Money-Metrics in Applied Welfare Analysis: A Saddlepoint Rehabilitation[*]

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Abstract

Once a popular tool to estimate welfare changes, the money metric of McKenzie (1957) and Samuelson (1974) gradually faded from use after welfare theorists and practioners argued that it led to inegalitarian recommendations. We present a rehabilitation articulated through Uzawa's saddle-point theorem of concave programming. Allowing for non-ordered preferences, we (i) prove that any competitive equilibrium allocation maximizes the money-metric sum at the associated competitive equilibrium price; and (ii) connect the results to Radner's local welfare measure and the behavioral welfare proposals of Bernheim-Rangel (2009). As a foundation for its use in local cost-benefit analysis, we also give conditions for the money-metric to be differentiable in consumption, without either transitivity or convexity of preferences.

Key words and phrases: Money-metric, saddlepoint inequalities, first fundamental welfare theorem, incomplete preferences, intransitive preferences, non-standard decision theories, behavioral welfare economics, benefit function, distance function, cost-benefit in the small

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1 Introduction

The *money-metric* of McKenzie (1957) and Samuelson (1974) gives the smallest income or wealth at given prices that leaves a consumer at least as well off as with a given consumption bundle.[1] In the quarter-century after Samuelson's 1974 essay popularized the money-metric, it was widely estimated, and the estimates used to evaluate the welfare effects of policies, especially taxation.[2] Following the publication of an important paper by Blackorby and Donaldson (1988), practitioners and welfare theorists came to argue that the money-metric sum would tend to produce inegalitarian policy recommendations,[3] and its use gradually waned in applied welfare economics. An exception is Angus Deaton who, while acknowledging the criticisms of welfare theorists, continues to recommend a first-order approximation to the money-metric for welfare analysis.[4]

These anxieties about the money metric were expressed in the setting of the standard neoclassical theory of the consumer, a theory in which choices are at a minimum rationalized by a complete and transitive preference relation, typically one representable by a utility function. Economists and other choice theorists have proposed models that drop either completeness or transitivity, while maintaining that choice is rationalized by a binary relation; and behavioral theorists often drop the idea of rationalizable choice altogether. The first complicates applied welfare analysis since a utility representation, and thereby an (individualistic) social welfare function, is impossible, and the second threatens to leave welfare economics "high and dry" altogether.[5] Undeterred, Bernheim and Rangel (2009) propose a generalization of revealed preference in the context of a choice-based approach to welfare economics for behavioral models. In a sentence, our goal is, first, to spell out some normative implications of using money-metrics for welfare in general, including the in classical setting of the literature in our first paragraph;

[1]Khan and Schlee (2016) review the influence and history of McKenzie's minimum income function.

[2]Prominent examples include King (1983), Varian (1982), Blundell, Preston, and Walker (1994), and Deaton (1980). The first starts with an expenditure function specification, the second famously uses revealed preference methodology, and the last two use *approximations* to the money-metric. Other important papers are Knoblauch (1992) and Blundell, Browning, and Crawford (2003). Varian (2012) explains that his original reason for pursuing revealed preference was to develop a method for estimating Samuelson's money-metric.

[3]Among empirical researchers, those disavowing or expressing reservations about using money metrics for applied welfare because of their possible inegalitarian implications include Diewert and Wales (1988, p. 305), Banks, Blundell, and Lewbel (1996, p. 1232-3), Deaton and Zaidi (2002, p. 11), and Bargain, Decoster, Dolls, Neumann, Peichl, and Siegloch (2013). Among welfare theorists, those expressing reservations include Blundell, Preston, and Walker (1994, p. 38), Slesnick (1998, p. 1241), and Hammond (1994, Section 3.1), Blackorby and Donaldson (1987), and the already-mentioned Blackorby and Donaldson (1988).

[4]Deaton and Zaidi (2002) compare the merits of the money-metric sum and a proposal of Blackorby and Donaldson (1987) but write somewhat defensively that "Our own choice is to stick with money metric utility" (p. 11). They develop aggregate consumption measures for general welfare evaluations based on it; see also Deaton (2003). Two other examples of money-metric applications include Creedy and Hérault (2012) and Aaberge, Colombino, and Strøm (2004).

[5]The quote is from Gorman (1971, p.82), who analyzes consumer theory with intransitive indifference.

and, second, to show the surprising usefulness of money-metrics for welfare analysis in non-standard and behavioral choice models.[6]

More concretely, a simple question that does not seem to have been posed, much less answered is this: specifically, what allocations *maximize* the money-metric sum, and what efficiency/equity properties do they satisfy (or fail to satisfy)? A second, related, question is this: what *changes* in allocations does the money-metric sum sanction? The second question certainly has been asked and studied; see for example Fleurbaey and Maniquet (2011, pp. 20-1), who also succinctly review some of the history of money-metrics in welfare and social choice analysis.

Starting with the second question, our twist is to reformulate it to this: what does the money-metric sum say about whether a change is a *potential Pareto improvement*, that is, the new allocation can be redistributed so that that everyone prefers it to the original allocation? We give two answers to this reformulation. For the first, fix a price p and consider the the money-metric sum evaluated at an allocation A. If the cost of another allocation B at p is less than the money-metric sum at A, then B *cannot* be a potential Pareto improvement over A (Proposition 2). For the second, we show that, if each consumer's money-metric is differentiable, then the derivative of the sum equals a local welfare measure proposed by Radner (1993 [1978]), and if the derivative is positive, a small-enough change of aggregate consumption in that direction is a potential Pareto improvement (Propositions 3 and 4). The last result gives a welfare foundation for those who, like Deaton, recommend a first-order approximation to the money-metric for welfare. A potential Pareto improvement can be regressive—the consumption of some who have more of every good than others might grow faster than poorer consumers—but it can also be progressive, or neither. As a foundation for local cost-benefit analysis, we prove that a money metric is differentiable under surprisingly-minimal assumptions (Section 9.3).

As for our first question—what allocations maximize a money-metric sum?—we answer it in the setting of a private-ownership, Arrow-Debreu-McKenzie model. We prove that, if the common price in each consumer's money metric is a competitive equilibrium price for *some* distribution of endowments and ownership shares, then the money-metric sum is maximized at *any* of the associated competitive equilibrium allocations (Theorem 2(a)). Since the first welfare theorem holds in our framework, each of these are Pareto optimal. Regarding equity, competitive equilibrium allocations clearly can be inequitable (under any reasonable notion): all of society's goods can be given to a single person. They can also be equitable (under some widely-held notions): in an economy with equal endowments and ownership shares, any competitive equilibrium allocation is

[6] In this regard we should mention that Fleurbaey and Schokkaert (2013) extend Bernheim and Rangel (2009) to include equity considerations.

envy-free as well as Pareto optimal: no consumer would rather have another's equilibrium bundle. In short, as far as maximizers of the money-metric sum go, our Theorem 2 suggests that they are *largely* neutral with respect to ethics. This, then, is the basis for our rehabilitation. There is by now a large social choice literature involving money metrics, and, in particular, Bosmans, Decancq, and Ooghe (2018) develop their own rehabilitation by axiomatizing a (generally) *non-additive* welfare function of individual money-metrics. We relate their work to ours in our Section 5, and elaborate on and qualify the ethical neutrality that Theorem 2 suggests.[7]

Our Theorem 2 is reminiscent of two classical welfare results. Let u_i represent consumer i's preferences over commodity bundles x_i and consider the linear combination $\sum_i \lambda_i u_i(x_i)$ for positive numbers λ_i. Consider the choice of a feasible allocation of goods to maximize this sum. First, any solution to this problem is Pareto optimal, and so if that optimal allocation is supportable as a competitive equilibrium, the conclusion is the same as our Theorem 2. The other is this: if each u_i is concave and each firm's production set is convex, then Negishi (1960, Theorem 3) proved that any competitive equilibrium allocation solves the celebrated *saddlepoint inequalities* from concave programming (Uzawa (1958)) for some choice of weights and with the multiplier on the resource constraint equal to the associated competitive equilibrium price. The choice of a price vector in the money-metric sum seems analogous to the choice of a vector of weights in the linear combination of utilities.

The facts in the last paragraph lead to these questions: what is new in our results? and why are they surprising? We think our results are new and surprising for two reasons. First, they hold even if money-metrics do not represent preferences;[8] even if preferences are incomplete or intransitive, so there is *no* preference representation at all; and indeed even if choices are not rationalized by *any* binary relation, as in some behavioral models; the last is our Theorem 3. Our results suggest that money-metrics can be an especially useful welfare tool in nonstandard or behavioral models.[9]

The second surprise has to do with what it was that Blackorby and Donaldson (1988) prove about the money metric, a remarkable fact that helped lead to its withering away in applied welfare economics: that it is concave in consumption for every price if and *only* if preferences are quasihomothetic (their Propositions 1 and 2).[10] In their set-

[7]And we relate both rehabilitations to an earlier theorem of Fleurbaey and Maniquet (2011). Their book as well as Decancq, Fleurbaey, and Schokkaert (2015, Chapter 2) touch on much of the associated literature. We also mention that Chambers and Hayashi (2012, p. 811) axiomatize the aggregate money-metric for social choices involving risk by conceiving of it as the sum of von Neumann-Morgenstern utilities.

[8]We know that money metrics need not represent consumer preferences even under standard assumptions on the consumption sets and preferences; we will elaborate in the body of the text, but a classic reference on this point is Weymark (1985a).

[9]For example, we can accommodate framing effects and "multiple selves" models. Such multiple-selves models figure prominently in Bernheim and Rangel (2009).

[10]They combine their non-concavity result with a theorem of Debreu and Koopmans (1982) that the quasi-

ting, if demand is single-valued, quasihomotheticity is equivalent to straight-line wealth-expansion paths of demand—a knife-edge condition that is empirically implausible.[11] The Blackorby-Donaldson propositions effectively imply that money-metrics are almost-never concave in consumption. The surprise, then, is this: even though the money metric is not concave, and even though it does not in general represent a consumer's preferences, any demand point solves the celebrated saddlepoint inequalities from *concave* programming for the problem of picking a consumption to maximize a (nonconcave) money metric on the budget set. This is our Theorem 1. Our Theorem 2 then extends this individual money-metric saddlepoint theorem to the economy as a whole. We stress that Negishi's Theorem 3 on saddlepoints and competitive allocations fails if concavity of the utilities is weakened to quasiconcavity—and that, even if preferences are convex, not every convex preference relation has a concave representation,[12]—whereas our Theorem 1 holds without preference convexity, transitivity, or completeness.

Samuelson's (1974) interest in the money metric was to propose new definitions of complementarity in consumption, but he sensed that economists would be ineluctably drawn to add individual money metrics and use the sum for welfare analysis. He concludes his essay with a disavowal.

> Since money can be added across people, those obsessed by Pareto-optimality in welfare economics as against interpersonal equity may feel tempted to add money-metric utilities across people and think that there is *ethical* warrant for maximizing the resulting sum. That would be an illogical perversion and any such temptation should be resisted. (p. 1266, emphasis added).

In the classical setting that Samuelson tacitly works in, individual money metrics do represent preferences, so the sum is an individualistic Bergson-Samuelson welfare function, and any maximizer of it is Pareto optimal. Samuelson's objection to the sum is that it is just about optimality and ignores ethics.[13] At one level our rehabilitation modestly seeks to restore the money-metric sum to the contempt that Samuelson held it in, precisely because of its neutrality with respect to ethics. And beyond that to restore it as a useful tool for those, who, if not obsessed by Pareto optimality, are at least keenly interested in it, a tool whose power is most evident in non-standard or behavioral choice models, where conventional questions of welfare economics are most vexing.

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concavity of a sum of functions implies that all but one of the individual functions is concave to conclude that money-metric sums are not generally quasiconcave in consumption. Fleurbaey (2009, Section 4.4) puts the Blackorby-Donaldson theorem in the broad context of social choice theory and welfare economics.

[11]Special cases include *homothetic*, *quasi-linear*, and *Stone-Geary* preferences; see Deaton and Muellbauer (1980, chapter 6), or Mas-Colell, Whinston, and Green (1995, chapter 4) for textbook treatments.

[12]For the last point, see Kannai (1977) and his references.

[13]To be clear, we *emphatically* agree with Samuelson that it would be a perversion to think that there is an *ethical* warrant for simply maximizing the money-metric sum and accepting the result. Just as we think there is no ethical warrant for stopping at mere Pareto optimality.

2 Preliminaries: Definitional, Notational and Conceptual

The next subsection introduces the standard definition of the money metric, and discusses ways to extend it to cover the non-standard preferences we consider. The rest of the section reviews background facts from the literature; a reader in a hurry to get to our main results can skip to Section 3 and refer back to subsections 2.2-2.4 as needed.

2.1 Money-metrics: The Background

We begin with the basic notation and definitions. There are $L > 1$ commodities, and the consumption set $X \subseteq \mathbb{R}^L_+$ is is assumed throughout to be non-empty and closed. The binary relation $\succsim \subseteq X \times X$ describing a consumer's preferences, is interpreted as weak preference. We write $x \succ y$ if $x \succsim y$ but not $y \succsim x$ (strict preference) and $x \sim y$ if $x \succsim y$ and $y \succsim x$ (indifference). We consider two sets of assumptions on $\succsim$.

Standard Assumptions. $\succsim$ *is complete, transitive, closed, and locally nonsatiated (LNS).*[14]

Note that neither $\succsim$, nor the set over which it is defined, is assumed here to be convex, and, in particular, "discrete" commodities are allowed, as in McKenzie (1957).

In the classic textbook case of $X = \mathbb{R}^L_+$ and the Standard Assumptions plus monotonicity ($x >> y$ implies $x \succ y$), the money-metric at price p assigns the wealth level w to the indifference set passing through any point x demanded at (p, w). In this classic textbook setting, this assignment of numbers to indifference sets represents $\succsim$ on X; see Figure 2 (but ignore for now the points a, b and x'). In this case the money-metric is related to the familiar expenditure function, defined as $e(p, \bar{u}) = \min_{u(x') \geq \bar{u}} p \cdot x'$, by substituting for $\bar{u}$ the utility level $u(x)$ attained by the commodity bundle x. This is to say that $M(x, p) = e(p, u(x))$.

The next set of assumptions is substantially weaker and accommodates non-standard decision theories which relax completeness or transitivity.[15]

Minimal Assumptions. $\succsim$ *is reflexive and locally nonsatiated, and* $\succ$ *is open.*[16]

[14] $\succsim$ is complete if, for every x and y in X, either $x \succsim y$ or $y \succsim x$. It is transitive if, for every x, y, and z in X with $x \succsim y$ and $y \succsim z$, $x \succsim z$. It is closed if the set $\{(x, y) \in X^2 | x \succsim y\}$ is closed; it is locally nonsatiated if, for every $x \in X$ and open neighborhood N of x in $\mathbb{R}^L$, there is a $y \in N \cap X$ such that $y \succ x$.

[15] It can accommodate, for example, Bewley's (2002) "multiple selves" model of incomplete preferences, along with many others; Fishburn (1991) reviews some classic models of intransitive preference.

[16] $\succsim$ is *reflexive* if $x \succsim x$ for every $x \in X$; $\succ$ is *open* if $\{(x, y) \in X^2 \,|\, x \succ y\}$ is an open relative to $X \times X$. That $\succ$ is open implies that the set $\{y \in B(p, w) | y \not\succ x\}$ of points that do not dominate x is closed for any $x \in B(p, w)$. Absent completeness and under our definition of demand as undominated affordable points, this seems an apt form of continuity. The only place we use it is in the proof of Corollary 1.

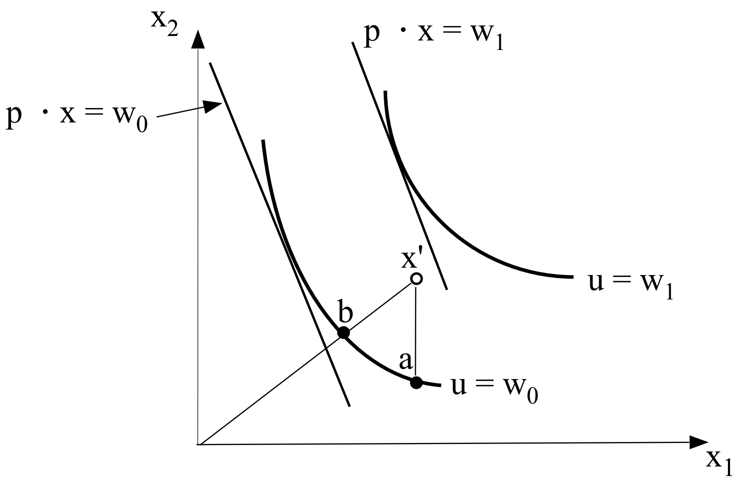

Figure 1: Monetary-labelled indifference curves

For a price-wealth pair $(p, w) \in \mathbb{R}^{L+1}_+$, with $p \neq 0$, the budget set and the demand correspondence are

$$B(p, w) = \{x \in X \mid p \cdot x \leq w\} \text{ and } d(p, w) = \{y \in B(p, w) \mid x \succ y \implies x \notin B(p, w)\}.$$

Under the Standard Assumptions, the money-metric $M(\cdot, \cdot)$ is the value function for the following wealth-minimization problem[17]

$$M(x, p) = \min_{x' \succsim x} p \cdot x'. \tag{1}$$

Since we also consider non-standard preferences, the question arises as to what the appropriate definition is for these cases. One possibility is to use the definition McKenzie (1958) proposes when $\succsim$ is possibly-not closed,

$$M_*(x, p) = \inf_{x' \succsim x} p \cdot x'. \tag{2}$$

If $\succsim$ is closed, then the two definitions agree.[18] Since $\succsim$ is reflexive, the constraint set in (2) is nonempty; and since X is a closed subset of $\mathbb{R}^L_+$, the infimum is nonnegative.

