Disclosure in Epidemics*

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Abstract

We study information disclosure as a policy tool to minimize welfare losses in epidemics through mitigating healthcare congestion. We first present a stylized model of the healthcare congestion game to show that congestion occurs only when individuals expect the disease to be sufficiently severe, and it leads to misallocation of scarce healthcare resources. Coarse information disclosure, compared with full transparency, can be welfare-improving, as it can help avoid some congestion when the true severity level is high. The optimal disclosure policy features a middle censorship rule, which censors an intermediate range of severity levels and fully reveals all other states. Under the optimal policy, when information is censored, the healthcare system is run at its full capacity without congestion and, thus, achieves ex post efficiency. In an epidemic outbreak, the optimal censorship range expands if the disease is more infectious; and the optimal disclosure policy censors (weakly) higher severity levels but fully reveals (strictly) lower severity levels if the healthcare system is more prepared.

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Q: *"Why weren't we told to wear masks in the beginning?"* **A:** *"The reason for that is that we were concerned [about] the public health community. ... It was at a time when personal protective equipment, including the N95 masks and the surgical masks, were in very short supply. And we wanted to make sure that the people – namely, the health care workers [had the necessary equipment.] We [did] not want them to be without the equipment that they needed. So, there was [no] enthusiasm about going out and everybody buying a mask and getting a mask, we were afraid that would deter the people who really needed it. "*

– Dr. Anthony Fauci, in an interview with TheStreet on June 15, 2020

1 Introduction

In an epidemic outbreak, healthcare resources (e.g., ventilators, personal protective equipment and ICU beds) are scarce, and what is worse, the limited amount of resources may not be used efficiently. As Dr. Fauci mentioned in the above quote, healthcare workers, who are most in need of face masks, may fail to get them when the masks are in short supply, and they are in high demand. Such *congestion* can have significant welfare consequences. As we have seen in the COVID-19 outbreak, government policies to fight the epidemic are largely targeted at reducing congestion (i.e., "flattening the curve").

Another distinctive feature of an epidemic outbreak is the shortage of information. Information about the novel virus, such as its modes of transmission and the severity of the disease, is difficult to get. At the early stage of the epidemic outbreak, the general populace relies on public authorities for that information. As evidenced by Dr.Fauci's response, despite the public demand for transparency, strategic information disclosure is adopted to manage the public's beliefs so as to avoid congestion in the COVID-19 outbreak.[1]

What role does information disclosure play in reducing congestion? Is full transparency, indeed, always the best disclosure policy in terms of protecting the general

[1]Public information disclosure about the disease severity and recommendations about social distancing are found effective in shaping individuals' beliefs and their behavior in different contexts during the COVID-19 outbreak (see Simonov et al. (2020), Bursztyn et al. (2020), and Allcott et al. (2020)).

public's interests ? If not, what would be the benefit from coarse information disclosure, and what is the optimal choice for government authorities?

To pursue our research questions, we first construct a stylized model to understand how the healthcare congestion happens and its welfare consequences. In the model, the healthcare system has a capacity constraint and can offer only a limited number ($\bar{n}$) of hospital admissions. However, there is a continuum mass of agents who are susceptible to the virus. An agent, if infected, will be treated only if he chooses to visit the hospital and is admitted; otherwise, the infected agent bears a cost. This cost s is common to all agents and it is referred to as the *severity of the disease*.[2] Each agent knows his own likelihood of infection, whereas all agents hold a common belief about the disease's severity.

Congestion happens if the total number of hospital visitors is greater than the fixed capacity of the healthcare system. In that case, hospital admissions will be assigned randomly to visitors. The distinctive feature of the congestion game is that one agent's hospital visit poses negative externalities on other visitors, as the chance of being admitted is lower when congestion occurs. We solve for the unique equilibrium, in which each agent chooses to pay a fixed cost to visit the hospital if and only if his likelihood of infection is above a threshold, which is determined by the expected severity level $\tilde{s}$.

The micro-founded model of healthcare congestion highlights that the congestion happens only if the expected severity level $\tilde{s}$ is sufficiently high – i.e., above the *exhaustion level* s_0 — but it does not depend on the true severity level directly. Congestion is never efficient. Whenever congestion occurs, some agents with a low likelihood of infection get access to hospital services, while some agents with a higher likelihood of infection are not admitted. This misallocation of healthcare resources, together with the wasted visiting costs, define the efficiency loss associated with congestion. As the magnitude of this efficiency loss varies as the expected severity level $\tilde{s}$ changes, we then analyze how should the government optimally disclose information about the true severity level which results in the public belief $\tilde{s}$.

[2] One interpretation of the disease severity is the expected mortality rate caused by the infectious disease.

Can full transparency be optimal?

First of all, information about the disease's severity is valuable, as it helps each agent make a better-informed decision on balancing the cost and benefit of using healthcare resources. If there is no congestion externality, such private decision making is socially efficient, and, thus, full transparency is optimal.

However, as long as congestion becomes a concern, agents do not take the congestion externality into account and, thus, tend to overuse hospital services. Under full transparency, whenever the true severity level s surpasses the exhaustion level s_0, congestion occurs, and the healthcare resources will not be efficiently allocated. Therefore, in those states, adopting some coarse information disclosure to manipulate agents' beliefs may be beneficial, as it may help to dissuade agents (with a relatively low infection risk) from visiting the hospital. In this sense, some congestion externalities may be corrected under certain disclosure policies. disclosure.

Consider the case in which the true severity level s is greater than the exhaustion level s_0. The public authority (or the *principal*) can mitigate congestion if she can let agents believe that the disease is less severe than it actually is. If she is able to enforce a posterior belief, under which the expected severity level is exactly s_0, then the congestion is avoided and ex-post efficiency is achieved. However, Rational agents would form that expectation only if the principal also reports s_0 at some severity levels that are lower than the exhaustion level s_0. The principal can always accomplish this by deviating slightly from full disclosure, and reporting s_0 to the agents whenever the true severity level s is in a sufficiently small interval around s_0 – i.e., $s \in [s_-, s_+]$, which admits a conditional mean of s_0.

We show that such a small deviation is always welfare-improving. The benefit from such coarse information provision stems from avoiding congestion when $s \in (s_0, s_+]$. It saves the unnecessary visiting costs and the efficiency losses from mis-allocation of healthcare resources. However, for $s \in [s_-, s_0)$, it makes agents believe that the disease is more severe than it actually is, thereby distorting their optimal decision making. This distortion is not significantly costly if the interval $[s_-, s_+]$ around s_0 is sufficiently small. Different from the congested case, in which some agents pay the visiting costs for nothing while their visits disrupt the healthcare system, this distortion will not lead to congestion, and, thus all visitors, if infected, will be treated. We show that such an interval always exists to make the deviation welfare-improving;

therefore, full transparency is never the optimal disclosure policy.

What is the optimal disclosure policy?

We fully characterize an optimal disclosure policy provided the distribution of the likelihood of infection is convex. This optimal policy pushes the above discussed intuition to the limit by identifying the largest interval $[s_-, s_+]$, which fully exploits the benefit from averting congestion. We can interpret this optimal disclosure policy as a simple *middle censorship rule*. It censors the all states s between s_- and s_+, while fully revealing all other states.[3] We further show that this middle censorship rule is essentially the unique optimal disclosure policy.[4]

Under the optimal policy, congestion never happens when information is censored. The censorship rule induces a posterior belief under which agents believe that the severity is at the the exhaustion level s_0 for any $s \in (s_0, s_+]$. That corrects the congestion externalities otherwise appear under full transparency and, thus, ensures that the healthcare system is run efficiently at its full capacity without congestion.

Somewhat surprisingly, the highest severity levels (i.e., $s > s_+$) are fully revealed under the optimal policy, as the benefit from avoiding congestion in those states should be the highest. This can be understood from analyzing the trade-offs associated with censoring. The benefit from avoiding congestion depends on efficiency loss induced by misallocation. Given any convex distribution of the likelihood of infection, the number of visitors with high likelihoods of infection is always higher than that of visitors with low likelihoods. Therefore, the extent of misallocation and the associated efficiency losses do not increase boundlessly with the number of visitors when the severity level further increases. Moreover, to avoid congestion, censoring higher severity levels must be accompanied by censoring states with lower severity, which always induces more distortions in the agents' efficient decision making.

[3] Depending on the prior belief of the severity of the disease, if the highest possible severity level $\bar{s}$ is lower than the upper limit of the optimal censoring range s_+, then the optimal disclosure rule degenerates to a upper censorship rule, which censors all states above s_-.

[4] Other optimal policies differ only in the states in which visiting the hospital is a strictly dominated strategy as the cost is strictly higher than the benefit, regardless of what others do. Pooling those states together does not change the allocation of healthcare resources or ex-ante welfare. The middle censorship policy is, in fact, the most informative of all optimal policies.

Key parameters of an epidemic outbreak

Preparedness of the healthcare system The conventional wisdom would say that, with a more prepared healthcare system, more transparency would be implemented in an epidemic outbreak. The healthcare system is more prepared if it is better equipped and/or the disease it faces is weaker. In either case, the congestion is less likely to happen. Within the theoretical framework, we test this intuition by conducting comparative statics analyses on the distribution of the disease severity and the capacity of the healthcare system.

Our analysis challenges the conventional wisdom. We show that, if the system becomes more prepared, either because the disease becomes slightly weaker or the system is slightly better equipped, the censoring range only moves to the right — some higher severity levels will be censored, whereas there will be more revelation of the low severity levels. Therefore, the optimal disclosure becomes less transparent on higher severity levels and more transparent on lower severity levels.[5]

To understand this result, consider any high severity level s that is above the exhaustion level and within the censoring range – i.e., $s \in (s_0, s_+]$. If the disease becomes weaker, then there is a smaller chance that the true severity level is as high as s, while if the the healthcare system becomes more resourceful, then the exhaustion level s_0 increases. In either case, the healthcare system becomes more prepared. To censor this severity level while keeping the posterior mean at s_0, the optimal disclosure policy must censor fewer low severity levels. Therefore, the cost of censoring this high severity level s decreases, and some higher severity levels will be censored in a better-prepared healthcare system.

