

Doubling Down on Debt: Limited Liability as a Financial Friction[‡]

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We investigate how a combination of limited liability and preexisting debt distort firms’ investment and equity payout decisions. We show that equity holders have incentives to “double-sell” cash flows in default, leading to overinvestment, provided that the firm has preexisting debt and the ability to issue new claims to the bankruptcy value of the firm. In a repeated version of the model, we show that the inability to commit to not double-sell cash flows leads to heterogeneous investment distortions, where high leverage firms tend to overinvest but low leverage firms tend to underinvest. Permitting equity payouts financed by new debt mitigates overinvestment for high leverage firms, but raises bankruptcy rates and exacerbates low leverage firms’ tendency to underinvest—as the anticipation of equity payouts from future debt raises their cost of debt issuance. Finally, we provide empirical evidence consistent with the model.

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1 Introduction

Debt-fuelled investment is often “cast as a villain” during corporate default events, with firms that engaged in equity buybacks receiving the harshest criticism.¹ We present a simple analytically tractable model of a firm with optimal default, and use it to analyze real investment, leverage, and debt-financed equity payouts (i.e. equity buybacks and dividends). We use our model to generate predictions on how and why over- or underinvestment happens, and analyze how policies such as restrictions on equity buybacks affect the social efficiency of investment for different firms.

We show that limited liability alone leads to financial frictions, with rich implications for real investment, leverage, and equity payouts. Limited liability is characterized by equity holders’ inability to commit to paying liabilities after firm default — with the exception of those liabilities that are directly secured by claims in liquidation. In our model, limited liability provides a dual role. It prevents firms from committing to not dilute preexisting unsecured debt, but it also helps equity holders commit to a lower default threshold by selling new secured debt.²

The model has three types of agents. Equity holders who operate the firm, preexisting debt investors, and new debt investors who competitively price defaultable consols issued by the firm. In our baseline model equity holders face a single investment opportunity. They simultaneously choose how much to invest, whether to finance this new investment with debt or equity, and the direct payout to themselves (i.e. equity buybacks or dividends). Direct equity payouts financed by debt are constrained to be at most a fixed fraction of firm cash flows. Equity holders can walk away from the firm at any time, and the optimal default decision can be characterized as a threshold on cash flows as in Leland (1994). Cash flows are assumed to evolve according to a continuous stochastic process.

Preexisting liabilities are a crucial ingredient in our baseline model. In the absence of preexisting liabilities, equity holders invest up to the efficient level that maximizes total firm value, and are indifferent between modes of financing as in Modigliani and Miller (1958). By contrast, with preexisting liabilities, they have incentives to overinvest and prefer to finance new investment with debt. This finding contrasts with the common intuition that debt represents a drag on investment and the classic debt-overhang model of Myers (1977). Our model differs because equity holders choose both how much to invest and whether to finance the new investment with debt or equity.

Our mechanism hinges on equity holders’ lack of commitment to a specific default threshold and their ability to “double-sell” some cash flows previously promised to preexisting debt holders. By financing new investment with debt, equity holders increase leverage and, as a result, commit to defaulting earlier and at higher cash flows. This transforms coupon payments promised to existing debt holders into new bankruptcy claims, which in turn can be partially sold to new debt holders. The incentive to dilute preexisting debt holders increases equity holders’ marginal benefit of debt financing, leading to overinvestment and, if permitted, direct payouts to equity holders.³ By contrast, if equity holders are constrained to

¹See Aguiar et al. (2019) for the sovereign debt analogy.

²The inability to commit to a future debt level is complementary, and largely orthogonal, to the inability to commit to not default. This latter role of limited liability is emphasized in the classic entrepreneurship models of Albuquerque and Hopenhayn (2004) and Vereshchagina and Hopenhayn (2009). In our model, the risk of investment is fixed to isolate our channel from risk-shifting.

³This dilution mechanism is related to, though different from, debt dilution in sovereign default literature

finance investment with equity we recover the standard underinvestment result, as the benefits from equity-financed investment are partly captured by existing debt holders through a greater distance to default.

We next show that allowing direct equity payouts financed by debt mitigates inefficient overinvestment in our baseline model when there are no future investment opportunities. This is because equity payouts provide a more efficient way of diluting existing debt holders than inefficient overinvestment. In the corner case with unconstrained equity payouts we can split the equity holders’ problem into two separate problems: (1) investment and (2) dilution of existing debt holders. Equity holders choose investment to maximize the net present value of the firm and then choose the level and financing of equity payouts to optimally dilute existing debt holders. As we restrict direct equity payouts, incentives to overinvest emerge. Thus, as suggested by Myers (1977), restrictions on equity payouts financed by new debt tend to increase investment. However such an increase in investment is undesirable in our baseline model since investment is already inefficiently high.

To illuminate the economic forces leading to overinvestment, we allow equity holders to separately issue unsecured debt and bankruptcy claims. We model unsecured debt as having no value following firm default, and bankruptcy claims as receiving a share of the value of the firm net of default costs in bankruptcy.⁴ When equity holders cannot sell bankruptcy claims they cannot dilute existing debt holders, so they choose to finance new investment with equity and underinvest. By contrast, if a sufficiently high fraction of the firm’s liquidation value can be promised through bankruptcy claims, the overinvestment result continues to hold. The finding that excessive restrictions on the issuance on new secured debt leads to underinvestment might partially explain why such covenants are rare in practice (Billett et al. (2007)).

We next extend our model to feature stochastic arrival of repeated investment opportunities, and show that a new underinvestment incentive arises that is particularly relevant for firms with low initial leverage. With repeated investments, debt investors anticipate equity holders’ lack of commitment to not dilute them in the future, raising the cost of debt financing at the time of investment. The resulting incentive to finance with equity and underinvest is particularly relevant for firms with low initial leverage and a high arrival rate of investment opportunities, because these firms have the largest capacity to dilute existing debt holders and frequent opportunities to do so. The prediction that low leverage and high investment opportunity firms tend to underinvest and finance with equity, while high leverage firms — just as in the one-shot investment model — tend to overinvest and finance with debt, is reminiscent of the role of zombie firms during Japan’s “lost decade” (Caballero et al. (2008)).

With repeated investment opportunities, equity payout restrictions have benefits and costs that affect the cross-section of firms differently. If new debt can be used to finance equity payouts (i.e. equity buybacks and dividends), debt holders’ concerns about future dilution are amplified by the anticipation that equity holders will be able to pay themselves directly, further exacerbating underinvestment for firms that have low initial leverage and

(see, for example, Arellano and Ramanarayanan (2012) and Hatchondo et al. (2016)).

⁴In the baseline model we consider a single debt instrument which consists of an unsecured component (i.e., coupon claims) and a bankruptcy claim (i.e., a claim to the liquidation value of the firm). Here we decouple those two claims and allow equity holders to issue them separately. We choose a deliberately stark model of unsecured debt to illustrate the mechanism. In reality the recovery rate for senior unsecured debt in default is close to 50% (Moody’s (2011)), so in practice unsecured bonds would correspond to a bundle of bankruptcy claims and unsecured debt in our model.

frequent investment opportunities. By contrast, for high leverage firms with infrequent investment opportunities, allowing equity payouts financed by new debt continues to move investment towards the first-best and mitigates the tendency to overinvest, though it also increases bankruptcy rates of high leverage firms.

Ensemble simulations of the repeated investment model show that modeling default optimally, as opposed to an exogenous default threshold, is crucial for how equity payout restrictions affect leverage, real investment, and bankruptcies. By mitigating debt investors' concerns about future dilution, equity payout restrictions lower the cost of debt finance and raise firms' default threshold, reducing bankruptcy rates. However, by reducing the bankruptcy rates of high leverage firms, equity payout restrictions also exacerbate these firms' tendency to overinvest.

In the final part of the paper we show that the model is consistent with several observations in the data. First, we show that debt issuance and payouts to equity holders are positively correlated both over time and across firms in Compustat data, as would be the case in our model if the availability of investment opportunities varies over the business cycle and across firms. This evidence speaks to our model mechanism, where an investment opportunity generates incentives for equity holders to issue new debt and partially consume the proceeds through equity buybacks or dividends. Further, we document in the data that high leverage firms had higher capital expenditures investment than low leverage firms since the mid-2000s, consistent with the model prediction that high leverage firms tend to overinvest whereas low leverage firms tend to underinvest.

While we present only an investment model of a single firm, it suggests that crisis government policies for the corporate sector, such as those during the Covid-19 crisis, affect firms heterogeneously with potentially important macroeconomic implications. In our model, an across-the-board increase in firm leverage (as caused, for example, by a policy of distributing crisis loans indiscriminately to all firms) inefficiently distorts investment away from firms with low leverage and plentiful investment opportunities, and towards firms with high leverage and few investment opportunities, which could potentially hamstring innovation and growth for years to come. With imperfect credit markets (e.g., with collateral constraints as in Buera (2009), Moll (2014), Khan and Thomas (2013), or Buera et al. (2015)), young firms are likely to have both non-zero but low leverage and high investment opportunities, leaving them particularly exposed to the underinvestment incentives highlighted by our model. Low leverage firms may further underinvest during credit crunches that disproportionately affect the price of debt (Bernanke and Gertler (1989) and Buera and Moll (2015)). To the extent that inefficiencies in our model are concentrated among the most highly levered firms, a targeted crisis policy of reducing leverage for those firms appears preferable. While in practice leverage reduction can take many forms, our analysis shows that the popular proposal to restrict equity buybacks and dividends is not an automatic cure-all, because it incentivizes inefficient overinvestment at highly levered zombie firms. As restrictions on equity payouts financed by new debt tend to be the most binding when investment opportunities are scarce, our model suggests that relaxing these restrictions during a subsequent economic recovery does little to defray their costs.

Literature Review — Most broadly our paper is related to the literature that studies investment distortions arising from different types of credit market frictions. Albuquerque and Hopenhayn (2004) and Clementi and Hopenhayn (2006) analyze investment distortions arising from adverse selection and moral hazard issues, respectively. See also Hopenhayn and

Werning (2008) and Clementi et al. (2010). Relative to these strands of the literature we examine an environment with perfect information and fully competitive debt markets and focus on understanding how limited liability distorts equity holders’ financing and investment decisions. Thus, we abstract from collateral constraints that arise in the presence of adverse selection and moral hazard issues, though such collateral constraints may provide a rationale for the preexisting leverage in our framework. In that sense our work is complementary to the literature that studies investment distortions arising from collateral constraints as analyzed by Buera (2009), Buera et al. (2011, 2015), Khan and Thomas (2013), or Moll (2014) (see Buera et al. (2015) for an excellent review of this literature). Our analysis puts the emphasis on the equity holders’ ability to transform coupons into collateral claims suggesting that frictions in collateral quality such as in Cole and Kocherlakota (2001) and Gorton and Ordoñez (2014) may further distort investment in the presence of limited liability.

Our paper is also related to the debt overhang literature, including the seminal contributions of Myers (1977), Hennessy (2004), Parrino and Weisbach (1999), and Moyen (2007). Relative to the debt overhang literature, we allow the firm to choose the investment size and finance new opportunities optimally with debt and equity, thereby obtaining new and rich implications for real investment. Our work complements Diamond and He (2014), He and Milbradt (2016), Milbradt and Oehmke (2015) and Brunnermeier and Oehmke (2013) who study the optimal maturity choice of debt issued to finance investment and challenge the traditional view that short-term debt necessarily mitigates debt overhang, and Donaldson et al. (2019) who challenge the view that collateralized borrowing necessarily mitigates underinvestment. Instead, we focus on equity holders’ choice between equity and debt financing of investment and challenge the view that debt overhang necessarily leads to underinvestment. We also complement Green and Liu (2020)’s work on competition in lending markets when there is limited commitment, and the “leverage ratcheting” literature (Admati et al. (2018), DeMarzo (2019), DeMarzo and He (2020)), which has emphasized implications for optimal leverage when firms are unable to commit to not issue more debt in the future. We contribute to this literature by focusing on investment, showing that both over- and underinvestment can happen in the cross-section of firms, and by analyzing how restrictions on equity payouts (such as dividends or equity buybacks) affect investment and financing choices.

Our mechanism is related but distinct from risk-shifting (see e.g., Jensen and Meckling (1976), Leland and Toft (1996), or Vereshchagina and Hopenhayn (2009)), whereby equity holders close to default substitute a riskier project even if it lowers the firm’s expected cash flows. By contrast, in our model equity holders cannot change the volatility of cash flows separately from the overall scale of the firm, and can commit to the current investment when seeking new debt financing. Finally, our mechanism is also related to debt dilution and debt overhang in the sovereign debt literature (see, for example, Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2013), Hatchondo et al. (2016), and Aguiar et al. (2009)). A key difference between these papers and our work is due to the institutional details that govern sovereign and corporate defaults. While sovereigns have no obligation to pay their creditors anything in default, following a corporate bankruptcy the creditors are entitled to seize the firm’s assets in accordance with national bankruptcy laws.

2 Model

There are three types of agents in the model: Equity holders that operate the firm, existing debt holders who hold debt issued in the past, and competitive outside creditors. Equity holders face a one-time investment opportunity at time 0, at which time they can issue new debt and equity and make direct payouts to themselves (i.e. dividends or equity buybacks). Debt is senior to equity but all debt, including newly issued debt, has equal priority.⁵ All actions are perfectly observable and there is complete information. To keep the model analytically tractable and to highlight the underlying intuition, in our baseline model we assume that equity holders have one-time investment opportunity. We consider a model with repeated investment in Section 5.

2.1 Firm State, Notation, and Laws of Motion

The state of a firm is summarized by its *cash flows*, Z , and the book value of its *liabilities*, L , defined as the present discounted value (PDV) of all promised cash flows to debt holders. Equity and bond holders discount future payoffs at the same constant rate $r > 0$.⁶ In the absence of new investment $Z(t)$ follows a geometric Brownian motion with risk-neutral drift μ and instantaneous volatility of $\sigma^2 > 0$

$$dZ(t) = \mu Z(t)dt + \sigma Z(t)d\mathbb{W}(t), \quad Z(0) > 0, \quad (1)$$

where $\mathbb{W}(t)$ is a standard Brownian motion. Liabilities, L , may have a one-time jump at time 0 (i.e., at the time of investment if equity holders decide to finance some of the investment with debt) but remain constant for all $t > 0$. Thus, we do not allow equity holders to issue new debt or repurchase existing debt after time 0.⁷

Real Investment and Payouts to Equity At time 0 equity holders have a one-time investment opportunity that expires immediately if not executed. In particular, at time 0 equity holders can increase the initial cash flows of the firm from $Z(0)$ to $\hat{Z} \equiv (1 + g)Z(0)$, where $g \geq 0$ captures equity holders' investment financed through a combination of new debt and equity. After the initial jump in cash flows, cash flows follow (1). Investment is costly, with the cost function given by $q(g)/Z \equiv \frac{\zeta g^2}{2}$.⁸

At the time of the investment, we also allow equity holders to make direct payouts to themselves, M . We interpret M as equity buybacks, dividends, or leveraged-buyouts financed by issuing new debt. Thus, M captures any payout to equity holders that equity holders

⁵In Section 4.3 we consider the case where new debt is junior. Allowing new debt to be senior would strengthen our results.

⁶Due to the absence of fixed costs in this model, cash flows are equivalent to EBITDA profits and proportional to both the assets-in-place and enterprise value. The book value of liabilities L does not take into account the equity holder's option to default. For example, if the firm's liabilities consist of one unit of defaultable consol that promises coupon c every instant of time then $L = \int_0^\infty ce^{-rs}ds$.

⁷We relax this assumption in Section 5 where we consider repeated investment. In that case we allow liabilities to change each time investment occurs, but we maintain the assumption that in the absence of investment opportunities, the firm's liabilities are constant. This captures the intuition that major capital structure adjustment often occur jointly with investment. For the analysis of the dynamic adjustments of capital structure in the absence of investment opportunities see Admati et al. (2018).

⁸We choose this simple function for numerical convenience. All our analytical results hold with a standard strictly increasing and strictly convex cost function that satisfies $q(g) = q'(g) = 0$ and $\lim_{g \rightarrow \infty} q'(g) = \infty$.

can use immediately for consumption. The presence of M allows us to consider proposals to limit share buybacks or dividend payments. We assume that equity holders can only consume $M \in [0, \kappa Z]$, where $\kappa \geq 0$ is the parameter capturing institutional constraints on financing equity buybacks with new debt, with $\kappa = 0$ being our baseline.⁹

Financing and Limited Liability Equity holders can fund the total cost of investment, $q(g)Z$, and the total equity payouts, M , with their own funds (equity financing), by issuing new debt (debt financing), or any linear combination of them. Equity holders are assumed to be deep-pocketed and hence able to finance investment or equity payouts with their own funds if they so choose. We denote the proportion of debt financing by $\psi \in [0, 1]$. If the firm issues only equity (i.e., $\psi = 0$) then the liabilities of the firm, L , have no jump at time 0. If $\psi > 0$, liabilities jump at time 0. In that case, let \hat{L} denote *post-investment liabilities*. \hat{L} is implicitly determined by equity holders' budget constraint $\Phi(\hat{L}, Z, L, g, \psi, M) = 0$. The equation $\Phi(\cdot) = 0$ is the budget constraint (to be derived below) imposed by equity holders' financing choices $\{g, \psi, M\}$ and equity holders' optimal default strategy, taking as given the pre-investment firm state $\{Z, L\}$. In our baseline case, we derive the budget constraint in Section 2.3 assuming that all debt takes the form of defaultable consols, which pay one coupon until the firm defaults and represent a claim to the assets of the firm in bankruptcy. In Section 4 we extend our analysis to unsecured and junior debt.

Equity holders are protected by limited liability. This means that after investment and financing choices have been made, equity holders can choose to default and walk away with nothing at any time, whereupon the firm is taken over by debt holders. Deep-pocketed equity holders are assumed to have sufficient funds (i.e. inject new equity) to keep the firm as a going concern, if they so choose, when promised debt payments exceed firm cash flows.

Equity Value At any point in time there are two states variables: current cash flows, Z and current liabilities, L . Let $V(Z, L)$ denote the post-investment value of equity (i.e., the value of operating the firm after the investment option was executed) when the current cash flows are Z and current liabilities are L . Similarly, let $V^*(Z, L)$ denote the pre-investment value of equity (i.e., the value of operating the firm to equity holders at the time they make their investment decision) when time 0 cash flows are Z and time 0 liabilities are L .¹⁰

It will prove useful to rescale the value of equity with cash flows. Thus, we denote the post- and pre-investment equity value relative to cash flows as $v(\cdot) \equiv V(\cdot)/Z$ and $v^*(\cdot) \equiv V^*(\cdot)/Z$, respectively. Similarly, we define current leverage as $\ell \equiv L/Z$ and the equity payouts per unit of Z as $m \equiv M/Z$.

Value in Default Upon default the firm is taken over by the debt holders who continue to operate it. However, default has real costs in the sense that immediately following default the firm's cash flows decrease from Z to $(1 - \theta)Z$, where $\theta \in [0, 1]$. The parameter θ captures deadweight costs associated with bankruptcy proceedings and debt holders' lower skill in

⁹The restriction that $M \geq 0$ is without loss of generality since equity holders would never choose $M < 0$ (which, in the model, corresponds to buying back debt).

¹⁰We only need to explicitly differentiate between pre-investment and post-investment states when discussing the actual investment decision, in which case we denote the post-investment states by \hat{L} and $Z(1 + g)$.

running the firm.¹¹ Consequently, the *liquidation value* that will be distributed among debt holders in default conditional on Z is $V((1 - \theta)Z, 0)$.

2.2 Equity Holders' Investment and Default Decisions

Equity holders face the following decisions. First, at time 0, they have to choose how much to invest, g , how much to pay out to themselves, M , and how to finance these choices, ψ . Having made these choices, at each instant of time they need to decide whether to keep operating the firm or default instead.

Investment Decision Given an initial state (Z, L) , equity holders choose the financing mix ψ and real investment g to maximize the post-investment equity value plus direct payouts to equity, net of new equity injected into the firm. The post-investment equity value is given by $V((1 + g)Z, \hat{L})$, where \hat{L} are the post-investment liabilities. Equity holders take the equilibrium budget constraint $\Phi(\cdot) = 0$ as given, and solve

$$V^*(Z, L) = \max_{\substack{g \geq 0 \\ \psi \in [0, 1] \\ 0 \leq M \leq \kappa Z}} \left\{ \overbrace{V((1 + g)Z, \hat{L})}^{\text{Post-Investment Equity}} - \overbrace{(1 - \psi)q(g)Z}^{\text{Equity Financed}} + \overbrace{M}^{\text{Payouts}} \right\} \quad (2)$$

$$\text{s.t. } \underbrace{\Phi(\hat{L}, Z, L, g, \psi, M)}_{\text{Equilibrium Budget Constraint}} = 0 \quad (3)$$

and subject to the feasibility of the payoffs in default embedded in L and \hat{L} . To understand how existing liabilities distort equity holders' investment choices we consider the following first-best benchmark.

Definition 1 (First-Best Investment). *We define the first-best undistorted investment, g^u , as investment that maximizes the net present value of the firm. That is,*

$$g^u(Z) \equiv \arg \max_g \left\{ \overbrace{V((1 + g)Z, 0)}^{\text{Post-Investment Equity}} - \overbrace{q(g)Z}^{\text{Equity Financed}} \right\} \quad (4)$$

Thus, first-best investment is the level that equity holders would choose if the firm had no preexisting debt and investment had to be fully financed with equity. Since both the payoffs and costs are linear in Z , we can show that g^u is independent of Z throughout our model. The homotheticity that leads to a constant g^u is not essential, but simplifies the analysis.

Default Decision Equity holders optimally choose to default when the equity value, $V(Z, L)$, reaches 0. Note that after investment only the cash flows fluctuate, and the equity

¹¹The presence of deadweight costs of default is not needed for our mechanism and, indeed, in the baseline version of our model we set $\theta = 0$. However, since the empirical literature estimates the bankruptcy costs to be non-trivial, we incorporate it into the model and devote Section 4.1 to investigating how equity holders' decisions are affected by the presence of such costs.

holders' *continuation* problem becomes a standard stopping problem (as in Leland (1994) with liabilities L as an additional state), which is given by

$$rV(Z, L) = Z - rL + \mu Z \partial_Z V(Z, L) + \frac{\sigma^2}{2} Z^2 \partial_{ZZ} V(Z, L) \quad (5)$$

$$V(\underline{Z}, L) = 0 \quad (6)$$

$$\partial_Z V(\underline{Z}, L) = 0, \quad (7)$$

where \underline{Z} is the endogenous default barrier. Here (6) and (7) are the standard value-matching and smooth pasting conditions, respectively. Defining

$$\eta \equiv \frac{(\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2 r}}{\sigma^2} > 0 \quad (8)$$

$$\chi \equiv \left(\frac{(r - \mu)\eta}{\eta + 1} \right)^\eta > 0 \quad (9)$$

$$s(\ell) \equiv \frac{\chi}{\eta + 1} \ell^\eta \quad (10)$$

we can characterize the solution to equity holders' default problem (5).

Proposition 1 (Post-Investment Equity and Default). *Suppose that the current state of the firms is (Z, L) . Then the equity value of the firm is given $V(Z, L) = v(\ell)Z$, where $\ell \equiv L/Z$ is firm's leverage and*

$$v(\ell) = \frac{1}{r - \mu} - \ell(1 - s(\ell)) \quad (11)$$

The endogenous default threshold \underline{Z} satisfies

$$\frac{\underline{Z}(L)}{r - \mu} = \frac{\eta}{1 + \eta} L \quad (12)$$

and the liquidation value per unit of liabilities is given by,

$$\frac{V((1 - \theta)\underline{Z}(L), 0)}{L} = \frac{(1 - \theta)\eta}{1 + \eta} \quad (13)$$

Proof. See Appendix A.1. □

Before moving on to how debt is priced, it is useful to visualize how firm cash flows are divided between equity and debt holders. Figure 1 depicts a possible path of Z and shows how cash flows are divided among all claimants (for simplicity we set $\theta = 0$). Default occurs at the threshold $\underline{Z}(L)$ optimally chosen by the firm, following Proposition 1. Because we are considering the corner case without bankruptcy costs ($\theta = 0$), firm cash flows are not impacted by the default event—instead they simply change claimants. Prior to default, coupons are paid and residual funds are distributed to equity holders. After default, all cash flows are owned by the default claimants. The price of any financial claim is simply the expected present discounted value of cash flows allocated to this claimant, with the expectation taken over all possible realizations of the $Z(t)$ process.

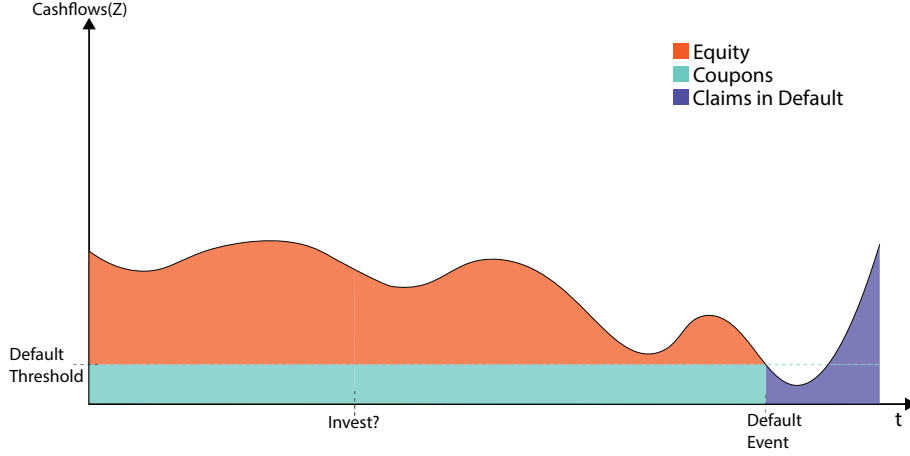


Figure 1: Cash flows to debt and equity holders under limited liability in the absence of investment opportunities when $\theta = 0$.