[17] $M(\cdot, \cdot)$ is the minimum income function in McKenzie (1957), who emphasized it as a function of p for fixed x. Samuelson (1974, pp. 1272-1273) clearly ascribed the concept to McKenzie but studied it as a function of x for a fixed p. Given his emphasis, he changed the nomenclature of an expenditure or a minimum income function to a money-metric, and it is this change that illuminates the inherent duality of the object and opened up new vistas and applications for it.

[18] The Minimal Assumptions require that $\succ$ is open and we explained the rationale for it in footnote 16. The reason we do not also assume that $\succsim$ is closed is the Theorem in Schmeidler (1971). It asserts that if X is a connected topological space, $\succsim$ is closed and transitive, $\succ$ is open and nontrivial ($x \succ y$ for some x, y in X), then $\succsim$ is complete. We want to allow the possibility that X is connected and $\succsim$ is transitive, but not complete.

For another possible definition, for $x \in X$, let $\mathcal{R}(x)$ be the *closure* of the set $\{x' \in X \mid x' \succ x\}$. Define

$$M(x,p) = \min_{x' \in \mathcal{R}(x)} p \cdot x' \text{ and } h(p,x) \equiv \text{argmin}_{x' \succsim x} p \cdot x'. \tag{3}$$

As with (2), definition (3) agrees with (1) under the Standard Assumptions. We will use (3) as our definition of the money metric. We do this in part because our Theorems hold for the Minimal Assumptions when we use this definition. In the Appendix (Section 9.1) we show that $M(x,p) \geq M_*(x,p)$. That inequality suggests the names *lower money metric* for (2) and *upper money metric* for (3); and we prove that the two are equal if we strengthen local nonsatiation to rule out "thick indifference sets" (Proposition 5). We refer to $h(\cdot,\cdot)$ as the (Hicksian) *compensated demand* to distinguish it from the (Marshallian) demands already defined.

To conclude this preliminary section, we remind the reader that the money metric is related to two other classical welfare measures: the *compensating variation* and the *equivalent variation.* Impose the Standard Assumptions with the classical consumption set $\mathbb{R}^L_+$, so that $M(\cdot,p)$ represent the consumer's preferences. The equivalent variation for a move from price-wealth pair (p^0,w^0) to (p^1,w^1) is (for $x^t \in d(p^t,w^t)$ for $t=0,1$)

$$EV(p^0,(p^1,w^1)) = M(x^1,p^0) - M(x^0,p^0);$$

and the compensating variation for that move is

$$CV(p^0,(p^1,w^1)) = M(x^1,p^1) - M(x^0,p^1).$$

The equality $M(x^1,p^0) = EV(p^0,(p^1,w^1)) + w^0$ reveals that the equivalent variation is a special case of the money metric, in that the money metric is defined for all consumption plans $x \in X$, not just those that are demanded at some price-wealth pair, that is, $x \in d(p,w)$ for some $(p,w) >> 0$.[19]

2.2 Example: Money-metrics and Quasi-linear Utility

We now calculate the money-metric for the familiar and historically-important special case of quasi-linear preferences defined on two goods, $X = \mathbb{R}^2_+$, and represented by the quasi-linear-in-good-2 form $u(x_1,x_2) = \phi(x_1) + x_2$, where ϕ is strictly concave and differentiable with $\phi' > 0$. The strict concavity of ϕ implies that preferences are strongly convex. For this subsection, we take good 2 to be *numeraire* and set $p_2 = 1$.

For $p_1 > 0$, define $f(p_1)$ to be the unique number (if any) that solves the Kuhn-Tucker

[19] With the proviso that p^0 is taken as an equilibrium price, the equivalent variation is what King (1983) proposes as a welfare measure; he calls it *equivalent income.*

conditions for the consumer's problem of choosing $x_1 \geq 0$ to maximize $\phi(x_1) + w - p_1 x_1$, i.e., by ignoring the non-negativity constraint on x_2. These conditions are

$$\phi'(f(p_1)) - p_1 f(p_1) \leq 0 \quad f(p_1)\left[\phi'(f(p_1)) - p_1 f(p_1)\right] = 0, \quad \text{and} f(p_1) \geq 0.$$

If there is no solution to these conditions, set $f(p_1) = \infty$. The demand for good 1 equals $f(p_1)$ if $f(p_1) \leq w/p_1$ (that is, the constraint $x_2 \geq 0$ does not bind), and equals w/p_1 otherwise. The indirect utility function for the consumer is $v(p_1, w) = \phi(f(p_1)) - p_1 f(p_1) + w$ if $w \geq p_1 f(p_1)$ and $\phi(w/p_1)$ otherwise. Note that the marginal utility of wealth equals 1 if and only if the non-negativity constraint on good 2 *does not bind*. For reference utility $\bar{u}$ in the range of u, the expenditure function equals $e(p_1, \bar{u}) = \bar{u} - \phi(f(p_1)) + p_1 f(p_1)$ if $\bar{u} \geq \phi(f(p_1))$ and equals $p_1 \phi^{-1}(\bar{u})$ otherwise. It follows that the money-metric is given by

$$M(x, p) = \begin{cases} u(x_1, x_2) - \left[\phi(f(p_1)) - p_1 f(p_1)\right] & \text{if } u(x_1, x_2) \geq \phi(f(p_1)) \\ p_1 \phi^{-1}(u(x_1, x_2)) & \text{otherwise} \end{cases} \tag{4}$$

In particular, for fixed p, $M(x, p) = u(x_1, x_2)$ up to an additive constant *if and only if the non-negativity constraint on x_2 in the expenditure minimization problem does not bind at p*; otherwise the two functions diverge.Of course, since x_2 enters the representation linearly and $X = \mathbb{R}^2_+$, the non-negativity constraint for x_2 must bind for a region of values for $(p, \bar{u})$, namely those for which the compensated demand for good 2 is 0. Applications that hope to preserve the simplicity of quasi-linearity then, often implicitly, restrict parameters to avoid corner solutions for the *numeraire* good. And these parameters from the viewpoint of the consumer are endogenous equilibrium variables from the viewpoint of the economy as a whole.[20]

The defining simplification of quasi-linearity, and the one that justifies consumer's surplus calculations, is Marshall's elusive "constancy of the marginal utility of money"– *provided* that the the commodity in which the preferences are quasi-linear in is consumed in positive quantities; see Samuelson (1942). An important stream of the last seventy years of applied welfare economics seeks to justify consumers' surplus calculations in economies that are not quasi-linear.[21] An attraction of the money-metric lies precisely in its guarantee of the sought-after constancy of the marginal utility of money without the strong assumption of quasi-linearity; and in the effortless way that it handles corner solutions, non-convexities, and even non-ordered preferences.

[20] One workaround is simply to drop the non-negativity constraint on the *numeraire* good altogether, as in Mas-Colell, Whinston, and Green (1995, Chapter 3, p. 44 and Section 10.C).

[21] For example, Harberger (1971), Willig (1976), Tirole (1988, pp. 7-13), Weitzman (1988), Vives (1999, chapter 3), Schlee (2013b), and Hayashi (2017).

2.3 The Non-concavity of the Money-metric

The surprise of our money-metric saddlepoint theorem is that money metrics are not generally concave. In order to help the reader *see* the non-concavity, we outline an alternative proof of the Blackorby-Donaldson (1988) theorem that uses the least-concave representation result of Debreu (1976); see Khan and Schlee (2017). The assertion is that a money-metric is concave in consumption at every price $p >> 0$ if and only if preferences are quasihomothetic (implying demand is affine in wealth). Let $X = \mathbb{R}^L_+$, suppose the Standard Assumptions on preferences hold, and that $x \succsim 0$ for every $x \in X$ (for example, $\succsim$ is monotone).[22] In this case the money metric itself is a preference representation; see Shafer (1980, Lemma) or Section 9.4 below. For any representation u, write the money-metric as $M(x,p) = e_u(p, u(x))$, where e_u is the expenditure function associated with u. There are two cases. Either there is *no* concave representation of preferences, in which case we are done;[23] or there is a concave representation u of preferences. In the second case, the indirect utility function is concave and strictly increasing in w for each p.[24] Since the expenditure function $e_u(p,\cdot)$ is the inverse of the indirect utility for fixed p, $e_u(p,\cdot)$ is convex. So $M(\cdot,p)$ is a *convex transformation* of any concave representation; which is to say that it is *less concave* than any concave representation.[25] By Debreu (1976), there is a *least concave representation*, $\tilde{u}$; that is, if u is a concave representation, then $u = T \circ \tilde{u}$ for some concave, strictly increasing function $T : Range(\tilde{u}) \to \mathbb{R}$. So either i) $M(\cdot,p)$ is affinely related to $\tilde{u}$ –that is $M(x,p) = a(p) + b(p)\tilde{u}(x)$ for every p for some functions a and b – and hence the money-metric is *also* a least concave representation; or ii) it is *not* concave for some p. Since i) implies that the indirect utility is affine in wealth, this completes the argument.

The literature is silent about the region of consumption for which the money-metric is not concave, but Samuelson (1974, p. 1276) makes an intriguing assertion. He writes that the money-metric $M(x,p^0)$ is "locally concave in all x that are near to x^0 [a point demanded at p^0] (and which may as well be restricted to x's that are at least as good as x^0)." But what he writes in symbols is

$$\nabla_x M(x^0,p^0)\cdot(x - x^0) \geq M(x,p^0) - M(x^0,p^0) \tag{5}$$

for all x in a neighborhood of x^0, that is, that $M(\cdot,p^0)$, is *concave precisely at the demanded*

[22] Unfortunately Khan and Schlee (2017) omit explicit mention of this last assumption.

[23] The Finetti-Fenchel-Kannai examples highlight how a convex preference relation may not have a concave representation for precisely such plausible preference orderings. See Kannai (1977) and Kannai (1981) for examples and the antecedent background.

[24] If the consumer has expected utility preferences and we interpret u as a von Neumann-Morgenstern utility, this fact implies that aversion for wealth risk follows from aversion for consumption risk; see Kreps (2013, Proposition 6.16, p. 136).

[25] For two continuous real-valued functions f and g on a convex set $D \subseteq \mathbb{R}^n$ that represent the same binary relation $\succsim$, g is *less concave* than f if there is a convex, and strictly increasing, real-valued function T on *Range*(f) that is convex with $g = T \circ f$.

point x^0.[26] Whereas (5) is certainly true if money metric is differentiable at a demanded point *and* the demanded point is in the interior of the consumption set, Samuelson's verbal assertion about concavity in a neighborhood of a demand point remains to this day an open question.[27]

2.4 Saddlepoints and Uzawa's Theorems

Here we recall for the reader Uzawa's (1958) saddlepoint theorems. Let $\emptyset \neq Z \subseteq \mathbb{R}^n_+$, and let $f : Z \to \mathbb{R}$ and $g : Z \to \mathbb{R}^m$ be functions, where n and m are positive integers. Consider the constrained optimization problem $\max_{z \in Z} f(z)$ subject to $g(z) \leq 0$, and the saddlepoint inequalities asserting the existence of $z^* \geq 0$ and $\lambda^* \geq 0$ such that

$$\mathcal{L}(z, \lambda^*) \leq \mathcal{L}(z^*, \lambda^*) \leq \mathcal{L}(z^*, \lambda)$$

for all $z \in Z$ and $\lambda \in \mathbb{R}^m_+$, and where $\mathcal{L}(z, \lambda) = f(z) - \lambda \cdot g(z)$.

We can now present

Theorems (Uzawa (1958))**.** *1. If (z^*, λ^*) satisfies the saddlepoint inequalities, then z^* solves the optimization problem. 2. If z^* solves the optimization problem, Z is convex, f is concave, g is convex, and there exists $\hat{z} \in Z$ such that $g(\hat{z}) << 0$ (Slater's constraint qualification), then there exists $\lambda^* \geq 0$ such that (z^*, λ^*) satisfies the saddlepoint inequalities.*

If the saddlepoint inequalities hold, then z^* maximizes $f(z) - \lambda^* g(z)$ on Z without directly imposing the constraint $g(z) \leq 0$. A difficulty in applying Uzawa's Theorem 1 is to verify that the saddlepoint inequalities hold. The deeper Theorem 2 gives sufficient conditions for the saddlepoint to hold; it applies to the consumer's problem if preferences are representable by a concave function. There are two well-known impediments for a successful application. First, the value of the multiplier, the marginal utility of money, depends on the parameters of the problem, and thereby complicates its use for comparative statics. Second, it requires a concave representation of preferences, and as already pointed out in the introduction, even if one imposes convexity of preferences, some convex preference relations are not representable by a concave function.[28] We remind the reader that concavity *cannot* be relaxed to quasi-concavity in Uzawa's Theorem 2 even if the constraint function g is linear. The Cobb-Douglas function $u(x_1, x_2) = x_1 x_2$ for the consumer's problem is a simple counterexample.

[26] By "concave at the point x" for a differentiable function f we mean that $\nabla f(x) \cdot (y - x) \geq f(y) - f(x)$ for all y in some neighborhood of x.

[27] For what it is worth, it was reflection on Samuelson's assertions about local concavity of money-metric that lead us to think about what *global* properties of concave functions, if any, that it preserves, independently of differentiability and interiority assumptions, and thereby to Uzawa's saddlepoint theorems.

[28] See the references in Footnote 23.

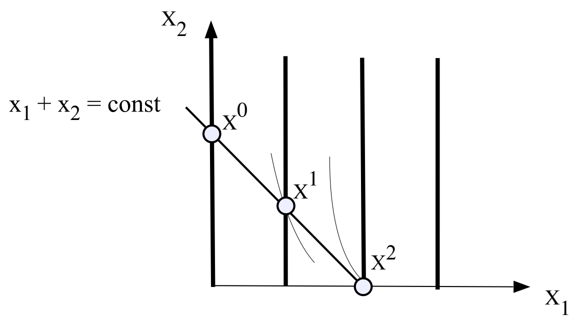

Figure 2: The consumption set equals the vertical bars, so the consumption set is not convex. Assume that preferences are strongly monotone. If $x^2 \succ x^1 \succsim x^0$, then $x = x^1$ violates the local cheaper point at $p^0 = (1/2, 1/2)$. The point x^2 is demanded at $(p^0, M(p^0, x^1))$, but there is no point in a small-enough neighborhood of x^2 that is cheaper at those prices. We have $M(p^0, x^1) = M(p^0, x^2)$, even though $x^2 \succ x^1$. The curves in the Figure are not indifference curves but are meant to be a visual aid.

3 A Money-Metric Saddlepoint Theorem

If $\succsim$ is transitive, then it follows that

$$x \succsim y \quad \text{implies that} \quad M(x,p) \geq M(y,p), \tag{6}$$

since the constraint set for the income minimization problem (2) at x is a subset of the constraint set for that problem at y. If $\succsim$ is complete and transitive, it follows *if* $x' \in d(p,w)$, then x' maximizes $M(\cdot, p)$ on the budget set $B(p,w)$. In this section we are after something bolder: that x' maximizes $M(x,p) - p \cdot x$ on X *without imposing* the budget constraint, and even if preferences are not complete or not transitive. Of course if $M(\cdot, p)$ represents $\succsim$, then any maximizer x' of $M(\cdot, p)$ on $B(p,w)$ is a demand. But without further restrictions on the consumption set, a money metric does not in general represent preferences even under the Standard Assumptions: in particular, we can have $x \succ y$, but $M(x,p) = M(y,p)$.[29] The possibility is depicted in Figure 2. The consumption set equals the vertical bars, preferences are strongly monotone, and $x^2 \succsim x^1 \succsim x^0$. At the price $p = (1/2, 1/2)$, $M(x^1, p) = M(x^2, p)$. But nothing prevents $x^2 \succ x^1$. In that case $M(\cdot, p)$ does not represent the consumer's preferences on X, and $x^1 \notin d(p, p \cdot x^1)$ even though it maximizes $M(x,p)$ on the budget set $B(p,w)$. We now introduce a variant of the condition that McKenzie (1957) uses in his derivation of the Slutsky equation.[30]

[29]Weymark (1985a) is the classic reference for money metrics as a preference representation for general consumption sets. Shafer (1980) proves that the money metric represents preferences for the classic consumption set $X = \mathbb{R}^L_+$ under what we call our Standard Assumptions if in addition $x \succsim 0$ for every $x \in X$.

