_t\right)\right) = -\xi_t\left(\mathcal{P}_t\left(T_t - G_t\right) + \Delta i_t\mathcal{M}_t\right) dt + \xi_t\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t}\left(\frac{d\left(\xi_t/\mathcal{P}_t\right)}{\xi_t/\mathcal{P}_t} + i_t dt\right),$$

or in integral form

$$\xi_T\frac{\mathcal{B}_T + \mathcal{M}_T}{\mathcal{P}_T} - \xi_t\frac{\mathcal{B}_t + \mathcal{M}_t}{\mathcal{P}_t} = -\int_t^T \xi_s\left(T_s - G_s\right) ds - \int_t^T \xi_s\Delta i_s\frac{\mathcal{M}_s}{\mathcal{P}_s} ds + \int_t^T \xi_s\frac{\mathcal{B}_s + \mathcal{M}_s}{\mathcal{P}_s}\left(\frac{d\left(\xi_s/\mathcal{P}_s\right)}{\xi_s/\mathcal{P}_s} + i_s dt\right).$$

Up to this point, we have merely rearranged and integrated the government budget constraint. To derive the fiscal theory equation, the literature proceeds by using two equilibrium conditions. First, if the nominal SDF $\xi/\mathcal{P}$ prices the government bonds, then its expected rate of change must be the negative of the nominal interest rate. Then, the last stochastic integral on

[52] Here we abstract from long-term bonds and the possibility of taxes, spending, and adjustments in $\mathcal{B}$ and $\mathcal{M}$ that are not absolutely continuous over time (e.g., lumpy adjustments in response to a Poisson shock). Such elements could be easily added, but require more complicated notation without generating additional insights for our purposes.

the right must be a martingale and disappears when taking conditional time-t expectations $E_t[\cdot]$. Second, a private-sector transversality condition is invoked to eliminate a terminal value of government debt when passing to the limit $T \to \infty$. We perform the first operation, but do not want to restrict attention to environments where transversality can rule out a nonzero discounted terminal value. When taking the limit $T \to \infty$, we therefore arrive at the more general equation (1).

A.2 Omitted Details in Section 3

In this section, we present some additional formal details about the two example models and their solution.

A.2.1 Example I: Perpetual Youth

Details on the Household Problem. The HJB equation for household i's problem is

$$\rho V_t\left(b_t^i, k_t^i\right) - \partial_t V_t\left(b_t^i, k_t^i\right) = \max_{c^i}\left\{\log c^i + \partial_b V_t\left(b_t^i, k_t^i\right)\left(r_t^f b_t^i + (1-\tau_t)\, a k_t^i - c^i\right) + \partial_k V_t\left(b_t^i, k_t^i\right)\left(-\delta k_t^i\right)\right\}.$$

This is a standard consumption-savings problem, so we conjecture a functional form $V_t\left(b^i, k^i\right) = \alpha_t + \frac{1}{\rho}\log\left(b^i + q_t^K k^i\right)$ for the value function, where α_t and q_t^K depend on (aggregate) investment opportunities, but not on individual state variables b^i and k^i. We verify this conjecture below.

Substituting the functional form guess into the HJB equation yields

$$\rho\alpha_t + \log\left(b^i + q_t^K k^i\right) - \dot{\alpha}_t - \frac{1}{\rho}\frac{\dot{q}_t^K k^i}{b^i + q_t^K k^i} = \max_{c^i}\left\{\log c^i - \frac{1}{\rho}\frac{1}{b^i + q_t^K k^i}c^i\right\} + \frac{1}{\rho}\frac{1}{b^i + q_t^K k^i}\left(r_t^f b^i + (1-\tau_t)\, a k^i - \delta q_t^K k^i\right). \quad (19)$$

The first-order condition for the maximization with respect to c^i is

$$0 = \frac{1}{c^i} - \frac{1}{\rho\left(b^i + q_t^K k^i\right)},$$

which immediately implies the consumption rule (6) stated in the main text.

Substituting the optimal consumption choice into equation (19) and canceling and rearranging terms implies

$$\left(\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho - \frac{r_t^f}{\rho}\right) b^i + \left(\left(\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho\right) q_t^K - \frac{(1-\tau_t)\, a - \delta q_t^K + \dot{q}_t^K}{\rho}\right) k^i = 0. \tag{20}$$

Clearly, the functional form guess can only be correct if equation (20) is satisfied for all states (b^i, k^i). But that means in particular,that α_t and q_t^K must satisfy the two equations

$$\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho = \frac{r_t^f}{\rho}, \tag{21}$$

$$\rho\alpha_t - \dot{\alpha}_t + 1 - \log\rho = \frac{(1-\tau_t)\, a - \delta q_t^K + \dot{q}_t^K}{\rho q_t^K}, \tag{22}$$

that are obtained by setting either b^i or k^i to zero in equation (20). Conversely, if there are functions α_t and q_t^K that satisfy equations (21) and (22), then clearly also (20) holds for all (b^i, k^i) (and all times t) and because (20) is equivalent to the original HJB equation (19), also that equation must then hold for all (b^i, k^i) (and all times t). To verify the conjectured functional form for V, it is thus sufficient to show that α_t, q_t^K satisfying (21) and (22) do indeed exist.

Equation (21) is a linear ordinary differential equation (ODE) in α_t. The general solution for this ODE is

$$\alpha_t = \beta e^{\rho t} + \int_t^\infty e^{-\rho(s-t)} \left(1 - \log\rho - \frac{r_s^f}{\rho}\right) ds,$$

where $\beta \in \mathbb{R}$ is an integration constant. This solution is well-defined as long as the given risk-free rate path r_t^f is sufficiently regular (e.g., continuous and bounded). Thus, a solution to (21) does indeed exist and with the specific choice $\beta = 0$, the resulting HJB solution function V_t also satisfies a transversality condition (which is necessary for optimality).

Next, combine equations (21) and (22) to substitute out the α_t-dependent terms, which yields the ODE

$$\frac{r_t^f}{\rho} = \frac{(1-\tau_t)\, a - \delta q_t^K + \dot{q}_t^K}{\rho q_t^K}$$

for q_t^K. Rearranging implies the equation

$$\dot{q}_t^K = -(1-\tau_t)\, a + \left(r_t^f + \delta\right) q_t^K, \tag{23}$$

which is a differential version of equation (7) in the main text. To derive the latter equation, use $k_s^i = k_t^i e^{-\delta(s-t)}$ and thus

$$e^{-\int_t^s r_{s'}^f ds'} q_s^K k_s^i = e^{-\int_t^s \left(r_{s'}^f + \delta\right) ds'} q_s^K k_t^i.$$

Taking the time-derivative of the product on the right yields

$$d\left(e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_s\right)=-\left(r^f_s+\delta\right)e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_s ds+e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}\dot{q}^K_s ds,$$

so that

$$\frac{d\left(e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_s k^i_t\right)}{ds}=e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}\underbrace{\left(\dot{q}^K_s-\left(r^f_s+\delta\right)q^K_s\right)}_{=-(1-\tau_s)a\text{ by (23)}}k^i_t.$$

Integrating both sides over $s\in[t,T]$ yields

$$\begin{aligned}e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}q^K_T k^i_t-q^K_t k^i_t&=-\int_t^T e^{-\int_t^s\left(r^f_{s'}+\delta\right)ds'}(1-\tau_s)ak^i_t ds\\&=-\int_t^T e^{-\int_t^s r^f_{s'}ds'}(1-\tau_s)ak^i_s ds\end{aligned}$$

Rearranging and taking the limit $T\to\infty$ implies equation (7).

Derivation of Equation (9). We use the notation

$$\mu^{q,B}_t:=\frac{\dot{q}^B_t}{q^B_t},\qquad \mu^{q,K}_t:=\frac{\dot{q}^K_t}{q^K_t},\qquad \mu^{\vartheta}_t:=\frac{\dot{\vartheta}_t}{\vartheta_t}.$$

These definitions are unproblematic as long as the denominator expressions are different from zero, that is as long as both bonds and human capital have positive value in equilibrium. By definition of ϑ_t, we have $\mu^{\vartheta}_t=(1-\vartheta_t)\left(\mu^{q,B}_t-\mu^{q,K}_t\right)$.

