Identification of Random Coefficient Latent Utility Models

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This Paper

- Identifies distribution of random coefficients for large class of models.

- Covers discrete choice, bundles models, consideration set models, among others.

- Key feature: need to identify average demand function.

- Get traction by exploiting envelope theorem.
Example: Discrete Choice

Discrete choice with linear random coefficients

\[ v_k = \beta'_k x_k + \varepsilon_k. \]

- \( \beta \) and \( \varepsilon \) are random

- Special case is random coefficients logit, in which \( \varepsilon \) has a known distribution up to location.
  - Studied in Fox, Il Kim, Ryan, Bajari (2012, JoE).
  - We differ by letting both \( \beta \) and \( \varepsilon \) have nonparametric distribution.

- Identify all moments of \( \beta = (\beta_1, \ldots, \beta_K) \).
These models (and others) can be written as perturbed utility models of the form

\[ Y(X, \beta, \varepsilon) \in \arg\max_{y \in B} \sum_{k=1}^{K} (\beta'_k X_k) y_k + D(y, \varepsilon). \]

- \( Y \) is quantity vector for \( K \) goods.
- \( X \) collects regressors.
- \( \varepsilon \) can be infinite dimensional.
- \( B \) is nonrandom however \( D(y, \varepsilon) \) can be \(-\infty\) for certain combinations.
  - Allows “consideration sets.”
This paper starts with average structural function

$$\bar{Y}(x) = \int Y(x, \beta, \varepsilon) d\tau(\beta, \varepsilon)$$

and asks what we can learn about distribution of $\beta$.

- When $X$ and $(\beta, \varepsilon)$ are independent,
  $$\bar{Y}(x) = \mathbb{E}[Y \mid X = x].$$

- Also identifiable with endogeneity.
  - We complement Berry and Haile (2014, ECTA), who identify $\bar{Y}(x)$ in a demand setting with instruments.
Let

\[ V(\beta'_1x_1, \ldots, \beta'_K x_K) = \int \left( \max_{y \in B} \sum_{k=1}^{K} y_k (\beta'_k x_k) + D(y, \varepsilon) \right) d\mu(\varepsilon). \]

Then

\[ \int Y(x, \beta, \varepsilon) d\mu(\varepsilon) = \nabla V(\beta'_1x_1, \ldots, \beta'_K x_K) \]

at any point of differentiability.

- Related to Williams-Daly-Zachary theorem of discrete choice.
Assumption

\( \beta \) and \( \varepsilon \) are independent so we can write

\[
\bar{Y}(x) = \int \int Y(x, \beta, \varepsilon) d\mu(\varepsilon) d\nu(\beta).
\]

Notation:

\[
\bar{Y}(x, \beta) = \int Y(x, \beta, \varepsilon) d\mu(\varepsilon)
\]

\[
\bar{Y}(x) = \int \bar{Y}(x, \beta) d\nu(\beta).
\]
Assume each $x_k$ is scalar for simplicity.

Write envelope theorem as

$$\overline{Y}_k(x, \beta) = \partial_k V(\beta_1 x_1, \ldots, \beta_K x_K).$$

Differentiate envelope theorem further to get

$$\partial_{x_j} \overline{Y}_k(x, \beta) = \partial_{j,k} V(\beta_1 x_1, \ldots, \beta_K x_K) \beta_j$$

and

$$\partial_{x_{\ell}} \partial_{x_j} \overline{Y}_k(x, \beta) = \partial_{\ell,j,k} V(\beta_1 x_1, \ldots, \beta_K x_K) \beta_\ell \beta_j.$$
Take expectations over $\beta$ and evaluate at $x = 0$ to get:

$$\partial_{x_\ell} \partial_{x_j} \overline{Y}_k(0) = \partial_{\ell,j,k} V(0) \int \beta_\ell \beta_j d\nu(\beta).$$

- Key feature: $\partial_{\ell,j,k} V(0)$ does not depend on $\beta$. 
Example: Identification of Second Moments

\[ \partial_{x_1} \partial_{x_1} \bar{Y}_2(0) = \partial_{1,1,2} V(0) \int \beta_1^2 d\nu(\beta) \]

\[ \partial_{x_1} \partial_{x_2} \bar{Y}_2(0) = \partial_{1,2,2} V(0) \int \beta_2 \beta_1 d\nu(\beta) \]

\[ \partial_{x_2} \partial_{x_1} \bar{Y}_1(0) = \partial_{2,1,1} V(0) \int \beta_1 \beta_2 d\nu(\beta) \]

\[ \partial_{x_2} \partial_{x_2} \bar{Y}_1(0) = \partial_{2,2,1} V(0) \int \beta_2^2 d\nu(\beta) \]

\[ \Rightarrow \]
Example: Identification of Second Moments

\[
\frac{\partial x_1 \partial x_1 \bar{Y}_2(0)}{\partial x_2 \partial x_1 \bar{Y}_1(0)} = \frac{\partial_{1,1,2} V(0) \int \beta_1^2 d\nu(\beta)}{\partial_{2,1,1} V(0) \int \beta_1 \beta_2 d\nu(\beta)}
\]

\[
= \frac{\int \beta_1^2 d\nu(\beta)}{\int \beta_1 \beta_2 d\nu(\beta)}
\]

- Uses symmetry \( \partial_{1,1,2} V(0) = \partial_{2,1,1} V(0) \).

- Symmetry has been used without random coefficients in Allen and Rehbeck (2019, ECTA).
Combining other equations identifies the ratio of any two second moments.

- Identification given a scale assumption \( \int \beta^2_1 d\nu(\beta) = 1 \).
Theorem

Assume $\int \beta_{1,1}^M d\nu(\beta)$ is known. Under regularity conditions, each $M$-th order moment of the form

$$\int \beta_{k_1,\ell_1} \cdots \beta_{k_M,\ell_M} d\nu(\beta)$$

is identified. In addition, for each $\gamma \in \{1, \ldots, K\}^{M+1}$,

$$\partial_{\gamma} V(0)$$

is identified.
Counterfactuals and Welfare

Can identify counterfactual/welfare objects under additional assumptions.

- Integrated indirect utility

$$V(\beta_1'x_1, \ldots, \beta_K'x_K) = \int \left( \max_{y \in B} \sum_{k=1}^{K} y_k(\beta_k'x_k) + D(y, \varepsilon) \right) d\mu(\varepsilon).$$
Identification of $V$ (Welfare)

How to identify $V$?

- Main result identified partial derivatives of $V$ at 0.

- If $V$ is real analytic we can identify the function globally from these derivatives.

- (Paper presents two other techniques.)
Once $V$ is identified, counterfactuals can be identified from envelope theorem.

\[ \overline{Y}(x, \beta) = \partial_k V(\beta_1 x_1, \ldots, \beta_K x_K). \]
Conclusion

- Identification of moments of linear random coefficient distribution in class of perturbed utility models.
- Covers several examples in a single framework.
- Requires only the average structural function.
- Exploits the envelope theorem.