[30]We warn the that the cheaper point assumption is being formulated in a slightly different way than in Walrasian general equilibrium theory where for each price, it is the existence of a cheaper point in the budget

Definition 1. *A point $x \in X$ satisfies the* **local cheaper point condition** *at $p \in \mathbb{R}^L_+$ if, for every $x' \in d(p, M(p,x))$ and every open neighborhood N of x', there is a point $x'' \in N \cap X$ such that $p \cdot x'' < p \cdot x'$.*

In Figure 2, if $x^2 \succ x^1 \succsim x^0$, then x^1 violates the local cheaper point condition since $x^2 \in d(p, p \cdot x^1)$, but there is no point in X that is cheaper than it at price $(1/2, 1/2)$. Of course x^2 itself violates the cheaper point condition as well, but this violation does not pose a problem for us since x^2 is itself demanded at price p and income $p \cdot x^2$: $x^2 \in d(p, p \cdot x^2)$. The difficulty for x^1 is that $x^1 \notin d(p, p \cdot x^1)$.

The role of the local cheaper point condition is spelled out in the next lemma. With it, compensated and uncompensated demands are equal; and the constraint $x' \succsim x$ in the wealth minimization problem (2) binds. We prove the Lemma in the Appendix.

Lemma 1. *Fix $p >> 0$. If $\succsim$ satisfies the Standard Assumptions, and $x \in X$ satisfies the local cheaper point assumption at p, then*
(a) $h(p,x) = d(p, M(x,p))$; and (b) $x' \sim x$ for any $x' \in h(p,x)$.

Of course in the classic case $X = \mathbb{R}^L_+$ the local cheaper point assumption holds for every $x \neq 0$. Again, the point $x = 0$ poses no problem for us since it is itself demanded at zero wealth at any price: $0 \in d(p, 0)$.

We now turn to our main result for a single consumer. It lays the foundation for our agenda on money metrics in consumer theory.

Theorem 1 (Money-Metric Saddlepoint)**.** *Fix $(p,w) > 0$. Define $\mathcal{L}(x,\lambda) = M(x,p) + \lambda[w - p \cdot x]$. Consider the saddlepoint inequalities for $(x^*, 1)$ with $x^* \in X$:*

$$\mathcal{L}(x,1) \leq \mathcal{L}(x^*,1) \leq \mathcal{L}(x^*,\lambda) \qquad \textit{for every } x \in X \textit{ and } \lambda \geq 0. \tag{7}$$

- *(a) If $\succsim$ satisfies the Minimal Assumptions and $x^* \in d(p,w)$, then the saddlepoint inequalities (7) hold.*
- *(b) If $\succsim$ satisfies the Standard Assumptions, $p >> 0$, the saddlepoint inequalities (7) hold, and if x^* satisfies the local cheaper-point condition at p, then $x^* \in d(p,w)$ for $w = p \cdot x^*$.*

We remind the reader that a money metric is not generally concave in consumption or even quasiconcave—and that quasiconcavity cannot replace concavity in Uzawa's Saddlepoint Theorem 2. And we emphasize that parts (a) and (b) hold even if the money metric does not represent preferences, as in the example in Figure 2, and (a) holds even if preferences are incomplete nor intransitive, so that there is no representation at all. We leave the proofs of Theorems 1(a) and 2(a) in the text since they are short and informative. The part-(b) proofs for each are in the Appendix.

set of each consumer that is being asserted.

Proof of (a): Let $x^* \in d(p,w)$ for some $w \geq 0$. Since $\succsim$ satisfies local nonsatiation, $p \cdot x^* = w$, and so $\mathcal{L}(x^*, \lambda) = 0$ for every $\lambda \geq 0$, including $\lambda = 1$. The inequalities "$\mathcal{L}(x,1) \leq \mathcal{L}(x^*,1)$ for all $x \in X$" is equivalent to "x^* maximizes $M(x,p) - p \cdot x$ on X." Since, by local nonsatiation of $\succsim$, any $x' \in X$ is in $\mathcal{R}(x')$, the closure of $\{x \in X \mid x \succ x'\}$, and so is itself feasible for the wealth minimization problem (3) at $x = x'$; it follows that $M(x,p) - p \cdot x \leq 0$ for every $x \in X$, implying that the difference is certainty maximized whenever it equals zero. Consider any $y \in X$ with $p \cdot y < p \cdot x^* = w$. Since $x^* \in d(p,w)$, $y \not\succ x^*$. We will show that $y \notin \mathcal{R}(x^*)$. Consider any convergent sequence x^n in X with $x^n \succ x^*$ for every integer n. Since $x^* \in d(p,w)$, $p \cdot x^n > w$ for every n. Since $p \cdot y < w$, it follows that $y \neq \lim x^n$, and so $y \notin \mathcal{R}(x^*)$. Then x^* solves problem (3), implying $M(x^*,p) - p \cdot x^* = 0$. □

We now specialize to the classic case of $X = \mathbb{R}^L_+$ to illustrate a use of Theorem 1. It asserts that the wealth expansion path at a price p *equals* the set of maximizers of $M(x,p) - p \cdot x$ on X. It has the implication that we can replace the standard utility maximization problem in classical consumer theory—maximize utility subject to a budget constraint—with an *un*constrained optimization problem using the money metric, whether or not it represents preferences. It illustrates that for some well-behaved consumption sets, we can dispense with completeness and transitivity to get the conclusion of Theorem 1(b). For Corollary 1, we don't need the full conclusion of Lemma 1, but simply that $x' \in d(p, p \cdot x')$ if and only if $p \cdot x' = M(x',p)$.[31]

Corollary 1. *Let $X = \mathbb{R}^L_+$ and $p >> 0$. If the Minimal Assumptions hold, then $x' \in d(p,w)$ for some $w \geq 0$ if and only if x' maximizes $M(x,p) - p \cdot x$ on X.*

The corollary puts consumer theory on the same footing as producer theory: maximizing benefit minus cost on a consumption or production set, at least for the classic case $X = \mathbb{R}^L_+$. In a sequel to this paper we use this formulation to simplify dramatically the comparative statics of the consumer's problem. A long-recognized difficulty in applying the lattice-theoretic comparative statics methodology of Topkis (1995) and Milgrom and Shannon (1994) to the consumer's problem is that different budget sets do not stand in the strong-set-order relation.[32] Corollary 1 bypasses this problem by dispensing with the budget constraint.

Proof of Corollary 1: If $x' \in d(p,w)$ then, by Theorem 1(a), x' maximizes $M(x,p) - p \cdot x$ on X under the Minimal Assumptions. Now suppose that x' maximizes $M(x,p) - p \cdot x$ on $X = \mathbb{R}^L_+$. As in the proof of Theorem 1(a), local nonsatiation implies that $M(x,p) \leq p \cdot x$ for any $x \in X$. Since $0 \in X$ and $M(0,p) = 0$, it follows that $0 = M(0,p) - p \cdot 0 \leq M(x',p) - p \cdot x' \leq 0$ which implies that $M(x',p) = p \cdot x'$. Set $w = M(x',p) \geq 0$. We will show that $y \in B(p,w)$ implies that $y \not\succ x'$. Let $y \in B(p,w)$. If $x' = 0$, then $w = 0$ and the budget set $B(p,w)$ is a singleton, so the conclusion holds since $\succsim$ is reflexive (implying $\succ$ is irreflexive). Let $x' \neq 0$, which since $X = \mathbb{R}^L_+$ implies that $x' > 0$. If $p \cdot y < w$, then $y \not\succ x'$ by the definition of

[31] Absent transitivity, it is well-known that the equality in Lemma 1(a) fails; see Fountain (1981).

[32] See for example Quah (2007) and Mirman and Ruble (2008).

$M(x', p) = \min_{z \in \mathcal{R}(x')} p \cdot z$ and $w = M(x', p)$. If $p \cdot y = w$, then since $x' \neq 0$ and $p >> 0$, we have $w > 0$, which in turn implies that $y > 0$. Consider the sequence $y^n = (1 - \frac{1}{n})y$. Fix n. Since $p \cdot y^n < w$, it follows that $y^n \in \{z \in X, \mid z \not\succ x'\}$, which is a closed set since $\succ$ is open. We then have $\lim_{n\to\infty} y^n = y \not\succ x'$, so $x' \in d(p, w)$. $\square$

4 Competitive Allocations and the Money-metric Sum

We now extend the framework of Section 2 to I consumers. This is to say that each consumer i has a consumption set $X_i \subseteq \mathbb{R}^L_+$ that is nonempty and closed; and a preference relation $\succsim_i \subseteq X_i \times X_i$ satisfying either the Standard or Minimal Assumptions. Define $M_i(x, p) = \inf_{\{x' \succsim_i x\}} p \cdot x'$ with $(p, x) \in \mathbb{R}^L_+ \times X_i$. For $(p, w) \in \mathbb{R}^{L+1}_+$ and $B(p, w) = \{x \in X \mid p \cdot x \leq w\}$, we let $d_i(p, w) = \{x' \in B(p, w) \mid x \succ_i x' \text{ implies } x \notin B(p, w)\}$.

Let there be $J \geq 1$ firms, with firm j endowed with a production set $Y_j \subseteq \mathbb{R}^L$ satisfying $0 \in Y_j$ (possibility of inaction). The aggregate production set is $Y = \sum_j Y_j$, assumed to be closed. Let

$$\pi_j(p) = \max_{y_j \in Y_j} p \cdot y_j$$

when it exists. Consumers are endowed with goods and ownership shares in firms. Consumer i's price-dependent wealth is $w_i(p) = p \cdot \omega_i + \sum_j \theta_{ij} \pi_j(p)$, where $\omega_i \in X_i$ is consumer i's endowment of goods and θ_{ij} is i's ownership share of firm j. We assume that $\omega = \sum_i \omega_i > 0$.

Definition 2. *Let $\mathcal{A} = \Pi_i X_i \times \Pi_j Y_j$. An **allocation** is a point $(x_1, ..., x_I, y_1, ..., y_J) \in \mathcal{A}$. An allocation $(x_1, ..., x_I, y^1, ..., y_J)$ is **feasible** if*

$$\sum_i x_i \leq \sum_i \omega_i + \sum_j y_j.$$

We will denote an *allocation* by $(\mathbf{x}, \mathbf{y})$. We denote aggregate consumption by $x = \sum_i x_i$, aggregate production by $y = \sum_j y_j$, and the aggregate endowment by $\omega = \sum \omega_i$.

Definition 3. *A **competitive equilibrium** is a feasible allocation $(\mathbf{x}^*, \mathbf{y}^*)$ and a nonzero price vector $p^* \in \mathbb{R}^L_+$ such that*

1. *For every $j = 1, ..., J$, $p^* \cdot y^*_j \geq p^* \cdot y_j$ for every $y_j \in Y_j$;*

2. *For every $i = 1, ..., I$, $x_i \succ_i x^*_i$ implies $x_i \notin B(p^*, w_i(p^*))$.*

For each consumer i and $p \in \mathbb{R}^L_+$, let $X^{lcp}_i(p) \subseteq X_i$ be the set of consumption plans that satisfy the local cheaper point condition at p.

We now show that any competitive equilibrium allocation maximizes the sum of money metric utilities on the set of feasible allocations—despite the non-concavity, and even though the money metric need not represent preferences.[33] In what follows we assume that competitive equilibrium prices are nonnegative. This would follow, for example, if, for every good ℓ, some consumer's preferences were strictly monotone in that good, or if the production set Y satisfies free disposal.

Theorem 2. *Let $\mathcal{L}_p(\mathbf{x}, \mathbf{y}, \mu) = \sum_i M_i(x_i, p) + \mu \cdot [\omega + y - x]$. Consider the following saddlepoint inequalities at $(\mathbf{x}^*, \mathbf{y}^*, p^*) \in \mathcal{A} \times \mathbb{R}^L_+$:*

$$\mathcal{L}_{p^*}(\mathbf{x}, \mathbf{y}, p^*) \leq \mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, p^*) \leq \mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, \mu) \quad \textit{for every } (\mathbf{x}, \mathbf{y}, \mu) \in \mathcal{A} \times \mathbb{R}^L_+. \tag{8}$$

(a) If $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium, and each consumer's preferences satisfy the Minimal Assumptions, then the saddlepoint inequalities (8) hold. It follows that $(\mathbf{x}^*, \mathbf{y}^*)$ maximizes $\sum_i M_i(x_i, p^*)$ on the set of feasible allocations.

(b) If the saddlepoint inequalities (8) hold at some $p^* >> 0$; if each consumer's preferences satisfy the Standard Assumptions; and if for every consumer i, $x_i^* \in X_i^{lcp}(p^*)$; then $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium for some distribution of endowments and ownership shares.

Proof of (a): Let $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ be a competitive equilibrium. Since for any $\mu \in \mathbb{R}^L_+$, $\mu \cdot [\omega + y^* - x^*] \geq 0$ and, by local nonsatiation, $p^* \cdot [\omega + y^* - x^*] = 0$, certainly $\mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, p^*) \leq \mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, \mu)$ for every $\mu \in \mathbb{R}^L_+$. Profit-maximization for each firm j at $p = p^*$ implies that $p^* \cdot y^* \geq p^* \cdot y$ for every $y \in Y$; and for each consumer i, $M_i(x_i, p^*) - p^* \cdot x_i \leq 0$ for every $x_i \in X_i$, with equality at x_i^* by Theorem 1(a). So $\mathcal{L}_{p^*}(\mathbf{x}, \mathbf{y}, p^*) \leq \mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, p^*) = p^* \cdot (\omega + y^*)$. Since the saddlepoint inequalities (8) hold, it follows from Theorem 1 in Uzawa (1958) (part 1. of the Theorem in Section 2.4 here) that the competitive allocation maximizes the sum of money metrics on the set of feasible allocations. □

Figure 2 helps illustrate a difference between parts (a) and (b). Suppose it describes a single-consumer economy with endowment x^0 and that the line $x_1 + x_2 = const$ describes the set of feasible consumption bundles with the economy's constant-returns technology. If $x^2 \succ x^1 \succ x^0$ then the unique equilibrium allocation is $x = x^2$. The plan x^2 maximizes the consumer's money metric at $p = (1/2, 1/2)$. The point x^2 is not covered by part (b) since it violates the local cheaper point assumption. But if $x^1 \succ x^2 \succ x^0$, then the equilibrium allocation is x^1, which satisfies the local cheaper point assumption and solves the saddlepoint inequalities at $p = (1/2, 1/2)$. It is also a competitive equilibrium, illustrating part (b).

We conclude this section with a series of remarks.

[33] Our results go through under two kinds of market incompleteness. The first is that there simply aren't markets for some goods, so consumers must consume their endowments of those goods. The second is that every consumer must consume the same physical good across a common *set* of states, locations, or dates. On the inclusion of public goods and non-priced commodities as parameters of the money-metric, see Hammond (1994), Fleurbaey and Maniquet (2011) and Fleurbaey and Blanchet (2013).

Remark 1 (The money-metric sum and (in)justice)**.** Theorem 2 asserts that, at a price p, any competitive equilibrium allocation associated with that price (for some distribution of endowments and ownership shares) maximizes the money-metric sum. Of course, a competitive equilibrium allocation can give everything to a single person. But it can also be, for example, an *egalitarian Walrasian allocation*, that is, a competitive equilibrium allocation in an economy in which everyone has the same wealth (and so is envy-free). As far as *maximizers* of the money-metric sum go, there does not seem to be an inherent tendency towards inequity *or* equity. It seems to be, as Samuelson (1974) intuited, and inveighed about, largely ethically *neutral*.[34] We elaborate on and qualify this point in our Section 5 □

Remark 2 (The saddlepoint and the wealth of nations)**.** The last sentence of Theorem 2(a) asserts that a competitive equilibrium allocation maximizes the money-metric sum *on the set of feasible allocations*. The saddlepoint inequalities (8) imply something more surprising: a competitive allocation maximizes the monetary sum $\sum_i M_i(x_i, p^*) - p^* \cdot x + p^* \cdot y$ on the set $\mathcal{A}$ of allocations, *feasible or not*. □

Related to this Remark, the next result is an immediate implication of Corollary 1 that we write out formally for completeness. The point is that we can solve for the allocations demanded at a price ignoring the budget constraint.