We first divide equation (23) by q^K_t and then plug in the risk-free rate expression from equation (8) in the main text. The resulting equation is

$$\mu^{q,K}_t=-(1-\tau_t)\frac{a}{q^K_t}-\breve{\mu}^{\mathcal{B}}_t+g+\mu^{q,B}_t+\delta.$$

Next, the government budget constraint (3) implies $\tau_t a=\mathfrak{g}-\breve{\mu}^{\mathcal{B}}_t q^B_t$ and we have

$$q^B_t=\vartheta_t\frac{a-\mathfrak{g}}{\rho},\qquad q^K_t=(1-\vartheta_t)\frac{a-\mathfrak{g}}{\rho}.$$

Substituting these expressions into the previous equation and rearranging yields

$$\mu^{q,K}_t-\mu^{q,B}_t=-\frac{\rho}{1-\vartheta_t}-\frac{\breve{\mu}^{\mathcal{B}}_t}{1-\vartheta_t}+g+\delta$$

and thus

$$\mu^{\vartheta}_t=\rho+\breve{\mu}^{\mathcal{B}}_t-(1-\vartheta_t)\left(g+\delta\right).$$

This is a backward equation for ϑ_t that has been derived under the assumption that bonds have a positive value ($q_t^{\mathcal{B}} > 0 \Leftrightarrow \vartheta_t > 0$). In particular, in these cases multiplying the equation by ϑ_t represents an equivalence transformation. Furthermore, if $\vartheta_t = 0$, then no arbitrage requires also $\dot{\vartheta}_t = 0$; otherwise, agents could earn an infinite risk-free return from investing into bonds. Consequently, the ODE

$$\dot{\vartheta}_t = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - (1 - \vartheta_t)(\delta + g)\right)\vartheta_t \tag{24}$$

must hold along any equilibrium path, regardless of whether bonds have positive value or not.

Equation (24) is the differential version of equation (9) stated in the main text. The latter equation can be derived from (24) by appropriate time integration along the same lines as the FTPL equation from the flow budget constraint (see Appendix A.1) and equation (7) from equation (23) (see household problem above). For this reason, we omit the proof here.

Steady-State Equilibria. In steady state, equation (24) simplifies to

$$\left(\rho + \breve{\mu}^{\mathcal{B}} - (1 - \vartheta)(\delta + g)\right)\vartheta = 0.$$

This is a second-order polynomial and has precisely two solutions, $\vartheta = 0$ and $\vartheta = 1 - \frac{\rho + \breve{\mu}^{\mathcal{B}}}{\delta + g}$. Only the latter solution can be associated with a positive value of government bonds, $q_t^{\mathcal{B}} > 0$, and it is if and only if

$$1 > \frac{\rho + \breve{\mu}^{\mathcal{B}}}{\delta + g}.$$

A.2.2 Example II: Uninsurable Idiosyncratic Risk

Details on the Household Problem. Because agents can freely adjust portfolios at each time instant in this model, the household problem can be written in terms of a single state variable, net worth $n_t^i = q_t^K k_t^i + b_t^i$. As in the main text, let $\theta_t^i := b_t^i / n_t^i$ denote the fraction of net worth invested into bonds. Then net worth evolves according to

$$\frac{dn_t^i}{n_t^i} = -\frac{c_t^i}{n_t^i}dt + dr_t^{\mathcal{B}} + \left(1 - \theta_t^i\right)\left(dr_t^{K,i}\left(\iota_t^i\right) - dr_t^{\mathcal{B}}\right), \tag{25}$$

where returns $dr_t^{\mathcal{B}}$ and $dr_t^{K,i}\left(\iota_t^i\right)$ are as stated in the main text.

The household chooses consumption c_t^i, real investment ι_t^i, and the portfolio share θ_t^i to maximize utility V_0^i subject to (25). The HJB equation for this problem is[53]

$$\rho V_t\left(n^i\right) - \partial_t V_t\left(n^i\right) = \max_{c^i,\theta^i,\iota^i}\left\{\log c^i + V_t'\left(n^i\right)\left[-c^i + n^i\left(\frac{dr_t^{\mathcal{B}}}{dt} + \left(1-\theta^i\right)\overbrace{\left(\frac{a-\mathfrak{g}-\iota^i}{q_t^K} + \Phi\left(\iota^i\right) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \check{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right)}^{=\frac{\mathbb{E}_t\left[dr_t^{K,i}\left(\iota_t^i\right)\right]}{dt} - \frac{dr_t^{\mathcal{B}}}{dt}}\right)\right]\right.$$
$$\left. + \frac{1}{2}V_t''\left(n^i\right)\left(n^i\right)^2\left(1-\theta^i\right)^2\tilde{\sigma}^2\right\}.$$

This is a standard consumption-portfolio-choice problem, so we conjecture again a functional form $V_t\left(n^i\right) = \alpha_t + \frac{1}{\rho}\log n_t^i$ for the value function, where as for the previous model, α_t depends on (aggregate) investment opportunities, but not on individual net worth n^i.

Substituting this guess into the HJB equation yields

$$\begin{aligned}\rho\alpha_t + \log\left(n^i\right) - \dot{\alpha}_t = \max_{c^i}&\left(\log c^i - \frac{c^i}{\rho n^i}\right)\\ &+ \frac{1}{\rho}\max_{\theta^i,\iota^i}\left(\left(1-\theta^i\right)\left(\frac{a-\mathfrak{g}-\iota^i}{q_t^K} + \Phi\left(\iota^i\right) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \check{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right) - \frac{\left(1-\theta^i\right)^2\tilde{\sigma}^2}{2}\right)\\ &+ \frac{1}{\rho}\frac{dr_t^{\mathcal{B}}}{dt}.\end{aligned} \tag{26}$$

The first-order conditions for the maximization with respect to c^i, ι^i and $1-\theta^i$ are

$$\begin{aligned}0 &= \frac{1}{c^i} - \frac{1}{\rho n^i},\\ 0 &= \frac{1-\theta^i}{\rho}\left(\Phi'(\iota^i) - \frac{1}{q_t^K}\right)\\ 0 &= \left(\frac{a-\mathfrak{g}-\iota^i}{q_t^K} + \Phi\left(\iota^i\right) - \Phi(\iota_t) - \frac{\mu_t^{\vartheta} - \check{\mu}_t^{\mathcal{B}}}{1-\vartheta_t}\right) - \left(1-\theta^i\right)\tilde{\sigma}^2\end{aligned}$$

These three equations immediately imply the three conditions stated in the main text.

Expressing q^B, q^K, ι in Terms of ϑ. Combining the aggregate resource constraint (10) with the optimal consumption rule (aggregated over all agents) relates total wealth to total consumption in each period,

$$q_t^B + q_t^K = \frac{1}{\rho}C_t/K_t = \frac{a-\mathfrak{g}-\iota_t}{\rho}.$$

[53]Here, we have used the government budget constraint (3) to eliminate $\tau_t a$ in the return on capital.