2.3 Pricing Debt Instruments

In this section, we derive how debt is priced and determine the equilibrium budget constraint $\Phi(\cdot)$. Debt is priced by outside creditors who are risk-neutral and who anticipate equity holders' optimal default decision (as characterized in Proposition 1). Let T denote the stopping time when cash flows first hit \underline{Z} and equity holders choose to default.¹²

For tractability, we assume that all debt takes the form of defaultable consols following Leland (1994, 1998). A defaultable consol pays 1 in perpetuity prior to default and receives a share of the bankruptcy value of the firm in default. Conditional on the current state of the firm (Z, L) and equity holders' optimal default decision, the market price of a such bond $P(Z, L)$ equals

$$P(Z, L) \equiv \frac{p(Z, L)}{r} = \underbrace{\mathbb{E}_T \left[\int_0^T e^{-r\tau} d\tau \right]}_{\text{PDV of promised coupons} = P^U(Z, L)} + \underbrace{\mathbb{E}_T \left[e^{-rT} \frac{V((1-\theta)\underline{Z}(L), 0)}{rL} \right]}_{\text{PDV of claims in bankruptcy} = P^B(Z, L)}, \quad (14)$$

where $p(Z, L)$ denotes the price of the defaultable consol bond relative to the risk-free rate.

Equation (14) emphasizes that a defaultable consol consists of an unsecured component (i.e., the coupon payments prior to default) with market price $P^U(Z, L)$, and a secured component (i.e., the value of claims in bankruptcy) with market price $P^B(Z, L)$. We use $p^U(\cdot, \cdot)$ and $p^B(\cdot, \cdot)$ to denote these prices relative to the risk-free rate, r . In Sections 4.2 and 4.3 we use this decomposition to analyze how our results change if new debt is unsecured or junior to existing debt.

Proposition 2 establishes that leverage ($\ell \equiv L/Z$) is the relevant state for pricing debt, solves for the prices of the defaultable consol and of its unsecured and secured components, and characterizes equity holders' budget constraint $\Phi(\cdot) = 0$.

Proposition 2 (Pricing Debt Instruments). *The relevant state for pricing debt is leverage*

¹²We assume that $Z_0 > \underline{Z}$.

$\ell \equiv L/Z$. Moreover,

$$p(\ell) = 1 - (1 + \theta\eta)s(\ell) \quad (15)$$

$$p^U(\ell) = 1 - (1 + \eta)s(\ell) \quad (16)$$

$$p^B(\ell) = (1 - \theta)\eta s(\ell) \quad (17)$$

Given these debt instruments, the budget constraint (3), normalized by Z , is given by

$$p(\hat{\ell}) \left((1 + g)\hat{\ell} - \ell \right) = \psi q(g) + m \quad (18)$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \quad (19)$$

where $\hat{\ell} \equiv \hat{L}/\hat{Z}$ is post-investment leverage.

Proof. See Appendix A.2. □

The budget constraint (18) is standard. The left hand side equals the total value of new debt issued, normalized by Z . The right hand side represents equity holders' need to raise new debt financing, and equals the debt-financed portion of investment costs plus equity payouts. Equation (19) ensures the feasibility of the default claim embedded in the defaultable consol and is satisfied as long as $Z_0 > \underline{Z}$, i.e. the firm is not initially in default.

In the case without bankruptcy costs (i.e., $\theta = 0$) the price of the defaultable consol relative to the risk-free rate simplifies to $p(\ell) = 1 - s(\ell)$. In this particularly simple case, $s(\ell)$ can be interpreted as the spread relative to the risk-free rate. The spread $s(\ell)$ also appears in the expression for $v(\ell)$ (the normalized value of equity) with the opposite sign, where it captures the value of equity holders' option to default. Thus, we see that a more valuable default option increases the value of equity at the expense of bond holders.

2.4 Equity Holders' Investment Problem and the First-Best

We now restate equity holders' investment problem, normalizing by current cash flows and substituting in the budget constraint associated with defaultable consols (18). Equity holders of a firm with pre-investment leverage ℓ choose $(g, m, \psi, \hat{\ell})$ such that

$$v^*(\ell) = \max_{\substack{g \geq 0 \\ \psi \in [0,1] \\ 0 \leq m \leq \kappa}} \left\{ \overbrace{(1 + g)v(\hat{\ell})}^{\text{Post-Investment Equity}} - \overbrace{(1 - \psi)q(g)}^{\text{Equity Financed}} + \underbrace{m}_{\text{Payouts}} \right\} \quad (20)$$

$$\text{s.t.} \quad \underbrace{p(\hat{\ell})}_{\text{Bond Price}} \underbrace{((1 + g)\hat{\ell} - \ell)}_{\text{New Bonds}} = \underbrace{\psi q(g)}_{\text{Debt Financed}} + \underbrace{m}_{\text{Payouts}} \quad (21)$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \quad (22)$$

The first-best investment from Definition 1 solves

$$g^u \equiv \arg \max_g \left\{ \overbrace{(1 + g)v(0)}^{\text{Post-Investment Equity}} - \overbrace{q(g)}^{\text{Equity Financed}} \right\} \quad (23)$$

The equity holders' objective function (20) is simply a normalization of (2). Similarly, the first-best investment in (23) is just the normalization of (4). The constraint (22) enforces the feasibility of the default payoff embedded in defaultable consols, meaning that equity holders cannot issue so much debt that the firm is strictly above its default threshold immediately after investment. While the constraint never binds when $\kappa = 0$, it may bind if direct payments to equity holders are allowed ($\kappa > 0$).

3 Analysis of Investment Decision

In this section, we analyze equity holders' investment and financing decisions in our baseline model with a single investment opportunity. We first characterize the equity holders' problem relative to the first-best. This comparison allows us to identify the sources of inefficiencies in equity holders' investment decisions.

3.1 Sources of Investment Distortions

We define the function $H(\hat{\ell})$

$$H(\hat{\ell}) \equiv \theta \eta s(\hat{\ell}) \quad (24)$$

representing the deadweight cost of default per unit of leverage, which is non-zero if a share of the firm is dissipated in default (i.e., $\theta > 0$). We then characterize the investment problem as follows.

Proposition 3. *Equity holders' investment problem can be written as*

$$v^*(\ell) = \max_{\substack{g, \hat{\ell} \geq 0 \\ \psi \in [0,1] \\ 0 \leq m \leq \kappa}} \left\{ \frac{1+g}{r-\mu} - q(g) - p(\hat{\ell})\ell - (1+g)H(\hat{\ell})\hat{\ell} \right\} \quad (25)$$

$$s.t. \ p(\hat{\ell})((1+g)\hat{\ell} - \ell) = \psi q(g) + m \quad (26)$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \quad (27)$$

The first-best investment, g^u , is the unique solution to

$$0 = \frac{1}{r-\mu} - q'(g^u) \quad (28)$$

Proof. See Appendix A.3. With the assumption of quadratic investment costs $q(g) = \zeta g^2/2$, optimal investment is given by $g^u = \frac{1}{\zeta(r-\mu)}$. \square

The reformulated objective function (25) shows that the post-investment value of equity equals the expected PDV of cash flows generated by the firm net of (i) the cost of investment ($q(g)$), (ii) the PDV of cash flows promised to the existing debt holders ($p(\hat{\ell})\ell$), and (iii) cash flows lost in default ($(1+g)H(\hat{\ell})\hat{\ell}$), all normalized by Z . Because new debt is fairly priced equity holders bear the full cost of the investment and any change in the expected deadweight cost of default. For the same reason equity payouts m do not appear directly in (25). However, ψ and m affect the post-investment value of equity indirectly through $\hat{\ell}$.

The above characterization of the equity holders' problem emphasizes the sources of inefficient investment. Compared to (23), we see that equity holders face two distortions. The first distortion is due to existing debt, as captured by $p(\hat{\ell})\ell$. Since $p(\hat{\ell})$ is a decreasing function of post-investment leverage, $\hat{\ell}$, equity holders have an incentive to increase leverage. This is the classic conflict between equity and debt holders pointed out by Myers (1977), who however in contrast to us assumes that investment is entirely financed with equity. The second distortion is due to bankruptcy costs and is captured by $(1 + g)H(\hat{\ell})\hat{\ell}$. Since $H(\cdot)$ is an increasing function, the presence of bankruptcy costs discourages equity holders from taking on additional leverage.

3.2 Investment Relative to First-Best

For the rest of this section we focus on the case without bankruptcy costs (i.e., $\theta = 0$), so the last term in (25) vanishes, and preexisting leverage is the only source of investment distortions.¹³ If in addition preexisting debt is zero (i.e., $\ell = 0$), it is immediate from the reformulated investment problem in Proposition 3 that equity holders' optimal investment satisfies

$$\frac{1}{r - \mu} - q'(g) = 0, \quad (29)$$

that is investment equals the first-best ($g = g^u$). When $\ell = 0$ equity holders are also indifferent between equity and debt financing and whether to make equity payouts (i.e., they are indifferent over any feasible choices of ψ and m). In other words, when $\ell = 0$ the Modigliani and Miller (1958) theorem holds, and the firm value is independent of financing.

Investment deviates from this simple benchmark when the firm has preexisting debt ($\ell > 0$). Without bankruptcy costs, it is immediate from (25) that equity holders' optimal investment satisfies the following first-order condition (FOC)

$$\frac{1}{r - \mu} - q'(g) - p'(\hat{\ell})\frac{\partial \hat{\ell}}{\partial g}\ell = 0 \quad (30)$$

Equation (30) is the key equation of our model.¹⁴ Compared to (29), which determines equity holders' investment choice when $\ell = 0$, the FOC now includes the additional term $-p'(\hat{\ell})\frac{\partial \hat{\ell}}{\partial g}\ell$. This new term captures the marginal change in the value of existing debt due to the change in the firm's distance to default. When this term is positive at $g = g^u$ equity holders have an incentive to invest beyond the level that would maximize the total value of the firm, while the opposite is true when the term is negative. Since the value of existing debt is decreasing in leverage (i.e., $p'(\hat{\ell}) < 0$) it follows that the sign of this distortion depends on the sign of $\frac{\partial \hat{\ell}}{\partial g}$. If optimal investment is associated with an increase in leverage, that is if $\frac{\partial \hat{\ell}}{\partial g} > 0$, equity holders *overinvest* relative to first-best. If optimal investment is associated with a decrease in leverage, that is if $\frac{\partial \hat{\ell}}{\partial g} < 0$, equity holders *underinvest*.

¹³We investigate how the presence of bankruptcy costs affects investment in Section 4.1.

¹⁴This FOC applies in the case when the constraint on equity payouts is binding, which is always the case when κ is not too large. The FOC when this constraint is not binding is given by (B.42).

3.3 Dilution Mechanism and Inefficient Investment

We now present our first main result that preexisting debt encourages overinvestment. We first characterize equity holders' choices without equity payouts financed by debt (i.e., $\kappa = 0$).

Proposition 4. *Suppose that $\kappa = 0$ and denote by g^* equity holders' optimal investment.*

1. *If equity holders can only use equity financing then they underinvest, that is $g^* < g^u$.*
2. *If equity holders can choose financing optimally then*
 - (a) *They finance all their investment with debt*
 - (b) *They overinvest, that is $g^* > g^u$*

Proof. See Appendix B.2 □

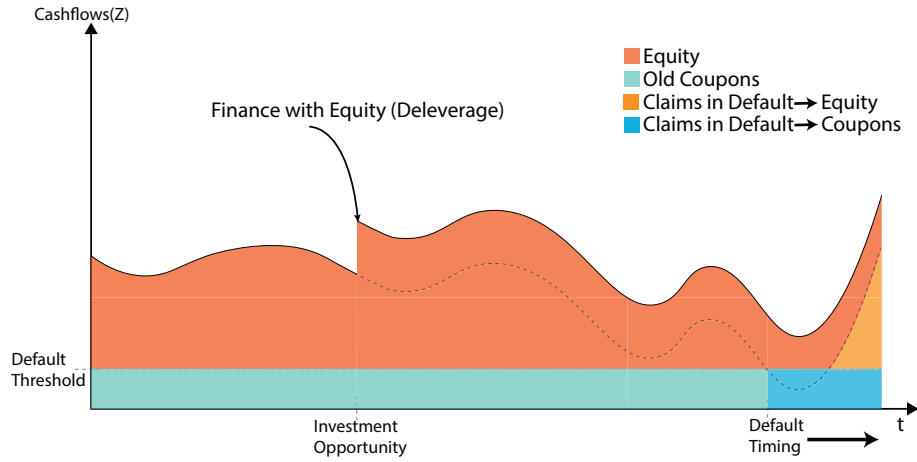


Figure 2: Equity financed investment, due to deleveraging, decreases the option value of default

The first part of Proposition 4 nests the classic underinvestment result of the debt overhang literature (Myers (1977)). Thus, our model makes precise that equity financing is a condition required for this classic result. Figure 2 visualizes the intuition. Investment financed with equity leads to deleveraging, leading equity holders to pay coupons for longer. As a result, a portion of the cash flows from the new investment is allocated to existing debt holders in form of coupon payments (the portion of the “Claims in Default \rightarrow Coupons” area above the dotted line). The benefit from new investment is hence partly captured by existing debt holders, implying that equity holders’ benefit of investing is less than the social benefit. In terms of the key equation (30), deleveraging implies that $-p'(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} < 0$, and hence a reduction of equity holder’s incentive to invest. This classic argument has been used to explain the historically low investment in the aftermath of the Great Recession in Europe (see, for example, Kalemli-Ozcan et al. (2018)).

By contrast, the second part of Proposition 4 shows that equity holders may want to overinvest if they can choose their financing method freely, provided that the firm has preexisting debt (i.e., $\ell > 0$). Note that the volatility of firm cash flows is assumed to remain the

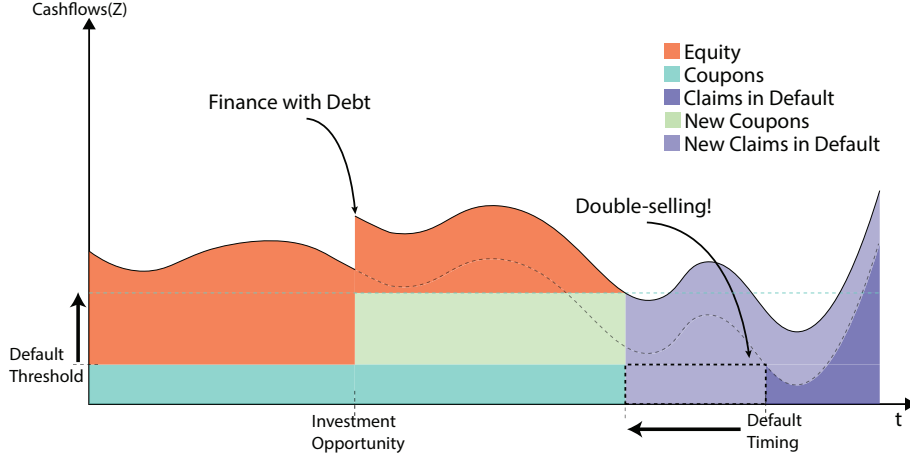


Figure 3: Debt financed investment, due to increased leverage, dilutes existing debt holders by double-selling some of their promised coupon payments.

same before and after investment, and thus the mechanism here is distinct from the potential incentive to invest in risky projects with negative present value, so-called risk-shifting (Jensen and Meckling (1976), Leland (1998)).

Why do equity holders overinvest? Figure 3 illustrates that a sufficiently large debt-financed investment leads to higher leverage and earlier default, transforming a portion of the coupon payments that would have been captured by existing debt holders (the rectangular area with dashed edges) to claims in default, which have to be shared with new debt holders. Thus, by issuing new debt and increasing leverage, equity holders can sell again claims to some of the cash flows that were previously promised to existing debt holders. The marginal benefit to equity holders of investing hence exceeds the social benefit, and equity holders overinvest relative to the first-best. In terms of (30), this additional benefit of financing investment with debt is captured by $-p'(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell > 0$.¹⁵

We can equivalently think of the overinvestment incentive in terms of equity holders' option to default. When equity holders issue new debt they sell claims to their cash flows and purchase an option to default, as the default threshold increases. However, old debt holders are not compensated for this increase in equity holders' option value to default. Thus, the third term on the left-hand side of (30) can also be interpreted as the marginal change in equity holders' option to default that is not priced by the market at the time of investment. Thus, if equity payouts are restricted, equity holders have an incentive to issue more debt and use these additional funds to invest more than the first-best investment.

¹⁵It is the coupon payments promised to existing debt holders that are double-sold by equity holders, not the existing debt holders' bankruptcy claims. To be precise, *before* the investment takes place, the value of existing debt holders' bankruptcy claims at the time of default is $\frac{\eta}{1+\eta}L$, the value of the firm in default (see (13)). *After* the investment takes place the value of existing debt holders' bankruptcy claims at the time of default is given by $V((1-\theta)\underline{Z}(L), 0) \times (L/\hat{L})$ since the post-investment firm's value in default ($\frac{\eta}{1+\eta}\hat{L}$) is divided proportionally between new and old debt holders. It follows that the value of existing bankruptcy claims at the time of default is unchanged. Moreover, since after investment default happens on average earlier, the PDV of existing bankruptcy claims actually goes up.

3.4 Equity Payouts

Having seen that the availability of debt financing can lead to overinvestment in the presence of preexisting debt, we now turn to analyzing how debt-financed payouts to equity holders, such as dividends and equity buybacks, affect real investment (i.e., $\kappa > 0$). We define $\bar{m}(\psi, g)$ as the highest equity payout that satisfies the budget constraint (21) given investment, g , and financing choice, ψ . Thus, given g, ψ , equity holders' choice of m has to satisfy $m < \min\{\kappa, \bar{m}(\psi, g)\}$. As shown Appendix B.3, equation (B.12), $\bar{m}(\psi, g) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g)$.

Proposition 5. *Suppose that $\kappa > 0$. Let g^*, m^*, ψ^* denote the equity holders' optimal choices of investment, payouts, and financing, respectively. Then there exists $\underline{\kappa} \in \mathbb{R}_+$ such that*

1. *If $\kappa < \underline{\kappa}$ then*

- (a) *Equity holders overinvest, that is $g^* > g^u$*
- (b) *Equity holders finance investment and equity payouts with debt, that is $\psi^* = 1$*
- (c) *Equity holders make payouts to themselves up to the constraint, that is $m^* = \kappa$*

Following investment equity holders strictly prefer to continue operating the firm.

2. *If $\kappa \geq \underline{\kappa}$ then*

- (a) *Equity holders invest the first-best amount, that is $g^* = g^u$*
- (b) *Equity holders finance their investment and equity payouts at least partially with debt, that is $\psi^* \in [\max\{\underline{\psi}_\kappa, 0\}, 1]$, where $\underline{\psi}_\kappa > 0$ is the unique solution to*

$$\kappa = \bar{m}(g^*, \underline{\psi}_\kappa) \quad (31)$$

- (c) *Equity holders make payouts to themselves $m^* = \bar{m}(g^*, \psi^*)$*

Following investment equity holders are indifferent between defaulting and continuing to operate the firm.

The threshold $\underline{\kappa}$ is decreasing in ℓ and r , and increasing in σ .

Proof. See Appendix B.3. □

Proposition 5 extends our overinvestment result to the case in which equity payouts financed with debt are permitted at the time of investment. As long as equity holders face sufficiently tight restrictions on equity payouts financed by debt ($\kappa < \underline{\kappa}$), we find that they continue to overinvest. Different from the case with $\kappa = 0$, equity holders accompany investment with direct equity payouts, further increasing post-investment leverage.

By contrast, when the constraint on direct equity payouts is lax ($\kappa \geq \underline{\kappa}$), equity holders invest the first-best amount. In this case, equity holders have a more efficient way of double-selling existing debt holders' claims without the need to resort to inefficient investment, decoupling equity holders' problem into two separate problems: (1) an investment problem and (2) a dilution of existing debt holders problem. Equity holders choose g to maximize the net present value of the firm and m to maximize the transfer from existing debt holders to themselves. The latter implies choosing the highest feasible m so that the firm defaults

right after investment. Thus, when $\underline{\kappa} \geq \kappa$, equity holders essentially sell the firm to the new debt holders.

Proposition 5 shows that restrictions on equity payouts can increase investment and reduce the probability of bankruptcy, in line with the intuition in Myers (1977). Different from Myers (1977), the resulting investment in our model is debt-financed and tends to be inefficiently high, reminiscent of highly-levered zombie firms. The conclusion that equity payout restrictions lead to inefficiently high investment needs to be tempered somewhat by the observation that this one-shot investment problem does not account for potential future debt dilutions. We will see in Section 5 that dynamic considerations restore the intuition that equity buyout restrictions can lead to desirable increases in investment for some firms, though the overinvestment incentive highlighted in the one-shot model tends to remain for the most highly levered firms.

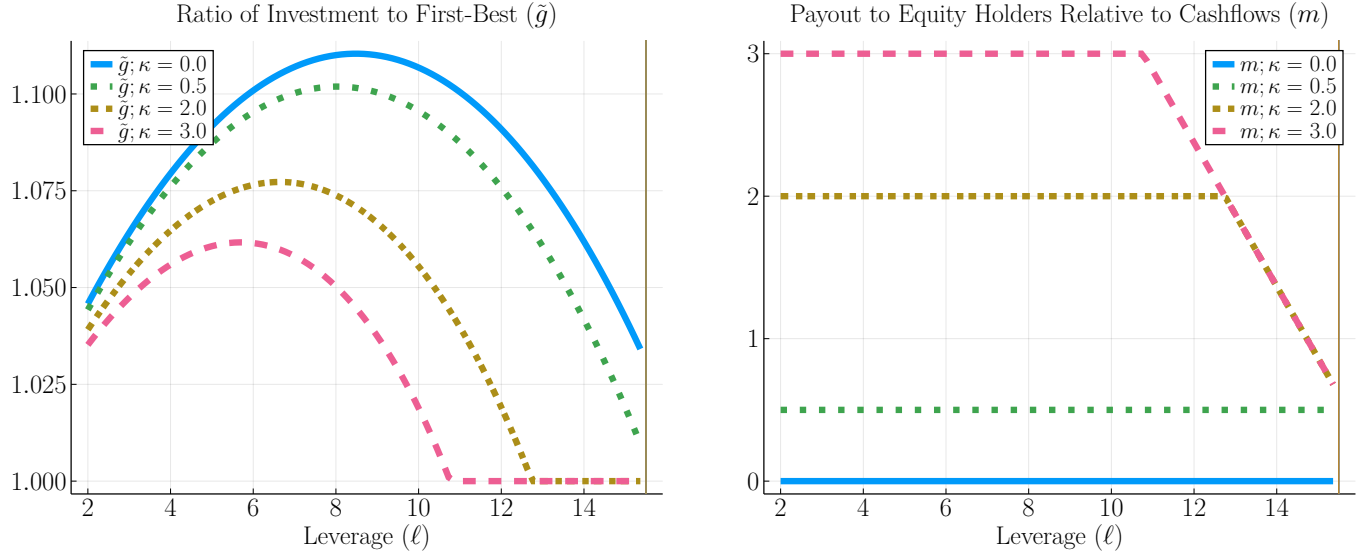


Figure 4: Investment relative to first-best ($\tilde{g} = g/g^u$, left panel) and payouts to equity holders (m , right panel) for different levels of initial leverage (ℓ) and for different values of the equity payout constraint ($\kappa = 0$ captures strict and $\kappa = 3$ lax constraints). The model parameters are discussed in Appendix E.1.

Figure 4 depicts equity holders' investment choices relative to first-best (i.e., $\tilde{g} \equiv g/g^u$) and equity payouts for different values of κ . We see that for low values of κ (i.e., $\kappa \in \{0, 0.5\}$) equity holders invest more than the first-best amount and make payouts to themselves up to the constraint. However, the amount of overinvestment decreases with κ , since financing direct equity payouts with debt is a more efficient way of diluting the claims of existing debt holders than inefficient overinvestment.

For $\kappa \in \{2, 3\}$ and leverage sufficiently high we see that equity holders invest the first-best amount and issue $m < \kappa$. In that case, the constraint on direct equity payouts is lax and equity holders can make sufficiently large equity payouts to push leverage to the default threshold. Equity holders invest the first-best amount and then sell the firm to the new debt holders. Thus, unconstrained equity payouts are one way to overcome the overinvestment problem in the one-shot investment model.

The left panel of Figure 4 shows that overinvestment is hump-shaped in preexisting leverage. Overinvestment initially increases with leverage, as equity holders can dilute more

claims. However, as leverage increases further, existing claims primarily represent default claims, limiting equity holders' ability to turn existing coupons into new default claims, and decreasing the incentive to overinvest. This non-monotonicity further distinguishes our mechanism from classic risk-shifting (Jensen and Meckling (1976)), which typically leads to a monotonic relationship between the distance to default and the incentive to gamble for resurrection.