Corollary 2. *Let $X_i = \mathbb{R}^L_+$ for every consumer i, and suppose that each consumer's preferences satisfy either the Standard or Minimal Assumptions. Then $\mathbf{x}^* \in (d_1(p, w_1), ..., d_I(p, w_I))$ for some $(w_1, ..., w_I) \geq 0$ if and only if $\mathbf{x}^*$ maximizes $\sum_i M_i(x_i, p) - p \cdot x$ on $\mathbb{R}^{LI}_+$.*

Theorem 2 shows that a competitive equilibrium allocation maximizes the sum of money metrics if the reference price is the associated competitive equilibrium price. That includes any other competitive equilibrium allocations, supportable at a possibly-different price. This distinguishes the sum of money metrics from the sum of compensating variations, since generally the sum of compensating variations is *positive* when moving from one competitive equilibrium to another.[35]

Remark 3 (The sum as a *individualistic* Bergson-Samuelson welfare function (B-S SWF))**.** Suppose that each consumer i's money metric represents $\succsim_i$. Then the money-metric sum is a a *individualistic* Bergson (1938)-Samuelson (1947) *social welfare function* (B-S SWF). Since terminology is not uniform, we clarify that we define a *B-S SWF* to be a real-valued function $\mathcal{W}$ on the set of allocations $\mathcal{A}$; we define an *individualistic B-S SWF* to be a B-S SWF such

[34] If p^* is not a competitive equilibrium price for some distribution of endowments and ownership shares, then our theorem is silent about what allocations maximize the money-metric sum. If a price fails to be an equilibrium price because of preference nonconvexities, then considering an economy with a *continuum* of consumers might expand the scope of Theorem 2.

[35] This possibility has become known as the Boadway (1974) paradox. Like the sum of equivalent variations, the sum of money metrics is nonpositive when moving from one competitive allocation to another. This fact was one consideration in King's (1983) adoption of the money-metric for welfare.

that, for every consumer i, and for each fixed consumption levels of the other consumers, $\mathcal{W}$ represents $\succsim_i$.[36] A classic example is the additive B-S SWF mentioned in the Introduction, $\sum \lambda_i u_i(x)$, where, for each i, $\lambda_i > 0$ and u_i represents $\succsim_i$. If $M_i(\cdot, p)$ represents $\succsim_i$ for every i, then the money-metric sum is obviously a special case of this form.

As with any other individualistic B-S SWF, it follows that any maximizer of the money-metric sum on the set of feasible allocations is Pareto optimal. This fact and our Theorem 2(a) give a new proof of the *first welfare theorem* that every competitive equilibrium allocation is Pareto optimal.[37] □

Remark 4 (The sum as a *family* of B-S SWFs)**.** One use of *families* of B-S SWFs in workaday economics is simply to calculate the set of Pareto optimal allocations. Consider for example, the family of additive B-S SWFs as in Remark 3 as the weights vary over all positive values. It is well-understood that a Pareto optimal allocation might not maximize any of these sums: in general one must allow some of the weights to be zero; the resulting sum is no no longer an *individualistic* B-S SWF, and some allocations which maximize it need not be Pareto optimal. It does follow that, if the utility-possibility set is convex, then the entire set of Pareto optimal allocations is in range of maximizers of an additive B-S SWF as the weights vary over all nonnegative, nonzero vectors; more broadly it is in the range of maximizers of $\mathcal{W}(\mathbf{x}) = W(u_1(x_1), ..., u_I(x_I))$ over all nondecreasing functions W on $\mathbb{R}^I$.[38]

If some consumer's preferences are not represented by the money metric, then the money-metric sum is no longer an individualistic B-S SWF. Still a similar result follows for the money-metric sum. Consider any economy in which the conclusion of the second welfare theorem holds, specifically in which any Pareto optimal allocation can be supported as a *compensated equilibrium* with prices $p > 0$. (For example, Y is convex and satisfies free disposal, and each $\succsim_i$ satisfies either the Minimal Assumption or the Standard Assumptions plus convexity.) Then as we vary p over all nonnegative elements of the unit simplex, the set of maximizers of the money-metric sum *equals* the set of Pareto optimal allocations.[39] □

Remark 5 (Ranking other allocations)**.** Theorem 2 and the preceding remarks consider allocations that *maximize* the money-metric sum. Here we consider how the money-metric sum

[36] Many authors define a Bergson-Samuelson social welfare function simply to be an increasing function mapping *utility profiles* to real numbers. See for example Mas-Colell, Whinston, and Green (1995, p. 117) and Kreps (2013, Section 8.4). The original articles suggest that our definition is historically more accurate. Fleurbaey and Mongin (2005) give a fascinating account of the history of the idea and its relationship to Arrow's social welfare function and his celebrated impossibility theorem. They refer to what we call a B-S SWF simply as a *Bergson-Samuelson function*.

[37] Bergstrom (1973), Gale and Mas-Colell (1977), Fon and Otani (1979), Weymark (1985b), and Rigotti and Shannon (2005) prove versions of the first welfare theorem with incomplete or intransitive preferences.

[38] See for example Kreps (2013, Section 8), especially Proposition 8.1 and his discussion of Figure 8.3.

[39] If in addition *the economy is* ***Gorman*** *(1953)*, that is, in addition to the Standard Assumptions, strong convexity, and monotonicity, each consumer i has an expenditure function of the quasihomothetic form $e_i(p, \bar{u}_i) = a_i(p) + b(p)\bar{u}_i$, where the function $b(\cdot) > 0$ is *common* across consumers, consumer i's money metric is $M_i(x, p) = a_i(p) + b(p)u_i(x)$, where u_i is a corresponding representation of i's preferences. Each money metric is concave and the set of maximizers of the sum does not depend on the reference price. In particular, the sum is maximized at a feasible allocation *if and only if* it is Pareto optimal.

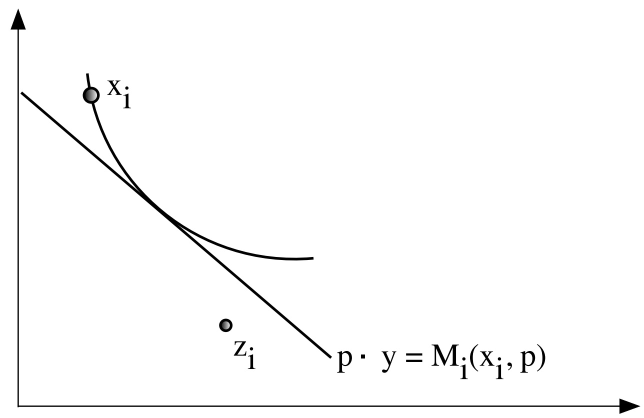

Figure 3: The curve is the boundary of the at-least-as-good-set at x_i. Since $p \cdot z_i < M_i(x_i, p)$, point z_i cannot be strictly preferred to x_i. If $\succsim_i$ is complete, then $x_i \succ z_i$. It follows that, in the notation of Bernheim and Rangel (2009), that $(z_i, x_i) \notin R'$; and if z_i and x_i are comparable under $\succsim_i$, then $x_i P^* z_i$ in their notation.

ranks other allocations. What can be said depends on preferences in the economy. If each consumer's preferences are represented by i's money metric, then of course the money-metric sum is an individualistic B-S SWF and so *Pareto consistent*: if the sum rises with an allocation change then at least one consumer is strictly better off. Two other cases are of interest.

1. *Each consumer's preferences are complete and transitive, but for some consumer i, $M_i(\cdot, p)$ does not represent i's preferences.* In this case the sum is no longer a *individualistic* B-S SWF. But the sum is still Pareto consistent. This fact follows from our equation (6).

2. *Some consumer's preferences are either incomplete or intransitive.* In this case the sum of money metrics need not be Pareto consistent. But the sum can still be used to identify aggregate consumption plans that are *not* a potential Pareto improvement relative to some allocation $\mathbf{x}$. The important fact here is that a consumer's money metric can be used to identify bundles that are not better than a given bundle (even though it of course does not represent preferences). Fix a consumer, a reference price p and a consumption plan x_i. If $p \cdot y_i < M_i(p, x_i)$, then $y_i \not\succ x_i$. Of course if $x_i \in d_i(p, p \cdot x_i)$, this is just the statement that x_i is directly strictly revealed preferred to y_i. But the money metric reveals more than this, since the point x_i need not be demanded at p; see Figure 2. In Section 7 we show that we can aggregate this result to rule out allocations that are not potential Pareto improvements (Proposition 2). □

We repeat that the conclusion of Theorem 2(a) holds even under cases 1 and 2 of Remark 5: a competitive equilibrium allocation maximizes the money-metric sum (at the associated competitive equilibrium price) even if the money metric does not represent preferences of some consumers. The sum can still be used to identify candidates for optimal allocations.

Remark 6 (Money-metrics and Bernheim-Rangel's (2009) *unambiguous choice* relations)**.** Bernheim and Rangel (2009) propose two *unambiguous choice* relations that extend the usual revealed-preference relation. Their purpose is to give a foundation for choice-based welfare analysis in non-standard (behavioral) decision theories in which choices violate the weak axiom of revealed preference—and so any rationalizing binary relation is either incomplete or intransitive. They work in a setting of a choice function defined over a family of constraint sets which includes all finite subsets of an underlying "consumption set," including all two-point constraint sets. In a general equilibrium setting, of course, budget sets are included in the family of constraint sets. They assume that choice in any constraint set in the family is nonempty. They define an *unambiguously chosen over* relation P^* by xP^*y if y is never chosen if x is available. They also define a *weakly unambiguously chosen over* relation R' by $xR'y$ if, whenever x is available and y is chosen, x is also chosen.[40]

The money metric in case 2. of our Remark 5 can be used to uncover whether alternatives do or do not stand in these relations. Specifically, if $p \cdot y < M(x, p)$, then (i) either $x \succ y$ and therefore xP^*y, or x and y are not comparable, in which case it *cannot* be that $yR'x$; and if in addition $\succsim$ is complete, then $x \succ y$ and therefore xP^*y. See Figure 2.

Not all behavioral models can be described as choosing undominated elements according to a possibly-incomplete or intransitive binary relation. (Richter (1971) characterizes the set of demands that can be rationalized by *some* binary relation.) It turns out that we can reformulate the money metric for these cases using the Bernheim-Rangel relation P_i^*. Let $\mathcal{R}_i^*(x) = \{x' \in X \mid x' P_i^* x\}$, and replace $\mathcal{R}_i(x)$ with $\mathcal{R}_i^*(x)$ in the definition of the money metric, (3). Define the *behavioral money metric* (or Bernheim-Rangel money metric) by $M_i^{br}(x, p) = \min_{x' \in \mathcal{R}_i^*(x)} p \cdot x'$. Let $d_i^*(p, w) = \{y \in B(p, w) \mid x \in B(p, w) \Rightarrow (x, y) \notin P_i^*\}$. Note that any choice of consumer i in the budget set $B(p, w)$ must be an element of $d_i^*(p, w)$. $\square$

The next theorem asserts that the conclusions of Theorem 1(a) and Theorem 2(a) holds for the behavioral money metric.

Theorem 3 (Behavioral money-metric)**.** *If for each consumer i P_i^* is locally nonsatiated (for every $x \in X$ and neighborhood N of x, there is a $y \in N \cap X$ with yP_i^*x), then the conclusions of Theorem 1(a) and Theorem 2(a) hold for the behavioral money metric, M_i^{br}.*

Since Bernheim and Rangel (2009, Theorem 7) prove a generalized first welfare theorem for a behavioral competitive equilibrium, this modified money-metric sum can be used to identify

[40] A motivation for the "unambiguously-chosen-over" relations is that one alternative is always chosen over another independently of any "frame" of the decision problem.

candidate *generalized* Pareto optima (on their definition).[41] Of course the demand d^* in the theorem statement is rationalized by the relation P^*, but the consumer's actual demand need not be.[42] And the conclusions hold for *any* selection from d_i^*.

Proof of Theorem 3: If the conclusion of Theorem 1(a) holds for the behavioral money metric, so does the conclusion of Theorem 2(a). But the conclusion of Theorem 1(a) follows from the proof of Theorem 1(a) word-for-word after substituting $\mathcal{R}^*$ for $\mathcal{R}$, M^{br} for M, and d^* for d.□

Remark 7 (A Representative Consumer?)**.** A natural question arises as to the sense which the sum of money-metrics gives us a "representative consumer." Remark 5 suggests a limitation of this interpretation. A *positive representative consumer* refers to a hypothetical consumer who owns all the economy's wealth and has a preference relation that generates the economy's *aggregate* demand at every price-aggregate wealth pair. Maximizing the sum of money-metrics at a price-aggregate wealth pair generates a *set* consisting of every *vector* of demands generated by any distribution of that aggregate wealth (Corollary 2). The italicized words indicate the differences. An example might help further illustrate the differences. Consider an exchange economy with two consumers and two goods. Consumer i only cares for good i, the more the better. Consumer i's endowment is a fraction α_i of the aggregate endowment $\omega_i = \alpha_i\omega$. (That is, endowments are *collinear*.) This economy has an *aggregate* demand generated by Cobb-Douglas preferences: $U(x_1, x_2) = x_1^{\alpha_1}x_2^{\alpha_2}$, so clearly there is a positive representative consumer for this economy. The representative consumer has homothetic preferences. The economy, however, is clearly is not Gorman: the consumers do not have wealth expansion paths that are straight lines *with common slope*. The sum of money-metric utilities is $M = p_1x_{11} + p_2x_{22}$ where x_{ii} is consumer i's consumption of good i. Maximizing this sum subject to $p_1x_1 + p_2x_2 \leq w = p_1\omega_1 + p_2\omega_2$ gives *all aggregate* consumption plans that meet the budget constraint (with consumer i getting all of good i). The "money-metric aggregate consumer" is indifferent between Pareto Optimal allocations at a given aggregate wealth level. That "consumer's" maximum value is simply aggregate wealth at any such allocation.[43] □

[41] As a highly-stylized framing example that satisfies our assumptions on P^* and $\mathcal{R}$, fix a bundle $x^0 \in X$. Suppose that a consumer i's demand at (p, w) maximizes $u_i(x)$ if $x^0 \in B(p, w)$, but maximizes $v_i(x)$, if $x^0 \notin B(p, w)$, where u_i and v_i are locally nonsatiated and strictly quasiconcave, but *not* ordinally equivalent.

[42] For example suppose that $L = 2$ and that $d^*(1, 1, 1) = \{(0, 1), (1, 0)\}$ and $d^*(2, 1, 1) = \{(0, 1), (1/2, 0\})$. This correspondence can be generated by a binary relation in which every interior consumption point along either budget line is dominated by one of the two endpoints. But the selection $f(1, 1, 1) = (0, 1)$ and $f(2, 1, 1) = (1/2, 0)$ is not. The selection violates Sen's α for choice functions, that a point chosen in some set remains chosen in any subset that contains it.

[43] To make the aggregate money-metric consumer look more like a representative consumer, one is tempted first to maximize the sum of money metric utilities with respect to an allocation x' subject to the constraint that the aggregate consumption plan is no larger than some fixed vector $(x_1, ..., x_L)$, and to interpret the resulting value function $\mathbf{M}(p; x_1, ..., x_L)$ as a representation of preferences of a representative consumer. In the example, this maximized sum-of-money-metrics reduces just to $\mathbf{M}(p; x_1, x_2) = p_1x_1 + p_2x_2$. Maximizing this subject to the aggregate budget set $p_1x_1 + p_2x_2 \leq w$ gives all affordable aggregate plans. With this procedure inefficient allocations are precluded by the "min" inequalities in the SP: in order for $\lambda = 1$ to minimize $\mathcal{L}(x, \lambda)$, it must be that $x_{21} = x_{12} = 0$.

5 Nonconcave Money-Metrics and (In)Equities

As we mentioned in our Introduction, one reason for the waning of the money metric in applied welfare analysis was the money-metric non-concavity. We briefly consider some of its implications here; we also relate it to recent developments in social choice theory that show the potential usefulness of money metrics for welfare and criteria for equity in allocations. As we point out in Remark 1 to Theorem 4, allocations that maximize the money-metric sum can be just or unjust (whatever one's notions of these qualities), *provided* that they are supportable as competitive equilibrium and that the associated competitive equilibrium price is used as the reference price. Any potential (in)justice there does not rely on the (non)concavity of the money metric.

To set the stage for our discussion, we begin with a simple example of how maximizers of the sum of *non*-concave money metrics might be inequitable.

Example 1 (Money-metrics and inequality)**.** *Consider an exchange economy consisting of two consumers with identical preferences $\succsim$ on $X = \mathbb{R}^L_+$ that satisfy the Standard Assumptions, and in addition are convex and* monotone *($x >> y$ implies that $x \succ y$), but* not *quasi-homothetic; and let $M(\cdot, p)$ be their common continuous,* non-*concave (at some p) money-metric.*[44] *Then we know that it is not mid-point concave at some p, which is to say,*

$$2M(\tfrac{1}{2}x + \tfrac{1}{2}y, p) < M(x, p) + M(y, p) \text{ for some } x \text{ and } y \in X. \tag{9}$$

Let the aggregate endowment be $x + y$. Then the money-metric sum is higher at the unequal allocation which gives one consumer x and the other y than if they both get $(x + y)/2$. □

Our rehabilitation obviously does not rule out this example. But we note that it depends on the reference price. By Theorem 2, if we consider a reference price p^* that supports equal division as a competitive equilibrium allocation in this exchange economy, then the sum of *those* money-metrics would *not* increase with a move to an unequal allocation.