Divide both equations by q_t^K, use $1-\vartheta_t = \frac{q_t^K}{q_t^B+q_t^k}$ on the left-hand side and $q_t^K = \frac{1}{\Phi'(\iota_t)} = 1+\phi\iota_t$ on the right hand side to obtain an equation that relates ϑ_t to the investment rate ι_t:

$$\frac{1}{1-\vartheta_t} = \frac{a-\mathfrak{g}-\iota_t}{1+\phi\iota_t}.$$

Solving for ι_t yields

$$\iota_t = \frac{(1-\vartheta_t)(a-\mathfrak{g})-\rho}{1-\vartheta_t+\phi\rho}$$

and substituting that equation into $q_t^K = 1+\phi\iota_t$ and $q_t^B = \frac{\vartheta_t}{1-\vartheta_t}q_t^K$ implies

$$q_t^B = \vartheta_t \frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho},$$
$$q_t^K = (1-\vartheta_t)\frac{1+\phi(a-\mathfrak{g})}{1-\vartheta_t+\phi\rho}.$$

Derivation of Equation (12). Bond market clearing and the fact that all households choose the same θ_t^i imply $\theta_t^i = \vartheta_t$ and substituting this and into the first-order condition for θ^i yields

$$\frac{a-\mathfrak{g}-\iota_t}{q_t^K} - \frac{\mu_t^\vartheta - \breve{\mu}_t^{\mathcal{B}}}{1-\vartheta_t} = (1-\vartheta_t)\tilde{\sigma}^2.$$

Now use $\frac{a-\mathfrak{g}-\iota_t}{q_t^K} = \frac{\rho}{1-\vartheta_t}$, multiply by $1-\vartheta_t$ and solve for μ_t^ϑ:

$$\mu_t^\vartheta = \rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2\tilde{\sigma}^2.$$

As in the perpetual youth model, this is a backward equation for ϑ_t that has been derived under the assumption that bonds have a positive value ($\vartheta_t > 0$). By the same arguments as there, the equivalent ODE

$$\dot{\vartheta}_t = \left(\rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t \tag{27}$$

remains valid on all equilibrium paths, even if $\vartheta_t = 0$. As before, this is the differential version of the integral equation (12) stated in the main text and the latter can be easily derived from the former by time integration.

Steady-State Equilibria. In steady state, equation (27) simplifies to

$$\left(\rho + \breve{\mu}_t^{\mathcal{B}} - (1-\vartheta_t)^2\tilde{\sigma}^2\right)\vartheta_t = 0.$$

This is a third-order polynomial and has precisely three solutions, $\vartheta = 0$ and $\vartheta = 1 \pm \frac{\sqrt{\rho+\breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$. The solution with $\vartheta = 0$ is associated with worthless government bonds, $q^B = 0$. The solution with "+" has always the property $\vartheta > 1$ and would therefore imply either a negative capital price (if $q^B > 0$) or a negative value of government bonds. Both cases violate free disposal and therefore this solution cannot be a valid equilibrium.

The remaining solution $\vartheta = 1 - \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}$ can be a valid equilibrium solution if it is nonnegative. If it is even positive, then the associated equilibrium features a positive value of bonds. This is the case if and only if

$$\tilde{\sigma} > \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}.$$

A.3 Omitted Details in Section 6

Derivation of Equation (14). Because all agents consume the same constant fraction ρ of their wealth, the consumption share c_t^i / C_t of agent i at time t must equal the agent's wealth share η_t^i. We can therefore write, using the aggregate resource constraint 10,

$$c_t^i = \eta_t^i C_t = \eta_t^i \left(a - \mathfrak{g} - \iota_t\right) K_t.$$

Thus, expected utility of agent i is given by

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log c_t^i dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\log \eta_t^i + \log\left(a - \mathfrak{g} - \iota_t\right) + \log K_t\right) dt\right]. \tag{28}$$

To compute the integrals in equation (28), note that if

$$\frac{dx_t}{x_t} = \mu_t^x dt + \sigma_t^x d\tilde{Z}_t,$$

then

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log x_t dt\right] = \frac{\log x_0}{\rho} + \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{\mu_t^x - \left(\sigma_t^x\right)^2/2}{\rho} dt\right]. \tag{29}$$

This follows from a simple calculation:

$$\begin{aligned}
\int_0^\infty e^{-\rho t} \left(\log x_t - \log x_0\right) dt &= \int_0^\infty e^{-\rho t} \int_0^t d\log x_s dt \\
&= \int_0^\infty e^{-\rho t} \left(\int_0^t \mu_s^x ds + \int_0^t \sigma_s^x d\tilde{Z}_s - \frac{1}{2}\int_0^t \left(\sigma_s^x\right)^2 ds\right) dt \\
&= \int_0^\infty \int_s^\infty e^{-\rho t} dt \left(\mu_s^x - \frac{1}{2}\left(\sigma_s^x\right)^2\right) ds + \int_0^\infty e^{-\rho t} \int_0^t \sigma_s^x d\tilde{Z}_s dt \\
&= \int_0^\infty e^{-\rho s} \frac{\mu_s^x - \left(\sigma_s^x\right)^2/2}{\rho} ds + \int_0^\infty e^{-\rho t} \int_0^t \sigma_s^x d\tilde{Z}_s dt.
\end{aligned}$$

When taking expectations, the second term disappears because it is a martingale. Thus, we obtain formula (29).

To apply formula (29), we need to determine $\frac{dK_t}{K_t}$ and $\frac{d\eta_t^i}{\eta_t^i}$. We know that

$$\frac{dK_t}{K_t} = \big(\Phi(\iota_t) - \delta\big)\, dt. \tag{30}$$

For η_t^i, we have

$$\begin{aligned}
\frac{d\eta_t^i}{\eta_t^i} &= \frac{dn_t^i}{n_t^i} - \frac{d\bar{q}_t}{\bar{q}_t} - \frac{dK_t}{K_t} \\
&= \left(-\rho dt + dr_t^{\mathcal{B}} + (1-\vartheta_t)\left(dr_t^{K,i}(\iota_t) - dr_t^{\mathcal{B}}\right)\right) - \mu_t^{\bar{q}} dt - \big(\Phi(\iota_t) - \delta\big)\, dt \\
&= \left(-\rho - \breve{\mu}_t^{\mathcal{B}} + \mu_t^{\vartheta}\right) dt + (1-\vartheta_t)\left(\frac{a - \mathfrak{g} - \iota_t}{q_t^K} + \frac{\breve{\mu}_t^{\mathcal{B}} - \mu_t^{\vartheta}}{1-\vartheta_t}\right) dt + (1-\vartheta_t)\,\tilde{\sigma} d\tilde{Z}_t^i \\
&= \left(-\rho + (1-\vartheta_t)\frac{\rho}{1-\vartheta_t}\right) dt + (1-\vartheta_t)\,\tilde{\sigma} d\tilde{Z}_t^i \\
&= (1-\vartheta_t)\,\tilde{\sigma} d\tilde{Z}_t^i
\end{aligned} \tag{31}$$

where $\bar{q}_t := q_t^B + q_t^K$ and $\mu_t^{\bar{q}} := \frac{\dot{\bar{q}}_t}{\bar{q}_t}$. Here, the third line uses the return expressions and the government budget constraint (3) and the fourth line the aggregate resource constraint (10).

Equations (30) and (31) together with formula (29) allow us to compute the integrals in (28):

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log \eta_t^i dt\right] &= \frac{\log \eta_0^i}{\rho} - \frac{1}{2\rho}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\,(1-\vartheta_t)^2\,\tilde{\sigma}^2 dt\right], \\
\mathbb{E}\left[\int_{t_0}^\infty e^{-\rho t} \log K_t dt\right] &= \frac{\log K_0}{\rho} + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\,\frac{\big(\Phi(\iota_t) - \delta\big)}{\rho} dt\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}\left[\int_{t_0}^\infty e^{-\rho t} \log c_t^i dt\right] &= \frac{\log \eta_0^i + \log K_0}{\rho} \\
&+ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log\left(a - \mathfrak{g} - \iota_t\right) + \frac{\big(\Phi(\iota_t) - \delta\big)}{\rho} - \frac{(1-\vartheta_t)^2\,\tilde{\sigma}^2}{2\rho}\right) dt\right]
\end{aligned}$$

After substituting ι_t as a function of ϑ_t (as stated in Section 3.2) and the functional form $\Phi(\iota) = \frac{1}{\phi}\log\big(1 + \phi\iota\big)$, we obtain equation (14).