Infectiousness Finally, we extend the model to incorporate the feature of infectiousness into our analysis. Assume that each infected agent, if untreated, can pass the virus to γ individuals afterwards. We treat γ as an exogenous parameter, which is determined by the nature of the virus.[6]

[5] However, if the healthcare system is already sufficiently prepared, then, under the optimal disclosure policy, all severity levels that can lead to congestion are censored, and, thus, congestion is completely eliminated. If the system becomes even more prepared, then the censoring range strictly shrinks, with more states fully revealed.

[6] In reality, it is possible that this contagion is determined by agents' other choices, such as social distancing, which may depend on their beliefs about the severity of the disease and may change with

The healthcare resources become more valuable when this contagion effect is considered, as treating one infected agent can now help to eliminate the additional welfare loss caused by the contagion. Therefore, congestion and the associated misallocation of healthcare resources become more costly. However, the agents never take the infection externality into account when choosing whether to visit the hospital. As a result, the principal has an incentive to let more agents visit the hospital when there is no congestion. That lowers the cost of the censorship rule, as the agents' decision making regarding hospital visits is distorted to favor the principal. Under some reasonable parameter choices, we find that introducing the contagion effect does not change the optimal information structure; in other words, the middle censorship rule remains the optimal choice for the principal. As the censorship is associated with a higher benefit and a lower cost, the censoring range expands when there is a contagion effect. If the virus is more infectious, the censoring range further expands until all high severity levels are censored.

Related Literature This paper contributes to a broader literature on policy interventions in epidemics. A recent growing economic literature looks at policy analysis within the SIR or SEIR framework (e.g., Acemoglu et al. (2020), Atkeson (2020), Alvarez, Argente and Lippi (2020), Berger, Herkenhoff and Mongey (2020), Eichenbaum, Rebelo and Trabandt (2020), Fajgelbaum et al. (2020) and Jones, Philippon and Venkateswaran (2020)). This strand of the literature emphasizes the impact of lockdown and quarantine policies on contagion dynamics. Other mitigation measures, such as disclosing information on the locations of infection cases (Argente, Hsieh and Lee, 2020) and group rotations (Ely, Galeotti and Steiner, 2020*b*), are also considered in the literature. The focus of this paper is different. We study healthcare congestion in epidemics by considering an incomplete information environment with heterogeneous agents differing in infection risk and investigating the optimal information disclosure policy in such an environment.[7] For the majority of our analysis, infection and contagion dynamics are taken as exogenous.[8]

different disclosure policies. We do not consider this channel in our simple extension.

[7]We provide a micro-foundation for healthcare congestion in a static setting, whereas congestion externality is taken as given in some dynamic macroeconomic models. See, for example, Berger, Herkenhoff and Mongey (2020) and Jones, Philippon and Venkateswaran (2020).

[8]In Section 4.3, we briefly consider contagion dynamics through the lens of how information disclosure would impact the efficiency of the healthcare system in offering treatments to cure the

Regarding incomplete information in epidemics, our model concerns the disclosure of the disease severity by taking the information of infection as given. Testing can endogenize the distribution of infection risk in our model. Deb et al. (2020), Ely, Galeotti and Steiner (2020*a*), Lipnowski and Ravid (2020) and Gans (2020), among others, recently looked into the optimal testing schemes in information production and facilitating quarantine decision-making.

In terms of methodology, we apply the information design approach, initiated by Rayo and Segal (2010) and Kamenica and Gentzkow (2011), to the healthcare congestion problem in epidemics. The information design problem we consider features a group of heterogeneous agents differing in infection risk, who play a congestion game with strategic substitution; and a benevolent principal who chooses the information structure on a continuous state space to address congestion externality and restore efficiency.[9] Under fairly general conditions, we clearly characterize the properties of the principal's objective function, which endogenously arises from the healthcare congestion game. Given these properties, we apply the method in Dworczak and Martini (2019) and demonstrate the optimality of the middle censorship rule.[10]

Disclosure policies involving various kinds of censorship rules have proven to be optimal in different economic contexts.[11] For example, in a recent work, Kolotilin, Mylovanov and Zapechelnyuk (2019) show that if the designer's indirect utility function is convex below some threshold and concave above the threshold – i.e., S-shaped – then and only then is an upper censorship disclosure rule optimal. In contrast, we find that the middle censorship rule is optimal when the principal's indirect utility function is a piecewisely convex with a kink – i.e., W-shaped.

Structure The rest of the paper is organized as follows. Section 2 presents the model of healthcare congestion. In Section 3, we set up the information disclosure

infected patients.

[9]Another related paper is Das, Kamenica and Mirka (2017). In a quite different setup, they investigate the optimal information structure in traffic networks to reduce road congestion.

[10]Dworczak and Martini (2019) find conditions for a disclosure rule to be optimal for a class of information design problems in which only the marginal distribution of conditional means matters for the principal.

[11]For example, see Alonso and Câmara (2016) for voting, Goldstein and Leitner (2018) for banks' risk sharing, Gehlbach and Sonin (2014) and Ginzburg (2019) for media control, Inostroza and Pavan (2020) for regime change, and Zapechelnyuk (2020) for quality control.

problem and solve for the optimal disclosure policy. Section 4 discusses how the optimal disclosure policy is dependent on the key parameters of an epidemic outbreak, and Section 5 concludes the paper. Proofs are provided in the Appendix.

2 A Simple Model of Healthcare Congestion

In this section, we develop a stylized model of healthcare congestion. We model scarce public healthcare resources as the limited number of treatments that the healthcare system can provide. We focus on the strategic behavior of demanding the public healthcare resource – namely, visiting a hospital – and link that to individual beliefs about the disease. This simple model helps us understand how healthcare congestion occurs, as well as its welfare consequences, and it serves as the baseline for the information design problem in the next section.

2.1 Model Setup

There is a unit mass of risk-neutral agents, indexed by $i \in [0, 1]$. During an epidemic outbreak, they are susceptible to the spread of a virus. Each agent decides whether or not to visit the hospital. Let $a_i \in [0, 1]$ denote the probability that agent i visits a hospital, and $n = \int_{i\in[0,1]} a_i \mathrm{d}i$ denote the total number of visitors.

Limited Capacity and Congestion The capacity of the healthcare system, such as the number of doctors and nurses, ICU beds, and ventilators, is limited. We assume that the healthcare system can admit, at most, $\bar{n} \in (0, 1)$ visitors. When the total number of visitors is no greater than this capacity, every visitor will be admitted to the hospital. If an agent is admitted, he will be treated if infected. When the total number of visitors exceeds the capacity of the hospital, congestion occurs. In this case, hospital visitors will be selected randomly to be admitted. Each visitor will be admitted with probability $\frac{\bar{n}}{n}$. Hence, we can write the probability of admission as

$$p(n, \bar{n}) = \min\{1, \frac{\bar{n}}{n}\}.$$

Likelihood of Infection Agents are uncertain about whether they are infected, as well as the severity of the disease if they are infected. Specifically, each agent

i has private information about his likelihood of being infected. This likelihood, or the agent's *type*, is denoted by $q_i \in [0,1]$.[12] The distribution of any individual's likelihood of infection q_i is characterized by a cumulative distribution function (CDF) $H : [0,1] \to [0,1]$.[13] The nature of the virus determines the distribution H, as it decides how easily one can detect an infection from related signs and symptoms. By the law of large numbers, the distribution of this likelihood in the population is also H. We assume that H is continuous and strictly increasing.

Severity of the Disease We use $s \in \mathcal{S} \equiv [0, \bar{s}]$ to denote the *severity of the disease* – that is, the welfare loss to any agent when he is infected but not treated. The value of s is common to all the agents. The larger s is, the more the agent suffers from the infection. Given the novelty of the virus, agents do not have precise information about the value of s. Rather, they rely on public disclosure to learn s, and have a posterior belief that is characterized by the expected severity level $\tilde{s} \in \mathcal{S}$. In this section, we solve the agent's problem for any given expected severity level $\tilde{s}$. We will explore the optimal disclosure rule about the severity s in the next section.[14]

Payoffs To visit the hospital, each agent has to pay a fixed visiting cost c. The cost can be interpreted, for instance, as the opportunity cost of time spent on one's visit, the costs associated with transportation and making an appointment, etc. For simplicity, we assume that the visiting cost is identical across all the agents.

In addition, each agent's payoff depends on whether he is infected, as well as, whether he is treated if he is infected. If an agent is not infected, his payoff, net of the visiting cost, is normalized to 0. This is independent of his visiting decision. If an agent is infected, he will be treated only if he visits the hospital and is admitted. We

[12] We view this private information as follows. Each agent has a binary prior belief about whether or not he is infected. Moreover, each agent also observes a private noisy signal about whether he is infected. For example, the agents observe their own symptoms that are linked to this disease. With this private signal, an agent forms his posterior belief and believes that he is infected with probability q_i.

[13] We assume that all the agents hold a common prior belief about infection. Because they face the same disease, they are also subject to the same noisy signal generating process. Therefore, this distribution H of q_i is common to all the agents.

[14] One can also interpret s as the *expected* payoff loss to an infected agent, which is determined by the mortality rate of this disease. In that case, government's disclosure about s can be understood as revealing the (historical) mortality rate of this disease.

assume that he will be cured after receiving the treatment. In this case, his payoff, net of the visiting cost, is also 0.[15] If an agent is infected but is not admitted to the hospital, his welfare loss is s — the severity of the disease. This can happen because either he does not visit the hospital, or he is not admitted due to hospital congestion. Table 1 summarizes each agent's ex post payoff.

		not infected	infected
not visit the hospital		0	$-s$
visit	admitted	$-c$	$-c$
	not admitted	$-c$	$-c-s$

Table 1: Ex post payoff

For any given likelihood of infection q_i, the expected severity level $\tilde{s}$, and the number of visitors n, agent i's expected payoff from his visiting decision a_i can be written as

$$v_i(a_i; q_i, \tilde{s}, n) \equiv a_i(-c - q_i(1 - p(n, \bar{n}))\tilde{s}) + (1 - a_i)(-q_i\tilde{s}). \tag{1}$$

Because $p(n, \bar{n})$ is decreasing and is strictly so when $n > \bar{n}$, agents' visits to the hospital create a negative externality on others' payoffs. We sometimes refer to this as *congestion externality*. It happens precisely due to the limited capacity of the healthcare system. Consequently, agents' hospital visiting decisions are *strategic substitutes*. Conditional on there being congestion, an agent's incentive to visit a hospital decreases when others are more likely to do so.