As mentioned already, Bosmans, Decancq, and Ooghe (2018) (BDO) present a social-choice rehabilitation of money metrics. It consists of a representation theorem for a *social preference relation*, that is, a relation which specifies, for each possible profile $\succsim = \{\succsim_1, ..., \succsim_I\}$ of preferences in an economy, a binary relation $R_\succsim$ over allocations. It asserts that social preferences satisfy six axioms if and only if there is a reference price vector p and a strictly increasing, *Schur-concave* function[45] W such that, for each preference profile ,

$$\mathbf{x}' R_\succsim \mathbf{x} \quad \text{if and only if} \quad W(M_1(x'_1, p), ..., M_I(x'_I, p)) \geq W(M_1(x_1, p), ..., M_I(x_I, p)).$$

[44] For our purpose here, preferences are *quasi-homothetic* if there is an indirect utility is of the form $V(p, w) = a(p) + b(p)w$ for some functions $a(\cdot)$ and $b(\cdot)$. For the continuity claim in Equation (9), we appeal to Weymark (1985a, Proposition 2).

[45] A function f on a nonempty convex set $D \subset \mathbb{R}^n$ is *Schur-concave* if $f(x) \geq f(y)$ whenever y *majorizes* x, that is, whenever $\sum_i^k y_{[i]} \geq \sum_i^k x_{[i]}$ for $k = 1, ..., n-1$, with equality for $k = n$, where $y_{[n]} \geq y_{[n-1]} \geq y_{[1]}$ and $x_{[n]} \geq x_{[n-1]} \geq x_{[1]}$.

The money-metric sum is intended to capture efficiency, the Schur-concave function W, equity. We note that BDO work in a classical setting, with monotone, convex preferences on $\mathbb{R}^L_+$. In this classical setting, a money metric represents a consumer's preferences.

Since the *sum* of money metrics is a special case of the representation theorem that they prove,[46] the BDO rehabilitation likewise does not rule out Example 1. Their novel axiom and the one that they use to capture equity considerations, they refer to as the *Efficiency-Preserving Transfer Principle.* To explain it, consider an economy in which two of the consumers in it have the same preferences, and, furthermore, an allocation in which one of these consumer has more of every good than the other. The efficiency-preserving transfer principle asserts that if the richer consumer transfers some consumption to the poorer (but remains weakly richer), *and the Scitovsky set for the economy remains unchanged by the transfer*, then the new allocation is weakly preferred in the social order.[47]

The question is, how does Example 1 escape this axiom? One of two ways. Either the preferences of the consumers are quasi-homothetic, in which case their money-metrics are concave and the example cannot arise; or, in the second case, they are not quasi-homothetic. In the second case, the Scitovsky sets typically *vary* as the allocation of a fixed stock of goods changes, and so the axiom does not apply. Indeed, Gorman (1953) proved that the Scitovsky set is globally preserved under transfers if and *only* if the preferences of consumers are quasi-homothetic with a common slope of their wealth-expansion paths.[48]

The BDO representation, including the special case of a *sum* of money metrics, violates what Fleurbaey and Maniquet (2011) call the *Transfer Among Equals* axiom (their Axiom 2.5, and henceforth TAE); namely if a society includes two consumers with the same preferences, and one of them has more of every good than the other, then a transfer of goods from the richer to the poorer without reversing their rank improves social welfare. This of course is the same as BDO's efficiency-preserving transfer principle, minus the requirement that the transfer preserve the Scitovsky set.

We want to emphasize that the TAE axiom can fail even when the money-metric is concave, having seen already that Example 1 directly exploits the lack of concavity of the money metric.

Example 2 (Concave money-metrics and inequality)**.** *Consider an economy with two consumers with identical preferences represented by the Cobb-Douglas function* $u(x_1, x_2) = \sqrt{x_1, x_2}$*. The money metric utility for this function is* $M(x,p) = p_1^{1/2} p_2^{1/2} \sqrt{x_1, x_2}$*. Set the reference price equal to* $p = (1,1)$*, so that the money metric equals the preference representation. Since the money-metric is concave, the construction in Example 1 cannot arise. Consider the unequal*

[46] The function $f(y) = \sum y_i$ is (weakly) Schur-concave.

[47] The *Scitovsky set* for a population of consumers $(1, ..., I)$ at a reference allocation $(x_1, ..., x_I)$ is the sum of at-least-as-good-as sets from this allocation. Formally, at an allocation $(x_1, ..., x_I)$ it equals $\{x' \in X \mid x' = \sum_{i=1}^{I} x'_i,$ $u_i(x'_i) \geq u_i(x_i),\ i = 1, ..., I\}$, where for each consumer i, u_i represents that consumer's preferences.

[48] To be sure, Bosmans, Decancq, and Ooghe (2018) write in their Footnote 14 that "This axiom does not cover all cases where the distributions before and after the transfer are both efficient (equal marginal rates of substitution). Indeed, in some such cases the Scitovsky boundaries do not coincide, but rather intersect at the societal bundle."

allocation $((1,1),(6,6))$. *The sum of money-metrics is 7. The sum of money-metrics at the more-equal allocation* $((3,2),(4,5))$ *is* ≈ 6.92. *This sum of identical, homothetic, concave money-metrics therefore violates the Fleurbaey-Maniquet* TAE *axiom.* $\square$

The example helps illustrate a fundamental point: ***some*** *inequities of the money-metric sum do not depend on a failure of concavity.* It is a general property of additive B-S SWF functions, even well-behaved ones, and indeed of the BDO non-additive generalization, short of the maximin limit

$$W\big(M_1(x_1,p), M_2(x_2,p), ..., M_I(x_I,p)\big) = \min\{M_1(x_1,p), M_2(x_2,p), ..., M_I(x_I,p)\},$$

which violates BDO's strong monotonicity axiom.

The Transfer Among Equals axiom has obvious normative appeal. Fleurbaey and Maniquet (2011, Theorem 3.1) prove a startling result. When combined with two other seemingly innocuous axioms, this transfer axiom implies what they call *Absolute Priority to the Worst Off*: if an economy has two consumers with the same preferences, and one of them has more of every good than the other, then social welfare rises with any feasible decrease in consumption of the richer and increase in consumption of the poorer, *even if some goods are thrown away.* For example, the "leaky bucket" transfer between equals from the allocation $\big((1,1),\ (100,100)\big)$ to $\big((1+\varepsilon,1+\varepsilon),\ (1+2\varepsilon,1+2\varepsilon)\big)$ is socially a (weak) improvement for any $\varepsilon \in (0,1)$, however close to 0. The Absolute Priority axiom, however, seems less appealing than the Transfer among Equals axiom since it justifies destruction of resources. One welfare function that satisfies the transfer axiom Fleurbaey-Maniquet call the *Walrasian Egalitarian* function. It is defined using the profile of money-metrics and is given by

$$\mathcal{W}(\mathbf{x}) = \max_{\{p \,|\, p\cdot\omega=1\}} \min\{M_1(x_1,p), ..., M_I(x_I,p)\}. \tag{10}$$

Note that the money-metric reference price potentially varies with the allocation. This way of choosing the reference price is analogous to putting more weight on the least-well-off person in this sense: for a fixed price p, normalized so that $p \cdot \omega = 1$, consider $d_i(p, p \cdot \lambda\omega)$ for $\lambda \in [0,1]$, consumer i's demand at price p and wealth equal to the value of a fraction λ of the economy's endowment; the value $\mathcal{W}(\mathbf{x})$ in (10) is the smallest value of λ such that some consumer i is indifferent between $d_i(p, p \cdot \lambda\omega)$ and x_i, i's consumption in the allocation; and any such indifferent consumers are deemed to be the worst off; see Fleurbaey and Maniquet (2011, Figure 5.3). They prove that, for exchange economies, the set of maximizers of $\mathcal{W}$ on $\mathcal{A}$ *equals* the set of egalitarian competitive allocations, that is, the set of competitive equilibrium allocations starting from equal endowments (Theorem 5.7). They go on to observe that the "min" function in equation (10) is the *only* function of the money-metric profile satisfying this equality that *also* satisfies the Transfer among Equals axiom.

Their theorem is presented in a classical environment with convex, monotone preferences defined on the consumption set $\mathbb{R}^L_+$, so that money-metrics represent consumer preferences. The main surprise of our theorems, to repeat, is that they hold in non-classical environments in which money metrics need not represent preferences, and indeed when there is no preference representation at all. Fleurbaey and Maniquet (2011, Chapter 5) consider extensions to nonconvexities in preferences and consumption sets, but maintain completeness and transitivity. As mentioned already, Fleurbaey and Schokkaert (2013) extend Bernheim and Rangel (2009) to allow for equity considerations when individual preferences are incomplete. An interesting open question is what properties the representation in Bosmans, Decancq, and Ooghe (2018) has in *non-classical* environments involving not just non-convexities, but incomplete or intransitive preferences, and indeed in behavioral models not rationalizable by any binary relation. That money-metrics are well-defined in these non-classical settings and that they have interesting welfare properties gives some hope for a useful answer.

6 Relationship to the Benefit and Distance functions

We now consider the relationship between money metric utility and two other functions: the benefit function of Luenberger (1992a) and the distance function analyzed by Gorman (1970).[49] Throughout this subsection, we impose the Standard Assumptions on preferences for each consumer and assume that each consumer i's preferences has a utility representation, u_i on X_i. We point out that versions of Theorems 1 and 2 hold for these functions. We refer the reader to Luenberger (1992a) and Gorman (1970) for proofs of the properties of the benefit and distance functions that we assert here.[50]

To elaborate, let $x_i \in X_i$, $\bar{u}_i \in Range(u_i)$, $g \in \cap_{i=1}^I X_i$ (assuming the intersection is nonempty). Define $b_i(x_i, \bar{u}_i)$, consumer i's *benefit function* at $(x_i, \bar{u})$, to be the number β which solves $u(x_i - \beta g) = \bar{u}_i$ if a solution exists, $-\infty$ otherwise. And define $f_i(x_i, \bar{u}_i)$, consumer i's *distance function* at $(x_i, \bar{u}_i)$ to be the number γ which solves $u_i(x_i/\gamma) = \bar{u}_i$ if a solution exists, $-\infty$ otherwise.[51] To convert these numbers into money units, define $B_i(x_i, \bar{u}_i, p) = b_i(x_i, \bar{u}_i, p) p \cdot g$ and $F_i(x_i, \bar{u}_i, p) = f_i(x_i, \bar{u}_i) e_i(p, \bar{u}_i)$.[52]

Parallel to Theorem 1, one can show that if $x_i^* \in d_i(p, w)$, then $(x_i^*, 1)$ solves the saddlepoint inequalities with either $B_i(x_i, V_i(p, w), p)$ or $F_i(x_i, V_i(p, w), p)$ replacing the consumer's money-metric in the Lagrangian. Luenberger (1992a) proved the first, without using the language of saddlepoints. As a corollary it follows that x_i^* maximizes both $B_i(x_i, V_i(p, w), p) - p \cdot x_i$ and $F_i(x_i, V_i(p, w), p) - p \cdot x_i$ on X_i—again ignoring the budget constraint. An important lemma in the proof of this fact spells out the relationship between the expenditure functions and the

[49] The distance function was introduced in producer theory by Shepard (1953) and in consumer theory by Malmquist (1953); Deaton (1979) reviews some of the history of this function.

[50] Gorman (1970) is reprinted in Blackorby and Shorrocks (1995); proofs of the properties we use here can also be found in Deaton (1979) and Luenberger (1995). Finally, we refer the reader to Fuchs-Seliger (1990a) and Fuchs-Seliger (1990b) for a reconsideration of these measures in the context of revealed preference theory.

[51] Gorman (1970) works in a setting for which the distance function is real-valued.

[52] Luenberger (1992a) normalizes prices so that $p \cdot g = 1$, so the extra multiplicative term drops out.

benefit and distance functions:

$$e_i(p, \bar{u}_i) = \min_{x_i \in X_i} \{p \cdot x_i / f_i(x_i, \bar{u}_i)\} = \min_{x_i \in X_i} \{p \cdot x_i - b_i(x_i, \bar{u}_i)\}, \tag{11}$$

where, for the first equality, prices are normalized so that $p \cdot g = 1$.

We know of no proof of the fact that if $x_i^* \in d_i(p, w_i)$ then x_i^* maximizes $F_i(x_i, V_i(p, w), p) - p \cdot x_i$ on X_i. We sketch it here. Clearly $p \cdot x_i^* = w$ and $f(x_i^*, V(p, w_i)) = 1$, so the objective equals 0 at x_i^*. Substitute $V_i(p, w_i)$ into the utility argument on both sides of the first equality in (11) to get $w = \min_{x_i \in X_i}(p \cdot x_i)/f(x_i, V_i(p, w_i)) \leq p \cdot x' / f(x', V_i(p, w_i))$. For any affordable x_i', $p \cdot x_i' \leq w$, so the preceding inequality implies that $f(x_i', V_i(p, w_i))w_i \leq p \cdot x_i'$, so the objective is *at most* 0 for any affordable $x_i' \in X_i$. Define $B(\mathbf{x}, \bar{\mathbf{u}}) = \sum_i B_i(x_i, \bar{u}_i)$ and $F(\mathbf{x}, \bar{\mathbf{u}}) = \sum_i F_i(x_i, \bar{u}_i)$, where $\bar{\mathbf{u}} = (\bar{u}_1, ..., \bar{u}_I)$ is a utility profile. Parallel to Theorem 2, if $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium, then $(\mathbf{x}^*, \mathbf{y}^*)$ maximizes $B(\mathbf{x}, \mathbf{u}^*) + p^* \cdot y - p^* \cdot x$ and $F(\mathbf{x}, \mathbf{u}^*) + p^* \cdot y - p^* \cdot x$ on $\mathcal{A}$, where $u_i^* = V_i(p^*, w_i(p^*))$ for $i = 1, ..., I$. Luenberger (1992b) proves this last fact for the aggregate benefit function.[53] We can now present

Proposition 1. *The conclusion of Theorem 2(a) continues to hold if each M_i is replaced by F_i; or each M_i is replaced by B_i.*

For special cases $F_i(x_i, V_i(p, w), p)$ and $B_i(x_i, V_i(p, w), p)$ equal the money-metric $M_i(x_i, p)$ (sometimes up to an additive constant). For $X_i = \mathbb{R}_+^L$, $F_i(x_i, V_i(p, w), p) = M_i(x_i, p)$ when i's preferences are homothetic (the consumer's wealth drops out of F_i when preferences are homothetic); and $B_i(x_i, V_i(p, w), p) = M_i(x_i, p) + w$ when i's preferences are quasi-linear with respect to some good ℓ and the reference bundle g equals the ℓ-th unit vector. Unlike the money-metric function, the benefit and distance functions are concave *whenever* preferences are convex, and so they avoid the inegalitarian implications of Example 1. Like the money-metric function, the benefit and distance functions need not represent preferences: as already mentioned, there are convex preference relations not representable by a concave function (Kannai (1977)).[54] They each also fail property (6) that the money metric satisfies. It is easy to show that the aggregate benefit and distance functions are not Pareto consistent: a change can increase the measure, but every person is worse off with the change. Such examples are easy to construct in single-consumer economies even for preferences that are Cobb-Douglas (for the benefit function) or quasilinear (for the distance function). In sum, the aggregate benefit and aggregate distance functions can do better than the money-metric sum on equity, they often do worse on efficiency grounds. We hope to explore these points more fully in future work.

[53] It is hard to believe that the aggregate distance function $F(x, \bar{\mathbf{u}}^*)$ has not been used by someone, and that the Proposition has not been proved for it, but we do not know a reference for it.

[54] Compare with the first paragraph in Chambers, Chung, and Färe (1996).

7 Potential Improvements and Local Cost-Benefit Analysis

We now show how the money-metric sum can either rule out or identify potential Pareto improvements. We start with the first issue. We say that an allocation $\mathbf{x}'$ is a *potential Pareto improvement* over allocation $\mathbf{x}$ if there is some allocation $\mathbf{x}''$ with $x'' = x'$, $x_i'' \succsim x_i'$ for every consumer i with the preference strict for at least one consumer i (where $x^\theta = \sum_i x_i^\theta$ for $\theta \in \{','' \}$). It is a *strict* potential Pareto improvement if the preference can be taken to be strict for each consumer i.