4 Bankruptcy Costs, Collateral Claims, and Seniority

In this section, we analyze how the results in the one-shot investment model are affected by the presence of bankruptcy costs (Section 4.1), the ability to issue separately unsecured claims and bankruptcy claims (Section 4.2), and the ability to issue only junior debt (Section 4.3). Throughout this section, for simplicity, we assume that there are no equity payouts (i.e., $\kappa = 0$).

4.1 Bankruptcy Costs

In the presence of default costs (i.e., $\theta \in (0, 1]$), debt financing is associated with the following trade-off. On the one hand, as before, issuing new debt allows equity holders to resell some of existing debt holders' claims, which we have seen encourages overinvestment. On the other hand, issuing new debt increases the deadweight cost of default and hence the cost of debt financing, encouraging underinvestment.¹⁶

From (25) we see that in the presence of bankruptcy costs the equity holders' optimal investment satisfies the FOC

$$\frac{1}{r - \mu} - q'(g) - p'(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell - \left[(1 + g) H'(\hat{\ell}) \hat{\ell} \frac{\partial \hat{\ell}}{\partial g} + (1 + g) H(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} + H(\hat{\ell}) \hat{\ell} \right] = 0 \quad (32)$$

The third term in (32) captures the positive marginal effect from increasing leverage associated with expropriating existing debt holders, as before. The fourth term in (32) (in square brackets) is new and captures the marginal increase in the deadweight cost of default. Depending on which effect dominates, equity holders may over- or underinvest. Not surprisingly, we find that as long as θ is not too large the first effect dominates and equity holders invest more than is socially optimal.

Proposition 6. *Suppose that $\theta > 0$.*

1. *If equity holders can only use equity financing then they underinvest, that is $g^* < g^u$.*
2. *If equity holders can choose financing optimally then for each ℓ there exists $\underline{\theta}(\ell) > 0$ such that for all $\theta \in [0, \underline{\theta}(\ell)]$ equity holders overinvest, that is $g^* > g^u$.*

The first part of Proposition 6 shows that the result that equity holders underinvest with equity financing extends to $\theta > 0$. The second part shows that equity holders continue to overinvest with optimal financing even in the presence of bankruptcy costs, as long as those

¹⁶With default costs the Modigliani–Miller theorem no longer holds even in the absence of preexisting debt (i.e., $\ell = 0$), as in that case equity holders would invest first-best amount but finance it entirely with equity.

costs are not too large. This is because for $\theta < \underline{\theta}(\ell)$ the marginal benefit associated with issuing additional debt dominates the marginal increase in deadweight cost.

Figure 5 shows optimal investment as a function of initial leverage ℓ at different bankruptcy costs θ . We see that the incentives for overinvestment continue to hold for empirically relevant values of θ .¹⁷ Only when bankruptcy costs are very large ($\theta = 0.95$ in Figure 5) firms underinvest regardless of their initial leverage.

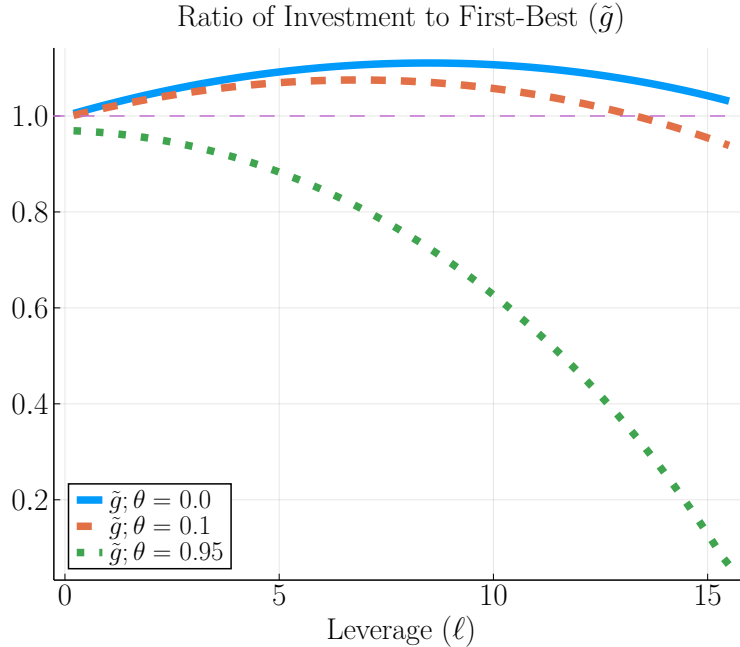


Figure 5: Investment relative to first-best (\tilde{g}) for different levels of initial leverage (ℓ) and for different values of deadweight cost of default θ , where $\theta = 0$ captures no default costs. The model parameters are discussed in Appendix E.1.

4.2 Financing with Unsecured and Collateral Claims

So far, we have considered a particular type of debt contract, that bundles coupon payments with a claim to a fraction of the liquidation value of the firm upon default. In this section, we “decouple” these two claims and allow equity holders to issue them separately.¹⁸ This analysis is directly motivated by our analysis in Section 3, which suggests that the ability to issue claims to the liquidation value of the firm is important.

As before, let L be the pre-investment PDV of the coupon payments promised to debt holders. However, assume now that holders of these claims have no claims to the default value of the firm so that the value of each such claim is now $P^U(\ell)$ rather than $P(\ell)$ (see

¹⁷The literature on costs of default typically estimates the costs to be between 2% and 25% (see, for example, Bris et al. (2006) or Glover (2016)).

¹⁸In the previous sections, the coupons and collateral claims were sold as a single asset. Financial intermediaries might then strip the default claims from the unsecured coupons, and create new assets such as a Credit Default Swap (CDS). Alternatively, equity holders could sell unsecured claims and collateral claims as separate assets.

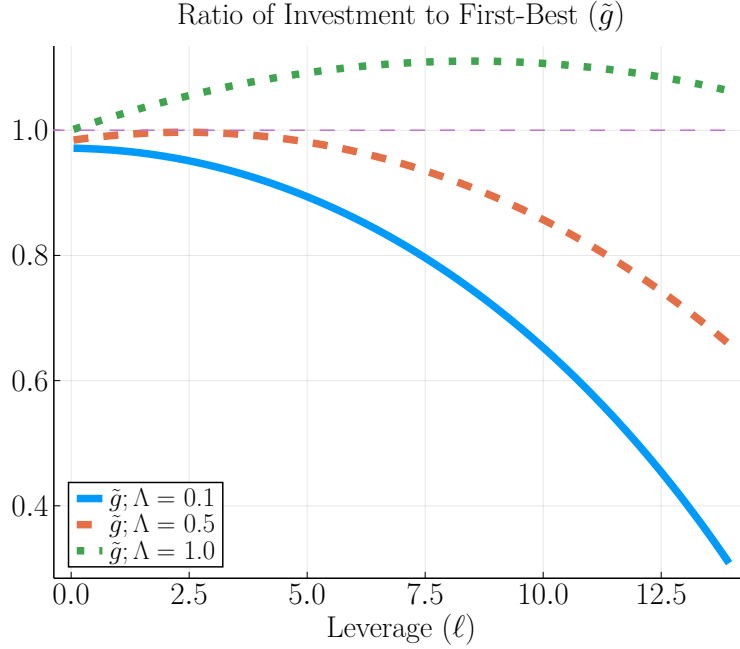


Figure 6: Investment relative to first-best (\tilde{g}) for different levels of initial leverage (ℓ) and for different values of the collateral friction (Λ , where $\Lambda = 1$ captures no frictions in issuing claims to collateral).

Section 2.3).¹⁹

Next, let B be the number of claims to the default value of the firm that are outstanding prior to investment. Each of these bankruptcy claims entitles its holder to one unit of funds upon the firm's default. If the liquidation value of the firm is lower than the number of bankruptcy claims, the value of the firm is divided proportionally across claimants. Finally, we assume that equity holders face a collateral constraint on the issuance of B , restricting B to a fraction Λ of the firm's liquidation value (derived in Proposition 1)

$$B \leq \Lambda \frac{\eta}{1 + \eta} L. \quad (33)$$

Investment now depends on the parameter Λ , capturing the firm's ability to issue new bankruptcy claims.

Proposition 7 (Investment with Collateral Frictions). *Suppose that at time 0 we have $B = \Lambda \frac{\eta}{1 + \eta} L$. Then for each $\ell > 0$ there exist $\underline{\Lambda}(\ell)$ and $\bar{\Lambda}(\ell)$ with $0 < \underline{\Lambda}(\ell) < \bar{\Lambda}(\ell)$ such that*

1. *Equity holders underinvest for all $\Lambda \in [0, \underline{\Lambda}(\ell)]$*

¹⁹The present value of promised coupon payments L is unchanged whether these coupon payments are coupled with claims in default or not. Thus, L in the setup considered in this section maps directly to L in the benchmark model. It is the market value of these promises that is affected and, hence, the value of L is now $P^U(\ell)L$ rather than $P(\ell)L$.

²⁰Otherwise, as argued in Lemma 15 in Appendix C.2, equity holders would like to issue an infinite amount of new claims in default. This is because equity holders get nothing in default and hence by selling additional claims they can dilute the existing holders of claims in default.

2. *Equity holders overinvest for all $\Lambda > \bar{\Lambda}(\ell)$*

Proof. See Appendix C.2.3. □

When Λ is sufficiently small, frictions to issuing new bankruptcy claims act like a tax on debt issuance and discourage equity holders from issuing new debt. As a consequence, when Λ is small equity holders switch to equity financing, generating an incentive to underinvest. On the other hand, when $\Lambda \geq 1$ this distortion is absent and the results of the benchmark analysis continue to hold.

Proposition 7 simplifies in the case when equity holders can finance investment with unsecured debt only (i.e., $\Lambda = 0$). We state this case as a separate corollary.

Corollary 1. *When equity holders can issue only unsecured debt then they underinvest.*

Corollary 1 has the simple but important implication that debt covenants against issuing new secured debt result in underinvestment, potentially explaining why such covenants are rare in practice (see Billett et al. (2007)). More generally, if perceived collateral quality varies over time as in Gorton and Ordoñez (2014), our model suggests that this could lead to cycles of over- and underinvestment.

4.3 Seniority

In the benchmark one-shot investment model we assumed that new debt is of equal seniority to preexisting debt (*pari-passu*). In this section, we consider how our results change if equity holders can issue junior debt only. If new debt is junior then new debt holders are paid in default only after the existing debt holders. Since the post-investment value of the firm in default is $\frac{\eta}{1+\eta}\hat{L}$ (see (13)) while the value of existing debt is L it follows that new debt holders receive a total of $\max\{0, \frac{\eta}{1+\eta}\hat{L} - L\}$ upon default. In Appendix C.3 we show that the price of a junior bond issued to finance investment is

$$P^J(\hat{\ell}, \ell) = P^U(\hat{\ell}) + \frac{\max\left\{0, \frac{\eta}{1+\eta}\hat{\ell}(1+g) - \ell\right\}}{r\hat{\ell} - r\ell} \chi \hat{\ell}^\eta < p(\hat{\ell}), \quad (34)$$

where $r\hat{\ell} - r\ell$ is the number of new junior bonds issued and $\frac{\eta}{1+\eta}\hat{\ell}(1+g)$ is the liquidation value of the firm (both normalized by Z).

Proposition 8. *Suppose that equity holders can only issue junior debt. Then equity holders underinvest.*

Proof. See Appendix C.3 □

Restricting equity holders to issuing only junior debt eliminates overinvestment incentives. Because equity holders lack the ability to dilute existing debt holders' promised coupon payments, existing debt holders always capture part of the benefits from new investment either in the form of lower default probability (if equity holders decide to decrease leverage) or in the form of an increase in the value of their claims in default (if equity holders decide to increase leverage).

5 Repeated Investments

So far, we have analyzed a one-shot investment model. In that setting, we have shown that equity holders have an incentive to “conspire” with new debt holders to dilute preexisting debt holders, thereby leading to debt-financed overinvestment. In this Section, we analyze a version of our model with repeated investment opportunities. We show that repeated investment opportunities have non-trivial consequences as buyers of new debt price in the likelihood of future dilution, thereby increasing the cost of debt financing. We find that this channel can lead to underinvestment especially among low leverage firms with frequent investment opportunities, and that direct equity payouts (i.e. equity buybacks and dividends) financed out of debt further exacerbate underinvestment among these firms.

5.1 Model with Repeated Investment

We consider the same setup as described in Section 2, but assume that investment opportunities arrive at a constant Poisson rate λ . As above, the state of the firm at any given point in time is $\{Z, L\}$. Upon arrival of an investment opportunity, equity holders have the choice to increase current cash flows from Z to $Z(1 + g)$ at cost $Zq(g)$. It follows that cash flows follow a jump diffusion

$$dZ(t) = \mu Z(t)dt + \sigma Z(t)dW(t) + g(Z(t^-), L(t^-))Z(t^-)dN(t), \quad Z(0) > 0, \quad (35)$$

where $N(t)$ is a Poisson process with intensity $\lambda \geq 0$ and $g(Z(t^-), L(t^-))$ is equity holders’ investment at time t conditional on the state of the firm $\{Z, L\}$ and the arrival of an investment opportunity. Note that when $\lambda = 0$ we are back to the model with a one-shot investment opportunity.

Investment can be financed by issuing defaultable consols via competitive debt markets (as described in Section 2.3) or equity. This implies that, in contrast to the model of Section 2, liabilities are no longer constant. Rather, $L(t)$ is now a pure jump process

$$dL(t) = (\hat{L}(t) - L(t^-))dN(t), \quad (36)$$

where $\hat{L}(t)$ —as defined in Section 2.1—denotes the value of liabilities immediately after investment implied by equity holders’ investment and financing decisions.

5.2 Optimal Investment Problem

Conditional on the arrival of an investment opportunity, the firm solves the natural analogue of the one-shot problem, except that both the $v(\cdot)$ and $p(\cdot)$ functions account for the possible arrival of future investment opportunities. The following proposition describes the equity and debt holders’ problems with repeated investment and derives the first-best solution.

Proposition 9 (Repeated Investment). *The solution to the investment problem with repeated investments consists of a normalized value of equity $v(\ell)$, price $p(\ell)$, equity holders’ policies $\{g(\ell), m(\ell), \psi(\ell), \hat{\ell}(\ell)\}$, and the default threshold for leverage, $\bar{\ell}$, such that*

1. *Given $v(\ell)$ and $p(\ell)$, the policies $\{g(\ell), m(\ell), \psi(\ell), \hat{\ell}(\ell)\}$ solve the firm’s investment problem in (20).*

2. Given $p(\ell)$ and the policies, $v(\ell)$ satisfies the differential variational inequality (DVI)

$$0 = \min\{(r - \mu)v(\ell) + \mu\ell v'(\ell) - \frac{\sigma^2}{2}\ell^2 v''(\ell) - \lambda(v(\hat{\ell}(\ell)) - v(\ell)) - (1 - r\ell), v(\ell)\} \quad (37)$$

3. The default threshold $\bar{\ell}$ is optimal and is determined by the indifference in (37)

4. Given $v(\ell)$ and the equity holders' policies, the price $p(\ell)$ solves the boundary value problem

$$rp(\ell) = r + (\sigma^2 - \mu)\ell p'(\ell) + \frac{\sigma^2}{2}\ell^2 p''(\ell) + \lambda(p(\hat{\ell}(\ell)) - p(\ell)) \quad (38)$$

$$p(\bar{\ell}) = \frac{(1 - \theta)v(0)}{\bar{\ell}} \quad (39)$$

Furthermore, the first-best investment choice as defined in Definition 1 is

$$g^u = \frac{1}{\zeta(r - \mu) \left(\frac{1}{2} \left(\sqrt{1 - \frac{2\lambda}{\zeta(r - \mu)^2}} - 1 \right) + 1 \right)} \quad (40)$$

Proof. See Appendix D. □

Unlike in the one-shot case, we do not have closed-form solutions when $\lambda > 0$. Thus, we need to solve the model numerically using upwind finite difference methods. To do so, we add artificial reflecting barriers to the stochastic process, $v'(\ell_{\min}) = 0$, $v'(\ell_{\max}) = 0$, and $p'(\ell_{\min}) = 0$. The absorbing boundary condition for $p(\cdot)$ comes from (39) (i.e. the liquidation value of the firm at the time of default).²¹

5.3 Analysis

Figure 7 shows that the repeated arrival of investment opportunities generates heterogeneous investment distortions, with low leverage firms tending to underinvest and high leverage firms tending to overinvest, as in the one-shot model. Figure 7 plots investment relative to first-best ($\tilde{g} \equiv g/g^u$) against firm leverage, with the investment arrival rate ranging from $\lambda = 0$ (i.e. the one-shot model) to $\lambda = 0.35$. With more frequent investment opportunities (i.e. higher λ), new debt investors expect their claims to be diluted sooner and require to be compensated, increasing the cost of debt finance to equity holders. Debt investors' anticipation of future dilution is more severe for low leverage firms, so these firms' equity holders face the strongest increase in the cost of debt finance, and the strongest tendency to underinvest. When the investment arrival rate λ is sufficiently high, the cost of new debt increases so much that equity holders choose to invest less than the socially efficient level for all but the highest levels of leverage.

²¹The numerical algorithm is to: (1) guess a $\hat{\ell}(\ell)$ policy, which manifests as a jump-diffusion process; (2) solve the firm's default problem conditional on this jump process as a DVI; (3) taking the jump-diffusion process and the default policy of the firm as given, solve the debt holder's pricing problem; (4) solve the optimal investment choice of g and $\hat{\ell}$ given the $v(\ell)$ and $p(\ell)$ functions from the previous steps; and (5) update the $\hat{\ell}(\ell)$ until the algorithm converges.

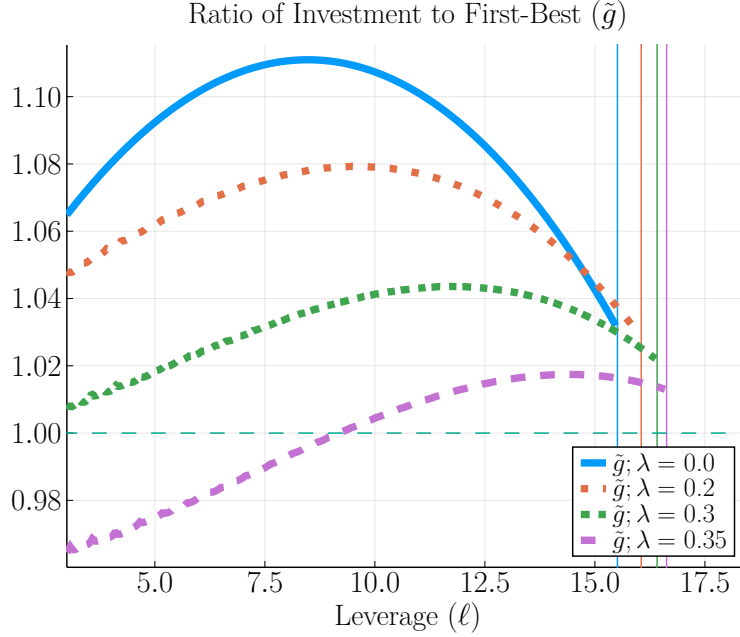


Figure 7: Investment relative to first-best, \tilde{g} for different values of the arrival rate of investment opportunities, λ (where $\lambda = 0$ corresponds to the one-shot model). Bankruptcy costs are set to zero ($\theta = 0$). Vertical lines indicate the default threshold for each value of λ . The calibration of the model parameters—including the $\lambda = 0.3$ baseline—is discussed in Appendix E.1.

Figure 8 shows that allowing equity holders to make direct payouts to themselves (i.e. dividends and equity buybacks) strengthens low leverage firms' tendency to underinvest in the repeated model. In contrast to the model with one-shot investment, allowing equity payouts financed with debt (i.e. $\kappa > 0$) need not improve the efficiency of firm investment and may even induce equity holders to switch from overinvestment to underinvestment.²² Comparing the left and right panels of Figure 8 shows that the ability to make direct payouts to equity holders ($\kappa > 0$) depresses investment more when investment opportunities are frequent (i.e. λ is high). Intuitively, debt holders' concerns about the future dilution of their claims are exacerbated when equity holders can make direct payouts to themselves financed with debt.

For the highest levels of leverage, Figure 8 shows that investment converges to the first-best as κ increases, similarly to the one-shot model. This is because for high enough ℓ and κ equity holders are able to issue so much debt that they optimally choose to default immediately after investment, which implies that debt holders immediately take over the firm and equity holders have no opportunities to dilute new debt holders' claims. Thus, as long as there are no deadweight bankruptcy costs, new debt holders incentives are again aligned, just as in the model with one-shot investment.

²²Grullon and Michaely (2002) show that payouts to equity holders (dividends plus equity repurchases) are around 50% of earnings. Because not all payouts in practice are financed by new debt issuance we consider this an upper bound on average direct equity payouts $= \lambda\kappa$ in our model, leading us to consider values for κ between 0 and 2.

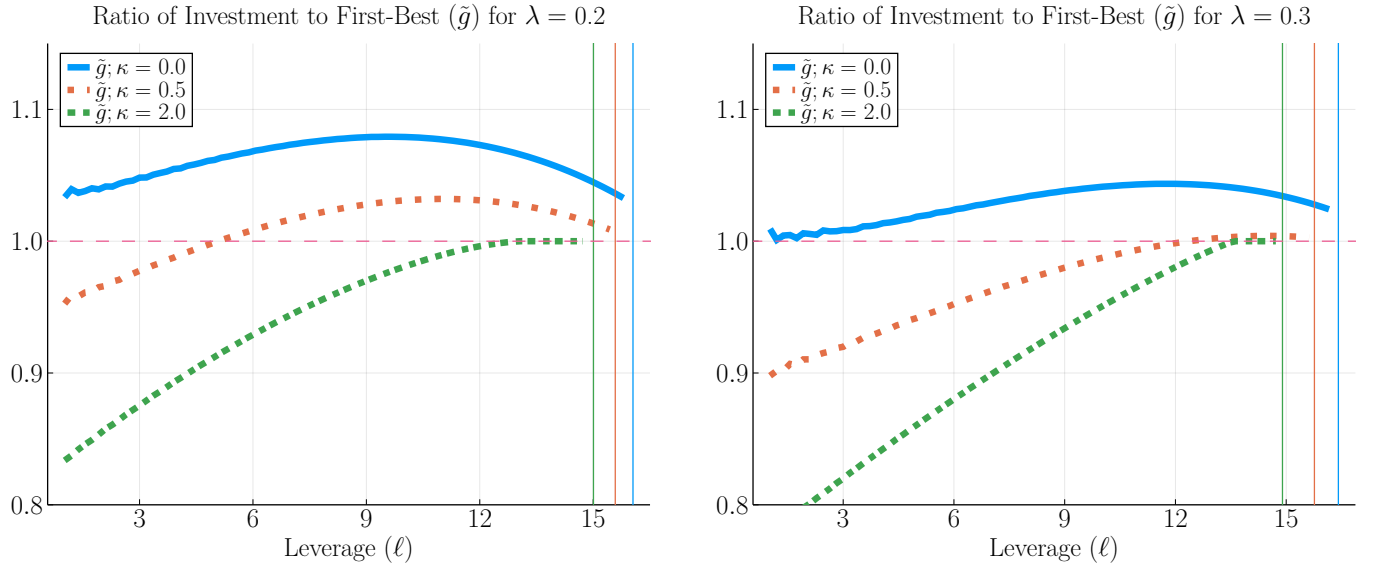


Figure 8: Investment relative to first-best (\tilde{g}) for different levels of initial leverage (ℓ) and different values of the constraint on payouts to equity out of debt κ (where $\kappa = 0$ is the baseline case of no equity payouts from debt). The two panels show this for an arrival rate of new investment opportunities $\lambda = 0.2$ (left) and $\lambda = 0.3$ (right). Bankruptcy costs are set to zero ($\theta = 0$). Vertical lines indicate the default thresholds for each value of κ . The calibration of the model parameters—including the $\lambda = 0.3$ baseline—is discussed in Appendix E.1.

5.4 Dynamic Leverage, Investment, and Bankruptcy

Figure 9 shows simulated paths (averaged over 500 paths) for firms with different initial leverage ratios. The left panels assume that equity payouts are prohibited ($\kappa = 0$), while the right panels permit equity buybacks ($\kappa = 1$). The top panels show simulated leverage, the middle panels simulated real investment, and the bottom panels show simulated cumulative bankruptcy rates.

The results from the prior comparative statics carry through, whereby high leverage firms' tendency to overinvest is exacerbated by restrictions on equity buybacks (left panels), whereas low leverage firms' tendency to underinvest is most pronounced when direct equity payouts from new debt are permitted (right panels). The leverage and investment gaps between firms with different initial leverage narrow over time, as bankruptcies of high leverage firms cumulate. As low leverage firms' leverage increases over time, due to the incentive to double-sell cash flows, high leverage firms' leverage is capped above by default.

The comparative statics of investment and bankruptcy rates with respect to equity payout restrictions crucially depend on the endogenous default threshold, showing that it is important to model the optimal default decisions of limited liability equity holders. Relaxing the equity payout constraint increases bond holders' fears of being diluted in the future and hence the cost of new debt financing to equity holders considering whether to keep the firm as a going concern, leading to a lower optimal default threshold (top panels). As firms default earlier and at lower levels of leverage, this mitigates overinvestment by highly-levered firms but increases bankruptcy rates.

