# Manufacturing Risk-free Government Debt*

Zhengyang Jiang
Northwestern Kellogg

Hanno Lustig
Stanford GSB, NBER

Stijn Van Nieuwerburgh
Columbia Business School, NBER, and CEPR

Mindy Z. Xiaolan
UT Austin McCombs

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**Abstract**

Governments face a trade-off between insuring bondholders and insuring taxpayers against output shocks. If they insure bondholders by manufacturing risk-free zero-beta debt, then they can only provide limited insurance to taxpayers. Taxpayers will pay more taxes in bad times regardless of whether output shocks are permanent or temporary. Permanent shocks impute long-run output risk to the debt while transitory shocks impute interest rate risk, all of which must be offset through taxation to keep the debt safe. Conversely, if governments insure taxpayers against adverse macro shocks, then the debt becomes risky. Convenience yields on government debt temporarily alleviate the trade-off.

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*First version: 07/08/2020. Jiang: Finance Department, Kellogg School of Management, Northwestern University; zhengyang.jiang@kellogg.northwestern.edu. Lustig: Department of Finance, Stanford Graduate School of Business, Stanford CA 94305; hlustig@stanford.edu; https://people.stanford.edu/hlustig/. Van Nieuwerburgh: Department of Finance, Columbia Business School, Columbia University, 3022 Broadway, New York, NY 10027; svnieuwe@gsb.columbia.edu. Xiaolan: McCombs School of Business, the University of Texas at Austin; mindy.xiaolan@mccombs.utexas.edu. We are grateful to Adrien Auclert, Aydogan Alti, Jonathan Berk, Saki Bigio, Luigi Boccola, Lorenzo Bretscher, Mike Chernov (discussant), Hal Cole, Pierre Collin-Dufresne, Alexandre Corhay (discussant), Pierre-Olivier Gourinchas, Peter DeMarzo, Sebastian DiTella, Francois Gourio, Patrick Kehoe, Ralph Koijen, Arvind Krishnamurthy, Matteo Maggiori, Ellen McGrattan, Clemens Sialm, Alp Simsek (discussant), Adi Sunderam (discussant), Alan Taylor, Sheridan Titman and seminar participants at Stanford University (macro and finance), Columbia University (macro and finance), the University of Pennsylvania, Duke-UNC University research triangle seminar, UT Austin, University of Lausanne, University of Rochester, University of Maryland, NBER IFM, 2021 JHU Carey Finance Conference, 2021 FIRS, 2021 WFA, and the European Winter Meetings of the Econometric Society for helpful comments. Yuan Tian and Jeremias Huber provided outstanding research assistance. We gratefully acknowledge financial support from NSF award 2049260.

Governments around the world responded to the pandemic with a massive increase in spending financed with new debt. OECD governments collectively borrowed USD 18 trillion from the bond markets in 2020, 29% of GDP, and 60% more than in 2019. Central government debt/GDP ratios for OECD countries are projected to increase further in 2021 (OECD, 2021).

This recent debt expansion raises a classic question: How much counter-cyclical fiscal policy can governments implement while keeping the debt safe? In this paper, we show that governments face a trade-off between insuring their bondholders–by making debt risk-free–and insuring their taxpayers–by spending more and taxing less– in the wake of adverse macroeconomic shocks. If a government provides more insurance to bondholders, it enjoys lower risk premia on its debt, but then it can provide less insurance to taxpayers. Making government debt safer requires raising more tax revenue as a fraction of GDP from taxpayers in bad times. The larger the sovereign debt burden, the steeper this trade-off becomes.

Our analysis focuses on the case of risk-free debt, because it is empirically relevant for countries like the U.S.[1] A country's government debt is risk-free if the government debt portfolio has a zero beta, meaning that its valuation is immune to fluctuations in the economy and financial markets. Default-free debt is not necessarily risk-free debt, since its valuation can still fluctuate before expiration. Governments, like the U.S., have an incentive to manufacture safe debt, because safe debt earns sizeable convenience yields, lowering the interest they must pay on their debt. Krishnamurthy and Vissing-Jorgensen (2012) estimate convenience yields on U.S. Treasuries of around 75 bps per year, while more recent estimates find even larger convenience yields of around 200 basis points (Jiang, Krishnamurthy, and Lustig, 2021, 2018; Koijen and Yogo, 2019).

We focus most of our attention on an economy with permanent output shocks. Manufacturing risk-free debt in the presence of permanent output risk requires a non-trivial feat of financial engineering. The government bond portfolio is backed by a long position in a claim to tax revenue and a short position in a claim to government spending. The Treasury's long position in the tax claim exceeds the short position in the spending claim by the value of outstanding debt. To ensure risk-free debt (a zero beta on government debt $\beta^D = 0$), the claim to tax revenues needs to have a lower beta than the spending claim: $\beta^T < \beta^G$. Since both claims have the same exposure to long-run output risk, this condition imposes tight restrictions on the process of primary surpluses, or equivalently on tax revenues given a process for government spending.[2]

The tax claim has a low beta if the present discounted value (PDV) of future tax revenues increases in bad times, times in which the investor's marginal utility is high. Since the taxpayers

[1] Updating an earlier calculation by Hall and Sargent (2011), Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019b) compute an average annual excess return of 1.16% on the portfolio of all U.S. Treasuries.

[2] Recast in the language of Modigliani-Miller, the claim to tax revenue can be regarded as the government's unlevered asset, which is divided into the government debt and the claim to government spending. To manufacture risk-free debt, the spending claim has to be a levered version of the government's asset. Therefore, just as the equity of a firm has to be riskier than its asset in order to generate risk-free debt, the government's spending beta has to be higher than its tax beta to ensure a zero-beta debt. The tax beta has to be low.

pay the taxes, they have a short position on the tax revenue claim. From their perspective, a low-beta tax claim is a risky tax liability. The government cannot insure taxpayers when it insures bondholders by keeping the debt risk-free. The larger the amount of outstanding debt, the more levered the government becomes, and the larger the gap between the tax beta and the spending beta needs to be to keep the debt risk-free. As the debt grows, the beta of the tax claim has to go to zero holding fixed the spending beta. The trade-off between insuring taxpayers and bondholders steepens.

Conversely, if the government insists on insuring the taxpayers by lowering tax rates in bad times, then the tax beta is high and the government debt becomes risky. The bondholders now bear the macroeconomic risk. The riskier the debt, the larger the welfare benefits of insurance that accrue to taxpayers.

We characterize the restrictions imposed on tax revenues when debt is made risk-free. They depend on the debt issuance policy the government follows. When the government keeps the debt/output ratio constant, there is no scope for insurance of taxpayers. The tax process has to be safer than the spending process at all horizons—short, intermediate and long horizons. When the government can issue more debt in response to a negative GDP growth shock rather than raise taxes, the tax claim is riskier than the spending claim over short horizons. Over intermediate and longer horizons, the surplus and tax revenue claims have to become sufficiently safe for investors (risky for taxpayers) to offset the long-run output risk priced into the debt process when debt and output share the same stochastic trend. We characterize the amount of insurance that can be provided to taxpayers over finite horizons by studying a sufficient statistic, the cash-flow beta of the surplus/tax process.

A corollary of these results is that the government can only run primary deficits on average if the tax claim is safer than the spending claim. Taxpayers insure bondholders by suffering high taxation at the wrong time, in high marginal-utility states. The negative covariance of marginal utility and surpluses is what creates fiscal capacity. The question of whether the risk-free rate $r$ is lower than the growth rate $g$ is neither necessary nor sufficient to gauge fiscal capacity.

Over the last two decades, the beta of U.S. government debt has turned negative (Baele, Bekaert, and Inghelbrecht, 2010; Campbell, Pflueger, and Viceira, 2020). When government debt carries a negative beta, the scope for insuring taxpayers shrinks even further. The zero-beta debt case is conservative in terms of the restrictions it implies on surpluses/taxes.

Our paper is the first one to analytically characterize the trade-off between insuring taxpayers and bondholders at different horizons in an environment with plausible asset pricing implications and debt dynamics. Modern asset pricing has consistently found that permanent shocks to output and consumption account for most of the variance of the pricing kernel, and receive a high price of risk in securities market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and

Chernov, 2018). Models without large permanent shocks counterfactually produce bond risk premia that exceed equity risk premia.[3] We show that the presence of permanent risk has important implications for the fiscal policy literature.

When output shocks are permanent, debt inherits from the long-run risk in output as along as debt and output are co-integrated. To keep the debt risk-free, the government must offset the long-run output risk in the debt by making the surplus safer over intermediate horizons. It now only has very limited ability to insure taxpayers over short horizons. This is costly to taxpayers because the welfare benefits of insurance are largest when shocks to output and consumption are permanent (Alvarez and Jermann, 2004).

In traditional neoclassical and (New-)Keynesian models, shocks to output and consumption are transitory, as output fluctuates around potential output. The business cycle models used in the literature on optimal fiscal policy also imply that equilibrium output and consumption do not have a unit root component.[4] Surprisingly, the trade-off between insuring taxpayers and bondholders worsens in these models. Models with only transitory shocks to the pricing kernel generate a great deal of interest rate risk in government debt. The long-term government bond is the riskiest asset in such economies (Bansal and Lehmann, 1997; Alvarez and Jermann, 2005; Backus, Chernov, and Zin, 2014). When output is below potential, the representative investor wants to borrow, pushing up interest rates when her marginal utility is high. To keep the debt risk-free in the environment with transitory shocks, the government has to offset the interest rate risk in the debt in the distant future by producing safer surpluses in the near future. This dramatically shortens the horizon over which governments can insure taxpayers. Under natural parameter conditions, the trade-off is steeper in the model with transitory shocks than in the model with permanent shocks. Hence, the results from the literature studying household consumption smoothing in the face of transitory idiosyncratic risk with a risk-free asset (see Chamberlain and Wilson, 2000) do not extend to governments smoothing consumption against transitory aggregate shocks with risk-free debt.

In our economy, the government provides insurance to taxpayers against aggregate shocks. In models developed by Bassetto and Cui (2018); Brunnermeier, Merkel, and Sannikov (2020); Reis (2021); Chien and Wen (2019, 2020); Kocherlakota (2021), government debt plays a key role in allowing agents to self-insure against idiosyncratic risk and by providing liquidity services. These features give rise to a convenience yield, which may contribute a bubble component to the valuation of public debt (Brunnermeier et al., 2020; Reis, 2021).

As our paper shows, it is hard to generate TVC violations when there is enough priced, per-

[3] See e.g. Borovička et al. (2016) who argue that investors receive a large additional risk premium for bearing long-run cash flow risk. Recently, van Binsbergen (2020) questions this view.

[4] These models have mean-reverting processes for productivity and government spending (see Chari, Christiano, and Kehoe, 1994; Debortoli, Nunes, and Yared, 2017; Bhandari, Evans, Golosov, and Sargent, 2017, for examples of calibrated economies with spending and productivity innovations).

manent output risk in the economy to match the equity risk premium in the data. When the government commits to a stationary debt/output policy, the TVC is satisfied as long as the discount rate for a claim to GDP exceeds the growth rate of the economy. Put differently, violations of the TVC for government debt may also result in violations of the TVC for the GDP claim (unlevered stock market).[5] The condition $r > g$, analyzed in models without aggregate risk, does not apply in economies with priced permanent output risk because there is a risk premium adjustment $\gamma\sigma$, where $\sigma$ is output vol and $\gamma$ is the market price of risk. The right condition is $r + \gamma\sigma > g + \sigma^2$.[6] Even when the TVC is satisfied, the government can sustain permanent deficits when $r < g$ , but this is not a free lunch. The government earns an insurance premium in exchange for insuring the bondholders against aggregate risk by increasing taxes in bad times. The larger the debt, the more insurance the government provides to bondholders.

There is a normative literature on optimal taxation which focuses on representative agent economies with distortionary taxes, following Barro (1979)'s seminal work on tax smoothing. The risk-return trade-off we highlight is present in the background, but is not explicitly analyzed. In this class of models, distortionary taxation provides the Ramsey planner with a motive to shift aggregate risk onto bondholders. When the government can issue a complete set of contingent claims, the planner favors shifting the risk entirely from taxpayers onto bond investors (Lucas and Stokey, 1983).[7] In related work, Du, Pflueger, and Schreger (2020) study the choice of currency denomination in the context of optimal debt management and Bigio, Nuno, and Passadore (2019) study optimal debt and maturity management in a model with segmented bond markets.

The normative analysis in this literature also has counterfactual implications when confronted with actual debt policies. In the face of incomplete markets, the Ramsey planner typically wants the government to accumulate assets in the long run (Bhandari et al., 2017). In doing so, the government can escape the trade-off between insuring bondholders and taxpayers. In the short run, the Ramsey planner has a fiscal hedging motive to issue risky debt when taxation is distortionary. The U.S. and most other developed economies chose not to go down this route.[8] This literature has not modeled the seigniorage revenue governments can only earn by manufacturing safe debt. This non-distortionary source of revenue may counteract the fiscal hedging motive for choosing risky debt in the short run and accumulating assets in the long run.

[5] Equilibrium models that generate violations of the transversality condition (TVC) typically imply TVC violations for all long-lived assets including stocks, not just government debt.

[6] Contemporaneous work by van Wijnbergen, Olijslagers, and de Vette (2020) highlights this point in a general equilibrium asset pricing model. In later work, Barro (2020) considers a model with production and disaster risk.

[7] By changing the maturity composition of debt, the government may be able to get closer to the optimal tax policy when markets are incomplete, essentially by making the debt riskier and shifting the risk onto the bondholders (Angeletos, 2002; Buera and Nicolini, 2004; Lustig, Sleet, and Yeltekin, 2008; Farhi, 2010; Arellano and Ramanarayanan, 2012; Bhandari et al., 2017). However, Buera and Nicolini (2004) show that this typically involves implausible bond portfolios. Moreover, many of the models in this literature may be misspecified because they do not have permanent output risk. Long-term government bond risk premia exceed all other risk premia in such models.

[8] As noted, over the past two decades, the beta of U.S. government debt has actually turned negative.

Our analysis contributes to a growing branch of the international economics literature that emphasizes the U.S. role as the world's safe asset supplier in explaining low U.S. rates.[9] Government debt can only earn safe asset convenience yields if the debt is in fact safe and risk-free. As the world's safe asset supplier, the U.S. government may have been able to temporarily escape the trade-off if government debt earns large and counter-cyclical convenience yields. We find that counter-cyclical seigniorage revenue only relaxes the trade-off in the short run. Convenience yields cannot alleviate the trade-off in the long-run because seigniorage revenue is also exposed to long-run output risk. Liu et al. (2019) provide a structural model of convenience yields and fiscal policy. Our paper is the first to analyze how convenience yields change the trade-off between insuring bondholders and taxpayers.

Regarding the riskiness of government debt, the focus in the literature has been mostly on countries' willingness and ability to repay.[10] The trade-off we focus on between bondholder and taxpayer insurance applies regardless of whether a country contemplates default and regardless of which securities the country decides to issue (e.g., maturity choice). Our work is not focused on how the maturity choice affects the riskiness of debt, but rather on how the fundamental cash-flows determine its riskiness. Again, the fact that long-term government debt has a negative beta in recent decades only reinforces our conclusions.

There is a growing literature at the intersection of asset pricing and macro which focuses mostly on the asset pricing effects of fiscal uncertainty (Gomes, Michaelides, and Polkovnichenko, 2012; Croce, Kung, Nguyen, and Schmid, 2012; Pástor and Veronesi, 2013). Finally, Mian, Straub, and Sufi (2021, 2020) examine the distributional implications of government debt issuance, pointing out that the wealthy buy a large share of government (and private) debt. To the extent that the Gini coefficient of government debt holdings exceeds that of taxes, the government is trading off insuring the rich versus insuring the middle class.

The paper is organized as follows. Section 1 derives the general trade-off between insuring bondholders and taxpayers. Section 2 introduces convenience yields as a way of relaxing this tradeoff. Section 3 characterizes the trade-off analytically in a canonical model with permanent shocks to output and marginal utility. Section 4 revisits the tradeoff in a model with transitory shocks to output and marginal utility. The appendix contains the proofs and several auxiliary results.

[9]See work by Gourinchas and Rey (2007); Caballero, Farhi, and Gourinchas (2008); Caballero and Krishnamurthy (2009); Maggiori (2017); He, Krishnamurthy, and Milbradt (2018); Gopinath and Stein (2018); Krishnamurthy and Lustig (2019); Jiang et al. (2021); Jiang, Krishnamurthy, and Lustig (2019a); Liu, Schmid, and Yaron (2019); Koijen and Yogo (2019). In related work, Farhi and Gourio (2018); Eggertsson, Robbins, and Wold (2018); Ball and Mankiw (2021) emphasize the role of increased market power in reconciling higher economic growth rates with low observed rates of return.

[10]See e.g. recent work by Aguiar and Gopinath (2006); Arellano (2008); Aguiar, Amador, Hopenhayn, and Werning (2019); Bocola and Dovis (2019); DeMarzo, He, and Tourre (2019) for examples.

# 1 The General Trade-off between Insuring Bondholders and Taxpayers

We use $T_t$ to denote government revenue, and $G_t$ to denote government spending. $M_t$ denotes the stochastic discount factor. We assume that debt is fairly priced and does not earn any convenience yields. Let $B_t$ denote the market value of outstanding government debt at the beginning of period $t$, before expiring debt is paid off and new debt is issued. The debt can be long-term or short-term, and it can be nominal or real. In fact, it can be any contingent claim. The value of the government debt equals the sum of the expected present values of future tax revenues minus future government spending:

$$B_t = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} - G_{t+j}) \right], \tag{1}$$

provided that there are no arbitrage opportunities in the bond market and a transversality condition holds $\lim_{k\to\infty} \mathbb{E}_t M_{t,t+k} B_{t+k} = 0$. This result does not rely on complete markets, and still applies even when the government can default on the debt. Let $P_t^T = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j} T_{t+j} \right]$ and $P_t^G = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j} G_{t+j} \right]$ denote the present values of the "cum-dividend" tax claim and spending claim. Value additivity then implies that $B_t = P_t^T - P_t^G$. The value of a claim to surpluses equals the value of a claim to taxes minus the value of a claim to spending.[11] For notational convenience, let $D_t = B_t - S_t$ denote the difference between the market value of outstanding government debt and the government surplus. By the government budget condition, $D_t$ is the market value of outstanding government debt at the end of period $t$, after expiring debt is paid off and new debt is issued.

## 1.1 Marginal Benefit of Fiscal Stabilization Policy for Taxpayers

We consider an incomplete markets economy in which households are finitely lived. To keep the analysis simple, we assume in this section that the government borrows at the margin from foreign investors.[12] Hence, we can think of the bondholders as foreign investors henceforth. The analysis in this section connects fiscal stabilization policy and debt valuations to household welfare through the marginal benefit of fiscal stabilization policy.

We use $U^h$ to denote the expected utility of an investor with horizon $h$. In the background, households face idiosyncratic income risk that cannot be perfectly insured away. This risk manifests itself as the idiosyncratic component in the (post-trade) equilibrium consumption process $\{C^i\} = \{C + \epsilon^i\}$, where $\epsilon^i$ denotes the idiosyncratic component of consumption.

[11] See Jiang et al. (2019b) for a proof.

[12] In an infinite horizon, closed economy with a representative agent, the bondholder is also the net transfer recipient. If the transfers are lump-sum, Ricardian equivalence holds, and net transfers are irrelevant. There is no trade-off between insuring bondholders and taxpayers. Any consideration that breaks Ricardian equivalence resuscitates the trade-off: finite horizons, borrowing constraints, etc. Foreign bond holders is just one consideration.

Let $\{T\}$ denote the tax and $\{G\}$ denote the spending process; $\{T, G\}$ can only depend on the history of aggregate shocks. We assume that the government balances the budget in expectation: $\mathbb{E}_0[G - T] = 0$. We focus on fiscal insurance against aggregate risk. We do not consider transfers that are contingent on idiosyncratic shocks. We analyze this case in section B of the Appendix. We consider distortionary taxation in section C of the Appendix.

We consider the set of all taxpayer households who participate in asset markets and who are unconstrained. We can infer their marginal benefit from fiscal policy from the market valuation of tax revenues and spending outlays, even though these households do not equalize their intertemporal marginal rates of substitution.

In the spirit of the cost of business cycles measures (Lucas, 1987), we define the benefit of government transfers, $\Omega^h_{net}(\alpha)$, to agent $i$ with horizon $h$ as follows:

$$U^h((1 + \Omega^{h,i}_{net}(\alpha))C + \epsilon^i) = U^h\left((1-\alpha)\{C\} + (\alpha)\{C + G - T\} + \epsilon^i\right),$$

where $\Omega^{h,i}(\alpha)$ denotes the fraction of systematic consumption this agent is willing to give up in exchange for a new systematic consumption profile that is a weighted average of the old one (with probability $1-\alpha$) and a new one that adds net government transfers $G - T$ (with probability $\alpha$). The total benefit of fiscal stabilization obtains for $\alpha = 1$. In the spirit of Alvarez and Jermann (2004), we study the marginal benefit at $\alpha = 0$, $\Omega^{h'}_{net}(0)$. We use $P^h_0[\{X\}]$ to denote $\mathbb{E}_0[\sum_{t=0}^{\infty} M_{0,t}X_t]$.

**Proposition 1.1.** *For households who participate in asset markets and who receive net transfers $\{G - T\}$, the marginal insurance benefit over horizon $h$ is given by:*

$$\Omega^{h'}_{net}(0) = \frac{P^h_0\left[\{G_t - T_t\}\right]}{P^h_0\left[\{C_t\}\right]} = \frac{P^h_0\left[\{C_t\}\right] + P^h_0\left[\{G_t - T_t\}\right]}{P^h_0\left[\{C_t\}\right]} - 1$$

To compute the marginal benefit of government fiscal intervention from the perspective of this household, we can compare the market valuations of net transfers and the household's consumption stream. $P^h_0\left[\{G_t - T_t\}\right]$ measures how much this household would be willing to pay for these net transfers as an insurance policy. The safer the net transfers compared to her actual consumption, the higher the marginal benefit of the net transfers to the household. A larger marginal benefit of fiscal stabilization policy to households implies a larger risk premium paid by the government:

$$P^h_0\left[\{G_t - T_t\}\right] = \mathbb{E}_0 \sum_{t=1}^{h} M_{0,t}\left[G_t - T_t\right] = \sum_{t=1}^{h} Cov_0(-M_{0,t}, T_t - G_t),$$

where we used $\mathbb{E}_0[G - T] = 0$. There is a one-to-one mapping between the riskiness of the surplus, $S_t = T_t - G_t$, and the marginal (and total) welfare benefits. The riskier the surplus, the larger the welfare benefit to the taxpayer. If the government provides a marginal insurance benefit to taxpayers at horizon $h$, $\Omega^{h'}_{net}(0) > 0$, and runs zero primary surpluses on average, the transfer policy has a positive cost: $P_0^h[\{G_t - T_t\}] > 0$.

Importantly, we do not need a representative agent for this marginal benefit approach to be valid. All we need is a pricing kernel for the marginal investors. The cash flows $\{T_t - G_t\}$ and $\{C_t\}$ have to be in the span of traded assets.[13] Since we are computing the marginal benefit, we are not actually changing their equilibrium consumption. The marginal benefit/cost is a first-order approximation of the total benefit/cost. When utility is increasing, homothetic, and concave, and the new allocation $\{C + G - T\}$ is preferred to the old one $\{C\}$, then the marginal benefit is an upper bound on the total benefit $\Omega_{net}^{h'}(0) \geq \Omega_{net}^{h}(1)$ (see Alvarez and Jermann, 2004). Hence, to get a positive total benefit, we need a positive marginal benefit, and the positive cost result goes through. But a positive marginal benefit does not imply a positive total benefit. The trade-off between insuring taxpayers and bondholders is more stringent when considering the total benefit.

