Measurement of Factor Strength: Theory and Practice*

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Abstract

This paper proposes an estimator of factor strength and establishes its consistency and asymptotic distribution. The estimator is based on the number of statistically significant factor loadings, taking multiple testing into account. Both cases of observed, and unobserved factors are considered. The small sample properties of the proposed estimator are investigated using Monte Carlo experiments. It is shown that the proposed estimation and inference procedures perform well, and have excellent power properties, especially when the factor strength is sufficiently high. Empirical applications to factor models for asset returns show that out of 146 factors recently considered in the literature, only the market factor is truly strong, while all other factors are at best semi-strong, with their strength varying considerably over time. Similarly, we only find evidence of semi-strong factors using a large number of U.S. macroeconomic indicators.

Keywords: Factor models, factor strength, measures of pervasiveness, cross-sectional dependence, market factor, macroeconomic shocks

JEL Classifications: C38, E20, G20

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1 Introduction

This paper is concerned with the characterisation and estimation of individual factor strengths in the context of multi-factor models, both when the factors are observed and when they are latent. We propose to measure the strength of a given factor by the degree of its pervasiveness identified by the number of its associated non-zero factor loadings. The degree of factor strength is measured by the rate at which the number of non-zero factor loadings rises with the total number of loadings. A factor is said to have maximum strength (equal to 1) if *all* its associated loadings are non-zero. A factor is deemed to be weak if the rate of non-zero factor loadings increase is less than 1/2, and factors with strength between 1/2 and 1 will be referred to as semi-strong in the sense that they are pervasive but not necessarily strong. More formally, for illustrative purposes, consider the following single factor model

$$x_{it} = c_i + \gamma_i f_t + u_{it},\ i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T, \tag{1}$$

where f_t, is a known factor, c_i is the unit-specific effect, $u_{it} \sim IID(0, \sigma_i^2)$ is an idiosyncratic error, and γ_i is the factor loading for unit i. In the standard factor literature, the strength of f_t is measured by the rate at which $\omega_n^2 = \sum_{i=1}^n \gamma_i^2$ rises with n. Denoting the expansion rate of ω_n^2 in terms of n by α, the standard large n and T latent factor models assume that $\alpha = 1$, as required, for example, by Assumption B in Bai and Ng (2002) and Bai (2003). At the other extreme, a factor is deemed to be weak if $0 \leq \alpha < 0.5$. This case is studied in Onatski (2012). Similar notions of factor strength are also used in recent financial studies by Lettau and Pelger (2018), and Anatolyev and Mikusheva (2019).

The rate α is determined by the number as well as the size of non-zero loadings, γ_i. In this paper we focus on the former whilst a number of papers in the literature that consider the case of weak factors, assume that $\gamma_i = \gamma_{in} = \delta_i / n^{(1-\alpha)/2}$, with bounded and non-zero δ_i for all i, which yields $\omega_n^2 = \left(n^{-1} \sum_{i=1}^n \delta_i^2\right) n^{\alpha}$. See, for example, Kleibergen (2009) or Onatski (2012) who consider factor models with $\alpha = 1/2$. This approach restricts all loadings, γ_{in}, to decline at the same rate with n, when $\alpha < 1$. In our view declining values for γ_i, as n increases, make little empirical sense. Our chosen setup where the main determinant of factor strength is the number of non-zero factor loadings is empirically more defensible and verifiable. Estimation of α under our formulation is also easier to implement as compared to the alternative formulation, $\gamma_{in} = \delta_i / n^{(1-\alpha)/2}$. To our knowledge there is no literature on how to estimate α under this alternative specification.

In most empirical applications, the value of α is unknown. Incorrectly setting it to $\alpha = 1$ can result in misleading inference. Also, as we shall see, without further *a priori* restrictions on the factor loadings, it is not possible to identify α when the factor in question is weak ($\alpha < 1/2$). But in most empirical applications in finance and macroeconomics, the values of α that are of interest and of consequence, are within the range $\alpha \in (0.5, 1]$. As recently shown by Pesaran and Smith (2019), factor strengths play a crucial role in the identification of risk premia in arbitrage asset pricing models, and determine the rates at which risk premia can be estimated. The strength of macroeconomic shocks is also of special interest, as its value has important bearing on forecasting and policy analysis. Contributions in terms of factor selection and factor model estimation when $\alpha \in (0.5, 1)$ include Freyaldenhoven (2019) and Uematsu and Yamagata (2019).

The analysis of this paper is also closely related to the literature on strong and weak cross-sectional dependence. One important example is the role of dominant units in production or financial networks and how to identify and measure their degree of dominance when interconnections are known (Acemoglu et al. (2012), Pesaran and Yang (2020)), or unknown (Kapetanios et al. (2020)). Bailey et al. (2016) (BKP hereafter) give a thorough account of the rationale and motivation behind the need for determining the extent of CSD, be it in finance, micro or macroeconomics. To estimate the degree of CSD in a panel dataset, BKP analyse the rate at which the variance of the cross section average of observations in that panel tends to zero and show that it depends on the degree or exponent of CSD which they denote by α. They explore a latent factor model setting as a vehicle for characterising strong and semi-strong covariance structures as defined in Chudik et al. (2011). They relate these to the degree of pervasiveness of factors in unobserved factor models often used in the literature to model CSD. In a follow up paper to BKP, Bailey et al. (2019) extend their analysis in two respects. First, they consider a more generic setting which does not require a common factor representation and holds more generally for both moderate and sizeable CSD. They achieve this by directly considering the significance of individual pair-wise correlations and base the estimation of α on the proportion of statistically significant correlations. Second, they show that their estimator also applies to the residuals obtained from panel data regressions.

The estimators developed in Bailey et al. (2016, 2019) are helpful as overall measures of CSD, but they do not provide information on the strength of *individual* factors which is of interest, for example, in the pricing of risk in empirical finance and in identifying dominant factors in macroeconomic fluctuations. In this paper we propose an estimator of factor strength and establish its consistency and asymptotic distribution when $\alpha > 1/2$. The proposed estimator is based on the number of statistically significant factor loadings, taking account of the multiple tests being carried out. We find that it is a powerful and highly accurate estimator, especially for higher levels of factor strength. Despite its simplicity, the distribution of the estimator, being based on sums of random variables that follow, potentially heterogeneous, Bernoulli distributions, is quite complicated and non-standard. While the parameters of these distributions are hard to pin down, they can be bounded in such a way as to provide both grounds for the validity of a central limit theorem for the asymptotically dominant part of the estimator and an upper bound for the asymptotic variance. These two elements allow for the construction of asymptotically conservative test statistics.

We focus mainly on the case where the factors are observed, which is of primary interest in tackling the financial empirical example mentioned earlier, among many others. We also consider using cross section averages as a proxy in the case of unobserved common factors. In practice, we face a significant factor identification issue when there are more than one unobserved common factors. In the case of multiple unobserved factor models, our contribution is best viewed as providing inferential information about the exponent of the strongest factor shared amongst the cross section units, even though we present some results on estimating the strength of weaker factors with $1/2 < \alpha < 1$.

We investigate the small sample properties of the proposed estimator by means of Monte Carlo experiments under a variety of scenarios. In general, we find that the estimator, and the associated inference, perform well. The test is conservative under the null hypothesis, but, nevertheless, has excellent power properties, especially when α is close to unity, even for moderate sample sizes.

We illustrate the relevance of our proposed estimator by means of two empirical applications, using well known datasets in finance and macroeconomics. First, we consider a large number of factors proposed in the finance literature for asset pricing. For example, Harvey and Liu (2019) document over 400 such factors, and Feng et al. (2020) consider the problem of factor selection using penalised regressions. In view of recent theoretical results in Pesaran and Smith (2019), our empirical contribution focuses on the estimation of factor strengths, since factor selection is only meaningful for asset pricing if the factors under consideration are sufficiently strong. We compute 10-year rolling estimates of α (together with their standard error bands) for the excess market return (as a measure of the market factor), and the remaining 145 factors considered by Feng et al. (2020). Out of the 146 factors considered, we find that only the market factor is sufficiently strong over all rolling windows, with its average strength estimated to be around 0.99 over the full sample (from September 1989 to December 2017). In contrast, none of the other factors achieve strengths exceeding 0.90 over the full sample, but over the sub-sample that includes the recent financial crisis as many as 48 (out of 145) have average strength estimated to lie between 0.9 and 0.94. Remarkably, the well-known size and value factors introduced in Fama and French (1993) are not particularly prominent as compared to cash and leverage factors. Further, of special interest is the high degree of time variation in the estimates of factor strengths, which cannot be attributed to sampling variation, considering the high precision with which the factor strengths are estimated, particularly when the true factor strength is close to unity.

Our second empirical application considers an unobserved factor model and asks if there exists any strong latent factor shared by the set of macroeconomic variables originally investigated by Stock and Watson (2012). In particular, we consider an updated version of Stock and Watson (SW) dataset covering 187 variables over the period 1988Q1-2019Q2. Although it is not possible to separately identify the strengths of individual latent factors, we are able to show that the strength of the strongest of the latent factors in the updated SW data set is around 0.94 which is sufficiently high for the factor to be important for macroeconomic analysis, but yet statistically different from 1, usually assumed in the literature.

The rest of the paper is organised as follows: Section 2 introduces our proposed measure of factor strength and develops the estimation and inference theory for the single factor case. A general multi-factor set up is then considered in Section 3 which includes the main theoretical results of the paper. Section 4 discusses the case of unobserved factors, and after highlighting the identification problem involved, considers first the estimation of the strength of the strongest factor implied by the model, and then estimation the strength of all sufficiently strong unobserved factors. Sections 5 and 6 provide extensive simulation and empirical evidence of the performance of our estimator. Section 7 provides some concluding remarks. Mathematical proofs and additional empirical and simulation results are contained in an online Appendix.

Notation: Generic positive finite constants are denoted by C_i, for $i = 1, 2, \ldots$. They can take different values at different instances. If $\{f_n\}_{n=1}^{\infty}$ is a real sequence and $\{g_n\}_{n=1}^{\infty}$ is a sequence of positive numbers, then $f_n = O(g_n)$, if there exists a positive finite constant C_0 such that $|f_n|/g_n \leq C_0$ for all n. $f_n = o(g_n)$ if $f_n/g_n \to 0$ as $n \to \infty$. If $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are both positive sequences of real numbers, then $f_n = \ominus(g_n)$ if there exist $N_0 \geq 1$ and positive finite constants C_0 and C_1, such that $\inf_{n \geq N_0}(f_n/g_n) \geq C_0$, and $\sup_{n \geq N_0}(f_n/g_n) \leq C_1$. $\to_d$ denotes convergence in distribution as $n, T \to \infty$.

2 Estimation strategy

To illustrate the basic idea behind our estimation strategy we begin with a single factor model where the factor is observed, and turn subsequently to the cases of observed or unobserved multiple factors. Suppose that T observations are given, on n cross section units, namely $\{x_{it}, i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, T\}$, and follow the single factor model (1), repeated here for convenience:

$$x_{it} = c_i + \gamma_i f_t + u_{it}, \tag{2}$$

where f_t, $t = 1, 2, \ldots, T$ is a known factor, c_i is the unit-specific effect, $u_{it} \sim IID(0, \sigma_i^2)$ is an idiosyncratic error, and γ_i is the factor loading for unit i. The factor loadings are assumed to be non-zero for the first $[n^\alpha]$ units, and zero for the rest, where $[\cdot]$ denotes the integer part function. More specifically, suppose that, for some $c > 0$,

$$\begin{aligned} |\gamma_i| &> c \text{ a.s. for } i = 1, 2, \ldots, [n^\alpha], \\ |\gamma_i| &= 0 \text{ a.s. for } i = [n^\alpha] + 1, [n^\alpha] + 2, \ldots, n, \end{aligned} \tag{3}$$

where α measures the strength of factor f_t, which in the case of the single factor model coincides with the *exponent of cross section dependence* discussed in BKP.[1] The exponent α measures the degree of pervasiveness or strength of the factor. It is important to reiterate that BKP focus on estimating an overall measure of cross-sectional dependence in x_{it}, without particular reference to a single specific factor. They base their estimator on the variance of the cross-sectional average, while noting the pros and cons of alternative approaches, based on other characteristics of x_{it}, such as, e.g., the maximum eigenvalue of the covariance of x_{it}. Given the prominence of this maximum eigenvalue as a basis for characterising CSD, they note existing work, as well as reasons for which a formal eigenvalue analysis may not be promising for this purpose.

As we noted above our aim is different. We wish to determine the strength or pervasiveness of particular factors and use α, as defined through (3), as a tool for that purpose. To estimate α we begin by running n least squares regressions of $\{x_{it}\}_{t=1}^T$ for each $i = 1, 2, \ldots, n$ on an intercept and f_t to obtain

$$x_{it} = \hat{c}_{iT} + \hat{\gamma}_{iT} f_t + \hat{\nu}_{it}, \quad t = 1, 2, \ldots, T$$

where $\hat{c}_{iT}$ and $\hat{\gamma}_{iT}$ are the Ordinary Least Squares (OLS) estimates of this regression. Denote by $t_{iT} = \hat{\gamma}_{iT} \,/\, \text{s.e.}\,(\hat{\gamma}_{iT})$ the t-statistic corresponding to γ_i:

$$t_{iT} = \frac{(\mathbf{f}'\mathbf{M}_\tau \mathbf{f})^{1/2}\, \hat{\gamma}_{iT}}{\hat{\sigma}_{iT}} = \frac{(\mathbf{f}'\mathbf{M}_\tau \mathbf{f})^{-1/2}\, (\mathbf{f}'\mathbf{M}_\tau \mathbf{x}_i)}{\hat{\sigma}_{iT}}, \tag{4}$$

where $\mathbf{M}_\tau = \mathbf{I}_T - T^{-1}\boldsymbol{\tau}_T\boldsymbol{\tau}_T'$, $\boldsymbol{\tau}_T$ is a $T \times 1$ vector of ones, $\mathbf{f} = (f_1, f_2, \ldots, f_T)'$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iT})'$, and $\hat{\sigma}_{iT}^2 = T^{-1}\sum_{t=1}^T \hat{\nu}_{it}^2$. Also assume that, for some $c > 0$, $T^{-1}\mathbf{f}'\mathbf{M}_\tau\mathbf{f} > c$, which is necessary for

[1] More generally, we can have $|\gamma_i| = c_1\gamma^{i-[n^\alpha]}$, with $|\gamma| < 1$ and $c_1 > 0$, for $i = [n^\alpha] + 1, [n^\alpha] + 2, \ldots, n$, in (3). But for simplicity of exposition, we opt for $|\gamma_i| = 0$ a.s. instead.

identification of γ_i. Consider the proportion of the n regressions with statistically significant coefficients γ_i:

$$\hat{\pi}_{nT} = n^{-1}\sum_{i=1}^{n} \hat{d}_{i,nT}, \tag{5}$$

where $\hat{d}_{i,nT} = \mathbf{1}\left[|t_{iT}| > c_p(n)\right]$, $\mathbf{1}(A) = 1$ if $A > 0$, and zero otherwise, and the critical value function, $c_p(n)$, is given by

$$c_p(n) = \Phi^{-1}\left(1 - \frac{p}{2n^{\delta}}\right). \tag{6}$$

Here p is the nominal size of the individual tests, $\delta > 0$ is the critical value exponent and $\Phi^{-1}(\cdot)$ denotes the inverse cumulative distribution function of the standard normal distribution.

Suppose that $\hat{\pi}_{nT} > 0$, and consider the following estimator of α

$$\tilde{\alpha} = 1 + \frac{\ln \hat{\pi}_{nT}}{\ln n}.$$

In the rare case where $\hat{\pi}_{nT} = 0$, we then set $\tilde{\alpha} = 0$. Overall

$$\hat{\alpha} = \begin{cases} \tilde{\alpha}, \text{ if } \hat{\pi}_{nT} > 0, \\ 0, \text{ if } \hat{\pi}_{nT} = 0. \end{cases} \tag{7}$$

Clearly $\hat{\alpha} \in [0, 1]$ a.s.; also, $\hat{\alpha}$ and $\tilde{\alpha}$ are asymptotically equivalent since for $\alpha > 0$ then $\mathbb{P}(n\ \hat{\pi}_{nT} = 0) \to 0$ as $n \to \infty$.

It is tempting to argue in favour of using the proportion of non-zero loadings, π, instead of the exponent α. The two measures are clearly related - $\pi = n^{\alpha-1}$, and coincide only when $\alpha = 1$. But when $\alpha < 1$, π becomes smaller and smaller as $n \to \infty$, and eventually tends to 0, for all values of $\alpha < 1$. The rate at which π tends to zero with n is determined by α, and hence α is a more discriminating measure of pervasiveness than π. It is also unclear how a particular value of π should be chosen as a measure of pervasiveness. It is also important to note that when π is set to $\pi^0 > 0$, a fixed value, then $\alpha = 1 + \ln(\pi^0)/\ln(n)$, and $\alpha \to 1$ as $n \to \infty$, if π^0 is fixed in n. Therefore, unlike α which can be chosen to be fixed in n, any choice of π which is fixed in n implies $\alpha \to 1$ as $n \to \infty$, albeit at the very slow $ln(n)$ rate.