2.2 Equilibrium

All agents simultaneously and independently decide whether to visit the hospital, given their type q_i and the public belief $\tilde{s}$. A strategy for agent i is a mapping $a_i(\,\cdot\,; \tilde{s}) : [0, 1] \to [0, 1]$.

[15] We adopt this assumption for simplicity. Alternatively, one can assume that an agent will recover only with some probability if he is infected and treated. Or, any agent who is infected and receives a treatment to cure the disease would need to pay an additional medical cost. None of these alternative assumptions will change our results qualitatively.

Definition 1. Suppose that it is publicly believed that the expected severity of the disease is $\tilde{s}$. A strategy profile $\{a_i^*(\,\cdot\,;\tilde{s})\}_i$ is a Bayesian Nash equilibrium if for every $i \in [0,1]$,

$$a_i^*(q_i;\tilde{s}) \in \underset{a_i \in [0,1]}{\arg\max}\ v_i(a_i; q_i, \tilde{s}, n(\tilde{s})),\ \forall q_i \in [0,1], \tag{2}$$

and

$$n(\tilde{s}) = \int_0^1 \left(\int_0^1 a_i^*(q;\tilde{s}) \mathrm{d}H(q) \right) \mathrm{d}i. \tag{3}$$

Recall that v_i, as defined in (1), is agent i's expected payoff function. Condition (2) requires that the visiting decision of every type of every agent is optimal, given the total number of visitors, $n(\tilde{s})$. Condition (3) requires that agents' visiting decisions, in turn, determine the total number of visitors to the hospital.

The following proposition fully characterizes the unique Bayesian Nash equilibrium of this game.

Proposition 1. *For any expected severity level $\tilde{s} \in [0, \bar{s}]$, there is a unique Bayesian Nash equilibrium. In this equilibrium, agents play a symmetric cutoff strategy:*

$$a^*(q_i;\tilde{s}) = \begin{cases} 1, & \text{if } q_i \geq H^{-1}(1-n(\tilde{s})), \\ 0, & \text{if } q_i < H^{-1}(1-n(\tilde{s})), \end{cases} \tag{4}$$

where the total number of visitors $n(\tilde{s})$ satisfies

$$n(\tilde{s}) = \begin{cases} 0, & \text{if } \ \tilde{0} \leq \tilde{s} \leq c, \\ 1 - H\left(\frac{c}{\tilde{s}}\right), & \text{if } \ c < \tilde{s} \leq \frac{c}{H^{-1}(1-\bar{n})}, \end{cases} \tag{5}$$

and is the unique solution to

$$1 - H\left(\frac{cn(\tilde{s})}{\tilde{s}\bar{n}}\right) = n(\tilde{s}), \tag{6}$$

if $\tilde{s} > \frac{c}{H^{-1}(1-\bar{n})}$. Congestion occurs if and only if $\tilde{s} > \frac{c}{H^{-1}(1-\bar{n})}$.

Given the strategies taken by others and, thus, the total number of visits $n(\tilde{s})$, an agent with likelihood of infection q_i visits the hospital if his expected payoff from visiting exceeds that from not visiting – i.e., $-c - q_i(1 - p(n(\tilde{s}), \bar{n}))\tilde{s} \geq -q_i\tilde{s}$, or, equivalently, $q_i \geq \frac{c}{p(n(\tilde{s}),\bar{n})\tilde{s}} \equiv \beta(\tilde{s})$. This, in turn, implies that the equilibrium total number of visitors is

$$n(\tilde{s}) = 1 - H(\beta(\tilde{s})) = 1 - H\left(\frac{c}{p(n(\tilde{s}),\bar{n})\tilde{s}}\right). \tag{7}$$

From this equation, we can write the cutoff $\beta(\tilde{s})$ as $H^{-1}(1 - n(\tilde{s}))$, as stated in (4). Equations (5) and (6) give a full characterization of the solution $n(\tilde{s})$ to (7).

To understand (5) and (6), it is helpful to think about the extreme case, in which $\bar{n} = 1$, as a reference point. In this case, the healthcare system has unlimited capacity and a hospital congestion never happens. Because every hospital visitor is admitted, there is no strategic interaction among the agents. An agent with a likelihood of infection q_i visits the hospital if his expected payoff from visiting exceeds that from not visiting: $-c \geq -q_i\tilde{s}$. When $\tilde{s} \leq c$, the severity of the disease is very mild in expectation, and no one chooses to visit the hospital. Otherwise, when $\tilde{s} > c$, an agent visits the hospital only if his likelihood of infection exceeds $\frac{c}{\tilde{s}}$. In this case, the total number of hospital visitors is $1 - H(\frac{c}{\tilde{s}})$.

Now, consider the case in which $\bar{n} < 1$. When $1 - H(\frac{c}{\tilde{s}}) \leq \bar{n}$, or, equivalently, $\tilde{s} \leq \frac{c}{H^{-1}(1-\bar{n})}$, the agents actually face the same situation as they did when $\bar{n} = 1$, and, thus, congestion never occurs. Therefore, they behave the same in equilibrium as they would if there were unlimited capacity. The equilibrium number of visitors is, thus, $n(\tilde{s}) = 0$ if $\tilde{s} \leq c$, and $n(\tilde{s}) = 1 - H(\frac{c}{\tilde{s}})$ if $c < \tilde{s} \leq \frac{c}{H^{-1}(1-\bar{n})}$, as (5) claims.

However, when $1 - H(\frac{c}{\tilde{s}}) > \bar{n}$, the situation becomes different. If the agents still behave as if there is unlimited capacity, then the total number of visitors will exceed $\bar{n}$. In this case, congestion takes place, and each hospital visitor's probability of admission is strictly lower than 1. This fact, in turn, lowers each agent's incentive to visit the hospital. Indeed, an agent with a likelihood of infection q_i will visit the hospital if $q_i > \frac{cn(\tilde{s})}{\tilde{s}\bar{n}}$, since, now, the probability of admission is only $\frac{\bar{n}}{n(\tilde{s})}$. Then, the total number of visitors must satisfy (6). In this case, it is easy to check that $n(\tilde{s}) < 1 - H(\frac{c}{\tilde{s}})$. That is, the equilibrium number of visitors is strictly lower than that when there is unlimited capacity. This happens because the limited capacity creates strategic substitutes among the agents.

Based on the equilibrium characterization, congestion occurs only when the agents believe that the disease is sufficiently severe, or $\tilde{s} > s_0 \equiv \frac{c}{H^{-1}(1-\bar{n})}$. We refer to s_0 as the *exhaustion level*, under which all healthcare resources are just exhausted based on the equilibrium strategies. It will play an important role in our analysis in the next section.

Lemma 1. *When $\tilde{s} \in [c, \bar{s}]$ increases, the number of hospital visitors $n(\tilde{s})$ increases and the cutoff $\beta(\tilde{s})$ decreases.*

2.3 Total welfare

Next, to understand the welfare consequences of congestion, we calculate the equilibrium total welfare of this economy based on Proposition 1. It is the aggregated equilibrium payoff of the population.

Lemma 2. *Suppose that the true severity level is s and the public belief is $\tilde{s}$. In the unique Bayesian Nash equilibrium, the ex post total welfare of this economy is*

$$U(\tilde{s}, s) \equiv -s\mathbb{E}q - cn(\tilde{s}) + s\min\{n(\tilde{s}), \bar{n}\}\mathbb{E}(q|q \geq \beta(\tilde{s})). \tag{8}$$

As shown in (8), the total welfare $U(\tilde{s}, s)$ has three components. The first component, $-s\mathbb{E}q$, is the total welfare loss if there is no treatment provided. The second component, $-cn(\tilde{s})$, represents the total cost of hospital visits. The last component measures how much welfare gain can be achieved by the treatments provided by the healthcare system with limited capacity $\bar{n}$.

To understand the last component, recall that an agent visits the hospital only when his likelihood of infection is $q \geq \beta(\tilde{s})$. Thus, the average likelihood of infection among those who visit the hospital is $\mathbb{E}(q|q \geq \beta(\tilde{s}))$. Given that the total number of admissions is $\min\{n(\tilde{s}), \bar{n}\}$, by the law of large numbers, we can interpret $\min\{n(\tilde{s}), \bar{n}\}\mathbb{E}(q|q \geq \beta(\tilde{s}))$ as the total number of treatments the healthcare system provides. Therefore, the total welfare gain from the healthcare system is $s\min\{n(\tilde{s}), \bar{n}\}\mathbb{E}(q|q \geq \beta(\tilde{s}))$.

The next proposition helps us understand how the public belief affects the total welfare of this economy.

Proposition 2. *Suppose that the true severity level is s, and the public belief $\tilde{s}$ maximizes the ex post total welfare. Then, the economy achieves its ex post efficiency*

i). if $s \leq c$, $\tilde{s} \in [0, c]$;

ii). if $c < s \leq s_0$, $\tilde{s} = s$; and *iii). if $s > s_0$, $\tilde{s} = s_0$.*

The result should be intuitive. For any agent, the benefit of a hospital visit is at most s. When the disease is very mild – i.e., $s \leq c$ – the cost of a hospital visit outweighs the benefit. Thus, ex post efficiency requires that no one go to the hospital. This outcome is achieved if $\tilde{s} \leq c$, since $n(\tilde{s}) = 0$, in this case, by Proposition 1. If $\tilde{s} > c$, then some agents visit the hospital in equilibrium, which leads to welfare losses.

When $s > c$, in the absence of limited hospital capacity, those agents with type $q_i \geq \frac{c}{s}$ should go to the hospital, since their benefit from visiting, $q_i s$, outweighs the cost c. If $s \leq s_0$, even if all these agents visit the hospital, the hospital capacity is not filled. Thus, in this case, ex post efficiency requires all agents with $q_i \geq \frac{c}{s}$ to visit the hospital. This outcome is achieved only if $\tilde{s} = s$. If $\tilde{s} < s$ or $\tilde{s} > s$, there will be welfare loss in equilibrium due to too few or too many hospital visits. However, if $s > s_0$, the situation changes. The number of agents with type $q_i \geq \frac{c}{s}$ is $1 - H(\frac{c}{s})$, which now exceeds the hospital capacity. If all these agents visit the hospital, hospital congestion occurs.