Existence, Uniqueness and Properties of ϑ^{opt}. Taking first order conditions for maximizing the time-t integrand in equation (14) with respect to ϑ_t implies

$$(1-\vartheta_t)^3\,\tilde{\sigma}^2 + \phi\rho\,(1-\vartheta_t)^2\,\tilde{\sigma}^2 + \rho\,(1-\vartheta_t) - \rho = 0. \tag{32}$$

This is a third-order polynomial equation in $1-\vartheta_t$ and has thus precisely three complex solutions. Because the coefficients on all monomials of positive order are nonnegative and the constant coefficient is negative, standard results on polynomial roots imply that precisely one of these complex solutions is real and that solution must be positive. Consequently, there is a unique real number $\vartheta^{\text{opt}} < 1$ such that $1-\vartheta^{\text{opt}}$ satisfies the first-order condition.[54] It is also easy to see that $\vartheta^{\text{opt}} > 0$ as otherwise the positive-sign terms in equation (32) would exceed the negative-sign term in absolute value. Therefore, there is a unique optimal $\vartheta^{\text{opt}} \in (0,1)$ that maximizes the time-t integrand in equation (14) with respect to ϑ_t. Because the coefficients in equation (32) just depend on the parameters $\tilde{\sigma}$, ρ and ϕ, so does ϑ^{opt}. By the implicit function theorem, ϑ^{opt} must be strictly increasing in $\tilde{\sigma}$.

A.4 Welfare and Optimal Bubble Mining in the Perpetual Youth Example

In this appendix, we briefly outline the welfare analysis for the perpetual youth model. We start by computing the expected utility of a single agent i born at time $t_0(i)$:

$$\mathbb{E}\left[\int_{t_0(i)}^{\infty} e^{-\rho t}\log c_t^i dt\right] = e^{-\rho t_0(i)}\frac{X^i_{t_0(i)} + \log\left(a-\mathfrak{g}\right) + \left(\vartheta g - (1-\vartheta)\,\delta\right)/\rho}{\rho} \tag{33}$$

with

$$X_t^i = \begin{cases} \log\left(1-\vartheta\right), & t>0 \\ \log\eta_0^i + \log K_0, & t=0 \end{cases}$$

where $\eta_0^i := \left(b_0^i + q_0^K k_0^i\right) / \left((q_0^B + q_0^K)K_0\right)$ is the initial wealth share at time $t=0$ of an agent i that is already alive at that time.

The derivation of equation (33) proceeds in analogy to the derivation of equation (14) for the idiosyncratic risk model: we use the consumption rule and the aggregate resource constraint to write

$$c_t^i = \eta_t^i\left(a-\mathfrak{g}\right)K_t,$$

[54] The objective is not generally concave, but this first-order condition nevertheless always corresponds to a global maximum as can be readily verified by studying its asymptotic properties as $1-\vartheta\to\infty$ and $1-\vartheta\to 0$.

then decompose

$$\mathbb{E}\left[\int_{t_0(i)}^{\infty} e^{-\rho t} \log c_t^i dt\right] = \int_{t_0(i)}^{\infty} e^{-\rho t} \left(\log \eta_t^i + \log\left(a - \mathfrak{g}\right) + \log K_t\right) dt$$

and then compute the three integrals separately using a version of formula (29):[55]

$$\int_{t_0(i)}^{\infty} e^{-\rho t} \log \eta_t^i dt = e^{-\rho t_0(i)} \left(\frac{\log \eta_{t_0(i)} - \int_{t_0(i)}^{\infty} e^{-\rho(t - t_0(i))} \left(1 - \vartheta_t\right) dt \cdot \left(\delta + g\right)}{\rho}\right),$$

$$\int_{t_0(i)}^{\infty} e^{-\rho t} \log\left(a - \mathfrak{g}\right) dt = e^{-\rho t_0(i)} \frac{\log\left(a - \mathfrak{g}\right)}{\rho},$$

$$\int_{t_0(i)}^{\infty} e^{-\rho t} \log K_t dt = e^{-\rho t_0(i)} \frac{\log K_{t_0(i)} + g/\rho}{\rho}.$$

Combining terms, imposing a steady state, and using the definition $X_{t_0(i)}^i := \log\left(\eta_{t_0(i)}^i K_{t_0(i)}\right)$ and the fact that for $t_0(i) > 0$, $b_{t_0(i)}^i = 0$ and thus

$$\eta_{t_0(i)}^i K_{t_0(i)} = \frac{b_{t_0(i)}^i + q_{t_0(i)}^K k_{t_0(i)}}{q_{t_0(i)}^B + q_{t_0(i)}^K} = 1 - \vartheta_{t_0},$$

we obtain equation (33).

Next, let $\lambda(i)$ denote some weighting function with the properties $\int_0^\infty \lambda(i) di < \infty$ and $\lambda(i) \geq 0$ for all i and consider the social welfare function

$$\mathbb{W} := \int_0^\infty \lambda(i) \mathbb{E}\left[\int_{t_0(i)}^{\infty} e^{-\rho t} \log c_t^i dt\right] di$$

Using equation (33), we can write this as

$$\begin{aligned}
\mathbb{W} &= \int_0^\infty \lambda(i) e^{-\rho t_0(i)} \frac{X_{t_0(i)}^i + \log\left(a - \mathfrak{g}\right) + \left(\vartheta g - \left(1 - \vartheta\right)\delta\right)/\rho}{\rho} di \\
&= \frac{1}{\rho} \int_0^\infty \lambda(i) e^{-\rho t_0(i)} X_{t_0(i)}^i di + \int_0^\infty \lambda(i) e^{-\rho t_0(i)} di \cdot \frac{\log\left(a - \mathfrak{g}\right) + \left(\vartheta g - \left(1 - \vartheta\right)\delta\right)/\rho}{\rho} \\
&= \frac{1}{\rho} \int_0^{L_0} \lambda(i) e^{-\rho t_0(i)} \left(\log \eta_0^i + \log K_0\right) di + \frac{1}{\rho} \int_{L_0}^{\infty} \lambda(i) e^{-\rho t_0(i)} \log\left(1 - \vartheta\right) di \\
&\qquad + \overline{\Lambda} \cdot \frac{\log\left(a - \mathfrak{g}\right) + \left(\vartheta g - \left(1 - \vartheta\right)\delta\right)/\rho}{\rho}
\end{aligned}$$

[55]Note that $d\eta_t^i / \eta_t^i = -\left(1 - \vartheta_t\right)\left(\delta + g\right) dt$, as a simple calculation shows.

$$= \underbrace{\frac{1}{\rho}\int_0^{L_0} \lambda(i) e^{-\rho t_0(i)} \left(\log \eta_0^i + \log K_0\right) di}_{=:\mathbb{W}_0} + \Lambda \cdot \frac{\log(1-\vartheta)}{\rho} + \overline{\Lambda} \cdot \frac{\log(a-\mathfrak{g}) + \left(\vartheta g - (1-\vartheta)\delta\right)/\rho}{\rho},$$

where in the last line, the constant $\mathbb{W}_0$ does not depend on ϑ and we define

$$\Lambda := \int_{L_0}^{\infty} \lambda(i) e^{-\rho t_0(i)} di$$
$$\overline{\Lambda} := \int_0^{\infty} \lambda(i) e^{-\rho t_0(i)} di$$

which are the total discounted welfare weights attached to all future generations and to all generations, respectively.

A planner with this welfare objective would therefore like to manipulate ϑ to maximize the sum of the last two terms in the expression for $\mathbb{W}$. The first of these terms captures the share of total output added by newborns that they consume themselves. This is decreasing in ϑ as a larger value of bonds means that newborns without financial wealth transfer a larger share of their labor income to previously existing generations as they gradually redeem those generations' bond holdings. This first term is multiplied by Λ because only newborn agents after the initial time are negatively affected. The second term is increasing in ϑ as a larger value of bonds allows for more inter-generational wealth transfer and thereby increases the consumption growth rate of all agents. This second term is multiplied by $\overline{\Lambda}$ because consumption growth of all generations is affected equally.

The first-order condition for maximizing $\mathbb{W}$ with respect to ϑ is

$$\Lambda \frac{1}{1-\vartheta^{\text{opt}}} = \overline{\Lambda} \frac{g+\delta}{\rho} \Rightarrow \vartheta^{\text{opt}} = 1 - \frac{\Lambda}{\overline{\Lambda}} \frac{\rho}{g+\delta}.$$

Thus, there is a unique optimal solution ϑ^{opt} that is strictly decreasing in ρ and strictly increasing in $g+\delta$.