2.1 Asymptotic distribution

Denote the true α by α_0, let $d_i^0 = \mathbf{1}\left[\gamma_i \neq 0\right]$ and note that $D_n^0 = \sum_{i=1}^{n} d_i^0 = n^{\alpha_0}$ (the integer part symbol is dropped for simplicity). Let

$$\hat{D}_{nT} = n\hat{\pi}_{nT} = \sum_{i=1}^{n} \hat{d}_{i,nT}, \tag{8}$$

and note that $\hat{D}_{nT}/D_n^0 = n^{\hat{\alpha}-\alpha_0}$. Taking logs, we obtain

$$
\begin{aligned}
(\ln n)(\hat{\alpha}-\alpha_0) &= \ln\left(\frac{\hat{D}_{nT}}{D_n^0}\right) = \ln\left(1+\frac{\hat{D}_{nT}-n^{\alpha_0}}{n^{\alpha_0}}\right) \\
&= \ln\left(1+A_{nT}+B_{nT}\right) \\
&= A_{nT}+B_{nT}+O_p\left(A_{nT}^2\right)+O\left(B_{nT}^2\right)+O_p\left(A_{nT}B_{nT}\right)+\ldots, \qquad (9)
\end{aligned}
$$

where

$$
A_{nT} = \frac{\sum_{i=1}^{n}\left[\hat{d}_{i,nT}-E\left(\hat{d}_{i,nT}\right)\right]}{n^{\alpha_0}}, \qquad (10)
$$

$$
B_{nT} = \frac{\sum_{i=1}^{n}E\left(\hat{d}_{i,nT}\right)-n^{\alpha_0}}{n^{\alpha_0}}. \qquad (11)
$$

To motivate the proposed estimator and to simplify the derivations, here we assume σ_i is known and u_{it} is Gaussian, and turn to the more general multi-factor case with non-Gaussian errors in Section 3. In this simple case we have the following lemmas proven in the online Appendix A.

Lemma 1 *Let the model be given by (2) where (3) holds, σ_i is known and u_{it} is a Gaussian martingale difference process for all i. Then, for some $C_1>0$,*

$$
B_{nT} = \frac{p\left(n-n^{\alpha_0}\right)}{n^{\delta+\alpha_0}} + O\left[\exp\left(-T^{C_1}\right)\right], \qquad (12)
$$

where p is the nominal size of the individual tests, and δ is the exponent of the critical value function defined in (6).

Lemma 2 *Let the model be given by (2) where (3) holds, σ_i is known and u_{it} is a Gaussian martingale difference process for all i. Then, in the case where $\alpha_0<1$, for some $C_1>0$,*

$$
Var\left(A_{nT}\right) = \psi_n(\alpha_0) + O\left[n^{-\alpha_0/2}\exp\left(-T^{C_1}\right)\right], \qquad (13)
$$

where

$$
\psi_n(\alpha_0) = p\left(n-n^{a_0}\right)n^{-\delta-2\alpha_0}\left(1-\frac{p}{n^{\delta}}\right). \qquad (14)
$$

If $\alpha_0=1$, for some $C_1>0$,

$$
Var\left(A_{nT}\right) = O\left[\exp\left(-T^{C_1}\right)\right]. \qquad (15)
$$

As we note from the above lemmas, we need to distinguish between the two cases where $\alpha_0=1$ and where $\alpha_0<1$. In the former case, $A_{nT}\to_p 0$ exponentially fast in T, and overall

$$
(\ln n)(\hat{\alpha}-1) = O_p\left[n^{-1}\exp\left(-C_2T\right)\right] + O\left[\exp\left(-C_1T\right)\right],
$$

for some positive constants C_1 and C_2. Furthermore, in the case where $\alpha_0<1$, using (13) and (14), it follows that

$$A_{nT} = O_p\left[\psi_n(\alpha_0)^{1/2}\right] + O\left[n^{-\alpha_0/2}\exp\left(-C_1T/2\right)\right]$$
$$= O_p\left(n^{1/2-\delta/2-\alpha_0}\right) + O\left[n^{-\alpha_0/2}\exp\left(-C_1T/2\right)\right].$$

Therefore, $A_{nT} = o_p(1)$ if $\delta > 1 - 2\alpha_0$, which is in turn met if $\delta > 0$, for all values of $\alpha_0 > 1/2$.

It is clear that the distribution of $\hat{\alpha}$ experiences a form of degeneracy when $\alpha_0 = 1$, and $\hat{\alpha}$ tends to its true value of 1 exponentially fast. We refer to this property as ultraconsistency to distinguish it from the more usual terminology of superconsistency that refers to rates of convergence that are faster than the usual one of the square root of the sample size. Usually faster rates are polynomial in the sample size and not exponential, and therefore the new term reflects this important difference.

The above results suggest the following scaling of $\hat{\alpha}$ when $\alpha_0 < 1$:

$$\psi_n^{-1/2}\left(\ln n\right)\left(\hat{\alpha} - \alpha_0\right) = \psi_n^{-1/2}A_{nT} + \psi_n^{-1/2}B_{nT} + o_p(1).$$

Also, using (A.6) from the online Appendix A, we have

$$B_{nT} = \frac{\sum_{i=1}^{n} E\left(\hat{d}_{i,nT}\right) - n^{\alpha_0}}{n^{\alpha_0}} = \frac{p\left(n - n^{\alpha_0}\right)}{n^{\delta+\alpha_0}} + O\left[\exp\left(-C_1T\right)\right].$$

It is also easily seen that $B_{nT} = o(1)$ if $\delta > 1 - \alpha_0$.

Since $1/2 < \alpha_0 < 1$ (recall that the case of $\alpha_0 = 1$ is treated separately), then for values of α_0 close to unity (from below) it is sufficient that $\delta > 0$, and for values of α_0 close to 1/2, we need $\delta > 1/2$. In the absence of a priori knowledge of α_0, it is sufficient to set $\delta = 1/2$. In practice, factors that are sufficiently strong with α_0 falling in the range $[2/3, 1]$ are likely to be of greater interest, and for precise estimation of such factors it would be sufficient to set $\delta = 1/4$. Our Monte Carlo results show that the estimates of factor strength are reasonably robust to the choice of δ, so long as it is not too small and lies in the range $1/4 - 1/2$. Alternatively, one can consider various cross-validation methods to calibrate δ.

Also, since $\left[\psi_n(\alpha_0)\right]^{-1/2} A_{nT} = O_p(1)$, then $\left[\psi_n(\alpha_0)\right]^{-1/2} A_{nT}^2 = O_p\left(A_{nT}\right) = o(1)$. Using these results we can now write

$$\left[\psi_n(\alpha_0)\right]^{-1/2}\left(\ln n\right)\left(\hat{\alpha} - \alpha_0 - \zeta_n\right) = \left[\psi_n(\alpha_0)\right]^{-1/2} A_{nT} + o_p(1),$$

where

$$\zeta_n\left(\alpha_0\right) = \frac{p\left(n - n^{\alpha_0}\right)}{\left(\ln n\right) n^{\delta+\alpha_0}}.$$

Finally, since u_{it} are independent across i, and $\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)$ have zero means, then by a standard martingale difference central limit theorem, we have (as n and $T \to \infty$)

$$[\psi_n(\alpha_0)]^{-1/2} A_{nT} = [\psi_n(\alpha_0)]^{-1/2} \frac{1}{n^{\alpha_0}} \sum_{i=1}^{n} \left[\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)\right] \to_d N(0,1).$$

Hence,

$$[\psi_n(\alpha_0)]^{-1/2} (\ln n) \left[\hat{\alpha} - \alpha_0 - \zeta_n(\alpha_0)\right] \to_d N(0,1), \tag{16}$$

where

$$\zeta_n(\alpha_0) = \frac{p\left(n - n^{\alpha_0}\right)}{(\ln n)\, n^{\delta+\alpha_0}}. \tag{17}$$

To test $H_0 : \alpha = \alpha_0$, we utilise the following score statistics where α_0 in the normalisation part of the test is replaced by its estimator, $\hat{\alpha}$:

$$z_{\hat{\alpha}:\alpha_0} = \frac{(\ln n)\left(\hat{\alpha} - \alpha_0\right) - p\left(n - n^{\hat{\alpha}}\right) n^{-\delta-\hat{\alpha}}}{\left[p\left(n - n^{\hat{\alpha}}\right) n^{-\delta-2\hat{\alpha}} \left(1 - \frac{p}{n^{\delta}}\right)\right]^{1/2}}. \tag{18}$$

The null will be rejected if $|z_\alpha| > cv$, where cv is the critical value of the standard normal distribution at the desired significance level (which need not be the same as p). For a two sided test at 5% level, $cv = 1.96$.

3 A general treatment with a multi-factor model

As a generalisation of the above set up consider the multi-factor regressions

$$x_{it} = c_i + \sum_{j=1}^{m} \gamma_{ij} f_{jt} + u_{it} = c_i + \boldsymbol{\gamma}_i' \mathbf{f}_t + u_{it}, \text{ for } i = 1, 2, \ldots, n \text{ and } t = 1, 2, \ldots, T \tag{19}$$

where $\boldsymbol{\gamma}_i = (\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{im})'$, and it is assumed that the m-dimensional vector, $\mathbf{f}_t = (f_{1t}, f_{2t}, \ldots, f_{mt})'$, is observed. We also assume that, for some unknown ordering of units over i,

$$|\gamma_{ij}| > c > 0 \text{ a.s. for } i = 1, 2, \ldots, [n^{\alpha_{j0}}],$$

$$|\gamma_{ij}| = 0 \text{ a.s. for } i = [n^{\alpha_{j0}}] + 1, [n^{\alpha_{j0}}] + 2, \ldots, n.$$

Throughout the paper we assume that $\alpha_{j0} > 0.5$, for $j = 1, 2, \ldots, m$. As discussed in the Introduction and also in Pesaran and Smith (2019) this is most relevant case, empirically.

Then the following strategy may be employed to provide inference on α_{j0}, for $j = 1, 2, \ldots, m$. For a given unit i, consider the least squares regression of $\{x_{it}\}_{t=1}^{T}$ on the intercept and $\mathbf{f}_t$. $\hat{c}_{iT}$ and $\hat{\boldsymbol{\gamma}}_{iT}$ are the OLS estimates of this regression. Denote by $t_{ijT} = \hat{\gamma}_{ijT} / \text{s.e.}(\hat{\gamma}_{ijT})$, the t-statistic corresponding to γ_{ij}:

$$t_{ijT} = \frac{\left(\mathbf{f}_{j\circ}' \mathbf{M}_{F_{-j}} \mathbf{f}_{j\circ}\right)^{-1/2} \left(\mathbf{f}_{j\circ}' \mathbf{M}_{F_{-j}} \mathbf{x}_i\right)}{\hat{\sigma}_{iT}}, \; j = 1, 2, \ldots, m; \; i = 1, 2, \ldots, n,$$

$\mathbf{f}_{j\circ} = (f_{j1}, f_{j2}, \ldots, f_{jT})'$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iT})'$, $\mathbf{M}_{F_{-j}} = \mathbf{I}_T - \mathbf{F}_{-j} \left(\mathbf{F}_{-j}' \mathbf{F}_{-j}\right)^{-1} \mathbf{F}_{-j}'$,

$\mathbf{F}_{-j} = (\tau_T, \mathbf{f}_{1\circ}, \ldots, \mathbf{f}_{j-1\circ}, \mathbf{f}_{j+1\circ}, \ldots, \mathbf{f}_{m\circ})'$, $\hat{\sigma}_{iT}^2 = T^{-1} \sum_{t=1}^{T} \hat{u}_{it}^2$, and $\hat{u}_{it} = x_{it} - \hat{c}_{iT} - \hat{\boldsymbol{\gamma}}_{iT}' \mathbf{f}_t$.

Consider the total number of factor loadings of factor j, γ_{ij}, that are statistically significant over $i = 1, 2, \ldots, n$:

$$\hat{D}_{nT,j} = \sum_{i=1}^{n} \hat{d}_{ij,nT} = \sum_{i=1}^{n} \mathbf{1}\left[|t_{ijT}| > c_p(n)\right],$$

where $\mathbf{1}(A) = 1$ if $A > 0$, and zero otherwise, and the critical value function that allows for the multiple testing nature of the problem, $c_p(n)$, is given by

$$c_p(n) = \Phi^{-1}\left(1 - \frac{p}{2n^{\delta}}\right).$$

As before, p is the nominal size, $\delta > 0$ is the critical value exponent and $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal distribution. Let $\hat{\pi}_{nT,j}$ be the fraction of significant loadings of factor j, and note that $\hat{\pi}_{nT,j} = \hat{D}_{nT,j}/n$. As in the single factor case, we consider the following estimator of α_{j0}, for $j = 1, 2, \ldots, m$

$$\hat{\alpha}_j = \begin{cases} 1 + \frac{\ln \hat{\pi}_{nT,j}}{\ln n}, \text{ if } \hat{\pi}_{nT,j} > 0, \\ \\ 0, \text{ if } \hat{\pi}_{nT,j} = 0. \end{cases} \tag{20}$$

We make the following assumptions:

Assumption 1 *The error terms, u_{it}, and demeaned factors $\mathbf{f}_t - E(\mathbf{f}_t)$, are martingale difference processes with respect to $\mathcal{F}_{t-1}^{u_i} = \sigma(u_{i,t-1}, u_{i,t-2}, \ldots)$ and $\mathcal{F}_{t-1}^{f} = \sigma(\mathbf{f}_t, \mathbf{f}_{t-1}, \ldots)$, respectively. u_{it} are independent over i, and of $\mathbf{f}_t$, and have constant variances, $0 < \sigma_i^2 < C < \infty$.*

Assumption 2 $E\left\{[\mathbf{f}_t - E(\mathbf{f}_t)][\mathbf{f}_t - E(\mathbf{f}_t)]'\right\} = \mathbf{\Sigma}$, *where $\mathbf{\Sigma}$ is some positive definite matrix.*

Assumption 3 *There exist sufficiently large positive constants C_0, C_1, and $s > 0$ such that*

$$\sup_{i,t} \Pr(|x_{it}| > \nu) \leq C_0 \exp(-C_1 \nu^s), \text{ for all } \nu > 0, \tag{21}$$

$$\sup_{j,t} \Pr(|f_{jt}| > \nu) \leq C_0 \exp(-C_1 \nu^s), \text{ for all } \nu > 0. \tag{22}$$

Then, we have the following theorem:

Theorem 1 *Consider model (19) with m observed factors and let Assumptions 1 - 3 hold. Then, for any $\alpha_{j0} < 1$, $j = 1, 2, \ldots, m$,*

$$\psi_n(\alpha_{j0})^{-1/2} (\ln n) (\hat{\alpha}_j - \alpha_{j0}) \to_d N(0, C) \tag{23}$$

for some $C < 1$, where

$$\psi_n(\alpha_{j0}) = p\,(n - n^{\alpha_{j0}})\, n^{-\delta - 2\alpha_{j0}} \left(1 - \frac{p}{n^{\delta}}\right). \tag{24}$$

The above theorem provides the inferential basis for testing hypotheses on the true value of α_j. The proof of the theorem is provided in the online Appendix B. In the remarks below we discuss operational matters concerning the above result and how to relax some of the assumptions of Theorem 1.

A test based on $\psi_n(\alpha_{j0})^{-1/2} (\ln n) (\hat{\alpha}_j - \alpha_{j0})$ will be conservative, in the sense that the rejection probability under the null hypothesis will be bounded from above by the significance level. The reason is that in general we cannot get an asymptotic approximation for the variance of $\hat{\alpha}_j - \alpha_{j0}$ but only an upper bound resulting in a conservative test.

Assumptions 1 and 3 can be relaxed. Rather than independence over i for u_{it} in Assumption 1, one can assume some spatial mixing condition, which would still allow the central limit theorem underlying (23), to hold. Further, the thin probability tails in Assumption 3 can be replaced with a suitable moment condition in order to derive the variance bound needed to construct a test statistic. We abstract from such complications by maintaining Assumption 3. The martingale difference assumption for $\mathbf{f}_t$ simplifies the analysis and allows the use of the theory in the main part of Chudik et al. (2018). Relaxing this to a mixing assumption is possible at the expense of further mathematical complexity using, e.g., the results in the online appendix of Chudik et al. (2018).