Compared to the situation in which only those agents with $q_i \geq \frac{c}{s_0}$, visit the hospital so that the hospital is just filled to capacity, a hospital congestion results in welfare loss for two reasons. First, there are unnecessary expenditures on visiting the hospital. This is because $n(\tilde{s}) > \bar{n}$, whereby the the number of admissions is bounded above by $\bar{n}$. Second, misallocation of healthcare resources occurs in congestion. This happens because the hospital assigns its admissions randomly to visitors whenever congestion occurs. As a result, healthcare resources are used on some visitors with a low likelihood of infection, while some visitors with a higher likelihood of infection do not have access to those resources. Therefore, misallocation results in a lower number of treatments provided by the healthcare system. To see this more clearly, the average likelihood of a visitor being admitted, $\mathbb{E}(q|q \geq \beta(\tilde{s}))$, is decreasing in $\tilde{s}$ since the cutoff $\beta(\tilde{s}) = H^{-1}(1 - n(\tilde{s}))$ decreases with $\tilde{s}$ (see Lemma 1).

Therefore, when the true severity level is above s_0, ex post efficiency requires that only those agents with a likelihood of infection above $\frac{c}{s_0}$ visit the hospital. This ex post efficient outcome can be achieved in equilibrium only if $\tilde{s} = s_0$.

3 Information Disclosure

3.1 The Principal's Problem

In the previous section, we analyzed the model of hospital congestion given the agents' beliefs about the true severity level. As we saw from Proposition 2, it is the agents' beliefs that determine their choices in visiting the hospital and the total welfare of this simple economy. Since rational agents form their beliefs based on all relevant information, what information is available to them is then crucial for the economy's well-being.

Proposition 2 shows that fully revealing the severity level maximizes ex-post welfare when there is no congestion – i.e., $s \leq s_0$. However, it also clearly demonstrates the benefit from manipulating agents' beliefs when the true severity level can cause congestion under full revelation – i.e., $s > s_0$. Then, the natural question is what the optimal disclosure policy that can maximize the ex-ante total welfare would be. We aim to answer this question in this section.

Specifically, we introduce a benevolent principal to the model, whose objective is to maximize the total welfare of this economy. She does not observe each agent's private type –i.e., their likelihood of infection – whereas she can commit to an information disclosure rule with regard to the true severity level s.[16] the true severity level s, which in turn impacts the agents' hospital visiting decisions.

We assume that it is common knowledge that the severity level s is distributed according to a continuous and strictly increasing cumulative distribution function G over the interval $[0, \bar{s}]$. We also assume that $\bar{s} > s_0$ to avoid the trivial case of no congestion.[17] A disclosure rule specifies a measurable mapping from $[0, \bar{s}]$ to some signal space. Given a disclosure rule, and the prior distribution G, the agents form their posterior belief about the severity level s after observing the realized signal. Based on this posterior belief, the agents then decide whether to visit the hospital.

[16] In reality, the rule of information disclosure during a national or public health emergency is often governed by laws, and we believe, in that way, the ex ante commitment can be granted. For example, in the U.S., "*states have also enacted reporting requirements beyond specific diseases that indicate a public health threat. These laws vary in coverage and detail.*". See "Public Health Collection, Use, Sharing, and Protection of Information," 2012, available at: http://www.astho.org/Programs/Preparedness/Public-Health-Emergency-Law/Public-Health-and-Information-Sharing-Toolkit/Collection-Use-Sharing-and-Protection-Issue-Brief/.

[17] If $\bar{s} \leq s_0$, then full information disclosure is optimal by Proposition 2.

As shown in the previous section, because agents are risk- neutral, their hospital visiting decisions depend on posterior beliefs only through the public belief $\tilde{s}$, which is the posterior mean.

Every disclosure rule induces a joint distribution of the true severity level s and the public belief $\tilde{s}$. By Lemma 2, under this disclosure rule, the principal's ex ante expected payoff is

$$\begin{aligned}\tilde{\mathbb{E}}U(\tilde{s},s) &= \tilde{\mathbb{E}}\Big[-s\mathbb{E}q - cn(\tilde{s}) + s\min\{n(\tilde{s}),\bar{n}\}\mathbb{E}(q|q\geq\beta(\tilde{s}))\Big]\\ &= -\tilde{\mathbb{E}}s\mathbb{E}q + \tilde{\mathbb{E}}\Big[-cn(\tilde{s}) + \tilde{\mathbb{E}}(s|\tilde{s})\min\{n(\tilde{s}),\bar{n}\}\mathbb{E}(q|q\geq\beta(\tilde{s}))\Big]\\ &= -\tilde{\mathbb{E}}s\mathbb{E}q + \tilde{\mathbb{E}}\Big[-cn(\tilde{s}) + \tilde{s}\min\{n(\tilde{s}),\bar{n}\}\mathbb{E}(q|q\geq\beta(\tilde{s}))\Big], \end{aligned} \tag{9}$$

where $\tilde{\mathbb{E}}$ is with respect to the joint distribution of s and $\tilde{s}$. The second equality comes from the fact that $\tilde{\mathbb{E}}(s|\tilde{s}) = \tilde{s}$. Notice that the first term in (9), $-\tilde{\mathbb{E}}s\mathbb{E}q$, is independent of the disclosure rule. It is the expected welfare loss caused by the disease if there were no healthcare system and is determined solely by the nature of the disease. For the second term, let

$$V(\tilde{s}) \equiv -cn(\tilde{s}) + \tilde{s}\min\{n(\tilde{s}),\bar{n}\}\mathbb{E}(q|q\geq\beta(\tilde{s})). \tag{10}$$

It is the *conditional value of the healthcare system* given public belief $\tilde{s}$. Since this conditional value depends only on the public belief characterized by $\tilde{s}$, the *expected value of the healthcare system*, $\tilde{\mathbb{E}}V(\tilde{s})$, depends only on the *marginal distribution* of the public belief $\tilde{s}$. Because the public belief $\tilde{s}$ is simply the posterior mean, it is well known that a distribution of the posterior mean $\tilde{G}$ is induced by some disclosure rule if and only if it is a mean-preserving contraction of G.[18] Therefore, the principal's problem can be succinctly written as choosing a mean-preserving contraction of G to maximize the expected value of the healthcare system:

$$\begin{aligned}&\max_{\tilde{G}}\int_0^{\bar{s}} V(\tilde{s})\mathrm{d}\tilde{G}(\tilde{s})\\ &\text{s.t. } \tilde{G} \text{ is a mean-preserving contraction of } G.\end{aligned} \tag{11}$$

With slight abuse of terminology, we also refer to $\tilde{G}$ as a mean-preserving contraction of G as a disclosure rule.

[18]See, for instance, Blackwell (1951), Gentzkow and Kamenica (2016), Kolotilin (2018) and Dworczak and Martini (2019). Distribution $\tilde{G}$ is a mean-preserving contraction of G if $\int_0^s \tilde{G}(\tilde{s})\mathrm{d}\tilde{s} \leq \int_0^s G(\tilde{s})\mathrm{d}s$ for all $s\in[0,\bar{s}]$ and $\int_0^{\bar{s}} \tilde{s}\mathrm{d}\tilde{G}(\tilde{s}) = \int_0^{\bar{s}} \tilde{s}\mathrm{d}G(\tilde{s})$.

3.2 Optimal Disclosure Rule

To have a full characterization of the optimal disclosure rule, we make an assumption on the distribution of the likelihood of infection q – that is, the convexity of H. We will discuss the implications of this assumption and, later, the optimal disclosure rule without this assumption. Under this assumption, the following proposition presents our main result. It states that the principal's optimal disclosure rule is a simple censorship rule.

Proposition 3. *Suppose that* H *is convex. There exists* $0 < s_- < s_0 < s_+ \leq \bar{s}$ *such that the following disclosure rule is optimal:*

$$\tilde{G}^*(\tilde{s}) = \begin{cases} G(\tilde{s}), & \text{if } 0 \leq \tilde{s} < s_-, \\ G(s_-), & \text{if } s_- \leq \tilde{s} < s_0, \\ G(s_+), & \text{if } s_0 \leq \tilde{s} < s_+, \\ G(\tilde{s}), & \text{if } s_+ \leq \tilde{s} \leq \bar{s}. \end{cases} \tag{12}$$

In words, Proposition 3 identifies a range $[s_-, s_+]$ of severity levels around the exhaustion level s_0 such that simply pooling the severity levels in this range and fully revealing them outside this range is a principal's optimal disclosure policy. This optimal policy can be interpreted as a *middle censorship rule*, which censors only the intermediate range of $[s_-, s_+]$. Under this disclosure policy, the agents' public beliefs after observing "no message" is precisely the exhaustion level s_0, in which case the healthcare system runs just at its full capacity. In view of Proposition 2, this optimal rule avoids the hospital congestion when $s \in (s_0, s_+]$. When the maximum severity level is low so that $s_+ = \bar{s}$, the middle censorship rule degenerates to an *upper censorship rule*, which censors all severity levels above s_-.

The proof of Proposition 3 is built on Theorem 1 in Dworczak and Martini (2019). To understand the underlying intuition, it is crucial to understand how the conditional value of the healthcare system, $V(\tilde{s})$, changes as the public belief $\tilde{s}$ varies. We explain this in Subsection 3.3 below and then discuss the logic behind Proposition 3 in Subsection 3.4.

3.3 Conditional Value of the Healthcare System $V(\tilde{s})$

The agents' equilibrium strategy in Proposition 1 implies that the conditional value is a piecewise function

$$V(\tilde{s}) = \begin{cases} 0, & \text{if } \tilde{s} \in [0, c], \\ \int_{\frac{c}{\tilde{s}}}^{1} (\tilde{s}q - c) \mathrm{d}H(q), & \text{if } \tilde{s} \in (c, s_0], \\ \int_{\frac{c}{x(\tilde{s})}}^{1} (x(\tilde{s})q - c) \mathrm{d}H(q), & \text{if } \tilde{s} \in (s_0, \bar{s}]. \end{cases}$$

where $x(\tilde{s}) = \tilde{s}\frac{\bar{n}}{n(\tilde{s})}$.[19]

The healthcare resources, or the treatments provided by the hospital, are valuable to the infected agents only if these agents choose to visit the hospital and get admitted. When the benefit of visiting the hospital for *an agent who is infected ex post*, $\tilde{s}$,[20] is less than the visiting cost, c, no one visits the hospital. In this case, the value $V(\tilde{s})$ is simply 0. When this benefit $\tilde{s}$ is greater than the visiting cost c but lower than the exhaustion level s_0, agents with type $q \geq \beta(\tilde{s}) = \frac{c}{\tilde{s}}$ choose to visit the hospital. Since there is no congestion, the net payoff to a type q hospital visitor is $\tilde{s}q - c$, and, thus, the value of the healthcare system is $V(\tilde{s}) = \int_{\frac{c}{\tilde{s}}}^{1} (\tilde{s}q - c) \mathrm{d}H(q)$.