The interpretation of this optimal policy prescription is straightforward when comparing ϑ^{opt} to the competitive equilibrium value of ϑ in the monetary steady state, $\vartheta = 1 - \frac{\rho + \breve{\mu}^{\mathcal{B}}}{\delta + g}$. In the special case that the planner does not care about existing generations, $\overline{\Lambda} = \Lambda$, the inter-generational resource transfer generated by the bubble in the competitive equilibrium without policy intervention is optimal. The size of the bubble is thus constrained efficient. It optimally trades off the lower initial value of consumption for newborns with the higher consumption growth rate that results from a larger bubble value. Therefore, zero primary surpluses and no bubble mining, $\breve{\mu}^{\mathcal{B}} = 0$, implement the optimal policy in this case.

Whenever the planner does care about the generation initially alive, $\overline{\Lambda} > \Lambda$, then she desires to transfer additional resources from future generations to initial generations than in the benchmark case where $\overline{\Lambda} = \Lambda$. This requires increasing the value of debt (lowering ϑ) and taxing future generations. Therefore, the government optimally runs positive primary surpluses and uses these surpluses to back its debt, $\breve{\mu}^{\mathcal{B}} < 0$, instead of mining the bubble. Consequently, a positive rate of bubble mining ($\breve{\mu}^{\mathcal{B}} > 0$) is never optimal in this model.

As in the idiosyncratic risk model, the optimal value of ϑ^{opt} is independent of the government spending need $\mathfrak{g}$ and $\mathfrak{g}$ does also not appear in equation (24), so that also the degree of bubble mining $\breve{\mu}^{\mathcal{B}}$ required to implement $\vartheta = \vartheta^{\text{opt}}$ must be independent of $\mathfrak{g}$. The reason for this result is similar to the idiosyncratic risk model: when government spending $\mathfrak{g}$ increases, the government must transfer a larger fraction of current output away from the private sector to itself and taxing current output is the most direct way that does not distort the inter-generational resource transfers. In contrast, funding additional spending by increasing primary deficits (and bubble mining) dilutes the bubble at a faster rate and lowers the interest rate. This increases not just the resources available to the government today, but also increases the initial consumption level of future newborn agents at the expense of consumption growth for all agents. As the optimal trade-off between initial consumption and consumption growth is independent of the aggregate level of private consumption $C_t = (a - \mathfrak{g})K_t$, the optimal distortion induced by $\breve{\mu}^{\mathcal{B}}$ should also be independent of the government spending need.

A.5 Missing Steps in Proof of Lemma 1

We provide the proof for the idiosyncratic risk model. The proof for the perpetual youth model is largely identical, but simpler as there are fewer conditions to be verified.

It is to show that any solution $\vartheta : [0, \infty) \to [0, 1]$ to (27) corresponds to a unique equilibrium of the model. For any such function, define ι, q^B, and q^K consistent with the expressions given in the main text, i.e.,

$$\begin{aligned}
\iota_t &= \frac{(1 - \vartheta_t)(a - \mathfrak{g}) - \rho}{1 - \vartheta_t + \phi\rho}, \\
q_t^B &= \vartheta_t \frac{1 + \phi(a - \mathfrak{g})}{1 - \vartheta_t + \phi\rho}, \\
q_t^K &= (1 - \vartheta_t)\frac{1 + \phi(a - \mathfrak{g})}{1 - \vartheta_t + \phi\rho}.
\end{aligned}$$

Because $\vartheta_t \in [0, 1]$ at all times, $q_t^B, q_t^K \geq 0$, so these expressions are consistent with free disposal of both bonds and capital.We now verify that ι_t, q_t^B, q_t^K and $\theta_t := \vartheta_t$ satisfy all household choice conditions and the aggregate resource constraint.

One immediately verifies that ι_t and q_t^K satisfy households' optimal investment choice condition, $q_t^K = \frac{1}{\Phi'(\iota_t)} = 1 + \phi\iota_t$. In addition, total wealth of all households is $\left(q_t^B + q_t^K\right) K_t$ and because individual consumption demand $c_t^i = \rho n_t^i$ implies an aggregate consumption demand of $C_t = \rho\left(q_t^B + q_t^K\right) K_t$, we obtain

$$\begin{aligned}
C_t + \mathfrak{g}K_t + \iota_t K_t &= \left(\rho\left(q_t^B + q_t^K\right) + \mathfrak{g} + \iota_t\right) K_t \\
&= \left(\rho\frac{1 + \phi\left(a - \mathfrak{g}\right)}{1 - \vartheta_t + \phi\rho} + \mathfrak{g} + \frac{\left(1 - \vartheta_t\right)\left(a - \mathfrak{g}\right) - \rho}{1 - \vartheta_t + \phi\rho}\right) K_t \\
&= \left(\left(\frac{\phi\rho}{1 - \vartheta_t + \phi\rho} + \frac{\left(1 - \vartheta_t\right)}{1 - \vartheta_t + \phi\rho}\right)\left(a - \mathfrak{g}\right) + \mathfrak{g}\right) K_t \\
&= aK_t,
\end{aligned}$$

so this equilibrium candidate satisfies the aggregate resource constraint (10).

It is left to show that at the asset prices q^B and q^K, agents' bond portfolio share $\theta_t = \vartheta_t$ is consistent with their optimal choice condition for θ_t. We consider two cases:

1. If $\vartheta_t > 0$, then equation (27) (that ϑ satisfies by assumption) is equivalent to $\mu_t^\vartheta = \rho + \breve{\mu}_t^{\mathcal{B}} - \left(1 - \vartheta_t\right)^2 \tilde{\sigma}^2$ and rearranging the latter equation and using $\theta_t = \vartheta_t$ yields

$$1 - \theta_t = \frac{1}{\tilde{\sigma}^2}\frac{\rho + \breve{\mu}_t^{\mathcal{B}} - \mu_t^\vartheta}{1 - \vartheta_t}. \tag{34}$$

   Next, by definition of ι_t and q_t^K

$$\begin{aligned}
\frac{a - \mathfrak{g} - \iota_t}{q_t^K} &= \frac{\left(a - \mathfrak{g}\right)\left(1 - \vartheta_t + \phi\rho\right) - \left(1 - \vartheta_t\right)\left(a - \mathfrak{g}\right) + \rho}{\left(1 - \vartheta_t\right)\left(1 + \phi\left(a - \mathfrak{g}\right)\right)} \\
&= \frac{\left(1 + \phi\left(a - \mathfrak{g}\right)\right)\rho}{\left(1 - \vartheta_t\right)\left(1 + \phi\left(a - \mathfrak{g}\right)\right)} = \frac{\rho}{1 - \vartheta_t},
\end{aligned}$$

   and substituting this into equation (34) yields

$$1 - \theta_t = \frac{1}{\tilde{\sigma}^2}\left(\frac{a - \mathfrak{g} - \iota_t}{q_t^K} - \frac{\mu^\vartheta - \breve{\mu}_t^{\mathcal{B}}}{1 - \vartheta_t}\right),$$

   which is precisely households' first-order condition with respect to θ_t as stated in the main text.

2. If $\vartheta_t = 0$, then $q_t^B = 0$, hence bonds have no value and the return on bonds is not well-defined. Consequently, the household portfolio choice condition as stated in the main text is not directly applicable. Instead, households demand a finite quantity of bonds (which

is consistent with equilibrium and $\theta_t = 0$), if and only if $\dot{q}^B_t \leq 0$, i.e., the value of bonds is expected to remain nonpositive in the infinitesimal future. Because $q^B_s \geq 0$ for all s, this condition reduces here to $\dot{q}^B_t = 0 \Leftrightarrow \dot{\vartheta}_t = 0$. We therefore have to show that $\vartheta_t = 0$ implies $\dot{\vartheta}_t = 0$. We do this in two steps:

First, at $\vartheta_t = 0$, equation (27) is misleading, because it appears that always $\dot{\vartheta}_t = 0$, but this ignores that by the government budget constraint, $\breve{\mu}^{\mathcal{B}}_t = \frac{\mathfrak{g}-\tau_t a}{q^B_t} = \frac{\mathfrak{g}-\tau_t a}{\vartheta_t}\frac{1-\vartheta_t+\phi\rho}{1+\phi(a-\mathfrak{g})}$, which diverges to $\pm\infty$ as $\vartheta_t \searrow 0$ unless $\mathfrak{g} - \tau_t a \to 0$. Nevertheless, the right-hand side of (27) remains well-defined even at the limit point and for $\mathfrak{g} - \tau_t a \nrightarrow 0$, if we plug in $\breve{\mu}^{\mathcal{B}}_t$,

$$\dot{\vartheta}_t = \left(\rho - (1-\vartheta_t)^2\,\tilde{\sigma}^2\right)\vartheta_t + (\mathfrak{g} - \tau_t a)\,\frac{1-\vartheta_t+\phi\rho}{1+\phi\,(a-\mathfrak{g})}. \tag{35}$$

We use this representation of ODE (27) in the remaining argument.

