The Flexible Inverse Logit Model (FIL) model, a structural inverse demand model for products that are differentiated in a way that is both observed and unobserved by the modeller. The FIL model has three main attractive features: (i) it is easy to estimate by linear IV regression; (ii) it provides rich substitution patterns; (iii) it is consistent with utility maximization by heterogeneous consumers.

General Setting

- Differentiated products \((j = 1, \ldots, J)\) and 1 outside good \((j = 0)\).
- \(x = (x_1, \ldots, x_J)\): vector of observed characteristics,
- \(\xi = (\xi_1, \ldots, \xi_J)\): vector of unobserved characteristics terms,
- \(p = (p_1, \ldots, p_J)\): vector of prices,
- \(s = (s_1, \ldots, s_J) \in \Delta^J_+\): vector of non-zero market shares.

Linear Index Restriction (Berry and Haile, 2014). Set \(x = (x^{(1)}, x^{(2)})\)

\[
\begin{align*}
\delta_j &= x^{(1)}_j \beta - \alpha p_j + \xi_j, \quad j > 0, \quad \text{and} \quad \delta_0 = 0. \\
\end{align*}
\]

Flexible Inverse Logit Model

The FIL model is the inverse demand function \(\sigma^{-1}: \Delta^* \rightarrow \mathbb{R}^J\)

\[
\sigma^{-1}(x, \mu) = \ln \left( \frac{s_j}{1 - \sum_{k \neq j} s_k} \right) - \sum_{i \in J} \mu_{ij} \ln \left( \frac{s_j}{s_i + s_j} \right) = \delta_j, \quad j > 0. 
\]

Economic Restrictions. Imply that the FIL model (2) is invertible, i.e.,

defines a demand function (rather than a correspondence).

Motivations. The FIL model

1. allows deviations from IIA thanks to its parameters \(\mu_{ij}\) (it reduces to the logit model when all \(\mu_{ij} = 0\)).
2. is a member of Fosgerau, Monardo and de Palma (2020)'s class of closed-form inverse demand models based on nesting (with a nest for each pair \((i, j)\) of products and a nest for \(j = 0\)).
3. is specific instance of the large class of models of consumer heterogeneity studied by Allen and Rehbeck (2019): \(\mu\) parametrizes the distribution of preferences.

Substitution Patterns.

- The FIL model is flexible in the sense of Diewelt (1974) in a large class of well-defined inverse demand functions.
- (R1) and (R2) imply that (i) \(\sigma_j\) is strictly increasing in \(p_j\), and (ii) does not restrict products to be substitutes in demand.
- \(\mu_{ij}\) governs the substitution between products \(i\) and \(j\). Example: \(j = 3, s_1 = 0.15, s_2 = 0.25, s_3 = 0.20, p_1 = p_2 = p_3 = 1\) and \(\mu_{23} = 0.2\).

\(\Rightarrow\) Higher \(\mu_{12}\) implies a higher cross-price elasticity.

- Use the distance-metric of Pinkse, Slade and Brett (2002) to obtain substitution patterns that depend on \(x^{(2)}\) directly:
  - Closer products in \(x^{(2)}\) tends to be more substitutable.
  - Example: projection into \(x \in [0, 1]\) with similarity measure \(d_{ij} = 1 - |x_i - x_j|\): specify \(\mu(\delta_{ij}; \gamma) = \sum_{k=0}^{M} \gamma_k (\delta_{ij})^k\).

Estimation and Identification

Estimation by Linear IV Regression with Aggregate Data.

\[
\ln \frac{s_j}{s_0} = x^{(1)}_j \beta - \alpha p_j + \sum_{i \in J} \mu_{ij} \ln \left( \frac{s_j}{s_i + s_j} \right) + \xi_j, \quad j > 0
\]

- Assumption: prices and log-shares are endogenous (i.e., correlated with \(\xi\)), but product characteristics are exogenous.

Identification = Identification of the parameters.

- Main identification assumption: existence of instruments \(z\).
- Identification of \(\delta\). Easy!
  - Higher market shares implies higher utility indexes: given \(\mu\), there is a one-to-one mapping between \(\delta\) and \(s\).
  - Parameters in \(\delta\): dealing with price endogeneity thanks to valid supply-side instruments (cost shifters and/or markup shifters).
- Identification of \(\mu_{ij}\)'s. More tricky (as random coefficients).
  - Requires exogenous variation in the relative share of product \(j\) with respect to product \(i\), \(\ln(s_j/(s_i + s_j)) = -\ln(s_i + 1)\).
  - Need instruments that reveal about the substitution patterns: variables that generate exogenous variation in the choice set (including changes in prices) are good candidates.

  - Stylized example: \(j = 3, s_1 = 0.15, s_2 = 0.25, s_3 = 0.20, p_1 = p_2 = p_3 = 1\) and \(\mu_{13} = 0.2\).
  - Variation in cost shifters: \(\Delta c_1 = 10\%\).

\(\Rightarrow\) Monotonic relationships: the way prices and relative shares change with \(\Delta c_1\) drives the estimate of \(\mu_{12}\).

Comparison to BLP

Simulated DGP based on Armstrong (2016).

- Simulate a fully structural static model of demand and supply.
  - Demand: RCL model with utility linear in income and prices and with one normally distributed coefficient on an exogenous continuous characteristic \(x_{(2)}^3; \beta_h \sim N(3,6)\).
  - Supply: price competition model with multi-product firms.

Results.

<table>
<thead>
<tr>
<th></th>
<th>True Distribution</th>
<th>True Parameters</th>
<th>Median Estimation</th>
<th>Median Estimated Parameters</th>
<th>Root Mean Square Error</th>
<th>Root Mean Square Error</th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>FIL</td>
<td>1 (0.49)</td>
<td>0.69</td>
<td>0.69</td>
<td>0.69</td>
<td>0.13</td>
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<tr>
<td>BLP</td>
<td>1 (0.49)</td>
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<td>0.13</td>
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Conclusion. Simulations show that the FIL is able to match the substitution patterns of the RCL model pretty well and to obtain quite right predictions of a merger/new product's price effects.

Perspectives. The FIL model allows for complementarity in demand: (i) simulation with Genzktow (2007)'s DGP; (ii) revisit his work on the substitution between offline and online channels with an application to the hotel industry.