When $\tilde{s} > s_0$, congestion occurs, and the benefit of visiting the hospital for an agent who is infected ex post is $x(\tilde{s}) = \tilde{s}\frac{\bar{n}}{n(\tilde{s})}$ instead of $\tilde{s}$. This is because hospital admission is random, and each visitor has only a $\frac{\bar{n}}{n(\tilde{s})}$ chance of being admitted. In equilibrium, only agents with type $q \geq \beta(\tilde{s}) = \frac{c}{x(\tilde{s})}$ choose to visit the hospital. The net payoff to a type q hospital visitor is $x(\tilde{s})q - c$, and, thus, the value of the healthcare system is $V(\tilde{s}) = \int_{\frac{c}{x(\tilde{s})}}^{1} (x(\tilde{s})q - c) \mathrm{d}H(q)$.

Figure 1 gives a graphical illustration of a typical V function under the assumption that H is convex. Appendix C.1 studies the detailed properties of V. As depicted in the figure, function V is increasing, piecewise convex, and has a kink at the exhaustion level s_0, among other things. The easiest way to see these properties is to assume that H has a continuous density. In this case, V is piecewise differentiable, and the marginal value of the healthcare system is

$$V'(\tilde{s}) = \begin{cases} \int_{\frac{c}{\tilde{s}}}^{1} q \mathrm{d}H(q), & \text{if } \tilde{s} \in (c, s_0), \\ x'(\tilde{s}) \int_{\frac{c}{x(\tilde{s})}}^{1} q \mathrm{d}H(q), & \text{if } \tilde{s} \in (s_0, \bar{s}]. \end{cases} \tag{13}$$

[19] See Lemma C.1 in Appendix C

[20] To clarify, conditional on the posterior belief $\tilde{s}$, the conditional expected value of a treatment to an agent who is infected ex post is $\tilde{s}$. Hence, the expected value to an agent with type q is $q\tilde{s}$.

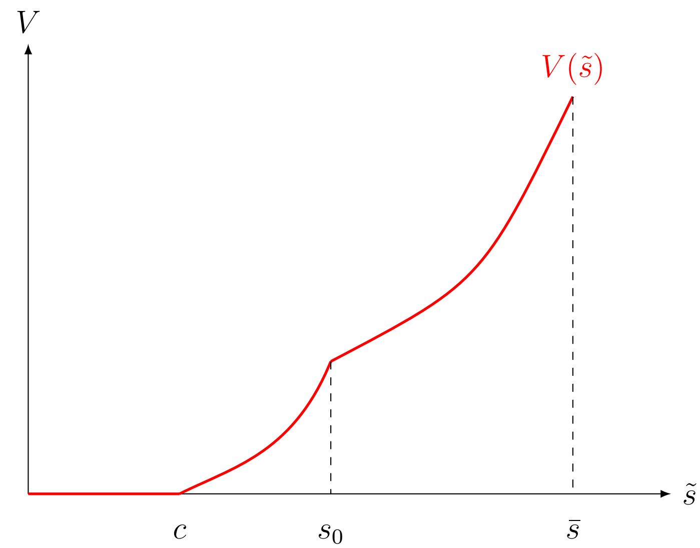


Figure 1: The expected value of the healthcare system

Intuitively, a change in $\tilde{s}$ has two effects on the value of the healthcare system. First, as discussed, it changes the benefit of a hospital visit for each infected agent. Second, it changes the number of agents who visit the hospital. Marginally, the second effect vanishes, as the value of the healthcare system to the marginal agent is always 0.[21] Consequently, the marginal value of the healthcare system measures only the first effect. It is equal to the marginal benefit of a hospital visit for an infected agent (1 for $\tilde{s} \in (c, s_0]$ and $x'(\tilde{s})$ for $\tilde{s} \in (s_0, \bar{s}]$), multiplied by the total number of hospital visitors who are infected ex post.

As can be seen from (13), the marginal value $V'(\tilde{s})$ over $(c, s_0]$ is positive and increasing. The property of $V'(\tilde{s})$ over $(s_0, \bar{s}]$ relies on $x(\tilde{s})$. In equilibrium, the benefit of a hospital visit for an infected agent, $x(\tilde{s})$, must be increasing in $\tilde{s}$; otherwise, fewer agents will visit the hospital, which, in turn, will increase the benefit of a hospital visit. Thus, $V'(\tilde{s})$ is also positive over $(s_0, \bar{s}]$. Under the assumption that H is convex, we show in Lemma C.4 of Appendix C.1 that the marginal benefit, $x'(\tilde{s})$, is also increasing in $\tilde{s}$.[22] This, in turn, implies that $V'(\tilde{s})$ is also increasing over $(s_0, \bar{s}]$.

From the above discussions, the convexity of H is not needed for the monotonicity of $V(\tilde{s})$ in $\tilde{s}$ or the convexity of V over $[c, s_0]$. But this assumption serves as a

[21] Recall that the marginal agent with type $\beta(\tilde{s})$ is indifferent between visiting the hospital and not, as $\tilde{s}\beta(\tilde{s}) = c$.

[22] We show that x is convex in $\tilde{s}$ in Lemma C.4.

sufficient condition for the convexity of V over $(s_0, \bar{s}]$. Recall that over the range of $\tilde{s}$ that admits congestion – i.e., $\tilde{s} \in (s_0, \bar{s}]$ – congestion becomes more intense when $\tilde{s}$ increases, as there are a greater number of visitors $n(\tilde{s})$ (see Lemma 1). Intuitively, under any convex distribution H, when $\tilde{s}$ increases from a higher level, a smaller mass of agents with lower likelihood of infection join the group of hospital visitors, thereby intensifying the congestion but contributing to the misallocation to a lesser extent. Therefore, when H is convex, the value of the healthcare system increases, and it increases at a faster pace when the disease is expected to be more severe. In other words, although congestion is costly, the convexity of H guarantees that it will not significantly undermine the functioning of the healthcare system.

3.4 Intuition

It is important to note that V is not globally convex.[23] The kink at s_0 arises precisely because hospital congestion lowers the marginal value of the healthcare system. It is this kink that makes the full transparency policy not optimal.[24] To see this, consider a simple disclosure rule that censors an interval $[s_1, s_2]$ around s_0 such that $\mathbb{E}_G(s|s_1 \leq s \leq s_2) = s_0$. Under this censorship rule, the agents' public belief is just s_0 when the severity level is in $[s_1, s_2]$, and they perfectly know the severity level when it is outside this interval.

Compared to the full transparency policy, the agents behave differently under this censorship rule when $s \in [s_1, s_2]$. In view of Proposition 2, on the one hand, they overreact when $s \in [s_1, s_0)$, as they expect that the severity level is at s_0. This results in ex post welfare loss because of unnecessary hospital visits (though does not lead to congestion). That defines the cost of this simple censorship rule.

On the other hand, when $s \in (s_0, s_2]$, the simple censorship rule makes agents believe that the severity is lower than it actually is. This induced belief ($\tilde{s} = s_0$) reduces the agents' incentive to visit the hospital. In fact, this policy helps to internalize the congestion externality in agents' decision making, thereby eliminating hospital congestion and implementing ex-post efficiency (see Proposition 2). That explains the benefit from adopting this simple censorship rule.

[23] See inequality C.5 in Lemma C.5 of Appendix C.1.

[24] In Proposition 5 in Section 4, we formally show that full disclosure is never an optimal disclosure policy under any distribution H that admits a continuous density.

The overall welfare difference between this censorship rule and the full transparency policy is measured by

$$\int_{s_1}^{s_2} V(s_0)\mathrm{d}G(\tilde{s}) - \int_{s_1}^{s_2} V(\tilde{s})\mathrm{d}G(\tilde{s}).$$

The kink at s_0 immediately suggests that this difference is positive, provided that the interval $[s_1, s_2]$ is sufficiently small. This can be easily seen from Figure 2. The kink at s_0 implies the existence of a straight line ℓ over $[s_1, s_2]$ that is everywhere above V, except for coinciding with V at s_0. Hence, the above welfare difference is $\int_{s_1}^{s_2}(\ell(\tilde{s}) - V(\tilde{s}))\mathrm{d}G(\tilde{s}) > 0$, showing that censoring $[s_1, s_2]$ does yield a higher ex ante welfare than full transparency.

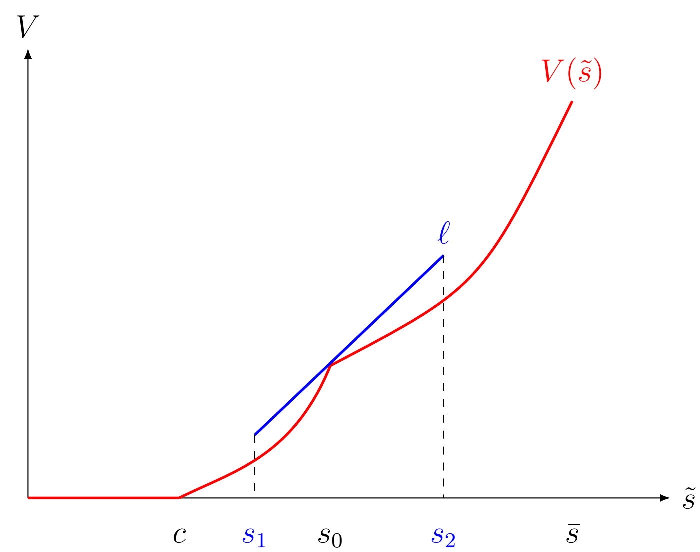
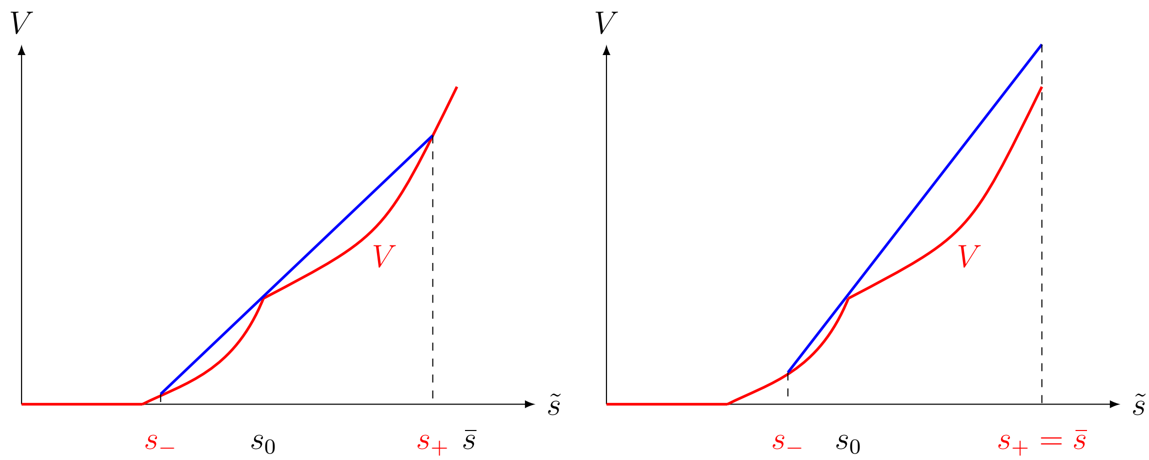