Our distributional result is stated only for $\alpha_{j0} < 1$. Similar arguments would apply for the variance $\hat{\alpha}_j - \alpha_{j0}$ when $\alpha_{j0} = 1$. But the upper bound for the variance of $\hat{\alpha}_j - \alpha_{j0}$ would be a function of nuisance parameters including γ_{ij}. This is the case since the dominant term in the variance is the one relating to units not affected by $\mathbf{f}_t$, when $\alpha_{j0} < 1$, and for these units, $\gamma_{ij} = 0$. But when $\alpha_{j0} = 1$, the probability bounds that are used to derive the variance bound will not have such a dominant term, and the remaining terms will contain γ_{ij}. However, testing under the null hypothesis that $\alpha_{j0} = 1$ is further complicated by the fact that $\alpha_{j0} = 1$ is at the boundary of the parameter space for α_{j0}. It is well known (see, e.g., Andrews (2001)) that such cases cannot be handled using standard asymptotic inference, and therefore this case is discussed separately, in the online Appendix C. Nevertheless, it is clear from the discussion of Section 2.1 that estimation when $\alpha_0 = 1$ has some very desirable properties, such as a very fast rate of convergence, which we have referred to as ultraconsistency. We conjecture that in the case where $\alpha_{j0} = 1$ for some values of j, and $\alpha_{j0} < 1$ for some values of j, the distributional results presented in Theorem 1 hold for factors for which $\alpha_{j0} < 1$.

4 Case of unobserved factors

When the factors are unobserved we can provide practical guidance on the strength of the strongest factor or factors, and estimating the strength of other factors encounters a significant identification problem. This is related to the known fact that latent factors are identified only up to a non-singular $m \times m$ rotation matrix, $\mathbf{Q} = (q_{ij})$, where m is the assumed number of factors.

It is instructive to review this fact. Consider the multi-factor model (19) with $\mathbf{f}_t$ unobserved. Without loss of generality suppose that $m = 2$ and assume that factors, $\mathbf{f}_t = (f_{1t}, f_{2t})'$, are unobserved with strengths $\alpha_{10} > 1/2$ and $\alpha_{20} > 1/2$. Denote the principal component (PC) estimates of these factors by $\hat{\mathbf{g}}_t = (\hat{g}_{1t}, \hat{g}_{2t})'$, and note that under standard regularity conditions in the literature (as n and $T \to \infty$)

$$f_{1t} = q_{11}\hat{g}_{1t} + q_{12}\hat{g}_{2t} + o_p(1), \tag{25}$$

$$f_{2t} = q_{21}\hat{g}_{1t} + q_{22}\hat{g}_{2t} + o_p(1). \tag{26}$$

Then the estimates of the loadings associated with these PCs are given by

$$\tilde{\gamma}_i = \begin{pmatrix} \tilde{\gamma}_{i1} \\ \tilde{\gamma}_{i2} \end{pmatrix} = \left(\hat{\mathbf{G}}'\mathbf{M}_\tau\hat{\mathbf{G}}\right)^{-1}\hat{\mathbf{G}}'\mathbf{M}_\tau\mathbf{x}_i = \left(\hat{\mathbf{G}}'\mathbf{M}_\tau\hat{\mathbf{G}}\right)^{-1}\hat{\mathbf{G}}'\mathbf{M}_\tau\mathbf{F}\boldsymbol{\gamma}_i + \left(\hat{\mathbf{G}}'\mathbf{M}_\tau\hat{\mathbf{G}}\right)^{-1}\hat{\mathbf{G}}'\mathbf{M}_\tau\mathbf{u}_i,$$

where $\hat{\mathbf{G}} = (\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \ldots, \hat{\mathbf{g}}_T)'$. Also since $\mathbf{Q}$ is non-singular, $\hat{\mathbf{G}} \to_p \mathbf{F}\mathbf{Q}^{-1}$, and using the above we have $\tilde{\boldsymbol{\gamma}}_i \to_p \mathbf{Q}\boldsymbol{\gamma}_i$. It is now easily seen that the strength of f_{1t} (or f_{2t}) computed using the estimates, $\tilde{\gamma}_{i1}$, $i = 1, 2, \ldots, n$ may not provide consistent estimates of the associated factor strengths. To see this write the result $\tilde{\boldsymbol{\gamma}}_i \to_p \mathbf{Q}\boldsymbol{\gamma}_i$ in an expanded format as

$$\tilde{\gamma}_{i1} = q_{11}\gamma_{i1} + q_{12}\gamma_{i2} + o_p(1),$$

$$\tilde{\gamma}_{i2} = q_{21}\gamma_{i1} + q_{22}\gamma_{i2} + o_p(1).$$

Squaring both sides and summing over i we have

$$\sum_{i=1}^n \tilde{\gamma}_{i1}^2 = q_{11}^2 \sum_{i=1}^n \gamma_{i1}^2 + q_{12}^2 \sum_{i=1}^n \gamma_{i2}^2 + 2q_{11}q_{12} \sum_{i=1}^n \gamma_{i1}\gamma_{i2} + o_p(1),$$
$$\sum_{i=1}^n \tilde{\gamma}_{i2}^2 = q_{21}^2 \sum_{i=1}^n \gamma_{i1}^2 + q_{22}^2 \sum_{i=1}^n \gamma_{i2}^2 + 2q_{21}q_{22} \sum_{i=1}^n \gamma_{i1}\gamma_{i2} + o_p(1).$$

Now using the definition of factor strength in (3) and assuming that $\alpha_{10} > \alpha_{20}$, in general we have[2]

$$\sum_{i=1}^n \tilde{\gamma}_{i1}^2 = \ominus(n^{\alpha_{10}}),\ \sum_{i=1}^n \tilde{\gamma}_{i2}^2 = \ominus(n^{\alpha_{10}}),$$

namely, using the estimated loadings of the principal components does not allow us to distinguish between

[2]Note that $\left|\sum_{i=1}^n \gamma_{i1}\gamma_{i2}\right| < \sup_i |\gamma_{i1}| \left(\sum_{i=1}^n |\gamma_{i2}|\right) = \ominus(n^{\alpha_2})$.

the strength of the two factors, and only the strength of the strongest factor can be identified. When $\alpha_{10} > \alpha_{20}$, identification of α_{20} requires setting $q_{21} = 0$, and conversely to identify α_{10} when $\alpha_{10} < \alpha_{20}$ requires setting $q_{12} = 0$. It is worth noting that using covariance eigenvalues does not help resolve this problem. There are two separate issues – ordering eigenvalues and how to identify the factors associated with ordered eigenvalues. The eigenvectors associated with the largest eigenvalues are not uniquely determined and therefore the identification issue remains. In conclusion, any estimate, $\hat{\alpha}_2$, is a function of the assumed rotation and the utility of such an estimate, given the above analysis, is unclear.[3]

One approach to dealing with this identification problem is to estimate $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$. The exponent α_0 can be estimated using the estimators proposed in Bailey et al. (2016) and Bailey et al. (2019). The approach of this paper can also be used to estimate α_0 by computing the strength of the first PC, or that of the simple cross section average, namely $\bar{x}_t = n^{-1} \sum_{i=1}^{n} x_{it}$. One can also use the weighted cross section average $\bar{x}_{t,\gamma} = \sum_{i=1}^{n} \hat{w}_i x_{it}$, where $\hat{w}_i$ is estimated as the slope of $\bar{x}_t$ in the OLS regression of x_{it} on an intercept and $\bar{x}_t$.[4]

Accordingly, we again emphasise that we assume that the m unobserved factors are strong and/or semi-strong with $1/2 < \alpha_{j0} \leq 1$, and focus on estimation of $\alpha_0 = \max_j(\alpha_{j0})$. In Section 4.1 we suggest how to identify, in theory, the strengths of weaker factors. Reintroducing a subscript 0 to denote true parameters, we assume that $\{x_{it},\ i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T\}$ are generated from the multi-factor model (19) where the factors are unobserved with strengths $\alpha_{10} > \alpha_{20} \geq \alpha_{30} \geq \cdots \geq \alpha_{m0} > 1/2$. Clearly $\alpha_0 = \alpha_{10}$. To emphasise the focus on the strongest factor we recast the model as follows:

$$x_{it} = c_i + \gamma_i f_t + v_{it}, \text{ for } i = 1, 2, \ldots, n \text{ and } t = 1, 2, \ldots, T \tag{27}$$

$$v_{it} = \sum_{j=2}^{m} \gamma_{ij} f_{jt} + u_{it}, \tag{28}$$

where the strongest factor f_t has strength α_0 while the rest of the factors have strengths $\alpha_{20} \geq \alpha_{30} \geq \cdots \geq \alpha_{m0} > 1/2$. We assume that the m-dimensional vector, $\mathbf{f}_t = (f_t, f_{2t}, \ldots, f_{m^0,t})'$, is unobserved. We also assume that, for some unknown ordering of units over i,

$$|\gamma_i| > c > 0 \text{ a.s. for } i = 1, 2, \ldots, [n^{\alpha_0}], \tag{29}$$

$$|\gamma_i| = 0 \text{ a.s. for } i = [n^{\alpha_0}] + 1, [n^{\alpha_0}] + 2, \ldots, n.$$

[3]It may be the case that using a rotation criterion can provide an interesting avenue for further research on this issue. See, for example, Kaiser (1958), Ročková and George (2016) and Freyaldenhoven (2019).

[4]In most applications, α can be estimated consistently using the simple average. But as shown in Pesaran (2015), pp. 452-454, the weighted average is more appropriate when the loadings of the strong factors have zero means. Also note that by construction $\sum_{i=1}^{n} \hat{w}_i = 1$.

$$
\begin{aligned}
|\gamma_{ij}| &> c > 0 \text{ a.s. for } i = 1, 2, \ldots, [n^{\alpha_{j0}}],\ j = 2, \ldots, m \\
|\gamma_{ij}| &= 0 \text{ a.s. for } i = [n^{\alpha_{j0}}] + 1, [n^{\alpha_{j0}}] + 2, \ldots, n,\ j = 2, \ldots, m.
\end{aligned} \tag{30}
$$

In what follows, we continue to consider that Assumptions 1 and 3 hold for the above representation, and use the simple cross section average, $\bar{x}_t$ to consistently estimate $\alpha_0 = \alpha_{10}$. Taking the first factor to be the strongest is made for convenience (with $\alpha_0 - \alpha_{j0} > 0$, for $j = 2, 3, \ldots, m$). The strength of the strongest factor, α_0, is defined by (with γ_i denoting the associated loadings)

$$
\sum_{i=1}^{n} |\gamma_i| = \ominus\left(n^{\alpha_0}\right),
$$

and the strengths of the remaining factors by

$$
\sum_{i=1}^{n} |\gamma_{ij}| = \ominus\left(n^{\alpha_{j0}}\right), \text{ for } j = 2, 3, \ldots, m.
$$

In addition, we assume that the non-zero factor loadings have non-zero means, namely

$$
\lim_{n\to\infty} n^{-\alpha_0} \sum_{i=1}^{n} \gamma_i \neq 0, \text{ and } \lim_{n\to\infty} n^{-\alpha_{j0}} \sum_{i=1}^{n} \gamma_{ij} \neq 0,
$$

and hence,

$$
\bar{\gamma} = \bar{\gamma}_1 = n^{-1} \sum_{i=1}^{n} \gamma_i = \ominus\left(n^{\alpha_0 - 1}\right),
$$

$$
\bar{\gamma}_j = n^{-1} \sum_{i=1}^{n} \gamma_{ij} = \ominus\left(n^{\alpha_{j0} - 1}\right), \text{ for } j = 2, \ldots, m.
$$

Note that we do not assume any ordering of the zero loadings across the units.

For each i, consider the least squares regression of $\{x_{it}\}_{t=1}^{T}$ on an intercept and the cross section average of x_{it}, $\bar{x}_t$, and denote the resulting estimators by $\hat{c}_{iT}$ and $\hat{\beta}_{iT}$, respectively. As in the single factor case, $\alpha_0 = \max_j(\alpha_{j0})$ is estimated by (7), except that when computing the t-statistics, t_{iT}, defined by (4), $\mathbf{f}$ is replaced by $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_T)'$. Denote by $\bar{t}_{iT} = \hat{\beta}_{iT} / \text{s.e.}\left(\hat{\beta}_{iT}\right)$, the t-statistic corresponding to γ_i:

$$
\bar{t}_{iT} = \frac{\left(\bar{\mathbf{x}}'\mathbf{M}_\tau \bar{\mathbf{x}}\right)^{-1/2} \left(\bar{\mathbf{x}}'\mathbf{M}_\tau \mathbf{x}_i\right)}{\hat{\sigma}_{iT}},
$$

$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iT})'$, and $\hat{\sigma}_{iT}^2 = T^{-1}\mathbf{x}_i'\mathbf{M}_{\bar{H}}\mathbf{x}_i$, where $\mathbf{M}_{\bar{H}} = \mathbf{I}_T - \bar{\mathbf{H}}\left(\bar{\mathbf{H}}'\bar{\mathbf{H}}\right)^{-1}\bar{\mathbf{H}}'$, with $\bar{\mathbf{H}} = (\boldsymbol{\tau}_T, \bar{\mathbf{x}})$.

As before, consider the number of regressions with significant slope coefficients:

$$
\bar{D}_{nT} = \sum_{i=1}^{n} \bar{d}_{i,nT} = \sum_{i=1}^{n} \mathbf{1}\left[|\bar{t}_{iT}| > c_p(n)\right],
$$

where the critical value function, $c_p(n)$, is as specified earlier. Then, setting $\bar{\pi}_{nT} = \bar{D}_{nT}/n$, we have

$$\hat{\alpha} = \begin{cases} 1 + \frac{\ln \bar{\pi}_{nT}}{\ln n}, \text{ if } \bar{\pi}_{nT} > 0, \\ \\ 0, \text{ if } \bar{\pi}_{nT} = 0. \end{cases}$$

To investigate the limiting properties of $\hat{\alpha}$ we first consider the value of $\bar{t}_{iT}$ under (19) and note that

$$\bar{\mathbf{x}} = \bar{c}\boldsymbol{\tau}_T + \mathbf{F}\bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}}, \text{ and } \mathbf{x}_i = c_i\boldsymbol{\tau}_T + \mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i,$$

where $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_T)'$, $\boldsymbol{\gamma}_i = (\gamma_i, \gamma_{i2}, \ldots, \gamma_{im})'$, $\bar{\boldsymbol{\gamma}} = n^{-1}\sum_{i=1}^{n} \boldsymbol{\gamma}_i$, $\mathbf{u}_i = (u_{i1}, u_{i2}, \ldots, u_{iT})'$ and $\bar{\mathbf{u}} = n^{-1}\sum_{i=1}^{n} u_i$. Using these results we have

$$\bar{t}_{iT} = \frac{T^{-1/2}\left(\bar{\mathbf{x}}'\mathbf{M}_\tau \mathbf{x}_i\right)}{\hat{\sigma}_{iT}\left(T^{-1}\bar{\mathbf{x}}'\mathbf{M}_\tau \bar{\mathbf{x}}\right)^{1/2}} = \frac{T^{-1/2}(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})'\mathbf{M}_\tau\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right)}{\hat{\sigma}_{iT}\left[T^{-1}(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})'\mathbf{M}_\tau(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})\right]^{1/2}}, \tag{31}$$

and

$$\hat{\sigma}_{iT}^2 = T^{-1}\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right)'\mathbf{M}_{\bar{H}}\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right). \tag{32}$$

Before proceeding, we slightly modify our assumptions to address the identification issue inherent in considering unobserved factors.

Assumption 4 $E\left\{\left[\mathbf{f}_t - E\left(\mathbf{f}_t\right)\right]\left[\mathbf{f}_t - E\left(\mathbf{f}_t\right)\right]'\right\} = \boldsymbol{I}_m$.

Lemma 3 below, which is of fundamental importance, is proven in the online Appendix A and provides probability bounds for $\bar{t}_{iT}$. It uses results from the auxiliary Lemma 4 (also stated and proved in the online Appendix A) in terms of the rates in probability and probability tail bounds for the constituent parts of $\bar{t}_{iT}$.

Lemma 3 *Consider model (27)-(28) with factor loadings given by (29)-(30), where $\mathbf{f}_t$ is unobserved, and let Assumptions 1, 3 and 4 hold. Then, as long as $\sqrt{T}n^{(\alpha_{20}-\alpha_0)} \to 0$, for some $C > 0$,*

$$\Pr\left[|\bar{t}_{iT}| > c_p(n)|\gamma_i \neq 0\right] > 1 - O\left[\exp(-T^C)\right], \tag{33}$$

and

$$\Pr\left[|\bar{t}_{iT}| > c_p(n)|\gamma_i = 0\right] \leq \frac{Cp}{n^\delta}. \tag{34}$$

Equations (33) and (34) provide the crucial ingredients for the main result given below, as (33) ensures that the t-statistic rejects with high probability when a unit contains a factor, while (34) ensures that the probability of rejection for a unit that does not contain a factor, is small.