Second, substituting $\vartheta_t = 0$ into equation (35) yields

$$\dot{\vartheta}_t = (\mathfrak{g} - \tau_t a)\,\frac{1-\vartheta_t+\phi\rho}{1+\phi\,(a-\mathfrak{g})} = 0,$$

where the last equality follows from the government budget constraint (3) in the limit $\mathcal{P}_t \to \infty$ (which is equivalent to $q^B_t = \vartheta_t = 0$) and the assumption that government policy is specified to be consistent with the government budget constraint.[56] Consequently, $\vartheta_t = 0$ implies indeed $\dot{\vartheta}_t = 0$.

A.6 Model Extension with Bubbles on Other Assets

The setup is as outlined in Section 7. We start by imposing on households the additional constraint that they have to invest a fraction x_t of their (bond) savings b^i_t into government bonds and a fraction $1 - x_t$ into the other bubble asset, where $x_t = q^B_t / \hat{q}^B_t$ is exogenously given. It is then easy to see that the household problem is precisely as in Section 3.1 for the perpetual youth model and as in Section 3.2 for the idiosyncratic risk model, except that b^i_t denotes savings in the bubble portfolio (whereas before we set $x_t \equiv 1$).

In addition, also the model solution is precisely as before with two exceptions:

First, we have to replace everywhere q^B_t with $\hat{q}^B_t$, ϑ_t with $\hat{\vartheta}_t$ as only the total bubble portfolio

[56] To be precise, our assumption only excludes negative primary surpluses $\mathfrak{g} > \tau_t a$ when the market is not willing to absorb the bond issuance necessary to finance these deficits. The government could always choose positive surpluses. However, this cannot be the case here as it would imply $\dot{\vartheta}_t < 0$ and thus not be consistent with the assumption that ϑ_t is contained in $[0, 1]$.

matters, not its individual components. In particular, we now obtain the equations

$$\hat{q}^B_t = \hat{\vartheta}_t \frac{a-\mathfrak{g}}{\rho},$$
$$q^K_t = \left(1-\hat{\vartheta}_t\right)\frac{a-\mathfrak{g}}{\rho}$$

for $\hat{q}^B_t$ and q^K_t in the perpetual youth model and the equations

$$\iota_t = \frac{\left(1-\hat{\vartheta}_t\right)(a-\mathfrak{g})-\rho}{1-\hat{\vartheta}_t+\phi\rho},$$
$$\hat{q}^B_t = \hat{\vartheta}_t \frac{1+\phi(a-\mathfrak{g})}{1-\hat{\vartheta}_t+\phi\rho},$$
$$q^K_t = \left(1-\hat{\vartheta}_t\right)\frac{1+\phi(a-\mathfrak{g})}{1-\hat{\vartheta}_t+\phi\rho}.$$

for $\hat{q}^B_t$, q^K_t, and ι_t in the idiosyncratic risk model (and, in both cases, $q^B_t = x_t\hat{q}^B_t$ by the exogenous portfolio split).

Second, the solution procedure that resulted in the valuation equation for ϑ_t in the main text has to take into account that only a fraction x_t of the bubble is now diluted by bubble mining. As a result, the return on the bubble portfolio is not $dr^{\mathcal{B}}_t$ but $d\hat{r}^{\mathcal{B}}_t$ as stated in the main text,

$$d\hat{r}^{\mathcal{B}}_t = \left(g_t + \mu^{\hat{q},B}_t - \frac{q^B_t}{\hat{q}^B_t}\breve{\mu}^{\mathcal{B}}_t\right)dt,$$

where $\breve{\mu}^{\mathcal{B}}_t$ enters with an additional coefficient $q^B_t/\hat{q}^B_t = x_t$. As a consequence, the derivation of the ODE for $\hat{\vartheta}_t$ changes relative to the baseline models:

1. For the perpetual youth model, when equating the return of the bubble portfolio with the risk-free rate, $d\hat{r}^{\mathcal{B}}_t = r^f_t dt$, we obtain

   $$r^f_t = g + \mu^{\hat{q},B}_t - \frac{q^B_t}{\hat{q}^B_t}\breve{\mu}^{\mathcal{B}}_t$$

   instead of equation (8).

2. For the idiosyncratic risk model, the portfolio choice first-order condition becomes

   $$\frac{a-\mathfrak{g}-\iota_t}{q^K_t} - \frac{\mu^{\vartheta}_t - \frac{q^B_t}{\hat{q}^B_t}\breve{\mu}^{\mathcal{B}}_t}{1-\vartheta_t} = \left(1-\theta^i_t\right)\tilde{\sigma}^2$$

   instead of the condition stated in Section 3.2.

In both cases, the only difference is that also in these conditions $\breve{\mu}_t^{\mathcal{B}}$ is replaced with $\frac{q_t^B}{\hat{q}_t^B}\breve{\mu}_t^{\mathcal{B}}$. Otherwise, the derivation of $\mu_t^{\hat{\vartheta}}$ proceeds precisely as in the baseline models (there for μ_t^{ϑ}) and thus the result must be

$$\mu_t^{\hat{\vartheta}} = \rho + \frac{q_t^B}{\hat{q}_t^B}\breve{\mu}_t^{\mathcal{B}} - f\left(1 - \vartheta_t\right).$$

This implies equation (16) stated in the main text.

In sum, the restricted model with the additional constraint on household portfolios has precisely the same equations as the baseline model, except that ϑ_t is replaced with $\hat{\vartheta}_t$, q_t^B with $\hat{q}_t^B$, and $\breve{\mu}_t^{\mathcal{B}}$ with $x_t\breve{\mu}_t^{\mathcal{B}}$. Therefore, any theoretical proposition about the baseline model holds equally also for the restricted model for any exogenous $x_t \in (0, 1]$. This includes Lemma 1, so that a time path $\hat{\vartheta}_t$ corresponds to a (unique) equilibrium of the restricted model if and only if it solved equation (16) and satisfies $0 \leq \hat{\vartheta}_t \leq 1$ for all t.

The results in the previous paragraph remain true in particular if we choose for x_t the process $q_t^B/\hat{q}_t^B$ that arises endogenously in the actual model where the portfolio choice between the two bubble assets is not constrained.[57] Then, there is an additional no-arbitrage condition between the two bubble assets because both assets must earn the risk-free rate for the household to be indifferent between the two.[58] One way to write this condition is as $dr_t^{\mathcal{B}} = d\hat{r}_t^{\mathcal{B}}$, which is equivalent to

$$g + \mu_t^{\hat{q},B} - \frac{q_t^B}{\hat{q}_t^B}\breve{\mu}_t^{\mathcal{B}} = g + \mu_t^{q,B} - \breve{\mu}_t^{\mathcal{B}}$$

$$\Leftrightarrow \qquad \underbrace{\mu_t^{\hat{q},B} - \mu_t^{\bar{q}}}_{=\mu_t^{\hat{\vartheta}}} + \frac{\hat{\vartheta}_t - \vartheta_t}{\hat{\vartheta}_t}\breve{\mu}_t^{\mathcal{B}} = \underbrace{\mu_t^{q,B} - \mu_t^{\bar{q}}}_{=\mu_t^{\vartheta}}$$

where $\bar{q}_t := \hat{q}_t^B + q_t^K$ and $\mu_t^{\bar{q}} := \frac{\dot{\bar{q}}_t}{\bar{q}_t}$. Solving for μ_t^{ϑ} and multiplying by ϑ_t yields

$$\dot{\vartheta}_t = \frac{\vartheta_t}{\hat{\vartheta}_t}\left(\dot{\hat{\vartheta}}_t + \left(\hat{\vartheta}_t - \vartheta_t\right)\breve{\mu}_t^{\mathcal{B}}\right),$$

which, after substituting in equation (16), is equivalent to equation (17) in the main text. This implies that equation (17) is necessary for any model equilibrium in the actual (unrestricted) model. It is also clearly the case that $0 \leq \vartheta_t \leq \hat{\vartheta}_t \leq 1$ for all t.