Figure 2: Welfare improvement of censorship around s_0

Along the same line of reasoning, Figure 2 also suggests that the total welfare can be further improved by extending the censoring range. Indeed, the optimal disclosure rule in Proposition 3 pushes this logic to the limit. It identifies the largest censoring range $[s_-, s_+]$ around s_0 that: i) satisfies the conditional mean $\mathbb{E}_G(s|s_- \leq s \leq s_+)$ being s_0; and ii) admits a straight line over $[s_-, s_+]$ that passes through $(s_0, V(s_0))$ and is above V elsewhere.

It is worth mentioning that in the optimal disclosure policy, censorship does not apply to severity levels above s_+ (if such states exist). Given the convexity of V over $(s_0, \bar{s}]$, censoring additional high severity levels while maintaining the posterior mean of all censored states at the exhaustion level s_0 becomes increasingly more costly. The reason is that this further distorts the agents' decision making for some states $s < s_0$ and, thus, induces increasingly more unnecessary hospital visits. In Section 4, we will discuss the optimal choice of disclosure rule if V is not convex over $(s_0, \bar{s}]$ when the convexity of H does not hold.

Figure 3 gives an illustration of the censoring range under the optimal disclosure rule. Figure 3a illustrates the case of middle censorship, whereas Figure 3b illustrates upper censorship. Recall that, given the function V, whether the optimal disclosure rule is a middle censorship or an upper censorship depends on how the severity level s is distributed – i.e., G. We will discuss this further when investigating how the optimal disclosure policy changes with the distribution G in the next Section.



(a) Middle censorship (b) Upper censorship

Figure 3: The optimal censoring range

3.5 Uniqueness

The optimal disclosure rule in Proposition 3 is not the principal's unique optimal disclosure rule. As we saw above, the value of the healthcare system, $V(\tilde{s})$, is constantly zero over $[0, \min\{c, s_-\}] \subset [0, c]$, because any public belief $\tilde{s}$ in this range results in no hospital visits at all. Trivially, those disclosure rules that release coarser information over this range, but share the same censorship structure over $[\min\{c, s_-\}, \bar{s}]$ as $\tilde{G}^*$, must achieve the same welfare. Our next proposition shows that these disclosure rules are the only optimal rules. In other words, the optimal disclosure rule is essentially unique, if we ignore this uninteresting multiplicity. Clearly, $\tilde{G}^*$ is the most informative of all these optimal disclosure rules.

Proposition 4. *Suppose that H is convex. If $\tilde{G}$ is an optimal disclosure rule, then $\tilde{G}$ is a mean-preserving contraction of $\tilde{G}^*$, and $\tilde{G}(\tilde{s}) = \tilde{G}^*(\tilde{s})$ if $\tilde{s} \geq \min\{c, s_-\}$.*

4 Discussion and Extension

From the discussions so far, we know that information disclosure policy can be effective in mitigating healthcare congestion and improving efficiency. Under the assumption that the distribution of likelihood of infection, H, is convex, we have a clear characterization of the optimal disclosure policy. It censors an optimal interval around the exhaustion level and ensures that the system is run just at its full capacity when information is censored.

Readers may wonder how the optimal disclosure policy would change if the public authority were to face a different virus in terms of the distribution of its severity G, the distribution of infection likelihood H, the infectiousness of the virus; or if healthcare system has a greater capacity $\bar{n}$. In this section, to address these questions, we extend the model by incorporating the infectiousness of the virus into our analysis, and conduct comparative statics analysis on the key parameters of the epidemic.

4.1 Full Transparency is Never Optimal

One of the key insights we get from the analysis in Section 3 is that, as compared with full transparency, censoring an interval around the exhaustion level s_0 can help to avoid some congestion, and, thus, it is welfare-improving. Our main result (Proposition 3) is derived by assuming the convexity of the distribution H. The following proposition shows that this insight is not restricted to that assumption.

Proposition 5. *Assume that H has a continuous density. There exists an interval $[s_-, s_+]$ with conditional mean s_0 such that censoring s in $[s_-, s_+]$ while fully revealing otherwise is better than full transparency.*

For any distribution H of severity levels, full transparency is not optimal, provided that H satisfies a regularity condition. It is dominated by a censorship rule that censors a small interval around the exhaustion level s_0. The underlying intuition is the same as that discussed in Section 3.4. Censoring a small interval around the exhaustion level s_0, while fixing the posterior belief to s_0, can prevent the congestion for some states above s_0 and, thus, is welfare-improving.

4.2 Preparedness of the Healthcare System

According to the conventional wisdom, when the healthcare system facing an epidemic outbreak is more prepared, the policy maker will be more confident and, thus, will implement more transparency. Our model is well suited to examine this intuition. We view the *preparedness* of the healthcare system as the ex-ante probability that congestion will occur under full transparency – i.e., $\mathbb{P}(s > s_0) = G(\frac{c}{H^{-1}(1-\bar{n})})$. Everything else equal, the healthcare system is more prepared if the disease it faces is less likely to be severe, or if it has a larger capacity $\bar{n}$. To test how the optimal disclosure policy changes with the preparedness of the healthcare system, we conduct comparative statics analyses on the distribution of disease severity G and the capacity of the healthcare system $\bar{n}$.

We first compare the optimal censoring ranges under two different distributions of severity levels, and the results from this comparison are summarized in the following proposition.

Proposition 6. *Suppose that H is convex. Consider two distributions G_1 and G_2 over $[0, \bar{s}]$. Let g_i be the density of G_i, for $i = 1, 2$. Assume that g_1 and g_2 satisfy the strict monotone likelihood ratio property (SMLRP): $g_2(s)/g_1(s)$ is strictly increasing. Then, the optimal censoring ranges ,$[s_-^1, s_+^1]$ under G_1 and $[s_-^2, s_+^2]$ under G_2, satisfy $s_-^1 > s_-^2$ and $s_+^1 \geq s_+^2$ with strict inequality if $s_+^2 < \bar{s}$.*

By SMLRP, the disease is less likely to be severe under G_1 than under G_2. The healthcare system is, thus, more prepared under the former. Proposition 6 claims that, with a more prepared healthcare system, the optimal censoring range moves to the right. Rather than revealing more states in which the disease is severe, it censors more high severity levels and reveals more states in which the severity levels are low. The intuition behind this result hinges on the cost from censoring. Given that the role of censorship is to avoid congestion, censoring high severity levels must be accompanied by censoring those low severity levels at which full transparency is efficient. The resulting distortions on those low severity states define the cost of this censorship. When high severity levels are less likely to occur, censoring those states becomes less costly, as it does not require censoring many low severity levels.

By Proposition 6, once the disease is already sufficiently weak (i.e., $s_+^2 = \bar{s}$), the censoring range shrinks if its severity level is weakened further. More specifically, the upper bound of the censoring range is unchanged and fixed at $\bar{s}$, whereas the lower bound moves to right – i.e., $s_-^1 > s_-^2$. In either case, under the optimal disclosure policy, all severity levels above the exhaustion level s_0 are censored, and no congestion happens ex-post.

Next, we study how the optimal disclosure policy changes with the capacity $\bar{n}$ of the healthcare system. Clearly, facing the same disease, a healthcare system with a larger capacity is better prepared for that disease. This comparative statics analysis cannot yield a tractable result for the general economic environment we consider, since both the exhaustion level s_0 and the function V change with $\bar{n}$. We choose to restrict our attention to uniform H. Our findings are summarized in the next proposition.

Proposition 7. *Suppose that H is uniform distribution over $[0,1]$ and $\bar{s} > c$. There exists $\bar{n}^* \in (0, 1 - \frac{c}{\bar{s}})$ such that*

1. *when $\bar{n} \in (0, \bar{n}^*)$, the optimal disclosure rule is middle censorship, and both $s_-(\bar{n})$ and $s_+(\bar{n})$ strictly increase as $\bar{n}$ increases over this range;*

2. *when $\bar{n} \in [\bar{n}^*, 1 - \frac{c}{\bar{s}})$, the optimal disclosure rule is upper censorship, and $s_-(\bar{n})$ strictly increases as $\bar{n}$ increases over this range.*[25]

When the healthcare system becomes more resourceful, the exhaustion level s_0 increases. Since the optimal disclosure policy censors an interval around the exhaustion level, it is intuitive that the censoring range moves to the right. As the capacity $\bar{n}$ increases to a sufficiently high level, the upper bound of the censoring range reaches the highest possible severity level $\bar{s}$. If $\bar{n}$ further increases from that level, the censoring range starts to shrink, with its lower bound moving further to right.[26]

The above two propositions offers a consistent result regarding the relationship between the transparency in information disclosure and the preparedness of the healthcare system. They suggest that the conventional wisdom is correct if the healthcare

[25]When $\bar{n} \geq 1 - \frac{c}{\bar{s}}$, full information disclosure is optimal, since congestion never occurs.

[26]In fact, this result is not restricted to uniform H. It is true for general convex H.

system is sufficiently prepared, either because the disease it faces is sufficiently weak or due to its sufficiently high capacity. In both cases, all severity levels that lead to congestion are censored and, thus, congestion is completely eliminated. As the disease becomes even weaker or the healthcare system is better equipped, the censoring range strictly shrinks, with more and more states fully revealed.