Overall, we have the following theorem, proven in the online Appendix B, justifying the proposed method for unobserved factors.

Theorem 2 *Consider model (27)-(28) with factor loadings given by (29)-(30), where $\mathbf{f}_t$ is unobserved, let Assumptions 1, 3 and 4 hold and denote by α_0 the true value of α (the strength of the strongest unobserved factor). Then, as long as $\sqrt{T}n^{(\alpha_{20}-\alpha_0)} \to 0$, for any $\alpha_0 < 1$,*

$$\psi_n(\alpha_0)^{-1/2}\,(\ln n)\,(\hat{\alpha}-\alpha_0) \rightarrow_d N(0,C)$$

for some $C<1$, *where* α_{20} *denotes the strength of the second strongest factor, and*

$$\psi_n(\alpha_0) = p\,(n-n^{\alpha_0})\,n^{-\delta-2\alpha_0}\left(1-\frac{p}{n^{\delta}}\right).$$

The above theorem provides the inferential basis for testing hypotheses on the true value of α, in the case of unobserved factors. Clearly, since $1 \geq \alpha_0 \geq \alpha_{20} \geq 0.5$, $T/n \rightarrow 0$ is a necessary condition and, of course, the actual sufficient condition may be more restrictive depending on the values of α_0 and α_{20}.

The above analysis readily extends to the case where two or more of the unobserved factors have the same strength. For example, suppose that $\alpha_0 = \max_j(\alpha_{j0}) = \alpha_{10} = \alpha_{20} > \alpha_{30} \geq \alpha_{40} \geq .... \geq \alpha_{m0}$. Then it is easily seen that α is consistently estimated by $\hat{\alpha}$, even though $\alpha_{10} = \alpha_{20}$. What matters for identification of α_0 in this case is that $\sqrt{T}n^{(\alpha_{30}-\alpha_0)} \rightarrow 0$. This case is further investigated below using Monte Carlo techniques.

4.1 Multiple unobserved factors of differing strengths

Our analysis has focused on $\alpha_0 = \alpha_{10} = \max_j(\alpha_{j0})$. A possible way to provide some information on α_{j0}, $j>1$, may be based on a sequential application of weighted cross section averages. In particular, once the least squares regression of $\{x_{it}\}_{t=1}^{T}$ on an intercept and the cross section average of x_{it}, $\bar{x}_t$, has been fitted, residuals can be obtained. Simple cross section averages of these residuals are easily seen to be identically equal to zero. However, weighted cross section averages can be constructed, along the lines discussed in Pesaran (2015), pp. 452-454, and the t-statistics of the relevant loadings can be used, in a similar way to that discussed above, to construct estimators for α_{20} and, sequentially via the construction of further sets of residuals, for α_{j0}, $j>2$. It is possible to show that, if $\sqrt{T}n^{(\alpha_{j+1,0}-\alpha_{j0})} \rightarrow 0$, $j>1$, a result similar to that of Theorem 2 holds for α_{j0}, $j>1$. This is stated formally in the following Theorem.

Theorem 3 *Consider model (27)-(28) with factor loadings given by (29)-(30), where* $\mathbf{f}_t$ *is unobserved. Suppose that Assumptions 1, 3 and 4 hold, and denote by* α_{j0} *the true value of* α_j. *Then, as long as* $\sqrt{T}n^{(\alpha_{j+1,0}-\alpha_{j0})} \rightarrow 0$, $j>1$, *for any* $0.5<\alpha_{j+1,0}<\alpha_{j0}<1$,

$$\psi_n(\alpha_{j0})^{-1/2}\,(\ln n)\,(\hat{\alpha}_j-\alpha_{j0}) \rightarrow_d N(0,C)$$

for some $C<1$, *and*

$$\psi_n(\alpha_{j0}) = p\,(n-n^{\alpha_{j0}})\,n^{-\delta-2\alpha_{j0}}\left(1-\frac{p}{n^{\delta}}\right).$$

The proof of the theorem is provided in the online Appendix B. However, this result clearly requires considerable differences to exist between the successive values of α's and/or very large values for n. The need for large values of n in the case of unobserved factors, contrasts to our results for the case of observed factors, where a less stringent condition on the relative expansion rates of n and T is required. The conditions of Theorem 3 must be born in mind when attempting to estimate second or third (semi) strongest unobserved factors. Estimation of factor strength in the case of unobserved factors involves the additional difficulty of how to distinguish between the strongest, the second strongest, the third strongest and so on factors. The condition $\sqrt{T}n^{(\alpha_{j+1,0}-\alpha_{j0})} \to 0$, $j > 1$ in Theorem 3 relates to this identification problem, and requires a sufficient degree of difference between successive factor strengths for consistent estimation. In practice, we can only hope to identify the first two or three strongest factors so long as their strengths are close to unity and at the same time not too close to one another.

Finally, one may wish to have some indication of the value of m^0 (the true number of factors), and to this end some preliminary investigation might be required. One possibility would be to consider various existing methods for selecting the number of factors with all the attendant, well known, performance issues such methods present. Of course, these issues are further exacerbated if factors under consideration are not sufficiently strong. In short, special care needs to be exercised when estimating factor strength in the case of unobserved factors. In practice, it might only be possible to identify and estimate the strengths of top 2 or 3 unobserved factors, at most. Also, when factors are unobserved and their strengths are not known *a priori*, the meaning of m^0 itself is ambiguous, and must be defined with reference to the strengths of the factors themselves. In our set up m^0 refers to the number of factors with $\alpha_{j0} > 1/2$. But condition $\sqrt{T}n^{(\alpha_{j+1,0}-\alpha_{j0})} \to 0$ of Theorem 3 suggests that only factors with α_{j0} sufficiently large can be identified. This contrasts to the standard factor literature that assumes all factors are *a priori* strong with $\alpha_{j0} = 1$, for $j = 1, 2, ..., m^0$. The concept of m^0 and its identification in the more general setting where $\alpha_{j0} \leq 1$ requires further investigation.

5 Monte Carlo study

5.1 Design

We investigate the small sample properties of the proposed estimator of α under both observed and unobserved factors using a number of Monte Carlo simulations. We consider the following two-factor data generating process (DGP):

$$x_{it} = c_i + \gamma_{i1} f_{1t} + \gamma_{i2} f_{2t} + u_{it}, \tag{35}$$

for $i = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, T$. We generate the unit specific effects as $c_i \sim IIDN(0, 1)$, for $i = 1, 2, \ldots, n$. The factors, $\mathbf{f}_t = (f_{1t}, f_{2t})'$, are generated as multivariate normal: $\mathbf{f}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_f)$, where

$$\mathbf{\Sigma}_f = \begin{pmatrix} \sigma_{f_1}^2 & \rho_{12}\sigma_{f_1}\sigma_{f2} \\ \rho_{12}\sigma_{f_1}\sigma_{f2} & \sigma_{f2}^2 \end{pmatrix},$$

with $\sigma_{f_1} = \sigma_{f2} = 1$, and $\rho_{12} = corr(f_{1t}, f_{2t})$, using the values $\rho_{12} = 0.0, 0.3$. The factors are generated as autoregressive processes (considering both stationary and unit root cases):

$$f_{jt} = \begin{cases} \rho_{f_j} f_{j,t-1} + \sqrt{1-\rho_{f_j}^2}\varepsilon_{jt}, & \text{if } \left|\rho_{f_j}\right| < 1 \\ f_{j,t-1} + \varepsilon_{jt}, & \text{if } \rho_{f_j} = 1 \end{cases}, \text{ for } t = -49, -48, \ldots, 1, \ldots, T$$

with $f_{j,-50} = 0$ and $\varepsilon_{jt} \sim i.i.d.N(0, 1)$, $j = 1, 2$. In the stationary case, we set $\rho_{f_1} = \rho_{f_2} = 0.5$.

For the innovations, u_{it}, we consider two cases: (i) Gaussian, where $u_{it} \sim IIDN(0, \sigma_i^2)$ for $i = 1, 2, \ldots, n$; (ii) non-Gaussian, where the errors are generated as $u_{it} = \frac{\sigma_i}{2}\left(\chi_{2,it}^2 - 2\right)$, where $\chi_{2,it}^2$ for $i = 1, 2, \ldots, n$ are independent draws from a chi-squared distribution with 2 degrees of freedom, and σ_i^2 are generated as $IID(1 + \chi_{2,i}^2)/3$.

In terms of the factor loadings, γ_{i1} and γ_{i2}, first we generate $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2$ (such that $E(v_{ij}) = \mu_{v_j}$). Next, we randomly assign $[n^{\alpha_{10}}]$ and $[n^{\alpha_{20}}]$ of these random variables as elements of vectors $\boldsymbol{\gamma}_j = (\gamma_{1j}, \gamma_{2j}, \ldots, \gamma_{nj})'$, $j = 1, 2$, respectively, where $[.]$ denotes the integer part operator.[5] For α_{10} and α_{20}, we consider values of $(\alpha_{10}, \alpha_{20})$ starting with 0.75 and rising to 1 at 0.05 increments, namely $0.75, 0.80, \ldots, 0.95, 1.00$, comprising of 36 experiments for all combinations of a_{10} and a_{20} in the range $[0.75, 1.00]$.[6] We set $\mu_{v_1} = \mu_{v_2} = 0.71$ so that both means are sufficiently different from zero. We then select the error variances, σ_i^2, so as to achieve an average fit across all units of around $\bar{R}_n^2 = n^{-1}\sum_{i=1}^n R_i^2 \approx 0.34$. This coincides with the average fits of regressions from our finance application. Scaling σ_i^2 by 3/4 achieves $\bar{R}_n^2 \approx 0.41$. To this end, we note that:

$$R_i^2 = \frac{\gamma_{i1}^2 + \gamma_{i2}^2}{\gamma_{i1}^2 + \gamma_{i2}^2 + \sigma_i^2} = \frac{\varpi_{i1}^2 + \varpi_{i2}^2}{1 + \varpi_{i1}^2 + \varpi_{i2}^2}, \text{ if for the } i^{th} \text{ unit: both } \gamma_{i1} \neq 0 \text{ and } \gamma_{i2} \neq 0,$$

where $\varpi_{ij}^2 = \gamma_{ij}^2/\sigma_i^2$, for $j = 1, 2$. Similarly, $R_i^2 = \varpi_{i1}^2/\left(1 + \varpi_{i1}^2\right)$, if $\gamma_{i1} \neq 0$ and $\gamma_{i2} = 0$, $R_i^2 = \varpi_{i2}^2/\left(1 + \varpi_{i2}^2\right)$, $\gamma_{i2} \neq 0$ and $\gamma_{i1} = 0$, and clearly $R_i^2 = 0$, if $\gamma_{i1} = \gamma_{i2} = 0$.

We consider the following experiments:

[5] The randomisation of loadings becomes important when analysing the case of unobserved factors, as discussed in Section 4.

[6] Results for combinations of α_{10} and α_{20} below 0.75 are available upon request.

EXP 1A: (observed single factor - Gaussian errors): Using (35) with $\gamma_{i2} = 0$, for all i, and Gaussian errors.

EXP 1B: (observed single factor - non-Gaussian errors): Using (35) with $\gamma_{i2} = 0$, for all i, and non-Gaussian errors.

EXP 2A: (two observed factors - Gaussian errors) A two-factor model with correlated observed factors ($\rho_{12} = 0.3$) and Gaussian errors.

EXP 2B: (two observed factors - non-Gaussian errors) A two-factor model with correlated observed factors ($\rho_{12} = 0.3$) and non-Gaussian errors.

EXP 3A: (unobserved single factor - non-Gaussian errors) Using (35) subject to $\gamma_{i2} = 0$, for all i, and non-Gaussian errors with $\alpha_0 = \alpha_{10}$ computed using the simple cross section average $\bar{x}_t = n^{-1} \sum_{i=1}^{n} x_{it}$.

EXP 3B: (two unobserved factors - non-Gaussian errors) Using (35) with $\rho_{12} = 0.0$ and non-Gaussian errors, $\alpha_{10} = 0.95, 1.00$, and $\alpha_{20} = 0.51, 0.75, 0.95, 1.00$. In this case $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated using the simple cross section average $\bar{x}_t = n^{-1} \sum_{i=1}^{n} x_{it}$.

EXP 3C: (two unobserved factors - Gaussian errors) Using (35) with $\rho_{12} = 0.0$, $\alpha_{10} = 0.90, 1.00$; $\alpha_{20} = 0.51, 0.75, 0.90$, and Gaussian errors. For this experiment α_{10} and α_{20} are estimated sequentially using weighted cross section averages of x_{it}, namely $\tilde{x}_t = n^{-1} \sum_{i=1}^{n} \hat{w}_i x_{it}$, where $\hat{w}_i = \sum_{t=1}^{T} \bar{x}_t x_{it} / \sum_{t=1}^{T} \bar{x}_t^2$, and then the weighted cross section averages of the residuals obtained from the first stage regression of x_{it} on $\bar{x}_t$.

Further, we consider an additional experiment that assumes a misspecified observed factor model that mirrors the analysis of our empirical finance example in Section 6.1:

EXP 4: (observed misspecified single factor - Gaussian errors) A misspecified single observed factor model, where the DGP is a two-factor model with correlated factors ($\rho_{12} = 0.3$) and Gaussian errors in (35), $\alpha_{10} = 1$, and $\alpha_{20} = 0.75, 0.80, \ldots, 0.95, 1.00$. For this experiment we report the estimates of α_{10} computed based on the misspecified single factor model $x_{it} = c_i + \gamma_{i1} f_{1t} + e_{it}$.

The factor strengths are estimated using (7), with the nominal size of the associated multiple tests set to $p = 0.10$, and the critical value exponent to $\delta = 1/4$.[7]

[7] We also consider other values of p and δ, namely $p = 0.05$ and $\delta = 1/3$ or $1/2$, and found the results to be qualitatively

For all experiments we report bias and RMSE of $\hat{\alpha}_j$, size and power of tests of $H_0 : \alpha_j = \alpha_{j0}$ against $\alpha_j = \alpha_{ja}$, $j = 1, 2$, using the test statistic given by

$$z_{\hat{\alpha}_j:\alpha_{j0}} = \frac{(\ln n)\left(\hat{\alpha}_j - \alpha_{j0}\right) - p\left(n - n^{\hat{\alpha}_j}\right) n^{-\delta-\hat{\alpha}_j}}{\left[p\left(n - n^{\hat{\alpha}_j}\right) n^{-\delta-2\hat{\alpha}_j}\left(1 - \frac{p}{n^{\delta}}\right)\right]^{1/2}},\ j = 1, 2. \tag{36}$$

We consider two-sided tests throughout. Empirical size is computed as

$$size_{j,R} = R^{-1}\sum_{r=1}^{R} I\left(\left|z_{\hat{\alpha}_j:\alpha_{j0}}\right| > cv\ \ |H_0\right),\ j = 1, 2.$$

where cv is the critical value of the two-sided normal distribution test which we set to $cv = 1.96$ (for 95% coverage). The empirical power of the tests of $H_0 : \alpha_j = \alpha_{j0}$ against the alternative $H_1 : \alpha_j = \alpha_{ja}$, are obtained for $\alpha_{ja} = \alpha_{j0} + \kappa$, $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$ (20 alternatives) for values of $\alpha_{j0} \in [0.75, 1.00)$. Here, DGP (35) is generated under H_1 and the rejection frequency is computed as

$$power_{j,R} = R^{-1}\sum_{r=1}^{R} I\left(\left|z_{\hat{\alpha}_j:\alpha_{j0}}\right| > cv\ \ |H_1\right),\ j = 1, 2,$$

where $z_{\hat{\alpha}_j:\alpha_{j0}}$ is given by (36). When α_{j0} and/or α_{ja} is equal to unity, we can compute size and power following the randomisation procedure proposed in the online Appendix C.

For all experiments we consider all combinations of $n = \{100, 200, 500, 1,000\}$ and $T = \{60, 120, 200, 500, 1,000\}$, and set the number of replications per experiment to $R = 2,000$. The parameter values of c_i and γ_{ij} in the DGP are redrawn at each replication.

5.2 MC findings

We start with the more general two factor model where the factors are observed (experiments 2A and 2B). Overall, the outcomes are very similar when the model is generated under a one or two factor specification, or under normal and non-normal errors. To save space, here we report the results for experiment 2B with moderately correlated factors ($\rho_{12} = 0.3$) and non-Gaussian errors.[8] Table 1 reports bias, RMSE and size for the estimator of the strength of factor f_{1t}, namely $\hat{\alpha}_1$, for different values of α_{10}, and different (n, T) combinations, when the strength of the second factor is set to $\alpha_{20} = 0.85$. As to be expected, bias and RMSE are universally low and gradually decrease as n, T and α_{10} rise. Especially when $\alpha_{10} = 1$, bias and RMSE are negligible even when $T = 60$. Similar results hold when α_{20} is set to different values in

very similar to those obtained when $p = 0.10$ and $\delta = 1/4$. See Tables S21-S25 in the online Appendix E which show bias, RMSE and size results for Experiment 2B corresponding to these values.

