THE CONVERSE ENVELOPE THEOREM

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paper: arxiv.org/abs/1909.11219
Envelope theorem: optimal decision-making \( \Rightarrow \) formula.

Textbook intuition: formula is consequence of FOC.

Modern envelope theorem of MS02\(^1\): almost no assumptions. \( \rightarrow \) FOC ill-defined, so need different intuition.

My theorem: with almost no assumptions, formula equivalent to generalised FOC.

\(-\) an envelope theorem: FOC \( \Rightarrow \)

\(-\) a converse: formula \( \Rightarrow \) FOC.

Application to mechanism design.

Environment

Agent chooses action $x$ from a set $\mathcal{X}$ to maximise objective $f(x, t)$, where $t \in [0, 1]$ is a parameter.

No assumptions on $\mathcal{X}$, almost none on $f$:

1. $f(x, \cdot)$ is differentiable for each $x \in \mathcal{X}$
2. $f(x, \cdot)$ is ‘not too erratic’. (definition: slide 12)

Decision rule: a map $X : [0, 1] \rightarrow \mathcal{X}$.

Associated value function: $V_X(t) := f(X(t), t)$. 
Envelope theorem

\[ X \text{ satisfies the } \bigotimes \text{ formula iff } \]

\[ V_X(t) = V_X(0) + \int_0^t f_2(X(s), s) \, ds \quad \text{for every } t \in [0, 1]. \]

Equivalently: \( V_X \) is absolutely continuous and

\[ V'_X(t) = f_2(X(t), t) \quad \text{for a.e. } t \in [0, 1]. \]

\( X \) is optimal iff \( \) for every \( t \), \( X(t) \) maximises \( f(\cdot, t). \)

Modern envelope theorem (MS02).
Any optimal decision rule satisfies the \( \bigotimes \) formula.
Textbook intuition

Differentiation identity:

\[ V'_X(t) = \frac{d}{dm} f(X(t + m), t) \bigg|_{m=0} + f_2(X(t), t). \]

'indirect effect' + 'direct effect'.

\[ V'_X(t) = \text{direct effect} \quad (\boxcheck \text{ formula}) \]

\[ \iff \text{indirect effect} = 0 \quad (\text{FOC}). \]

Problem: ‘indirect effect’ (hence FOC) ill-defined!

- \( f(\cdot, t) \) & \( X \) need not be differentiable.

- actions \( \mathcal{X} \) need have no convex or topological structure.
The outer first-order condition

*Disjuncture:* in general, \( \Box \) formula \( \iff \) FOC.

- one solution: add strong ‘classical’ assumptions. (slide 13)
- my solution: find the correct FOC!

Decision rule \( X \) satisfies the outer FOC iff

\[
\frac{d}{dm} \int_r^t f(X(s + m), s) ds \bigg|_{m=0} = 0 \quad \text{for all } r, t \in (0, 1).
\]

‘Integrated’ version of classical FOC.

- always well-defined
- equiv’nt to classical FOC when latter well-defined. (slide 13)
Theorem

Envelope theorem & converse.
For a decision rule $X : [0, 1] \rightarrow \mathcal{X}$, the following are equivalent:

(1) $X$ satisfies the oFOC

$$\frac{d}{dm} \left. \int_r^t f(X(s + m), s) \, ds \right|_{m=0} = 0 \quad \text{for all } r, t \in (0, 1),$$

and $V_X(t) := f(X(t), t)$ is absolutely continuous.

(2) $X$ satisfies the $\blacksquare$ formula

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s) \, ds \quad \text{for every } t \in [0, 1].$$

(proof idea: slide 14)
Mechanism design application: environment

Agent with preferences $f(y, p, t)$ over physical outcome $y \in \mathcal{Y}$ and payment $p \in \mathbb{R}$.

- type $t \in [0, 1]$ is agent’s private info
- assume single-crossing.

What’s new:
- outcome space $\mathcal{Y}$ is an abstract partially ordered set
- preferences not assumed quasi-linear in payment.