Under this approach, it is straightforward to compute the marginal cost of taxation for a household who pays taxes but does not receive transfers:

$$\Omega_{tax}^{h'}(0) = -\frac{P_0^h[\{T_t\}]}{P_0^h[\{C_t\}]} = \frac{P_0^h[\{C_t\}] - P_0^h[\{T_t\}]}{P_0^h[\{C_t\}]} - 1$$

Similarly, we can take the perspective of a transfer recipient who does not pay taxes, and compute the marginal benefit of transfers as:

$$\Omega_{trans}^{h'}(0) = \frac{P_0^h[\{G_t\}]}{P_0^h[\{C_t\}]} = \frac{P_0^h[\{C_t\}] + P_0^h[\{G_t\}]}{P_0^h[\{C_t\}]} - 1$$

**Infinite Horizon Case** We now let the horizon over which the government provides insurance go to $\infty$. The marginal benefit of net transfers to taxpayers becomes:

$$\Omega_{net}^{\infty'}(0) = \frac{P_0^{\infty}[\{G_t - T_t\}]}{P_0^{\infty}[\{C_t\}]} = \frac{(P_0^G - G_0) - (P_0^T - T_0)}{P_0^C - C_0} = \frac{-D_0}{P_0^C - C_0},$$

where we used that the market value of debt equals the expected present-discounted value of future surpluses, assuming the TVC is satisfied. If the government provides a positive marginal insurance benefit to taxpayers at horizon $\infty$, $\Omega_{net}^{\infty'}(0) > 0$, then the government needs to endow a sovereign wealth fund of size $-D_0 = P_0^{\infty}[\{G_t - T_t\}] > 0$. Conversely, if debt $D_0 \geq 0$, the government can only provide a negative marginal insurance benefit $\Omega_{net}^{h'}(0) < 0$ to taxpayers at horizon $\infty$ with average zero primary surpluses. Providing insurance to risk averse taxpayers through a zero-average net transfer policy has a positive cost to the government of $P_0^G - P_0^T > 0$. If the government has positive debt outstanding, then it cannot provide insurance to the taxpayers when balancing the budget on average. Instead, the government loads aggregate risk onto its

[13] The entire government debt portfolio is a claim to $\{T_t - G_t\}$. Unlevered equity will be akin to a claim to aggregate consumption.

taxpayers.

The government can only run deficits in expectation if it is providing insurance to bondholders and offloading aggregate risk onto taxpayers. This is true regardless of the difference between the risk-free rate and the growth rate in the economy. The more valuable the debt, i.e., the safer the cash flows to bondholders, the smaller the insurance benefit to taxpayers, i.e. the smaller the marginal benefit of fiscal stabilization to taxpayers. Taxpayers always prefer riskier debt. This defines the trade-off between insuring bondholders and taxpayers.[14]

**Gordon Growth Intuition** Let $r^i$ denote the expected log return on an asset $i$ and $g$ the average growth rate of output. To develop more intuition, we use Gordon's growth formula:

$$\Omega^{\infty\prime}_{net}(0) \approx \frac{\frac{G_0}{r^G-g} - \frac{T_0}{r^T-g}}{\frac{C_0}{r^c-g}} \approx \frac{G_0/C_0}{(r^G-g)/(r^C-g)} - \frac{T_0/C_0}{(r^T-g)/(r^C-g)} = \frac{G_0}{C_0}\frac{(r^C-g)(r^T-r^G)}{(r^G-g)(r^T-g)}$$

The second equality says that the safer the spending process (lower $r^G$) and the riskier the tax process (higher $r^T$) relative to the consumption process ($r^C$), the higher the marginal benefit of fiscal stabilization to the taxpayer. The last equality considers the case in which the government runs a zero surplus ($G_0 = T_0$). Even when the government provides zero net transfers, the household benefits when the net transfers arrive in bad states of the world, i.e., when fiscal policy provides insurance against business cycle risk. This is the case if the tax process is riskier than the spending process ($r^T > r^G$). The larger the gap between the discount rate of the tax and the spending claim, the larger the welfare benefit for the transfer recipient. However, the government needs savings to fund this zero transfer insurance in the amount of:

$$-D_0 = \frac{G_0}{r^g-g} - \frac{T_0}{r^T-g} \approx G_0\frac{(r^T-r^G)}{(r^G-g)(r^T-g)} > 0$$

Alternatively, to start this insurance scheme without initial assets, the government needs to run primary surpluses. The higher the benefits from the perspective of the net transfer recipients, the higher the cost of funding for the government.

**Corollary 1.1.** *Assume that the transversality condition (TVC) is satisfied. If $D_0 \geq 0$, the government runs surpluses on average, and the government provides insurance to its taxpayers, then the debt is risky.*

## 1.2 Characterizing the Trade-Off with Return Betas

The results in the previous section motivate why we use the risk properties of the discounted surplus claim as a sufficient statistic for measuring the marginal benefit of taxpayer insurance. To

[14]In section D of the Appendix, we connect this measure to the marginal cost of the business cycle measure developed by Alvarez and Jermann (2004).

examine its risk property, we define the return to this claim and its return beta.

Let $R^D_{t+1}$, $R^T_{t+1}$ and $R^G_{t+1}$ denote the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively:

$$R^D_{t+1} = \frac{B_{t+1}}{B_t - S_t}, \qquad R^T_{t+1} = \frac{P^T_{t+1}}{P^T_t - T_t}, \qquad R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t}.$$

In Jiang et al. (2019b), we show that the government debt portfolio return is the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right] \;=\; \frac{P^T_t - T_t}{D_t}\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] - \frac{P^G_t - G_t}{D_t}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right]. \tag{2}$$

This result only relies on equation (1) and additivity. By rearranging equation (2), we derive the following expression for the risk premium on the tax claim:

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{P^G_t - G_t}{D_t + (P^G_t - G_t)}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] + \frac{D_t}{D_t + (P^G_t - G_t)}\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right]. \tag{3}$$

Governments typically want a counter-cyclical spending claim, i.e. they want to spend more in recessions. On the other hand, they also want a risky tax claim, because they want to reduce the tax burden in recessions. As a result, the tax claim's risk premium $\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right]$ is high and the spending claim's risk premium $\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right]$ is low. When the debt value $D_t$ is positive, the fraction $\frac{P^G_t - G_t}{D_t + (P^G_t - G_t)}$ is between 0 and 1. Then, for equation (3) to hold, it requires a high risk premium $\mathbb{E}_t\left[R^D_{t+1} - R^f_t\right]$ on the government debt portfolio. As the debt risk premium is a measure of the risk premium or insurance premium charged by bondholders, the government's debt portfolio has to be risky.

According to equation (3), the tax revenue claim is the unlevered version of the spending claim, or, equivalently, the spending claim is the levered version of the tax claim. This result is analogous to the Miller-Modigliani relation between the unlevered return on equity (the return on the tax claim) and the levered return on equity (the return on the spending claim).

We define the beta of an asset $i$ as: $\beta^i_t = \frac{-cov_t\left(M_{t+1}, R^i_{t+1}\right)}{var_t(M_{t+1})}$. By the investor's Euler equation, $\beta^i_t \lambda_t$ determines the conditional risk premium of this asset $\mathbb{E}_t\left[R^i_{t+1} - R^f_t\right] = \beta^i_t \cdot \gamma_t$, where the market price of risk is $\gamma_t = R^f_t \cdot var_t(M_{t+1})$. Let $\beta^D_t$, $\beta^T_t$ and $\beta^G_t$ denote the beta of the bond portfolio, the tax claim, and the spending claim, respectively. We assume $\beta^Y_t > 0$, so that the output claim has a positive risk premium. The following proposition characterizes the relationship of these risk exposures.

**Proposition 1.2.** *The beta on the tax claim is a weighted average of the beta of the spending claim and the* *beta of the debt:*

$$\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}\beta_t^G + \frac{D_t}{D_t + (P_t^G - G_t)}\beta_t^D.$$

Governments want to provide insurance to transfer recipients by choosing $\beta_t^G < \beta_t^Y$, but they also want to provide insurance to taxpayers by choosing $\beta_t^T > \beta_t^Y$. However, the following corollary states that $\beta_t^G < \beta_t^Y < \beta_t^T$ is impossible if the government debt is risk-free.

**Corollary 1.2.** *In order for debt to be risk-free ($\beta_t^D = 0$), the beta of the tax claim needs to equal the unlevered beta of the spending claim: $\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t)}\beta_t^G$.*

If the government has a positive amount of risk-free debt $D_t > 0$, there is no scope to insure taxpayers. Instead, the taxpayers provide insurance to the rest of the economy.

Consider the first case in which the spending claim has a positive beta $(\beta_t^G > 0)$. Then, the government engineers risk-free debt by lowering the beta of the tax claim relative to that of the spending claim: $\beta_t^T < \beta_t^G$. A low beta for the tax claim means that tax revenue must fall by less than GDP in a recession. Tax rates must rise in recessions. The more debt there is outstanding, the lower the beta of the tax claim needs to be relative to that of the spending claim. With more debt, the trade-off between insuring bondholders and taxpayers becomes steeper. The restriction on the betas holds true regardless of the specific dynamics of the tax and spending process. In the next sections, we will derive restrictions on the underlying cash flows by committing to particular processes for debt/output and spending/output.

The only way the government can provide insurance to debt holders, while keeping the debt risk-free, is by saving—choosing $D_t < 0$. In other words, the government can only insure taxpayers at the expense of bondholders.

Consider the second case in which the spending claim has a negative beta $(\beta_t^G < 0)$. To ensure risk-free debt, the tax claim must also have a negative beta when $D_t > 0$ ($\beta_t^T < 0$). The taxpayers have large tax payments during recessions; they are insuring the bondholders.

## 1.3 Characterizing the Trade-Off with Cash Flow Betas

Thus far, we have characterized the return betas of the tax and spending claims. We can get further insight on what restrictions risk-free debt imposes on surplus dynamics by studying cash-flow betas for the surplus claim.

**Proposition 1.3.** *When the transversality condition is satisfied, the debt return innovation reflects news about the present discounted value of future government surpluses: $D_t(\mathbb{E}_{t+1} - \mathbb{E}_t)[R_{t+1}^D] = (\mathbb{E}_{t+1} -$*

$\mathbb{E}_t)[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}]$. *When the debt is risk-free, there is no news about future surpluses:* $(\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}] = 0$.

This proposition implies a restriction on the dynamics of future surpluses in response to any shock that arrives at time $t+1$. If a shock raises surpluses in the near-term future, then either the surpluses in the long-term or the discount rates have to adjust.

Define the cash flow beta of cumulative future discounted surpluses as:

$$\beta_t^{S,CF}(h) \equiv -\frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{j=1}^{h} M_{t+1,t+j}S_{t+j}\right)}{D_t var_t(M_{t+1})}.$$

It follows immediately from the previous proposition that if debt is risk-free, $\beta_t^{S,CF}(\infty) = 0$. That is, for the government debt to be risk-free, the cash flow beta of the entire discounted surplus stream must be zero. The following proposition connects the return beta on the debt to the cash-flow beta of cumulative surpluses.

**Proposition 1.4.** *The return beta of debt equals the cash-flow beta of the discounted surpluses over h periods minus the return beta of debt outstanding h periods from now:*

$$\beta_t^D \quad = \quad \beta_t^{S,CF}(h) - \frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+h}D_{t+h}\right)}{D_t var_t(M_{t+1})}.$$

*When debt is risk-free ($\beta_t^D = 0$), then the cash flow beta is determined by the return beta of debt outstanding h periods from now:*

$$\beta_t^{S,CF}(h) \quad = \quad \frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+h}D_{t+h}\right)}{D_t var_t(M_{t+1})}.$$

*When debt has negative risk premium ($\beta_t^D < 0$), then the cash flow beta is smaller than the return beta of debt outstanding h periods from now:*

$$\beta_t^{S,CF}(h) \quad \leq \quad \frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+h}D_{t+h}\right)}{D_t var_t(M_{t+1})}.$$

As the horizon tends to $\infty$, the debt issuance covariance tends to zero, and $\beta_t^D \to \beta_t^{S,CF}(\infty)$.

As long as the debt is risk-free, the risk properties of the government surpluses over a finite horizon $h$ are completely determined by the riskiness of the debt issuance process at time $t+h$. This implies that the cash-flow beta of the surplus process does not depend on the spending and tax revenue dynamics during those $h$ periods. The only source of state-contingency is the debt issuance process itself. When hit by a bad shock, the government can respond by issuing more debt $h$ period from now.

$\beta_t^{S,CF}(h)$ is our summary statistic for how much insurance the government provides to taxpayers with at horizon $h$. Over short horizons, it can insure taxpayers and produce a risky surplus process with high $\beta_t^{S,CF}(h)$, by choosing a counter-cyclical debt issuance process with $cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+h}D_{t+h}\right)$. When the debt is risk-free, there can be no insurance over long horizons. When the government debt has a negative risk premium, this constraints the government even more: the debt issuance beta puts an upper bound on the cash-flow beta of the surplus. We will evaluate this constraint quantitatively by letting the government adopt a specific debt policy.

# 2 Relaxing the Trade-off with Convenience Yields

Some governments are endowed with the ability to issue safe government debt at prices that exceed their fair market value. The resulting "convenience yields" relax the trade-off between insuring bondholders and taxpayers. Typically, the debt of such government serves the role of a special, safe asset for domestic or foreign investors. U.S. Treasuries currently fill the role of the world's safe asset. In order to collect convenience yields, the government needs to manufacture safe debt. This justifies our emphasis on the $\beta^D = 0$ case throughout this paper (or $\beta^D < 0$, which makes all results stronger). The convenience yield $\kappa_t$ is defined as a wedge in the investors' Euler equation for government bonds: $\mathbb{E}_t\left[M_{t,t+1}R_t^D\right] = \exp(-\kappa_t)$. Krishnamurthy and Vissing-Jorgensen (2012) estimate convenience yields on U.S. Treasuries of around 75 bps per year. Using the deviations from CIP in Treasury markets, Jiang et al. (2021, 2018); Koijen and Yogo (2019) estimate convenience yields that foreign investors derive from their holdings of dollar safe assets; these estimates exceed 200 bps.

## 2.1 The Trade-off With Return Betas

Let $K_{t+j} = (1 - e^{-\kappa_{t+j}})D_{t+j}$ be the amount of interest the government does not need to pay in period $t+j$ thanks to the convenience yield. The current value of government debt reflects the present value of all convenience yields earned on future debt. We refer to this value as the Treasury's seigniorage revenue: $P_t^K = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(1 - e^{-\kappa_{t+j}})D_{t+j}\right]$. Jiang et al. (2019b) show that the value of the government debt equals the sum of the present value of future tax revenues plus future seigniorage revenues minus future government spending:

$$B_t = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}(T_{t+j} + (1 - e^{-\kappa_{t+j}})D_{t+j} - G_{t+j})\right] = P_t^T + P_t^K - P_t^G,$$

provided that a transversality condition holds.

Extending the Modigliani-Miller approach to the world with convenience yields, government debt is equivalent to a portfolio that goes long in the tax claim and the seigniorage claim and short in the spending claim. The government debt risk premium becomes:

$$\mathbb{E}_t\left[R^D_{t+1}-R^f_t\right] = \frac{P^T_t-T_t}{B_t-S_t}\mathbb{E}_t\left[R^T_{t+1}-R^f_t\right]+\frac{P^K_t-T_t}{B_t-S_t}\mathbb{E}_t\left[R^K_{t+1}-R^f_t\right]-\frac{P^G_t-G_t}{B_t-S_t}\mathbb{E}_t\left[R^G_{t+1}-R^f_t\right],$$

where $R^D_{t+1}$, $R^T_{t+1}$,$R^K_{t+1}$ and $R^G_{t+1}$ are the holding period returns on the bond portfolio, the tax claim, the seigniorage claim, and the spending claim, respectively.

We take government spending process, and the debt return process as given, and explore the implications for the properties of the tax claim. Next, we impose risk-free debt, because only safe debt earns convenience yields.

**Proposition 2.1.** *In the absence of arbitrage opportunities, if the TVC holds and the debt is risk-free ($\beta^D = 0$), then the expected excess return on the tax claim is the unlevered expected excess return on the spending claim and the seigniorage claim:*

$$\mathbb{E}_t\left[R^T_{t+1}-R^f_t\right] = \frac{(P^G_t-G_t)\mathbb{E}_t\left[R^G_{t+1}-R^f_t\right]-(P^K_t-K_t)\mathbb{E}_t\left[R^K_{t+1}-R^f_t\right]}{D_t+(P^G_t-G_t)-(P^K_t-K_t)},$$

*with the beta of the tax claim given by* $\beta^T_t = \frac{(P^G_t-G_t)\beta^G_t-(P^K_t-K_t)\beta^K_t}{D_t+(P^G_t-G_t)-(P^K_t-K_t)}$.

Consider the special case where the convenience yield seigniorage process has a zero beta ($\beta^K = 0$); seigniorage revenues are a-cyclical. Then the implied beta of the tax revenue process exceeds the beta without seigniorage revenue because $P^K_t - K_t > 0$:

$$\beta^T_t = \frac{P^G_t-G_t}{D_t+(P^G_t-G_t)-(P^K_t-K_t)}\beta^G_t > \frac{P^G_t-G_t}{D_t+(P^G_t-G_t)}\beta^G_t,$$

A higher tax beta means that more insurance to taxpayers is now possible. If the seigniorage revenue is sufficiently counter-cyclical ($\beta^K < 0$), then the proposition shows that $\beta^T_t$ is higher still so that even more taxpayer insurance is possible.

## 2.2 The Trade-off With Cash-Flow Betas

We now explore how the trade-off over finite horizons is affected by the presence of convenience yields. We do so under the assumption that seigniorage revenue from convenience is proportional to the debt outstanding.

**Assumption 1.** *The convenience yield $\kappa$ is constant.*

We define the cash flow beta of future discounted seigniorage revenue as:

$$\beta_t^{K,CF}(h) \equiv \frac{-(1-e^{-\kappa}) \cdot cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{h} M_{t+1,t+j} D_{t+j}\right)}{D_t var_t(M_{t+1})}.$$

**Proposition 2.2.** *The return beta of debt equals the cash-flow beta of the discounted surpluses and seigniorage revenue over h periods minus the beta of debt outstanding h periods from now:*

$$\beta_t^D \quad = \quad \beta_t^{S,CF}(h) + \beta_t^{K,CF}(h) - \frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) M_{t+1,t+h} D_{t+h}\right)}{D_t var_t(M_{t+1})}.$$

*When debt is risk-free ($\beta_t^D = 0$), then the cash flow beta is determined by the seigniorage beta and beta of the debt outstanding h periods from now:*

$$\beta_t^{S,CF}(h) \quad = \quad \frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) M_{t+1,t+h} D_{t+h}\right)}{D_t var_t(M_{t+1})} - \beta_t^{K,CF}(h).$$

*When debt has negative risk premium ($\beta_t^D < 0$), then the cash flow beta is smaller than the the seigniorage beta and beta of debt outstanding h periods from now:*

$$\beta_t^{S,CF}(h) \quad \leq \quad \frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) M_{t+1,t+h} D_{t+h}\right)}{D_t var_t(M_{t+1})} - \beta_t^{K,CF}(h).$$

To keep the debt risk-free ($\beta_t^D = 0$) while delivering a risky surplus over short horizons ($\beta_t^{S,CF}(h) > 0$), the government can resort to issuing more debt when marginal utility growth is high ($cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) M_{t+1,t+h} D_{t+h}\right) > 0$). When it earns seignorage revenue, this debt issuance produces a safe seigniorage revenue stream over short horizons ($\beta_t^{K,CF}(h) < 0$), increasing $\beta_t^{S,CF}(h) > 0$ and expanding taxpayer insurance possibilities. If the convenience yields are large enough, this can quantitatively alter the trade-off. We explore this possibility below. However, over long horizons, $\beta_t^{K,CF}(h)$ turns positive when debt is co-integrated with output.

## 2.3 Marginal Benefit of Fiscal Stabilization Policy Revisited

Note that $D_0 = (P_0^T - T_0) + (P_0^K - K_0) - (P_0^G - G_0)$. We now let the horizon over which the government provides insurance go to $\infty$. The marginal benefit of net transfers to taxpayers becomes:

$$\Omega_{net}^{\infty'}(0) = \frac{P_0^{\infty}\left[\{G_t - T_t\}\right]}{P_0^{\infty}\left[\{C_t\}\right]} = \frac{(P_0^G - G_0) - (P_0^T - T_0)}{P_0^C - C_0} = \frac{(P_0^K - K_0) - D_0}{P_0^C - C_0},$$

where we used that the market value of debt–inclusive of seigniorage revenue–equals the expected present-discounted value of future surpluses, assuming the TVC is satisfied.

**Corollary 2.1.** *If debt $D_0 \geq 0$, the government can provide a positive marginal insurance benefit $\Omega_{net}^{h'}(0) >$*

*0 to taxpayers at horizon $\infty$ with on average zero primary surpluses, provided that the value of seignorage exceeds the value of debt: $(P_0^K - K_0) > D_0$.*

Providing insurance to risk averse taxpayers through a zero-average net transfer policy no longer necessarily has a positive cost to the government, because the Treasury now collects seignorage revenue. Some insurance provision to taxpayers may be possible with positive debt outstanding and zero-average surpluses, depending on the value of the seigniorage revenue claim. This is true regardless of the risk-free rate and the growth rate in the economy.

# 3 The Trade-off in a Benchmark Economy with Permanent Risk

We characterize the trade-off between insuring debtholders and taxpayers in a canonical macro-finance model in the tradition of Lucas (1978). We reverse-engineer the revenue process $T$ that keeps the debt risk-free. We do so under simple spending and debt policies at first and more complex policies in the next step.

## 3.1 Setup

To derive closed form-solutions, we adopt an exogenous stochastic discount factor (SDF) with plausible asset pricing implications. This SDF prices payoffs from the perspective of domestic and foreign investors buying government debt.

First, we consider an economy with permanent output shocks and a homoscedastic SDF:

**Assumption 2.** *(a) Let $Y_t$ and $y_t = \log Y_t$ denote output and its log. All output shocks are i.i.d. and permanent: $y_{t+1} = \mu + y_t + \sigma\varepsilon_{t+1}$, where $\varepsilon_{t+1}$ denotes the innovation to output growth that is i.i.d. normally distributed with mean zero and standard deviation one. (b) The log SDF is given by: $m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}$. (c) The government only issues one-period real risk-free debt.*

Note that the one-period risk-free rate in this model is constant and equal to $\rho$.

## 3.2 Characterizing the Trade-off with Constant Debt-Output

To build intuition for the general trade-off between insurance of bondholders and taxpayers, we start by considering the simplest case of constant spending/output and debt/output ratio policies.

**Assumption 3.** *(a) The government commits to a constant spending/output ratio $x = G_t/Y_t$.*
*(b) The government commits to a constant debt/output ratio $d = D_t/Y_t$.*

Under Assumption 3, the government budget constraint implies a counter-cyclical process for tax revenue-to-GDP (the tax rate):

$$\frac{T_t}{Y_t} = \frac{G_t}{Y_t} - \frac{D_t}{Y_t} + R_{t-1}^f \frac{D_{t-1}}{Y_t} = x - d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right).$$

To perfectly insure the bondholders by keeping the debt risk-free, the government must make the tax revenue claim counter-cyclical: $\partial(T/Y)\ \partial\varepsilon < 0$. When the growth rate of output is low ($\varepsilon < 0$), tax revenue needs to increase as a fraction of GDP. Tax rates must rise in recessions. The magnitude of the counter-cyclical exposure is increasing in the debt-to-GDP ratio $d$.

Similarly, the primary surplus/output ratio is counter-cyclical:

$$s_t = \frac{S_t}{Y_t} = \frac{T_t - G_t}{Y_t} = -d\left(1 - \exp\left\{-(\mu - \rho + \sigma\varepsilon_t)\right\}\right). \tag{4}$$

We have that $\partial s_t / \partial \varepsilon_t < 0$. When the unconditional growth rate of output exceeds the risk-free rate ($\mu > \rho$), the government runs a primary deficit on average. But when shocks are negative enough ($\mu - \rho < -\sigma\varepsilon$), the government must run a primary surplus.