[8]Corresponding results when factors are uncorrelated ($\rho_{12} = 0.0$) or under Gaussian errors are given in the online Appendix E.

the range 0.75 to 1.00. These are available in the online Appendix E.

Moving on to the rejection probabilities under the null hypothesis, we note that since the variance of our proposed estimator is quite small, the rejection probabilities are sensitive to the bias of $\hat{\alpha}_1$. Hence, for smaller values of α_{10} the test is considerably oversized, which is to be expected. However, as the sample size and α_{10} increase, the size distortion reduces considerably, resulting in a well behaved test under the null hypothesis. For $\alpha_{10} = 0.95$ correct empirical size is achieved even for moderate values of T, while, as mentioned earlier, when $\alpha_{10} = 1$ our estimator has an exponential rate of convergence and rapidly converges to its true value. Next, we turn to the power of the test and consider the rejection probabilities under a sequence of alternative hypotheses. Figure 1 depicts power functions corresponding to the strength of factor f_{1t} under non-Gaussian errors, for values of $\alpha_{10} = 0.80, 0.85, 0.90$ and 0.95 when $\alpha_{20} = 0.85$, $T = 200$, and as n increases from 100 to $1,000$. This figure clearly shows that the proposed estimator is very precisely estimated for all values of α_{10} considered, and for all (n, T) combinations. Also as α_{10} rises towards unity the power approaches unity even for very small deviations from the null. We do not report power results for $\alpha_{10} = 1$, due to the ultraconsistency of the estimator in this case.

Similar findings hold when we consider models with one observed factor (experiments 1A and 1B), irrespective of whether the errors are Gaussian. Bias, RMSE and size results under Gaussian and non-Gaussian errors are shown in Tables S1a and S1b of the online Appendix E. Corresponding power functions are shown in Figures S1a and S1b of the same appendix, and give a similar picture as the one we discussed for the two factor case.

We now consider experiments where at the estimation stage the number and/or the identity of factors are assumed unknown. In the case of experiment 3A, the DGP is generated with a single factor, whilst under experiments 3B and 3C the DGP is generated with two uncorrelated factors. In the first of these experiments the factor strength α_{10} is computed with respect to the pervasiveness of the simple cross section average, $\bar{x}_t$. This case is analysed in Section 4. The results corresponding to experiment 3A when errors are non-Gaussian are summarised in Table 2 with the associated power functions in Figure 2. As can be seen, the small sample performance of the estimator of factor strength deteriorates somewhat as compared to when the factor is known, particularly for values of α_0 that are not sufficiently close to unity. The empirical size is particularly elevated for values of $\alpha_0 \leq 0.9$ when compared to the case of observed factors. However, for large sample sizes and values for α_0 close to unity, the proposed estimator seems to be reasonably well behaved even if the factor is unobserved.

In the case of two unobserved factors (experiment 3B), we estimate $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$, again using the simple cross section average, $\bar{x}_t$, first when $\alpha_{10} = 1$ and $\alpha_{20} = 0.51, 0.75, 0.95, 1$. As shown in the top panel of Table 3 under non-Gaussian errors, when α_{20} is set to the lower bound ($= 0.51$), then bias and RMSE results are again universally very low and match the results of the case of one unobserved factor, which is expected. Some deterioration in results can be detected as α_{20} is increased towards unity, for small values of T, e.g. $T = 60$ or 120, but again the size distortions vanish as T increases. The ultraconsistency of our estimator when $\alpha_{10} = 1$ is evident by the values for both bias and RMSE measures which are so small that we have scaled them up by $10,000$ in the top panel of Table 3. When $\alpha_{10}, \alpha_{20} < 1$, estimating α_0 becomes more challenging. This is clear from the bias and RMSE results shown in the bottom panel of Table 3, when $\alpha_{10} = 0.95$ and α_{20} is set to the same values as before (here the scaling of all bias and RMSE values is returned to 100). In line with the conditions of Theorem 2, namely $\sqrt{T} n^{(\alpha_{20}-\alpha_{10})} \to 0$, results worsen for values of α_{20} relatively close to α_{10}, but improve as the distance between α_{10} and α_{20} widens, for any given value of n and T. When $\alpha_{20} = 1$, then the estimator of $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ becomes ultraconsistent, as was the case in the top panel of Table 3.[9]

Experiment 3C continues with the case of two unobserved factors. In this case, we estimate both α_{10} and α_{20}, using the sequential weighted cross section average (CSA) procedure set out in Section 4.1. Table 4 presents bias and RMSE results for α_{10} and α_{20} over $2,000$ replications when $\alpha_{10} = \{0.90, 1\}$, and $\alpha_{20} = \{0.51, 0.75, 0.90\}$, with $\alpha_{10} > \alpha_{20}$. From these findings it is evident that the stronger factor strength, α_{10}, is accurately estimated universally using this approach as well, especially so when $\alpha_{10} = 1$. For the weaker factor with exponent α_{20}, the estimates show a larger bias and RMSE, as to be expected, but continue to be clustered around the true values as n and T rise, and α_{20} is sufficiently distinct from α_{10}, namely when the gap, $\alpha_{10} - \alpha_{20}$, is relatively large. Given the challenges associated with the latent multi-factor setting in terms of identifying and estimating the true factor strengths, the sequential weighted CSA approach produces encouraging initial results.

Finally, consider experiment 4 designed to reflect the setting of the empirical finance application presented in subsection 6.1. Here we focus on a DGP with two factors that are correlated, but a single observed factor model is used for estimating the strength of the first factor, f_{1t}. The results for $\alpha_{10} = 1$ are shown in Table S20a of the online Appendix E, and as can be seen, omitting a second relevant and correlated factor in this case does not unduly affect the performance of the estimator of the strength of

[9] Using the first principal component (PC) of x_{it} instead of the cross section average (CSA) produces similar results when $\alpha_{j0} = 1$, $j = 1, 2$, but under performs in comparison to CSA when $\alpha_{j0} < 1.0$. These results are available in the online Appendix E. See also Section 19.5.1 of Pesaran (2015) where the asymptotic properties of cross section average and the first PC are compared.

the first factor.[10] This seems to be the case for all (n, T) combinations and for different values of α_{20}.[11] However, misspecification is likely to be consequential if the first factor is not sufficiently strong.

6 Empirical applications to finance and macroeconomics

6.1 Identifying risk factors in asset pricing models

The asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), and its multi-factor extension in the context of the Arbitrage Pricing Theory (APT) developed by Ross (1976) are the leading theoretical contributions implemented widely in modern empirical finance to analyse the cross-sectional differences in expected returns. Both approaches imply that expected returns are linear in asset betas with respect to fundamental economic aggregates, and the Fama-MacBeth two-pass procedure (Fama and MacBeth (1973)) is one of the most broadly used methodologies to assess these linear pricing relationships. The first stage in this approach entails choosing the risk factors to be included in the asset pricing model. Given the upsurge in the number of factors deemed relevant to asset pricing in the past few years, a rapidly growing area of the finance literature has been concerned with evaluating the contribution of potential factors to these models. Harvey and Liu (2019) document over 400 such factors published in top ranking academic journals. The primary focus of this literature has been on factor selection on the basis of performance metrics such as the Gibbons,Ross and Shanken statistic of Gibbons et al. (1989), or the maximum squared Sharpe ratio of Fama and French (2018) among many others. More recent contributions further allow for the possibility of false discovery when the number of potential factors is large and multiple testing issues arise - see Feng et al. (2020).

Our application focuses on determining the strength of these factors as a means of evaluating whether their risk can be priced correctly and abstracts from the question of factor selection as such. As shown by Pesaran and Smith (2019), the APT theory requires that risk factors should be sufficiently strong if their associated risk premium is to be estimated consistently. The risk premium of a factor with strength α can be estimated at the rate of $n^{-a/2}$, where n is the number of individual securities under consideration. As a result, $\sqrt{n}$ consistent estimation of the risk premium of a given factor requires the factor in question to be strong with its α equal to unity. Factors with strength less than 0.5 cannot be priced and are absorbed in pricing errors. But in principle, it should be possible to identify the risk premium of semi-strong factors

[10] The bias and RMSE values for this experiment are negligible so that in Table S20a they are reported after scaling them up by the factor of 10,000.

[11] Corresponding results for the case of uncorrelated factors ($\rho_{12} = 0.0$) are also available in the online Appendix E.

(factors whose α lies in the range $1 > \alpha > 1/2$), but very large number of securities are needed for this purpose. In practice, where n is not sufficiently large, at best only factors with strength sufficiently close to unity can be priced.[12] As an illustration of their theoretical results, Pesaran and Smith (2019) consider the widely used Fama and French (1993) three-factor model applied to the constituents of the S&P500 index and assess the strength of each of the factors included in the model, namely the market, size and value factors. In what follows we carry out a more comprehensive investigation of this topic, by assessing the strength of a total of 146 factors.

6.1.1 Data

We consider monthly excess returns of the securities included in the S&P 500 index over the period from September 1989 to December 2017. Since the composition of the index changes over time, we compiled returns on all 500 securities at the end of each month and included in our analysis only those securities that had at least 10 years of history in the month under consideration. On average, we ended up with $n = 442$ securities at the end of each month. The one-month US treasury bill rate (in percent) was chosen as the risk-free rate (r_{ft}), and excess returns were computed as $\tilde{r}_{it} = r_{it} - r_{ft}$, where r_{it} is the return on the i^{th} security between months $t-1$ and t in the sample, inclusive of dividend payments (if any).[13] In addition to the market factor (measured as the excess market return) we consider the 145 factors considered by Feng et al. (2020), which are largely constructed as long/short portfolios capturing a number of different characteristics.[14] In order to account for time variations in factor strength, we use rolling samples (340 in total) of 120 months (10 years) each. The choice of the rolling window is guided by the balance between T and n, and follows the usual practice in the finance literature.[15]

6.1.2 Factor models for individual securities

We commence with the following regressions:

[12] In an early critique of tests of asset pricing theory, Roll (1977) argued that for a test to be valid, it is required that all assets traded in the economy are included in the empirical analysis. In effect requiring n to be very large, and much larger than the number of securities traded on exchanges.

[13] Further details relating to the construction of this dataset can be found in the online Appendix D and in Bailey et al. (2016, 2019).

[14] The authors would like to thank Dacheng Xiu for providing the dataset that covers all the 146 factors, inclusive of the market factor. Apart from 15 factors obtained from specific websites, the remaining factors are constructed using only stocks for companies listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. Moreover, financial firms and firms with negative book equity are excluded. For each characteristic, stocks are sorted using NYSE breakpoints based on their previous year-end values, then long-short value-weighted portfolios (top 30% - bottom 30% or 1-0 dummy difference) are built and rebalanced every June for a 12-month holding period. Further details regarding the construction of this dataset can be found in Feng et al. (2020).

[15] We also consider rolling samples of size 60 months (5 years). The results are shown in the online Appendix D.

$$r_{it} - r_{ft} = a_i + \beta_{im}(r_{mt} - r_{ft}) + \sum_{j=1}^{k} \beta_{ij} f_{jt} + u_{it}, \text{ for } i = 1, 2, \ldots, n_\tau, \tag{37}$$

where n_τ are the number of securities in 10-year rolling samples from September 1989 to December 2017, with $\tau = 1, 2, \ldots, 340$. r_{mt} denotes the return on investing in the market portfolio, which here is approximated by a value weighted average of all CRSP firms incorporated in the US and listed on the NYSE, AMEX, or NASDAQ that have data for month t. As such, this definition of the market portfolio is wider than one which assumes an average of the 440 or so S&P500 securities considered in this study. The excess market return, $(r_{mt} - r_{ft})$, then approximates the market factor. f_{jt} for $j = 1, 2, \ldots, 145$ represent the potential risk factors in the active set under consideration. As explained in Section 5 of Pesaran and Smith (2019), the strength of factor j is defined by $\sum_{i=1}^{n} \left(\beta_{ij} - \bar{\beta}_j\right)^2 = \circleddash\left(n^{\alpha_j}\right)$, and once the market factor is included in (37), it is the case that the coefficients are expressed as deviations of the factor loadings from their means, as required.

Initially, we set $k = 0$ and consider the original CAPM specification of Sharpe (1964) and Lintner (1965),

$$r_{it} - r_{ft} = a_{im} + \beta_{im}(r_{mt} - r_{ft}) + u_{it,m}. \tag{38}$$

We apply our estimator (7) to the loadings β_{im}, $i = 1, 2, \ldots, n_\tau$, and obtain estimates of the strength of the market factor across the rolling windows, $\hat{\alpha}_{m,\tau}$, $\tau = 1, 2, \ldots, 340$.[16]

Next, in order to assess the effect on the market factor strength estimates of adding more factors to (38), as well as to quantify the strength of these additional factors, we add the 145 factors to the CAPM regression, (38), one at a time; namely we run the regressions

$$r_{it} - r_{ft} = a_{is} + \beta_{im|s}(r_{mt} - r_{ft}) + \beta_{is} f_{st} + u_{it,s}, \; i = 1, 2, \ldots, n_\tau \tag{39}$$

for each $s = 1, 2, \ldots, 145$, and each rolling window $\tau = 1, 2, \ldots, 340$. Our choice of model is motivated by the fact that once we have conditioned on the market factor, we can use the One Covariate at the time Multiple Testing (OCMT) methodology of Chudik et al. (2018) as an additional step for selecting the factors that ought to be included in our final asset pricing model. Again, we compute the strength of the market factor with the s^{th} factor included, which we denote by $\hat{\alpha}_{m,\tau|s}$, as well as the strength of each of the additional factors, which we denote by $\hat{\alpha}_{s,\tau}$, for all 340 rolling windows, $\tau = 1, 2, \ldots, 340$. As with the Monte Carlo experiments, in the computation of factor strength we set the nominal size of the associated multiple tests to $p = 0.10$, and the critical value exponent to $\delta = 1/4$.

[16] A similar analysis using the simple CAPM model was conducted in the empirical application of Bailey et al. (2016) where a preliminary suggestion of our estimator of factor strength was originally made. This accompanied the main empirical analysis of quantifying the degree of cross-sectional dependence inherent in the rolling panels of S&P500 security excess returns studied, making use of the estimator formally developed in that paper.

6.1.3 Estimates of factor strengths

First, we consider the rolling estimates obtained for the strength of market factor, α_m, when using the CAPM and the augmented CAPM specifications given by (38) and (39). Figure 3 displays $\hat{\alpha}_{m,\tau}$, $\tau = 1, 2, \ldots, 340$; the 10-year rolling estimates obtained using the CAPM regressions over the period September 1989 to December 2017. As can be seen, all $\hat{\alpha}_{m,\tau}$ are quite close to unity, and it can be safely concluded that the market factor is strong and its risk premium can be estimated consistently at the usual rate of $\sqrt{n}$. There is some evidence of departure from unity over the period between December 1999 to January 2011 which saw a number of sizeable financial events such as the Long-Term Capital Management (LTCM) crisis, the burst of the dot-com bubble and, more recently, the global financial crisis. $\hat{\alpha}_{m,\tau}$ records its minimum value of 0.958 in August 2008, around the time of the Lehman Brothers collapse.[17] As implied by our theoretical results of Section 3, standard errors around these estimates are extremely tight and hard to distinguish graphically from the point estimates.[18] It is also interesting that the estimates of market factor strength are generally unaffected if we consider the augmented CAPM regressions. For each rolling window we now obtain 145 estimates of α_m, denoted by $\hat{\alpha}_{m,\tau|s}$ for $s = 1, 2, \ldots, 145$. We display the average of these estimates, namely, $\bar{\hat{\alpha}}_{m,\tau} = (1/145) \sum_{s=1}^{145} \hat{\alpha}_{m,\tau|s}$, in Figure 3. It is clear that $\bar{\hat{\alpha}}_{m,\tau}$ closely track $\hat{\alpha}_{m,\tau}$. The two series are almost identical during the periods September 1989 to December 1999 and January 2011 to December 2017. There are some minor deviations between $\hat{\alpha}_{m,\tau|s}$ and $\hat{\alpha}_{m,\tau}$ during the period December 1999 to January 2011, when they both deviate marginally from unity, with a maximum deviation of 0.011 in September 2008. The average estimates of $\alpha_{m,\tau}$ also have very narrow confidence bands, with an average standard error of 0.0038 over the full sample, taking its maximum value of 0.0099 in September 2008. Overall, it is evident that the inclusion of an additional factor in (39) has little effect on estimates of the market factor strength, which is in line with the Monte Carlo evidence for experiment 4 summarised in the previous Section.

