## Introduction

While huge theoretical literature studies how the physical skewness of an asset’s dollar return (i.e., the ratio of later value to earlier value) affects investor behavior and asset prices, only few empirical studies convincingly test the main predictions of that literature. The reason for this gap is that it is challenging to empirically estimate skewness with realistic amounts of data, especially over the long return horizons the theoretical literature focuses on. Only recently, a handful of studies including Fama and French (2018) and Farago and Hjalmarsson (2019) have started addressing the estimation issue head on, proposing estimators relying on fewer data. Yet, those estimators often deliver biased estimates under realistic assumptions.

This paper develops a new parametric estimator of the skewness of an asset’s return over an arbitrary horizon under the assumption that the asset’s value can be modelled using a stochastic process from the affine stochastic volatility (ASV) model class.

The simulation exercise shows that our estimator is close to unbiased and efficient. In a further contrast to other estimators, it is also able to forecast skewness.

## Theoretical Framework

We first define ASV models in a way consistent with Duffie et al. (2000, 2003), Chernov et al. (2003), etc. In our definition:

1. the joint conditional moment generating function (MGF) $M(u, w, h; V)$, where $\mu \equiv E[e^{\mu X}]$, $\sigma^2 \equiv \text{Var}(X)$, and $\kappa$ is the observed log-value process, $V_t$ is the associated d-dimensional variance process, and $\lambda_t$ is $X_t - \mu_t$.

2. the unconditional MGF also exists, and $M(u, w, h; V) \equiv E[M(u, w, h; V)]$.

3. the third moments of $X_t$ do not explode over any finite horizon $h$.

Denote dollar return as $R_{A,t}$. By definition, $R_{A,t} \equiv c^{\mu_t}$, thus $E[R_{A,t}^1] = E[c^{\mu_t}] = M(k, 0, h)$ and $E[R_{A,t}^3] = M(0, k, h)$. Let $M(u, w, h; V) \equiv M(u, 0, h)$ and $M(u, h, h; V) \equiv M(u, 0, 0, h)$. Substitute them for the expected moments of dollar return in the skewness coefficient formula and simplify:

$$\text{Skew}(R_{A,t}) = \frac{M(3, h) - 3M(1, h)M(2, h) + 2M(1, h)^3}{M(2, h) - M(1, h)^2}$$

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Since the MGFs depend on the parameter values of the stochastic process, we can then obtain consistent estimates of the two skewness versions from consistent estimates of the parameters, assuming that the MGFs are smooth functions of the parameter vector.

## Example: the Heston (1993) Model

The model assumes the following processes:

$$dX_t = (\mu - \frac{1}{2} \kappa \sigma^2)dt + \sqrt{\kappa \sigma^2}dW_t,$$

$$dV_t = \kappa (\omega - V_t)dt + \xi \sqrt{V_t}dB_t,$$

where $\mu$, $\kappa$, $\sigma$, and $\xi$ respectively denote drift, mean reversion, long-run variance, and volatility-of-volatility. $W_t$ and $B_t$ are Brownian motions with $[W, B]_t = rt$. When $\kappa < 0$, leverage effect appears. The model is simple yet flexible, popular, and has closed-form MGF available in papers like Bates (2006), Andersen (2008), etc.

## Results: Strong Leverage Effect

Our estimator always yields the smallest mean absolute percent error and MSE regardless of horizon, and the outperformance is economically large.

## Empirical Results: S&P 500

Based on its 1950-2020 daily returns and a rolling 10-year estimation window, our application suggests:

- The evolutions of the skewness estimates for different return horizons are highly correlated over time. Leverage effect is strong (weekly estimates often > monthly estimates) and time-varying.
- The skewness of long-horizon dollar returns is probably much lower than suggested in recent works.

## Conclusions

Assuming that asset values can be modelled by ASV models, we derive a novel parametric estimator of the skewness of dollar returns. The simulation exercise based on the Heston process shows that our estimator strongly outperforms the others in most parameter settings and even under model misspecification. Empirical application on stock indexes shows an important time-varying leverage effect in that asset class and refutes the idea that the skewness of long-horizon returns is usually too high to be useful.