This simple model places tight restrictions on the persistence of surpluses. The conditional auto-covariance of the surplus/output ratio is zero: $cov_t(s_t, s_{t-1}) = 0$. The government cannot run persistent deficits. When $\sigma \to 0$, the government always runs deficits. But $\mu > \rho$ now implies a violation of the TVC, as we show below. This result is more general. With risk-free debt, the autocorrelation of surpluses tends to zero as the persistence of the debt/output ratio tends to one.

The restrictions on the surplus and tax processes described above were independent on the SDF model. Next, we turn to valuing the debt as the expected present-discounted value of future surpluses.

**Proposition 3.1.** *Under Assumptions 2 and 3, if the transversality condition holds and the primary surplus satisfies (4), the government debt value is the sum of the values of the surplus strips:*

$$D_t \quad = \quad \mathbb{E}_t\left[\sum_{k=1}^{\infty} M_{t,t+k} S_{t+k}\right] = dY_t.$$

This proposition confirms that the (ex-dividend) value of outstanding debt in period $t$ is indeed a constant fraction of output. The proof solves for the price of a claim to a single future surplus realization (a surplus strip), and adding up the surplus strip prices at all horizons. The result implies that there is no news about the present discounted value of future surpluses since output is already known at time $t$.

Note that in this equation, the government surpluses are not discounted at the risk-free rate even though the debt is risk-free. To see why, consider the valuation equation for debt as a function of surplus/output ratios:

$$D_t = \mathbb{E}_t\left[\sum_{j=0}^{T} M_{t,t+j} Y_{t+j} s_{t+j}\right] + \mathbb{E}_t\left[M_{t,t+T} Y_{t+T} \frac{D_{t+T}}{Y_{t+T}}\right].$$

Under Assumption 3, the debt/output ratio $\frac{D_{t+T}}{Y_{t+T}} = d$ in the second term is constant. The correct TVC for government debt in this model is given by:

$$\lim_{T\to\infty} \mathbb{E}_t \left[M_{t,t+T} D_{t+T}\right] = \lim_{T\to\infty} \exp\left\{T(\mu - \rho + \frac{1}{2}\sigma^2 - \gamma\sigma)\right\} dY_t. \tag{5}$$

This TVC is satisfied if and only if $-\rho + \mu + \frac{1}{2}\sigma^2 - \gamma\sigma < 0$. The textbook condition $\rho < \mu$ is neither necessary nor sufficient for a TVC violation. A necessary and sufficient condition is that there is enough permanent, priced risk in output: $\gamma\sigma > \mu - \rho + \frac{1}{2}\sigma^2$. The output risk premium (unlevered equity risk premium) must be high enough. This ensures that this term $\mathbb{E}_t \left[M_{t,t+T} Y_{t+T}\right] \to 0$ as $T \to \infty$.

This TVC highlights the importance of modeling the dynamics of outstanding debt $D_{t+T}$. While the debt at $t$ is risk-free under Assumption 2, meaning that its value does not change in response to news revealed between $t$ and $t+1$, the value of debt outstanding at some future date $t+T$, $D_{t+T}$, is a stochastic variable, even when the debt-to-output ratio is a constant. In fact, as long as the debt quantity and the output are co-integrated, the debt inherits the long-run risk properties of otuput. To facilitate our discussion of the risk property of the debt quantity, we define a *debt strip* as a claim that pays off exactly $D_{t+T}$ dollars at time $t+T$. As evident in Eq. (5), the risk premium associated with the debt strip is crucial in determining the TVC.

To summarize, if GDP growth has a permanent component, which modern macro and econometrics recognizes to be the case, then the surplus process in levels $S_t$ inherits that permanent component from $Y_t$. Surpluses have long-run risk. Therefore, even when the entire debt portfolio is risk-free, in the sense that there is no news about current or future surpluses, the risk-free rate is not the right discount rate for surpluses in the presence of permanent output risk.

Note that $\rho < \mu$ implies a violation of TVC only as $\sigma \to 0$. In an economy with permanent GDP risk, comparing the risk-free rate to the average growth rate of the economy, as in Blanchard (2019), sheds no light on the fiscal cost of deficits.[15] In general, the output risk premium matters even when debt is risk-free. The risk-free rate is not the correct discount rate for surpluses even when the debt is risk-free, in the presence of permanent output shocks.

Next, we turn to the main result characterizing the expected return and beta of the tax claim.

**Proposition 3.2.** *(a) The ex-dividend values of the spending and revenue claims are given by:*

$$P_t^G - G_t = x\frac{\xi_1}{1-\xi_1} Y_t, P_t^T - T_t = \left(d + x\frac{\xi_1}{1-\xi_1}\right) Y_t,$$

*with* $\xi_1 = \exp\left\{-\rho - \gamma\sigma + \mu + \frac{1}{2}\sigma^2\right\}$*. (b) The risk premia and betas on the tax claim and the spending*

[15] See Bohn (1995) for an early reference on why discounting at the risk-free may fail.

*claim satisfy:*

$$\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right], \beta^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\beta^G < \beta^G. \tag{6}$$

The constant $\xi_1$ is the price/dividend ratio of a one-period output strip, a claim to GDP next year. The expected return on this output strip is given by $\mathbb{E}_t\left[R^Y_{t+1}\right] = \frac{\exp(\mu+0.5\sigma^2)}{\exp(-\rho-\gamma\sigma+\mu+0.5\sigma^2)} = \exp(\rho+\gamma\sigma)$. Hence, the (log of the multiplicative) output risk premium is constant and equal to $\gamma\sigma$. Since spending is a constant fraction of output, the risk premium on the spending claim equals that of the output claim: $\mathbb{E}[R^G - R^f] = \mathbb{E}[R^Y - R^f]$. The beta of the spending claim equals the beta of the output claim: $\beta^G = \beta^Y > 0$. In section E of the Appendix, we explicitly solve for these risk premia.

The investor in government debt is long in a tax revenue claim and short in a spending claim. To make the debt risk-free, as long as the debt/output ratio $d$ is positive, we need to render the government tax revenue process safer than the spending process. A positive $d$ implies the fraction $\frac{x\frac{\xi_1}{1-\xi_1}}{d+x\frac{\xi_1}{1-\xi_1}}$ is between 0 and 1, which requires the return on the tax claim to be less risky than the return on the output claim: $0 < \beta^T < \beta^Y$. When output falls, tax revenues must fall by less. The tax rate increases. In other words, there is no scope to insure taxpayers. As the debt/output ratio $d$ increases, the government needs to make the tax revenue increasingly safe. The tax claim is really a portfolio of a claim to government spending and risk-free debt. The larger the debt/output ratio $d$, the safer the tax claim needs to be. As the debt/output ratio approaches infinity, the beta of the tax claim tends to 0.

### 3.3 Quantifying the Trade-off with Constant Debt-Output

Panel A of Table 1 proposes a calibration of the model that matches basic features of post-war U.S. data. We set $\gamma$ to 1. This parameter measures the maximum Sharpe ratio in the economy. A long asset pricing literature suggests that this is a reasonable value given high average excess returns on a broad set of risky assets. The standard deviation of annual output growth is set to $\sigma = 5\%$. The growth rate of real GDP is set to its observed value: $\mu = 3.1\%$. The real risk-free rate $\rho$ is set to 2%. Spending accounts for 10% of GDP in post-war data: $x = 0.10$.

Note that this calibration features a risk-free rate below the growth rate of output. However, per our discussion above, the TVC is satisfied because $-\rho + \mu + \frac{1}{2}\sigma^2 - \gamma\sigma = \log(\xi_1) = -0.0418 < 0$. The government cannot simply roll over the debt. The surpluses need to satisfy tight restrictions.

Figure 1 plots the risk premia on the tax and the spending claim as we vary the debt/output ratio $d$. The risk premium on the spending claim is 5% per annum. This is also the output risk

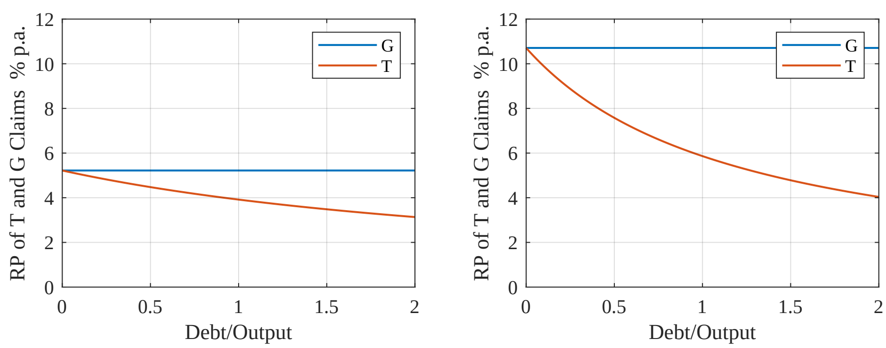
Table 1: Benchmark Calibration for U.S.

| Panel A: Preferences and Output Dynamics | | |
|---|---|---|
| $\gamma$ | 1 | maximum annual Sharpe ratio |
| $\rho$ | 2.0% | real risk-free rate |
| $\mu$ | 3.1% | mean of growth rate of output |
| $\sigma$ | 5.0% | std. of growth rate of output |
| **Panel B: Debt/Output Ratio Dynamics** | | |
| $\lambda$ | $1.94 \times \sigma$ | sensitivity of debt/output to output innovations |
| $d = \exp\{\phi_0/(1-\phi_1-\phi_2)\}$ | 0.43 | mean of debt/output |
| $\phi_1$ | 1.40 | AR(1) coeff of debt/output |
| $\phi_2$ | -0.48 | AR(2) coeff of debt/output |
| **Panel C: Government Spending/Output Ratio Dynamics** | | |
| $\beta^g$ | $1.53 \times \sigma$ | sensitivity of spending/output to output innovations |
| $\varphi_1^g$ | 0.88 | AR(1) coeff of spending/output |
| $x = \exp\left\{\varphi_0^g/(1-\varphi_1^g)\right\}$ | 0.10 | mean of govt. spending/output |

premium, which we can think of as an unlevered equity premium. By Corollary 3.2, the risk premium on the tax claim is given by (6). The risk premium on the tax claim is 5% when $d = 0$. It falls to 4% when $d = 1$, and close to 3% when $d = 2$. As the government becomes more levered, the tax claims needs to become safer for debt to remain risk-free. The scope for taxpayer insurance disappears. This trade-off steepens when we increase the maximum Sharpe ratio $\gamma$ from 1 to 2. When $\gamma = 2$, the risk premium on the spending claim is 10% per annum. The risk premium on the tax claim falls to 6% when $d = 1$ and close to 4% when $d = 2$.

Figure 1: Risk Premium of T and G Claims with $\gamma = 1$ or 2

The figure plots the implied risk premium of the T and G claims when the debt/output ratio and spending/output ratio are constant. The figure plots two values for the maximum Sharpe ratio $\gamma$ of 1 (left panel) and $\gamma$ of 2 (right panel). The other parameters are given in Table 1.



### 3.4 Characterizing the Trade-off with State-Contingent Debt-Output

The previous section showed that there is no scope for insuring taxpayers at any horizon in the presence of permanent output shocks when the debt/output ratio is constant. Next, we assume that the government commits to a state-contingent policy for the debt/output ratio which features persistence and counter-cyclicality. We show that this enables limited opportunities for the government to insure taxpayers over short horizons. The debt policies we analyze are fitted to the data and hence empirically plausible. However, as Appendix C shows, a Ramsey planner in an incomplete markets environment would also choose a mean-reverting and counter-cyclical process for debt, albeit with a negative long-run debt target.

We allow the government to vary the debt/output ratio counter-cyclically. We consider a flexible class of $AR(p)$ processes for the debt/output ratio.

**Assumption 4.** *The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ given by:*

$$\log d_t = \sum_{p=1}^{P} \phi_p \log d_{t-p} + \phi_0 - \lambda \varepsilon_t - \frac{1}{2}\lambda^2,$$

*where $\lambda > 0$ so that the debt-output ratio increases in response to a negative output shock $\varepsilon_t$.*

The results in Section 1 still apply and are a straightforward generalization of the results from the simple benchmark model of Section 3. The value of the spending is unchanged and the value of the tax claim now depends on the time-varying debt/output ratio $d_t$:

$$P_t^G - G_t = x\frac{\xi_1}{1-\xi_1}Y_t, \qquad P_t^T - T_t = \left(d_t + x\frac{\xi_1}{1-\xi_1}\right)Y_t.$$

The tax claim's conditional beta satisfies: $\beta_t^T = \frac{x\frac{\xi_1}{1-\xi_1}}{d_t + x\frac{\xi_1}{1-\xi_1}}\beta_t^G$.

Can the government systematically issue more risk-free debt, instead of raising taxes, when the economy is hit by a permanent, adverse shock, in order to break the restriction on insurance of taxpayers? We consider two special cases for the debt/output dynamics.

**Case 1: AR(1)** Assume that the debt/output ratio evolves according to an $AR(1)$-process:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2.$$

There are two sub-cases. First, when $0 < \phi_1 < 1$, the debt/output process is stationary. Second, when $\phi_1 = 1$ and $\phi_0 = 0$, the debt/output process is a martingale (non-stationary). In both cases, a positive $\lambda$ means that the debt/output ratio increases when the shock $\varepsilon_t$ is negative, implying a counter-cyclical debt policy. The Ramsey planner prefers tax rates that are smooth, and will resort to choosing counter-cyclical debt policy: $\lambda > 0$.[16] First, we need to make sure the transversality (TVC) is satisfied. How persistent can debt be without violating TVC?

**Proposition 3.3.** *Under Assumptions 2 and 4 with the maximal lag $P = 1$, (a) when $0 < \phi_1 < 1$, the TVC condition is satisfied if and only if $\log(\xi_1) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. (b) When $\phi_1 = 1$ and $\phi_0 = 0$, then the TVC condition is satisfied if and only if: $\log(\xi_1) + \lambda(\gamma - \sigma) = -\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) + \lambda(\gamma - \sigma) < 0$.*

For the case of $0 < \phi_1 < 1$, the TVC is satisfied whenever the price-dividend ratio of a claim to next period's output is less than one. That is, when investors are willing to pay less than $Y_t$ today for a claim to $Y_{t+1}$. This requires the discount rate to exceed the growth rate of GDP (modulo a Jensen adjustment). This condition can be satisfied even when $\rho < \mu$, as long as the risk premium $\gamma\sigma$ is large enough.[17] Even though $\rho < \mu$, the TVC is satisfied provided that the risk premium is larger than the gap between the growth rate and the risk-free rate: $\sigma\gamma > \mu - \rho + \frac{1}{2}\sigma^2$. In this case, the steady-state surplus is negative: $d = -\frac{s}{(1-\exp(\rho-\mu))}$. The government can run steady-state deficits, even though the TVC is satisfied, because the government earns an insurance risk premium in exchange for insuring bondholders against aggregate risk by raising taxes in bad times. So, this is not a free lunch: the larger the steady-state deficits, the larger the increase in taxes required in bad times, i.e. the more insurance is being provided.

For the random walk case in which $\phi_1 = 1$, the same condition ensures that the TVC is satisfied when the government does not pursue counter-cyclical stabilization ($\lambda = 0$). If the government does pursue counter-cyclical stabilization ($\lambda > 0$), then the TVC is only satisfied if:

$$\gamma\sigma - \lambda(\gamma - \sigma) > -\rho + \mu + \frac{1}{2}\sigma^2 \Leftrightarrow \lambda < \frac{\rho + \gamma\sigma - \mu - \frac{1}{2}\sigma^2}{\gamma - \sigma}.$$

The left-hand side of the first inequality is now lower than before when the Sharpe ratio of the economy exceeds the volatility of output ($\gamma > \sigma$). When debt issuance is sufficiently counter-cyclical, $\lambda > \sigma$, the expression on the left-hand side is decreasing in the economy's maximum Sharpe ratio $\gamma$. For high enough $\gamma$, the TVC is violated. Intuitively, when investors are risk averse enough, the insurance provided by the counter-cyclical debt issuance policy is so valuable that the price of a claim to the debt outstanding in the distant future $d_{t+T}Y_{t+T}$ fails to converge to zero. This claim is a terrific hedge. This is the first important insight contributed by asset pricing theory. If output is subject to permanent, priced risk and we want to rule out arbitrage opportunities then there have to be limits to the government's ability to pursue counter-cyclical debt issuance. This bound on $\lambda$ is shown in the second inequality. When the government exceeds this bound, it has granted itself an arbitrage opportunity.

[16] Recall that tax rates are countercyclical if the debt/output ratio is constant and debt is risk-free.

[17] We formally derive the expression for the risk premium in section E of the Appendix.

**Case 2: AR(2)** As we show below, a better description of the debt/output ratio is the data is an AR(2) process:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2. \tag{7}$$

When the roots of the characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle, the debt/output process is mean-reverting. The result of part (a) of Proposition 3.3 applies. If one or both roots are smaller than one, the result in part (b) of Proposition 3.3 applies.

**Response of the Surplus to Adverse Shock** We can compute the impulse-response functions (IRF) of the surpluses with respect to an output shock in closed form when the government issues risk-free debt. These moments are particularly powerful because they do not depend on the properties of the SDF. We start from the expression for the surplus/output ratio in period $t + j$ for $j \geq 1$: $s_{t+j} = \frac{S_{t+j}}{Y_{t+j}} = d_{t+j-1} \exp(\rho - \mu - \sigma \varepsilon_{t+j}) - d_{t+j}$, which follows directly from the government's static budget constraint. If we assume that the risk-free rate equals the growth rate of the economy ($\mu = \rho$), we obtain closed-form expression for the IRF of the surplus with respect to an output shock. Specifically, the IRF is evaluated at $\varepsilon_\tau = 0$ for all $\tau$, and hence $d_t = \exp(\frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) = \exp(\bar{d})$.

**Proposition 3.4.** *If Assumptions 2 and 4 hold, the TVC is satisfied, and $\rho = \mu$, when the debt/output ratio follows an AR(3) process, the IRF of the surplus output ratio is given by:*

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= -\sigma \exp(\bar{d}) + \lambda \exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 1, \\
&= -\lambda \exp(\bar{d}) + \lambda \phi_1 \exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 2, \\
&= -\lambda \psi_1 \exp(\bar{d}) + \lambda(\phi_1 \psi_1 + \phi_2) \exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 3, \\
&= -\lambda \psi_{j-2} \exp \bar{d} + \lambda \psi_{j-1} \exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j > 3.
\end{aligned}$$

*where $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2} + \phi_3 \psi_{j-3}, j > 3; \psi_3 = \phi_3 + \phi_2 \psi_1 + \phi_1 \psi_2; \psi_2 = \phi_2 + \phi_1 \psi_1; \psi_1 = \phi_1$.*

This result can easily be generalized to any AR process, with $\psi_j$ denoting the autocorrelation coefficient.

For an $AR(1)$ ($\phi_2 = \phi_3 = 0$), the initial response of the surplus is positive in the empirically relevant case where $\lambda > \sigma$. That is, a negative shock to output is countered with a large enough government debt issuance that the surplus in the initial period can be negative without jeopardizing the risk-free nature of the debt. However, the deficit must turn to a surplus starting in the second year since $\phi_1 < 1$. Surpluses remain in the years that follow. As the persistence of the debt/output process $\phi_1$ increases, the response of the surplus/output ratio converges to zero in year 2 and beyond.

For an $AR(2)$ ($\phi_3 = 0$), by choosing $\phi_1 > 1$, the government can run a deficit in the year of the shock (year 1) as well as in the following year. In year 3, the IRF equals $\lambda(\psi_2 - \psi_1) = \lambda(\phi_2 + \phi_1(\phi_1 - 1))$. This expression can be positive or negative depending on parameter values but is smaller than the response in year 2. In other words, the government's ability to run a third year of deficits in response to the negative output shock is either limited or gone. The IRF flips sign in year 3 or 4. The government must revert to running surpluses as the ACFs decline: $\psi_{j-1} < \psi_{j-2}$.

With higher-order $AR(p)$ models for debt/output, the government is able to run deficits for longer before a reversal. For example, for an AR(3), there is an additional year of deficits possible while keeping debt risk-free. These deficits must be made up by several years of surpluses afterwards. The surplus dynamics can display more pronounced hump-shaped IRFs. However, as shown below, there is no empirical support for higher-order $AR(p)$ dynamics (i.e., $p > 2$) in the observed US debt/output process.

### 3.5 Quantifying the Trade-Off with Counter-cyclical Debt/Output

The calibration of the debt/output dynamics is reported in Panel B of Table 1. When we fit an AR(1), we use an estimate of $\phi_1 = 0.985$. An AR(2) with the estimated coefficients $\phi_1 = 1.4$ and $\phi_2 = -0.48$ provides the best fit. We compare the dynamics of the debt/output ratio in model and data in section F of the Appendix. There is no evidence of higher-order dynamics.

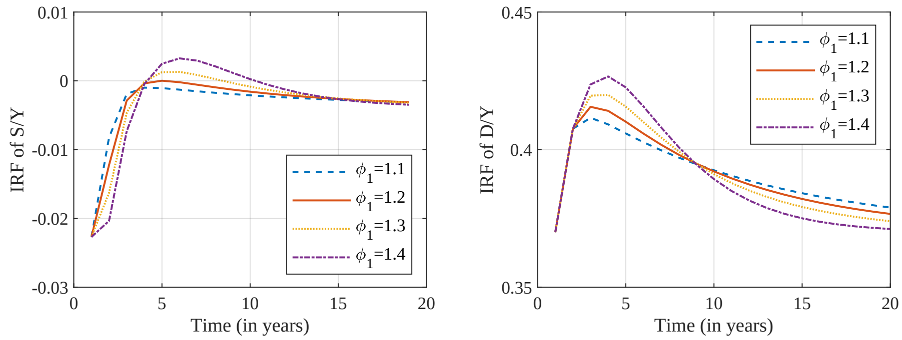
We choose $\phi_0$ to match the unconditional mean of the debt/output ratio of 0.43. We set $\lambda = 1.953 \times \sigma$ equal to match the slope coefficient in a regression of the debt/output ratio innovations on GDP growth in the post-war U.S. sample. A one percentage point increase in GDP growth lowers the debt/output ratio by 1.95 percentage points.

Given our values of $\sigma = 0.05$, $\gamma = 1$, and $\lambda = 1.96\sigma = 0.098$, we have $\gamma\sigma - \lambda(\gamma - \sigma) < 0$. If debt/output were non-stationary (have roots inside the unit circle), then this much counter-cyclicality would result in a violation of the TVC condition. The coefficient $\lambda$ would need to remain below $0.85\sigma$, which is only half of its empirical value, for the TVC to be satisfied in this case. Once we exceed this upper bound, the value of outstanding debt explodes. To be clear, the data suggest that the debt/output ratio is stationary, in which case the TVC is satisfied irrespective of the value for $\lambda$. Indeed, the parameter restriction in part (a) of Proposition 3.3 is satisfied. This is the case despite the risk-free interest rate being below the growth rate of output, because the risk premium $\gamma\sigma$ is large enough.

Figure 2 plots the IRF when debt/output follows an AR(2), our preferred empirical specification. We vary $\phi_1$ from 1.1 to 1.4 and choose $\phi_2$ to match the first-order autocorrelation of debt/output. With $\phi_1 = 1.1$, the IRF looks similar to the $AR(1)$ case with $\phi_1$ close to 1. However, with $\phi_1 = 1.4$ and $\phi_2 = -0.48$, the point estimates from the data, the IRF for the surplus/output ratio displays a hump-shaped pattern. Consistent with the results in Proposition 3.4, a state-

Figure 2: IRF of Surplus/Output and Debt/Output in Model

The figure plots the IRF of $S/Y$ and $D/Y$ for an $AR(2)$. We vary $\phi_1$ and choose $\phi_2$ to match the first-order autocorrelation. The other parameters are given in Table 1.




contingent and persistent debt issuance policy enables the government to delay the fiscal adjustment. The deficit/output ratio in the year of the shock is followed by an even larger deficit in year 2. However, the deficit must shrink dramatically in year 3 and turn into a surplus starting in year 4 and beyond. The surplus eventually converges back to $\bar{s}$ from above. Keeping debt risk-free still imposes severe restrictions on the size of the S-shaped surplus dynamics. Running sizeable deficits for more than two years is incompatible with risk-free debt.