We can safely conclude that the market factor is strong with the exception of a short period during the recent financial crisis. We now consider the 10-year rolling estimates of the strength of the remaining factors, denoted by $\alpha_{s,\tau}$, using the augmented CAPM regressions. These estimates together with their 90% confidence bands are shown in Figures A1 to A10 of the online Appendix D. They show considerable time variation, especially during December 1999 to January 2011. However, even though a rise in the

[17] Any deviations of $\hat{\alpha}_m$ from unity are not necessarily viewed as signs of market inefficiency. Factor strength could deviate from unity even during non-crises periods.

[18] The corresponding plot of $\hat{\alpha}_{m,\tau}$ estimates under (38) which includes its standard errors is shown at the top left corner of Figure A1 in the online Appendix D.

average pair-wise correlations between the 146 factors is evident in the build up to the 1999 crisis, at no point during the full sample (September 1989 to December 2017) do any of these factors become strong in the sense that $\hat{\alpha}_{s,\tau}$ is clearly below 1, for all s and τ. The market factor dominates all other factors in strength. Indeed, in Figure 4 we observe that the proportion of factors (out of the 145 in total) whose strength exceeds the threshold values of $0.85, 0.90$ and 0.95 in each rolling window progressively drops so that there are no factors left whose strength exceeds 0.95 throughout our sample period. This suggests that only the market factor can be considered to be a risk factor whose risk premium can be estimated consistently at the standard $\sqrt{n}$ rate. The role of the remaining 145 factors in the asset pricing models (39) could be to filter out the effects of any additional semi-strong cross-dependence in asset returns in order to achieve weak enough cross-sectional dependence in the errors u_{it}, required for $\sqrt{n}$ consistent estimation of market risk premia.

Next, we rank the 145 factors (plus the market factor) from the strongest to the weakest in terms of the percentage of months in our sample period (340 in total) that their strength exceeds the threshold value of 0.90. As shown in Table A1 of the online Appendix D, there are 65 factors that meet this criterion at least in some instances during the sample period. As expected, the market factor ranks first with an average estimated strength of 0.99, followed by factors associated with leverage, and the ratios of sales to cash, cash flow to price, net debt to price and earnings to price. The second ranking factor, leverage, has average strength of 0.827, with only 37.9% of the time being above 0.9. Interestingly, the Fama French value factor (high minus low) ranks 34th in our table while the size factor (small minus big) does not even enter the group of 65 factors, recording values of $\hat{\alpha}$ below 0.90 across all rolling windows. For completeness, Table A1 also includes time averages of each factor strength over the full sample (September 1989 - December 2017), and the three sub-samples: September 1989 - August 1999, September 1999 - August 2009, and September 2009 - December 2017. While on average, the strengths of these factors are around 0.80 in the first and the last decade in our sample, in the period between September 1999 to August 2009, the strength of many factors rises to around 0.91. This rise could be due to non-fundamental factors gaining importance over the fundamental factors during the recent financial crisis, and can be viewed as evidence of market decoupling.

Finally, it is of interest to investigate whether the strength of the strongest latent factor implied by the panel of S&P 500 securities' excess returns coincides with that of the market risk factor, which we identified as the strongest observed factor under our previous analysis. In line with the discussion of Section 4, the strength of the strongest unobserved factor will be captured by the strength of the cross section average of the excess returns in each rolling window, noting the stricter conditions on the (n,T) dimensions of the panel implied by Theorem 2. Figure 5 plots the 10-year rolling $\hat{\alpha}_{csa,\tau}$ estimates implied by the cross section average of excess returns against the 10-year rolling $\hat{\alpha}_{m,\tau}$ estimates implied by the simple CAPM regression (38). It is evident that the two series are almost identical throughout our sample period except for the period between September 1999 to January 2011 where they deviate from each other to some extent. The average correlation between $\hat{\alpha}_{csa,\tau}$ and $\hat{\alpha}_{m,\tau}$ over $\tau = 1, 2, \ldots, 340$ stands at 0.93. On this basis, we also computed the rolling correlation coefficients between the cross section average of individual securities' excess returns and the observed market risk factor again over the rolling windows $\tau = 1, 2, \ldots, 340$. These are consistently close to unity with an average value across all the rolling windows of 0.95, and with the lowest value of 0.85 obtained for the period between September 1999 to January 2011.

6.2 Strength of common macroeconomic shocks

Similar considerations apply to macroeconomic shocks and their pervasive effects on different parts of the macroeconomy. As discussed in Giannone et al. (2017) and references therein, the advent of 'high-dimensional' datasets has led to the development of predictive models that are either based on shrinkage of useful information inherent across the whole set of data into a finite number of latent factors (e.g. Stock and Watson (2015) and references therein), or assume that all relevant information for prediction is captured by a small subset of variables from the larger pool of regressors implied by these data (e.g. Hastie et al. (2015), Belloni et al. (2011) among others). Such methods are appealing in macroeconomics since they tend to provide more reliable impulse responses and forecasts over traditional models, when used for macroeconomic policy analysis and forecasting. However, as argued in Giannone et al. (2017), it is not evident that either approach is always clearly supported by the (unknown) structure of the given data and that model averaging might be preferable.

To measure the pervasiveness of the macroeconomic shocks, we make use of an updated version of the macroeconomic dataset compiled originally by Stock and Watson (2012) and subsequently extended by McCracken and Ng (2016). Here, we assume that the macroeconomic shocks are unobserved and estimate the strength of the strongest of such shocks from the updated dataset which consists of balanced quarterly observations over the period $1988Q1 - 2019Q2$ ($T = 126$) on $n = 187$ out of the 200 macroeconomic variables used in Stock and Watson (2012).[19] Ten out of the 200 macroeconomic variables used in Stock

[19]The raw data, which include both high-level economic and financial aggregates as well as disaggregated

and Watson (2012) are no longer available in the updated version of the dataset.[20] Further details on this dataset can be found in the online Appendix D.

6.2.1 How strong is the strongest of the unobserved macroeconomic shocks?

As discussed in Section 4, identifying and estimating the strengths of unobserved factors of varying strengths becomes challenging due to the fact that, in general, factors are identified only up to a non-singular rotation matrix. However, as argued above we are still able to identify and estimate the strength (α) of the strongest shock using the cross section average of the variables in the dataset.[21] We computed estimates of α for the pre-crisis period, 1988Q1 to 2007Q4, as well as for the full sample period ending on 2019Q2. The factor strength estimates are shown in Table 5. They are clustered around 0.94, and are quite robust to the choice of the parameters p and δ in the critical value function (6), as well as to the time period considered. These estimates are consistently below 1, and suggest that whilst there exist strong macroeconomic shocks, the effects of such shocks are not nearly as pervasive as have been assumed in the factor literature applied to macro variables. This finding is further corroborated by the estimates of the exponent of cross-sectional dependence of BKP, also shown in Table 5.[22]

7 Conclusions

Recent work by Bailey et al. (2016, 2019) has focused on the rationale and motivation behind the need for determining the extent of cross-sectional dependence, be it in finance or macroeconomics, and has provided a conceptual framework and tools for estimating the strength of such interdependencies in economic and financial systems. However, this literature does not address the problem of estimating the strength of individual factors that underlie such cross dependencies, which can be of interest, for example, for pricing of risk in empirical finance, or for quantifying the pervasiveness of macroeconomic shocks. The

components, are updated regularly and can be found on the Federal Reserve Bank of St Louis website at: https://research.stlouisfed.org/econ/mccracken/static.html. All variables were screened for outliers and transformed as required to achieve stationarity. Details about variable definitions, descriptions and transformations can be found in the accompanying FRED-QD appendix to McCracken and Ng (2016) which links to Stock and Watson (2012) and is downloadable from the aforementioned website.

[20] These are: (1) Construction contracts, (2) Manufacturing and trade inventories, (3) Index of sensitive materials prices (disc), (4) Spot market price index BLS&CRB: all commodities, (5) NAPM commodity price index, (6) 3m Eurodollar deposit rate, (7) MED3-TB3MS, (8) GZ-spread, (9) GZ Excess bond premium, and (10) DJIA.

[21] Again, one needs to take into consideration the stricter conditions on the (n, T) dimensions of the panel, as implied by Theorem 2.

[22] Using the Sequential Multiple Testing (SMT) detection procedure developed in Kapetanios et al. (2020), we also checked to see if any of the unit(s) in the macro dataset can be viewed as pervasive, namely sufficiently influential to affect all other variables. The SMT procedure could not detect any such variables for all choices $p_{\max} = 0, 1, \ldots, 6$, where p_{max} denotes the assumed maximum number of potential factors in the dataset.

current paper addresses this gap. It proposes a novel estimator of factor strength based on the number of statistically significant t-statistics in a regression of each unit in the panel dataset on the factor under consideration, and provides inferential theory for the proposed estimator. Detailed and extensive Monte Carlo and empirical analyses showcase the potential of the proposed method.

The current paper considers estimation and inference when the panel regressions are based on a finite number of observed factors. Some theoretical evidence is also provided for the case when the model contains unobserved factors. Further research is required to link our analysis to the problem of factor selection discussed by Feng et al. (2020). Also, it would be of interest to address the identification problem when there are multiple unobserved factors. One possibility would be to exploit the approach recently developed in Kapetanios et al. (2020) to see whether the unobserved factors can be associated with dominant units or some other observable components.

References

Acemoglu, D., V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi (2012). The network origins of aggregate fluctuations. *Econometrica 80*(5), 1977–2016.

Anatolyev, S. and A. Mikusheva (2019). Factor models with many assets: Strong factors, weak factors, and the two-pass procedure. *Available at arXiv: https://arxiv.org/abs/1807.04094*.

Andrews, D. W. K. (2001). Testing when a parameter is on the boundary of the maintained hypothesis. *Econometrica 69*(3), 683–734.

Bai, J. (2003). Inferential theory for factor models of large dimensions. *Econometrica 71*(1), 135–171.

Bai, J. and S. Ng (2002). Determining the number of factors in approximate factor models. *Econometrica 70*(1), 191–221.

Bailey, N., G. Kapetanios, and M. H. Pesaran (2016). Exponent of cross-sectional dependence: Estimation and inference. *Journal of Applied Econometrics 31*(6), 929–960.

Bailey, N., G. Kapetanios, and M. H. Pesaran (2019). Exponent of cross-sectional dependence for residuals. *Sankhya B 81*(1), 46–102.

Bailey, N., M. H. Pesaran, and L. V. Smith (2019). A multiple testing approach to the regularisation of sample correlation matrices. *Journal of Econometrics 208*(2), 507–534.

Belloni, A., V. Chrenozhukov, and C. Hansen (2011). Inference for high-dimensional sparse econometric models. *In Advances in Economics and Econometrics, 10th World Congress of Econometric Society 2011*.

Chudik, A., G. Kapetanios, and M. H. Pesaran (2018). A one covariate at a time, multiple testing approach to variable selection in high-dimensional linear regression models. *Econometrica 86*(4), 1479–1512.

Chudik, A., M. H. Pesaran, and E. Tosetti (2011). Weak and strong cross section dependence and estimation of large panels. *The Econometrics Journal 14*(1), C45–C90.

Corradi, V. and N. Swanson (2006). Predictive density and conditional confidence interval accuracy tests. *Journal of Econometrics 135*(1-2), 187–228.

Fama, E. F. and S. K. R. French (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics 33*(1), 3–56.

Fama, E. F. and S. K. R. French (2018). Choosing factors. *Journal of Financial Economics 128*(2), 234–252.

Fama, E. F. and J. MacBeth (1973). Risk, returns and equilibrium: Empirical tests. *Journal of Political Economy 81*(3), 607–636.

Feng, G., S. Giglio, and D. Xiu (2020). Taming the factor zoo: A test of new factors. *Journal of Finance 75*(3), 1327–1370.

Freyaldenhoven, S. (2019). A generalized factor model with local factors. *FRBP Working Paper No. 19-23*.

Giannone, D., M. Lenza, and G. Primiceri (2017). Economic predicitions with large data: The illusion of sparsity. *CEPR Discussion Paper No. DP12256 (revised July 2020)*.

Gibbons, M., S. Ross, and J. Shanken (1989). A test of the efficiency of a given portfolio. *Econometrica 57*(5), 1121–1152.

Harvey, C. R. and Y. Liu (2019). A census of the factor zoo. *Available at SSRN: https://ssrn.com/abstract=3341728 or http://dx.doi.org/10.2139/ssrn.3341728*.

Hastie, T., R. Tibshirani, and M. Wainwright (2015). *Statistical Learning with Sparsity*. CRC Press.

Kaiser, H. (1958). The varimax criterion for analytic rotation in factor analysis. *Psychometrika 23*(3), 187–200.

Kapetanios, G., M. H. Pesaran, and S. Reese (2020). Detection of units with pervasive effects in large panel data models. *Journal of Econometrics, forthcoming*.

Kleibergen, F. (2009). Tests of risk premia in linear factor models. *Journal of Econometrics 149*(1-2), 149–173.

Lettau, M. and M. Pelger (2018). Estimating latent asset-pricing factors. *Journal of Econometrics 218*(1), 1–31.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics 47*(1), 13–37.

McCracken, M. and S. Ng (2016). FRED-MD: A monthly database for macroeconomic research. *Journal of Business & Economic Statistics 34*(4), 574–589.

Onatski, A. (2012). Asymptotics of the principal components estimator of large factor models with weakly influential factors. *Journal of Econometrics 168*(2), 244–258.

Pearson, E. (1950). On questions raised by the combination of tests based on discontinuous distributions. *Biometrika 37*(3-4), 383–398.

Pesaran, M. H. (2006). Estimation and inference in large heterogeneous panels with a multifactor error structure. *Econometrica 74*(4), 967–1012.

Pesaran, M. H. (2015). *Time Series and Panel Data Econometrics*. Oxford University Press, Oxford.

Pesaran, M. H. and R. Smith (2019). The role of factor strength and pricing errors for estimation and inference in asset pricing models. *CESifo Working Paper No. 7919*.

Pesaran, M. H. and T. Yamagata (2017). Testing for alpha in linear factor pricing models with a large number of securities. *USC-INET Research Paper No. 17-13 (revised January 2018)*.

Pesaran, M. H. and C. F. Yang (2020). Econometric analysis of production networks with dominant units. *Journal of Econometrics 219*, 507–541.

Roll, R. (1977). A critique of the asset pricing theory's tests part i: On past and potential testability of the theory. *Journal of Financial Economics 4*(2), 129–176.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory 13*(3), 341–360.

Ročková, V. and E. George (2016). Fast Bayesian factor analysis via automatic rotations to sparsity. *Journal of the American Statistical Association 111*(516), 1608–1622.

Sharpe, W. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance 19*(3), 425–442.

Stock, J. and M. Watson (2012). Disentangling the channels of the 2007-2009 recession. *Brooking Papers on Economic Activity*, 81–135.

Stock, J. and M. Watson (2015). Factor models in macroeconomics. In J. Taylor and H. Uhlig (Eds.), *Handbook of Macroeconomics, Vol. 2*. North Holland.

Trapani, L. (2018). A randomized sequential procedure to determine the number of factors. *Journal of the American Statistical Association 113*(523), 1341–1349.