Conversely, if we have time paths ϑ_t, $\hat{\vartheta}_t$ with $0 \leq \vartheta_t \leq \hat{\vartheta}_t \leq 1$ for all t that satisfy equations

[57] This is simply, because if x_t happens to be the process $q_t^B/\hat{q}_t^B$ from the actual model, then the constrained portfolio choice is already unconstrained optimal and imposing the constraint becomes redundant.

[58] This statement remains true even if we allow for stochastic "bubble fluctuations" between the two assets. The reason is that the aggregate bubble portfolio has a return $d\hat{r}_t^{\mathcal{B}}$ that is free of any risk, thus agents are perfectly hedged against such "bubble fluctuations" and, at the margin, require a zero risk premium for holding these risks.

(16) and (17), then the previous arguments show that (a) there is an (unique) equilibrium of the restricted model with $x_t := \vartheta_t/\hat{\vartheta}_t$ such that $\hat{\vartheta}_t = \hat{q}_t^B/(\hat{q}_t^B + q_t^K)$ in equilibrium, (b) for $q_t^B := \vartheta_t/\hat{\vartheta}_t\hat{q}_t^B$, the return on government bonds $dr_t^{\mathcal{B}}$ as defined by equation (11)[59] satisfies the no-arbitrage condition $dr_t^{\mathcal{B}} = d\hat{r}_t^{\mathcal{B}}$,[60] and (c) both government bonds and the other bubble asset have a nonnegative value at all times (due to $0 \leq q_t^B \leq \hat{q}_t^B$). Consequently, the equilibrium of the restricted model for $x_t = \vartheta_t/\hat{\vartheta}_t$ is even an equilibrium of the actual, unrestricted, model, as the no arbitrage condition ensures that agents are willing to hold a fraction x_t of the bubble portfolio in government debt and due to nonnegative valuations, there is no deviation strategy that relies on disposing certain assets (or accumulating them indefinitely, which is equivalent).

To sum up, the following version of Lemma 1 holds in the extended model:

Lemma 2. *Absolutely continuous functions $\vartheta, \hat{\vartheta} : [0, \infty) \to \mathbb{R}$ correspond to a (unique) model equilibrium of the extended model with bubbles on other assets, if and only if they satisfy equations (16) and (17) and, for all $t \geq 0$, $0 \leq \vartheta_t \leq \hat{\vartheta}_t \leq 1$.*

We conclude this appendix by remarking that in the case of the threshold policy discussed in the main text, the two ODEs for $\hat{\vartheta}_t$ and ϑ_t can be written as

$$\dot{\hat{\vartheta}}_t = \begin{cases} \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t + \breve{\mu}^{\mathcal{B}}\vartheta_t, & \vartheta_t \geq \underline{\vartheta} \\ \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t - (\bar{\tau}a - \mathfrak{g})\, h\left(1 - \hat{\vartheta}_t\right), & \vartheta_t < \underline{\vartheta} \end{cases}, \tag{36}$$

$$\dot{\vartheta}_t = \begin{cases} \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t + \breve{\mu}^{\mathcal{B}}\vartheta_t, & \vartheta_t \geq \underline{\vartheta} \\ \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t - (\bar{\tau}a - \mathfrak{g})\, h\left(1 - \hat{\vartheta}_t\right), & \vartheta_t < \underline{\vartheta} \end{cases}. \tag{37}$$

where the function h is as in the main text.

A.7 Proof of Proposition 1

In this appendix, we can limit attention to the case in which $\breve{\mu}^{\mathcal{B}} \geq 0$. If $\breve{\mu}^{\mathcal{B}} < 0$, then the government always runs positive primary surpluses that are bounded away from zero under the threshold policy considered here.[61] In this case, it is necessarily true that $r^f > g$ and thus there is no space for bubbles, whether on government debt or other assets. It must therefore be

[59]This is the equation for the idiosyncratic risk model. For the perpetual youth model, one simply replaces $\Phi(\iota_t) - \delta$ with g.

[60]This follows immediately from the derivation of (17) above, as all transformations have been equivalence transformations.

[61]In fact, even the surplus-capital ratio is bounded away from zero which is what is ultimately relevant in a growing economy.

that $\hat{\vartheta}_t = \vartheta_t$ and the arguments in the main text for the baseline model without bubbles on other assets are sufficient to establish that $\vartheta_t = \vartheta^*$ is the unique equilibrium.

By the results derived in Appendix A.6, all model equilibria under the threshold policies are associated with ϑ and $\hat{\vartheta}$ paths that satisfy equations (36) and (37) and the additional condition that $0 \le \vartheta_t \le \hat{\vartheta}_t \le 1$ for all t. We show that such solution paths ϑ and $\hat{\vartheta}$ must satisfy all the assertions made in Proposition 1. We start by proving the following auxiliary lemma:

Lemma 3. *Under the threshold policy, for any equilibrium and all $t \ge 0$*

$$\vartheta_t \in [\underline{\vartheta}, \vartheta^*], \qquad \hat{\vartheta}_t \in [\vartheta^*, \vartheta_0^*],$$

where $\vartheta^*, \vartheta_0^* \ge 0$ are defined by the equations

$$f(1-\vartheta^*) = \rho + \breve{\mu}^{\mathcal{B}}$$
$$f(1-\vartheta_0^*) = \rho$$

and satisfy $\vartheta^* \le \vartheta_0^*$.

Proof. The proof proceeds by showing that if for any initial time t, ϑ_t or $\hat{\vartheta}_t$ are outside the asserted intervals, then the mathematical solution to ODEs (36) and (37) must cross the boundaries 0 or 1 in finite time and thus cannot correspond to an equilibrium solution by Lemma 2. We repeatedly use the simple fact that if the right-hand side of an ODE for a function x_t is continuous in x_t, strictly negative for all $x_t \in [0, \underline{x})$, and strictly positive for all $x_t \in (\overline{x}, 1]$, then all solution paths with $x_t < \underline{x}$ for some t reach 0 in finite time and all solution paths with $x_t > \bar{x}$ for some t reach 1 in finite time.[62]

We start by ruling out $\vartheta_t < \underline{\vartheta}$ for any t. For $\vartheta_t \in [0, \underline{\vartheta})$, ODE (37) implies a negative value for $\dot{\vartheta}_t$ regardless of the value of $\hat{\vartheta}_t \in [0,1]$. To see this, distinguish two cases:

1. If $\hat{\vartheta}_t \le \vartheta_0^*$, then $\rho - f\left(1-\hat{\vartheta}_t\right) \ge 0$. Thus, by $\vartheta_t \le \hat{\vartheta}_t$ and monotonicity of f and h,

$$\begin{aligned}\dot{\vartheta}_t \le \dot{\hat{\vartheta}}_t &= \left(\rho - f\left(1-\hat{\vartheta}_t\right)\right)\hat{\vartheta}_t - (\bar{\tau}a - \mathfrak{g})\, h\left(1-\hat{\vartheta}_t\right) \\ &\le \left(\rho - f\left(1-\vartheta_0^*\right)\right)\vartheta_0^* - (\bar{\tau}a - \mathfrak{g})\, h\left(1-\vartheta_0^*\right) \\ &= -\left(\bar{\tau}a - \mathfrak{g}\right) h\left(1-\vartheta_0^*\right) < 0,\end{aligned}$$

where the last inequality follows from the fact that h is bounded away from 0 for nonnegative arguments.