Finally, to make the model's implications for tax revenues as comparable to the data as possible, we posit a more realistic process for spending/output than the one we have worked with hitherto. Specifically, we assume that the government commits to a policy for the spending/output ratio $x_t = G_t/Y_t$ given by:

$$\log x_t = \varphi_0^g + \varphi_1^g \log x_{t-1} - b_g \varepsilon_t - \frac{1}{2} b_g^2. \tag{8}$$

When $b_g > 0$, the spending/output ratio rises in response to a negative output shock. We estimate $(\varphi_0^g, \varphi_0^g, \beta^g)$ from the post-war U.S. data. The parameter estimates are reported in Panel C of Table 1. Spending/output is counter-cyclical in the data. A 1% point decline in output coincides with a 1.53% point increase in the spending/output ratio. The persistence of spending/output matches that in the data with an AR(1) coefficient of 0.88. With this spending process in hand, we compute the model-implied tax revenue/output.

### 3.6 The Insurance Trade-off over Finite Horizons

How much smoothing can the U.S. government achieve for taxpayers when debt is risk-free by issuing more debt in response to bad shocks? It depends on the horizon. Section 1.3 showed that we can gauge the welfare implications by examining the riskiness of the net transfer process $G_t - T_t$ at different horizons. This section applies the general result to the specific asset pricing model and debt/output processes from the previous section. In the presence of permanent shocks, the government can only insure taxpayers over a limited period of time.

### 3.6.1 Cash-Flow Betas with Risk-free Debt

The general Proposition 1.4 specializes to the following result:

**Proposition 3.5.** *Under Assumptions 2 and 4, when debt is risk-free and debt/output follows an autoregressive process as in (7), the cash-flow beta of the discounted surpluses over h periods is given by the beta of debt h periods from now:*

$$\begin{aligned}\beta_t^{S,CF}(h) &= \frac{cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+h}D_{t+h}\right)}{D_t var_t(M_{t+1})} \\ &= \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t(M_{t+1})}\mathbb{E}_t[M_{t+1,t+h}d_{t+h}Y_{t+h}](\exp\{\gamma(\psi_{h-1}\lambda - \sigma)\} - 1).\end{aligned}$$

*where* $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}, j > 2; \psi_2 = \phi_2 + \phi_1\psi_1; \psi_1 = \phi_1;\ \psi_0 = 1$, *and* $sign\left(\beta_t^{S,CF}(h)\right) = sign\left(\gamma(\psi_{h-1}\lambda - \sigma)\right)$.

This result can easily be generalized to any AR process, with $\psi_j$ denoting the coefficients in the characteristic equation. The risk properties of the government surpluses over a given horizon are completely determined by riskiness of the debt issuance process, as long as the debt is risk-free. The cash-flow beta of the surplus at various horizons does not depend on the spending and tax revenue dynamics.

Analogously, we define the cash-flow beta of discounted government spending and of tax revenues.

**Corollary 3.1.** *Under Assumptions 2 and 4, and when debt is risk-free and debt/output follows an* $AR(2)$ *and the government commits to a policy of spending-to-output ratio following equation (8), the cash flow beta of spending and tax revenues have to satisfy the following restrictions:*

$$\begin{aligned}\beta_t^{G,CF}(h) &= \sum_{j=1}^{h}\frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]}\mathbb{E}_t[M_{t+1,t+j}x_{t+j}Y_{t+j}](\exp\left\{\gamma(\varphi_g^{j-1}b_g - \sigma)\right\} - 1). \\ \beta_t^{T,CF}(h) &= \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]}\mathbb{E}_t[M_{t+1,t+h}d_{t+h}Y_{t+h}](\exp\{\gamma(\psi_{h-1}\lambda - \sigma)\} - 1) \\ &+ \sum_{j=1}^{h}\frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]}\mathbb{E}_t[M_{t+1,t+j}x_{t+j}Y_{t+j}](\exp\left\{\gamma(\varphi_g^{j-1}b_g - \sigma)\right\} - 1).\end{aligned}$$

The properties of the $\beta_t^{G,CF}(h)$ depend on the persistence and cyclicality of the exogenous spending/GDP process in equation (8). The properties of $\beta_t^{T,CF}(h)$ depend on the risk properties of both the debt claim and the spending claim.

#### 3.6.2 The Trade-off over Finite Horizons with Constant Debt-Output

When debt/output is constant ($\lambda = 0$), Proposition 3.5 implies:

$$\beta_t^{S,CF}(h) = \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]}\mathbb{E}_t[M_{t+1,t+h} dY_{t+h}](\exp\{-\gamma\sigma\} - 1). \tag{9}$$

The cash-flow beta of the surplus is negative at all horizons since $\gamma\sigma > 1$. In bad times, the surplus/output ratio goes up. When spending/output is constant (or also goes up), tax revenues/output must go up. The government cannot insure taxpayers against adverse output shocks. Rather, the taxpayers insure the bondholders.

Panel A of Figure 3 plots the risk premium on a claim to cumulative surpluses over the next $h$ periods in the left panel. It is the spending beta multiplied by the market prices of risk, $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$. The cumulative risk premium at horizon $h$ is the sum of the individual strip risk premia up until horizon $h$. The negative risk premium over short horizons indicates that surpluses are a hedge. Since taxpayers are short the surplus claim, their tax-minus-transfer liability is risky. When debt/output is constant and there is no possibility to raise the debt in response to an adverse shock, the surplus/output ratio must rise on impact. This makes the one-period surplus claim a hedge. The year-2 surplus claim in contrast earns a small positive risk premium, reflecting the underlying output risk, so that the cumulative 2-period surplus risk premium is higher than the 1-period surplus risk premium. As $h \to \infty$, the sum of discounted surpluses converges to the current value of debt $D_t$. Insisting on risk-free debt ($\beta_t^D = 0$) implies that $\beta_t^{S,CF}(h) \to 0$. The red line in the left panel converges to zero from below for large $h$.

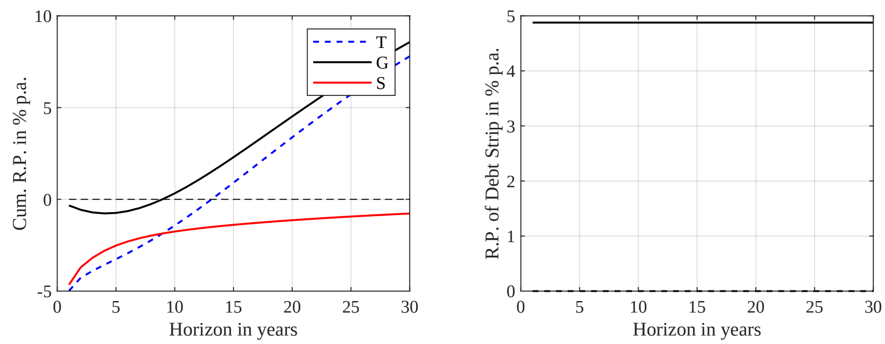
This risk premium on cumulative surpluses is inversely related to the risk premium on a debt strip, which is $\gamma\sigma > 0$ at all horizons. This debt strip risk premium is plotted in the right figure of Panel A. When the debt/output ratio is constant, the debt strip has the same risk as the output strip at all horizons. To offset the output risk in debt, the risk premium on the surplus has to be negative.

The solid black line in the left panel plots the risk premium on the claim to cumulative government spending over the next $h$ periods. It equals the cash-flow beta of the $h$-period spending claim multiplied by the market price of risk. Since the spending/output dynamics are exogenously given, the spending beta does not depend on the debt policy. The countercyclical nature of spending/output makes the risk premium negative at short horizons. At longer horizons, the spending risk premium turns positive reflecting the long-run output risk in the spending claim,

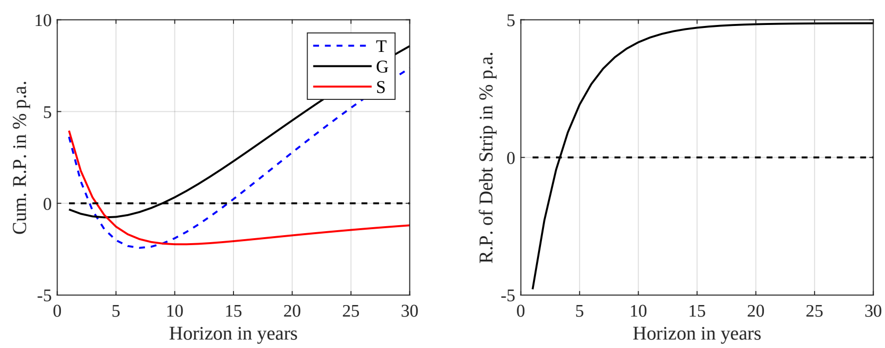
Figure 3: Risk Premia Across Horizons

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{i,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon $h$. The right panel plots the risk premium on the debt strips: $1 - \exp\{\gamma(\phi^{h-1}\lambda - \sigma)\}$. The parameters are given in Table 1, except for the debt dynamics in the first two panels.

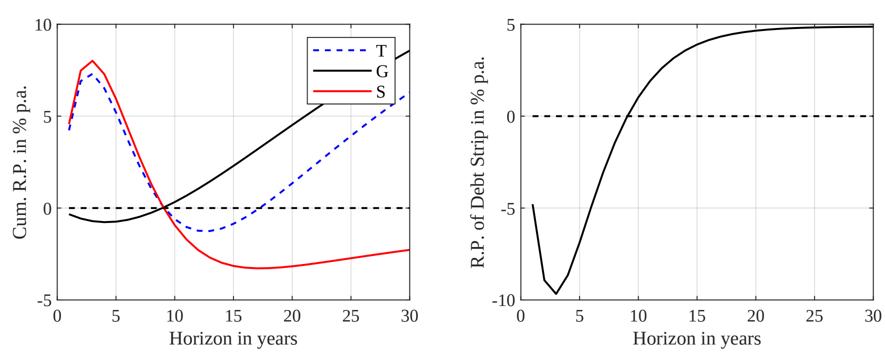
*Panel A: Constant Debt/Output ($\lambda = 0$)*




*Panel B: $AR(1)$ Debt/Output ($\phi_1 = 0.75$)*




*Panel C: $AR(2)$ Debt/Output*




since the spending/output ratio is stationary.

The extent of taxpayer insurance is captured by $\beta_t^{T,CF}(h)$. The blue dashed line in the left panel plots $\beta_t^{T,CF}(h)$ multiplied by the market price of risk, the risk premium on a claim to the next $h$ periods of tax revenue. When this risk premium is negative, taxpayers are providing insurance to the government rather than receiving insurance. The risk premium is negative until year 13 for our parameters. It then turns positive. The positive risk premium on longer-dated tax strips reflects cointegration between tax revenues and output and a positive risk premium for output risk.

Note that the tax beta $\beta_t^{T,CF}(h)$ in the left panel is below the spending beta $\beta_t^{G,CF}(h)$ at all horizons. As $h \to \infty$, these cash-flow betas converge to the return betas $\beta_t^T$ and $\beta_t^G$. As we discussed in Corollary 1, $\beta_t^T < \beta_t^G$ was the condition to keep the debt risk-free.

On the right-hand side of Panel A, we report a scaled version of the risk premium on the debt strips. Specifically, it is the valuation of the debt strip scaled by its expectation at time $t$ and then multiplied by the market price of risk $\frac{var_t(M_{t+1})}{\mathbb{E}_t[M_{t+1}]}$:

$$RP_t^{Dstrip}(h) \quad = \quad -\frac{cov_t\left(M_{t+1}, \frac{(\mathbb{E}_{t+1}-\mathbb{E}_t)M_{t+1,t+h}D_{t+h}}{\mathbb{E}_t[M_{t+1,t+h}D_{t+h}]}\right)}{var_t(M_{t+1})} \cdot \frac{var_t(M_{t+1})}{\mathbb{E}_t[M_{t+1}]}.$$

When the debt/output ratio is a constant, this risk premium measure for the debt strip is also a constant.

### 3.6.3 AR(1) for debt/output

The sign and magnitude of the cash-flow beta of the surplus $\beta_t^{S,CF}(h)$ are now governed by $\gamma(\phi_1^{h-1}\lambda - \sigma)$, where $\phi_1$ is the autocorrelation coefficient of the debt/output ratio.

If debt/output is pro-cyclical ($\lambda \leq 0$), $\beta_t^{S,CF}(h) < 0$ at all horizons. We are back in the previous case. In other words, when the government repays debt in bad times, it cannot provide any insurance to its taxpayers.

In the empirically relevant case of $\lambda > \sigma > 0$, the initial $\beta_t^{S,CF}(1) > 0$. By issuing more debt in response to an adverse shock, the government prevents the tax rate and the surplus from going up. This provides insurance to the taxpayers $\beta_t^{T,CF}(1) > 0$. The left plot in Panel B of Figure 3 shows the positive risk premium on cumulative surplus and tax claims at horizon $h = 1$. The one-period debt strip has a negative risk premium, $1 - \exp\{\gamma(\lambda - \sigma)\}$, due to the counter-cyclical nature of debt issuance, as shown in the right panel. The 1-period surplus can be risky because that risk is offset by the safety of the debt issuance at time $t + 1$.

However, due to its AR(1) nature, the debt/output ratio starts to revert back to its mean the very next period. The risk premium on the cumulative two-period surplus depends on $\gamma(\phi_1\lambda - \sigma)$ which is still positive but not as large as the one-period risk premium since $\phi_1 < 1$. Conversely, the cumulative two-period debt strip risk premium is not as negative as the one-period debt strip risk premium. The risk premium on the strip that pays the annual surplus two years from now is negative, pulling down the cumulative strip risk premium. The same is true for the two-year tax strip.

The surplus beta $\beta_t^{S,CF}(h)$ inherits the dynamics of the $AR(1)$ process for the debt/output ratio. As $h$ increases, the surplus beta eventually switches signs. This occurs at the first time $h$ for which $\phi^{h-1}\lambda < \sigma$. If the rate of mean-reversion in debt is high ($\phi_1$ is small), this switch occurs sooner. If the debt/output ratio is more persistent, the sign switch occurs later.

Given the counter-cyclical nature of government spending, the tax beta $\beta_t^{T,CF}(h)$ must cross over into negative territory sooner than the surplus beta. There is only a very limited amount of taxpayer insurance that the government can provide when debt is risk-free and follows AR(1) dynamics. This insurance is further curtailed due to the counter-cyclical nature of spending.

As the right panel shows, the risk premium on the debt strip increases with the horizon. As $h \to \infty$, it converges to the risk premium on a long-dated output strip. Again, this reflects the fact that debt is co-integrated with output. It is common in the literature to assume that this risk premium is zero at long horizons, because this allows discounting at the risk-free rate. In the presence of permanent shocks, this is incorrect. Similarly, the risk premia on the long-dated T-strip and G-strip also converge to risk premium on the long-dated output strip as $h \to \infty$.

When output shocks are i.i.d. and permanent, far-out surpluses are risky as they inherit the permanent output risk. Medium-term surpluses must be safe and have negative risk premia to offset both the positive risk premium of the short-run surpluses (short-run insurance provision to the taxpayer) and the positive risk premium of the long-run surpluses (output risk). Equivalently, the cash-flow betas of the tax strip must be below those of the spending strip at medium horizons. The cash-flow beta at $h = \infty$ equals the return beta, and so $\beta_t^T < \beta_t^G$ ensures that $\beta_t^D = 0$. Permanent output risk rules out insurance provision to taxpayers over long horizons.

### 3.6.4 AR(2) for debt/output

In our preferred case of an AR(2) for debt/output, the sign of the cash flow beta of the surplus is determined by $\gamma(\psi_{j-1}\lambda - \sigma)$. If $\lambda > \sigma$, the initial surplus beta is positive. The second beta is larger since $\psi_1 = \phi_1 > 1$. The third beta remains positive and is larger than the second beta if $\psi_2 > \psi_1$ or $\phi_1(\phi_1 - 1) + \phi_2 > 0$. This condition is satisfied for our point estimates $\phi_1 = 1.40$ and $\phi_2 = -0.48$. For these parameter values, the fourth beta is lower than the third, the fifth lower than the fourth, etc. Eventually the surplus cash-flow beta crosses over into negative territory. Panel C of Figure 3 shows this occurs in year 9. The cash-flow beta for tax revenue follows a similar pattern. The cash-flow betas inherit the hump-shaped pattern from the debt/output ratio.

What allows the government to provide temporary insurance to taxpayers is a debt issuance policy with more history dependence. Risk premia on debt strips, shown in the right panel, are more negative than in the AR(1) model and remain negative for longer (9 versus 3 years). The slow expansion and repayment of the debt in response to an adverse shock allows the government to postpone fiscal rectitude. But as $h$ increases, the expression $\gamma(\sigma - \psi_{j-1}\lambda)$ turns positive and converges to $\gamma\sigma$, the risk premium on the output strip. In sum, the cumulative surplus can be risky over a horizon $h$ (providing insurance to the tax payer) only if this risk is offset by the safety of debt issuance at time $t + h$. Insurance provision to the tax-payer is necessarily short-lived because of the long-run risk in debt.

### 3.7 Seigniorage Revenue

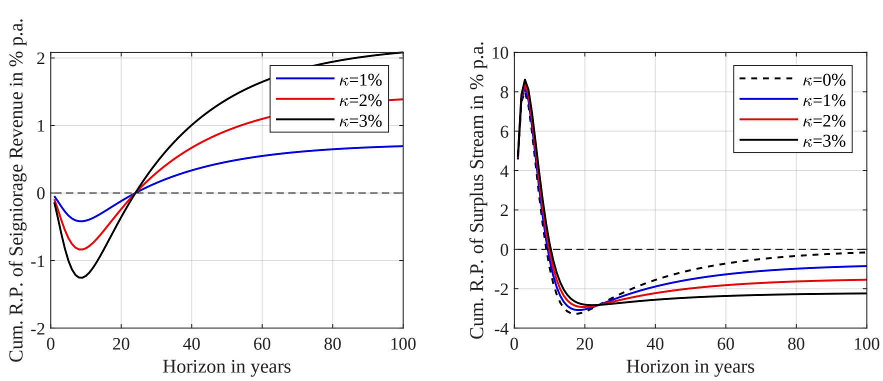
We now introduce convenience yields and apply the general analysis of Section 2 to the specific asset pricing model with permanent risk and the AR(2) dynamics for debt/output of this section. Under the proportional convenience yields Assumption 1, the seigniorage revenue beta becomes:

$$\beta_t^{K,CF}(h) \equiv -(1 - e^{-\kappa}) \sum_{j=1}^{h} \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t(M_{t+1})} \mathbb{E}_t[M_{t+1,t+j} d_{t+j} Y_{t+j}](\exp\{\gamma(\psi_{j-1}\lambda - \sigma)\} - 1).$$

Figure 4 plots the risk premium on the cumulative seigniorage claim in the left panel. It is the product of $\beta_t^{K,CF}(h)$ and the market price of risk. The three lines refer to different values for the convenience yield $(1 - e^{-\kappa})$, ranging from 1% to 3%. In the short run, the seignorage revenue claim is safe and hence earns a negative risk premium. The larger $\kappa$, the more negative the seigniorage risk premium at short horizons. As a result, the seigniorage revenue relaxes the trade-off between insuring bondholders and taxpayers over short horizons. This is shown in the right panel, which plots the risk premium on the cumulative surplus claim $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$. The latter is more positive in the short run, the higher the convenience yield.

Figure 4: Convenience Yield Seigniorage Betas

The left panel plots the risk premium of cumulative discounted seigniorage revenue, $\beta_t^{K,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, against the horizon $h$. The right panel plots the risk premium of cumulative discounted surpluses, $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, against the horizon $h$. The parameters are given in Table 1 with the debt/output ratio following an AR(2). Convenience yields $(1 - e^{-\kappa})$ range from 0% to 3%.



Over longer horizons, the seigniorage revenue is risky. Since seignorage revenue is proportional to debt outstanding, and debt is cointegrated with output, the long-run risk premium on the seigniorage revenue claim is dominated by long-run output risk. Seigniorage revenue inevitably adds long-run output risk to the debt, and more so the higher the convenience yield. This worsens the insurance possibilities to taxpayers over intermediate horizons. The return beta of the seigniorage revenue stream equals its cash-flow beta at horizon $h = \infty$, which is positive, thereby lowering the return beta of the tax claim $\beta^T$. In sum, convenience yields, even large ones, are no panacea. They allow for more taxpayer insurance over short horizons but less insurance in the longer-run.

### 3.8 Robustness

In section G of the Appendix, we analyze the effect of the counter-cyclicality of government spending on the trade-off. More counter-cyclical spending steepens the trade-off. We also analyze the case in which the government saves. Recall that the Ramsey planner chooses to accumulate savings in the long run (see section C of the Appendix). We show that the government can now insure taxpayers at all horizons. This is the only case in which the government fully escapes the trade-off.

## 4 Quantifying the Trade-off in Model with Transitory Output Shocks

We now study the insurance tradeoff in a model where output only experiences transitory shocks. This is the standard assumption in most macro-economic models. In models with only transitory shocks, long-term bonds are the riskiest assets; see Appendix H for a formal derivation. Government debt is now subject to substantial interest rate risk. To keep the debt risk-free in the presence of this interest rate risk, the government needs to deliver an even safer surplus process. Interest rate risk reduces the scope for insurance of taxpayers. Under natural parameter conditions, the trade-off between insuring taxpayers and bondholders is even steeper than in the model with permanent shocks.

We consider an internally consistent model with transitory shocks to both output and the pricing kernel.[18] Most of the equilibrium models in the literature on optimal taxation fit into this class of models. For an example, see Bhandari et al. (2017, pp. 653), which features a mean-reverting process for productivity growth and government spending. See Chari et al. (1994); Debortoli et al. (2017) for other examples. Our qualitative results would go through if we used the equilibrium pricing kernels implied by these models, but we adopt a flexible pricing kernel to derive closed-form solutions.

[18]In section I of the Appendix, we introduce transitory shocks to output but keep our original SDF with permanent shocks to the level of marginal utility. This model is misspecified. In any structural model with transitory output (or productivity) risk, marginal utility will only have a transitory component as well.

**Assumption 5.** *(a) The shocks to output are transitory. The log output process is given by:* $y_{t+1} = \xi_0 + \xi y_t + \sigma\varepsilon_{t+1}$ *where* $\varepsilon_{t+1}$ *denotes the innovation to log output which is i.i.d. normally distributed. (b) The log SDF is:* $m_{t,t+1} = -\rho - \frac{1}{2}\gamma^2 - \frac{\gamma}{\sigma}(\sigma\varepsilon_{t+1} + (\xi - 1)y_t)$.

This specific modification of the SDF is motivated by the fact that if the agent's consumption is equal to the output and has CRRA preferences with a relative risk aversion of $\gamma/\sigma$, the marginal utility growth is $m_{t,t+1} = -\tilde{\rho} - (\gamma/\sigma)(\xi_0 + (\xi - 1)y_t + \sigma\varepsilon_{t+1})$, where $\tilde{\rho} = \rho + \frac{1}{2}\gamma^2 - +(\gamma/\sigma)\xi_0$ In this case, the marginal utility of wealth can be written as: $\Lambda_{t+1} = \exp(-\tilde{\rho}(t+1) - (\gamma/\sigma)y_{t+1})$. There are no permanent shocks to the marginal utility of wealth.

The log of the risk-free rate is given by: $r_t^f = \rho + \gamma\frac{(\xi-1)y_t}{\sigma}$. Output risk drives interest rate risk. This model has counterfactual asset pricing implications. Interest rate risk will make the long bond, a zero coupon bond with the longest maturity, the riskiest asset in the economy.[19]

Surprisingly, even when there are no permanent shocks to output and the pricing kernel, the government cannot insure taxpayers over longer horizons. In fact, the trade-off worsens.