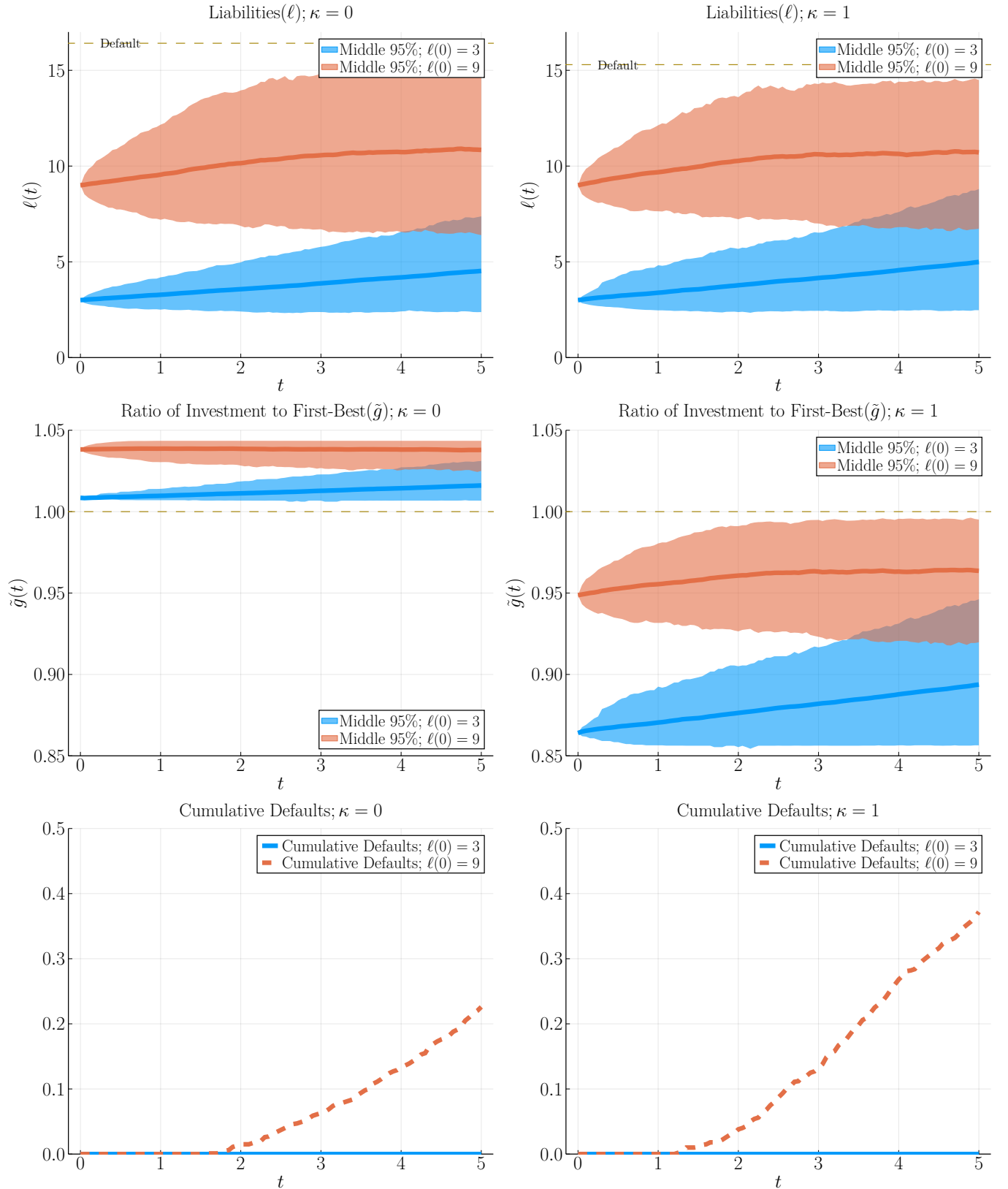


Figure 9: Simulation with $\lambda = 0.3$ of an ensemble of 1000 paths of leverage ℓ (top), investment relative to first-best \tilde{g} (middle), and cumulative defaults (bottom). Each panel shows the ensemble starting from $\ell(0) \in \{3, 9\}$ corresponding to interest coverage ratios of 4 and 1.5 respectively. The left panels use $\kappa = 0$ and the right panels use $\kappa = 1$. The central line is the mean, the red shaded area shows the 2.5th and 97.5th percentiles. Moments are based on non-defaulted firms.

6 Empirical Applications and Policy Discussion

This section describes two empirical results in firm-level data that are qualitatively consistent with the central model mechanism. First, in Section 6.1, we show empirical evidence that equity holders tend to increase direct equity payouts when issuing new debt, mirroring a central model prediction in the case when direct equity payouts are not prohibited. Second, in Section 6.2, we show empirical evidence that high leverage firms' capital expenditure investment has exceeded that of low leverage firms since the mid-2000s. This empirical finding provides validation for our model, which predicts that high leverage firms have the strongest incentive to double-sell claims to default cash flows. Finally, we use simple model comparative statics to understand how crisis policies for the corporate sector are likely to affect the cross-section of firm investment and bankruptcies.

6.1 Empirical Equity Payoffs and Debt Issuance

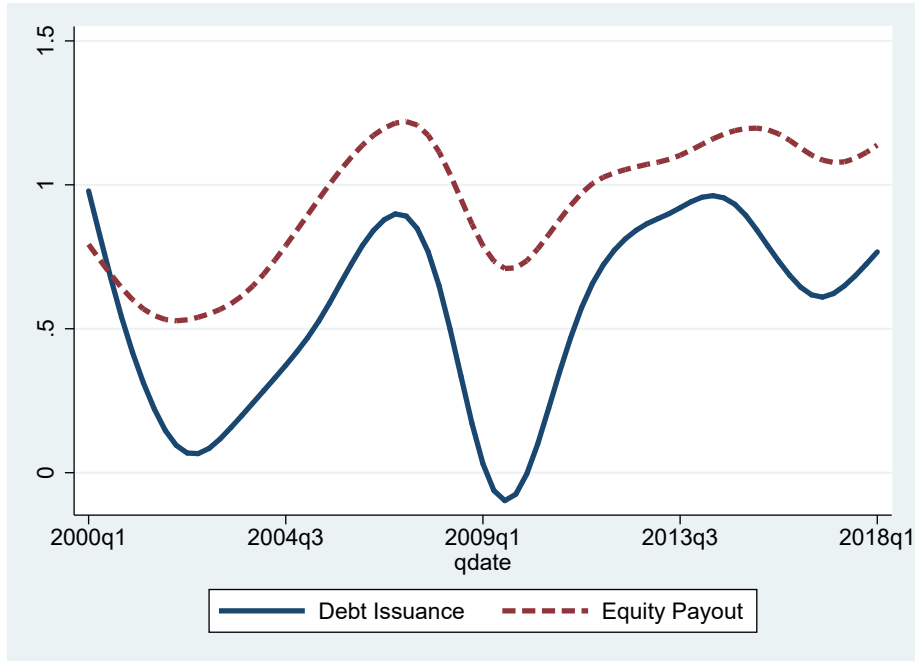


Figure 10: Debt Issuance and Equity Payouts in the Data

A key prediction of the model is that firms that increase their debt relative to assets (i.e. increase ℓ) have incentives to pay out directly to equity holders. Consistent with this incentive, Figure 10 shows that average debt issuance and equity payouts, defined as the sum of dividends and equity buybacks, comove over time. Furthermore, Table 1 shows that the positive relationship between debt issuance and equity payouts is not just driven by aggregate time series events.²³ The coefficients in Table 1 do not change as we include

²³We measure debt issuance and equity payouts similarly to Begenau and Salomao (2019) using Compustat data, though we are interested in gross equity payouts rather than net equity payouts to match the variable m in the model. Debt issuance equals the change in the book value of debt (long-term debt plus short-term debt in current liabilities), divided by the 8-quarter moving average of total firm assets. Gross equity payout equals dividends plus equity repurchases divided by the 8-quarter moving average of total firm assets. Figure 10

firm and time fixed effects, showing that the association holds when we isolate variation across firms. These regressions show that a one dollar increase in debt is associated with a highly statistically significant and economically meaningful 4 cent increase of the gross equity payout. These empirical results speak to our model, where equity holders financing new investment with debt have an incentive to partially pay out the proceeds directly to equity, but are constrained from doing so fully.

Within our model, it is natural to rationalize the empirical comovement between debt issuance and equity payouts with differences in the arrival rates of investment opportunities, λ , both over the business-cycle and across firms. If the arrival rate of new investment opportunities increases towards the beginning of an economic expansion, this would simultaneously increase the aggregate payouts to equity holders and the total amount of new debt issued by firms.²⁴ Our model also naturally generates cross-firm comovement between debt issuance and equity payouts, as only a fraction of firms have investment opportunities at any given time, and those firms optimally choose to simultaneously finance this investment with debt and increase equity payouts.

Table 1: Empirical Regressions of Debt Issuance and Equity Payouts

Equity payout			
Delta Debt	0.04** (0.002)	0.03** (0.002)	0.04** (0.002)
No. Obs.	136697	136697	136697
Time FE	No	Yes	Yes
Firm FE	No	No	Yes

Regression equation: $\text{Equity payout}_{i,t} = b_0 + b_1 \text{Delta Debt}_{i,t} + \varepsilon_{i,t}$. This table shows a panel regression of gross equity payouts onto the change in debt while controlling for time and firm fixed effects. The sample covers 2000Q1-2018Q1. Standard errors in parentheses are double-clustered by firm and quarter. * and ** denote statistical significance at the 5% and 1%-level.

6.2 Empirical Investment for High and Low Leverage Firms

We next show that the model implications for investment are consistent with the data both in the time-series and across firms. Figure 11 shows the time-series for capital expenditures investment to total assets for high leverage and low leverage firms in Compustat data. Focusing first on the aggregate time-series pattern common to high and low leverage firms, we note that periods of high capital expenditures are precisely those when new debt issuance and payouts to equity holders are also high, as shown in Figure 10, consistent with the inter-

reports the time series of the equal-weighted average across all firms each quarter. Covas and Den Haan (2011) and Begenau and Salomao (2019) study business cycle variation in debt and equity issuance. We differ from these studies because we study gross equity payouts (dividends plus equity buybacks) rather than equity issuance, to match direct equity payouts in the model.

²⁴Gourio and Kashyap (2007) provide direct plant-level empirical evidence that frequency of lumpy investment is pro-cyclical, consistent with this interpretation.

pretation that the availability of investment opportunities drives the positive comovement between debt issuance and equity payouts in the data.

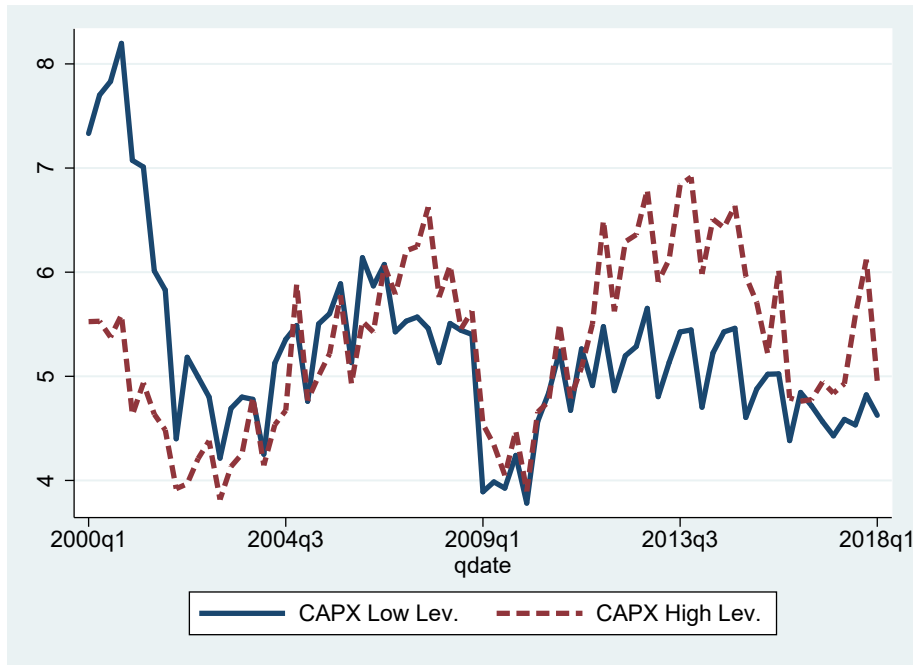


Figure 11: Real Investment by Leverage in the Data

Turning to the cross-section of firms, Figure 11 documents that, maybe surprisingly, the investment rates of high leverage firms were persistently higher than those of low leverage firms for more than a decade.²⁵ By contrast, during the early 2000s we observed the standard pattern of higher investment rates for low leverage firms, potentially due to unmodeled differences in credit market access. The relatively higher investment rates of high vs. low leverage firms during the latter half of our sample are suggestive of our model mechanism, whereby highly-levered firms tend to overinvest and low leverage firms tend to underinvest. Our model easily rationalizes the otherwise surprising empirical finding that high leverage firms invested more than low leverage firms during the latter half of our sample, especially to the extent that this was a period when direct payouts to equity holders were relatively unconstrained.²⁶ It can also rationalize the observation that the investment gap between high leverage and low leverage firms narrowed during the recession of 2008-2009, and again opened up even stronger during the subsequent long expansion, as in our model the relative underinvestment incentive of low leverage firms is strongest when the arrival rate of investment opportunities is high.

²⁵High leverage is defined as being in the top quartile of total debt/assets in each quarter. Low leverage is defined as being in the bottom quartile of total debt/assets in each quarter. We plot the equal-weighted average ratio of quarterly CAPX to lagged total assets for each group of firms for the period 2000.Q1-2018.Q1. See Appendix E for more details on the construction of the data. See Kahle and Stulz (2013) and Xiao (2019) for a detailed comparison of investment by firms with high and low debt during the acute phase of the 2008-09 financial crisis.

²⁶The conspicuous practice of equity buybacks in the airline industry prior to the Covid-19 crisis is consistent with a high value of κ . See e.g. “U.S. Airlines Spent 96% of Free Cash Flow on Buybacks”, Wall Street Journal, March 16, 2020.

Taken together, a simple story emerges from viewing the data through the lens of our model. As investment opportunities vary, either over the business cycle or across firms, firms issue debt to finance new investment and increase payouts to equity holders. During the second half of our sample when implicit restrictions on paying equity holders were lax, the gap between low leverage firms' incentive to underinvest and high leverage firms' incentive to overinvest was particularly strong, leading to relatively higher investment among highly-levered firms compared to low leverage firms. These patterns in investment and financing decisions likely have aggregate and cross-sectional implications for policy. We turn to these next.

6.3 Policy Discussion

The unprecedented shock and economic contraction from the Covid-19 virus have led to equally unprecedented government support for corporations, which, unlike during the financial crisis of 2008-2009, are at the center of the economic crisis. These developments have raised key design questions for government support of the corporate sector.²⁷ Which firms should be targeted? Should government assistance take the form of traditional loans, or should it be in the form of equity? Should there be restrictions on equity buybacks, and should these be permanent or temporary? Given that Covid-19 related government programs for the corporate sector are on the order of trillions of dollars, each of these questions is highly relevant, and plausibly affected by corporate limited liability as a financial friction.

Within our model, we interpret a policy of traditional loans to the corporate sector as a one-time increase in leverage l , whereas the government taking an equity position corresponds to a decrease in l . Either of these policies can be combined with restrictions on equity payouts ($\kappa = 0$) or permit direct equity payouts ($\kappa > 0$). To simulate the fact that investment opportunities are likely to be scarce during a crisis, we simulate leverage, investment, and bankruptcy rates for a lower investment arrival rate.²⁸

Figure 12 shows that in our model firms with high initial leverage invest more, potentially even beyond the socially efficient level, and are more likely to default than low leverage firms. By contrast, firms with low initial leverage are less prone to overinvestment and may even underinvest (if $\kappa > 0$, right panels). These cross-sectional effects in our partial equilibrium model suggest that a policy of giving hard loans to all firms without distinction may have adverse reallocative effects, shifting investment from young and low-levered firms towards highly-levered zombie firms, thereby impeding innovation and creative destruction, and ultimately, economic growth. On the other hand, within our model a targeted policy that reduces leverage for highly-levered firms, potentially through debt restructurings or new equity, appears most effective at preventing bankruptcies and inefficient overinvestment.²⁹

Comparing the left and right panels of Figure 12 shows that in our model equity buyback restrictions have benefits and costs that are heterogeneous across firms. Prohibiting equity buybacks ($\kappa = 0$, left panel) is beneficial in that it may raise investment towards the efficient

²⁷There is a large and rapidly growing literature on the effects and design of Covid-19 related government programs. Hanson et al. (2020) summarize some of the key policy design questions. Elenev et al. (2020) provide a quantitative theoretical analysis.

²⁸Figure 12 is analogous to Figure 9 except that it uses $\lambda = 0.2$ instead of $\lambda = 0.3$.

²⁹The prediction that some highly-levered firms invest beyond their efficient level might appear surprising, but it is in line with Granja et al. (2020)'s recent evidence that firms participating in the Paycheck Protection Program tend to hold a portion of the proceeds in cash.

level among low leverage firms, and reduces bankruptcy rates among high leverage firms. Such a policy, however, also has costs in that it may incentivize inefficient overinvestment among high leverage firms. Buyback restrictions have a lagged effect on bankruptcies in our model, predicting that the bankruptcy benefits of equity buyback restrictions may not be immediately visible and instead take time to manifest.

Finally, one might wonder whether equity buyback restrictions are less costly if they are temporary and allowed to expire once economic recovery has taken hold. If we continue to interpret an economic recovery as an increase in the arrival rate of investment opportunities, Figure 9 shows that inefficient overinvestment becomes less frequent for a higher arrival rate of investment opportunities, suggesting that the inefficient overinvestment caused by equity buyback restrictions is concentrated during recessions when investment opportunities are scarce. Our model therefore suggests that making equity buyback restrictions temporary may provide only very limited relief from the investment distortions that they cause.

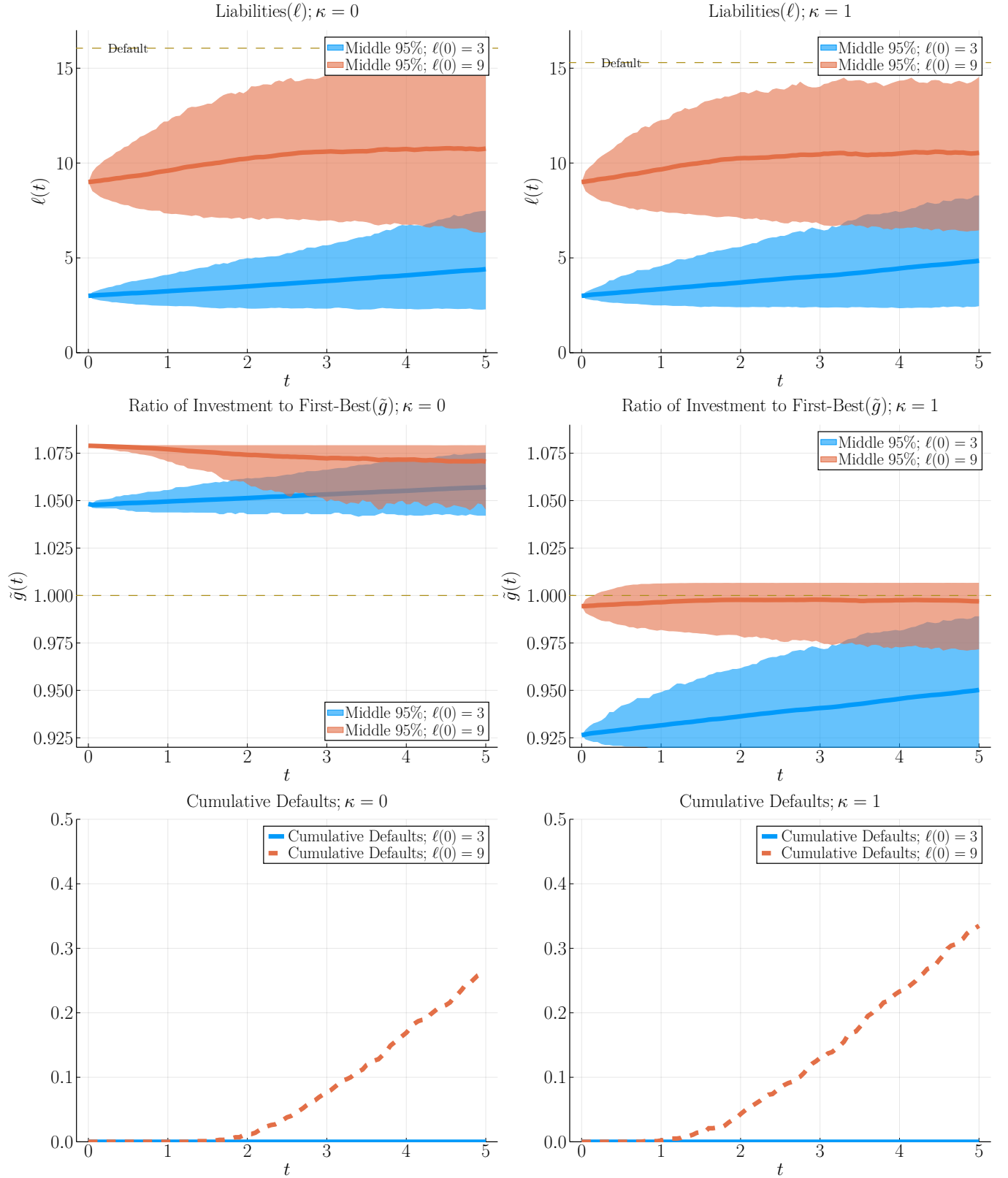


Figure 12: Simulation with $\lambda = 0.2$ of an ensemble of 1000 paths of leverage ℓ (top), investment relative to first-best \tilde{g} (middle), and cumulative defaults (bottom). Each panel shows the ensemble starting from $\ell(0) \in \{3, 9\}$ corresponding to interest coverage ratios of 4 and 1.5 respectively. The left panels use $\kappa = 0$ and the right panels use $\kappa = 1$. The central line is the mean, the red shaded area shows the 2.5th and 97.5th percentiles. Moments are based on non-defaulted firms.

7 Conclusion

In this paper we investigate how limited liability affects equity holders' incentives to invest, issue debt, and buy back equity. We show that when equity holders can finance their investment with a combination of debt and equity and when investment is a continuous choice, equity holders have incentives to overinvest. In a tractable model with one-shot investment, allowing equity holders to pay dividends or buy back equity directly when issuing new debt moves investment towards the first-best and mitigates overinvestment. In the one-shot investment model, the incentive to overinvest prevails even in the presence of default costs or restrictions on the issuance of secured debt.

A framework with repeated investment shows that the overinvestment result continues to hold for highly-levered firms, and generates important additional insights. With repeated investment, debt holders' anticipation of future dilution increases the cost of debt financing, leading to underinvestment for low leverage firms. Allowing for direct equity payouts in the repeated setup is no longer necessarily efficient, as it further increases debt holders' fears of being diluted in the future and the cost of debt financing. Simulations show that allowing equity buybacks has the benefit of mitigating overinvestment of highly-levered firms, at the cost of exacerbating underinvestment of low leverage firms and increasing bankruptcy rates.

Our model could be extended to allow for information frictions in the quality of collateral (Cole and Kocherlakota (2001), Gorton and Ordoñez (2014), Gorton and Ordoñez (2016), and Ordoñez et al. (2019)). Such frictions would cause significant distortions in our model, and further deviations from the Modigliani-Miller theorem, since our model mechanism works by transforming coupons into default claims. If collateral quality or quantity is hidden or cyclical, debt financing would likely be further distorted relative to our benchmark. This could manifest as possibly stochastic and cyclical bankruptcy costs or with hidden information in the amount of pledged collateral.

Our findings have potentially important macroeconomic implications for the investment consequences of financial frictions, both for the average firm and across the firm distribution (e.g. Buera (2009), Khan and Thomas (2013), Moll (2014), and Buera et al. (2015)) as well the interaction of insolvency and aggregate fluctuations (e.g. Atkeson et al. (2017)). We believe that investigating these general equilibrium consequences of our partial equilibrium model will be fruitful.

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Appendix A Proofs for Section 2

A.1 Proof of Proposition 1

Proof of Proposition 1. To find the post-investment value of equity $V(Z, L)$ that solves equity holders' default problem as described by (5)- (7) we use the method of undetermined coefficients with the guess

$$V(Z, L) = \frac{1}{r - \mu} \left(Z + \frac{\omega}{\eta} Z^{-\eta} \right) - L \quad (\text{A.1})$$

Using this guess in (5) and equating undetermined coefficients we arrive at the equation

$$2(r + \eta\mu) = \eta(1 + \eta)\sigma^2 \quad (\text{A.2})$$

We solve the above quadratic equation for η and note that the smaller of the two roots is explosive and, hence, it violates the transversality condition. Therefore,

$$\eta = \frac{(\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2 r}}{\sigma^2} \quad (\text{A.3})$$

Next, we substitute the guess (A.1) into (7) to find that

$$\omega = \underline{Z}^{\eta+1} \quad (\text{A.4})$$

Then we use (A.1) and (A.4) in (6) to find

$$\underline{Z} = \frac{(r - \mu)\eta}{\eta + 1} L \quad (\text{A.5})$$

(A.5) defines the default threshold and completes derivations of $V(Z, L)$. Before continuing we note that the default threshold directly depends on equity holders' liabilities L and, hence, in what follows we denote it by $\underline{Z}(L)$.