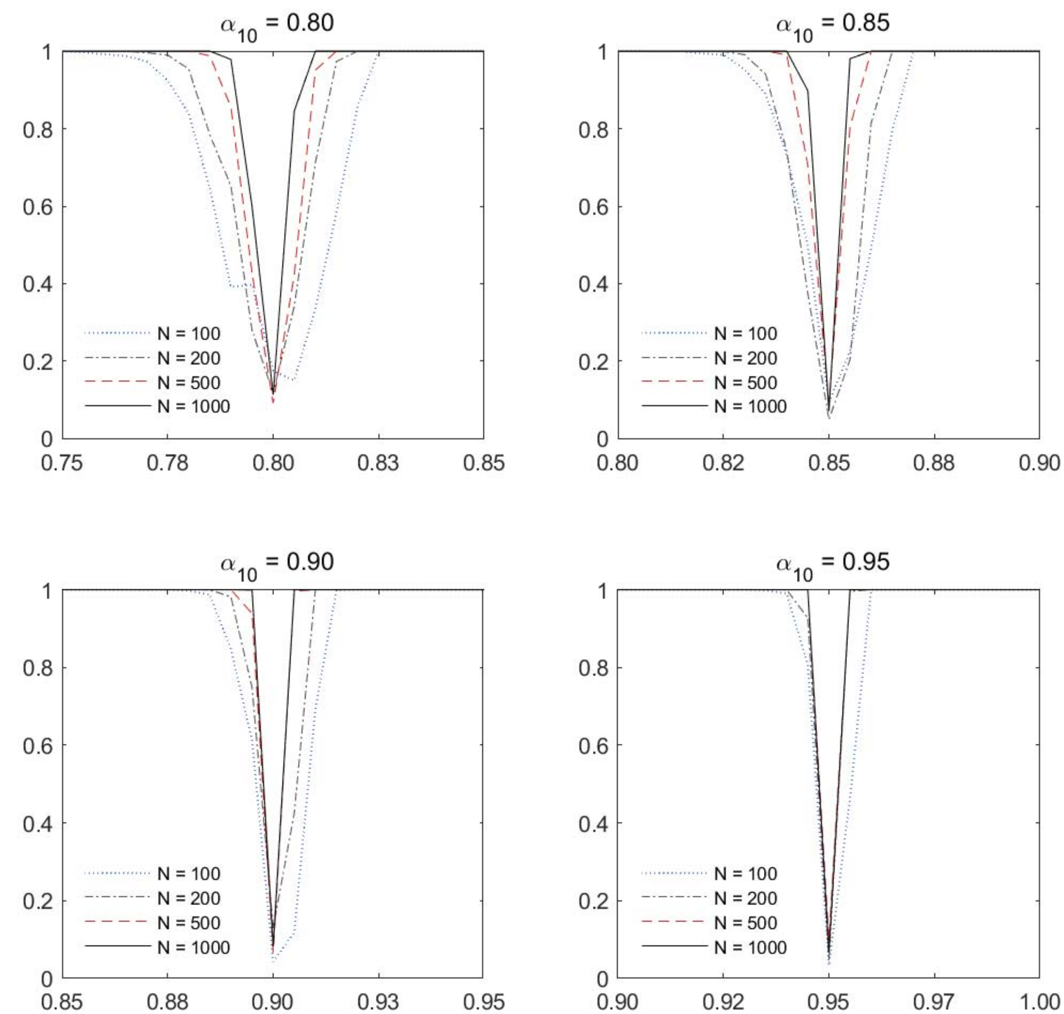
Uematsu, Y. and T. Yamagata (2019). Estimation of weak factor models. *ISER Discussion Paper No. 1053 (revised September 2020)*.

Table 1: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.85

	Bias (×100)					RMSE (×100)					Size (×100)				
$n\backslash$T	60	120	200	500	1000	60	120	200	500	1000	60	120	200	500	1000
	$\alpha_{10}=0.75$, $\alpha_{20}=0.85$														
100	1.13	1.18	1.15	1.07	1.03	1.65	1.54	1.52	1.43	1.40	9.00	4.10	3.65	2.40	2.25
200	1.46	1.46	1.39	1.32	1.32	1.68	1.62	1.55	1.47	1.47	14.50	9.50	8.30	7.10	6.60
500	1.28	1.30	1.21	1.15	1.13	1.41	1.37	1.28	1.22	1.20	22.40	13.55	10.00	8.05	8.20
1000	1.27	1.25	1.20	1.12	1.10	1.36	1.30	1.24	1.16	1.14	26.30	15.00	11.45	7.70	6.40
	$\alpha_{10}=0.80$, $\alpha_{20}=0.85$														
100	0.63	0.67	0.65	0.61	0.58	1.13	1.00	1.00	0.95	0.92	27.60	18.10	18.55	17.95	19.70
200	0.90	0.97	0.91	0.89	0.86	1.11	1.10	1.05	1.01	0.98	20.45	12.60	11.45	9.75	7.75
500	0.82	0.90	0.85	0.82	0.80	0.94	0.96	0.90	0.87	0.86	22.00	12.45	8.30	7.00	7.10
1000	0.78	0.85	0.81	0.76	0.75	0.87	0.88	0.84	0.79	0.77	28.35	17.05	11.60	8.40	6.65
	$\alpha_{10}=0.85$, $\alpha_{20}=0.85$														
100	0.51	0.66	0.68	0.62	0.61	0.87	0.85	0.87	0.81	0.79	21.60	9.65	9.15	7.65	6.90
200	0.49	0.61	0.59	0.55	0.54	0.70	0.71	0.69	0.65	0.64	14.15	5.45	4.15	3.15	3.45
500	0.38	0.52	0.49	0.46	0.46	0.51	0.57	0.53	0.51	0.50	29.35	11.95	7.40	8.45	7.10
1000	0.37	0.49	0.48	0.44	0.43	0.46	0.52	0.50	0.47	0.45	31.50	11.00	7.65	5.45	4.10
	$\alpha_{10}=0.90$, $\alpha_{20}=0.85$														
100	0.21	0.39	0.39	0.37	0.36	0.58	0.54	0.54	0.52	0.51	23.60	4.15	3.60	3.10	2.85
200	0.10	0.27	0.26	0.24	0.24	0.38	0.38	0.35	0.33	0.34	36.30	15.10	12.60	12.25	12.55
500	0.11	0.29	0.28	0.27	0.26	0.29	0.32	0.31	0.30	0.29	42.40	11.05	7.05	7.90	7.40
1000	0.10	0.27	0.28	0.26	0.25	0.24	0.29	0.29	0.27	0.27	48.70	11.25	9.85	5.55	6.70
	$\alpha_{10}=0.95$, $\alpha_{20}=0.85$														
100	-0.16	0.06	0.08	0.06	0.06	0.44	0.25	0.24	0.22	0.22	38.35	7.20	3.65	2.25	2.35
200	-0.10	0.10	0.11	0.10	0.10	0.30	0.19	0.17	0.18	0.17	46.80	8.85	4.40	4.75	3.95
500	-0.11	0.10	0.11	0.11	0.10	0.25	0.13	0.13	0.13	0.13	68.20	14.45	8.65	7.55	7.80
1000	-0.12	0.09	0.10	0.09	0.09	0.23	0.10	0.11	0.10	0.10	77.10	11.45	5.60	4.60	5.05
	$\alpha_{10}=1.00$, $\alpha_{20}=0.85$														
100	-0.28	-0.02	0.00	0.00	0.00	0.41	0.06	0.02	0.00	0.00	-	-	-	-	-
200	-0.25	-0.02	0.00	0.00	0.00	0.33	0.05	0.01	0.00	0.00	-	-	-	-	-
500	-0.26	-0.02	0.00	0.00	0.00	0.32	0.03	0.01	0.00	0.00	-	-	-	-	-
1000	-0.25	-0.02	0.00	0.00	0.00	0.31	0.03	0.00	0.00	0.00	-	-	-	-	-

Notes: Parameters of DGP (35) are generated as follows: for unit specific effects, $c_i \sim IIDN(0,1)$, for $i=1,2,\dots,n$. The factors, (f_{1t}, f_{2t}), are multivariate normal with variances $\sigma^2_{f_1}=\sigma^2_{f2}=1$ and correlation given by $\rho_{12}=corr(f_1,f_2)=0.3$. Each factor assumes an autoregressive process with correlation coefficients $\rho_{f_j}=0.5$, $j=1,2$. The factor loadings are generated as $v_{ij} \sim IIDU(\mu_{v_j}-0.2, \mu_{v_j}+0.2)$, for $[n^{\alpha_{j0}}]$ units, $j=1,2$, respectively, and zero otherwise. We set $\mu_{v_1}=\mu_{v_2}=0.71$. Both α_{10} and α_{20} range between $[0.75, 1.00]$ with 0.05 increments. The innovations u_{it} are non-Gaussian, such that $u_{it}=\frac{\sigma_i}{2}\left(\chi^2_{2,it}-2\right)$, with $\sigma^2_i \sim IID(1+\chi^2_{2,i})/3$, for $i=1,2,\dots,n$. In the computation of $\hat{\alpha}_j$, $j=1,2$, we use $p=0.10$ and $\delta=1/4$ when setting the critical value. Size is computed under H_0: α_j=α_{j0}, for $j=1,2$, using a two-sided alternative. The number of replications is set to $R=2000$.

Figure 1: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.85, $n = 100, 200, 500, 1000$ and $T = 200$



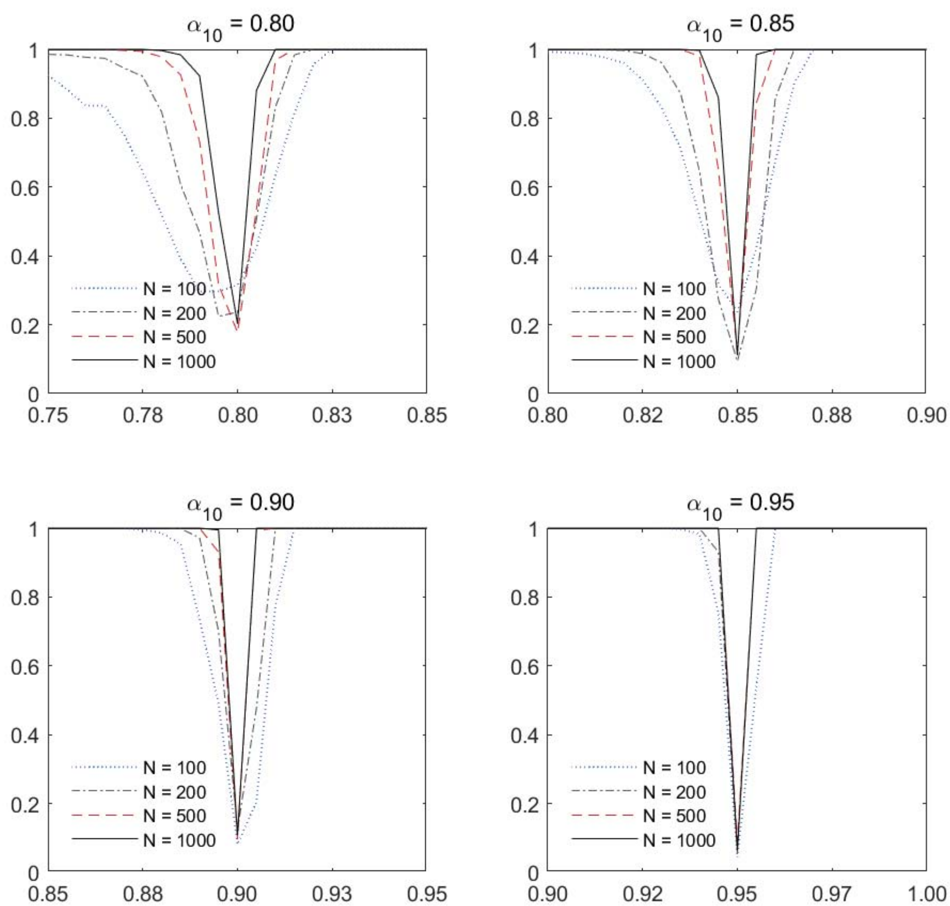
Notes: See the notes to Table 1 for details of the data generating process. Power is computed under H_1: $\alpha_{1a}{=}\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table 2: Bias, RMSE and Size (×100) of estimating the strength of strongest factor in the case of experiment 3A (unobserved single factor - non-Gaussian errors) using cross section average

	Bias (×100)					RMSE (×100)					Size (×100)				
$n\backslash$T	60	120	200	500	1000	60	120	200	500	1000	60	120	200	500	1000
	$\alpha_{10} = 0.75$														
100	2.22	2.40	2.70	4.34	6.69	2.76	2.84	3.12	4.70	6.98	25.40	28.15	35.35	73.55	98.15
200	2.07	2.02	2.10	2.60	3.49	2.41	2.26	2.33	2.80	3.69	31.40	29.80	32.10	51.80	82.70
500	1.66	1.61	1.56	1.61	1.81	1.88	1.75	1.67	1.72	1.92	33.00	29.30	28.05	30.15	43.60
1000	1.54	1.45	1.39	1.36	1.43	1.72	1.55	1.45	1.42	1.49	38.30	30.00	26.65	24.85	30.00
	$\alpha_{10} = 0.80$														
100	1.21	1.28	1.40	2.17	3.32	1.65	1.64	1.73	2.46	3.58	33.10	28.25	30.40	55.55	85.55
200	1.22	1.22	1.21	1.40	1.78	1.46	1.37	1.36	1.54	1.93	29.30	25.35	23.70	34.00	55.95
500	1.02	1.03	0.99	0.99	1.04	1.16	1.11	1.05	1.05	1.09	26.50	20.60	16.90	17.65	21.05
1000	0.92	0.92	0.87	0.85	0.86	1.02	0.97	0.91	0.88	0.89	32.40	24.45	19.95	17.35	17.70
	$\alpha_{10} = 0.85$														
100	0.87	0.93	0.96	1.27	1.72	1.15	1.14	1.16	1.46	1.91	25.90	20.80	22.05	37.40	63.10
200	0.68	0.72	0.69	0.75	0.89	0.86	0.83	0.80	0.86	1.00	15.70	10.65	8.90	12.50	20.25
500	0.46	0.57	0.55	0.53	0.53	0.62	0.62	0.59	0.57	0.58	25.35	13.60	10.75	9.50	11.00
1000	0.46	0.52	0.49	0.47	0.47	0.54	0.55	0.51	0.50	0.50	27.20	14.75	9.30	7.85	7.25
	$\alpha_{10} = 0.90$														
100	0.41	0.50	0.50	0.61	0.78	0.66	0.64	0.66	0.76	0.93	17.55	7.25	7.20	12.15	22.55
200	0.24	0.31	0.30	0.32	0.36	0.44	0.40	0.39	0.41	0.44	28.10	12.75	12.85	12.35	12.90
500	0.20	0.30	0.30	0.28	0.28	0.32	0.34	0.33	0.32	0.31	30.75	10.65	8.40	7.20	8.65
1000	0.17	0.29	0.28	0.27	0.26	0.26	0.31	0.29	0.28	0.28	38.65	14.25	9.45	7.65	7.75
	$\alpha_{10} = 0.95$														
100	0.00	0.10	0.11	0.14	0.19	0.36	0.26	0.26	0.29	0.34	25.30	5.40	4.95	6.75	10.20
200	0.00	0.12	0.11	0.12	0.13	0.25	0.19	0.18	0.19	0.20	32.90	7.20	4.85	5.75	6.15
500	-0.02	0.11	0.12	0.11	0.11	0.18	0.14	0.14	0.13	0.14	54.55	11.65	7.10	7.60	9.65
1000	-0.05	0.10	0.10	0.09	0.09	0.16	0.11	0.11	0.11	0.10	64.50	8.60	5.60	5.90	4.75
	$\alpha_{10} = 1.00$														
100	-0.15	-0.01	0.00	0.00	0.00	0.26	0.04	0.00	0.00	0.00	-	-	-	-	-
200	-0.16	-0.01	0.00	0.00	0.00	0.23	0.03	0.00	0.00	0.00	-	-	-	-	-
500	-0.18	-0.01	0.00	0.00	0.00	0.23	0.02	0.00	0.00	0.00	-	-	-	-	-
1000	-0.18	-0.01	0.00	0.00	0.00	0.23	0.02	0.00	0.00	0.00	-	-	-	-	-

Notes: Parameters of DGP (35) are generated as follows: for unit specific effects, $c_i \sim IIDN(0,1)$, for $i = 1, 2, \dots, n$. The factor, f_{1t}, is normally distributed with variance $\sigma^2_{f_1} = 1$. The factor assumes an autoregressive process with correlation coefficient $\rho_{f_1} = 0.5$. The factor loadings are generated as $v_{i1} \sim IIDU(\mu_{v_1} - 0.2, \mu_{v_1} + 0.2)$, for $[n^{\alpha_{10}}]$ units, and zero otherwise. $v_{i2} = 0$, for all i. We set $\mu_{v_1} = 0.71$. α_{10} ranges between $[0.75, 1.00]$ with 0.05 increments. The innovations u_{it} are non-Gaussian, such that $u_{it} = \frac{\sigma_i}{2}\left(\chi^2_{2,it} - 2\right)$, with $\sigma^2_i \sim IID(1 + \chi^2_{2,i})/3$, for $i = 1, 2, \dots, n$. $\alpha_0 = \alpha_{10}$ is estimated by regressing observations, x_{it}, on an intercept and the cross section average of x_{it}, $\bar{x}_t = n^{-1}\sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \dots, T$. In the computation of $\hat{\alpha}_1$ we use $p = 0.10$ and $\delta = 1/4$ when setting the critical value. The number of replications is set to $R = 2000$.