**Proposition 4.1.** *Under Assumption 5, the cash flow beta of the surpluses over j periods is given by:*

$$\beta_t^{S,CF}(h) = \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]}\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}]\left[\exp\left(\gamma(\phi_1^{j-1}\lambda - \xi^{j-1}\sigma - \gamma(1-\xi^{j-1})\right) - 1\right]$$

*when* $j \geq 1$*. The sign of the cash flow beta is determined by* $sign(\phi_1^{j-1}\lambda - \xi^{j-1}\sigma - \gamma(1-\xi^{j-1}))$.

The first component of the debt strip risk premium, $\gamma(\xi^{j-1}\sigma - \phi_1^{j-1}\lambda)$, compensates for output risk. The second component, $\gamma^2(1-\xi^{j-1})$, compensates for interest rate risk. Because the innovations are temporary, the output component of this risk premium converges to zero as we increase the horizon. The interest rate risk does not converge to zero, but rather to $\gamma^2$.[20] Hence, in the long-run the entire debt strip risk premium of $(1 - \exp(-\gamma^2))$ is due to interest rate risk. It is large and positive since the long bond is the riskiest asset in an economy with only transitory risk. In sum, while the transitory nature of output risk broadens the scope for insurance of taxpayers, this is more than offset by the rising interest rate risk. Compared to the permanent risk case, we have replaced long-run output risk with more long-run interest rate risk. The trade-off worsens as a result.

Figure 5 plots the trade-off for our calibration. In Panel A, the debt/output process is equally persistent as the output process. In the year of the shock, the surplus beta is still positive. In

[19] Modern asset pricing has consistently found that permanent cash flow shocks receive a high price of risk in the market. This model has no permanent priced risk.

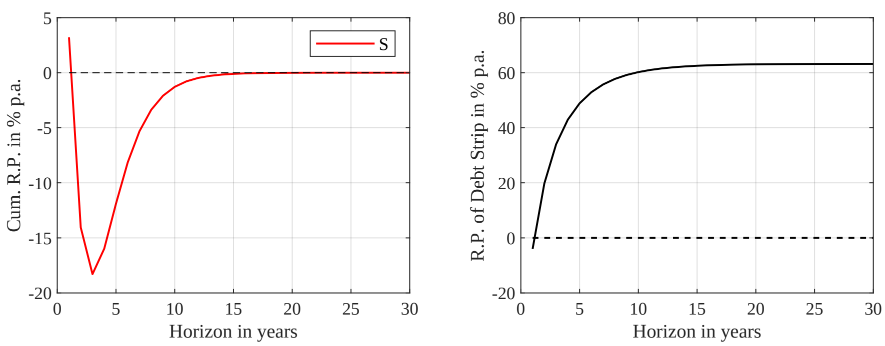
[20] If $M$ is conditionally log-normal, the expected log excess return on a long position in the longest-maturity bond reaches the entropy bound, one half of the variance of the log SDF: $h_t(\infty) = \lim_{k\to\infty}\mathbb{E}_t rx_{t+1}^k = (1/2)Var_t(m_{t+1})$ (Backus et al., 2014). In our setting, the expected log return on the longest maturity bond is $.5\gamma^2$. After adding a Jensen term, we recover the interest rate risk premium given by $\gamma^2$. This risk premium exceeds the output risk premium as long as the conditional volatility of the pricing kernel exceeds the conditional volatility of output: $\gamma > \sigma$.

the following year, the government already has to produce very safe surpluses, dragging the 2-year cumulative surplus risk premium into negative territory. At the 4-year horizon, the risk premium on the cumulative surplus declines to -17%. To keep the debt risk-free, the interest rate risk (shown in the right panel) has to be offset by very negative surplus and tax betas. As a result, the government can only insure taxpayers over very short periods even though the shocks to the economy are transitory. In Panel B, we consider a more persistent debt/output process. This mitigates the decline in the risk premium on the cumulative surplus, but the government can still not insure taxpayers beyond 2 years.

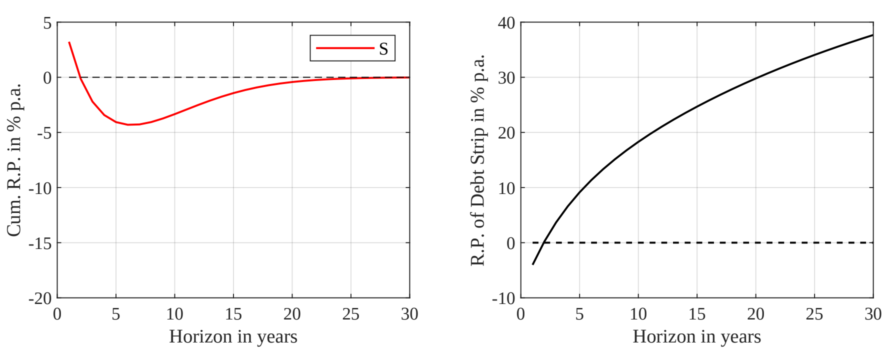
Figure 5: Risk Premia with Transitory Shocks to Output and SDF

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon $h$. The right panel plots the risk premium on the debt strips: $1-\exp(-\gamma(\sigma\xi^{j-1}-\phi^{j-1}\lambda+\gamma(1-\xi^{j-1})))$. In top panel, $\phi$ is 0.75 and $\xi$ is 0.75. In bottom panel, $\phi$ is 0.75 and $\xi$ is 0.98. Other parameters follow benchmark calibration in Table 1.

*Panel A: Output AR(1) with $\xi$ is 0.75*




*Panel B: Output AR(1) with $\xi$ is 0.98*




This result does not hinge on the specific SDF we use. In the absence of arbitrage opportunities, if the SDF is not subject to permanent innovations, the zero-coupon bond with the longest maturity will always earn the highest expected log return (see section H in the appendix for details). In any model with only transitory shocks, the cash flow beta of the $j$-period cumulative surplus converges to:

$$\lim_{j\to\infty} \beta_t^{S,CF}(j) \quad = \quad \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]} \mathbb{E}_t[M_{t+1,t+j} d_{t+j} Y_{t+j}] \left[\exp\left(-var_t(m_{t+1})\right) - 1\right],$$

where $var_t(m_{t+1})$ governs the interest rate risk premium. To keep the debt risk-free, the government has to offset the interest rate risk by generating safe surpluses, or equivalently, risky taxpayer liabilities.

## 5 Conclusion

The government engineers risk-free debt by choosing the exposure of the tax claim to output risk judiciously. The more debt there is outstanding, the lower this exposure must become, and hence the more output risk must be borne by taxpayers. There is no scope for insurance of both taxpayers and debt holders over long horizons in the presence of priced shocks to output, be they permanent or transitory in nature. Convenience yields on the debt, which induce the government to follow a safe debt strategy, alleviate the insurance tradeoff but only temporarily. The only way the government can provide insurance to taxpayers over all horizons while keeping the debt risk-free is by saving. Since U.S. surplus and tax dynamics look quite different from the ones that are needed to induce risk-free debt, U.S. Treasury debt may not be as safe as it looks.

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# A Proofs

## A.1 Proof of Proposition 1.1

*Proof.* Our derivation follows Alvarez and Jermann (2004) by assuming that $(1 + \Omega^h_{net}(\alpha))$ does not change the initial consumption $C_0$. By differentiating w.r.t. $\alpha$, we can back out the marginal benefit/cost of government intervention as follows:

$$\Omega^{h,i'}_{net}(0) = \frac{\mathbb{E}_0 \sum_{t=1}^h U^i_{c,t} \left[G_t - T_t\right]}{\mathbb{E}_0 \sum_{t=1}^h U^i_{c,t} \left[C_t\right]}.$$

When evaluating at $\alpha = 0$, we are not actually changing the household's consumption process. When the investors can invest in Treasury markets and do not face any binding constraints, the agents agree on the valuation of aggregate payoffs from the government debt:

$$\mathbb{E}_0 \sum_{t=1}^h M_{0,t} \left[G_t - T_t\right] = \mathbb{E}_0 \sum_{t=1}^h \frac{U^i_{c,t}}{U^i_{c,0}} \left[G_t - T_t\right].$$

For any investment horizon $h$, the following individual optimality condition has to hold, absent binding borrowing and short-sales constraints:

$$\mathbb{E}_0 \sum_{t=1}^h \frac{U^i_{c,t}}{U^i_{c,0}} \left[G_t - T_t\right] + \mathbb{E}_0 \frac{U^i_{c,h}}{U^i_{c,0}} \left[D_h\right] - D_0 = 0 = \mathbb{E}_0 \sum_{t=1}^h M_{0,t} \left[G_t - T_t\right] + \mathbb{E}_0 M_h D_h - D_0.$$

In addition, the transversality condition implies that, as $h \to \infty$, $\mathbb{E}_0 M_h D_h \to 0$. This implies that, for all $h$, the following equality holds:

$$\mathbb{E}_0 \sum_{t=1}^h M_{0,t} \left[G_t - T_t\right] = \mathbb{E}_0 \sum_{t=1}^h \frac{U^i_{c,t}}{U^i_{c,0}} \left[G_t - T_t\right].$$

By the same token, if these agents can trade claims to aggregate consumption, provided that the transversality condition holds, we know that for all $h$, the following equality holds:

$$\mathbb{E}_0 \sum_{t=1}^h M_{0,t} \left[C_t\right] = \mathbb{E}_0 \sum_{t=1}^h \frac{U^i_{c,t}}{U^i_{c,0}} \left[C_t\right].$$

By combining these 2 equalities, we obtain the following result, for each $h$:

$$\Omega^{h'}_{net}(0) = \frac{\mathbb{E}_0 \sum_{t=1}^h M_{0,t} \left[G_t - T_t\right]}{\mathbb{E}_0 \sum_{t=1}^h M_{0,t} \left[C_t\right]} = \frac{\mathbb{E}_0 \sum_{t=1}^h U^i_{c,t} \left[G_t - T_t\right]}{\mathbb{E}_0 \sum_{t=1}^h U^i_{c,t} \left[C_t\right]}.$$

□

## A.2 Proof of Corollary 1.1

*Proof.* We start from the marginal cost at infinite horizon:

$$\Omega^{\infty}_{net}(0) = \frac{P^{\infty}_0 \left[\{G_t - T_t\}\right]}{P^{\infty}_0 \left[\{C_t\}\right]} = \frac{(P^G_0 - G_0) - (P^T_0 - T_0)}{P^C_0 - C_0} = -\frac{D_0}{P^C_0 - C_0}.$$

Start from the following equation:

$$P^{\infty}_0 \left[\{G_t - T_t\}\right] = \mathbb{E}_0 \sum_{t=1}^{\infty} M_{0,t} \left[G_t - T_t\right] = \sum_{t=1}^{\infty} Cov_0(M_{0,t}, G_t - T_t) + \sum_{t=1}^{\infty} \mathbb{E}_0 M_{0,t} \mathbb{E}_0 \left[G_t - T_t\right].$$

Since the risk premium component for debt is positive: $RP_0 = -\sum_{t=1}^{\infty} Cov_0(M_{0,t}, T_t - G_t) > 0$, we need positive average surpluses $\mathbb{E}_0 \left[G_t - T_t\right] < 0$, to obtain $D_0 = P^{\infty}_0 \left[\{T_t - G_t\}\right] > 0$. □

## A.3 Proof of Proposition 1.2

*Proof.* From the investor's Euler equation $\mathbb{E}_t[M_{t+1}(R^i_{t+1} - R^f_t)]$=0, we know that the expected excess return on the tax claim, spending claim, and debt claims are given by

$$\mathbb{E}_t \left[R^T_{t+1} - R^f_t\right] = \frac{-cov_t\left(M_{t+1}, R^T_{t+1}\right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t\left(M_{t+1}, R^T_{t+1}\right)}{var_t[M_{t+1}]} \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]} = \beta^T_t \lambda_t,$$

$$\mathbb{E}_t\left[R^G_{t+1}-R^f_t\right] = \frac{-cov_t\left(M_{t+1},R^G_{t+1}\right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t\left(M_{t+1},R^G_{t+1}\right)}{var_t[M_{t+1}]}\frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]} = \beta^G_t\lambda_t,$$
$$\mathbb{E}_t\left[R^D_{t+1}-R^f_t\right] = \frac{-cov_t\left(M_{t+1},R^D_{t+1}\right)}{\mathbb{E}_t[M_{t+1}]} = \frac{-cov_t\left(M_{t+1},R^D_{t+1}\right)}{var_t[M_{t+1}]}\frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]} = \beta^D_t\lambda_t.$$

□

## A.4 Proof of Proposition 1.3

*Proof.* We start from the one-period government budget constraint:

$$T_t = G_t - (D_t - R_t D_{t-1}).$$

This implies that the return on government debt can be stated as follows:

$$\begin{aligned} R_t D_{t-1} &= S_t + D_t = S_t + \mathbb{E}_t[M_{t,t+1}R_{t+1}D_t], \\ &= S_t + \mathbb{E}_t\left[M_{t,t+1}[S_{t+1} + M_{t+1,t+2}R_{t+2}D_{t+1}]\right], \\ &= \mathbb{E}_t[\sum_{k=0}^{1} M_{t,t+k}S_{t+k}] + \mathbb{E}_t[M_{t,t+1}D_{t+1}], \end{aligned}$$

where we have used $R_{t+1}D_t = S_{t+1} + \mathbb{E}_{t+1}[M_{t+1,t+2}R_{t+2}D_{t+1}]$. By continued forward substitution, we obtain the following expression for the return on debt:

$$R_t D_{t-1} = \mathbb{E}_t[\sum_{k=0}^{h} M_{t,t+k}S_{t+k}] + \mathbb{E}_t[M_{t,t+h}D_{t+h}].$$

Replace the time index $t$ by $t+1$,

$$R_{t+1}D_t = \mathbb{E}_{t+1}[\sum_{j=1}^{h} M_{t+1,t+j}S_{t+j}] + \mathbb{E}_{t+1}[M_{t+1,t+h}D_{t+h}]. \tag{10}$$

Next, we take the limit of $h \to \infty$, and consider the case of risk-free debt. We start again from the one-period budget constraint:

$$T_t = G_t - (D_t - R^f_{t-1}D_{t-1}).$$

When the transversality condition holds, repeated forward substitution produces the following expression:

$$R^f_{t-1}D_{t-1} = S_t + D_t = S_t + \mathbb{E}_t[M_{t,t+1}R^f_t D_t] = S_t + \mathbb{E}_t[M_{t,t+1}(S_{t+1} + M_{t+1,t+2}R^f_{t+1}D_{t+1})] = \mathbb{E}_t[\sum_{k=0}^{\infty} M_{t,t+k}S_{t+k}].$$

Replace the time index $t$ by $t+1$, to obtain the following expression:

$$R^f_t D_t = \mathbb{E}_{t+1}[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}].$$

Since the left-hand side is known at time $t$, we obtain the result:

$$(\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{j=1}^{\infty} M_{t+1,t+j}S_{t+j}] = 0.$$

□

## A.5 Proof of Proposition 1.4

*Proof.* We start from eqn. 10. We take the innovations on both sides:

$$D_t(\mathbb{E}_{t+1} - \mathbb{E}_t)R_{t+1} = (\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{j=1}^{h} M_{t+1,t+j}S_{t+j}] + (\mathbb{E}_{t+1} - \mathbb{E}_t)[M_{t+1,t+h}D_{t+h}].$$

We obtain the following result:

$$
\begin{aligned}
D_t Cov_t(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) R_{t+1}) &= Cov_t(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)[\sum_{j=1}^{h} M_{t+1,t+j} S_{t+j}]) \\
&+ Cov_t(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)[M_{t+1,t+h} D_{t+h}]).
\end{aligned}
$$

□

## A.6 Proof of Proposition 2.1

*Proof.* We start from the value of the government debt equals the sum of the expected present values of future tax revenues plus future seigniorage revenues minus future government spending:

$$
B_t = \mathbb{E}_t \left[ \sum_{j=0}^{\infty} M_{t,t+j} (T_{t+j} + (1 - e^{-\kappa_{t+j}}) D_{t+j} - G_{t+j}) \right] = P_t^T + P_t^K - P_t^G,
$$

provided that a transversality condition holds. The government debt portfolio return equals the return on a portfolio that goes long in the tax claim and short in the spending claim:

$$
\mathbb{E}_t \left[ R_{t+1}^D - R_t^f \right] = \frac{P_t^T - T_t}{B_t - S_t} \mathbb{E}_t \left[ R_{t+1}^T - R_t^f \right] + \frac{P_t^K - K_t}{B_t - S_t} \mathbb{E}_t \left[ R_{t+1}^K - R_t^f \right] - \frac{P_t^G - G_t}{B_t - S_t} \mathbb{E}_t \left[ R_{t+1}^G - R_t^f \right],
$$

where $R_{t+1}^D$, $R_{t+1}^T$,$R_{t+1}^K$ and $R_{t+1}^G$ are the holding period returns on the bond portfolio, the tax claim, and the spending claim, respectively. We take government spending process, and the debt return process as exogenously given, and we explore the implications for the properties of the tax claim. In the absence of arbitrage opportunities, if the TVC holds, the expected excess return on the tax claim is the unlevered return on the spending claim and the debt claim:

$$
\mathbb{E}_t \left[ R_{t+1}^T - R_t^f \right] = \frac{(P_t^G - G_t) \mathbb{E}_t \left[ R_{t+1}^G - R_t^f \right]}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} + \frac{D_t \mathbb{E}_t \left[ R_{t+1}^D - R_t^f \right]}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} - \frac{(P_t^K - K_t) \mathbb{E}_t \left[ R_{t+1}^K - R_t^f \right]}{D_t + (P_t^G - G_t) - (P_t^K - K_t)}
$$

If we want the debt to be risk-free, then the following equation holds for expected returns:

$$
\mathbb{E}_t \left[ R_{t+1}^T - R_t^f \right] = \frac{(P_t^G - G_t) \mathbb{E}_t \left[ R_{t+1}^G - R_t^f \right]}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} - \frac{(P_t^K - K_t) \mathbb{E}_t \left[ R_{t+1}^K - R_t^f \right]}{D_t + (P_t^G - G_t) - (P_t^K - K_t)}
$$

and there is a similar equation for the betas:

$$
\beta_t^T = \frac{P_t^G - G_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \beta_t^G - \frac{P_t^K - K_t}{D_t + (P_t^G - G_t) - (P_t^K - K_t)} \beta_t^K.
$$

□

## A.7 Proof of Proposition 2.2

*Proof.* This result follows directly from the proof of Proposition 1.4. □

## A.8 Proof of Proposition 3.1

*Proof.* To verify the expression, first conjecture the pricing of the output strip is $\mathbb{E}_t [M_{t,t+k} Y_{t+k}] = \xi_k Y_t$, for $k \geq 0$. Then $\xi_0 = 1$ and

$$
\begin{aligned}
\xi_k Y_t = \mathbb{E}_t [M_{t,t+k} Y_{t+k}] &= \mathbb{E}_t [M_{t,t+1} \xi_{k-1} Y_{t+1}] = \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) \xi_{k-1} Y_t, \\
\xi_k Y_t &= \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) \xi_{k-1} Y_t,
\end{aligned}
$$

which verifies the conjecture and implies $\xi_k = \xi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)$. Similarly, we define a $k$-period surplus strip as a claim to $S_{t+k}$, with price given by $\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \chi_k Y_t$. The pricing of the first surplus strip is given by the following expression:

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+1}S_{t+1}\right] &= \mathbb{E}_t\left[M_{t,t+1}\{-dY_{t+1}\left(1 - R_t^f\exp[-(\mu+\varepsilon_{t+1})]\right)\}\right], \\
&= -d\mathbb{E}_t\left[M_{t,t+1}Y_{t+1}\right] + dY_t R_t^f \mathbb{E}_t\left[M_{t,t+1}\right], \\
&= -d\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t + dY_t, \\
&= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right] dY_t. \\
\chi_1 &= \left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right] d.
\end{aligned}$$

where the first equality uses the definition of the surplus and the government budget constraint. Then, similarly, the pricing of the $k$th surplus strip is given by:

$$\begin{aligned}
\chi_k Y_t &= \mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \mathbb{E}_t\left[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+k}S_{t+k}]\right] = \mathbb{E}_t\left[M_{t,t+1}\chi_{k-1}Y_{t+1}\right], \\
&= \chi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)Y_t.
\end{aligned}$$

Note that this calculation also implies that we cannot simply price these strips off the risk-free yield curve, even though the entire debt is risk-free. The solution is given by:

$$\begin{aligned}
\chi_1 &= d\left[1 - \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)\right]. \\
\chi_k &= \chi_{k-1}\exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2).
\end{aligned}$$

which implies that $\lim_{j\to\infty}\mathbb{E}_t\left[M_{t,t+j}S_{t+j}\right] = \sum_{k=1}^{\infty}\chi_k Y_t = \chi_1(1+\xi_1+\xi_1^2+\ldots)Y_t = \frac{1-\xi_1}{1-\xi_1}dY_t = dY_t$, where $\xi_1 = \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma-\sigma)^2)$.