Next, we show that the value of equity can be expressed as $V(Z, L) = v(\ell)Z$ and derive the expression for $v(\cdot)$. First, we note that

$$\frac{\underline{Z}(L)}{Z} = \frac{(r - \mu)\eta}{\eta + 1} \frac{L}{Z} = \frac{(r - \mu)\eta}{\eta + 1} \ell, \quad (\text{A.6})$$

where $\ell \equiv L/Z$. Next, we use (A.4) in (A.1) to find

$$V(Z, L)/Z = \frac{1}{r - \mu} \left(1 + \frac{1}{\eta} (\underline{Z}(L)/Z)^{\eta+1} \right) - \ell \quad (\text{A.7})$$

We substitute for \underline{Z}/Z the expression obtained in (A.6) to obtain

$$V(Z, L)/Z = \frac{1}{r - \mu} \left(1 + \frac{1}{\eta} \left[\frac{(r - \mu)\eta}{\eta + 1} \ell \right]^{\eta+1} \right) - \ell = \frac{1}{r - \mu} + \frac{\chi}{\eta + 1} \ell^{\eta+1} - \ell, \quad (\text{A.8})$$

where

$$\chi \equiv \left(\frac{(r - \mu)\eta}{\eta + 1} \right)^\eta \quad (\text{A.9})$$

Finally, we set

$$v(\ell) = \frac{1}{r - \mu} - \ell(1 - s(\ell)), \quad (\text{A.10})$$

where $s(\ell) = \frac{\chi}{\eta+1}\ell^\eta$, which implies that

$$V(Z, L)/Z = v(\ell) \quad (\text{A.11})$$

To find the liquidation value of the firm we note that equity holders walk away when cash flows Z reach the default threshold $\underline{Z}(L)$. At that time debt holders take over the firm so its liabilities are reset to $L = 0$ but the firm loses fraction $\theta \in [0, 1]$ of its value. It follows that the liquidation value of the firm, from creditors' perspective, is given by

$$V((1 - \theta)\underline{Z}(L), 0) = \frac{(1 - \theta)\underline{Z}(L)}{r - \mu} \quad (\text{A.12})$$

Thus, the liquidation value per unit of liabilities is given by

$$\frac{V((1 - \theta)\underline{Z}(L), 0)}{L} = \frac{(1 - \theta)}{r - \mu} \frac{\underline{Z}(L)}{L} \quad (\text{A.13})$$

Using the definition of $\underline{Z}(L)$ (see (A.5)) the above expression simplifies to

$$\frac{V((1 - \theta)\underline{Z}(L), 0)}{L} = \frac{(1 - \theta)\eta}{\eta + 1} \quad (\text{A.14})$$

□

A.2 Proof of Proposition 2

Proof of Proposition 2. (Derivations of debt prices) As explained in Section 2.3 we have

$$P^U(Z, L) \equiv \frac{p^U(Z, L)}{r} = \mathbb{E}_T \left[\int_0^T e^{-r\tau} d\tau \right] \quad (\text{A.15})$$

$$P^B(Z, L) \equiv \frac{p^B(Z, L)}{r} = \mathbb{E}_T \left[e^{-rT} \frac{V((1 - \theta)\underline{Z}(L), 0)}{rL} \right] \quad (\text{A.16})$$

$$P(Z, L) \equiv \frac{p(Z, L)}{r} = \frac{p^U(Z, L)}{r} + \frac{p^B(Z, L)}{r}, \quad (\text{A.17})$$

where T is the first-time cash flows, Z , reach the default threshold $\underline{Z}(L)$. Since Z follows a geometric Brownian motion (see (1)) we have

$$\mathbb{E}_T [e^{-rT}] = \exp \left(\frac{-(\mu - \sigma^2/2) - \sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2 r}}{\sigma^2} (\log Z - \log \underline{Z}(L)) \right) \quad (\text{A.18})$$

as shown, for example, in Jeanblanc et al. (2009). Using the definition of η and χ (see (A.3) and (A.9), respectively) and the expression obtained in (A.5) we conclude that

$$\mathbb{E}_T \left[e^{-rT} \right] = (Z/\underline{Z}(L))^{-\eta} = \chi \ell^\eta \quad (\text{A.19})$$

Using (A.19) we see that

$$P^U(Z, L) \equiv \frac{p^U(Z, L)}{r} = \frac{1}{r} [1 - \chi \ell^\eta] = \frac{1}{r} [1 - (1 + \eta)s(\ell)], \quad (\text{A.20})$$

where $s(\ell) = \chi/(1 + \eta)\ell^\eta$ and $\ell = L/Z$. Note that the above equation implies that the relevant state variable is ℓ . Hence, we can express the price of the unsecured component of a defaultable consol as

$$P^U(\ell) \equiv \frac{p^U(\ell)}{r} = \frac{1}{r} [1 - (1 + \eta)s(\ell)] \quad (\text{A.21})$$

Next, we consider the price of a bankruptcy claim $P^B(Z, L)$. Note that

$$P^B(Z, L) \equiv \frac{p^B(Z, L)}{r} = \frac{V((1 - \theta)\underline{Z}(L), 0)}{rL} \mathbb{E}_T \left[e^{-rT} \right] \quad (\text{A.22})$$

As shown in the proof of Proposition 1 (see (A.14)) we have

$$\frac{V((1 - \theta)\underline{Z}(L), 0)}{L} = \frac{(1 - \theta)\eta}{\eta + 1} \quad (\text{A.23})$$

Therefore,

$$P^B(Z, L) \equiv \frac{p^B(Z, L)}{r} = \frac{1}{r} \frac{(1 - \theta)\eta}{\eta + 1} \mathbb{E}_T \left[e^{-rT} \right] = \frac{1}{r} \frac{(1 - \theta)\eta}{\eta + 1} \chi \ell^\eta = \frac{1}{r} (1 - \theta)\eta s(\ell), \quad (\text{A.24})$$

where we used first (A.19) and then the definition of $s(\ell)$. We see again that the relevant state variable is ℓ and, thus, we can write the price of claims bankruptcy by

$$P^B(\ell) \equiv \frac{p^B(\ell)}{r} = \frac{1}{r} [(1 - \theta)\eta s(\ell)] \quad (\text{A.25})$$

From the above discussion it follows that the leverage ℓ is the relevant state for pricing defaultable consols so that we can write

$$P(Z, L) = P(\ell) \quad \text{and} \quad p(Z, L) = p(\ell) \quad (\text{A.26})$$

Finally, putting together (A.21) and (A.25) we obtain

$$P(\ell) = \frac{p(\ell)}{r} = \frac{1}{r} [1 - (1 - \theta)\eta s(\ell)] \quad (\text{A.27})$$

(The Budget Constraint) We now derive the budget constraint faced by equity holders. Equity holders issue debt to finance their equity payouts, M , and a fraction ψ of the investment cost $Zq(g)$. Let K denote the quantity of new bonds issued by equity holders to finance $\psi Zq(g) + M$. Then, K has to satisfy the following budget constraint

$$P(\hat{\ell})K = \psi Zq(g) + M, \quad (\text{A.28})$$

where $\hat{\ell}$ is the post-investment leverage. Next, we relate K to the change in leverage $\hat{L} - L$. Recall that L is defined as the present discounted value (PDV) of liabilities. Since each unit of debt promises a payment of a constant coupon of 1 and agents discount these payments at a rate r it follows the PDV of the cash flows promised to new debt holders is given by

$$K \int_0^\infty e^{-r\tau} d\tau = \frac{K}{r} \quad (\text{A.29})$$

Therefore, the post-investment liabilities are given by

$$\hat{L} = L + \frac{K}{r} \quad (\text{A.30})$$

It follows that

$$K = r(\hat{L} - L) \quad (\text{A.31})$$

Substituting the above expression for K into the budget constraint (A.28) and dividing both sides of the resulting equation by Z we obtain

$$P(\hat{\ell}) (r\hat{\ell}(1+g) - r\ell) = \psi q(g) + m, \quad (\text{A.32})$$

where, $\hat{\ell} = \hat{L}/(Z(1+g))$. Finally, using the definition of $p(\hat{\ell})$ (see (A.17)) we obtain

$$p(\hat{\ell}) (\hat{\ell}(1+g) - \ell) = \psi q(g) + m, \quad (\text{A.33})$$

which corresponds to (18) in the text. \square

A.3 Proof of Proposition 3

Proof of Proposition 3. To derive (25) consider equity holders' objective function (20) given by

$$(1+g)v(\hat{\ell}) - (1-\psi)q(g) + m$$

Using the budget constraint (see, for example, (A.33)) to eliminate $\psi q(g) + m$ from the above equation we obtain

$$(1+g)v(\hat{\ell}) - q(g) + p(\hat{\ell})(\hat{\ell}(1+g) - \ell) \quad (\text{A.34})$$

From Proposition 1 we know that

$$v(\hat{\ell}) = \frac{1}{r-\mu} - \hat{\ell}(1-s(\hat{\ell})) \quad (\text{A.35})$$

Therefore, (A.34) can be written as

$$\frac{1+g}{r-\mu} - (1+g)\hat{\ell}(1-s(\hat{\ell})) - q(g) + p(\hat{\ell})(\hat{\ell}(1+g) - \ell) \quad (\text{A.36})$$

Using the observation $p(\hat{\ell}) = 1 - (1 + \theta\eta)s(\hat{\ell})$ we note that

$$-(1+g)\hat{\ell}(1-s(\hat{\ell})) + p(\hat{\ell})(1+g)\hat{\ell} = -\theta\eta s(\hat{\ell})\hat{\ell}(1+g) \quad (\text{A.37})$$

Therefore, we can simplify (A.36) to

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) - \theta\eta s(\hat{\ell})\hat{\ell}(1+g) \quad (\text{A.38})$$

Defining

$$H(\hat{\ell}) = \theta\eta s(\hat{\ell}) \quad (\text{A.39})$$

and using this definition in (A.38) we obtain the equity holders' objective function (25) in Proposition 3.

To obtain (28) recall from Section 2.4 (Equation (23)) that the first-best unconstrained equation is defined as

$$g^u \equiv \arg \max_g (1+g)v(0) - q(g) \quad (\text{A.40})$$

Since $v(0) = \frac{1}{r-\mu}$ it follows that the F.O.C. that determines g^u is given by

$$0 = \frac{1}{r-\mu} - q'(g^u) \quad (\text{A.41})$$

That g^u is uniquely determined follows from the strict convexity of the cost function. \square

Appendix B Proofs for Section 3

In this section, we provide proofs of propositions stated in Section 3 (Propositions 4 and 5). We begin by establishing a number of useful preliminary results and by discussing feasibility of equity holders' choice of $\{g, \hat{\ell}, m, \psi\}$ (Appendix B.1). In Appendices B.2 and B.3 we then provide proofs of Propositions 4 and 5, respectively. Throughout this section we assume that $\theta = 0$.

B.1 Preliminary Results

Lemma 2. *Define $\bar{\ell}$ as the unique solution to*

$$1 = \chi \bar{\ell}^\eta, \quad (\text{B.1})$$

Then,

$$p(\bar{\ell})\bar{\ell} = \frac{1}{r-\mu} \quad (\text{B.2})$$

Proof. Plugging $\bar{\ell}$ into the expression for $p(\ell)$ (see (15)) we obtain

$$p(\bar{\ell}) = \frac{\eta}{1 + \eta} \quad (\text{B.3})$$

From the definitions of $\bar{\ell}$ and χ (Equations (B.1) and (9), respectively) we obtain

$$\bar{\ell} = \frac{1 + \eta}{\eta} \frac{1}{r - \mu} \quad (\text{B.4})$$

Combining (B.3) and (B.4) we obtain

$$p(\bar{\ell})\bar{\ell} = \frac{1}{r - \mu} \quad (\text{B.5})$$

□

In the proof of Lemma 2 we derived expressions for $\bar{\ell}$ and $p(\bar{\ell})$. Since we make use of these expressions repeatedly in what follows we report them in the following Corollary.

Corollary 3. *We have*

$$\bar{\ell} = \frac{1}{r - \mu} \frac{1 + \eta}{\eta} \quad \text{and} \quad p(\bar{\ell}) = \frac{\eta}{1 + \eta} \quad (\text{B.6})$$

Next, we show the constraint $p(\ell) \geq p^B(\ell)$ is satisfied if and only if $\ell \in [0, \bar{\ell}]$.

Corollary 4. *We have $p(\ell) \geq p^B(\ell)$ if and only if $\ell \in [0, \bar{\ell}]$.*

Proof. From the definitions of $p(\ell)$ and $p^B(\ell)$ (see Proposition 2) we have

$$p(\ell) - p^B(\ell) = 1 - \chi\ell^\eta$$

Since $\chi > 0$ and $\eta > 0$ it follows that $p(\ell) - p^B(\ell)$ is strictly decreasing in ℓ . Moreover, from the definition of $\bar{\ell}$ we see that $p(\bar{\ell}) - p^B(\bar{\ell}) = 0$. This establishes the claim. □

Having characterized the highest feasible leverage, $\bar{\ell}$, we now establish two useful results regarding the value of outstanding debt.

Lemma 5. *For all $\hat{\ell} < \bar{\ell}$ we have*

$$\frac{\partial}{\partial \hat{\ell}} [p(\hat{\ell}) (\hat{\ell}(1 + g) - \ell)] = p'(\hat{\ell}) (\hat{\ell}(1 + g) - \ell) + p(\hat{\ell})(1 + g) > 0 \quad (\text{B.7})$$

Proof. We have

$$\frac{\partial}{\partial \hat{\ell}} [p(\hat{\ell}) (\hat{\ell}(1 + g) - \ell)] = p'(\hat{\ell}) (\hat{\ell}(1 + g) - \ell) + p(\hat{\ell})(1 + g) \quad (\text{B.8})$$

$$= [1 - \chi\hat{\ell}^\eta] (1 + g) + \frac{\eta\chi}{1 + \eta} \hat{\ell}^{\eta-1} \ell, \quad (\text{B.9})$$

where we used the definition of $p(\ell)$ (see (15) with $\theta = 0$). The claim follows from the observation that, since $\hat{\ell} < \bar{\ell}$, we have $[1 - \chi\hat{\ell}^\eta] > 0$. □

Lemma 6. *The value of outstanding debt $p(\ell)\ell$ is strictly increasing in ℓ for all $\ell \in [0, \bar{\ell}]$.*

Proof. We have

$$\frac{\partial}{\partial \ell} p(\ell)\ell = p'(\ell)\ell + p(\ell) = 1 - \chi\ell^\eta > 0, \quad (\text{B.10})$$

where the last inequality follows since $\ell < \bar{\ell}$. \square

Finally, we discuss which equity holders' choices of g, ψ, m , and $\hat{\ell}$ are feasible (i.e., satisfy equity holders' budget constraint). Note that the equity holders' choices of g, ψ, m , and $\hat{\ell}$ have to jointly satisfy the equity holders' budget constraint

$$p(\hat{\ell})((1+g)\hat{\ell} - \ell) = \psi q(g) + m \quad (\text{B.11})$$

The budget constraint implies that once the equity holders make choices of g, ψ , and m the post-investment leverage $\hat{\ell}$ is determined implicitly by (B.11). Thus, we can treat the post-investment leverage as an implicit function of g, ψ , and m , and denote it by $\hat{\ell}(g, \psi, m)$. This leads to the following definition of feasibility of equity holders choices.

Definition 2. *The equity holders choices of g, ψ, m are feasible if $\hat{\ell}(g, \psi, m) \leq \bar{\ell}$*

We now derive a feasibility constraint on issuance of m .

Lemma 7. *Fix g and ψ such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$. Then m is feasible if $m \in [0, \bar{m}(g, \psi)]$, where*

$$\bar{m}(g, \psi) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g) \quad (\text{B.12})$$

Moreover, at $m = \bar{m}(g, \psi)$ we have $\hat{\ell}(g, \psi, m) = \bar{\ell}$.

Proof. Since ψ and g are such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$ then, given choices of g and ψ there exists $m > 0$ that satisfies

$$p(\hat{\ell}) \left(\hat{\ell}(1+g) - \ell \right) = \psi q(g) + m \quad (\text{B.13})$$

By applying the implicit function theorem to the above equation we see that

$$\frac{\partial \hat{\ell}}{\partial m} = \frac{1}{p'(\hat{\ell}) \left(\hat{\ell}(1+g) - \ell \right) + p(\hat{\ell})(1+g)} > 0, \quad (\text{B.14})$$

where the inequality follows from Lemma 5. It follows that at the highest feasible m , which we denote by $\bar{m}(g, \psi)$, we have $\hat{\ell} = \bar{\ell}$. Setting $\hat{\ell} = \bar{\ell}$ in (B.13) and rearranging, we obtain

$$\bar{m}(g, \psi) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g)$$

\square

With the above results in hand, we now prove Propositions 4 and 5 stated in Section 3.

B.2 Proof of Proposition 4 ($\kappa = 0$)

Proof of Proposition 4. (Part 1) Equity financing implies that $\psi = 0$ so that the post-investment leverage $\hat{\ell}$ is given by

$$\hat{\ell} = \frac{\ell}{1+g}$$

Thus, the equity holders' problem simplifies to

$$\max_g \frac{1+g}{r-\mu} - p\left(\frac{\ell}{1+g}\right)\ell - q(g)$$

The first-order condition associated with the above problem is given by

$$\frac{1}{r-\mu} + p'\left(\frac{\ell}{1+g}\right)\left(\frac{\ell}{1+g}\right)^2 - q'(g) = 0 \quad (\text{B.15})$$

Recall that g^u denotes the first-best investment (see Definition 1) and let g_e^* denote the equity holders' optimal investment when they finance their investment only with equity. Then, (28) and (B.15) imply that

$$\frac{1}{r-\mu} - q'(g^u) = 0 = \frac{1}{r-\mu} + p'\left(\frac{\ell}{1+g_e^*}\right)\left(\frac{\ell}{1+g_e^*}\right)^2 - q'(g_e^*) < \frac{1}{r-\mu} - q'(g_e^*), \quad (\text{B.16})$$

where the inequality follows from the observation that $p'(\ell) < 0$ for all ℓ . Since the cost function q is strictly increasing in g , (B.16) implies that $g_e^* < g^u$. This establishes the first part of the proposition.

(Part 2) We now allow the equity holders to choose their financing of investment optimally so that $\psi \in [0, 1]$. In this case, the equity holders' problem is given by

$$\max_{\substack{g, \ell \geq 0 \\ \psi \in [0, 1]}} \frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) \quad (\text{B.17})$$

$$\text{s.t. } p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) \quad (\text{B.18})$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \quad (\text{B.19})$$

where $m = 0$ since $\kappa = 0$. Recall that $\hat{\ell}(g, \psi, m)$ denotes the level of post-investment leverage implied by equity holders choices via the budget constraint. Since $m = 0$ in what follows we slightly abuse notation and write $\hat{\ell}(g, \psi)$ instead to $\hat{\ell}(g, \psi, 0)$.

It is easy to see that, as long as $\ell < \bar{\ell}$ the equity holders will never choose g, ψ such that $\hat{\ell}(g, \psi) = \bar{\ell}$. This is because when $\hat{\ell}(g, \psi) = \bar{\ell}$ then equity holders' post-investment value of equity is 0 (and the equity holders immediately default) while $\ell < \bar{\ell}$ implies that the pre-investment value of equity is strictly positive. It follows that the constraint (B.19) (which, as shown in Corollary 4 is equivalent to the constraint $\hat{\ell} \leq \bar{\ell}$) is not binding. Hence, the equity holders' problem can be written as

$$\max_{\substack{g \geq 0 \\ \psi \in [0, 1]}} \frac{1+g}{r-\mu} - p\left(\hat{\ell}(g, \psi)\right)\ell - q(g), \quad (\text{B.20})$$

subject to $\psi \in [0, 1]$, where $\hat{\ell}(g, \psi)$ is implicitly defined by (B.18). Note that the first-order derivative of equity holders' objective function (B.20) w.r.t. ψ is given by

$$-p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial \psi} > 0 \quad (\text{B.21})$$

since $p'(\hat{\ell}) < 0$ and $\partial \hat{\ell} / \partial \psi$ (obtained by applying the implicit function theorem to (B.18)) is given

$$\frac{\partial \hat{\ell}}{\partial \psi} = \frac{q(g)}{p'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + p(\hat{\ell})(1+g)} > 0 \quad (\text{B.22})$$

It follows that equity holders find it optimal to finance all of their investment with debt, that is, $\psi^* = 1$.

Consider next the optimal choice of g . Given the convexity of the cost function and the fact that cash flows increase linearly in g it follows that the optimal choice of g , which we denote by g^* , is interior. That is, g^* is a solution to the following first-order condition

$$\frac{1}{r - \mu} - p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial g} \Big|_{\substack{\psi=1 \\ g=g^*}} - q'(g^*) = 0, \quad (\text{B.23})$$

where

$$\frac{\partial \hat{\ell}}{\partial g} = - \frac{p(\hat{\ell})\hat{\ell} - \psi q'(g)}{p'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + p(\hat{\ell})(1+g)} \quad (\text{B.24})$$

We now argue that $\partial \hat{\ell} / \partial g$ evaluated at $g = g^*, \psi = \psi^* = 1$ is strictly positive. To see this, note that (B.23) implies that

$$-q'(g^*) = -\frac{1}{r - \mu} + p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial g} \Big|_{\substack{\psi=1 \\ g=g^*}}$$

Using the above expression in (B.24) evaluated at $g = g^*, \psi = 1$ yields

$$\frac{\partial \hat{\ell}}{\partial g} \Big|_{\substack{\psi=1 \\ g=g^*}} = - \frac{p(\hat{\ell})\hat{\ell} - \frac{1}{r-\mu} + p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial g} \Big|_{\substack{\psi=1 \\ g=g^*}}}{p'(\hat{\ell})(\hat{\ell}(1+g^*) - \ell) + p(\hat{\ell})(1+g^*)}$$

Rearranging the above equation, we obtain

$$\frac{\partial \hat{\ell}}{\partial g} \Big|_{\substack{\psi=1 \\ g=g^*}} \left[1 + \frac{p'(\hat{\ell})\ell}{p'(\hat{\ell})(\hat{\ell}(1+g^*) - \ell) + p(\hat{\ell})(1+g^*)} \right] = \frac{-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu}}{p'(\hat{\ell})(\hat{\ell}(1+g^*) - \ell) + p(\hat{\ell})(1+g^*)} \quad (\text{B.25})$$

From Lemma 2 and Lemma 6 we know that $-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu} \geq 0$ with a strict inequality if $\hat{\ell} < \bar{\ell}$. However, as we argued above, choosing $\hat{\ell} = \bar{\ell}$ is not optimal. Thus $-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu} > 0$. Next, note that by Lemma 5 the denominator on the RHS of (B.25) is strictly positive. Thus, it follows that the RHS of (B.25) is strictly positive.

Next, we simplify the expression in the square brackets at the LHS of (B.25) to obtain

$$\left[\frac{(1 + g^*) (p(\hat{\ell}) - p'(\hat{\ell})\hat{\ell})}{p'(\hat{\ell})(\hat{\ell}(1 + g^*) - \ell) + p(\hat{\ell})(1 + g^*)} \right] \quad (\text{B.26})$$

Lemmas 5 and 6 imply that the above expression is strictly positive. Therefore, we conclude that

$$\left. \frac{\partial \hat{\ell}}{\partial g} \right|_{\substack{\psi=1 \\ g=g^*}} > 0 \quad (\text{B.27})$$

Having established that $\partial \hat{\ell} / \partial g|_{\{g=g^*, \psi=1\}} > 0$ we consider again (B.23). Since, $p'(\hat{\ell}) < 0$ we have

$$0 = \frac{1}{r - \mu} - q'(g^*) - p'(\hat{\ell}) \ell \left. \frac{\partial \hat{\ell}}{\partial g} \right|_{\substack{\psi=1 \\ g=g^*}} > \frac{1}{r - \mu} - q'(g^*) \quad (\text{B.28})$$

Now, recall that the first-best investment satisfies

$$\frac{1}{r - \mu} - q'(g^u) = 0 \quad (\text{B.29})$$

Since the cost function q is strictly increasing, (B.28) and (B.29) imply that $g^* > g^u$. \square

B.3 Proof of Proposition 5 ($\kappa > 0$)

Before we prove Proposition 5, we establish an important intermediate result.

Lemma 8. *The equity holders' choose to issue as much dividend as they can. That is, given g and ψ such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$, the equity holders choose*

$$m^* = \min\{\kappa, \bar{m}(g, \psi)\}, \quad (\text{B.30})$$

where $\bar{m}(g, \psi)$ is defined in (B.12).

Proof. Recall that, when $\theta = 0$, the equity holders' objective function is given by

$$\frac{1 + g}{r - \mu} - p(\hat{\ell})\ell - q(g), \quad (\text{B.31})$$

where $\hat{\ell} = \hat{\ell}(g, \psi, m)$ (see Proposition 3).³⁰ Thus, the first-derivative of equity holders' objective function w.r.t. m is given by

$$-p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial m} > 0$$

since $p'(\hat{\ell}) < 0$ and $\partial \hat{\ell} / \partial m > 0$ (see the proof of Lemma 7). Thus, it follows that

$$m^* = \min\{\kappa, \bar{m}(g, \psi)\} \quad (\text{B.32})$$

\square

³⁰As explained in Appendix B.1 we can treat post-investment leverage as a function of $\{g, \psi, m\}$ defined implicitly by equity holders' budget constraint.