Proposition 2. *Consider a consumption allocation* $\mathbf{x} = (x_1, .., x_I)$ *and fix a price* $p >> 0$. *If* $p \cdot x' < \sum_i M_i^*(x_i, p)$ *for an allocation* $\mathbf{x}'$ *and each consumer's preferences satisfy the Minimal Assumptions, then* $\mathbf{x}'$ *is not a strict potential Pareto improvement over* $\mathbf{x}$.[55]

Proof: Consider a consumption allocation $\mathbf{x}$ and a price $p >> 0$. and an aggregate consumption x'. If $p \cdot x' < \sum_i M_i(x_i, p)$, then the aggregate consumption plan x' cannot be a potential Pareto improvement over $\mathbf{x}$. If $x_i' \succ x_i$ for $i = 1, ..., I-1$, then $p \cdot x_i' \geq M_i(x_i, p)$ for $i = 1, ..., I-1$. Sum over i to find $p \cdot \sum_1^{I-1} x_i' \geq \sum_1^{I-1} M_i(x_i, p)$. Since $p \cdot x' < \sum_i M_i(x_i, p)$, it follows that $p \cdot x_I' < M_I(x_I, p)$, so $x_I' \not\succ x_I$ and $\mathbf{x}'$ is not a strict potential Pareto improvement over $\mathbf{x}$. $\square$

7.1 Radner's Local Welfare Measure

We now establish a partial converse to Proposition 2. Toward that end, we remind the reader of Radner's (1993 [1978]) local welfare measure. He considers an economy in which each consumer i's preferences are represented by a C^2, concave, nonsatiated function u_i on $X_i = \mathbb{R}_+^L$. Let $\tilde{y} : [0, 1] \to Y$, where $Y \subseteq \mathbb{R}^L$ is a connected, nonempty production set. Radner (1993) refers to the scalar α as indexing *projects*. Suppose that $\tilde{y}$ is differentiable at $\alpha = 0$. The total supply for project α is $\omega + \tilde{y}(\alpha)$. A feasible *consumption allocation* of the total supply $\omega + \tilde{y}(\alpha)$ is a point $(x_1, ..., x_I) \in \mathbb{R}_+^{IL}$ such that $\sum_i x_i \leq \omega + \tilde{y}(\alpha)$. A feasible consumption allocation $\mathbf{x} = (x_1, ..., x_I)$ is a *valuation equilibrium* relative to a price $p \in \mathbb{R}_{++}^L$ if, for each $i = 1, ..., I$, $x_i \in d_i(p, p \cdot x_i)$. We stress that the model is silent about how the aggregate production plan is chosen; in particular it need not be part of a competitive equilibrium. The valuation equilibrium assumption assures that the *consumption allocation* of a given supply is Pareto optimal: the source of any inefficiency in the economy is the aggregate production plan.

Radner (1993 [1978]) proposed this local measure of welfare,

$$p^0 \cdot \tilde{y}'(0), \tag{12}$$

where the feasible allocation $\tilde{\mathbf{x}}(0) = (\tilde{x}_1(0), ..., \tilde{x}_I(0))$ of the initial supply $\omega + \tilde{y}(0)$ is a valuation equilibrium with respect to p^0. He assumes that each $\tilde{x}_i(0)$ is in the interior of i's consumption set, so the consumer's first-order conditions hold as an equality. Equation (12) is the derivative

[55] Under the additional assumption that preferences are strongly upper nonsatiated—Definition 4 in the next subsection—the allocation $\mathbf{x}'$ cannot be even a potential Pareto improvement.

of $p^0 \cdot (\tilde{y}(\alpha) - \tilde{y}(0))$ at $\alpha = 0$; in words, it is the change in the value of the aggregate production plan measured by the initial valuation equilibrium prices. He goes on to give conditions under which, if (12) is positive, then for some $\bar{\alpha} > 0$, the supply $\omega + y(\alpha)$ is a *potential Pareto improvement* over the supply $\omega + y(0)$ for every $\alpha \in (0, \bar{\alpha})$: the total supply of $\omega + y(\alpha)$ can be allocated so that each consumer i's prefers its consumption in this allocation to $\tilde{x}_i(0)$.

7.2 The Money-metric Sum and Radner's Measure

In what follows let $\tilde{\mathbf{x}}(\alpha) = \omega + \tilde{y}(\alpha)$ be a consumption allocation of the supply $\omega + \tilde{y}(\alpha)$. Define $\mathcal{M}(\alpha) = \sum_i M_i(\tilde{x}_i(\alpha), p^0)$, the sum of money-metric utilities. The next result asserts that $\mathcal{M}'(0)$ equals Radner's local welfare measure.

Proposition 3 (The money-metric sum and Radner's measure)**.** *Suppose that (a) each $\succsim_i$ satisfies Minimal assumptions on $X = \mathbb{R}^L_+$; (b) $\tilde{\mathbf{x}}(0) >> 0$ and $\tilde{\mathbf{x}}(\cdot)$ is differentiable at $\alpha = 0$; (c) each $M_i(\cdot, p^0)$ is differentiable at $x_i = \tilde{x}_i(0)$; and (d) the allocation $\tilde{\mathbf{x}}(0)$ is a valuation equilibrium relative to some $p^0 \in \mathbb{R}^L_{++}$. Then*

$$\mathcal{M}'(0) = p^0 \cdot \tilde{y}'(0).$$

We use the next easy fact in the proof.

Lemma 2. *Suppose that $\succsim_i$ satisfies the Minimal assumptions. Fix $x^0 \mathbb{R}^L_{++}$ and $p^0 \in \mathbb{R}^L_{++}$ with $x^0 \in d_i(p, p \cdot x^0)$ If $M_i(\cdot, p^0)$ is differentiable at $x = x^0$, then*

$$D_x M_i(x^0, p^0) = p^0. \tag{13}$$

Proof of Lemma 2: Since $x^0 \in d_i(p^0, p^0 \cdot x^0)$, x^0 maximizes $M_i(x, p^0) - p^0 \cdot x$ on $\mathbb{R}^L_+$ by Theorem 1(a). And since $M_i(\cdot, p^0)$ is differentiable at x^0_i, the necessary Kuhn-Tucker conditions must hold at $x_i = x^0_i$. Since $x^0_i >> 0$, the Kuhn-Tucker conditions hold as equalities, and (13) holds. □.

Proof of Proposition 3: By Lemma 2, and (a)-(d), each $M_i(\tilde{x}_i(\alpha), p^0)$ is differentiable at $\alpha = 0$ and the derivative equals $p^0 \cdot \tilde{x}'_i(0)$. Sum over all consumers and use feasibility to find $\mathcal{M}'(0) = \sum_i p^0 \cdot \tilde{x}'_i(0) = p^0 \cdot \tilde{y}'(0)$. □

Under the additional assumptions that for each $\alpha \in [0, 1]$, $(\tilde{x}(\alpha), \tilde{p}(\alpha))$ is a competitive equilibrium of an exchange economy in which consumer i's endowment is $\tilde{x}_i(0) + (1/I)(\tilde{y}(\alpha) - \tilde{y}(0))$; and $\tilde{p}(\cdot)$ differentiable at $\alpha = 0$, Radner (1993 [1978], p. 136) proved that if $p^0 \cdot \tilde{y}'(0) > 0$, then for all small-enough $\alpha > 0$, each consumer i strictly prefers $\tilde{x}_i(\alpha)$ to $\tilde{x}_i(0)$: the new supply $\omega + \tilde{y}(\alpha)$ is a strict potential Pareto improvement over the initial allocation; that is, the new supply can be allocated so that each consumer i prefers its new consumption to $\tilde{x}_i(0)$ and that preference is strict for at least one consumer i. By exploiting properties of the money metric, we come to the potential Pareto improvement conclusion while dispensing with the assumptions that $\mathbf{x}(\alpha)$ is a valuation equilibrium with respect to some price $p(\alpha)$; and that $p(\cdot)$ is differentiable at $\alpha = 0$.

Under the Standard Assumptions the conclusion foll by equation (6). With intransitive preferences, (6) can fail. In the proof we construct a local version of equation (6) that holds without transitivity. In place of transitivity, we require that preferences satisfy a strengthening of local nonsatiation used by Danan (2008).[56]

Definition 4. *Let $X \subset \mathbb{R}^L_+$ be nonempty and closed. A reflexive binary relation $\succsim \subset X \times X$ is* ***strongly upper nonsatiated*** *if, for every $x \sim y$ and every open neighborhood N of y, there is a $z \in N \cap X$ with $z \succ x$.*

The assumption rules out "thick" indifference sets; see Figure 4 in Section 9.1.

Proposition 4 (Local Potential Pareto Improvements)**.** *In addition to (b)-(d) of Proposition 3, assume that (e) for each consumer i, $\succsim_i$ is complete, closed, and LNS; and for some consumer j, $\succsim_j$ is strongly upper nonsatiated. If $\mathcal{M}'(0) = p^0 \cdot y'(0) > 0$ then for every α' in some nonempty interval $(0, \bar{\alpha})$, the total supply at α' is a potential Pareto improvement over the allocation at $\alpha = 0$. If each consumer's preferences are strongly upper nonsatiated, then the is a strict potential Pareto improvement.*

Proof of Proposition 4: Assume throughout that conditions (b)-(e) hold and that $p^0 \cdot y'(0) > 0$. Define $\tilde{x}_i(\alpha) = x_i^0 + \frac{1}{I}(y(\alpha) - y(0))$ for every $\alpha \in [0, 1]$ and $i = 1, ..., I$. The allocation $\tilde{\mathbf{x}}(\alpha)$ is clearly feasible for the economy with total supply $\omega + y(\alpha)$ for every $\alpha \in [0, 1]$. Let $\mathcal{M}_i(\alpha) = M_i(\tilde{x}_i(\alpha), p^0)$. By Lemma 2, $\mathcal{M}_i'(0) = p^0 \cdot \tilde{y}'(0)/I > 0$, which implies $\mathcal{M}_i(\alpha) > \mathcal{M}_i(0)$ for consumer i for every α in some interval $(0, \alpha_i)$. By Theorem 1(a), $M_i(x_i^0, p^0) = p^0 \cdot x_i^0$, and so $M_i(\tilde{x}_i(\alpha), p^0) > p^0 \cdot x_i^0$ for every $\alpha \in (0, \alpha_i)$. It follows that x_i^0 is not feasible for problem (3) at $(\tilde{x}_i(\alpha), p^0)$, that is, $x_i^0 \notin \mathcal{R}_i(x_i^*(\alpha))$ for every $\alpha \in (0, \alpha_i)$. Since $\succsim_i$ is complete, $\tilde{x}_i(\alpha) \succsim_i x_i^0$ for every $\alpha \in (0, \alpha_i)$. Now suppose that $\succsim_j$ is strongly upper nonsatiated. Then $x_j^0 \notin \mathcal{R}_j(x_j^*(\alpha))$ implies $\tilde{x}_j(\alpha) \not\sim x_j^0$, so $\tilde{x}_j(\alpha) \succ_j x_j^0$ for every $\alpha \in (0, \alpha_j)$. Set $\bar{\alpha} = \min\{\alpha_1, ..., \alpha_I)$. If each consumer's preferences are strongly upper nonsatiated, then clearly, for some $\alpha^* > 0$, $\tilde{x}_i(\alpha) \succ_j x_i^0$ for each consumer i and every $\alpha \in (0, \alpha^*)$. $\square$

Proposition 4 extends Radner's (1993 [1978]) theorem to intransitive preferences.[57] We do not extend it to incomplete preferences for two reasons. First, with incompleteness, potential Pareto improvements are harder to generate.[58] More seriously, the differentiability assumption

[56] As we show in the proof of Proposition 5 in Section 9.1, two sufficient conditions for it are local nonsatiation plus either transitivity *or* strong convexity.

[57] Fountain (1981) points out difficulties for standard welfare measures for changes "in the large" when consumers have intransitive preferences, but does not touch on local changes, the subject of Proposition 4.

[58] This first problem can be finessed by changing the definition of a Pareto improvement to "no consumer strictly prefers its initial consumption to the new consumption, and some consumer strictly prefers its new consumption to the old."

on the money-metrics are likely to fail with incomplete preferences: typically, with incomplete preferences, for each price there is more than one demand; and for each consumption plan there is more than one (normalized) price that supports it as a demand. And it is easy to find examples of economies with incomplete preferences for which the Radner measure is positive, but the conclusion fails. It fails, for example, in the economy that Rigotti and Shannon (2005) use to illustrate consumer inertia and indeterminacy: their consumers all have incomplete preferences of the class proposed by Bewley (2002) that has "kinks everywhere."

7.3 Other Welfare Measures in the Small

Schlee (2013a) shows that Radner's local measure equals the derivatives of four other welfare measures, including aggregate consumers' surplus. The three other measures are the Slutsky change in real income, the nominal Divisia price index, and a nominal version of Debreu's (1951) coefficient of resource utilization. As Schlee (2013a) points out, the derivative of the five measures also equal the derivatives of the compensating and equivalent variations, bringing the total of locally equivalent measures to seven. Schlee (2018) points out that the local equivalence between Radner's measure and the coefficient of resource utilization is false.[59] Proposition 3 and Corollary 4 extend these equivalences to the sum of money-metric utilities.

One can also show that the local equivalence extends to the aggregate benefit and distance functions of Section 6. This follows since the conclusion of Lemma 2 holds for these functions: $\nabla_{x_i} B_i(x_i, V_i(p,w), p) = p = \nabla_{x_i} F_i(x_i, V_i(p,w), p)$ whenever an interior point x_i is demanded at (p, w). The last paragraph of Schlee (2018) gives the details.

As already mentioned, both Blackorby and Donaldson (1987) and Hammond (1994) develop new welfare measures as alternatives to the sum of money metrics, at least partly in response to the non-concavity of individual money metrics. Blackorby and Donaldson (1987) propose an alternative that they call *welfare ratios*. For consumer i with utility representation u_i it is given by $r_i(p,w) = \frac{w_i}{e(p,\bar{u}_i)}$, where $\bar{u}_i$ is taken to be a poverty level of utility for consumer i. It can be converted into money units by multiplying r_i by $e(p^0, \bar{u}_i)$ for some reference price p^0 (following Deaton (2003, equation (8))). Letting $R_i(p, w, p^0)$ denote the resulting product, it follows that $\nabla_{x_i} R_i(p, p \cdot x_i, p^0) = p^0$ when the derivative is evaluated at $(p^0, p^0 \cdot x_i, p^0)$ and the local equivalence extends to the sum of such nominal welfare ratios. Hammond's (1994) aggregate money-metric measure is the number μ given by

$$W(V_1(p, w_1), ..., V_I(p, w_I)) = W(V_1(p^0, w_1^0 + \mu), ..., V_I(p^0, w_I^0 + \mu))$$

where V_i is consumer i's indirect utility function, W is some strictly increasing function of the utility profile, and the price-wealth pairs to the right of the equality describe some baseline

[59]The correct equivalence is between Radner's measure and a measure related to the coefficient that takes as a reference point a production plan that need not be a part of a Pareto optimal allocation, rather than the entire aggregate production set.

allocation. This measure is generally *not* locally equivalent to the derivative of the sum of money-metrics.[60] This failure is not an oversight of the author's: The function W is intended to incorporate equity concerns and so, unlike the money-metric sum, rank some non-equilibrium allocations higher than competitive equilibrium allocations.

8 Concluding Remarks on Future Work

We conclude with two remarks on future directions that are opened up by our rehabilitation of the money-metric. The first concerns the notion of *complementarity;* and the second, comparative-static analysis of the consumer's problem. Both will draw on our money-metric saddlepoint theorem presented here.

Samuelson (1974) used the money-metric to offer two new definitions of complementary commodities (his fifth and sixth definitions), and thereby connect to basic and classical themes in consumer theory. This program we carry forward in future work. Thus our next step is to reconsider his definitions of complementarity in light of our Corollary 1, and relate it to other proposed definitions in the literature. Of particular interest are those suggested or explored by Chipman (1977) and Kannai (1980), and more recently by Chambers, Echenique, and Shmaya (2010).

Of perhaps greater interest is the use of the money-metric in reconsidering results in monotone comparative statics of Quah (2007), Mirman and Ruble (2008), and Barthel and Sabarwal (2018). A important tool to carry this out is our Corollary 1, which formulates the consumer's problem as one subject *only* to non-negativity constraints, bypassing the budget constraint altogether. This point is of obvious importance since one stumbling block in applying lattice methods to the consumer's problem is that the budget set is not a lattice; more to the point, different budget sets are *not* comparable in the strong-set order.

9 Appendix

9.1 Relationship between the lower and upper money-metrics.

We first point out that the upper money metric (3) is at least as large as the lower money metric (2), that is,

$$M(x,p) \geq M_*(x,p) \quad \text{for every } p >> 0, w > 0, \text{ and } x \in X. \tag{14}$$

Let x^* be any solution to the problem defining the upper money metric, namely $M(x,p) = \min_{x' \in \mathcal{R}(x)} p \cdot x'$. Since $x^* \in \mathcal{R}(x) = cl\{x' \succ x\}$, there is a sequence x^n with $x^n \succ x$ for every

[60]This follows from his equation (39) after suppressing the public good and imposing budget balance on the right side. Only in the special in which his ω_i weights are equal does the local equivalence go through.

n and $\lim_{n\to\infty} x^n = x^*$. Since $x^n \succ x$, $p \cdot x^n \geq M_*(x,p) = \inf_{x' \succsim x} p \cdot x'$ for every n. Then $M(x,p) = \lim_{n\to\infty} p \cdot x^n \geq M_*(x,p)$, establishing (14).