□

## A.9 Proof of Proposition 3.2

*Proof.* From the gross risk-free rate expression $R_{t+1}^f = \exp(\rho)$ and the one-period government budget constraint, we obtain that:

$$\frac{T_t}{Y_t} = x - d\left(1 - R_{t-1}^f\frac{Y_{t-1}}{Y_t}\right).$$

The return on the tax claim can be stated as:

$$R_{t+1}^T = \frac{P_{t+1}^T}{P_t^T - T_t} = \frac{(d + x\frac{\xi_1}{1-\xi_1})Y_{t+1} + (x - d\left(1 - R_t^f\frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} = \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d + x\frac{\xi_1}{1-\xi_1})}.$$

Similarly, the return on the spending claim can be stated as:

$$R_{t+1}^G = \frac{P_{t+1}^G}{P_t^G - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t} = \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}.$$

Armed with these expressions, we get the following expression for the covariance:

$$cov(R_{t+1}^T, M_{t,t+1}) = \frac{x\frac{\xi_1}{1-\xi_1}}{(d + x\frac{\xi_1}{1-\xi_1})}cov(R_{t+1}^G, M_{t,t+1}),$$

which also translates to $\mathbb{E}_t\left[R_{t+1}^T - R_t^f\right] = \frac{x\frac{\xi_1}{1-\xi_1}}{d + x\frac{\xi_1}{1-\xi_1}}\mathbb{E}_t\left[R_{t+1}^Y - R_t^f\right]$. □

## A.10 Proof of Proposition 3.3

### A.10.1 Case of $AR(1)$

*Proof.* From the government budget constraint in the case of risk-free debt:

$$T_t = G_t - (D_t - R^f_{t-1} D_{t-1}).$$

Hence, the surplus process is given by:

$$\frac{S_t}{Y_t} \quad = \quad -\left(d_t - R^f_{t-1} d_{t-1} \frac{Y_{t-1}}{Y_t}\right) = d_{t-1} R^f_{t-1} \exp[-(\mu + \sigma\varepsilon_t)] - d^{\phi_1}_{t-1} \exp(\phi_0 - \lambda\varepsilon_t - \frac{1}{2}\lambda^2).$$

We conjecture that the price of the surplus strips is given by:

$$\mathbb{E}_t \left[M_{t,t+k} S_{t+k}\right] \quad = \quad (\chi_{k,t} - \psi_{k,t}) Y_t.$$

The pricing of the first surplus strip is given by:

$$\begin{aligned}
\mathbb{E}_t \left[M_{t,t+1} S_{t+1}\right] &= \mathbb{E}_t \left[M_{t,t+1} \{-Y_{t+1} \left(d_{t+1} - R^f_t d_t \exp[-(\mu + \sigma\varepsilon_{t+1})]\right)\}\right], \\
&= \mathbb{E}_t \left[-\exp(\phi_1 \log d_t + m_{t,t+1} + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}\right] + d_t Y_t, \\
&= -\exp(\phi_1 \log d_t + \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) Y_t + d_t Y_t, \\
(\chi_{1,t} - \psi_{1,t}) Y_t &= \left[d_t - \exp(\phi_0 + \phi_1 \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)\right] Y_t.
\end{aligned}$$

So, we define:

$$(\chi_{1,t}) Y_t = d_t Y_t, (\psi_{1,t}) Y_t = \exp(\phi_0 + \phi_1 \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) Y_t.$$

Similarly the price of the $k$-th surplus strip is given by:

$$\begin{aligned}
\mathbb{E}_t \left[M_{t,t+k} S_{t+k}\right] &= \mathbb{E}_t \left[M_{t,t+1} \mathbb{E}_{t+1}[M_{t+1,t+k} S_{t+k}]\right], \\
(\chi_{k,t} - \psi_{k,t}) Y_t &= \mathbb{E}_t \left[M_{t,t+1} (\chi_{k-1,t+1} - \psi_{k-1,t+1}) Y_{t+1}\right],
\end{aligned}$$

where the $\chi$'s are defined by the following recursion:

$$\begin{aligned}
\chi_{2,t} Y_t &= \mathbb{E}_t \left[M_{t,t+1} \chi_{1,t+1} Y_{t+1}\right], \\
\chi_{2,t} &= \mathbb{E}_t \left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}) \exp(-\lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) \exp(\mu + \sigma\varepsilon_{t+1})\right] \exp(\phi_1 \log d_t + \phi_0), \\
&= \exp(\phi_0 + \phi_1 \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) = \psi_{1,t},
\end{aligned}$$

and the $\psi$'s are defined by the following recursion:

$$\begin{aligned}
\psi_{2,t} Y_t &= \mathbb{E}_t \left[M_{t,t+1} \psi_{1,t+1} Y_{t+1}\right], \\
\psi_{2,t} &= \mathbb{E}_t \left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi_0 + \phi_1 \log d_{t+1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + \sigma\varepsilon_{t+1})\right], \\
\psi_{2,t} &= \exp(-2\rho + \phi_0 + \phi_1\phi_0 + \phi_1^2 \log d_t - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2), \\
&- \frac{1}{2}(\gamma^2 + \lambda^2) + 2\mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \frac{1}{2}(\gamma + \lambda\phi_1 - \sigma)^2), \\
&= \psi_{1,t} \exp(-\rho + \phi_1\phi_0 + (\phi_1^2 - \phi_1) \log d_t - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda\phi_1 - \sigma)^2).
\end{aligned}$$

More generally, we note that $\chi_{k+1,t} = \psi_{k,t}$, so that $\sum_{k=1}^{\infty} \mathbb{E}_t \left[M_{t,t+k} S_{t+k}\right] = \chi_{1,t} Y_t = D_t$. For some $0 < \phi_1 < 1$, we can solve for the price of the debt strips:

$$\mathbb{E}_t[M_{t,t+1} D_{t+1}] = \mathbb{E}_t[M_{t,t+1} Y_{t+1} d_{t+1}] = d^{\phi_1}_t \mathbb{E}_t[\exp(\phi_0 + m_{t,t+1} - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}],$$

$$= d_t^{\phi_1} \exp(\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) Y_t = \exp(\kappa_1) \exp(\phi_1 \log d_t) Y_t,$$

where $\kappa_1 = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2$. Similarly, the price of the two-period ahead debt strip is given by:

$$\begin{aligned}
\mathbb{E}_t[M_{t,t+2} D_{t+2}] &= \mathbb{E}_t[M_{t,t+1} \mathbb{E}_{t+1}[\exp(m_{t+1,t+2}) D_{t+2}]] = \mathbb{E}_t[M_{t,t+1} \exp(\kappa_1) \exp(\phi_1 \log d_{t+1}) Y_{t+1}], \\
&= \mathbb{E}_t[M_{t,t+1} \exp(\kappa_1) \exp(\phi_1^2 \log d_t + \phi_1 \phi_0 - \phi_1 \lambda \varepsilon_{t+1} - \frac{1}{2} \phi_1 \lambda^2) \exp(\mu + \sigma \varepsilon_{t+1})] Y_t, \\
&= \exp(\kappa_1 + \kappa_2) \exp(\phi_1^2 \log d_t) Y_t,
\end{aligned}$$

where $\kappa_2 = \phi_1 \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \phi_1 \lambda^2) + \mu + \frac{1}{2}(\gamma + \phi_1 \lambda - \sigma)^2$. Then, by induction,

$$\begin{aligned}
\lim_{j \to \infty} \mathbb{E}_t[M_{t,t+j} D_{t+j}] &= \lim_{j \to \infty} \exp(\sum_{k=1}^{j} \kappa_k) \exp(\phi_1^j \log d_t) Y_t, \\
&= \lim_{j \to \infty} \exp(\frac{\phi_0}{1 - \phi_1} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1 - \phi_1}) + \mu j + \sum_{k=1}^{j} \frac{1}{2}(\gamma + \lambda \phi_1^{k-1} - \sigma)^2) Y_t, \\
&= \lim_{j \to \infty} \exp(\frac{\phi_0}{1 - \phi_1} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1 - \phi_1}) + \mu j + j \frac{1}{2}(\gamma - \sigma)^2 + C) Y_t,
\end{aligned}$$

where $C = \sum_{k=1}^{j} [(\gamma - \sigma) \lambda \phi_1^{k-1} + \lambda^2 \phi_1^{2(k-1)}]$. The above equation is 0 if and only if $-\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. This condition does not depend on $\phi_1$ and $\lambda$.

□

### A.10.2 Case of Random Walk

*Proof.* Now, assume $\phi_1 = 1$ and $\phi_0 = 0$. Then $\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2$.
The TVC is

$$\lim_{j \to \infty} \mathbb{E}_t[M_{t,t+j} D_{t+j}] = \lim_{j \to \infty} \exp(\sum_{k=1}^{j} \kappa_k) \exp(\log d_t) Y_t,$$

which is 0 if and only if $-\rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 < 0$. □

### A.10.3 Case of $AR(2)$

*Proof.* From the government budget constraint in the case of risk-free debt:

$$T_t = G_t - (D_t - R_{t-1}^f D_{t-1}),$$

it follows that the surpluses are given by:

$$\begin{aligned}
S_t &= -\left(d_t Y_t - R_{t-1}^f d_{t-1} Y_{t-1}\right), \\
&= d_{t-1} R_{t-1}^f Y_{t-1} - \exp(\phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} - \lambda \varepsilon_t - \frac{1}{2}\lambda^2) Y_t.
\end{aligned}$$

We conjecture the price of the surplus strips is given by the following expression:

$$\mathbb{E}_t\left[M_{t,t+k} S_{t+k}\right] = (\chi_{k,t} - \psi_{k,t}) Y_t.$$

The pricing of the first surplus strip then as follows:

$$\begin{aligned}
\mathbb{E}_t\left[M_{t,t+1} S_{t+1}\right] &= \mathbb{E}_t\left[M_{t,t+1}\{-Y_{t+1}\left(d_{t+1} - R_t^f d_t \exp[-(\mu + \sigma \varepsilon_{t+1})]\right)\}\right], \\
&= \mathbb{E}_t\left[-\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + m_{t,t+1} + \phi_0 - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}\right] + d_t Y_t, \\
&= -\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2) Y_t + d_t Y_t, \\
(\chi_{1,t} - \psi_{1,t}) Y_t &= \left[d_t - \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)\right] Y_t.
\end{aligned}$$

We define the following objects:

$$
\begin{aligned}
(\chi_{1,t})Y_t &= d_t Y_t, \\
(\psi_{1,t})Y_t &= \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t.
\end{aligned}
$$

Similarly the pricing of the $k$-th surplus strip is

$$
\begin{aligned}
\mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] &= \mathbb{E}_t\left[M_{t,t+1}\mathbb{E}_{t+1}[M_{t+1,t+k}S_{t+k}]\right], \\
(\chi_{k,t} - \psi_{k,t})Y_t &= \mathbb{E}_t\left[M_{t,t+1}(\chi_{k-1,t+1} - \psi_{k-1,t+1})Y_{t+1}\right],
\end{aligned}
$$

where the $\chi$'s are defined by the following recursion:

$$
\begin{aligned}
\chi_{2,t}Y_t &= \mathbb{E}_t\left[M_{t,t+1}\chi_{1,t+1}Y_{t+1}\right], \\
\chi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(-\lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)\exp(\mu + \sigma\varepsilon_{t+1})\right]\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0), \\
&= \exp(\phi_0 + \phi_1 \log d_t + \phi_2 \log d_{t-1} - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2).
\end{aligned}
$$

and the $\psi$'s are defined by the following recursion:

$$
\begin{aligned}
\psi_{2,t}Y_t &= \mathbb{E}_t\left[M_{t,t+1}\psi_{1,t+1}Y_{t+1}\right], \\
\psi_{2,t} &= \mathbb{E}_t\left[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi_0 + \phi_1 \log d_{t+1} + \phi_2 \log d_t - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 \; + \; \mu + \sigma\varepsilon_{t+1})\right], \\
\psi_{2,t} &= \exp(-2\rho + \phi_0 + \phi_1\phi_0 + (\phi_1^2 + \phi_2)\log d_t + \phi_1\phi_2 \log d_{t-1} - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2), \\
&- \frac{1}{2}(\gamma^2 + \lambda^2) + 2\mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \frac{1}{2}(\gamma + \lambda\phi_1 - \sigma)^2).
\end{aligned}
$$

We note that $\chi_{k+1,t} = \psi_{k,t}$, so this expression can be simplified as follows:

$$
\sum_{k=1}^{\infty} \mathbb{E}_t\left[M_{t,t+k}S_{t+k}\right] = \chi_{1,t}Y_t = D_t.
$$

Also note that: $d_t = \exp(\phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} + \phi_0 - \lambda\varepsilon_t - \frac{1}{2}\lambda^2)$. Using this expression, we find that:

$$
\begin{aligned}
\mathbb{E}_t[M_{t,t+1}D_{t+1}] &= \mathbb{E}_t[M_{t,t+1}Y_{t+1}d_{t+1}], \\
&= d_t^{\phi_1} d_{t-1}^{\phi_2}\mathbb{E}_t[\phi_0 + \exp(m_{t,t+1} - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2)Y_{t+1}], \\
&= d_t^{\phi_1} d_{t-1}^{\phi_2}\exp(\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2)Y_t, \\
&= \exp(\kappa_1)\exp(\phi_1 \log d_t + \phi_2 \log d_{t-1})Y_t,
\end{aligned}
$$

Define $\kappa_1 = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2$.

$$
\begin{aligned}
\mathbb{E}_t[M_{t,t+2}D_{t+2}] &= \mathbb{E}_t[M_{t,t+1}\mathbb{E}_{t+1}[\exp(m_{t+1,t+2})D_{t+2}]], \\
&= \mathbb{E}_t[M_{t,t+1}\exp(\kappa_1)\exp(\phi_1 \log d_{t+1} + \phi_2 \log d_t)Y_{t+1}], \\
&= \mathbb{E}_t[M_{t,t+1}\exp(\kappa_1)\exp((\phi_1^2 + \phi_2)\log d_t + \phi_1\phi_2 \log d_{t-1} + \phi_1\phi_0 - \phi_1\lambda\varepsilon_{t+1} - \frac{1}{2}\phi_1\lambda^2)\exp(\mu + \sigma\varepsilon_{t+1})]Y_t \\
&= \exp(\kappa_1 + \kappa_2)\exp((\phi_1^2 + \phi_2)\log d_t + \phi_1\phi_2 \log d_{t-1})Y_t.
\end{aligned}
$$

Define $\kappa_2 = \phi_1\phi_0 - \rho - \frac{1}{2}(\gamma^2 + \phi_1\lambda^2) + \mu + \frac{1}{2}(\gamma + \phi_1\lambda - \sigma)^2$.

$$
\begin{aligned}
\lim_{j\to\infty} \mathbb{E}_t[M_{t,t+j}D_{t+j}] &= \lim_{j\to\infty} \exp(\sum_{k=1}^{j} \kappa_k)\exp(\psi_j \log d_t)Y_t, \\
&= \lim_{j\to\infty} \exp(\frac{\phi_0}{1 - \phi_1 - \phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1 - \phi_1 - \phi_2}) + \mu j + \sum_{k=1}^{j} \frac{1}{2}(\gamma + \lambda\psi_{k-1} - \sigma)^2)Y_t,
\end{aligned}
$$

$$= \lim_{j\to\infty} \exp(\frac{\phi_0}{1-\phi_1-\phi_2} - \rho j - \frac{1}{2}(\gamma^2 j + \frac{\lambda^2}{1-\phi_1-\phi_2}) + \mu j + j\frac{1}{2}(\gamma-\sigma)^2 + C)Y_t,$$

where $\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \phi_3\psi_{j-3}, j > 3; \psi_3 = \phi_3 + \phi_2\psi_1 + \phi_1\psi_2; \psi_2 = \phi_2 + \phi_1\psi_1; \psi_1 = \phi_1$, and $C = \sum_{k=1}^{j}[(\gamma-\sigma)\lambda\psi_{k-1} + \lambda^2\psi_{k-1}^2]$. The above equation is 0 if and only if $-\rho + \mu + \frac{1}{2}\sigma(\sigma - 2\gamma) < 0$. This condition does not depend on $\phi_1$, $\phi_2$ and $\lambda$. So this case is similar to the i.i.d. debt case $\phi = 0$. More extremely, when $\lambda = 0$, $d_t = \exp(\phi_0)$ is a constant. Now, assume $\phi = 1$. Then

$$\kappa_j = \phi_0 - \rho - \frac{1}{2}(\gamma^2 + \lambda^2) + \mu + \frac{1}{2}(\gamma + \lambda - \sigma)^2,$$

and $\lim_{j\to\infty} \mathbb{E}_t[M_{t,t+j}D_{t+j}] = \lim_{j\to\infty} \exp(\sum_{k=1}^{j}\kappa_k)\exp(\log d_t)Y_t$, which is 0 if and only if $\phi_0 - \rho - \frac{1}{2}(\gamma^2+\lambda^2) + \mu + \frac{1}{2}(\gamma+\lambda-\sigma)^2 < 0$. □

## A.11 Proof of Proposition 3.4

### A.11.1 Case of $AR(1)$

*Proof.* When the log of the debt/output process follows an $AR(1)$, the surplus/output ratio is given by:

$$\begin{aligned}\frac{S_{t+1}}{Y_{t+1}} &= d_t R_t^f \exp[-(\mu + \sigma\varepsilon_{t+1})] - d_t^{\phi_1}\exp(\phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) \\ &= \exp(r_t^f - \mu - \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi_1}) - \exp(\phi_1(-\sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}) + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).\end{aligned}$$

We assume that $r_t^f = \mu$. This expression for the surplus/output ratio can be restated as:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}) - \exp(-\sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+1-j} + \frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$:

$$\frac{\partial\frac{S_{t+1}}{Y_{t+1}}}{\partial\varepsilon_{t+1}} = (\lambda)\exp(g + \sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+1-j} + \frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}) - \sigma\exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial\frac{S_{t+1}}{Y_{t+1}}}{\partial\varepsilon_{t+1}} = (\lambda - \sigma)\exp(\frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}).$$

Next, we compute the derivative of the surplus/output ratio at $t+2$, given by

$$\frac{\partial\frac{S_{t+2}}{Y_{t+2}}}{\partial\varepsilon_{t+1}} = -\lambda\exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+1-j} + \frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}) + \lambda\phi_1\exp(-\sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+2-j} + \frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial\frac{S_{t+2}}{Y_{t+2}}}{\partial\varepsilon_{t+1}} = -\lambda\exp(\frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}) + \lambda\phi_1\exp(\frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}).$$

This generalizes to the following expression. For $j \geq 2$, we obtain:

$$\frac{\partial\frac{S_{t+j}}{Y_{t+j}}}{\partial\varepsilon_{t+1}} = -\lambda\phi_1^{j-1}\exp(\frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}) + \lambda\phi_1^j\exp(\frac{\phi_0-\frac{1}{2}\lambda^2}{1-\phi_1}).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = \lambda \phi_1^{j-1}(\phi_1 - 1)d, j > 1, \frac{\partial \frac{S_{t+1}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = (\lambda - \sigma)d, j = 1.$$

□

### A.11.2 Case of $AR(2)$

*Proof.* We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume that $r_t^f = \mu$. When the log of the debt/output process follows an $AR(2)$, the surplus/output ratio is given by:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty} \phi_1^j \lambda \varepsilon_{t-j} + \overline{d}) - \exp(\overline{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-1-j}) - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0:$

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = -\sigma \exp(\overline{d}) + \lambda \exp(\overline{d} - \frac{1}{2}\lambda^2).$$

The surplus/output ratio at $t+2$ is given by:

$$\frac{S_{t+2}}{Y_{t+2}} = \exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty} \phi_1^j \lambda \varepsilon_{t+1-j} + \overline{d}) - \exp(\overline{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j}) - \lambda \varepsilon_{t+2} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda \exp(\overline{d}) + \lambda(\phi_1) \exp(\overline{d} - \frac{1}{2}\lambda^2).$$

The surplus/output ratio at $t+3$ is given by:

$$\frac{S_{t+3}}{Y_{t+3}} = \exp(-\sigma\varepsilon_{t+3} - \sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+2-j} + \overline{d}) - \exp(\overline{d} + \phi_1(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t+1-j}) - \lambda \varepsilon_{t+3} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+3}}{Y_{t+3}}}{\partial \varepsilon_{t+1}} = -\psi_1 \lambda \exp(\overline{d}) + \lambda(\phi_1 \psi_1 + \phi_2) \exp(\mu + \overline{d} - \frac{1}{2}\lambda^2).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda \psi_{j-2} \exp(\overline{d}) + \lambda \psi_{j-1} \exp(\overline{d} - \frac{1}{2}\lambda^2).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\begin{aligned} \frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= -\sigma \exp(\overline{d}) + \lambda \exp(\overline{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 1, \\ &= -\lambda \exp(\overline{d}) + \lambda \phi_1 \exp(\overline{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 2, \\ &= -\lambda \psi_{j-2} \exp(\overline{d}) + \lambda \psi_{j-1} \exp(\overline{d} - \frac{1}{2}\lambda^2), \textit{ for } j > 2. \end{aligned}$$

□

### A.11.3 Case of AR(3)

*Proof.* We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. We assume $r^f = \mu$. When the log of the debt/output process follows an AR(3), the surplus/output ratio is given by:

$$\frac{S_{t+1}}{Y_{t+1}} = \exp(-\sigma\varepsilon_{t+1} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t-j} + \bar{d}) - \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-1-j} + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-2-j}) - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2).$$

Next, we compute the derivative of the surplus/output ratio at $t+1$, and we evaluate this derivative at $\varepsilon_{t+j} = 0$:

$$\frac{\partial \frac{S_{t+1}}{Y_{t+1}}}{\partial \varepsilon_{t+1}} = -\sigma\exp(\bar{d}) + \lambda\exp(\bar{d} - \frac{1}{2}\lambda^2).$$

The surplus/output ratio at $t+2$ is given by:

$$\frac{S_{t+2}}{Y_{t+2}} = \exp(-\sigma\varepsilon_{t+2} - \sum_{j=0}^{\infty}\phi_1^j\lambda\varepsilon_{t+1-j} + \bar{d}) - \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-1-j}) - \lambda\varepsilon_{t+2} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain: $\frac{\partial \frac{S_{t+2}}{Y_{t+2}}}{\partial \varepsilon_{t+1}} = -\lambda\exp(\bar{d}) + \lambda(\phi_1)\exp(\bar{d} - \frac{1}{2}\lambda^2)$. The surplus/output ratio at $t+3$ is given by:

$$\frac{S_{t+3}}{Y_{t+3}} = \exp(-\sigma\varepsilon_{t+3} - \sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j} + \bar{d}) - \exp(\bar{d} + \phi_1(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+2-j}) + \phi_2(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t+1-j} + \phi_3(-\sum_{j=0}^{\infty}\psi_j\lambda\varepsilon_{t-j}) - \lambda\varepsilon_{t+3} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+3}}{Y_{t+3}}}{\partial \varepsilon_{t+1}} = -\psi_1\lambda\exp(\bar{d}) + \lambda(\phi_1\psi_1 + \phi_2)\exp(\bar{d} - \frac{1}{2}\lambda^2).$$

We evaluate this derivative at $\varepsilon_{t+j} = 0$ to obtain:

$$\frac{\partial \frac{S_{t+4}}{Y_{t+4}}}{\partial \varepsilon_{t+1}} = -\psi_2\lambda\exp(\bar{d}) + \lambda(\phi_1\psi_2 + \phi_2\psi_1 + \phi_3)\exp(\bar{d} - \frac{1}{2}\lambda^2).$$

This generalizes to the following expression. For $j > 2$, we obtain:

$$\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} = -\lambda\psi_{j-2}\exp(\bar{d}) + \lambda\psi_{j-1}\exp(\bar{d} - \frac{1}{2}\lambda^2).$$

Assume $r^f = \mu$. Then we obtain the IRF:

$$\begin{aligned}
\frac{\partial \frac{S_{t+j}}{Y_{t+j}}}{\partial \varepsilon_{t+1}} &= -\sigma\exp(\bar{d}) + \lambda\exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 1, \\
&= -\lambda\exp(\bar{d}) + \lambda\phi_1\exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 2, \\
&= -\lambda\psi_1\exp(\bar{d}) + \lambda(\phi_1\psi_1 + \phi_2)\exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j = 3, \\
&= -\lambda\psi_{j-2}\exp\bar{d} + \lambda\psi_{j-1}\exp(\bar{d} - \frac{1}{2}\lambda^2), \textit{ for } j > 3.
\end{aligned}$$

□

## A.12 Proof of Proposition 3.5

### A.12.1 Case of *AR(1)*

*Proof.* As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$\log d_t = -\sum_{j=0}^{\infty} \phi^j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1-\phi}.$$

Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$S_{t+1} \quad = \quad d_t Y_t \exp(r_t^f) - \exp(\phi \log d_t + \phi_0 - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}.$$

We get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) = -\mathbb{E}_t[M_{t+1}d_{t+1}Y_{t+1}] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}] \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \mu + y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1)\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}] \\
&= -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$\begin{aligned}
& cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2})] = cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}]) \\
= \quad & -\mathbb{E}_t[M_{t+1}]E_t[M_{t+1,t+2}d_{t+2}Y_{t+2}](\exp(-\gamma(\sigma - \phi\lambda)) - 1).
\end{aligned}$$

Check the proof of Prop. 1.3 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. We get the following expression for the covariance of the discounted surpluses over $j$ periods:

$$\begin{aligned}
& cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j}S_{t+j}]) = cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]) \\
= \quad & -\mathbb{E}_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma - \phi^{j-1}\lambda)) - 1).
\end{aligned}$$

□

### A.12.2 Case of *AR(2)*

*Proof.* We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. As a result, we can solve for an expression of the log debt/output ratio as a function of the past shocks:

$$\log d_t = -\sum_{j=0}^{\infty} \psi_j \lambda \varepsilon_{t-j} + \frac{\phi_0 - \frac{1}{2}\lambda^2}{1 - \phi_1 - \phi_2}.$$

where $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}$. Consider a government that only issues risk-free debt. Note that the surplus at $t+1$ is given by:

$$S_{t+1} \quad = \quad d_t Y_t \exp(r_t^f) - \exp(+\phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \lambda \varepsilon_{t+1} - \frac{1}{2}\lambda^2) Y_{t+1}.$$

As a result, we get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}), \\
&= -\mathbb{E}_t[M_{t+1}d_{t+1}Y_{t+1}] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}], \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \mu + y_t + \phi_1 \log d_t + \phi_2 \log d_{t-1} + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1)\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}],
\end{aligned}$$

$$= \quad -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma-\lambda))-1).$$

By the same token, we get the following expression for the covariance of the discounted surpluses over two periods:

$$\begin{aligned} & cov_t(M_{t+1}, S_{t+1} + E_{t+1}[M_{t+1,t+2}S_{t+2})], \\ = \quad & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+2}d_{t+2}Y_{t+2}]), \\ = \quad & -\mathbb{E}_t[M_{t+1}]E_t[M_{t+1,t+2}d_{t+2}Y_{t+2}](\exp(-\gamma(\sigma-\psi_1\lambda))-1) \end{aligned}$$

Check the proof of Prop. 1.3 to see why the sum of the discounted surpluses drop out, and only the debt issuance term remains. And we get the following expression for the covariance of the discounted surpluses over $j$ periods:

$$\begin{aligned} & cov_t(M_{t+1}, \sum_{k=1}^{j} E_{t+1}[M_{t+1,t+j}S_{t+j}]), \\ = \quad & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]), \\ = \quad & -\mathbb{E}_t[M_{t+1}]E_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1). \end{aligned}$$

□

## A.13 Proof of Corollary 3.1

### A.13.1 Case of *AR(1)*

*Proof.* Start from the restriction:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = \quad & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) + cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}x_{t+k}Y_{t+k}\right), \\ = \quad & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\phi^{j-1}\lambda))-1) + \sum_{k=1}^{j} cov_t\left(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+k}x_{t+k}Y_{t+k}]\right), \end{aligned}$$

where $\log x_t$ follows equation (8).