Lemma 8 tells us that the equity holders' problem can be simplified to

$$\max_{\substack{g \geq 0 \\ \psi \in [0,1]}} \frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) \quad (\text{B.33})$$

$$\text{s.t. } p(\hat{\ell}) (\hat{\ell}(1+g) - \ell) = \psi q(g) + m^* \quad (\text{B.34})$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \quad (\text{B.35})$$

$$m^* = \min\{\kappa, \bar{m}(g, \psi)\} \quad (\text{B.36})$$

In other words, we can think of equity holders' problem as choosing first g and ψ and then setting m to the highest value they can choose.

Proof of Proposition 5. The proof consists of four parts. First, we characterize the solution when $\kappa = \infty$. We refer to this solution as the “unconstrained” solution. Next, we determine $\bar{\kappa}$ such that if $\kappa \geq \bar{\kappa}$ then the unconstrained solution is attainable. Thus, for all $\kappa \geq \bar{\kappa}$ the equity payout constraint, $m \leq \kappa$, is not binding. We then show that there exists $\underline{\kappa}$ with $\underline{\kappa} < \bar{\kappa}$ such that if $\kappa \in [\underline{\kappa}, \bar{\kappa})$ then the equity holders can still attain the same payoff as in the case of $\kappa = \infty$ but with an additional restriction on their financing choices. Finally, we determine the equity holders' choices when $\kappa < \underline{\kappa}$.

When $\kappa = \infty$ then $m^* = \bar{m}(g, \psi)$ (see Lemma 8). Then the first-order derivative of equity holders' objective function w.r.t. ψ is given by

$$-p'(\hat{\ell}) \left[\frac{\partial \hat{\ell}}{\partial \psi} + \frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^*}{\partial \psi} \right] \quad (\text{B.37})$$

Since $m^* = \bar{m}(g, \psi)$, we have that

$$\frac{\partial \bar{m}(\psi, g)}{\partial \psi} = -q(g) \quad (\text{B.38})$$

Moreover,

$$\frac{\partial \hat{\ell}}{\partial \psi} = \frac{q(g)}{p(\hat{\ell})(1+g) + p'(\hat{\ell})(\hat{\ell}(1+g) - \ell)} \quad (\text{B.39})$$

$$\frac{\partial \hat{\ell}}{\partial m} = \frac{1}{p(\hat{\ell})(1+g) + p'(\hat{\ell})(\hat{\ell}(1+g) - \ell)} \quad (\text{B.40})$$

Therefore, it follows that

$$p'(\hat{\ell}) \left[\frac{\partial \hat{\ell}}{\partial \psi} + \frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^*}{\partial \psi} \right] = 0 \quad (\text{B.41})$$

That is, when equity holders' equity payout choices are unconstrained they are indifferent between any $\psi \in [0, 1]$.

Fix $\psi^* \in [0, 1]$. (As we showed equity holders are indifferent between any choice of $\psi \in [0, 1]$). Consider now the first-order condition that determines the optimal investment, g^* , and which is given by

$$\frac{1}{r-\mu} + p'(\hat{\ell}) \ell \left[\frac{\partial \hat{\ell}}{\partial g} \Big|_{\substack{\psi=\psi^* \\ g=g^*}} + \frac{\partial \hat{\ell}}{\partial m} \Big|_{\substack{\psi=\psi^* \\ g=g^*}} \frac{\partial m^*}{\partial g} \Big|_{\substack{\psi=\psi^* \\ g=g^*}} \right] - q'(g^*) = 0 \quad (\text{B.42})$$

Since $m^* = \bar{m}(g, \psi)$, it follows that

$$\frac{\partial m^*}{\partial g} = \frac{1}{r - \mu} - \psi q'(g) \quad (\text{B.43})$$

Moreover, since $m^* = \bar{m}(g, \psi)$ we know that $\hat{\ell} = \bar{\ell}$ (see Lemma 7). Therefore,

$$\frac{\partial \hat{\ell}}{\partial g} \Big|_{\psi=\psi^*, g=g^*} + \frac{\partial \hat{\ell}}{\partial m} \Big|_{\psi=\psi^*, g=g^*} \frac{\partial m^*}{\partial g} \Big|_{\psi=\psi^*, g=g^*} = \frac{-p(\bar{\ell})\bar{\ell} + \psi^* q'(g^*) + \frac{1}{r-\mu} - \psi^* q'(g^*)}{p(\bar{\ell})(1+g^*) + p'(\bar{\ell})(\bar{\ell}(1+g^*) - \ell)} = 0, \quad (\text{B.44})$$

where the last equality follows from the fact that $p(\bar{\ell})\bar{\ell} = 1/(r - \mu)$ (see Lemma 6). Thus, the first-order condition determining g^* simplifies to

$$\frac{1}{r - \mu} - q'(g^*) = 0 \quad (\text{B.45})$$

implying that $g^* = g^u$. Thus, we conclude that if $\kappa = \infty$ then the solution to the equity holders' problem is given by $g^* = g^u$, $\psi^* \in [0, 1]$, and $m^* = \bar{m}(g^u, \psi^*)$. We refer to this solution as the “unconstrained solution.”

Next, we investigate for which κ the above unconstrained solution is feasible. This is the case if $\kappa \geq \bar{m}(g^u, \psi)$ for all $\psi \in [0, 1]$. Note that $\bar{m}(g^u, \psi)$ is decreasing in ψ . Therefore, if we define $\bar{\kappa} \equiv \bar{m}(g^u, 0)$ then for all $\kappa \geq \bar{\kappa}$ the “unconstrained solution” is attainable.

Before proceeding further, note for all $\kappa \geq \bar{\kappa}$ equity holders' payoff is given by

$$v^*(\ell) = \frac{1 + g^u}{r - \mu} - \frac{\eta}{1 + \eta} \ell - q(g^u) \quad (\text{B.46})$$

(B.46) shows us that the equity holders capture all the value of new investment and extract as much as possible from the old debt holders given the constraint that $\hat{\ell} \leq \bar{\ell}$. Moreover, note that the post-investment value of equity is independent of m , ψ , and $\hat{\ell}$.

Next, consider a situation where $\kappa < \bar{m}(g^u, 0)$ but $\kappa \geq \bar{m}(g^u, 1)$. In this case, for some choices of ψ the unconstrained solution characterized above is not feasible. However, the equity holders can still attain the payoff defined in (B.46) by choosing $g^* = g^u$, $\psi^* \in [\underline{\psi}_\kappa, 1]$, $m^* = \bar{m}(g^*, \psi^*)$, where $\underline{\psi}_\kappa$ is the unique solution to

$$\kappa = \bar{m}(g^u, \underline{\psi}_\kappa) \quad (\text{B.47})$$

Therefore, if we define $\underline{\kappa} \equiv \bar{m}(g^u, 1)$ then the above discussion implies that for all $\kappa \geq \underline{\kappa}$ the equity holders invest the first-best amount. Finally, note that from the definition of $\bar{m}(g^u, 1)$ we have

$$\frac{\partial \underline{\kappa}}{\partial \ell} < 0, \quad \frac{\partial \underline{\kappa}}{\partial \sigma^2} > 0, \quad \frac{\partial \underline{\kappa}}{\partial r} < 0 \quad (\text{B.48})$$

It remains to determine the equity holders' choices when $0 < \kappa < \underline{\kappa}$ (the case of $\kappa = 0$ is covered in Proposition 4). We first argue, by contradiction, that in this case the equity holders' optimal choices $\{g^*, \psi^*, m^*\}$ are such that $m^* = \kappa < \bar{m}(g, \psi)$. To see this assume, to the contrary, that $\{g^*, \psi^*, m^*\}$ are such that $m^* = \bar{m}(g^*, \psi^*) < \kappa$. Then, from Lemma 7 we know that $\hat{\ell}(g^*, \psi^*, m^*) = \bar{\ell}$, which implies that the equity holders' payoff is given by

$$\frac{1 + g^*}{r - \mu} - \frac{\eta}{1 + \eta} \ell - q(g^*) \quad (\text{B.49})$$

Note that $\frac{1+g}{r-\mu} - q(g)$ is strictly increasing in g for all $g < g^u$ and strictly decreasing in g for all $g > g^u$ and recall that since $\kappa < \underline{\kappa}$ it must be the case that $g^* \neq g^u$. Furthermore, note that if $g^* < g^u$ then the equity holders would have incentives to increase their investment from g^* to $g^* + \varepsilon$ for small $\varepsilon > 0$. This is feasible by setting

$$m' = \frac{1 + (g^* + \varepsilon)}{r - \mu} - \frac{\eta}{1 + \eta} \ell - \psi^* q(g^* + \varepsilon) \quad (\text{B.50})$$

as long as ε is small enough so that $m' \leq \kappa$. Hence $g < g^u$ cannot be optimal. By a similar argument, if $g^* > g^u$ then the equity holders would find it optimal and feasible to decrease their investment. Thus, we conclude that $\{g^*, \psi^*, m^*\}$ such that $m^* = \bar{m}(g^*, \psi^*) < \kappa$ are not optimal.

Next, suppose that $\{g^*, \psi^*, m^*\}$ are such that $m^* = \bar{m}(g^*, \psi^*) = \kappa$. In this case, the budget constraint implies that

$$\kappa = \frac{1 + g^*}{r - \mu} - \frac{\eta}{1 + \eta} \ell - \psi^* q(g^*) \quad (\text{B.51})$$

while the equity holders' payoff is given by

$$\frac{1 + g^*}{r - \mu} - \frac{\eta}{1 + \eta} \ell - q(g^*) \quad (\text{B.52})$$

Using (B.51) in (B.52) we see that the equity holders' payoff can be expressed as

$$\kappa - (1 - \psi^*) q(g^*) \quad (\text{B.53})$$

We now argue that the equity holders can attain a strictly higher payoff than the payoff in (B.53) by choosing $\psi' = 1$, $m' = \kappa$, and an investment g' such that g' solves the budget constraint

$$p(\hat{\ell}) (\hat{\ell}(1 + g') - \ell) = \kappa + q(g') \quad (\text{B.54})$$

with $\hat{\ell}(g', \psi', \kappa) < \bar{\ell}$. If equity holders make such choices then their payoff would be given by

$$\frac{1 + g'}{r - \mu} - p(\hat{\ell}) \ell - q(g') = \frac{1 + g'}{r - \mu} - p(\hat{\ell}) \hat{\ell}(1 + g') + \kappa > \kappa, \quad (\text{B.55})$$

where the first equality follows from (B.54), while the final inequality follows from the observation that $p(\hat{\ell}) \hat{\ell} < \frac{1+g}{r-\mu}$ for all $\hat{\ell} < \bar{\ell}$ (see Lemma 2 and Lemma 6). Therefore, we conclude that choice of $\{g^*, \psi^*, m^*\}$ such that $m^* = \bar{m}(g^*, \psi^*) = \kappa$ cannot be optimal. It follows that we must have $m^* = \kappa < \bar{m}(g^*, \psi^*)$.

We argued above that if $\kappa \in (0, \underline{\kappa})$ then $m^* = \kappa < \bar{m}(g^*, \psi^*)$. Therefore, we have

$$\left. \frac{\partial m^*}{\partial \psi} \right|_{g=g^*, \psi=\psi^*} = 0 \quad \text{and} \quad \left. \frac{\partial m^*}{\partial g} \right|_{g=g^*, \psi=\psi^*} = 0 \quad (\text{B.56})$$

This implies that, when $\kappa < \underline{\kappa}$, the first-order conditions that determine equity holders' choices of g and ψ are identical to those when $\kappa = 0$. Thus, using the same argument as in the proof of Proposition 4 we conclude that $g^* > g^u$ and $\psi^* = 1$. This concludes the proof. \square

Appendix C Proofs for Section 4

In this section of the appendix, we provide proofs of the results stated in Section 4. Throughout this section we assume that $\kappa = 0$.

C.1 Bankruptcy costs

In this section, we prove Proposition 6. We start with a number of preliminary results mostly of technical nature. A reader may wish to skip these intermediate claims and go straight to the proof of Proposition 6.

Lemma 9. *For all $\theta \in [0, 1]$ $p(\hat{\ell})((1+g)\hat{\ell} - \ell)$ is a strictly concave function of $\hat{\ell}$ and achieves its maximum for some $\hat{\ell} > \frac{\ell}{1+g}$.*

Proof. Differentiating twice $p(\hat{\ell})((1+g)\hat{\ell} - \ell)$ w.r.t. $\hat{\ell}$ we obtain

$$p''(\hat{\ell}) \left(\hat{\ell}(1+g) - \ell \right) + 2p'(\hat{\ell})(1+g) = -\eta(1+\theta\eta)\chi\hat{\ell}^{\eta-2} \left[\hat{\ell}(1+g) - \frac{\eta-1}{\eta+1}\ell \right] < 0$$

since $\eta, \theta, \chi > 0$ and $\left(\hat{\ell}(1+g) - \ell \right) \geq 0$ establishing strict concavity. Next, note that the first derivative of $p(\hat{\ell})((1+g)\hat{\ell} - \ell)$ w.r.t. $\hat{\ell}$ evaluated at $\hat{\ell}(1+g) = \ell$ is positive, which implies that the maximum must occur for some $\hat{\ell} > \frac{\ell}{1+g}$. \square

Next, we note that when $\theta > 0$ it is possible that investment g is consistent with two different levels of leverage $\hat{\ell}$. This is because when $\theta > 0$ then the LHS of the budget constraint is a single-peaked concave function. The next results, shows that if $\hat{\ell}_1$ and $\hat{\ell}_2$ with $\hat{\ell}_1 < \hat{\ell}_2$ both satisfy the budget constraint then the equity holders strictly prefer the smaller leverage.

Lemma 10. *Fix $g > 0$ and suppose that both $\hat{\ell}_1$ and $\hat{\ell}_2$ both satisfy the budget constraint and that $\hat{\ell}_1 < \hat{\ell}_2$. Then equity holders strictly prefer $\hat{\ell}_1$ to $\hat{\ell}_2$.*

Proof. This claim follows immediately from the fact the payoff to equity holders (as shown in Proposition 3) is given by

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\hat{\ell}(1+g) - H(\hat{\ell})\hat{\ell}(1+g),$$

which is strictly decreasing in $\hat{\ell}$ holding g constant. \square

Corollary 11. *The post-investment $\hat{\ell}$ satisfies*

$$p'(\hat{\ell})((1+g)\hat{\ell} - \ell) + p(\hat{\ell})(1+g) \geq 0 \tag{C.1}$$

Proof. The equity holders would never choose $\hat{\ell}$ such that $p'(\hat{\ell})((1+g)\hat{\ell} - \ell) + p(\hat{\ell})(1+g) < 0$ since then there exists $\hat{\ell}' < \hat{\ell}$, which also satisfies the equity holders' budget constraint and which is preferred by the equity holders (Lemma 10). Such $\hat{\ell}'$ exists since the LHS of the budget constraint is concave and increasing for small values of $\hat{\ell}$ (see Lemma 9 and the discussion that follows). \square

The next claim establishes that for small θ equity holders it is feasible for equity holders to finance the first-best investment g^u by debt.

Lemma 12. *There exists $\underline{\theta}_u$ such that g^u is strictly feasible under debt financing for all $\theta \leq \underline{\theta}_u$.*

Proof. Since the firm is solvent before investment, from definition of g^u , and convexity of q , we have

$$0 < \frac{1}{r - \mu} - p(\ell)\ell - H(\ell)\ell < \frac{1 + g^u}{r - \mu} - q(g^u) - p(\ell)\ell - H(\ell)\ell < \frac{1 + g^u}{r - \mu} - q(g^u) - \frac{\eta}{1 + \eta}\ell, \quad (\text{C.2})$$

Next, let \bar{g} be the value of g at which post-investment leverage is given by $\hat{\ell} = \bar{\ell}$. Therefore, \bar{g} satisfies

$$(1 - \theta) \left(\frac{1 + \bar{g}}{r - \mu} - \frac{\eta}{1 + \eta}\ell \right) - q(\bar{g}) = 0 \quad (\text{C.3})$$

(C.2) and (C.3) imply that there exists $\underline{\theta}_u > 0$ such that for all $\theta \in [0, \underline{\theta}_u]$ we have $\bar{g} > g^u$. \square

In what follows we will assume that θ is such that g^u is feasible under debt financing.

Lemma 13. *Suppose that investment is financed fully with debt. Then $\partial \hat{\ell} / \partial g \geq (>) 0$ if and only if $g \geq (>) g_0$, where g_0 is the unique solution to*

$$p(\hat{\ell})\hat{\ell} - q'(g) = 0 \quad (\text{C.4})$$

Proof. Recall that

$$\frac{\partial \hat{\ell}}{\partial g} = - \frac{p(\hat{\ell})\hat{\ell} - q'(g)}{p'(\hat{\ell}) (\hat{\ell}(1 + g) - \ell) + p(\hat{\ell})(1 + g)} \quad (\text{C.5})$$

From Corollary 11 we know that the denominator in (C.5) is positive. Therefore, $\partial \hat{\ell} / \partial g \geq 0$ if and only if

$$p(\hat{\ell})\hat{\ell} - q'(g) \leq 0 \quad (\text{C.6})$$

Note that at $g = 0$ the LHS of (C.6) is positive implying that for small values of g we have $\partial \hat{\ell} / \partial g < 0$. Similarly, for sufficiently high g the LHS of (C.6) is negative since $\lim_{g \rightarrow \infty} q'(g) = \infty$. Finally, let g_0 be a solution to (C.4). Then at $g = g_0$ we have

$$\frac{\partial}{\partial g} [p(\hat{\ell})\hat{\ell} - q'(g)] = [p'(\hat{\ell})\hat{\ell} + p(\hat{\ell})] \frac{\partial \hat{\ell}}{\partial g} - q''(g) = -q''(g) < 0 \quad (\text{C.7})$$

implying that g_0 is unique. \square

With these preliminary results, we can now prove Proposition 6.

Proof of Proposition 6. (Part 1): From Proposition 3 it is immediate that the first-order condition that determines optimal investment is given by

$$\frac{1}{r - \mu} - q'(g) - \partial_{\ell} p(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell - \left[(1 + g) H'(\hat{\ell}) \hat{\ell} \frac{\partial \hat{\ell}}{\partial g} + (1 + g) H(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} + H(\hat{\ell}) \hat{\ell} \right] = 0 \quad (\text{C.8})$$

If investment is financed with equity (that is, when $\psi = 0$) then $\hat{\ell} = \ell/(1 + g)$ and so $\partial \hat{\ell} / \partial g = -\ell/(1 + g)^2$. Using this observation in (C.8), we conclude that optimal investment under equity financing, which we denote by g_e^* , has to satisfy

$$\frac{1}{r - \mu} - q'(g_e^*) - H(\hat{\ell}) \hat{\ell} + \left[\partial_{\ell} p(\hat{\ell}) \ell + (1 + g_e^*) H'(\hat{\ell}) \hat{\ell} + (1 + g_e^*) H(\hat{\ell}) \right] \frac{\ell}{(1 + g_e^*)^2} = 0, \quad (\text{C.9})$$

where $\hat{\ell}$ in the above equation is the leverage implied by the equity holders' choices, and is given by $\hat{\ell} = \ell/(1 + g_e^*)$. Using this observation, we obtain

$$\partial_{\ell} p(\hat{\ell}) \ell + (1 + g_e^*) H'(\hat{\ell}) \hat{\ell} + (1 + g_e^*) H(\hat{\ell}) = -\eta \chi \frac{1}{1 + \eta} \hat{\ell}^{\eta} (1 + g_e^*) (1 - \theta) < 0$$

Therefore, we conclude that at $g = g_e^*$ we have

$$-H(\hat{\ell}) \hat{\ell} + \left[\partial_{\ell} p(\hat{\ell}) \ell + (1 + g_e^*) H'(\hat{\ell}) \hat{\ell} + (1 + g_e^*) H(\hat{\ell}) \right] \frac{\ell}{(1 + g_e^*)^2} < 0 \quad (\text{C.10})$$

It follows that

$$\frac{1}{r - \mu} - q'(g_e^*) > 0 = \frac{1}{r - \mu} - q'(g^u), \quad (\text{C.11})$$

where we used the characterization of g^u established in (28). Since q is strictly increasing it follows that $g_e^* < g^u$.

(Part 2): We establish this results in three steps. (1) We show that for a sufficiently small θ equity holders find it optimal to finance their investment fully with debt. (2) We next show that under debt financing at the optimal investment we have $\partial \hat{\ell} / \partial g > 0$ and provide a lower bound for this derivative. In light of Lemma 13 this implies that $g^* > g_0$. (3) Finally, we show that for any $\ell > 0$ there exists small enough θ such that for all $g \in [g_0, g^u]$, under debt financing, equity holders' marginal benefit from investing always exceed their marginal benefit when $\ell = 0$, which implies that $g^* > g^u$ for sufficiently low θ .³¹

The derivative of equity holders' objective function (derived in Proposition 3) w.r.t. ψ is given by

$$\left[-\partial_{\ell} p(\hat{\ell}) \ell - (1 + g) H'(\hat{\ell}) \hat{\ell} - (1 + g) H(\hat{\ell}) \right] \frac{\partial \hat{\ell}}{\partial \psi}, \quad (\text{C.12})$$

³¹Recall that equity holders invest first-best amount when $\ell = 0$.

where $\partial \hat{\ell} / \partial \psi > 0$ by the analogous argument to the one used in the proof of Proposition 4. Moreover,

$$-\partial_{\ell} p(\hat{\ell}) \ell - (1+g)H'(\hat{\ell})\hat{\ell} - (1+g)H(\hat{\ell}) = \eta \frac{1+\theta\eta}{1+\eta} \chi \hat{\ell}^{\eta-1} \ell - \eta \theta \chi \hat{\ell}^{\eta-1} \hat{\ell} (1+g) \quad (\text{C.13})$$

First, we note that $\psi = 0$ is never optimal. This is because $\psi = 0$ implies that $\hat{\ell}(1+g) = \ell$ and so

$$\eta \frac{1+\theta\eta}{1+\eta} \chi \hat{\ell}^{\eta-1} \ell - \eta \theta \chi \hat{\ell}^{\eta-1} \hat{\ell} (1+g) = \frac{\eta(1-\theta)}{1+\eta} \chi \hat{\ell}^{\eta} > 0 \quad (\text{C.14})$$

Therefore, equity holders always finance their investment at least partially with debt (and this also means that g_e^* is not optimal). Next, note that for all $g \in (0, g^u)$ we have

$$\frac{1+\theta\eta}{1+\eta} \chi \hat{\ell}^{\eta-1} \ell - \eta \theta \chi \hat{\ell}^{\eta-1} \hat{\ell} (1+g) > \chi \hat{\ell}^{\eta-1} \left[\frac{1}{1+\eta} \ell - \theta \eta \bar{\ell} (1+g^u) \right], \quad (\text{C.15})$$

where $\bar{\ell}$ is the maximum leverage the firm can have (see Corollary 4). The expression in square brackets is strictly decreasing in θ and positive at $\theta = 0$. It follows that there exists $\underline{\theta}_1(\ell) > 0$ such that for all $\theta \in [0, \underline{\theta}_1(\ell)]$ we have

$$\frac{1+\theta\eta}{1+\eta} \chi \hat{\ell}^{\eta-1} \ell - \eta \theta \chi \hat{\ell}^{\eta-1} \hat{\ell} (1+g) > 0 \quad (\text{C.16})$$

Therefore, we conclude that for all $\theta \in [0, \underline{\theta}_1(\ell)]$ the equity holders would finance any $g \in [0, g^u]$ fully with debt. Next, suppose that $\theta \in [0, \underline{\theta}_1(\ell)]$ and that the optimal investment g^* (which can be larger than g^u) is financed partially with equity, that is, $\psi^* \in (0, 1)$. Then,

$$\left[\partial_{\ell} p(\hat{\ell}) \ell + (1+g)H'(\hat{\ell})\hat{\ell} + (1+g)H(\hat{\ell}) \right] = 0 \quad (\text{C.17})$$

When (C.17) holds, the F.O.C. (C.8) simplifies to

$$\frac{1}{r-\mu} - q'(g) - H(\hat{\ell})\hat{\ell} = 0, \quad (\text{C.18})$$

which implies that $g^* < g^u$ as $H(\hat{\ell})\hat{\ell} > 0$. But this is a contradiction since we showed that if $\theta \in [0, \underline{\theta}_1(\ell)]$ then for all $g \in [0, g^u]$ optimal financing is $\psi = 1$. Thus, we conclude that optimal investment has to be financed fully with debt. This concludes the first part of our argument.

Above we established that $\theta \in [0, \underline{\theta}_1(\ell)]$ the optimal investment has to be fully financed with debt. Therefore, we can follow the same steps as in the proof of Proposition 4 to show that

$$\frac{\partial \hat{\ell}}{\partial g} = \frac{\frac{1}{r-\mu} - p(\hat{\ell})\hat{\ell} - H(\hat{\ell})\hat{\ell}}{p^U(\hat{\ell})(1+g)} > 0, \quad (\text{C.19})$$

which implies that $g^* > g_0$ (where g_0 is defined in Lemma 13). If $g^u < g_0$ then we are done (i.e., we conclude that equity holders overinvest). Thus, in what follows we assume that $g^u > g_0$.