Figure 2: Empirical power functions associated with testing different strengths of strongest factor in the case of experiment 3A (unobserved single factor - non-Gaussian errors) using cross section average, when $n = 100, 200, 500, 1000$ and $T = 200$



Notes: See the notes to Table 2 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table 3: Bias and RMSE of estimating the strength of strongest factor in the case of experiment 3B (two unobserved factors - non-Gaussian errors) using cross section average, when $\alpha_{10} = 1.00$ and $\alpha_{10} = 0.95$

	Bias ($\times 10,000$)					RMSE ($\times 10,000$)				
$n\backslash$T	60	120	200	500	1000	60	120	200	500	1000
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.51$									
100	-16.43	-0.76	-0.02	0.00	0.00	28.19	4.20	0.69	0.00	0.00
200	-18.70	-1.02	-0.06	0.00	0.00	26.79	3.43	0.76	0.00	0.00
500	-19.00	-1.09	-0.07	0.00	0.00	24.47	2.37	0.48	0.07	0.00
1000	-19.25	-1.24	-0.07	0.00	0.00	24.16	2.09	0.34	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$									
100	-17.02	-0.94	-0.01	0.00	0.00	28.75	4.68	0.49	0.00	0.00
200	-18.06	-1.16	-0.09	0.00	0.00	26.22	3.74	0.90	0.00	0.00
500	-18.99	-1.10	-0.08	0.00	0.00	24.71	2.38	0.52	0.00	0.00
1000	-19.65	-1.24	-0.08	0.00	0.00	24.80	2.05	0.34	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.95$									
100	-19.08	-1.80	-0.10	0.00	0.00	34.54	6.72	1.46	0.00	0.00
200	-20.83	-2.07	-0.16	0.00	0.00	31.80	5.33	1.22	0.00	0.00
500	-21.20	-2.07	-0.21	0.00	0.00	29.65	3.76	0.89	0.00	0.00
1000	-22.34	-2.24	-0.25	0.00	0.00	29.65	3.76	0.89	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$									
100	-1.16	-0.01	0.00	0.00	0.00	5.49	0.49	0.00	0.00	0.00
200	-1.48	-0.02	0.00	0.00	0.00	4.25	0.42	0.00	0.00	0.00
500	-1.55	-0.02	0.00	0.00	0.00	3.30	0.27	0.00	0.00	0.00
1000	-1.63	-0.03	0.00	0.00	0.00	2.81	0.23	0.03	0.00	0.00
	Bias ($\times 100$)					RMSE ($\times 100$)				
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.51$									
100	0.02	0.17	0.22	0.39	0.59	0.38	0.34	0.39	0.54	0.72
200	0.01	0.16	0.16	0.22	0.30	0.28	0.24	0.24	0.30	0.38
500	-0.03	0.13	0.13	0.14	0.17	0.19	0.16	0.16	0.17	0.20
1000	-0.06	0.10	0.11	0.11	0.11	0.17	0.12	0.12	0.12	0.13
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.75$									
100	0.68	1.25	1.58	1.72	1.80	0.97	1.40	1.67	1.79	1.87
200	0.47	1.00	1.26	1.51	1.54	0.70	1.11	1.33	1.54	1.57
500	0.23	0.60	0.84	1.19	1.26	0.43	0.71	0.91	1.21	1.27
1000	0.10	0.42	0.58	0.95	1.07	0.31	0.51	0.66	0.97	1.08
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.95$									
100	3.51	3.99	4.05	4.05	4.05	3.56	4.01	4.07	4.07	4.07
200	3.35	3.88	3.95	3.96	3.96	3.39	3.89	3.96	3.97	3.96
500	3.17	3.73	3.82	3.82	3.83	3.20	3.74	3.82	3.83	3.83
1000	3.02	3.62	3.71	3.73	3.72	3.05	3.63	3.71	3.73	3.72
	$\alpha_{10} = 0.95$, $\alpha_{20} = 1.00$									
100	-0.19	-0.02	0.00	0.00	0.00	0.32	0.07	0.02	0.00	0.00
200	-0.21	-0.02	0.00	0.00	0.00	0.30	0.06	0.01	0.00	0.00
500	-0.21	-0.02	0.00	0.00	0.00	0.29	0.04	0.01	0.00	0.00
1000	-0.21	-0.02	0.00	0.00	0.00	0.29	0.04	0.01	0.00	0.00

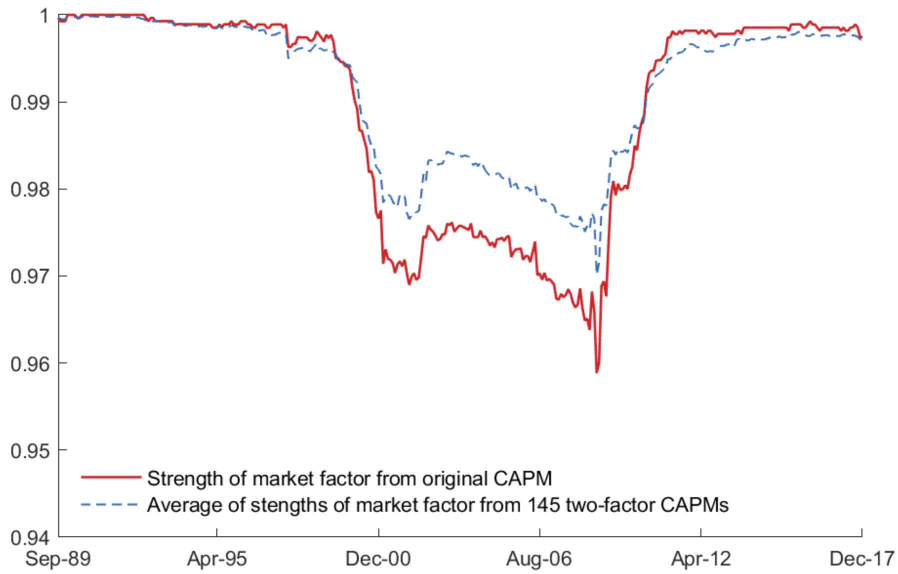
Notes: Parameters of DGP (35) are generated as described in Table 1, with $\rho_{12} = corr(f_1, f_2) = 0.0$. $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated by regressing observations, x_{it}, on an intercept and the cross section average of x_{it}, $\bar{x}_t = n^{-1}\sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \ldots, T$.

Table 4: Bias and RMSE (×100) of estimating factor strengths α_{10} and α_{20} in the case of experiment 3C (two unobserved factors - Gaussian errors) using sequential weighted cross section averages

	Factor strength estimate $\hat{\alpha}_1$					Factor strength estimate $\hat{\alpha}_2$				
	Bias (×100)									
n\T	60	120	200	500	1000	60	120	200	500	1000
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.51$									
100	0.996	1.195	1.347	1.798	2.254	12.730	20.574	25.966	34.521	39.083
200	0.694	0.792	0.834	1.058	1.326	9.877	15.166	20.563	29.687	35.508
500	0.592	0.660	0.647	0.720	0.824	8.021	10.638	14.151	22.454	29.073
1000	0.528	0.570	0.568	0.581	0.633	8.397	9.185	11.134	17.323	23.462
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.75$									
100	2.993	3.786	4.020	4.209	4.330	-5.110	2.158	7.557	14.542	17.867
200	2.330	3.135	3.465	3.625	3.644	-7.216	-0.726	4.606	12.061	16.293
500	1.712	2.404	2.764	3.045	3.046	-8.871	-3.609	0.979	8.403	13.103
1000	1.332	1.940	2.344	2.671	2.706	-9.177	-5.525	-1.390	5.549	10.499
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.51$									
100	-0.079	-0.002	0.000	0.000	0.000	12.077	19.252	25.094	33.634	38.489
200	-0.092	-0.004	0.000	0.000	0.000	9.113	14.259	19.118	28.637	34.609
500	-0.103	-0.005	0.000	0.000	0.000	7.447	9.901	13.289	21.315	27.951
1000	-0.103	-0.005	0.000	0.000	0.000	7.983	8.815	10.406	16.356	22.332
	$\alpha_{10} = 01.00$, $\alpha_{20} = 0.75$									
100	-0.081	-0.002	0.000	0.000	0.000	-6.237	2.250	7.548	14.589	18.003
200	-0.090	-0.004	0.000	0.000	0.000	-8.562	-2.124	3.532	11.930	16.289
500	-0.103	-0.005	0.000	0.000	0.000	-10.215	-5.456	-0.362	7.617	13.186
1000	-0.105	-0.005	0.000	0.000	0.000	-10.539	-6.710	-3.091	4.172	9.980
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.90$									
100	-0.119	-0.008	0.000	0.000	0.000	-21.706	-13.511	-8.373	-0.725	2.667
200	-0.121	-0.007	0.000	0.000	0.000	-22.487	-14.950	-9.211	-1.550	2.301
500	-0.127	-0.008	-0.001	0.000	0.000	-21.648	-16.020	-10.902	-2.487	1.872
1000	-0.123	-0.008	-0.001	0.000	0.000	-21.291	-15.452	-11.191	-2.793	1.719
	RMSE (×100)									
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.51$									
100	1.180	1.375	1.525	1.940	2.360	14.674	21.440	26.417	34.664	39.150
200	0.808	0.892	0.935	1.152	1.402	12.238	16.306	21.201	29.895	35.598
500	0.640	0.697	0.683	0.763	0.869	10.412	12.150	15.091	22.781	29.219
1000	0.555	0.587	0.587	0.602	0.656	10.490	10.831	12.283	17.761	23.660
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.75$									
100	3.205	3.874	4.081	4.270	4.388	11.910	9.298	10.643	15.286	18.165
200	2.525	3.215	3.503	3.654	3.674	13.402	10.567	10.189	13.523	16.790
500	1.888	2.494	2.800	3.056	3.057	13.661	11.711	10.644	11.785	14.376
1000	1.491	2.041	2.384	2.677	2.712	13.481	12.141	11.114	11.234	12.885
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.51$									
100	0.169	0.021	0.005	0.000	0.000	13.965	19.960	25.414	33.738	38.542
200	0.153	0.021	0.004	0.000	0.000	11.304	15.227	19.585	28.772	34.665
500	0.147	0.014	0.002	0.000	0.000	9.454	11.272	14.030	21.510	28.011
1000	0.138	0.011	0.002	0.000	0.000	9.976	10.336	11.403	16.640	22.417
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$									
100	0.173	0.021	0.005	0.000	0.000	12.450	9.811	11.082	15.586	18.399
200	0.153	0.021	0.004	0.000	0.000	13.747	10.820	10.143	13.800	17.034
500	0.147	0.014	0.002	0.000	0.000	14.327	11.897	10.356	11.575	14.875
1000	0.144	0.011	0.002	0.000	0.000	13.904	12.024	10.690	10.349	12.878
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.90$									
100	0.233	0.044	0.008	0.000	0.000	24.177	16.428	11.648	5.455	4.641
200	0.200	0.028	0.004	0.000	0.000	25.540	18.983	14.128	7.860	5.957
500	0.186	0.020	0.004	0.000	0.000	25.045	21.147	17.453	10.804	8.146
1000	0.175	0.016	0.003	0.000	0.000	24.746	20.884	18.527	12.820	9.856

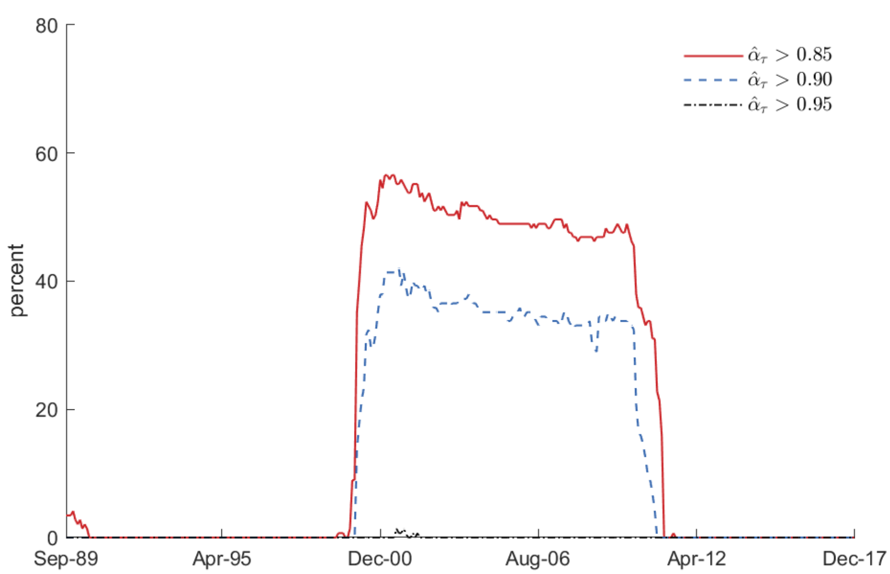
Notes: Parameters of DGP (35) are generated as described in Table 1, with $\rho_{12} = corr(f_1, f_2) = 0.0$. α_{10} and α_{20} are estimated first by regressing observations, x_{it}, on the weighted cross section average of x_{it}, $\tilde{x}_t = n^{-1}\sum_{i=1}^{n} \hat{w}_i x_{it}$, where $\hat{w}_i = \sum_{t=1}^{T} \bar{x}_t x_{it} / \sum_{t=1}^{T} \bar{x}_t^2$, for $t = 1, 2, \ldots, T$. Next, the same regression is run using residuals obtained from the first stage.

Figure 3: Comparison of the market factor strength estimates obtained from the original single factor CAPM ($\hat{\alpha}_{m,\tau}$) and the average estimates of its strength when computed using 145 two-factor asset pricing models ($\bar{\hat{\alpha}}_{m,\tau}$), over 10-year rolling windows



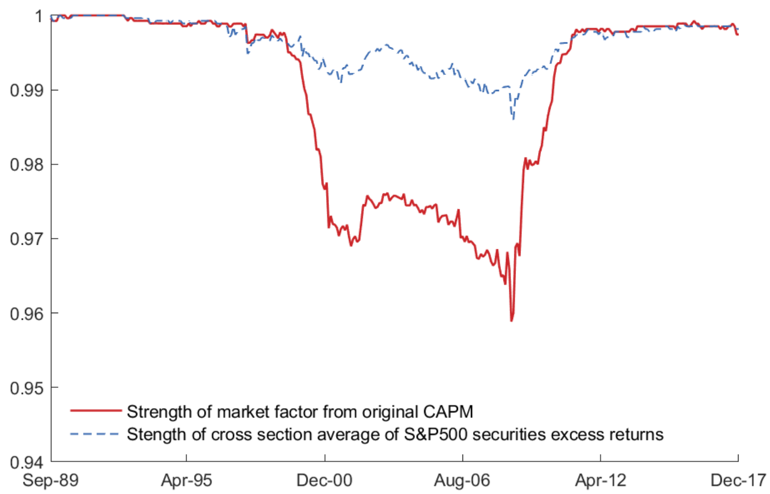
Notes: The market factor strength rolling estimates are computed using (7). The market factor strength average estimates produced from the 145 two-factor CAPMs are computed as $\bar{\hat{\alpha}}_{m,\tau} = (1/145)\sum_{s=1}^{145}(\hat{\alpha}_{s,\tau})$, for $\tau = 1, 2, \ldots, 340$ rolling windows.

Figure 4: Percentage of factors (out of 145) whose estimated strength ($\hat{\alpha}_{s,\tau}$), $\tau = 1, 2, \ldots, 340$ exceeds the thresholds of 0.85, 0.90 and 0.95, in each 10-year rolling window



Notes: The 145 factor strength estimates, $\hat{\alpha}_{s,\tau}$, $s = 1, 2, \ldots, 145$, are computed using (7).

Figure 5: Comparison of the market factor strength estimates obtained from the original single factor CAPM ($\hat{\alpha}_{m,\tau}$) and those from using the cross section average (CSA) of S&P500 securities' excess returns ($\hat{\alpha}_{csa,\tau}$), over 10-year rolling windows



Notes: The market factor and CSA of S&P500 securities' excess returns strength estimates over $\tau = 1, 2, \ldots, 340$ rolling windows are computed using (7).

Table 5: Strength estimates of the strongest unobserved factor using the cross section average (CSA) of the Stock and Watson (2012) dataset ($n = 187$ variables) and the corresponding exponent of cross section dependence (CSD)

	Q1 1988 - Q4 2007 ($T = 80$)			Q1 1988 - Q2 2019 ($T = 126$)		
	$\hat{\alpha}^*_{0.05}$	$\hat{\alpha}$	$\hat{\alpha}^*_{0.95}$	$\hat{\alpha}^*_{0.05}$	$\hat{\alpha}$	$\hat{\alpha}^*_{0.95}$
$p = 0.10$						
Strength of CSA ($\delta = 1/4$)	0.962	0.964	0.966	0.928	0.930	0.933
Strength of CSA ($\delta = 1/2$)	0.957	0.958	0.959	0.918	0.920	0.922
Exponent of CSD	0.833	0.873	0.913	0.858	0.920	0.981
$p = 0.05$						
Strength of CSA ($\delta = 1/4$)	0.962	0.964	0.966	0.927	0.929	0.931
Strength of CSA ($\delta = 1/2$)	0.957	0.958	0.959	0.912	0.914	0.915
Exponent of CSD	0.833	0.873	0.913	0.856	0.918	0.979

Notes: *90% confidence bands. In the computation of the strength of CSA, parameters p and δ are used when setting the critical value (6).
The exponent of CSD corresponds to the most robust estimator of cross-sectional dependence proposed in Bailey et al. (2016) and corrects for both serial correlation in the factors and weak cross-sectional dependence in the error terms.