Next, we compute the $cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}x_{t+k}Y_{t+k}\right)$:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}x_{t+k}Y_{t+k}\right) \\ = \quad & \sum_{k=1}^{j} cov_t\left(M_{t+1}, \mathbb{E}_{t+1}[M_{t+1,t+k}x_{t+k}Y_{t+k}]\right) \\ = \quad & \sum_{k=1}^{j} [\mathbb{E}_t[M_{t+1}M_{t+1,t+k}x_{t+k}Y_{t+k}] - \mathbb{E}_t[M_{t+1}]E_t[M_{t+1,t+k}x_{t+k}Y_{t+k}]] \\ = \quad & \sum_{k=1}^{j} \mathbb{E}_t[M_{t+1}]E_t[M_{t+1,t+k}x_{t+k}Y_{t+k}]\left(\exp(-\gamma(\sigma-\phi_g^{k-1}b_g))-1\right). \end{aligned}$$

We then obtain the cash flow beta $\beta_t^{G,CF}(h)$ from the definition.

□

### A.13.2 Case of *AR(2)*

*Proof.* We use $\psi(L)$ to denote the infinite MA representation of the debt/output process. Start from the restriction:

$$\begin{aligned} & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\ = \quad & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1) + cov_t\left(M_{t+1}, (\mathbb{E}_{t+1}-\mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}x_{t+k}Y_{t+k}\right) \\ = \quad & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\sigma-\psi_{j-1}\lambda))-1) + \sum_{k=1}^{j} \mathbb{E}_t[M_{t+1}]E_{t+1}[M_{t+1,t+k}x_{t+k}Y_{t+k}]\left(\exp(-\gamma(\sigma-\phi_g^{k-1}b_g))-1\right). \end{aligned}$$

□

### A.13.3 Proof of Proposition 4.1

*Proof.* We start from the one-period government budget constraint:

$$S_{t+1} = d_t Y_t \exp(r_t^f) - d_{t+1} Y_{t+1},$$

to obtain the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}) \\
&= -\mathbb{E}_t[M_{t+1}d_{t+1}Y_{t+1}] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}], \\
&= -\exp(-\rho - \frac{\gamma}{\sigma}(\psi - 1)y_t - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \xi_0 + (\xi - \gamma/\sigma * (\xi - 1))y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2), \\
&+ \exp(-\rho - \frac{\gamma}{\sigma}(\xi - 1)y_t)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1)\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}], \\
&= -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over $j \geq 2$ periods:

$$\begin{aligned}
& cov_t(M_{t+1}, E_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]), \\
= \; & -\mathbb{E}_t[M_{t+1}M_{t+1,t+j}d_{t+j}Y_{t+j}] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}], \\
= \; & -\mathbb{E}_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi - 1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\ldots - \frac{\gamma(\xi - 1)}{\sigma}(1 + \xi + \ldots + \xi^{j-2})y_{t+1}) \\
& \exp(\phi^j \log d_t - \phi^{j-1}\lambda\varepsilon_{t+1} + \ldots)\exp(\xi^j y_t + \xi^{j-1}\sigma\varepsilon_{t+1} + \ldots)] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda - \gamma(\xi - 1)\frac{1 - \xi^{j-1}}{1 - \xi})) - 1), \\
= \; & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \gamma(1 - \xi^{j-1}))) - 1).
\end{aligned}$$

We can also derive restrictions on the covariances with the tax process. The first equation follows from

$$\begin{aligned}
& cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}T_{t+k}\right) \\
= \; & cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}\right) + cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\textstyle\sum_{k=1}^{j} M_{t+1,t+k}G_{t+k}\right), \\
= \; & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda + \gamma(1 - \xi^{j-1}))) - 1) \\
+ \; & x \cdot cov_t\left(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)\sum_{k=1}^{j} M_{t+1,t+k}Y_{t+k}\right).
\end{aligned}$$

where the covariance of the pricing kernel with the output strip price is given by:

$$\begin{aligned}
& cov_t\,(M_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t)M_{t+1,t+k}Y_{t+k}) \\
= \; & \mathbb{E}_t[M_{t+1}M_{t+1,t+k}Y_{t+k}] - \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+k}Y_{t+k}], \\
= \; & \mathbb{E}_t[\exp(-\rho - \gamma(\xi - 1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})M_{t+1,t+k}\exp(\xi^k y_t + \xi^{k-1}\sigma\varepsilon_{t+1} + \ldots)] - \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+k}Y_{t+k}], \\
= \; & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+k}Y_{t+k}](\exp(-\gamma(\xi^{k-1}\sigma + \gamma(1 - \xi^{k-1}))) - 1).
\end{aligned}$$

Next, we conjecture

$$\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] = \exp(\sum_{k=1}^{j} \bar{\kappa}_k)\exp(\phi^j \log d_t + f_j y_t).$$

Note

$$\mathbb{E}_t[M_{t,t+j}d_{t+j}Y_{t+j}] = \mathbb{E}_t[M_{t,t+1}\exp(\sum_{k=1}^{j-1} \kappa_k)\exp(\phi^{j-1}\log d_{t+1} + f_{j-1}y_{t+1})],$$

$$
\begin{aligned}
&= \mathbb{E}_t[\exp(-\rho - \frac{\gamma}{\sigma}(\xi-1)y_t - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\sum_{k=1}^{j-1}\tilde{\kappa}_k) \\
&\quad \exp(\phi^{j-1}(\phi\log d_t + \phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) + f_{j-1}(\xi_0 + \xi y_t + \sigma\varepsilon_{t+1}))].
\end{aligned}
$$

So we confirm the conjecture,

$$
\begin{aligned}
\exp(\tilde{\kappa}_j) &= \mathbb{E}_t[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1} + \phi^{j-1}(\phi_0 - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2) + f_{j-1}(\xi_0 + \sigma\varepsilon_{t+1}))] \\
\tilde{\kappa}_j &= -\rho - \frac{1}{2}\gamma^2 + \phi^{j-1}(\phi_0 - \frac{1}{2}\lambda^2) + f_{j-1}\xi_0 + \frac{1}{2}(-\gamma - \phi^{j-1}\lambda + f_{j-1}\sigma)^2
\end{aligned}
$$

and

$$
\begin{aligned}
f_j &= -\frac{\gamma}{\sigma}(\xi - 1) + f_{j-1}\xi \\
&= \xi^j + \frac{\gamma}{\sigma}(1 - \xi^j) = \frac{\sigma-\gamma}{\sigma}\xi^j + \frac{\gamma}{\sigma}
\end{aligned}
$$

So, for $j > 1$,

$$
\begin{aligned}
&\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \;& \mathbb{E}_t[\exp(\sum_{k=1}^{j-1}\tilde{\kappa}_k)\exp(\phi^{j-1}\log d_{t+1} + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})y_{t+1})], \\
= \;& \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \frac{1-\phi^{j-1}}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\
+ \;& \sum_{k=1}^{j-1}\frac{1}{2}(-\gamma - \phi^{k-1}\lambda + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 \\
+ \;& \phi^{j-1}(\phi\log d_t + \phi_0 - \frac{1}{2}\lambda^2) + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(-\phi^{j-1}\lambda + (\sigma-\gamma)\xi^{j-1} + \gamma)^2).
\end{aligned}
$$

By a similar logic,

$$
\begin{aligned}
&\mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}] \\
= \;& \exp((-\rho - \frac{1}{2}\gamma^2)(j-1) + \left(\frac{1-\xi^{j-1}}{1-\xi}\frac{\sigma-\gamma}{\sigma} + \frac{\gamma}{\sigma}(j-1)\right)\xi_0 \\
+ \;& \sum_{k=1}^{j-1}\frac{1}{2}(-\gamma + ((\sigma-\gamma)\xi^{k-1} + \gamma))^2 + (\frac{\sigma-\gamma}{\sigma}\xi^{j-1} + \frac{\gamma}{\sigma})(\xi_0 + \xi y_t) + \frac{1}{2}(((\sigma-\gamma)\xi^{j-1} + \gamma))^2).
\end{aligned}
$$

So

$$
\begin{aligned}
&\mathbb{E}_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \;& \mathbb{E}_{t+1}[M_{t+1,t+j}Y_{t+j}]\exp(\frac{1-\phi^j}{1-\phi}(\phi_0 - \frac{1}{2}\lambda^2) + \sum_{k=1}^{j-1}((\gamma-\sigma)\xi^{k-1}\phi^{k-1}\lambda + \frac{1}{2}(\phi^{k-1}\lambda)^2) \\
+ \;& \phi^j\log d_t - \phi^{j-1}\lambda((\sigma-\gamma)\xi^{j-1} + \gamma) + \frac{1}{2}(\phi^{j-1}\lambda)^2).
\end{aligned}
$$

□

# B Idiosyncratic Insurance

We now consider transfers that are contingent on idiosyncratic shocks.

**Proposition B.1.** *For households who participate in asset markets and who receive net transfers $\{G-T\} = \{G^a - T^a\} + \{G^i - T^i\}$, the marginal insurance benefit over horizon h is given by:*

$$\Omega^{h'}_{net}(0) > \frac{P^h_0\left[\{G^a_t - T^a_t\}\right]}{P^h_0\left[\{C_t\}\right]} = \frac{P^h_0\left[\{C_t\}\right] + P^h_0\left[\{G^a_t - T^a_t\}\right]}{P^h_0\left[\{C_t\}\right]} - 1$$

*Proof.* The government can provide insurance against idiosyncratic shocks without increasing the riskiness of its debt.

$$\mathbb{E}_0 \sum_{t=1}^{h} \frac{U^i_{c,t}}{U^i_{c,0}} \left( \left[ G^i_t - T^i_t \right] + \left[ G^a_t - T^a_t \right] \right) = \mathbb{E}_0 \sum_{t=1}^{h} \frac{U^i_{c,t}}{U^i_{c,0}} \left[ G^i_t - T^i_t \right] + \mathbb{E}_0 \sum_{t=1}^{h} M_{0,t} \left[ G^a_t - T^a_t \right].$$

Provided that the net transfers provide insurance against idiosyncratic risk:

$$\sum_{t=1}^{h} Cov_0 \left( \frac{U^i_{c,t}}{U^i_{c,0}}, \left[ G^i_t - T^i_t \right] \right) > 0,$$

we obtain the following inequality:

$$\mathbb{E}_0 \sum_{t=1}^{h} \frac{U^i_{c,t}}{U^i_{c,0}} \left( \left[ G^i_t - T^i_t \right] + \left[ G^a_t - T^a_t \right] \right) > \mathbb{E}_0 \sum_{t=1}^{h} M_{0,t} \left[ G^a_t - T^a_t \right].$$

This implies that the marginal benefit of insurance, including idiosyncratic insurance:

$$\Omega^{h'}_{net}(0) > \frac{P^h_0\left[\{G^a_t - T^a_t\}\right]}{P^h_0\left[\{C_t\}\right]}$$

However, the cost to the government of financing these transfers is given by:

$$\mathbb{E}_0 \sum_{t=1}^{\infty} M_{0,t} \left[ G^i_t - T^i_t \right] + \mathbb{E}_0 \sum_{t=1}^{\infty} M_{0,t} \left[ G^a_t - T^a_t \right] = \mathbb{E}_0 \sum_{t=1}^{\infty} M_{0,t} \left[ G^a_t - T^a_t \right],$$

where we have used that idiosyncratic transfers are orthogonal to the stochastic discount factor. □

This result shows that the government can provide insurance against idiosyncratic risk with making the debt riskier. Chien and Wen (2019, 2020) analyze the Ramsey planner's problem in environments with idiosyncratic and aggregate risk. Government debt allows households to self-insure against idiosyncratic risk. Because the interest rate is lower than the time discount factor, this self-insurance motive dominates the tax smoothing motive.

# C Fiscal Hedging Demand in Economy with Distortionary Taxation

In a class of dynamic models with distortionary taxation going back to Lucas and Stokey (1983), the government chooses the tax rate optimally to hedge shocks to government spending. If the government can issue state-contingent debt, the optimal tax rate inherits the serial correlation of government spending.To the extent that the government's debt securities do not span all the shocks that hit the economy, maturity choice plays an important role. In a model in which only spending shocks drive the term structure, Angeletos (2002) and Buera and Nicolini (2004) show how the government can choose the maturity of non-state-contingent government debt to mimic the complete markets allocations in Lucas and Stokey (1983), thus creating an explicit role for the maturity structure. In general, the government will not try to replicate the complete markets allocation if variation in interest rates is largely explained by non-spending shocks, as is the case in the data. When governments only issue risk-free debt, the market incompleteness imputes more persistence to the optimal tax rates, as shown by Aiyagari, Marcet, Sargent, and Seppälä (2002), unless the government can accumulate savings. Lustig et al. (2008) examine the optimal maturity structure when the government issues nominal non-state-contingent debt. A general version of the incomplete markets optimal taxation problem is analyzed by Bhandari et al. (2017).

When taxes are distortionary, the government has an additional fiscal hedging motive to issue risky debt. In a representative agent economy with distortionary taxation, Bhandari et al. (2017) show that the planner will choose the long-run target debt level to minimize the variance of its financing requirements:

$$Var_0 \left[ R^D_1 B M_{0,1} - \sum_{j=1}^{\infty} M_{0,j} S^{\overline{\tau}}_j \right] = B^2 Var_0 [R^D_1 M_{0,1}] - B cov_0 \left( R^D_1 M_{0,1}, \sum_{j=1}^{\infty} M_{0,j} S^{\overline{\tau}}_j \right) + \ldots,$$

where $S_{t+j}^{\overline{\tau}}$ denotes the surplus evaluated at a constant tax rate $\overline{\tau}$ such that $B = \mathbb{E}_0 \sum_{j=1}^{\infty} M_{0,j} S_j^{\overline{\tau}}$. Given the constant tax rate, this surplus stream will be risky, and the covariance term will tend to be negative:

$$cov_0 \left( R_1^D M_{0,1}, \sum_{j=1}^{\infty} M_{0,j} S_j^{\overline{\tau}} \right) < 0.$$

With positive debt outstanding $B > 0$, minimizing the variance of financing requirements is achieved by choosing a debt instrument with return $R_1^D = 1/M_{0,1}$. The optimal debt instrument is the riskiest one, the one with the maximum squared Sharpe ratio $Var_0(M_{0,1})$. In environments with only transitory shocks, the riskiest security is the longest-maturity debt instrument. (In environments with permanent shocks to the SDF, the riskiest asset would be more like equity.) The fiscal hedging motive would lead the government to prefer risky debt.

In general, the solution to the variance minimization problem in Bhandari et al. (see 2017, p.650) implies a long-run debt target:

$$B^* = \frac{cov_0 \left( R_1^D M_1, \sum_{j=1}^{\infty} M_{0,j} S_j^{\overline{\tau}} \right)}{var_0(R_1^D M_1)}.$$

and a rate at which the debt reverts to the target given by $\frac{1}{1+e^{-2\rho} var_0(M_1)}$, where $\rho$ denotes the rate of time preference.

In the case of risk-free debt, the optimal target debt level is negative:

$$B^* = \frac{cov_0 \left( M_1, \sum_{j=1}^{\infty} M_{0,j} S_j^{\overline{\tau}} \right)}{var_0(M_1)} < 0$$

Given that the surplus evaluated at constant tax rates is exposed to short-run and long-run output risk, the planner will want to accumulate assets because debt does not provide a fiscal hedge. This rationalizing the prescription in Aiyagari et al. (2002) that the government should save. Having savings (a sovereign wealth fund) is what allows the government to choose $\beta_t^T > \beta_t^G$ and insure taxpayers against macro shocks. In the limit, by accumulating sufficient assets, the government can implement the Lucas and Stokey (1983) complete markets allocation. Even when the debt is risky, Bhandari et al. (2017) find that the planner wants to accumulate assets in the long-run because $cov_0 \left( R_1^D M_{0,1}, \sum_{j=1}^{\infty} M_{0,j} S_j^{\overline{\tau}} \right) < 0$. Importantly, in an incomplete markets environment with idiosyncratic and aggregate risk, the Ramsey planner has an incentive to accumulate debt because the debt allows agents to self-insure (Chien and Wen, 2019, 2020) .

# D Marginal Cost of Business Cycles

In the spirit of Lucas (1987), the government could choose to remove all business cycle variation through fiscal policy. This would result in a deterministic consumption path $\boldsymbol{C} = C + G - T$. We obtain the marginal cost of business cycles from Alvarez and Jermann (2004):

$$\Omega_{net}^{h'}(0) = \frac{P_0^h [\{\boldsymbol{C}\}]}{P_0^h [\{C_t\}]} - 1.$$

If we remove all aggregate consumption risk through fiscal policy, then we obtain the following marginal cost to the government:

$$\Omega_{net}^{\infty'}(0) = \frac{P_0^{\infty} [\{\boldsymbol{C}\}]}{P_0^{\infty} [\{C_t\}]} - 1 = \frac{\frac{C_0}{y_0 - g}}{\frac{C_0}{r^C - g}} = \frac{r^C - g}{y_0 - g} - 1 = \frac{r^C - y_0}{y_0 - g},$$

where $y_0$ is the discount rate for a growing perpetuity without any cash flow risk. This is the maximum marginal benefit the government can achieve for its taxpayers. This upper bound will be higher in models with more permanent risk and/or with a larger price of permanent risk since the consumption risk premium, $r^C - y_0$, is larger in such models. Modern asset pricing has consistently found that permanent shocks to output and consumption account for most of the variance of the pricing kernel, and receive a high price of risk in securities market (e.g., Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Bansal and Yaron, 2004; Borovička et al., 2016; Backus et al., 2018). Conversely, models without large permanent shocks produce bond risk premia that exceed equity risk premia, which is counter-factual. Hence, in models with realistic asset pricing, the marginal benefit of fiscal stabilization to taxpayers is high. Equivalently, the marginal cost to the taxpayers of keeping the debt risk-free is high.

# E Return Betas and Cash Flow Betas

What is the relationship between return betas and cash flow betas? In the simple case with constant debt/output and spending/output ratios, there is a one-to-one mapping:

**Corollary E.1.** *The expected returns can be expressed as a function of the cash flow betas:*

$$
\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= \frac{x}{d(1-\xi_1) + x\xi_1} \frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{\mathbb{E}_t(M_{t+1})}, \\
&= \frac{x}{d(1-\xi_1) + x\xi_1} \exp(\mu + \frac{1}{2}\sigma^2)(1 - \exp(-\gamma\sigma)) \\
\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= \frac{1}{\xi_1} \frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{\mathbb{E}_t(M_{t+1})} \\
&= \frac{1}{\xi_1} \exp(\mu + \frac{1}{2}\sigma^2)(1 - \exp(-\gamma\sigma)),
\end{aligned}
$$

*where* $\xi_1 = \exp(-\rho - \gamma\sigma + \mu + 0.5\sigma^2)$.

*Proof.* From $R^f_{t+1} = \exp(\rho)$ and $\frac{T_t}{Y_t} = x - d\left(1 - R^f_{t-1}\frac{Y_{t-1}}{Y_t}\right)$, we have that the return on the tax claim can be stated as:

$$
\begin{aligned}
R^T_{t+1} &= \frac{P^T_{t+1}}{P^T_t - T_t} = \frac{(d + x\frac{\xi_1}{1-\xi_1})Y_{t+1} + (x - d\left(1 - R^f_t \frac{Y_t}{Y_{t+1}}\right))Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t}, \\
&= \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{(d + x\frac{\xi_1}{1-\xi_1})Y_t} + \frac{d\exp(\rho)}{(d + x\frac{\xi_1}{1-\xi_1})}.
\end{aligned}
$$

Similarly, we have an expression for the return on the spending claim:

$$
R^G_{t+1} = \frac{P^G_{t+1}}{P^G_t - G_t} = \frac{x\frac{\xi_1}{1-\xi_1}Y_{t+1} + xY_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t} = \frac{x\frac{1}{1-\xi_1}Y_{t+1}}{x\frac{\xi_1}{1-\xi_1}Y_t}.
$$

As a result, we can state the risk premium as follows:

$$
\begin{aligned}
\mathbb{E}_t\left[R^T_{t+1} - R^f_t\right] &= -\frac{cov\left(M_{t+1}, R^T_{t+1}\right)}{\mathbb{E}_t(M_{t+1})} = \frac{x}{d(1-\xi_1) + x\xi_1} \frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{\mathbb{E}_t(M_{t+1})}, \\
\mathbb{E}_t\left[R^G_{t+1} - R^f_t\right] &= -\frac{cov\left(M_{t+1}, R^G_{t+1}\right)}{\mathbb{E}_t(M_{t+1})} = \frac{1}{\xi_1} \frac{-cov\left(M_{t+1}, Y_{t+1}/Y_t\right)}{\mathbb{E}_t(M_{t+1})},
\end{aligned}
$$

where we have used that $\xi_1 = \exp(-\rho - \frac{1}{2}\gamma^2 + \mu + \frac{1}{2}(\gamma - \sigma)^2) = \exp(-\rho - \gamma\sigma + \mu + \frac{1}{2}\sigma^2)$.

Then plug in

$$
\begin{aligned}
\frac{-cov_t\left(M_{t+1}, Y_{t+1}/Y_t\right)}{\mathbb{E}_t(M_{t+1})} &= \frac{-cov_t\left(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}), \exp(\mu + \sigma\varepsilon_{t+1})\right)}{\mathbb{E}_t(\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1}))} \\
&= \frac{-cov_t\left(\exp(-\gamma\varepsilon_{t+1}), \exp(\sigma\varepsilon_{t+1})\right)}{\exp(-\rho)} \exp(-\rho - \frac{1}{2}\gamma^2 + \mu), \\
&= -(\exp(\frac{1}{2}(\gamma^2 + \sigma^2))(\exp(-\gamma\sigma) - 1)) \exp(-\frac{1}{2}\gamma^2 + \mu), \\
&= \exp(\mu + \frac{1}{2}\sigma^2)(1 - \exp(-\gamma\sigma)).
\end{aligned}
$$

□

# F Persistence of Fiscal Processes in the Data and the Model

Panel A of Figure 6 plots the sample autocorrelation function (ACF) of the log government debt/output ratio as a function of the number of annual lags. The top right panel plots the partial autocorrelation function (PACF). They are estimated on the post-war U.S. sample (1947–2019). The PACF function indicates that an AR(2) process fits the data well. Lags beyond two years in the PACF are not statistically different from zero. The point estimates for $\phi_1$ and $\phi_2$ are 1.40 and -0.48, respectively. Both roots lie outside the unit circle (1.66 and 1.25), so that the debt/output process is stationary. While the AR(2) is our preferred specification, if we were to fit an AR(1), the point estimate for $\phi_1$ would be 0.986.