Denote by $\hat{\ell}(g, \theta)$ the post-investment leverage when equity holders invest g (and finance their investment with debt) and bankruptcy costs are θ . Then, under debt financing, for all $g \in [g_0, g^u]$ we have

$$\hat{\ell}(g_0, 0) < \hat{\ell}(g, \theta) < \hat{\ell}(g^u, \theta) \quad (\text{C.20})$$

Therefore,

$$\frac{\partial \hat{\ell}}{\partial g} \geq \frac{\frac{1}{r-\mu} - p(\hat{\ell}(g, \theta)) \hat{\ell}(g, \theta) - H(\hat{\ell}(g, \theta)) \hat{\ell}(g, \theta)}{p^U(\hat{\ell}(g_0, 0))(1 + g^u)} \equiv \Delta_{\hat{\ell}}(\theta) > 0, \quad (\text{C.21})$$

where the last inequality follows from the observation that for all $\hat{\ell} < \bar{\ell}$ we have

$$p(\hat{\ell})\hat{\ell} + H(\hat{\ell})\hat{\ell} < \frac{1}{r - \mu}$$

Observe that $\Delta_{\hat{\ell}}(\theta)$ is decreasing in θ . Next, consider (C.8), the F.O.C. for optimal investment g . We want to show that for small enough θ if $g \in [g_0, g^u]$ then

$$-\partial_{\ell} p(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell - \left[(1 + g) H'(\hat{\ell}) \hat{\ell} \frac{\partial \hat{\ell}}{\partial g} + (1 + g) H(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} + H(\hat{\ell}) \hat{\ell} \right] > 0 \quad (\text{C.22})$$

We note that

$$H(\hat{\ell}) \ell \leq \frac{\theta \eta}{1 + \eta} \bar{\ell} \quad (\text{C.23})$$

Therefore, for all $\theta \in [0, \underline{\theta}_1(\ell)]$ and all $g \in [0, g^u]$ we have

$$\begin{aligned} & -\partial_{\ell} p(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell - \left[(1 + g) H'(\hat{\ell}) \hat{\ell} \frac{\partial \hat{\ell}}{\partial g} + (1 + g) H(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} + H(\hat{\ell}) \hat{\ell} \right] \\ & > \frac{\partial \hat{\ell}}{\partial g} \left[-\partial_{\ell} p(\hat{\ell}) - (1 + g) H'(\hat{\ell}) \hat{\ell} - (1 + g) H(\hat{\ell}) \right] - \frac{\theta \eta}{1 + \eta} \bar{\ell} \\ & > \Delta_{\hat{\ell}}(\theta) \left[\frac{\eta}{1 + \eta} \chi^{\ell^n} - \theta \eta \hat{\ell}(g^u, \theta) \left((1 + g^u) \hat{\ell}(g^u, \theta) - \frac{\eta}{1 + \eta} \ell \right) \right] - \frac{\theta \eta}{1 + \eta} \bar{\ell}, \end{aligned} \quad (\text{C.24})$$

We note that both $\Delta_{\hat{\ell}}(\theta)$ and

$$\left[\frac{\eta}{1 + \eta} \chi^{\ell^n} - \theta \eta \hat{\ell}(g, \theta) \left((1 + g^u) \hat{\ell}(g^u, \theta) - \frac{\eta}{1 + \eta} \ell \right) \right]$$

are strictly decreasing in θ . Moreover,

$$\lim_{\theta \rightarrow 0} \left\{ \Delta_{\hat{\ell}}(\theta) \left[\frac{\eta}{1 + \eta} \chi^{\ell^n} - \theta \eta \hat{\ell}(g, \theta) \left((1 + g^u) \hat{\ell}(g^u, \theta) - \frac{\eta}{1 + \eta} \ell \right) \right] - \frac{\theta \eta}{1 + \eta} \bar{\ell} \right\} = \Delta_{\hat{\ell}}(0) \frac{\eta}{1 + \eta} \chi^{\ell^n} > 0 \quad (\text{C.25})$$

It follows that there exists $\underline{\theta}_2(\ell) > 0$ such that for all $\theta \in [0, \underline{\theta}_2(\ell)]$ and all $g \in [g_0, g^u]$ inequality (C.22) holds. As a consequence, for all $\theta \in [0, \underline{\theta}_2(\ell)]$ and all $g \in [g_0, g^u]$ we have

$$\frac{1}{r - \mu} - q'(g) - \partial_{\ell} p(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} \ell - \left[(1 + g) H'(\hat{\ell}) \hat{\ell} \frac{\partial \hat{\ell}}{\partial g} + (1 + g) H(\hat{\ell}) \frac{\partial \hat{\ell}}{\partial g} + H(\hat{\ell}) \hat{\ell} \right] > \frac{1}{r - \mu} - q'(g) \quad (\text{C.26})$$

Since g^u satisfies $\frac{1}{r - \mu} - q(g^u) = 0$ and q is increasing, we conclude that equity holders optimal investment exceeds g^u if $\theta \leq \min\{\underline{\theta}_1, \underline{\theta}_2\}$, which establishes the claim. \square

C.2 Unsecured debt and decoupled seniority

In this section, we provide a detailed analysis of a version of our model where the equity holders can finance their investment with unsecured debt and claims to firm's liquidation value (bankruptcy claims). Let B denote the total number of outstanding claims in default at $t = 0$ and L , as before, denote the present value of promised coupon payments at $t = 0$. We refer to this version of the model as the model with a “decoupled debt structure.”

C.2.1 Preliminary Claims and Pricing of Bankruptcy Claims

Proposition 10. *The post-investment value of equity and equity holders' default decision are the same as in the benchmark model.*

Proof. Note that both in the model with decoupled seniority and in the benchmark model L corresponds to the PDV of the promised coupon payments. Moreover, since the equity holders receive nothing in default, they do not care (once debt has been issued) who holds claims to the liquidation value of the firm. It follows that the equity holders' default problem is unchanged. \square

Corollary 14. *After investment took place, equity holders never find it optimal to repurchase claims to liquidation value.*

Proof. This follows from the fact that post-investment value of equity, given by $v(\ell)Z$ is independent of B . \square

The next proposition establishes the price of bankruptcy claims. Recall that each of outstanding bankruptcy claims has claim to one unit of the liquidation value of the firm, $V((1 - \theta)\underline{Z}(L), 0)$. If the number of claims exceeds the liquidation value (i.e. $B > V((1 - \theta)\underline{Z}(L), 0)$) then the liquidation value of the firm is divided proportionally among holders of bankruptcy claims. Note that unlike the benchmark model, debt holders that receive coupon payments do not have any claims to $V((1 - \theta)\underline{Z}(L), 0)$ (they hold unsecured debt).

Proposition 11. *Let $\ell \equiv L/Z$ and $b \equiv B/Z$. The price of a single bankruptcy claim given $\{\ell, b\}$ is*

$$p^B(\ell, b) = \min \left\{ 1, \frac{\eta}{1 + \eta} \frac{\ell}{b} \right\} \chi^{\ell^\eta} \quad (\text{C.27})$$

Proof. The value of a single bankruptcy claim is equal to the discounted payoff in default, that is

$$\mathbb{E}_T \left[e^{-rT} \min \left\{ 1, \frac{V((1 - \theta)\underline{Z}(L), 0)}{B} \right\} \right], \quad (\text{C.28})$$

where T denotes the time that equity holders default (i.e., the first time cash flows reach \underline{Z}) and expectations are taken over possible values of T . Recall that $V((1 - \theta)\underline{Z}(L), 0) = \frac{\eta}{1 + \eta} L$ and that $\mathbb{E}_T [e^{-rT}] = \chi \left(\frac{L}{Z} \right)^\eta$ (see (A.18)). Combining these observations, we conclude that the price of a single bankruptcy claim can be expressed as a function of ℓ and b and is given by

$$p^B(\ell, b) = \min \left\{ 1, \frac{\eta}{1 + \eta} \frac{\ell}{b} \right\} \chi^{\ell^\eta} \quad (\text{C.29})$$

\square

We assume that at the time of issuance the number of bankruptcy claims has to be lower than a multiple Λ of the firm's liquidation value (see (33) in Section 4.2). We make this assumption because otherwise, as we show below, the equity holders would like to issue infinite amount of such claims.³² It follows that the equity holders are able to finance part of their investment by issuing new claims in default only if

$$B \leq \Lambda \frac{\eta}{1+\eta} \hat{L} \quad (\text{C.30})$$

where $\frac{\eta}{1+\eta} \hat{L}$ is the post-investment liquidation value of the firm.

Let \hat{B} denote post-investment claims in default issued by the equity holders. Then since the number of outstanding claims has to be non-negative, \hat{B} has to satisfy

$$-B \leq \hat{B} - B \leq \max \left\{ \Lambda \frac{\eta}{1+\eta} \hat{L} - B, 0 \right\} \quad (\text{C.31})$$

Note that the equity holders have capacity to issue new liquidation claims only if $\Lambda \frac{\eta}{1+\eta} \hat{L} - B > 0$. Dividing both sides of (C.31) by Z and defining $\hat{b} \equiv \hat{B}/\hat{Z}$, we obtain

$$-b \leq \hat{b}(1+g) - b \leq \max \left\{ \Lambda \frac{\eta}{1+\eta} \hat{\ell}(1+g) - b, 0 \right\}, \quad (\text{C.32})$$

which is the constrained on issuance of liquidation claims faced by the equity holders at the time of investment.

Since in the model with decouple seniority the equity holders can finance their investment with either unsecured debt (i.e., defaultable consols with no claims in default) or liquidation claims (collateral claims) their budget constraint is given by

$$p^U(\hat{\ell})(\hat{\ell}(1+g) - \ell) + p^B(\hat{\ell}, \hat{b})(\hat{b}(1+g) - b) = \psi q(g), \quad (\text{C.33})$$

where $\hat{b}(1+g) - b$ has to satisfy constraint (C.32) and $p^B(\hat{\ell}, \hat{b})$ is defined in Proposition 11.

We now state equity holders' investment problem.

$$\max_{g, \hat{\ell}, \hat{b}, \psi} \frac{1+g}{r-\mu} - p(\hat{\ell})\hat{\ell} - (1-\psi)q(g) \quad (\text{C.34})$$

s.t.

$$p^U(\hat{\ell})(\hat{\ell}(1+g) - \ell) + p^B(\hat{\ell}, \hat{b})(\hat{b}(1+g) - b) = \psi q(g) \quad (\text{C.35})$$

$$-b \leq \hat{b}(1+g) - b \leq \max \left\{ \Lambda \frac{\eta}{1+\eta} \hat{\ell}(1+g) - b, 0 \right\} \quad (\text{C.36})$$

Compared to the benchmark model, here, the equity holders' problem can finance their investment either with unsecured defaultable consols or claims in default. However, equity holders' capacity to issue additional debt is limited by the constraint (C.36).

Next, we prove an important result, which implies that the equity holders always issue as many claims in default as possible when financing their investment.

³²However, we do not require the equity holders to repurchase these any of the claims in default if, for whatever reason, at some point *after* liquidation claims have been issued, B exceeds multiple Λ of firm's liquidation value. If that happens then the equity holders will simply lack the ability to issue new bankruptcy claims till either firm's liquidation value increases or the number of outstanding claims in default decreases so (33) is again satisfied.

Lemma 15. *Given b , the equity holders always choose \hat{b} such that*

$$\hat{b}(1+g) - b = \max\left\{\Lambda \frac{\eta}{1+\eta} \hat{\ell}(1+g) - b, 0\right\} \quad (\text{C.37})$$

Proof. Consider the equity holders' problem described above. Substituting the budget constraint (C.35) into the equity holders' objective function (C.34) we obtain

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\hat{\ell} - q(g) + p^U(\hat{\ell})(\hat{\ell}(1+g) - \ell) + p^B(\hat{\ell}, \hat{b})(\hat{b}(1+g) - b) \quad (\text{C.38})$$

Since the objective function is strictly increasing in \hat{b} it follows that the equity holders would always choose the highest feasible value of \hat{b} . \square

Intuitively, all else equal, issuing additional claims in default (if feasible) is beneficial since the proceeds from issuance of claims in default can be used to decrease usage of funds from other sources (either equity or unsecured debt) while equity holders would have to surrender all the cash flows following default, anyway.

Remark 1. *Note that Lemma 15 implies that in the absence of any constraints on issuance of liquidation claims the equity holders would choose $b \rightarrow \infty$. This observation is the reason why we impose constraint (33) in Section 4.2 (which implies constraint (C.30)).*

C.2.2 Simplifying Assumption and Its Consequences

To simplify the subsequent analysis, we make the following natural assumption.

Assumption 1. *We assume that the initial B satisfy*

$$B = \Lambda V((1-\theta)\underline{Z}(L), 0)L \quad (\text{C.39})$$

The above assumption states that, in the past, the equity holders issues the maximum amount of claims they could. In light of Lemma 15 this is a natural assumption in the sense that if $B < \Lambda V((1-\theta)\underline{Z}(L), 0)L$ then the equity holders would have strict incentives to issue additional claims in default if they were allowed to adjust their balance sheet. As shown in the next Lemma, this assumption substantially simplifies the analysis.

Lemma 16. *Suppose that at the time of investment $B = \Lambda V((1-\theta)\underline{Z}(L), 0)$. Then, equity holders' choices of \hat{b} , $\hat{\ell}$ and g are such that*

$$\hat{b}(1+g) - b = \Lambda \frac{\eta}{1+\eta} (\hat{\ell}(1+g) - \ell) \quad (\text{C.40})$$

implying that

$$\hat{b} = \Lambda \frac{\eta}{1+\eta} \hat{\ell} \quad (\text{C.41})$$

Proof. Under Assumption 1 we have

$$\Lambda \frac{\eta}{1+\eta} \hat{\ell}(1+g) - b = \Lambda \frac{\eta}{1+\eta} \hat{\ell}(1+g) - \Lambda \frac{\eta}{1+\eta} \ell = \Lambda \frac{\eta}{1+\eta} (\hat{\ell}(1+g) - \ell) \geq 0 \quad (\text{C.42})$$

Using this in (C.37) leads to desired result. \square

With the above result in hand, we can simplify the equity holders' budget constraint.

Lemma 17. *Under Assumption 1 equity holders' budget constraint can be written as*

$$p(\hat{\ell})(\hat{\ell}(1+g) - \ell) + d(\hat{\ell}) (\hat{\ell}(1+g) - \ell) = \psi q(g), \quad (\text{C.43})$$

where

$$d(\hat{\ell}) \equiv \min\{0, \Lambda - 1\} p^B(\hat{\ell}, \hat{\ell}) \quad (\text{C.44})$$

Proof. From Proposition 2 we know that when $\theta = 0$

$$\begin{aligned} p^U(\hat{\ell}) &= 1 - \chi \ell^\eta \\ p(\hat{\ell}) &= 1 - \frac{1}{1+\eta} \chi \ell^\eta \end{aligned}$$

while Proposition 2 implies that

$$p^B(\hat{\ell}, \hat{\ell}) = \frac{\eta}{1+\eta} \chi \ell^\eta$$

Therefore, we see that

$$p^U(\hat{\ell}) = p(\hat{\ell}) - p^B(\hat{\ell}, \hat{\ell})$$

Using this observation in equity holders' budget constraint (C.35) we obtain

$$(p(\hat{\ell}) - p^B(\hat{\ell}, \hat{\ell})) (\hat{\ell}(1+g) - \ell) + p^B(\hat{\ell}, \hat{b}) (\hat{b}(1+g) - b) = \psi q(g) \quad (\text{C.45})$$

From Lemma 16 we know that

$$\hat{b}(1+g) - b = \Lambda \frac{\eta}{1+\eta} (\hat{\ell}(1+g) - \ell)$$

and

$$p(\hat{\ell}, \hat{b}) = \min \left\{ 1, \frac{1}{\Lambda} \right\} \chi \hat{\ell}^\eta$$

Using these two observations in (C.45) we obtain

$$(p(\hat{\ell}) - p^B(\hat{\ell}, \hat{\ell})) (\hat{\ell}(1+g) - \ell) + \min\{\Lambda, 1\} \frac{\eta}{1+\eta} \chi \hat{\ell}^\eta (\hat{\ell}(1+g) - \ell) = \psi q(g) \quad (\text{C.46})$$

Recognizing that $\frac{\eta}{1+\eta} \chi \hat{\ell}^\eta = p^B(\hat{\ell}, \hat{\ell})$ and rearranging the above equation we obtain

$$p(\hat{\ell}) (\hat{\ell}(1+g) - \ell) + \min\{\Lambda - 1, 0\} p^B(\hat{\ell}, \hat{\ell}) (\hat{\ell}(1+g) - \ell) = \psi q(g) \quad (\text{C.47})$$

Finally, we define $d(\hat{\ell})$ as

$$d(\hat{\ell}) \equiv \min\{\Lambda - 1, 0\} p^B(\hat{\ell}, \hat{\ell}) \quad (\text{C.48})$$

to obtain desired result. \square

Lemma 17 implies that equity holders' problem can be simplified as follows.

Proposition 12. *Equity holders' investment problem can be written as*

$$v^*(\ell, b) = \max_{\substack{g \geq 0 \\ \psi \in [0,1]}} \frac{1+g}{r-\mu} - q(g) - p(\hat{\ell})\ell + d(\hat{\ell})(\hat{\ell}(1+g) - \ell) \quad (\text{C.49})$$

where $\hat{\ell}$ is determined explicitly by the budget constraint

$$p(\hat{\ell})(\hat{\ell}(1+g) - \ell) + d(\hat{\ell})(\hat{\ell}(1+g) - \ell) = \psi q(g) \quad (\text{C.50})$$

Proof. The equity holders' objective function is given by

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\hat{\ell}(1+g) - (1-\psi)q(g) \quad (\text{C.51})$$

Substitute (C.50) into the above objective function to obtain

$$\frac{1+g}{r-\mu} - q(g) - p(\hat{\ell})\hat{\ell}(1+g) + p(\hat{\ell})(\hat{\ell}(1+g) - \ell) + d(\hat{\ell})(\hat{\ell}(1+g) - \ell) \quad (\text{C.52})$$

Simplifying the above equation we obtain the desired result. Finally, note that the constraint (C.36) is subsumed by $d(\hat{\ell})$ in the budget constraint. \square

Proposition 12 tells us that compared to the benchmark model equity holders face now an additional distortion, as captured by $d(\hat{\ell})(\hat{\ell}(1+g) - \ell)$, which arises due to equity holders' limited ability to issue bankruptcy claims. In particular, when $\Lambda < 1$, the equity holders can sell liquidation claims only to a fraction of the liquidation value of their firm. This distorts equity holders choice of financing relative to the benchmark model. In contrast, if $\Lambda \geq 1$ then $d(\hat{\ell}, g) = 0$ so this distortion is not present and equity holders get fully compensated when issuing new liquidation claims for all the cash flows they give up following default. Hence, their investment decision is the same as in the benchmark model.

C.2.3 Main Result

In this section we prove the main result of Section 4.2 in the paper. Throughout this section we maintain Assumption 1. We begin by establishing two intermediate claims.

Lemma 18. *We have $\partial \hat{\ell} / \partial \psi \geq 0$ with a strict inequality whenever $g > 0$.*

Proof. By applying the implicit function theorem to the equity holders' budget constraint we obtain

$$\frac{\partial \hat{\ell}}{\partial \psi} = \frac{q(g)}{(p'(\hat{\ell}) + d'(\hat{\ell}))(\hat{\ell}(1+g) - \ell) + (p(\hat{\ell}) + d(\hat{\ell}))(1+g)} \geq 0 \quad (\text{C.53})$$

The numerator in (C.53) is always non-negative and strictly positive whenever $g > 0$. That the denominator in (C.53) is positive follows by an analogous argument to the one used to establish Corollary 11. \square

Lemma 19. *Suppose $\Lambda \in [0, 1)$. Then the sufficient condition for equity holders to under-invest (i.e., for $g^* < g^u$) is that $\psi^* \in [0, 1)$.*

Proof. Suppose that equity holders' optimal financing choice is $\psi^* \in (0, 1)$. Note that the first derivative of equity holders' objective function with respect to ψ is given by

$$\left[-p'(\hat{\ell})\ell + d'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + d(\hat{\ell})(1+g) \right] \frac{\partial \hat{\ell}}{\partial \psi}, \quad (\text{C.54})$$

where, by Lemma 18, we have $\partial \hat{\ell} / \partial \psi > 0$ (since equity holders always choose positive investment as $q'(g) = 0$). Thus, $\psi^* \in (0, 1)$ if and only if

$$\left[-p'(\hat{\ell})\ell + d'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + d(\hat{\ell})(1+g) \right] = 0 \quad (\text{C.55})$$

Next, note that the equity holders' optimal choice of investment satisfies

$$\frac{1}{r - \mu} - q'(g) + d(\hat{\ell})\hat{\ell} + \left[-p'(\hat{\ell})\ell + d'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + d(\hat{\ell})(1+g) \right] \frac{\partial \hat{\ell}}{\partial g} = 0, \quad (\text{C.56})$$

From (C.55), we know that if $\psi^* \in (0, 1)$ then the term in square brackets is zero and hence, (C.56) simplifies to

$$\frac{1}{r - \mu} - q'(g) + d(\hat{\ell})\hat{\ell} = 0 \quad (\text{C.57})$$

Since $d(\hat{\ell})\hat{\ell} < 0$ for all $\Lambda \in [0, 1)$ we conclude that $g^* < g^u$.

Next, consider the case $\psi^* = 0$. In this case $\hat{\ell} = \ell / (1 + g)$ implying that $\partial \hat{\ell} / \partial g = -\ell / (1 + g)^2 < 0$. Thus, in this case (C.56) simplifies to

$$\frac{1}{r - \mu} - q'(g) + d(\hat{\ell})\hat{\ell} + \left[-p'(\hat{\ell})\ell + d(\hat{\ell})(1+g) \right] \frac{-\ell}{(1+g)^2} \quad (\text{C.58})$$

Observe that

$$d(\hat{\ell})\hat{\ell} - d(\hat{\ell})(1+g) \frac{\ell}{(1+g)^2} = d(\hat{\ell})\hat{\ell} - d(\hat{\ell}) \frac{\ell}{(1+g)} = 0 \quad (\text{C.59})$$

It follows that (C.56) simplifies too

$$\frac{1}{r - \mu} - q'(g) + p'(\hat{\ell})\ell \frac{\ell}{(1+g)^2} = 0 \quad (\text{C.60})$$

Since $p'(\hat{\ell}) < 0$ it follows that if $\psi = 0$ then $g^* < g^u$. □

We are now ready to prove Proposition 7.

Proof of Proposition 7. Part (1): Suppose that $\Lambda \in [0, 1)$. From Lemma 19 we know that it is sufficient to show that for small values of Λ the equity holders never find it optimal to finance investment $g \geq g^u$ fully with debt. That is, it is enough to show that for all $g \geq g^u$ and $\psi = 1$ we have

$$-p'(\hat{\ell})\ell + d'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + d(\hat{\ell})(1+g) \leq 0 \quad (\text{C.61})$$

if Λ is sufficiently small.

Suppose that the equity holders find it optimal to finance investment $g \geq g^u$ fully with debt. Using the definition of $p(\hat{\ell})$ and $d(\hat{\ell})$ the LHS of (C.61) can be written as

$$\frac{\eta}{1+\eta} \chi^{\hat{\ell}^\eta} \left[\Lambda \ell + (1+\eta)(\Lambda-1)(\hat{\ell}(1+g) - \ell) \right], \quad (\text{C.62})$$

where $\frac{\eta}{1+\eta} \chi^{\hat{\ell}^\eta} > 0$. Note that $\hat{\ell}(1+g) = \hat{L}/Z$ and, thus, $\hat{\ell}(1+g)$ is increasing in g . Let $\hat{\ell}^u$ be the post-investment leverage associated with investment of g^u . Then, if $\psi = 1$ and $g > g^u$ we have

$$\left[\Lambda \ell + (1+\eta)(\Lambda-1)(\hat{\ell}(1+g) - \ell) \right] < \left[\Lambda \ell + (1+\eta)(\Lambda-1)(\hat{\ell}^u(1+g^u) - \ell) \right] \quad (\text{C.63})$$

Note that

$$\lim_{\Lambda \nearrow 1} \left[\Lambda \ell + (1+\eta)(\Lambda-1)(\hat{\ell}^u(1+g^u) - \ell) \right] = \Lambda \ell > 0 \quad (\text{C.64})$$

$$\lim_{\Lambda \searrow 0} \left[\Lambda \ell + (1+\eta)(\Lambda-1)(\hat{\ell}^u(1+g^u) - \ell) \right] = -(1+\eta)(\hat{\ell}^u(1+g^u) - \ell) < 0 \quad (\text{C.65})$$

Applying the implicit function theorem to (C.50) it is easy to show that $\hat{\ell}$ is decreasing in Λ . Therefore, it follows that there exists a unique $\underline{\Lambda}(\ell) > 0$ such that

$$\left[\Lambda \ell + (1+\eta)(\Lambda-1)(\hat{\ell}^u(1+g^u) - \ell) \right] < 0 \quad (\text{C.66})$$

if $\Lambda \in [0, \underline{\Lambda}(\ell)]$. Then, (C.63) implies that for all $\Lambda \in [0, \underline{\Lambda}(\ell)]$, if $\psi = 1$ and $g \geq g^u$ then

$$\left[\Lambda \ell + (1+\eta)(\Lambda-1)(\hat{\ell}(1+g) - \ell) \right] < 0 \quad (\text{C.67})$$

It follows that for all $\Lambda \in [0, \underline{\Lambda}(\ell)]$ the equity holders never find it optimal to overinvest.