However, the conventional wisdom does not hold when the healthcare system is not well prepared. That is, the system is better prepared if the disease becomes slightly weaker or it has a slightly higher capacity, but neither circumstance necessarily leads to more transparency. In both cases, the censoring range only moves to the right, censoring more high severity levels while revealing more low severity levels.

4.3 Infectiousness

So far, we have treated the epidemic outbreak as a static event, with all infections determined ex-ante by nature. In this subsection, we incorporate the infectiousness of the virus into our analysis. This allows us to consider how the contagion dynamics can be affected by the information disclosure, from the angle of healthcare resource allocation (based on hospital visits and admissions), thereby better connecting our model to the SEIR framework.

To introduce the infectiousness of the virus into the model, let us consider a simple dynamic setting with dates 0 and 1. Our model of healthcare congestion in Section 2 captures the date 0 decision making of all agents who have had some exposure to the infected group at $t = 0$. Following the terminology in the SEIR framework, we call these agents the *exposed* group. Individuals in the *susceptible* group have not been exposed to the virus and, thus, are not infected with certainty at date t. However, as long as there are some agents in the exposed group who are infected but not treated at the end of date 0, then they pass the virus to anyone in the susceptible group at a later date 1. The number of individuals that one infected person will pass on a virus to, on average, is denoted by γ.

The parameter γ captures the infectiousness of the virus. If a virus is more infectious, then γ is greater. An infected individual, if not treated, bears a cost s at date t. However, this also incurs an expected social cost of γs after date 0 because of the future contagion that can be caused by this infected agent. Agents in the exposed group never take this social cost into consideration when deciding whether to visit the hospital; therefore, this *infection externality* does not change the equilibrium strategy of the exposed group. However, welfare losses that are linked to the infection externality are valued by the benevolent principal. Accordingly, we can write the ex post total welfare for any given expected severity level $\tilde{s}$ and true severity level s as

$$U_\gamma(\tilde{s}, s) \equiv -(1+\gamma)s\mathbb{E}q - cn(\tilde{s}) + (1+\gamma)s\min\{n(\tilde{s}), \bar{n}\}\mathbb{E}(q|q \geq \beta(\tilde{s})).$$

The ex post total welfare in (8) can be taken as a special case of U_γ without the infection externality (i.e., $\gamma = 0$). When there is infection externality, one treatment can generate an expected welfare gain of γs from avoiding further infections, in addition to recovering the infected agent's loss s. Consequently, from the principal's perspective, the conditional value of the healthcare system is

$$V_\gamma(\tilde{s}) \equiv -cn(\tilde{s}) + (1+\gamma)\tilde{s}\min\{n(\tilde{s}), \bar{n}\}\mathbb{E}(q|q \geq \beta(\tilde{s})).$$

The optimal information design problem is not different from the one in (11), except for replacing the principal's objective function $V(\tilde{s})$ with $V_\gamma(\tilde{s})$. For tractability, we consider a uniform distribution H, and identify the optimal disclosure policy under that assumption. We further investigate how the optimal policy changes with the infectiousness parameter γ. The results are summarized in the following proposition.

Proposition 8. *Suppose that H is uniform distribution over $[0, 1]$.*

1. *For every $\gamma \in [0, 1)$, there exists $0 < s_-(\gamma) < s_+(\gamma) \leq \bar{s}$ with conditional mean s_0 such that censoring s in $[s_-(\gamma), s_+(\gamma)]$ while fully revealing otherwise is optimal.*

2. *Suppose that $\gamma_2 > \gamma_1$. Then, $s_-(\gamma_2) < s_-(\gamma_1)$ and $s_+(\gamma_2) > s_+(\gamma_1)$, if $s_+(\gamma_1) < \bar{s}$; $s_-(\gamma_2) = s_-(\gamma_1)$ and $s_+(\gamma_2) = s_+(\gamma_1)$, if $s_+(\gamma_1) = \bar{s}$.*

If q is uniformly distributed, V_γ maintains similar qualitative properties as in the case without infection externality. Therefore, the middle censorship rule is still the optimal disclosure policy, as Proposition 8 claims. If the upper limit of the censoring range $s_+(\gamma)$ reaches $\bar{s}$, then the middle censorship rule degenerates to an upper censorship rule.

As the infectiousness of the virus increases, the optimal censoring range expands until its upper limit $s_+(\gamma)$ reaches $\bar{s}$. This is because, , as the virus becomes more infectious, the cost of censoring low severity levels becomes lower, and the benefit of censoring high severity levels becomes higher. Without infection externality, censoring the severity levels below s_0 is costly since it induces unnecessary hospital visits. Censoring the severity levels above s_0 is beneficial, since it prevents hospital congestion and improves the allocation of limited healthcare system resources. As infection externality increases, censoring low states becomes less costly because more hospital visits in the absence of hospital congestion clearly mitigates the infection externality. Meanwhile, censoring high states becomes more beneficial, as the agents who are treated under the censorship are not only the ones who need the treatment most, but also the ones who are most contagious. When the infectiousness of the virus is already high, the upper limit of the optimal censoring range reaches the upper bound $\bar{s}$. Any further increase of the infectiousness of the virus will not change the censoring range, because the censorship rule can not avert congestion further.

5 Concluding Remarks

A main theme of economics concerns the efficient allocation of scarce resources. The epidemic outbreak, as a public health crisis, features a situation in which the free market with pricing competition cannot function to guarantee efficient resource allocation. At the same time, central planning cannot achieve efficiency since the social planner does not have enough information about individuals' demand. In such an environment, congestion can pose a critical threat to economic efficiency.

This paper investigates how to use information disclosure as a policy tool to mitigate the congestion and to maximize the ex ante efficiency of the healthcare system in an epidemic outbreak. We identify the optimal design of information disclosure and discuss how this optimal policy depends on the key parameters in an epidemic outbreak.

Results derived from this study can be applied to to determine the allocation of public resources in other economic environments in which the pricing mechanism is absent. For instance, suppose that government has a limited amount of research funding to support private companies' R&D activities on a certain technology. There is a tremendous amount of uncertainty associated with the future profitability of this innovation and, thus, government support is needed to advance this technology. The funding support is open to a group of companies that privately know their ability to advance this innovation. Our results shed light on how the government can adopt an information disclosure policy (about the future profitability of this technology) to manipulate the applicants' beliefs and regulate their choices in funding applications, so as to distribute the funding more efficiently.

In this study, we restrict our attention to how the disclosure of the severity information mitigates the healthcare congestion problem. In reality, there are also other dimensions of information that are important for the social welfare in epidemic outbreaks. Apart from informing the public about the severity of the ongoing infectinous disease, strategic information disclosure can also be adopted, for example, to warn individuals about the ongoing outbreak, to explain why certain social distancing measures are needed, or to reveal the contact tracing of infected patients, and so on. The disclosure of these information may also change the dynamics of infection through mechanisms other than hospital visits and admissions. In addition, other policies, such as testing and screening patients for hospital admission, may endogenously produce valuable information for a more efficient allocation of healthcare resources. In future work, it would be interesting to extend the analysis of information disclosure in epidemics in these directions.

Appendix A Proof of Proposition 1

We first establish an auxiliary lemma which will be used for the proof of both Propositions 1 and 3.

Lemma A.1. *For every* $\tilde{s} > 0$*, the equation*

$$m = 1 - H\Big(\frac{cm}{\tilde{s}\bar{n}}\Big) \tag{A.1}$$

for unknown m *has a unique solution* $m(\tilde{s}) \in (0, 1)$*. Moreover,* m *as a function of* $\tilde{s}$ *over* $(0, \infty)$ *is continuous, strictly increasing,* $\lim_{\tilde{s}\downarrow 0} m(\tilde{s}) = 0$*, and* $m\Big(\frac{c}{H^{-1}(1-\bar{n})}\Big) = \bar{n}$*.*

Proof. Fix $\tilde{s} > 0$. The function $m \mapsto m - 1 + H\big(\frac{cm}{\tilde{s}\bar{n}}\big)$ is strictly increasing over $[0, 1]$. When $m = 0$, its value is -1. When $m = 1$, its value is $H(\frac{c}{\tilde{s}\bar{n}}) > 0$. Therefore, there exists a unique $m(\tilde{s}) \in (0, 1)$ such that $m(\tilde{s}) - 1 + H\Big(\frac{cm(\tilde{s})}{\tilde{s}\bar{n}}\Big) = 0$. This $m(\tilde{s})$ is the unique solution to (A.1).

Pick $0 < \tilde{s}_1 < \tilde{s}_2$. We have

$$m(\tilde{s}_1) + H\bigg(\frac{cm(\tilde{s}_1)}{\tilde{s}_1\bar{n}}\bigg) = m(\tilde{s}_2) + H\bigg(\frac{cm(\tilde{s}_2)}{\tilde{s}_2\bar{n}}\bigg) < m(\tilde{s}_2) + H\bigg(\frac{cm(\tilde{s}_2)}{\tilde{s}_1\bar{n}}\bigg),$$

where the inequality comes from strict monotonicity of H. This implies $m(\tilde{s}_1) < m(\tilde{s}_2)$, i.e., $m(\tilde{s})$ is strictly increasing.

Consider a monotone sequence $\{\tilde{s}_k\}$ such that either $\tilde{s}_k \uparrow \tilde{s}$ or $\tilde{s}_k \downarrow \tilde{s}$ for some $\tilde{s} > 0$. Because m is monotone, $\lim_k m(\tilde{s}_k)$ exists. Since H is continuous and

$$m(\tilde{s}_k) = 1 - H\bigg(\frac{cm(\tilde{s}_k)}{\tilde{s}_k\bar{n}}\bigg), \ \forall k,$$

we have

$$\lim_k m(\tilde{s}_k) = 1 - H\bigg(\frac{c \lim_k m(\tilde{s}_k)}{\tilde{s}\bar{n}}\bigg).$$

By the uniqueness of $m(\tilde{s})$, we know $\lim_k m(\tilde{s}_k) = m(\tilde{s})$, proving the continuity of m.

Moreover, notice $m(\tilde{s}) = \frac{\tilde{s}\bar{n}}{c} H^{-1}(1 - m(\tilde{s}))$ for all $\tilde{s} > 0$. Therefore, $\lim_{\tilde{s}\downarrow 0} m(\tilde{s}) = 0$.