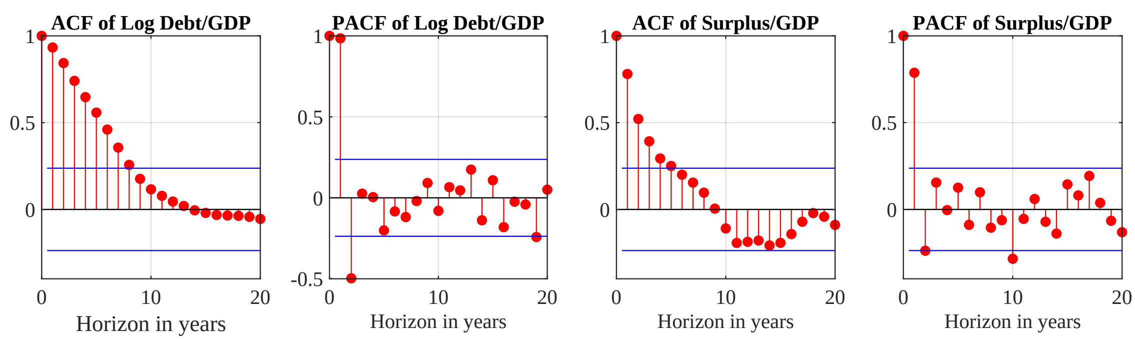
Panel A of Figure 6 plots the sample ACF and PACF for the primary surplus/output ratio in the data. The dynamics of surplus/output are well described by an AR(1). The surplus is quite persistent, with an $AR(1)$ coefficient around 0.81. The model cannot quite replicate the strong autocorrelation in surplus/output observed in the data. In the case of the AR(2), the ACF converges too quickly to zero, compared to the observed one plotted in Panel A of Figure 6. The ACF is no longer different from zero past two years, while in the data the ACF remains significantly positive for five years. Furthermore, the model produces a PACF(2) coefficient of -0.5, which is larger in absolute value than the one estimated in the data.

Figure 6 also plots the ACF and PACF of the debt/output and surplus/output ratios implied by the model of risk-free debt. Panel B is for the case where debt/output follows an AR(1) with the estimated persistence $\phi_1 = 0.985$. Panel C is for the case where debt/output follows an AR(2) with the estimated coefficients $\phi_1 = 1.4$ and $\phi_2 = -0.48$. The ACF and PACF for debt/output match the data by construction. As argued above, the AR(2) fits the ACF and PACF of the observed debt/output ratio the closest.

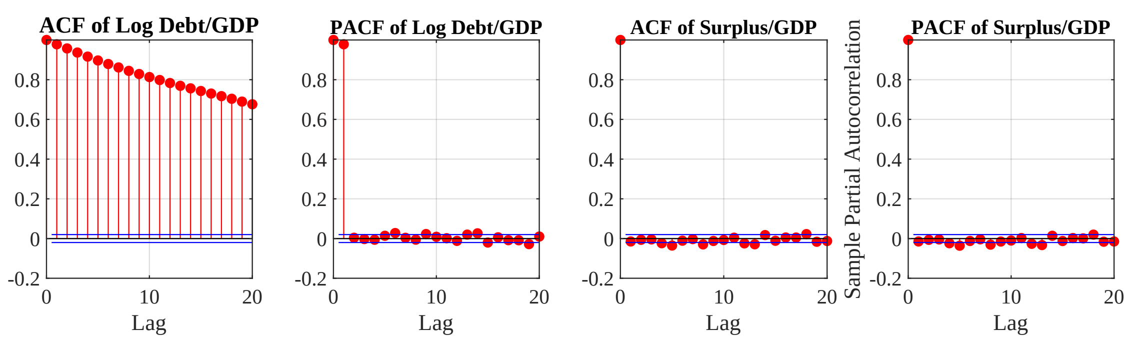
Figure 6: Autocorrelation in Debt/Output and Surplus/Output

Panel A plots the sample autocorrelation of the U.S. log government debt/output ratio, the U.S. government surplus/output ratio, the tax/output ratio and the spending/output ratio against GDP. Sample is annual, 1947—2019. Panel B plots the ACF and PACF of $S/Y$ and $D/Y$ for an $AR(1)$ with parameters $\phi_1 = 0.985$ and $\phi_2 = 0$. Panel C plots the ACF and PACF of $S/Y$ and $D/Y$ for an $AR(2)$. The parameters are listed in Table 1.

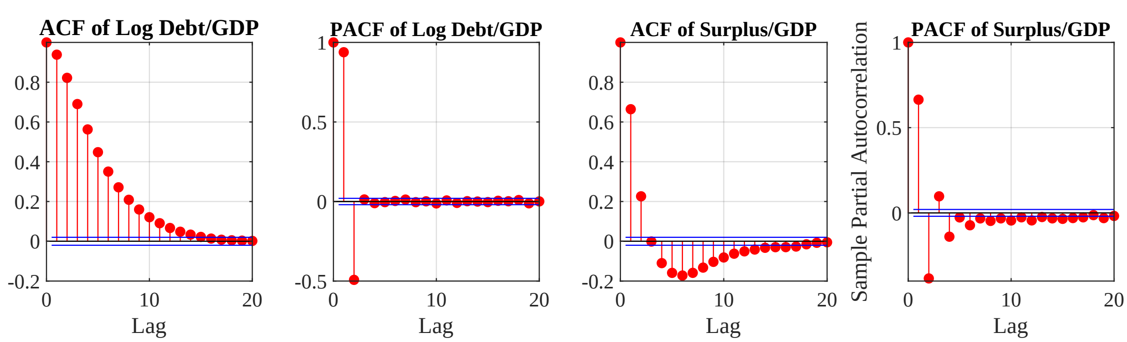
*Panel A: Data*




*Panel B: $AR(1)$ Model*




*Panel C: $AR(2)$ Model*



# G Robustness

We analyze the effects of the counter-cyclicality of spending, and the effects of saving.

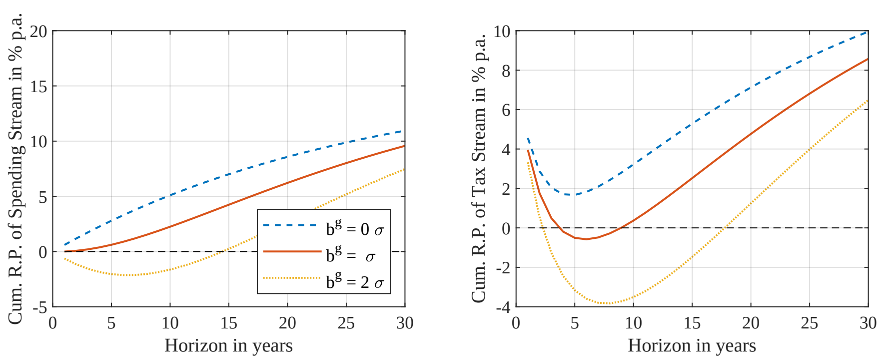
## G.1 Counter-cyclical Spending

The government insures transfer recipients by spending a larger fraction of GDP in recession. We now show that the more counter-cyclical spending becomes, the steeper the trade-off between insurance of bondholders and taxpayers.

Figure 7 plots scaled cash-flow betas for spending, $\beta_t^{G,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the top panel, and the implied tax revenue betas, $\beta_t^{T,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the bottom panel, for a range of values of the cyclicality of spending $b_g$ in equation (8). As government spending becomes more counter-cyclical, the short-term risk premium on the spending claim declines. The short-term risk premium on the tax claim has to decline as well in order to keep the government debt risk-free. As the tax claim becomes more counter-cyclical, taxpayers face a riskier tax liability proposition. As the governments provides more insurance to transfer recipients, this reduces the scope for insurance of taxpayers.

Figure 7: Varying the Counter-cyclicality of Spending

This figure plots the scaled cash-flow beta of spending $\beta_t^{G,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$ in the top panel and the implied tax revenue betas, $\beta_t^{T,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the bottom panel for a range of values of the cyclicality of spending $b_g$.



## G.2 Government Saving

Finally, we consider a government which saves instead of borrows.

We consider a government which saves at the risk-free rate: $D < 0$. Savings in levels is given by: $D_t = -\exp(\log d_t)$. We use the following stochastic AR-process in logs:

$$\log d_t = \phi_0 + \phi_1 \log d_{t-1} + \phi_2 \log d_{t-2} + \lambda \varepsilon_t - \frac{1}{2}\lambda^2, \tag{11}$$

where $\lambda$ now enters with a positive sign. In this case, the results in Prop. 3.5 go through. Of course, because $D < 0$, the government now has a short position in permanent output risk (because the value of taxes is smaller than the value of spending), and as a result, to keep its savings risk-free, the government surpluses have to contribute enough long-run output risk. We start with the simplest case in which the savings/output ratio is constant ($\lambda = 0$), Proposition 3.5 implies:

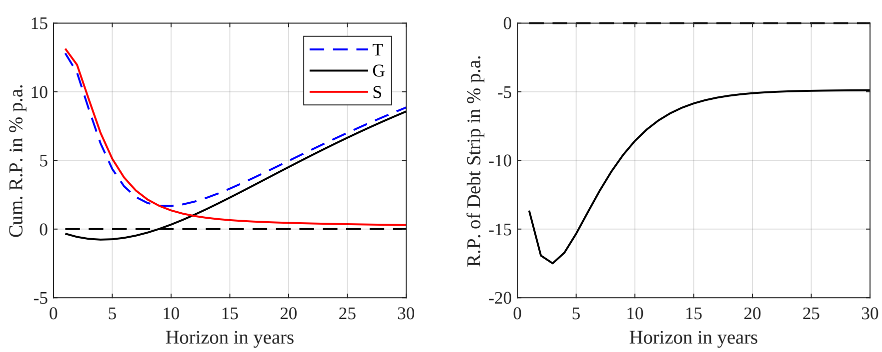
$$\beta_t^{S,CF}(h) = \frac{\mathbb{E}_t[M_{t+1}]}{-D_t var_t[M_{t+1}]} \mathbb{E}_t[M_{t+1,t+h} dY_{t+h}](1 - \exp\{-\gamma\sigma\}). \tag{12}$$

The cash-flow beta of the surplus is positive at all horizons since $\gamma\sigma > 0$. In bad times, the surplus/output ratio declines. When spending/output is constant (or also goes up), tax revenues/output must also decline. The government can insure taxpayers against adverse output shocks at all horizons. In fact, it has to do so, because its savings are risk-free. Figure 8 plots the risk premium on a claim to cumulative surpluses over the next $h$ periods in the left panel. The parameters are given in Table 1. The government now targets a savings/output ratio of 43%. The cumulative risk premium at horizon $h$ is the sum of the individual strip risk premia up until horizon $h$. The positive risk premium indicates that surpluses are risky over all horizons. Since taxpayers are short the surplus claim, their tax-minus-transfer liability is risky at all horizons. As $h \to \infty$, the sum of discounted surpluses converges to the current value of savings $D_t$. Insisting on risk-free savings ($\beta_t^D = 0$) implies that $\beta_t^{S,CF}(h) \to 0$. The red line in the left panel converges to zero from above, not from below, for large $h$.

### Figure 8: Risk Premia Across Horizons with Saving

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{i,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon $h$. The right panel plots minus the risk premium on the debt/savings strips: $1 - \exp\left\{\gamma(\phi^{h-1}\lambda - \sigma)\right\}$. The parameters are given in Table 1.

*$AR(2)$ Savings/Output*




In Panel B and C of Figure 8, we allow for pro-cyclical variation in the savings/output ratio $\lambda = 1.94 \times \sigma$. With pro-cyclical savings/output ratios, the government can provide even more insurance to taxpayers over short horizons by rendering the surplus and the tax revenue process riskier. The risk premium more than doubles at short horizons, which implies the marginal welfare benefit does too.

# H Entropy Bounds

**Proposition H.1.** *In the absence of arbitrage, the entropy of the SDF $M_{t+1} = \frac{\Lambda_{t+1}}{\Lambda_t}$ puts an upper bound on the expected log excess returns:*

$$L_t\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right) \geq \mathbb{E}_t \log R_{t+1} - \log R_{t+1,1}.$$

*Proof.* See Backus et al. (2014) for a comprehensive proof. We start from the definition of the conditional entropy of the SDF:

$$\begin{aligned} L_t\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right) &= \log \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} - \mathbb{E}_t \log \frac{\Lambda_{t+1}}{\Lambda_t} \\ &= -\mathbb{E}_t \log \frac{\Lambda_{t+1}}{\Lambda_t} - \log R_{t+1,1} \\ &\geq \mathbb{E}_t \log R_{t+1} - \log R_{t+1,1}. \end{aligned}$$

No arbitrage implies that:

$$\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1}\right] = 1.$$

Using Jensen's inequality, we obtain that:

$$0 = \log \mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1}\right] \geq \mathbb{E}_t \log\left(\frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1}\right).$$

As a result, the return $1/M_{t+1}$ the highest log return:

$$\mathbb{E}_t \log \frac{\Lambda_t}{\Lambda_{t+1}} \geq \mathbb{E}_t \log R_{t+1}.$$

□

**Proposition H.2.** *If the pricing kernel is not subject to permanent innovations, then the highest expected log return asset is the longest maturity zero-coupon bond.*

*Proof.* The return on a k-period zero coupon bond is given by:

$$\lim_{k\to\infty} \frac{\mathbb{E}_{t+1}\Lambda_{t+k+1}}{\Lambda_{t+1}} / \frac{\mathbb{E}_t \Lambda_{t+k}}{\Lambda_t}.$$

The expected log bond return is given by:

$$\lim_{k\to\infty} \mathbb{E}_t \log \frac{\mathbb{E}_{t+1}\Lambda_{t+k+1}}{\Lambda_{t+1}} / \frac{\mathbb{E}_t \Lambda_{t+k}}{\Lambda_t}.$$

The pricing kernel has no permanent innovations if and only if:

$$\lim_{k\to\infty} \mathbb{E}_t \log \frac{\mathbb{E}_{t+1}[\Lambda_{t+k}]}{\mathbb{E}_t[\Lambda_{t+k}]} = 0.$$

If there are no permanent innovations, then the expected log return on the long bond is given by:

$$\mathbb{E}_t \log \frac{\Lambda_t}{\Lambda_{t+1}}.$$

□

If $M$ is conditionally log-normal, the expected log excess return on a long position in the longest maturity bonds is (Alvarez and Jermann, 2005)

$$h_t(\infty) = \lim_{k\to\infty} \mathbb{E}_t rx^k_{t+1} = (1/2) Var_t(m_{t+1}).$$

# I Transitory Output Shocks and Permanent Shocks to Marginal Utility

This section introduces transitory shocks to output but keep our original SDF with permanent shocks to the level of marginal utility. In this setting, the government can insure taxpayers at all horizons while keeping the debt risk-free. We call this the goldilocks economy. However, this model is misspecified. In any structural model with transitory output (or productivity) risk, marginal utility will only have a transitory component as well. Hence, this model is merely an expositional device.

**Assumption 6.** *(a) The shocks to output are transitory. The log output process is given by:*

$$y_{t+1} \quad = \quad \xi_0 + \xi y_t + \sigma \varepsilon_{t+1}$$

*where $\varepsilon_{t+1}$ denotes the innovation to log output which is i.i.d. normally distributed.*
*(b) The log pricing kernel is:*

$$m_{t,t+1} \quad = \quad -\rho - \frac{1}{2}\gamma^2 - \gamma \varepsilon_{t+1}.$$

*(c) The government commits to a policy for the debt/output ratio $d_t = D_t/Y_t$ given by:*

$$\log d_t = \phi_1 \log d_{t-1} + \phi_0 - \lambda \varepsilon_t - \frac{1}{2}\lambda^2,$$

*where $\lambda > 0$ so that the debt-output ratio increases in response to a negative output shock $\varepsilon_t$.*

This asset pricing model is misspecified. This SDF does not reflect the mean-reversion in output and hence cannot be micro-founded. In this setting, the government faces no trade-off between insuring taxpayers and bondholders. The government can insure taxpayers over all horizons.

**Proposition I.1.** *The cash flow beta of the surpluses over j periods is given by:*

$$\beta_t^{S,CF}(j) \quad = \quad \frac{\mathbb{E}_t[M_{t+1}]}{D_t var_t[M_{t+1}]} \mathbb{E}_t[M_{t+1,t+j} d_{t+j} Y_{t+j}] (\exp(\gamma(\phi_1^{j-1}\lambda - \xi^{j-1}\sigma)) - 1)$$

*when $j \geq 1$. The sign of the cash flow beta is determined by sign $\left(\phi_1^{j-1}\lambda - \xi^{j-1}\sigma\right)$.*

Hence, the sign of the surplus cash-flow beta is determined by the sign of $\left(\phi_1^{j-1}\lambda - \xi^{j-1}\sigma\right)$. If $\lambda > \sigma$, the initial surplus cash-flow beta is positive. If the rate of mean-reversion in the output process is higher than in the debt/output ratio, $\phi_1 > \xi$, the surplus cash-flow beta stays positive for all $j > 1$. The positive surplus beta at all horizons indicates that the government can insure taxpayers at all horizons. This was not feasible in the case of permanent innovations.

For intuition, recall that the cumulative surplus risk premium is the inverse of the risk premium on a debt strip. The debt risk premium compensates investors for output risk. Because the output shocks are temporary, the output component of this risk premium converges to zero as the horizon grows. The transitory nature of output risk expands the scope for insurance of taxpayers. As $\xi \to 1$, we revert back to the expression derived in the benchmark model with permanent output risk: $\left(\phi_1^{j-1}\lambda - \sigma\right)$.

Figure 9 plots the risk premium on the cumulative surplus claim for the model with transitory output risk, $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$. The calibration is the same as in the benchmark model, except that the output process no longer has a unit root. In Panel A, we consider a case in which the debt/output ratio and the output processes are equally persistent: $\phi = \xi = 0.75$. At all horizons, the surplus claim is risky, contributing positive risk premium across all horizons. The tax claim is also risky across all horizons. In this goldilocks scenario, the government can insure taxpayers at all horizons while keeping the debt risk-free, insuring bondholders.

The right panel of Figure 9 plots the risk premium on the debt strips, which pay off $d_{t+j}Y_{t+j}$, given by

$$\gamma(\sigma\xi^{k-1} - \phi_1^{k-1}\lambda) \approx 1 - \exp\left\{-\gamma(\sigma\xi^{k-1} - \phi_1^{k-1}\lambda)\right\}.$$

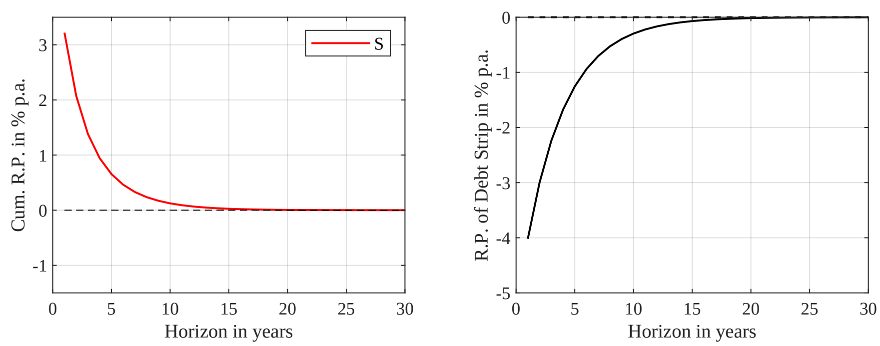
Given that $\lambda > \sigma$, the risk premium on the debt strips is negative at each horizon $j$. As $j \to \infty$, this debt strip risk premium converges to the risk premium on the output strips. The latter is 0% because the output innovations are transitory and the pricing kernel does not have a transitory component which contributes interest rate risk. The government can insure taxpayers over long horizon because the debt strip risk premium is negative at all horizons.

Simultaneous insurance of taxpayers and bondholders only works if the governments commits to a debt policy that is at least as persistent as the output process ($\phi_1 > \xi$). Panel B in Figure 9 plots a scenario where the debt/output ratio is less persistent than the debt/output ratio. In this case, the government has to produce safer surplus claims over longer horizons. The trade-off between insuring bondholders and taxpayers re-emerges.

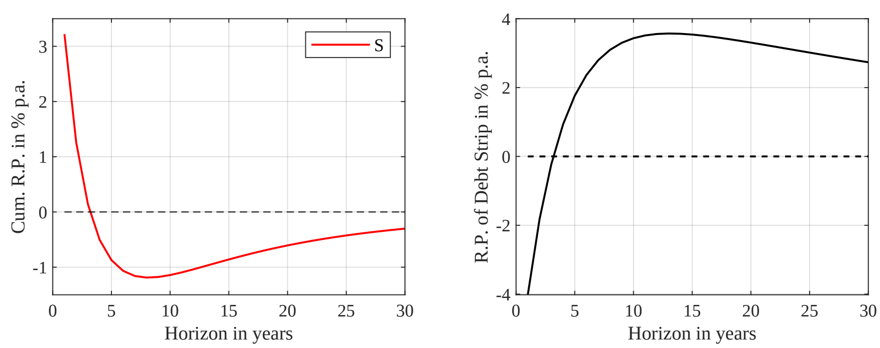
Figure 9: Risk Premia with Transitory Shocks to Output and Permanent Shocks to SDF

The figure plots the risk premium of cumulative discounted cash flows, $\beta_t^{S,CF}(h) \times \frac{var_t[M_{t+1}]}{\mathbb{E}_t[M_{t+1}]}$, in the left panel against the horizon $h$. The right panel plots the risk premium on the debt strips: $1 - \exp(-\gamma(\sigma\xi^{j-1} - \phi^{j-1}\lambda))$. In top panel, $\phi$ is 0.75 and $\xi$ is 0.75. In bottom panel, $\phi$ is 0.75 and $\xi$ is 0.98. Other parameters follow benchmark calibration in Table 1.

*Panel A: Output AR(1) with* $\xi = 0.75$




*Panel B: Output AR(1) with* $\xi = 0.98$



**Proof of Proposition I.1:**

*Proof.* Starting from the government budget constraint,

$$S_{t+1} \quad = \quad d_t Y_t \exp(r_t^f) - d_{t+1} Y_{t+1},$$

we get the following expression for the covariance:

$$\begin{aligned}
cov_t(M_{t+1}, S_{t+1}) &= cov_t(M_{t+1}, -d_{t+1}Y_{t+1}), \\
&= -\mathbb{E}_t[M_{t+1}d_{t+1}Y_{t+1}] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}], \\
&= -\exp(-\rho - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&+ \exp(-\rho)\exp(\frac{1}{2}(\lambda - \sigma)^2 + \xi_0 + \xi y_t + \phi \log d_t + \phi_0 - \frac{1}{2}\lambda^2) \\
&= -(\exp(-\frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma + \lambda - \sigma)^2 - \frac{1}{2}(\lambda - \sigma)^2) - 1)\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}], \\
&= -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[d_{t+1}Y_{t+1}](\exp(-\gamma(\sigma - \lambda)) - 1).
\end{aligned}$$

By the same token, we get the following expression for the covariance of the discounted surpluses over $j \geq 2$ periods:

$$\begin{aligned}
& cov_t(M_{t+1}, E_{t+1}[\sum_{k=1}^{j} M_{t+1,t+k}S_{t+k}]) \\
= \; & cov_t(M_{t+1}, -E_{t+1}[M_{t+1,t+j}d_{t+j}Y_{t+j}]), \\
= \; & -\mathbb{E}_t[M_{t+1}M_{t+1,t+j}d_{t+j}Y_{t+j}] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}], \\
= \; & -\mathbb{E}_t[\exp(-\rho - \frac{1}{2}\gamma^2 - \gamma\varepsilon_{t+1})\exp(\ldots - \frac{\gamma(\xi - 1)}{\sigma}(1 + \xi + \ldots + \xi^{j-2})y_{t+1}), \\
& \exp(\phi^j \log d_t - \phi^{j-1}\lambda\varepsilon_{t+1} + \ldots)\exp(\xi^j y_t + \xi^{j-1}\sigma\varepsilon_{t+1} + \ldots)] + \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}] \\
= \; & -\mathbb{E}_t[M_{t+1}]\mathbb{E}_t[M_{t+1,t+j}d_{t+j}Y_{t+j}](\exp(-\gamma(\xi^{j-1}\sigma - \phi^{j-1}\lambda)) - 1)
\end{aligned}$$

□