To show that the inequality can be strict, let $X = \mathbb{R}^2_+$, let $u_0(x) = x_1 x_2$, and let $u_1(x) = x_1 + x_2$. Let $x \succ y$ if and only if $u_t(x) > u_t(y)$ for $t = 0, 1$. If $u_{t'}(x) \geq u_{t'}(y)$ and $u_t(x) \leq u_t(y)$ for $t, t' \in \{0, 1\}$, with at least one inequality strict, then $x \sim y$. Figure 4 illustrates that $M(x^0, p) > M_*(x^0, p)$, where $x^0_1 = x^0_2$ and $p_1 < p_2$. Note that, if we replaced indifference with incomparability when the two functions disagree about ranking two points, then the two money metrics would be equal (the upper money metric is unchanged, but the lower increases).

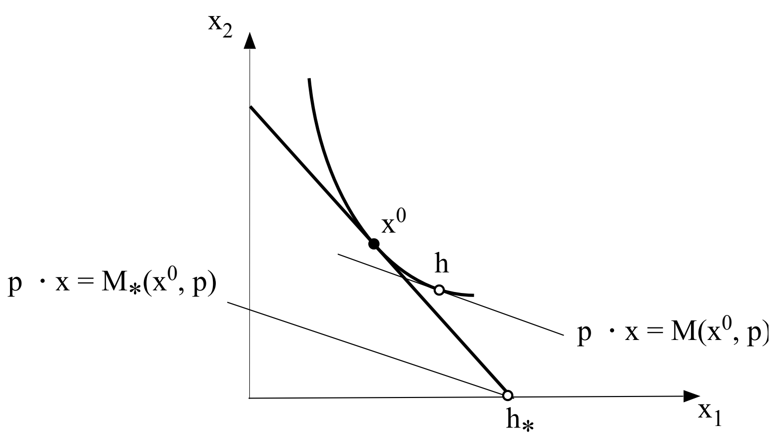

Figure 4: The upper and lower money metrics. h is the compensated demand for the upper- and h_* the compensated demand for the lower money metric when every point in the region between the two thicker curves is indifferent to x^0. If every point in this region is *not comparable* to x^0, then $M_*(x^0, p)$ would increase to $M(x^0, p)$.

The next proposition shows that the strengthening of local nonsatiation to strongly upper nonsatiation (Definition 4) implies the equivalence between the upper and lower money metrics.

Proposition 5. *Suppose that the Minimal Assumptions hold.*

1. *If $\succsim$ is strongly upper nonsatiated, then $M(x', p) = M_*(x', p)$ for every $(x', p) \in X \times \mathbb{R}^L_{++}$.*

2. *If either $\succsim$ is transitive or $\succsim$ is strongly convex, then $\succsim$ is strongly upper nonsatiated.*

Proof of Proposition 5: Fix $(x, p) \in X \times \mathbb{R}^L_{++}$. Suppose $\succsim$ is strongly upper nonsatiated. By the definition of $M_*(x, p)$, there is a sequence x^n with $x^n \succsim x$ for every n and $\lim p \cdot x^n = M_*(x, p)$. There are two cases. First there is a subsequence $x^{n(k)}$ of x^n with $x^{n(k)} \succ x$ for every k, so that $p \cdot x^{n(k)} \geq M(x, p)$ for every k. Then $M_*(x, p) = \lim_k p \cdot x^{n(k)} \geq M(x, p) \geq M_*(x, p)$ and the conclusion follows. Or there is a subsequence $x^{n(m)}$ of x^n with $x^{n(m)} \sim x$ for every m. For every m let B_m denote an open ball in $\mathbb{R}^L$ with radius $1/m$ centered on $x^{n(m)}$. By strong upper nonsatiation, for every m, there is a $y^m \in B_m \cap X$, with $y^m \succ x$. By the triangle inequality, the sequence $p \cdot y^m$ is convergent with the same limit as $p \cdot x^{n(m)}$. It follows that $M_*(x,p) = \lim_m p \cdot x^{n(m)} = \lim_m p \cdot y^m \geq M(x,p) \geq M_*(x,p)$, and part 1. follows. Strong upper nonsatiation follows immediately if $\succsim$ is LNS and transitive, or if $\succsim$ is strongly convex. $\square$

The next corollary is immediate.

Corollary 3. *Suppose that the Minimal Assumptions hold. If in addition $\succsim$ is strongly upper nonsatiated, then the conclusion of Theorems 1(a) and 2(a) hold for the lower money metric, M_*.*

9.2 Omitted proofs

Proof of Lemma 1: Fix $x \in X$ satisfying the local cheaper point condition at $p >> 0$. Since $p >> 0$, $\succsim$ is closed, and X is closed and bounded from below, both $h(p,x)$ and $d(p, M(x,p))$ are nonempty. Let $x' \in h(p,x)$ and $x'' \in d(p, M(x,p))$. Since $p \cdot x' = M(x,p)$ and $\succsim$ is complete, $x'' \succsim x'$. By $x' \succsim x$ and transitivity, $x'' \succsim x$. Since by local nonsatiation, $p \cdot x'' = M(x,p)$, $x'' \in h(p,x)$. And since x satisfies the local cheaper point assumption, there is a sequence x^n in X with limit x'' and $p \cdot x^n < p \cdot x'' = M(x,p)$, so, by completeness, $x \succ x^n$ for every n. By closedness of $\succsim$, $x \succsim \lim x^n = x''$, so $x'' \succsim x' \succsim x \succsim x''$ and both $x' \in d(p, p \cdot x)$ and $x' \sim x$ follow from transitivity. $\square$

Proof of Theorem 1(b): If the Saddlepoint inequalities (7) hold, then by Theorem 1 in Uzawa (1958), x^* maximizes $M(\cdot, p)$ on $B(p,w)$. If $M(\cdot, p)$ represents preferences on X—for example if $X = \mathbb{R}^L_+$—then we would be done by Uzawa's (1958) Theorem 1. Since we consider more general consumption sets, we proceed using Lemma 1. Let $x' \in h(p, x^*)$, so by Lemma 1, $x' \sim x^*$ and $x' \in d(p, M(p, x^*))$. The inequalities $\mathcal{L}(x^*, 1) \leq \mathcal{L}(x^*, \lambda)$ for every $\lambda \geq 0$ imply that $p \cdot x^* = w$, so $x^* \in B(p,w)$. And the inequality $\mathcal{L}(x', 1) \leq \mathcal{L}(x^*, 1)$ implies that $M(x', p) - p \cdot x' \leq M(x^*, p) - p \cdot x^*$. Since $x' \sim x^*$, $M(x', p) = M(x^*, p)$, so $\mathcal{L}(x, 1) \leq \mathcal{L}(x^*, 1)$ reduces to $w = p \cdot x^* \leq p \cdot x'$. Since $x^* \sim x' \succsim x$ for every $x \in B(p,w)$ and $\succsim$ is transitive, $x^* \in d(p,w)$. $\square$

Proof of Theorem 2(b): Now suppose that the saddlepoint inequalities (8) hold at some $p^* >> 0$. That $\mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, p^*) \leq \mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, \mu)$ for every $\mu \in \mathbb{R}_+$ and $p^* >> 0$ implies that $\omega + y^* - x^* = 0$. That $\mathcal{L}_{p^*}(\mathbf{x}, \mathbf{y}, p^*) \leq \mathcal{L}_{p^*}(\mathbf{x}^*, \mathbf{y}^*, p^*)$ for every $(\mathbf{x}, \mathbf{y}) \in \mathcal{A}$ immediately implies that $p^* \cdot y_j^* \geq p^* \cdot y_j$ for every $y_j \in Y_j$ for every firm j. Since $p^* >> 0$, $\omega > 0$, and $0 \in Y$, $p^*(\omega + y^*) > 0$. Let ν_i be given by $p^* \cdot x_i^* = \nu_i(p^* \cdot \omega + p^* \cdot y^*)$ and set $w_i^* = \nu_i(p^* \cdot \omega + p^* \cdot y^*)$. Set consumer i's endowment of goods equal $\nu_i \omega$ and i's ownership share of each firm equal to ν_i, so i's wealth is w_i^*. That $x_i^* \in d_i(p^*, w_i^*)$ now follows from Theorem 1 (b). $\square$

9.3 On the Differentiability of a Money-metric

We give preference conditions for $M_i(\cdot, p)$ to be differentiable at a point $x \in d(p, p \cdot x)$. We use such differentiability in Section 7. Let $g_i(x) = \{p \in \mathbb{R}^L_{++} \mid \sum p_\ell = 1 \text{ and } x \in d_i(p, p \cdot x)\}$, the normalized *inverse demand* correspondence for consumer i.

Proposition 6 (Money metric differentiability)**.** *Suppose that $X_i = \mathbb{R}^L_+$, and that $\succsim_i$ is complete, closed, and strongly upper nonsatiated. Let $x_i^0 >> 0$ and suppose that $\{x_i^0\} = d_i(p^0, p \cdot x^0)$, that is x_i^0 is the unique demand at (p^0, x^0). Then $M_i(\cdot, p^0)$ is differentiable at $x = x_i^0$ if in addition*

(a) $g_i(x)$ is nonempty and single-valued for every x in some open neighborhood $N_a \subset X$ of x_i^0;

*(b) $g_i(x)$ is Lipschitz continuous **at the point** x_i^0: namely, for some open neighborhood $N \subset N_a$ of x_i^0, there is a real number $K > 0$ such that, for every $x \in N$*

$$\|g_i(x_i^0) - g_i(x)\| \leq K\|x_i^0 - x\|$$

Before turning to a proof, we make a few remarks. First, we do not assume that preferences are transitive or convex, though we do assume that the demand is single-valued at (p^0, x_i^0). Single-valuedness is not necessary as the case of perfect-substitute preferences represented by $u(x) = \sum_i x_i$ illustrates; the money metric is $\min\{p_1, ..., p_L\} \sum_\ell x_\ell$. We extend this example in Corollary 3 below.

We can of course replace (b) with the assumption that $g_i(\cdot)$ is *differentiable* in a neighborhood of x_i^0. If the inverse demand $g_i(x_i^0)$ is not unique, then the conclusion can fail.[61] It remains to be seen how far Lipschitz continuity of the inverse demand at a demand point can be relaxed. In a follow-up paper on comparative statics with the money metric, we confirm that if $\succsim$ has an increasing C^1 utility representation with no critical point on the consumption set $\mathbb{R}^L_{++}$, then the money metric is monotone and C^1 with no critical point. In this case g_i is clearly continuous but not necessarily Lipschitz-continuous-at-a-point.

We use the next fact in the proof of Proposition 6. The conclusion is standard, but we do not assume transitivity or convexity.[62]

[61] The case of homothetic preferences in which the homogenous-of-degree-1 representation is not differentiable makes this point clear: the money metric takes the form $u(x)b(p)$ where u is homogenous of degree 1. If u is not differentiable at a demand point, then neither is the money-metric (e.g. Leontief preferences).

[62] Honkapohja (1987) proves that the compensated demand is upper hemicontinuous in (p, x) for transitive preferences. Transitivity is used in an essential way in his proof. He uses this Lemma in his proof that the money metric is jointly continuous in (p, x). Shafer (1980) had already done one better by showing that the money metric is jointly continuous in $(p, x, \succsim)$, where the space of preferences is endowed with the Hausdorff topology of closed convergence.

Lemma 3. *If $\succsim_i$ is complete, closed, and strongly upper nonsatiated, then the compensated demand correspondence $h_i(p, x)$ defined in (3) is upper hemicontinuous at every $(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{+}$.*

Proof of Lemma 3: Suppress the consumer subscript throughout. We will show that the lemma is a consequence of Berge's theorem.[63] Since X is closed, $p >> 0$ and $\succsim$ is closed, $h(p, x)$ is nonempty and sfor every $(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{+}$. Fix $(p^0, x^0) \in \mathbb{R}^L_{++} \times X$. Let $K = \{x \in X \,|\, x_\ell \leq 1 + p^0 \cdot x^0/p^0_\ell\,, \ell = 1, ..., L\}$. Since X is closed, K is compact and $x^0 \in K$. Moreover for some neighborhood N of x^0 and P of p^0, $h(p, x) \subseteq K$. This follows since for any $y \in h(p, x)$, $y_\ell \leq M(x, p)/p_\ell \leq p \cdot x/p_\ell$. For $x \in N$, let $R(x) = \{y \in K | y \succsim x\}$. Then for $(p, x) \in P \times N$, $h(p, x) = \text{argmin}_{x' \in R(x)} p \cdot x'$. We now show that the correspondence $R(\cdot) : N \rightrightarrows K$ is continuous at x^0. Since $\succsim$ is closed, R is upper hemicontinuous. To prove that it is lower hemicontinuous, let $y^0 \in R(x^0)$ and consider any sequence x^n in X converging to x^0. We need to construct a sequence y^n converging to y^0 with $y^n \in R(x^n)$ for every n. If $y^0 \succ x^0$, then, since $\succ$ is open, there is an n^* such that $y^0 \succ x^n$ for all $n > n^*$. Take $y^n = x^n$ for $n \leq n^*$ and $y^n = y^0$ for all $n > n^*$. If $y^0 \sim x^0$, then, by strong upper nonsatiation, there is a sequence z^m with limit y^0 and $z^m \succ x^0$ for all m. For each n, consider $\bar{m}(n) = \sup\{m \in \mathbb{N} \cup \{-\infty, \infty\} \mid z^m \succsim x^n\}$. If $\bar{m}(n) = -\infty$ (that is, the set is empty), then set $y^n = x^n$; if $\bar{m}(n) < \infty$, then set $y^n = z^{\bar{m}(n)}$; if $\bar{m}(n) = \infty$ then set $y^n = z^m$ for the *smallest* $m \geq n$ with $z^m \succsim x^n$. By construction, $y^n \in R(x^n)$ for every n. Since for every fixed m, there is an n^* with $z^m \succsim x^n$ for all $n \geq n^*$, it follows that $\lim_{n\to\infty} y^n = y^0$. $\square$

Proof of Proposition 6: We omit the consumer i subscript throughout. By Lemma 2, if $M(\cdot, p)$ is differentiable at $x' \in d(p, p \cdot x')$ with $x' >> 0$, then $D_x M(x', p) = p$. We will show that

$$\lim_{\|x - x^0\| \to 0} \frac{|M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x)|}{\|x - x^0\|} = 0. \tag{15}$$

From the Saddlepoint inequalities of Theorem 1 we have

$$M(x^0, p^0) - M(x, p^0) \geq p^0 \cdot (x^0 - x), \tag{16}$$

for every $x \in X$, so for any $x \in X$ with $x \neq x^0$

$$M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x) \geq 0. \tag{17}$$

For $x \in N$, we have $M(x, g(x)) = g(x) \cdot x$ and—recalling that $g(x)$ is a singleton on N—$M(x^0, g(x)) < g(x) \cdot x^0$ whenever $x \neq x^0$. From these inequalities find that

$$M(x^0, g(x)) - M(x, g(x)) < g(x) \cdot (x^0 - x) \tag{18}$$

[63] For a statement of Berge's theorem, see for example Kreps (2013, p. 476).

which, adding and subtracting the same terms, is the same as

$$M(x^0,p^0) - M(x,p^0) - p^0 \cdot (x^0 - x) < (g(x) - p^0) \cdot (x^0 - x) - \Gamma(x) \tag{19}$$

where $\Gamma(x) = [M(x^0, g(x)) - M(x^0, p^0)] + [M(x,p^0) - M(x, g(x)]$.

Since $M(x,\cdot)$ is concave, $M(x',p') \leq M(x,p) + y_{(p,x)} \cdot (p' - p)$ where $y_{(p,x)}$ is any element of $h(p,x)$.[64] If follows that

$$\Gamma(x) \geq (g(x) - p^0) \cdot y_{(g(x),x^0)} + (p^0 - g(x)) \cdot y_{(p^0,x)} \tag{20}$$

for any selections from the compensated demand correspondences $h(g(x), x^0)$ and $h(p^0, x)$. Insert (20) into (19) to find

$$M(x^0,p^0) - M(x,p^0) - p^0 \cdot (x^0 - x) < (g(x) - p^0) \cdot (x^0 - x + y_{(g(x),x^0)} - y_{(p^0,x)}) \tag{21}$$

Combine (17) and (21) to find

$$|M(x^0,p^0) - M(x,p^0) - p^0 \cdot (x^0 - x)| \leq \Omega(x) \tag{22}$$

where $\Omega(x) = (g(x) - p^0) \cdot (x^0 - x + y_{(g(x),x^0)} - y_{(p^0,x)})$. Since h is upper hemicontinuous by Lemma 3, and $h(p^0, x^0)$ is single-valued by Corollary 1 and the single-valuedness of $d(p^0, p^0 \cdot x^0)$, it follows that $\lim_{x \to x^0} y_{(g(x),x^0)} = x^0 = \lim_{x \to x^0} y_{(p^0,x)}$. The Lipschitz condition (b) and the Cauchy-Schwartz inequality imply that

$$\lim_{\|x - x^0\| \to 0} \frac{\Omega(x)}{\|x - x^0\|} = 0, \tag{23}$$

so (15) follows. □

Proposition 6 requires that the demand $d(p^0, p^0 \cdot x^0)$ be single-valued. The already-mentioned perfect-substitutes example $u(x) = \sum x_i$ demonstrates that demand single-valuedness is not necessary. The next corollary extends this example to preferences that are locally quasi-homothetic around a demand point x^0, without requiring $d(p^0, p^0 \cdot x^0)$ to be single-valued.