A physical allocation is $Y : [0, 1] \rightarrow \mathcal{Y}$.

$Y$ is implementable iff $\exists$ payment rule $P : [0, 1] \rightarrow \mathbb{R}$ s.t. $(Y, P)$ is incentive-compatible.

\[
\left( \text{viz.} \quad f(Y(t), P(t), t) \geq f(Y(r), P(r), t) \quad \text{for all } r, t. \right)
\]
Implementability theorem.
Under weak regularity assumptions,
any increasing physical allocation is implementable.

Argument:
- fix an increasing physical allocation $Y : [0, 1] \rightarrow \mathcal{Y}$
- choose a payment rule $P$ so that $\Box$ holds
- then by converse envelope theorem, $\circ$FOC holds
  $\iff$ mechanism $(Y, P)$ is locally IC.
- finally, local IC $\implies$ global IC by single-crossing.
Mechanism design application: example

Monopolist selling information.

Physical allocations $\mathcal{Y}$:
distributions of posterior beliefs, ordered by Blackwell.

By the implementability theorem, any information allocation that gives higher types Blackwell-better signals can be implemented.
Thanks!
Definition of ‘not too erratic’

A family \( \{ \phi_x \}_{x \in \mathcal{X}} \) of functions \([0, 1] \to \mathbb{R}\) is uniformly absolutely continuous (UAC) iff the family

\[
\left\{ t \mapsto \sup_{x \in \mathcal{X}} \left| \frac{\phi_x(t + m) - \phi_x(t)}{m} \right| \right\}_{m > 0}
\]

is uniformly integrable.

‘\( f(x, \cdot) \) not too erratic’ (slide 3)
means precisely that \( \{ f(x, \cdot) \}_{x \in \mathcal{X}} \) is UAC.

- a sufficient condition (maintained by MS02):
  \[ t \mapsto \sup_{x \in \mathcal{X}} |f_2(x, t)| \] dominated by an integrable function.

- a stronger sufficient condition: \( f_2 \) bounded.

\[ \hookrightarrow \text{back to environment (slide 3)} \]
Classical assumptions

*Classical assumptions:*
- $\mathcal{X}$ is a convex subset of $\mathbb{R}^n$
- action derivative $f_1$ exists & is bounded
- only Lipschitz continuous decision rules $X$ are considered.

(Bad for applications. Especially the Lipschitz restriction!)

\[
\text{Classical FOC: } \left. \frac{d}{dm} f(X(t+m), t) \right|_{m=0} = 0 \quad \text{for a.e. } t.
\]

Classical envelope theorem and converse.
Under the classical assump’ns, classical FOC $\iff \square$ formula.

Housekeeping lemma.
\(\text{oFOC } \iff \text{ classical FOC} \quad \text{whenever the latter is well-defined.}\)

$\rightarrow$ back to oFOC (slide 6)
Proof idea

Textbook intuition was based on differentiation identity:

\[ V'_X(s) = \frac{d}{dm} f(X(s + m), s) \bigg|_{m=0} + f_2(X(s), s), \]

\[ \text{‘indirect effect’} \quad \text{‘direct effect’} \]

or (integrating)

\[ V_X(t) - V_X(r) = \int_r^t \frac{d}{dm} f(X(s + m), s) \bigg|_{m=0} \, ds + \int_r^t f_2(X(s), s) \, ds. \]

I prove that the ‘outer’ version is always valid:

\[ V_X(t) - V_X(r) = \frac{d}{dm} \int_r^t f(X(s + m), s) \, ds \bigg|_{m=0} + \int_r^t f_2(X(s), s) \, ds. \]

\[ \text{‘indirect effect’} \quad \text{‘direct effect’} \]

The rest is easy:

\[ V_X(t) - V_X(r) = \text{direct effect} \quad (\blacklozenge \text{ formula}) \]

\[ \iff \quad \text{indirect effect} = 0 \quad (\text{oFOC}). \]