Part (2): To establish the second part of the proposition simply note that when $\Lambda \geq 1$ then $d(\hat{\ell}) = 0$ and, hence, the equity holders' face the same problem as in the benchmark model with $\kappa = 0$ and $\theta = 0$. Therefore, Proposition 4 that the equity holders overinvest for all $\Lambda \geq 1$. Setting $\bar{\Lambda} = 1$ establishes the desired results. \square

C.3 Seniority

Assume $\theta = 0$ for simplicity. Recall that given the equity holders' optimal default decision, the post-investment default value of the firm is given by (13). Hence, upon default $\frac{\eta}{1+\eta} \hat{L}$ is divided between senior and junior debt holders with junior debt holders being paid only after senior debt holders.

Assuming that the existing debt holders are senior to new debt holders, the price of a newly issued junior debt is given by

$$P^J(\hat{\ell}, \ell) = \frac{1}{r} \left[1 - \chi^{\hat{\ell}^\eta} \right] + \frac{1}{r} \frac{\max \left\{ 0, \frac{\eta}{1+\eta} \hat{\ell}(1+g) - \ell \right\}}{\hat{\ell}(1+g) - \ell} \chi^{\hat{\ell}^\eta}, \quad (\text{C.68})$$

where $\hat{\ell}(1+g) - \ell$ is the number of new junior bonds issued (normalized by Z). Next, let $p^J(\hat{\ell}, \ell) = rP^J(\hat{\ell}, \ell)$. If equity holders can finance their investment only with junior debt then their budget constraint becomes

$$p^J(\hat{\ell}, \ell) \left(\hat{\ell}(1+g) - \ell \right) = \psi q(g) \quad (\text{C.69})$$

With the above preliminaries we now prove Proposition 8.

Proof of Proposition 8. As explained in Section 4.3, the equity holders' problem is now given by

$$\max_{g, \hat{\ell}, \psi} \frac{1+g}{r-\mu} - p(\hat{\ell})\hat{\ell}(1+g) - (1-\psi)q(g) \quad (\text{C.70})$$

$$\text{s.t. } p^J(\hat{\ell}) (\hat{\ell}(1+g) - \ell) = \psi q(g) \quad (\text{C.71})$$

Substituting (C.71) in (C.70) we see that the equity holders' objective function becomes

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\hat{\ell}(1+g) - q(g) + p^J(\hat{\ell})(\hat{\ell}(1+g) - \ell), \quad (\text{C.72})$$

where

$$p^J(\ell) = \begin{cases} p^U(\hat{\ell}) & \text{if } \frac{\eta}{1+\eta}\hat{\ell}(1+g) \leq \ell \\ p^U(\hat{\ell}) + \frac{\frac{\eta}{1+\eta}\hat{\ell}(1+g) - \ell}{\hat{\ell}(1+g) - \ell} \chi \hat{\ell}^\eta & \text{if } \frac{\eta}{1+\eta}\hat{\ell}(1+g) > \ell \end{cases} \quad (\text{C.73})$$

and p^U is defined in Proposition 2.

We now argue that

$$\frac{\eta}{1+\eta}\hat{\ell}(1+g) - \ell > 0 \quad (\text{C.74})$$

cannot be optimal. For suppose that equity holders chose g and $\hat{\ell}$ such that (C.74) is satisfied. Then,

$$p^J(\hat{\ell}) (\hat{\ell}(1+g) - \ell) = p^U(\hat{\ell}) (\hat{\ell}(1+g) - \ell) + \left[\frac{\eta}{1+\eta}\hat{\ell}(1+g) - \ell \right] \chi \hat{\ell}^\eta \quad (\text{C.75})$$

Recognizing that $p^U(\hat{\ell}) + p^B(\hat{\ell}) = p(\hat{\ell})$ (see Proposition 2) and recalling that $p^B(\hat{\ell}) = \frac{\eta}{1+\eta}\chi \hat{\ell}^\eta$, we see that

$$p^J(\hat{\ell})(\hat{\ell}(1+g) - \ell) = p(\hat{\ell}) (\hat{\ell}(1+g) - \ell) - \frac{1}{1+\eta}\chi \hat{\ell}^\eta \ell \quad (\text{C.76})$$

Using (C.76) in (C.72) we see that the equity holders' objective function simplifies in this case to

$$\frac{1+g}{r-\mu} - \ell - q(g) \quad (\text{C.77})$$

However, if equity holders financed the same investment g with their own funds then their payoff would be

$$\frac{1+g}{r-\mu} - p\left(\frac{\ell}{1+g}\right) \ell - q(g) > \frac{1+g}{r-\mu} - \ell - q(g) \quad (\text{C.78})$$

since $p(\ell/(1+g)) < 1$ for all $g \geq 0$. It follows that the equity holders would never choose $\{g, \hat{\ell}\}$ s.t. (C.74) holds.

Next, note that if $\frac{\eta}{1+\eta}\hat{\ell}(1+g) - \ell \leq 0$ then from (C.73) we know that $p^J(\hat{\ell}) = p^U(\hat{\ell})$. But then the equity holders' problem is equivalent to their problem in the case of decoupled seniority structure with $\Lambda = 0$ in which case Corollary 1 implies that the equity holders underinvest. This completes the proof. \square

Appendix D Repeated Investment Derivations

This section derives the ODEs for a repeated investment decisions, which introduces a controlled jump-process.

Assume that upon an arrival of an investment opportunity, the state jumps to a deterministic function of the current state, $\hat{\ell}(\ell)$. Define the jump size as $\tilde{g}(\ell) \equiv \hat{\ell}(\ell) - \ell$. Then the SDE for ℓ is

$$d\ell_t = (\sigma^2 - \mu)\ell_t dt + \sigma\ell_t d\mathbb{W}_t + (\hat{\ell}(\ell) - \ell) d\mathbb{N}_t \quad (\text{D.1})$$

where \mathbb{N}_t is homogeneous Poisson process with arrival rate $\lambda \geq 0$.

Firm's HJBE First, we will derive the HJBE in ℓ -space without the jumps, and add them.

$$V(Z, L) = Zv(L/Z) \quad (\text{D.2})$$

And differentiate with respect to Z

$$\partial_Z V(Z, L) = v(L/Z) - \frac{L}{Z} \partial_\ell v(L/Z) = v(\ell) - \ell \partial_\ell v(\ell) \quad (\text{D.3})$$

$$\partial_{ZZ} V(Z, L) = \frac{L^2}{Z^3} \partial_{\ell\ell} v(L/Z) = \frac{1}{Z} \ell^2 \partial_{\ell\ell} v(\ell) \quad (\text{D.4})$$

Use the ODE in (5)

$$rV(Z, L) = Z - rL + \mu Z \partial_Z V(Z, L) + \frac{\sigma^2}{2} Z^2 \partial_{ZZ} V(Z, L) \quad (\text{D.5})$$

Divide by Z and use the definition of $v(\ell)$,

$$rv(\ell) = 1 - r\ell + \mu \partial_Z V(Z, L) + \frac{\sigma^2}{2} Z \partial_{ZZ} V(Z, L) \quad (\text{D.6})$$

Now substitute from the derivatives above

$$(r - \mu)v(\ell) = 1 - r\ell - \mu\ell \partial_\ell v(\ell) + \frac{\sigma^2}{2} \ell^2 \partial_{\ell\ell} v(\ell) \quad (\text{D.7})$$

Default Decision The notation denotes $\cdot|_\ell$ as the evaluation of a function at ℓ .

For the firm's boundary conditions, add in artificial reflecting barriers at some ℓ_{\min} and ℓ_{\max} , which will be the sizes of the grids. We will ensure that the equilibrium $\ell_{\min} < \bar{\ell}$ so it is never binding in the solution, the ℓ_{\max} will be chosen large enough to not effect the solution. With these, the boundary conditions are

Then, for the firm, we can write the DVI for their stopping problem as

$$u(c) \equiv 1 - r\ell \quad (\text{D.8})$$

$$\mathcal{L}_v \equiv r - \mu + \mu\ell \partial_\ell - \frac{\sigma^2}{2} \ell^2 \partial_{\ell\ell} - \lambda \left(\cdot|_{\ell+\tilde{g}(\ell)} - \cdot|_\ell \right) \quad (\text{D.9})$$

$$0 = \min\{\mathcal{L}_\ell v(\ell) - u(\ell), v(\ell)\} \quad (\text{D.10})$$

$$\partial_\ell v(\ell_{\min}) = 0 \quad (\text{D.11})$$

$$\partial_\ell v(\ell_{\max}) = 0 \quad (\text{D.12})$$

We would numerically find a $\bar{\ell}$ which fulfills the indifference point, and then find the value of liquidation per unit of PV of liabilities is

$$v^{\text{liq}} \equiv (1 - \theta) \frac{\lim_{\ell \rightarrow 0} v(\ell)}{\bar{\ell}} \quad (\text{D.13})$$

Bond Pricing The price of a bond, $P(Z, L)$ pays 1 unit until default. The ODE in the continuation region without jumps is

$$rP(Z, L) = 1 + \mu Z \partial_Z P(Z, L) + \frac{\sigma^2}{2} Z^2 \partial_{ZZ} P(Z, L) \quad (\text{D.14})$$

Take the definition

$$rP(Z, L) \equiv p(L/Z) \quad (\text{D.15})$$

And differentiate with respect to Z

$$r \partial_Z P(Z, L) = -\frac{1}{Z} \ell \partial_\ell p(\ell) \quad (\text{D.16})$$

$$r \partial_{ZZ} P(Z, L) = \frac{1}{Z^2} \left(2\ell \partial_\ell p(\ell) + \ell^2 \partial_{\ell\ell} p(\ell) \right) \quad (\text{D.17})$$

Multiply by r and substitute the derivatives into (D.14)

$$rp(\ell) = r + (\sigma^2 - \mu) \ell \partial_\ell p(\ell) + \frac{\sigma^2}{2} \ell^2 \partial_{\ell\ell} p(\ell) \quad (\text{D.18})$$

In default, the bond is entitled to a share of rL units of the liquidation value $V((1 - \theta)\underline{Z}(L), 0)$, hence $P(\bar{\ell}) = \frac{V((1-\theta)\underline{Z}(L), 0)}{rL}$. Divide by r and use the definitions of v^{liq} and $p = P/r$ to find that the boundary condition is $p(\bar{\ell}) = v^{\text{liq}}$.

Summarizing, bond pricers take v^{liq} and $\bar{\ell}$ as given, and then solve

$$\mathcal{L}_p \equiv r - (\sigma^2 - \mu) \ell \partial_\ell - \frac{\sigma^2}{2} \ell^2 \partial_{\ell\ell} - \lambda \left(\cdot|_{\ell+\bar{g}(\ell)} - \cdot|_\ell \right) \quad (\text{D.19})$$

$$\mathcal{L}_p p(\ell) = r \quad (\text{D.20})$$

$$\partial_\ell p(\ell_{\min}) = 0 \quad (\text{D.21})$$

$$p(\bar{\ell}) = v^{\text{liq}} \quad (\text{D.22})$$

where the lower boundary is an artificial reflecting barrier and the upper boundary is the liquidation absorbing barrier.

Investment Finally, the objective function of the firm at every arrival point λ remains to maximize the equity value. Given an equilibrium $p(\ell)$ and $v(\ell)$ functions—consistent with the optimal jump process, the agent solves

First-Best The first-best is derived through a guess-and-verify approach. First, guess that the user would choose a constant g due to the homotheticity of the problem. With that, the unnormalized Bellman equation (with jumps) is

$$rV(Z) = Z + \mu ZV'(Z) + \frac{\sigma^2}{2} Z^2 V''(Z) + \lambda \max_g \left\{ V((1+g)Z) - V(Z) - \zeta \frac{g^2}{2} \right\} \quad (\text{D.23})$$

Take the first-order condition

$$\zeta g Z = ZV'((1+g)Z) \quad (\text{D.24})$$

Guess the solution to the problem is $V(Z) = AZ$ for an undetermined Z , and substitute into the (D.23) and solve for A to find,

$$A = \frac{1 - \frac{1}{2}\zeta g^2 \lambda}{-g\lambda - \mu + r} \quad (\text{D.25})$$

Similarly, substitute the guess into (D.24) to find

$$g = \frac{A}{\zeta} \quad (\text{D.26})$$

Substitute for A from (D.26) into (D.25), solve the quadratic for g , and choose the positive root to find,

$$g^u = \frac{1}{\zeta(r - \mu) \left(\frac{1}{2} \left(\sqrt{1 - \frac{2\lambda}{\zeta(r - \mu)^2}} - 1 \right) + 1 \right)} = \frac{2}{\sqrt{\zeta(\zeta(r - \mu)^2 - 2\lambda)} + \zeta(r - \mu)} \quad (\text{D.27})$$

In addition, from the $V(Z) = AZ$ guess, and noting that $V(Z, 0)$ is the first-best if unconstrained in using equity investment

$$v(0) = \frac{V(Z, 0)}{Z} = A \quad (\text{D.28})$$

Consequently, for a default threshold $\bar{\ell}$, the liquidation value per unit of defaultable console in (D.13) is,

$$v^{\text{liq}} = \frac{1 - \theta}{\bar{\ell}} \frac{1 - \frac{1}{2}\zeta(g^u)^2 \lambda}{r - \mu - g^u \lambda} \quad (\text{D.29})$$

When $\lambda = 0$, these all nest the $g^u = \frac{1}{\zeta(r - \mu)}$ case.

Appendix E Summary of Data and Calibration

E.1 Calibration

To discipline parameters, we calibrate to moments from the firm dynamics (Pugsley et al. (2020)) and investment spikes (Gourio and Kashyap (2007)) literature.

Discount Rate We base our discount rate of cash flows on long-term real risk-free rate, measured as the 10-year nominal Treasury rate (from FRED) minus 1-year Survey of Professional Forecasters inflation expectations for the GDP deflator. This proxy for the long-term real rate averages 3.55% for the period 1979-2012, matching the sample period in Pugsley et al. (2020). We choose a long-term real rate because bonds in our model are perpetuities and hence extremely long-term, as well as the long-term real rate being closer to the opportunity cost of equity holders.

In addition, since we do not have exit in our model (but empirical investment rates would reflect the exit probability) we add the estimated 4.1% exogenous exit rate estimated in Pugsley et al. (2020) (i.e. a crude residual exit rate as a perpetual youth model not connected to firm fundamentals). Since equity and debt holders would be equally effected by the exogenous death shocks on cash flow claims (e.g. a perpetual youth model), they share the same discount rate.

Adding the exit probability to the baseline real risk-free rate, we use a discount rate of $r = 0.0355 + 0.04 = 0.0765$. Since we have a model of endogenous investment with infinitely lived firms, too low a discount rate would leave the present discounted value of firm cash flows undefined (see footnote 33 for necessary conditions).

First-Best Dynamics Assuming that the first-best, g^u , calculated from (40) exists, the dynamics of the unconstrained $Z(t)$ follow (35).³³ Since g^u is independent of $Z(t)$, the process is a simple jump diffusion process with constant drift, volatility, and jump sizes. Following standard techniques and using Ito's Lemma, the solution to this SDE from $Z(0)$ is

$$\log \left(\frac{Z_t}{Z_0} \right) = \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t + Y_t \quad (\text{E.1})$$

Where W_t is the integral of the Wiener process to time t , and Y_t is a compound jump process given N_t arrivals of the Poisson process with rate λ ,

$$Y_t = \sum_{k=1}^{N_t} \log(1 + g^u) \quad (\text{E.2})$$

By Wald's identity,

$$E[Y_t] = E \left[\sum_{k=1}^{N_t} \log(1 + g^u) \right] = E[N_t] \log(1 + g^u) = \lambda t \log(1 + g^u) \quad (\text{E.3})$$

Calculate the expectation of (E.1) and use (E.3)

$$E \left[\log \left(\frac{Z_t}{Z_0} \right) \right] = \left(\mu - \frac{1}{2} \sigma^2 \right) t + \lambda \log(1 + g^u) t \quad (\text{E.4})$$

Differentiate and then swap the order of differentiation to find the mean drift of the solution

$$\mathbb{E} [d \log Z_t] = \frac{\partial}{\partial t} E \left[\log \left(\frac{Z_t}{Z_0} \right) \right] = \mu - \frac{1}{2} \sigma^2 + \lambda \log(1 + g^u) \quad (\text{E.5})$$

³³A necessary condition for the present discounted value of firms to be defined is $\lambda < \frac{\zeta}{2}(r - \mu)^2$ as implied by (40). Intuitively, if λ is high without a compensating increase in ζ (to make investment more difficult), the mean growth rate of firms is too high relative to the discount rate to converge.

Similarly, we can look at the 2nd moment of (E.1) solution and calculate the mean instantaneous volatility as

$$\mathbb{V}[d \log Z_t] \equiv \mathbb{E}[(d \log Z_t)^2] - \mathbb{E}[d \log Z_t]^2 = \sigma^2 + \lambda \log(1 + g^u)^2 \quad (\text{E.6})$$

Note that in the case of the one-shot model, with $\lambda = 0$, $\mathbb{E}[d \log Z_t] = \mu - \frac{1}{2}\sigma^2$ and $\mathbb{V}[d \log Z_t] = \sigma^2$, the familiar formulas for the GBM process in (1).

Firm Dynamics For disciplining parameters, we will compare moments in the data to the dynamics given the first-best solutions from the calibration. This is done because (1) they are analytically tractable; and (2) we expect them to be fairly close to the version with financial distortions after averaging out across firms (i.e. distortions in the $g \neq g^u$ would partially cancel out for different leverage levels); and (3) the results of the calibration are not not very sensitive to the g^u target within the range of possible systematic bias.³⁴

For targets, we take moments from Pugsley et al. (2020). The authors estimate firm dynamics using covariances from the Longitudinal Business Database (LBD), an administrative panel covering nearly all private employers in the United States from 1976 to 2012. They find that—controlling for ex-ante heterogeneity—the volatility of a random walk using the 1-year growth rate for a balanced panel of LBD firms provides a target of $\sqrt{\mathbb{V}[d \log Z_t]} = 0.1846$.

While the covariance matrices in Pugsley et al. (2020) provides the ideal moments to target for volatility, due to the structure of their data, they cannot provide guidance on the drift. Furthermore, any drift estimates are compounded by the standard problem that firm drift has a bias towards survival, which necessitates a structural model. For example, models of firm dynamics and growth such as Arkolakis (2016) choose the drift to directly target the exit rates, and are usually significantly negative to rationalize exit rates and a stationary firm size distribution.

Since we do not have a model with exit selection, we instead target a backwards drift equal to 0.7% from the average United States TFP growth rate from 1976 to 2012 (ie. $\mathbb{E}[d \log Z_t] = -0.007$). This is in line with structural models with heterogeneous firms and monopolistic competition, where the interpretation is that if a firm has constant technology then its profits will fall backwards at roughly the average growth rate of the economy in relative terms. This number is less than zero—in order to match numbers found in structural heterogeneous firm models and reconcile with existence of a stationary firm-size distribution—but is chosen to be of a conservative magnitude to ensure it does not drive our dynamics.

Investment Arrival Rates and Size For the size and frequency of investments, we connect our large investment jumps to the investment spikes literature and in particular the empirical evidence of Gourio and Kashyap (2007).³⁵ They find that in close to 30% of US plant-years have an investment spike of 12% or more of total assets. In particular, as a proportion of investment relative to assets, 11.6% invest between 0.12 and 0.2, 8% invest between 0.2 and 0.35, and 8.3% invest more than 0.35). For the size of the investment, we take the weighted average of these over the minimum of each bin. i.e. $g^u = (0.12 \times 11.6\% +$

³⁴For example, take (E.5) and assume that the g^u was wrong by 10% due to systematic bias. This would only lead to a distortion of 0.5% since $\lambda \log\left(\frac{1+1.1g^u}{1+g^u}\right) = 0.005$ given our other parameters.

³⁵Table 1 of the NBER working paper version of Gourio and Kashyap (2007).

$0.2 \times 8\% + 0.35 \times 8.3\%)/(11.6\% + 8\% + 8.3\%) \approx 0.21$. Hence, fixing the arrival rate as $\lambda = 0.3$, the target jump-size for calibration is $g^u = 0.21$.

Joint Calibration Given the above, we fix $r = 0.0765$ and $\lambda = 0.3$ and baseline $\theta = 0$, $\kappa = 0$, and $\Lambda = 1$. We then need to jointly choose the three parameters ζ , σ , and μ to match our three target moments for g^u , $\mathbb{E}[d \log Z_t]$, and $\mathbb{V}[d \log Z_t]$. To do this, we use the system of equations (40), (E.5) and (E.6) along with the targets $g^u = 0.21$, $\mathbb{E}[d \log Z_t] = -0.007$ and, $\mathbb{V}[d \log Z_t] = 0.1846^2$. The solution provides our baseline calibration: $\sigma = 0.1534$, $\mu = -0.0514$, and $\zeta = 50.036$.

Guidance on Equity Payout Constraints Data on the total of equity payouts (including dividends, repurchases, etc. as described in Grullon and Michaely (2002)) can provide an upper bound on the conceivable κ constraint. It is an upper bound because not all dividends/equity buybacks in practice are financed through new debt issuance as in our model.

Given our calibrated $\lambda = 0.3$, an upper bound of reasonable κ is approximately $0.5/\lambda = 1.7$ based on Grullon and Michaely (2002)’s equal-weighted ratio of equity payouts divided by earnings, or $0.8/\lambda = 2.7$ for the value-weighted analogue.

Guidance on Interest Coverage Ratios A firm with state (Z, L) in our model has flow profits of Z and a present value of liabilities of L . At interest rate r , they pay a flow of rL in interest. Putting this together, the interest rate coverage ratio is $Z/(rL) = 1/(r\ell)$, where as before $\ell \equiv L/Z$.

Palomino et al. (2019) report an average interest coverage ratio of around 4 for the period 1970-2017. They also find that an interest coverage ratio of 1.5 is associated with a default probability of 3%, that 30% of creditors had an interest coverage ratio of 2 or less, and about 10% of borrowers had an interest coverage ratio of 1 or less. Further, Blume et al. (1998) report that the average interest coverage ratio for BBB-rated (i.e. low investment grade) companies is 4. We therefore consider an interest coverage ratio of 4 to capture an average firm and an interest coverage ratio of 1.5 for a highly-levered firm.

E.2 Firm Data

Quarterly firm accounting data is from Compustat 2000.Q1 to 2018.Q1. We exclude financial firms (SIC codes 6000-6999), utilities (SIC codes 4900-4949), and quasi-government companies (SIC codes 9000-9999). We also drop firms with total assets of less than \$100m or missing, as well as firms with negative or missing sales, negative or missing cash holdings, or firms with less than 20 quarterly observations for the investment rate. If there are multiple observations per quarter for one firm due to a change in the reporting period, we use the latest one. All items are winsorized at the 1% level.

The investment rate, leverage, and interest coverage are computed from Compustat. We compute quarterly CAPX taking into account that capital expenditures are reported cumulated over the year. [Investment rate=CAPX/Lagged ATQ; Leverage=Book value of debt/(Book value of debt+Market value of equity); Book value of debt=DLTTQ+DLCQ; Market value of equities=CSHOQ·PRCCQ]

Table 2: Compustat Summary Statistics

	Capex / total assets	Leverage	log(total assets)	Delta debt	Equity payout
mean	5.46	0.25	6.98	0.54	0.92
sd	6.49	0.25	1.53	5.38	1.87
p25	1.66	0.04	5.77	-0.75	0.00
p50	3.32	0.18	6.75	0.00	0.11
p75	6.50	0.38	7.92	0.52	0.95
min	0.00	0.00	4.61	-14.29	0.00
max	38.23	0.98	13.01	32.10	11.36
No. Obs.	136697	136697	136697	136697	136697

This table shows summary statistics for the Compustat sample measured at quarterly frequency 2000.Q1-2018.Q1. *Capex / total assets* are quarterly capital expenditures divided by previous quarter total assets expressed in annualized percent. *Leverage* is book value of debt divided by the sum of book value of debt and market value of equity. Market value of equity is computed using CRSP share price times CRSP shares outstanding where available, aggregated by GVKEY, and Compustat share price times Compustat shares outstanding otherwise. *log(total assets)* is the natural logarithm of total assets. *Delta debt* is the one quarter change in the book value of debt divided by the 8-quarter moving average of the firm’s total assets in percent. *Equity payout* equals dividends plus equity repurchases (DVY+PRSTKCY) divided by the 8-quarter moving average of the firm’s total assets in percent.

To compare high- versus low leverage firms, we sort firms by their prior-quarter leverage each quarter. We re-do the sort each quarter to ensure that portfolios reflect firms with constant characteristics and to control for time fixed effects that might drive all firms’ leverage. We sort by prior quarter leverage to ensure that the sorting variable is not mechanically affected by the current quarter investment decision. Figure 11 shows the equal-weighted investment rate for firms in the bottom leverage quartile in solid blue and the equal-weighted investment rate for firms in the top leverage quartile in dashed red.

In Section 6.1 we examine the correlation between changes in firm debt and equity payouts in our Compustat data. We measure debt issuance and equity payouts similarly to Begenau and Salomao (2019). Debt issuance equals the change in the book value of debt (long-term debt plus short-term debt in current liabilities), divided by the 8-quarter moving average of total firm assets. Gross equity payout equals dividends plus equity repurchases divided by the 8-quarter moving average of total firm assets. We divide by the moving average of firm assets to ensure that the denominator does not induce cyclicity in the ratios. We use gross, rather than net, equity payouts as a proxy for the m in our model to distinguish from the possibility that the firm can finance new real investment by issuing equity. Figure 10 shows the equal-weighted average debt issuance and gross equity payoff across firms for each quarter. We apply a Hodrick-Prescott filter with smoothing parameter 50 to both of these quarterly times series to smooth out quarterly fluctuations.