[62] The proof of this simple fact is omitted, but the basic idea is that close to the boundaries, $\dot{x}_t$ is bounded away from zero and thus boundaries must be reached in finite time as opposed to asymptotically.

2. If $\hat{\vartheta}_t \geq \vartheta_0^*$, then $\left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t \leq 0$ and hence

$$\dot{\vartheta}_t \leq -\left(\bar{\tau}a - \mathfrak{g}\right) h\left(1 - \hat{\vartheta}_t\right) < 0.$$

So in any case, we obtain $\dot{\vartheta}_t < 0$ and can thus rule out solution paths with $\vartheta_t < \underline{\vartheta}$ at any time t. In particular, we can from now on assume that always $\vartheta_t \geq \underline{\vartheta}$ (and it suffices to show that the lower boundary $\underline{\vartheta}$ instead of 0 is reached in finite time).

Next, we show that $\dot{\hat{\vartheta}}_t > 0$ for any $\hat{\vartheta}_t \in (\vartheta_0^*, 1]$, regardless of the value of $\vartheta_t \in [0, 1]$. This rules out $\hat{\vartheta}_t > \vartheta_0^*$ for any t. Indeed, we have $\breve{\mu}^{\mathcal{B}}\vartheta_t \geq 0$ (by the assumption $\breve{\mu}^{\mathcal{B}} \geq 0$) and thus

$$\dot{\hat{\vartheta}}_t \geq \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t.$$

For $\hat{\vartheta}_t > \vartheta_0^*$, $\rho - f\left(1 - \hat{\vartheta}_t\right) > 0$ and $\hat{\vartheta}_t > 0$, so also the product on the right must be strictly positive.

It is left to show $\vartheta_t \leq \vartheta^* \leq \hat{\vartheta}_t$ for all t. Suppose first that $\underline{\vartheta} \leq \vartheta_t \leq \hat{\vartheta}_t < \vartheta^*$. Then, $\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right) < 0$ and thus

$$\dot{\hat{\vartheta}}_t \leq \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t < 0,$$

$$\vartheta_t = \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t < 0.$$

Consequently, ϑ_t would have to cross the lower bound $\underline{\vartheta}$ in finite time, in contradiction to $\vartheta_t \geq \underline{\vartheta}$ for all t. Thus, we must have $\hat{\vartheta}_t \geq \vartheta^*$ for all t. Next, suppose that $\vartheta^* < \vartheta_t \leq \hat{\vartheta}_t \leq \vartheta_0^*$. Then $\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right) > 0$ and thus

$$\dot{\vartheta}_t = \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t > 0.$$

Consequently, ϑ_t would have to cross the upper bound ϑ_0^* in finite time, in contradiction to $\vartheta_t \leq \hat{\vartheta}_t \leq \vartheta_0^*$ for all t. Thus, we must also have $\vartheta_t \leq \vartheta^*$ for all t.

This concludes the proof of the Lemma (the additional inequality $\vartheta^* \leq \vartheta_0^*$ is obvious). □

Lemma 3 implies additional restrictions that simplify the proof of the proposition substantially. First, due to $\vartheta_t \geq \underline{\vartheta}$, the first case in equations (36) and (37) is always true in equilibrium. Second, due to $\vartheta^* \leq \hat{\vartheta}_t < \vartheta_0^*$, we have

$$\rho - f\left(1 - \hat{\vartheta}_t\right) < 0 \leq \rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right). \tag{38}$$

Define now $\vartheta^o_t := \hat{\vartheta}_t - \vartheta_t$, which is the share of wealth that is due to the other bubble asset. Combining ODEs (36) and (37) implies

$$\dot{\vartheta}^o_t = \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\left(\hat{\vartheta}_t - \vartheta_t\right) = \left(\rho - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta^o_t \tag{39}$$

along any equilibrium path. By inequality (38), the first factor is always negative and thus $\dot{\vartheta}^o_t \leq 0$.

Next, we can write equation (36) as

$$\begin{aligned}\dot{\hat{\vartheta}}_t &= \overbrace{\left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\hat{\vartheta}_t}^{=:F(\hat{\vartheta}_t)} - \breve{\mu}^{\mathcal{B}}\left(\hat{\vartheta}_t - \vartheta_t\right) \\ &= f(\hat{\vartheta}_t) - \breve{\mu}^{\mathcal{B}}\vartheta^o_t\end{aligned}$$

where due to inequality (38), F is a strictly increasing function in the relevant domain for $\hat{\vartheta}$. We show that then $\hat{\vartheta}$ must be nonincreasing over time. Indeed, if this was not the case, then there would be $t_0 < \infty$ such that $\dot{\hat{\vartheta}}_{t_0} > 0$ and thus $F(\hat{\vartheta}_{t_0}) > \breve{\mu}^{\mathcal{B}}\vartheta^o_{t_0}$. But due to monotonicity of f and ϑ^o, then also for all $\hat{\vartheta} \geq \hat{\vartheta}_{t_0}$ and all $t \geq t_0$

$$F(\hat{\vartheta}) \geq F(\hat{\vartheta}_{t_0}) > \breve{\mu}^{\mathcal{B}}\vartheta^o_{t_0} \geq \breve{\mu}^{\mathcal{B}}\vartheta^o_t.$$

This implies that $\dot{\hat{\vartheta}}_t \geq \dot{\hat{\vartheta}}_{t_0} > 0$ for all $t \geq t_0$ and thus $\hat{\vartheta}_t$ would have to grow without bounds contradicting $\hat{\vartheta}_t < \vartheta^*_0$ by Lemma 3.

Because $\hat{\vartheta}$ is nonincreasing over time and bounded below by ϑ^* by Lemma 3, the limit $\lim_{t\to\infty}\hat{\vartheta}_t$ exists and is $\geq \vartheta^*$. We show that the inequality cannot be strict. If it was, then there would be some $\vartheta^{**} > \vartheta^*$ and some $T < \infty$ such that $\hat{\vartheta}_t \geq \vartheta^{**}$ for all $t \geq T$. For such t, we would then have (by equation (37))

$$\begin{aligned}\dot{\vartheta}_t &= \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right)\right)\vartheta_t \\ &\geq \left(\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \vartheta^{**}\right)\right)\vartheta_t, \\ &= \left(f\left(1 - \vartheta^{**}\right) - f\left(1 - \vartheta^{*}\right)\right)\vartheta_t > 0\end{aligned}$$

so ϑ_t must grow without bounds contradicting $\vartheta_t \leq \vartheta^*$ by Lemma 3. This completes the proof that $\hat{\vartheta}_t \to \vartheta^*$ as $t \to \infty$.

The remaining results in the Proposition then easily follow:

With the limit result for $\hat{\vartheta}_t$ already shown, the limit result $\vartheta_t \to \vartheta^*$ is equivalent to $\vartheta^o_t \to 0$.

This is a direct implication of equation (39) because for $\breve{\mu}^{\mathcal{B}} > 0$, $\vartheta^* < \vartheta_0^*$, thus the (decreasing) function $t \to \hat{\vartheta}_t$ is bounded away from 0 and consequently the coefficient in equation (39) is also bounded away from zero. As a consequence, $0 \leq \vartheta_t^o \leq \vartheta_0^o e^{-\alpha t}$ for some positive decay rate $\alpha > 0$, and thus $\vartheta_t^o \to 0$.

If no bubbles on other assets are permitted, then $\vartheta_t = \hat{\vartheta}_t$. Due to Lemma 3, $\vartheta_t = \hat{\vartheta}_t \geq \vartheta^* \geq \vartheta_t$. Thus, equality must hold everywhere.

Similarly, if $\underline{\vartheta} = \vartheta^*$, then Lemma 3 implies $\vartheta_t = \vartheta^*$ for all t. In particular, $\dot{\vartheta}_t = 0$ and thus ODE (37) implies for all t:

$$\rho + \breve{\mu}^{\mathcal{B}} - f\left(1 - \hat{\vartheta}_t\right) = 0 \Leftrightarrow \hat{\vartheta}_t = \vartheta^*.$$

This concludes the proof of the proposition.