Corollary 4. *Suppose that $X_i = \mathbb{R}^L_+$, and that $\succsim_i$ is complete, closed, strongly upper non-satiated. Let $(x^0, p^0) \in d(p^0, p^0 \cdot x^0)$ and suppose that preferences are locally quasihomothetic in the sense that $M_i(x,p) = a_i(p) + b_i(p)u(x)$ on some open neighborhood $Q \subset X \times \mathbb{R}^L_{++}$ of (x^0, p^0). Then $M(\cdot, p^0)$ is differentiable at $x = x^0$ if (a) and (b) of Proposition 6 hold and b_i is Lipschitz at the point p^0.*

Proof: The function Γ in the proof of Proposition 6 here equals $\Gamma(x) = [b(g(x)) - b(p^0)] \cdot [u(x^0) - u(x)]$. By Lemma 3 u is continuous. And since b is Lipschitz at the point p^0, the

[64] For example, the proof of Proposition 9.24f in Kreps (2013) applies word-for-word here, with just a change in notation.

conclusion follows by combining (17) and (19) for this Γ, then taking limits. $\square$

9.4 A Representation Theorem

We slightly extend the representation theorem in Khan and Schlee (2016). This theorem differs from those in Weymark (1985a) in that we do not impose convexity of the consumption set. Fix $p >> 0$. corollary. Let $X^d(p) \subseteq X$ be the set of consumption plans satisfying $x \in d(p, p\cdot x)$, that is, plans x that are demanded at $(p, p \cdot x)$.

Theorem 4. *Let $X \subseteq \mathbb{R}^L_+$ be nonempty and closed. If $\succsim \in X \times X$ satisfies the Standard Assumptions, and $p >> 0$, then $M(\cdot, p)$ represents $\succsim$ on $X^{lcp}(p) \cup X^d(p)$. If $X = \mathbb{R}^L_+$ and $x \succsim 0$ for every $x \in X$, then $M(\cdot, p)$ represents $\succsim$ on X.*

Lemma 4. *If $x \in X^d(p)$, and $p >> 0$, then the conclusion of Lemma 1 holds for (x, p); that is, (a) $d(p, M(x, p)) = h(p, x)$, and (b) if $y \in d(p, M(x, p))$ then $y \sim x$.*

Proof of Lemma 4: Suppose that $y \in h(p, x)$, so that $p \cdot y = M(x, p)$ and $y \succsim x$; it follows that $y \in d(p, M(x, p))$. Suppose that $y \in d(p, M(x, p))$, so $y \succsim x$. By local nonsatiation, $p \cdot y = M(x, p)$, so $y \in h(p, x)$. Since $x \in d(p, M(x, p))$, it follows that $y \sim x$. $\square$

We can now complete the proof of Theorem 3.

Proof of Theorem 4. Suppose that $x \succsim y$. By transitivity, $\{z \in X \mid z \succsim y\} \subseteq \{z \in X \mid z \succsim y\}$, so $M(x, p) \geq M(y, p)$. Now suppose that $y \succ x$. By Lemmata 1 and 4, $z' \in d(p, M(z, p))$ implies that $z' \sim z$ for $z = x, y$. So $y' \sim y \succ x \sim x'$. By transitivity, $y' \succ x'$. By LNS, $M(y, p) > M(x, p)$. If $X = \mathbb{R}^L_+$ and $x \succsim 0$ for every $x \in X$, then $X = X^{lcp}(p) \cup X^d(p)$. $\square$

References

Aaberge, R., U. Colombino, and S. Strøm (2004): "Do more equal slices shrink the cake? An empirical investigation of tax-transfer reform proposals in Italy," *Journal of Population Economics*, 17(4), 767–785.

Banks, J., R. Blundell, and A. Lewbel (1996): "Tax reform and welfare measurement: do we need demand system estimation?," *Economic Journal*, 106(438), 1227–1241.

Bargain, O., A. Decoster, M. Dolls, D. Neumann, A. Peichl, and S. Siegloch (2013): "Welfare, labor supply and heterogeneous preferences: evidence for Europe and the US," *Social Choice and Welfare*, 41(4), 789–817.

Barthel, A.-C., and T. Sabarwal (2018): "Directional monotone comparative statics," *Economic Theory*, 66(3), 557–591.

Bergson, A. (1938): "A reformulation of certain aspects of welfare economics," *The Quarterly Journal of Economics*, 52(2), 310–334.

Bergstrom, T. (1973): "Competitive equilibrium without transitivity, monotonicity or free disposal," *Washington University (mimeo).*

Bernheim, B. D., and A. Rangel (2009): "Beyond revealed preference: choice-theoretic foundations for behavioral welfare economics," *Quarterly Journal of Economics*, 124(1), 51–104.

Bewley, T. F. (2002): "Knightian decision theory. Part I," *Decisions in economics and finance*, 25(2), 79–110.

Blackorby, C., and D. Donaldson (1987): "Welfare ratios and distributionally sensitive cost-benefit analysis," *Journal of Public Economics*, 34(3), 265–290.

——— (1988): "Money metric utility: A harmless normalization?," *Journal of Economic Theory*, 46(1), 120–129.

Blackorby, C., and A. Shorrocks (eds.) (1995): *Collected Works of WM Gorman: Separability and Aggregation*, vol. 1. Oxford University Press.

Blundell, R., M. Browning, and I. A. Crawford (2003): "Nonparametric Engel curves and revealed preference," *Econometrica*, 71(1), 205–240.

Blundell, R., I. Preston, and I. Walker (1994): "Introduction to applied welfare analysis," *The Measurement of Household Welfare*, pp. 1–50.

Boadway, R. W. (1974): "The welfare foundations of cost-benefit analysis," *Economic Journal*, 84(336), 926–939.

Bosmans, K., K. Decancq, and E. Ooghe (2018): "Who's afraid of aggregating money metrics?," *Theoretical Economics*, 13(2), 467–484.

Chambers, C. P., F. Echenique, and E. Shmaya (2010): "On behavioral complementarity and its implications," *Journal of Economic Theory*, 145(6), 2332 – 2355.

Chambers, C. P., and T. Hayashi (2012): "Money-metric utilitarianism," *Social Choice and Welfare*, 39(4), 809–831.

Chambers, R. G., Y. Chung, and R. Färe (1996): "Benefit and distance functions," *Journal of Economic theory*, 70(2), 407–419.

Chipman, J. S. (1977): "An empirical implication of Auspitz-Lieben-Edgeworth-Pareto complementarity," *Journal of Economic Theory*, 14(1), 228–231.

Creedy, J., and N. Hérault (2012): "Welfare-improving income tax reforms: a microsimulation analysis," *Oxford Economic Papers*, 64(1), 128–150.

DANAN, E. (2008): "Revealed preference and indifferent selection," *Mathematical Social Sciences*, 55(1), 24–37.

DEATON, A. (1979): "The distance function in consumer behaviour with applications to index numbers and optimal taxation," *Review of Economic Studies*, 46(3), 391–405.

——— (1980): "Measurement of welfare : theory and practical guidelines," Living standards measurement study working paper, no. LSM 7.

——— (2003): "Household surveys, consumption, and the measurement of poverty," *Economic Systems Research*, 15(2), 135–159.

DEATON, A., AND J. MUELLBAUER (1980): *Economics and Consumer Behavior*. Cambridge University Press.

DEATON, A., AND S. ZAIDI (2002): *Guidelines for constructing consumption aggregates for welfare analysis*, vol. 135. World Bank Publications.

DEBREU, G. (1976): "Least concave utility functions," *Journal of Mathematical Economics*, 3(2), 121–129.

DEBREU, G., AND T. C. KOOPMANS (1982): "Additively decomposed quasiconvex functions," *Mathematical Programming*, 24(1), 1–38.

DECANCQ, K., M. FLEURBAEY, AND E. SCHOKKAERT (2015): "Inequality, income, and well-being," in *Handbook of income distribution*, vol. 2, pp. 67–140. Elsevier.

DIEWERT, W. E., AND T. J. WALES (1988): "Normalized quadratic systems of consumer demand functions," *Journal of Business & Economic Statistics*, 6(3), 303–312.

FISHBURN, P. C. (1991): "Nontransitive preferences in decision theory," *Journal of risk and uncertainty*, 4(2), 113–134.

FLEURBAEY, M. (2009): "Beyond GDP: The quest for a measure of social welfare," *Journal of Economic literature*, 47(4), 1029–75.

FLEURBAEY, M., AND D. BLANCHET (2013): *Beyond GDP: Measuring Welfare and Assessing Sustainability*. Oxford University Press, Oxford.

FLEURBAEY, M., AND F. MANIQUET (2011): *A Theory of Fairness and Social Welfare*, vol. 48. Cambridge University Press.

FLEURBAEY, M., AND P. MONGIN (2005): "The news of the death of welfare economics is greatly exaggerated," *Social Choice and welfare*, 25(2-3), 381–418.

Fleurbaey, M., and E. Schokkaert (2013): "Behavioral welfare economics and redistribution," *American Economic Journal: Microeconomics*, 5(3), 180–205.

Fon, V., and Y. Otani (1979): "Classical welfare theorems with non-transitive and non-complete preferences," *Journal of Economic Theory*, 20(3), 409–418.

Fountain, J. (1981): "Consumer surplus when preferences are intransitive: Analysis and interpretation," *Econometrica*, pp. 379–394.

Fuchs-Seliger, S. (1990a): "An axiomatic approach to compensated demand," *Journal of Economic Theory*, 52(1), 111–122.

——— (1990b): "Compensated and direct demand without transitive and complete preferences," *Annals of Operations Research*, 23(1), 299–310.

Gale, D., and A. Mas-Colell (1977): "On the role of complete, transitive preferences in equilibrium theory," in *Equilibrium and disequilibrium in economic theory*, pp. 7–14. Springer.

Gorman, W. (1953): "Community preference fields," *Econometrica*, pp. 63–80.

——— (1970): "Quasi-separable preferences, costs, and technologies," *University of North Carolina, Chapel Hill, mimeo.*

Gorman, W. M. (1971): "Preference, revealed preference and indifference," *Preferences, utility and demand. New York: Harcourt and Brace Jovanovich.*

Hammond, P. J. (1994): "Money metric measures of individual and social welfare allowing for environmental externalities," in *Models and Measurement of Welfare and Inequality*, pp. 694–724. Springer, Berlin.

Harberger, A. C. (1971): "Three basic postulates for applied welfare economics: An interpretive essay," *Journal of Economic Literature*, 9(3), 785–797.

Hayashi, T. (2017): *General Equilibrium Foundation of Partial Equilibrium Analysis*. Springer.

Honkapohja, S. (1987): "On continuity of compensated demand," *International Economic Review*, pp. 545–557.

Kannai, Y. (1977): "Concavifiability and constructions of concave utility functions," *Journal of Mathematical Economics*, 4(1), 1–56.

——— (1980): "The ALEP definition of complementarity and least concave utility functions," *Journal of Economic Theory*, 22(1), 115–117.

——— (1981): "Concave utility functions – existence, constructions and cardinality," in *Generalized Concavity in Optimization and Economics*, ed. by S. Schailble, and W. T. Ziemba, pp. 543–595. Academic Press, New York.

Khan, M. A., and E. E. Schlee (2016): "On Lionel McKenzie's 1957 intrusion into 20th-century demand theory," *Canadian Journal of Economics/Revue Canadienne d'Economique*, 49(2), 589–636.

——— (2017): "The nonconcavity of money-metric utility: a new formulation and proof," *Economics Letters*, 154(1), 10–12.

King, M. A. (1983): "Welfare analysis of tax reforms using household data," *Journal of Public Economics*, 21(2), 183–214.

Knoblauch, V. (1992): "A tight upper bound on the money metric utility function," *American Economic Review*, 82(3), 660–663.

Kreps, D. M. (2013): *Microeconomic Foundations I: Choice and Competitive Markets*. Princeton University Press, Princeton, New Jersey.

Luenberger, D. G. (1992a): "Benefit functions and duality," *Journal of Mathematical economics*, 21(5), 461–481.

——— (1992b): "New optimality principles for economic efficiency and equilibrium," *Journal of optimization theory and applications*, 75(2), 221–264.

——— (1995): *Microeconomic Theory*. McGraw-Hill, Inc., New York.

Malmquist, S. (1953): "Index numbers and indifference surfaces," *Trabajos de estadística*, 4(2), 209–242.

Mas-Colell, A., M. Whinston, and J. Green (1995): *Microeconomic Theory*. Oxford University Press, New York.

McKenzie, L. (1957): "Demand theory without a utility index," *The Review of Economic Studies*, 24(3), 185–189.

——— (1958): "Further comments," *Review of Economic Studies*, 25(3), 200–200.

Milgrom, P., and C. Shannon (1994): "Monotone comparative statics," *Econometrica*, pp. 157–180.

Mirman, L. J., and R. Ruble (2008): "Lattice theory and the consumer's problem," *Mathematics of Operations Research*, 33(2), 301–314.

NEGISHI, T. (1960): "Welfare economics and existence of an equilibrium for a competitive economy," *Metroeconomica*, 12(2-3), 92–97.

QUAH, J. K.-H. (2007): "The comparative statics of constrained optimization problems," *Econometrica*, 75(2), 401–431.

RADNER, R. (1993 [1978]): "A note on the theory of cost-benefit analysis in the small," *Capital, Investment and Development*, pp. 129–41.

RICHTER, M. (1971): "Rational choice," in *Preferences, Utility and Demand*, ed. by J. Chipman. Harcourt Brace Jovanovich, New York, 1971.

RIGOTTI, L., AND C. SHANNON (2005): "Uncertainty and risk in financial markets," *Econometrica*, 73(1), 203–243.

SAMUELSON, P. A. (1942): "Constancy of the Marginal Utility of Income," in *Studies in Mathematical Economics and Econometrics, in Memory of Henry Schultz*, ed. by O. Lange, pp. 75–91. University of Chicago Press.

——— (1947): *Foundations of Economic Analysis*. Harvard University Press, Cambridge, MA.

——— (1974): "Complementarity: An essay on the 40th anniversary of the Hicks-Allen revolution in demand theory," *Journal of Economic Literature*, 12(4), 1255–1289.

SCHLEE, E. E. (2013a): "Radner's cost-benefit analysis in the small: an equivalence result," *Economics Letters*, 120(3), 570–72.

——— (2013b): "Surplus maximization and optimality," *American Economic Review*, 103(6), 2585–2611.

——— (2018): "Radner's cost–benefit analysis in the small: A correction and new equivalences," *Economics Letters*, 172(C), 31–33.

SCHMEIDLER, D. (1971): "A condition for the completeness of partial preference relations," *Econometrica*, 39(2), 403.

SHAFER, W. J. (1980): "On the existence and interpretation of value allocation," *Econometrica*, 48(2), 467–476.

SHEPARD, R. W. (1953): *Cost and Production Functions*. Princeton University Press, Princeton, New Jersey.

SLESNICK, D. T. (1998): "Empirical approaches to the measurement of welfare," *Journal of Economic Literature*, 36(4), 2108–2165.

Tirole, J. (1988): *The Theory of Industrial Organization.* MIT Press, Cambridge, MA.

Topkis, D. M. (1995): "Comparative statics of the firm," *Journal of Economic Theory*, 67(2), 370–401.

Uzawa, H. (1958): "The Kuhn-Tucker theorem in concave programming," in *Studies in Linear and Non-Linear Programming*, ed. by K. J. Arrow, L. Hurwicz, and H. Uzawa, pp. 32–37. Stanford University Press.

Varian, H. R. (1982): "The Nonparametric Approach to Demand Analysis," *Econometrica*, 50(4), 945–973.

——— (2012): "Revealed preference and its applications," *Economic Journal*, 122(560), 332–338.

Vives, X. (1999): *Oligopoly Pricing: Old Ideas and New Tools.* MIT Press, Cambridge, MA.

Weitzman, M. L. (1988): "Consumer's surplus as an exact approximation when prices are appropriately deflated," *Quarterly Journal of Economics*, 103(3), 543–553.

Weymark, J. A. (1985a): "Money-Metric utility functions," *International Economic Review*, 26(1), 219–232.

——— (1985b): "Remarks on the first welfare theorem with nonordered preferences," *Journal of Economic Theory*, 36(1), 156–159.

Willig, R. D. (1976): "Consumer's surplus without apology," *American Economic Review*, 66(4), 589–597.