Finally, observe

$$\bar{n} = 1 - H\left(\frac{c\bar{n}}{\frac{c}{H^{-1}(1-\bar{n})}\bar{n}}\right).$$

Since the solution to (A.1) is unique as we have shown, we know $m\Big(\frac{c}{H^{-1}(1-\bar{n})}\Big) = \bar{n}$. ∎

Lemma A.2. *For every* $\tilde{s} > 0$, *the equation*

$$n = 1 - H\left(\frac{c}{p(n, \bar{n})\tilde{s}}\right) \tag{A.2}$$

for unknown n *has a unique solution:*

$$n(\tilde{s}) = \begin{cases} 0, & \text{if } \tilde{s} \le c, \\ 1 - H\left(\frac{c}{\tilde{s}}\right), & \text{if } c < \tilde{s} \le \frac{c}{H^{-1}(1-\bar{n})}, \\ m(\tilde{s}), & \text{if } \tilde{s} > \frac{c}{H^{-1}(1-\bar{n})}, \end{cases} \tag{A.3}$$

where $m(\tilde{s})$ *is the unique solution to (A.1) in Lemma A.1. Moreover,* $n(\tilde{s}) > \bar{n}$ *if and only if* $\tilde{s} > \frac{c}{H^{-1}(1-\bar{n})}$.

Proof. Consider the function $f(n) = n - 1 + H\left(\frac{c}{p(n,\bar{n})\tilde{s}}\right)$ for $n \ge 0$. When $\tilde{s} \le c$, $\frac{c}{p(n,\bar{n})\tilde{s}} \ge 1$ for all $n \ge 0$, since $p(n, \bar{n}) \le 1$ for all $n \ge 0$. In this case, $f(n) = n$ for $n \ge 0$. Therefore, the unique solution to (A.2) is $n(\tilde{s}) = 0$.

When $\tilde{s} > c$, f function takes the following form

$$f(n) = \begin{cases} n - 1 + H(\frac{c}{\tilde{s}}), & \text{if } n \le \bar{n}, \\ n - 1 + H(\frac{cn}{\tilde{s}\bar{n}}), & \text{if } \bar{n} < n \le \frac{\tilde{s}}{c}\bar{n}, \\ n, & \text{if } n > \frac{\tilde{s}}{c}\bar{n}. \end{cases}$$

Because f is strictly increasing, $f(0) < 0$ and $f(\frac{\tilde{s}}{c}\bar{n}) = \frac{\tilde{s}}{c}\bar{n} > 0$, f has a unique zero $n(\tilde{s}) > 0$. This is the unique solution to (A.2). When $f(\bar{n}) \ge 0$, or equivalently $\tilde{s} \le \frac{c}{H^{-1}(1-\bar{n})}$, we know $n(\tilde{s}) \le \bar{n}$. Therefore, $f(n(\tilde{s})) = 0$ implies $n(\tilde{s}) = 1 - H(\frac{c}{\tilde{s}})$. When $f(\bar{n}) < 0$, or equivalently $\tilde{s} > \frac{c}{H^{-1}(1-\bar{n})}$, we know $n(\tilde{s}) > \bar{n}$. Therefore, $f(n(\tilde{s})) = 0$ implies $n(\tilde{s}) = 1 - H\left(\frac{cn(\tilde{s})}{\tilde{s}\bar{n}}\right)$. By Lemma A.1, $n(\tilde{s}) = m(\tilde{s})$ for $\tilde{s} > 1 - H\left(\frac{c}{\tilde{s}}\right)$. This completes the proof. ∎

Proof of Proposition 1. Suppose the strategy profile $\{a_i^*(\cdot\,; \tilde{s})\}_{i=1}$ is a Bayesian Nash equilibrium. Consider agent i with probability of infection q_i. Agent i's expected payoff from his visiting decision a_i is

$$v_i(a_i; q_i, \tilde{s}, n(\tilde{s})) = a_i(-c - q_i(1 - p(n(\tilde{s}), \bar{n}))\tilde{s}) + (1 - a_i)(-q_i\tilde{s}),$$

where $n(\tilde{s})$, which is the total number of hospital visits, satisfies (3). Therefore, we have

$$a_i^*(q_i; \tilde{s}) = \begin{cases} 1, & \text{if } q_i > \frac{c}{p(n(\tilde{s}),\bar{n})\tilde{s}}, \\ 0, & \text{if } q_i < \frac{c}{p(n(\tilde{s}),\bar{n})\tilde{s}}. \end{cases} \tag{A.4}$$

Since i is arbitrary, we immediately know that the equilibrium is symmetric and every agent uses the same strategy as in (A.4). By (3), we know that $n(\tilde{s})$ satisfies

$$n(\tilde{s}) = 1 - H\left(\frac{c}{p(n(\tilde{s}), \bar{n})\tilde{s}}\right).$$

By Lemma A.2, we know that the unique solution $n(\tilde{s})$ is given by (A.3). This fully characterizes the unique Bayesian Nash equilibrium. Finally, Lemma A.2 also states that hospital congestion occurs, i.e., $n(\tilde{s}) > \bar{n}$, if and only if $\tilde{s} > \frac{c}{H^{-1}(1-\bar{n})}$. ∎

Appendix B Proofs of Lemmas 2 and 2

Proof of Lemma 2. By Proposition 1, the aggregate welfare is

$$\begin{aligned} U(\tilde{s}, s) &= \int_{q<\beta(\tilde{s})} (-qs)\mathrm{d}H(q) + \int_{q>\beta(\tilde{s})} (-c - q(1 - p(n(\tilde{s}), \bar{n}))s)\mathrm{d}H(q) \\ &= -s\mathbb{E}q - c(1 - H(\beta(\tilde{s}))) + p(n(\tilde{s}), \bar{n})s \int_{q>\beta(\tilde{s})} q\mathrm{d}H(q) \\ &= -s\mathbb{E}q - cn(\tilde{s}) + s\min\{n(\tilde{s}), \bar{n}\}\mathbb{E}(q|q > \beta(\tilde{s})), \end{aligned}$$

where the last equality comes from $n(\tilde{s}) = 1 - H(\beta(\tilde{s}))$. ∎

Proof of Lemma 2. Suppose the true severity level is s. Consider the planner's problem of deciding who go to the hospital to maximize the ex post total welfare. It is

$$\max_{a:[0,1]\to\{0,1\}} \int_0^1 a(q)(-c - q(1 - p(n, \bar{n}))s) + (1 - a(q))(-qs)\mathrm{d}H(q),$$
$$\text{s.t. } n = \int_0^1 a(q)\mathrm{d}H(q).$$

It is straightforward to see that the optimal allocation must take a cut-off form $a(q) = 1$ if and only if $q \geq q^*$ for some $q^* \in [0, 1]$. Restricting to such a cut-off form, the planner's problem can be written as

$$\max_{q^*\in[0,1]} -s\mathbb{E}q + \int_{q^*}^1 (-c + q\min\{1, \frac{\bar{n}}{1 - H(q^*)}\}s)\mathrm{d}H(q).$$

If $\bar{n} < 1 - H(q^*)$, or equivalently, $q^* < H^{-1}(1 - \bar{n})$, the objective function becomes

$$-s\mathbb{E}q - c(1 - H(q^*)) + \mathbb{E}(q\,|\,q \geq q^*),$$

which is clearly strictly increasing in q^*. Therefore, $q^* < H^{-1}(1-\bar{n})$ is never optimal. Thus, the planner's problem is equivalent to

$$\max_{q^*\in[H^{-1}(1-\bar{n}),1]} \quad -s\mathbb{E}q + \int_{q^*}^{1}(-c+qs)\mathrm{d}H(q).$$

The solution to this problem is $q^* = \max\{H^{-1}(1-\bar{n}), \frac{c}{s}\}$, or equivalently

$$q^* = \begin{cases} 1, & \text{if } s \le c, \\ \frac{c}{s}, & \text{if } c < s \le s_0, \\ H^{-1}(1-\bar{n}), & \text{if } s > s_0, \end{cases}$$

where, recall, $s_0 = \frac{c}{H^{-1}(1-\bar{n})}$ is the exhaustion level.

On the other hand, by Theorem 1, we can see that the equilibrium cut-off is

$$\beta(\tilde{s}) = \begin{cases} 1, & \text{if } \tilde{s} \le c, \\ \frac{c}{\tilde{s}}, & \text{if } c < \tilde{s} \le \frac{c}{H^{-1}(1-\bar{n})}. \end{cases}$$

Therefore, the equation $\beta(\tilde{s}) = q^*$ has a solution

$$\tilde{s} = \begin{cases} \text{any number in } [0,c], & \text{if } s \le c, \\ s, & \text{if } c < s \le s_0, \\ s_0, & \text{if } s > s_0. \end{cases}$$

This proves desired results. ■

Appendix C Proof of Propositions 3 and 4

In this section, we maintain the assumption that H is convex. The complete proof of Propositions 3 and 4 is lengthy, since it involves a detailed understanding of the function V in (10). For this, we first construct two ancillary functions, $\tilde{V}$ and $\hat{V}$, with which we will mainly work throughout the whole proof.

Define function $\tilde{V} : [0,\infty) \to \mathbb{R}$ as

$$\tilde{V}(\tilde{s}) \equiv \begin{cases} 0, & \text{if } \tilde{s} \in [0,c], \\ \int_{\frac{c}{\tilde{s}}}^{1}(\tilde{s}q-c)\mathrm{d}H(q), & \text{if } \tilde{s} \in (c,\infty). \end{cases} \tag{C.1}$$

Define function $x : (0, \infty) \to \mathbb{R}$ as

$$x(\tilde{s}) \equiv \frac{\tilde{s}\bar{n}}{m(\tilde{s})}, \ \forall \tilde{s} > 0, \tag{C.2}$$

where $m(\tilde{s})$ is the unique solution to (A.1) in Lemma A.1. Define function $\hat{V} : (0, \infty) \to \mathbb{R}$ as the composition of $\tilde{V}$ and x:

$$\hat{V}(\tilde{s}) \equiv \tilde{V}(x(\tilde{s})), \ \forall \tilde{s} > 0. \tag{C.3}$$

By the construction of x, we have

$$H\left(\frac{c}{x(\tilde{s})}\right) = 1 - m(\tilde{s}) \in (0,1), \ \forall \tilde{s} > 0.$$

This implies that $x(\tilde{s}) > c$ for all $\tilde{s} > 0$. Consequently, we have

$$\hat{V}(\tilde{s}) \equiv \int_{\frac{c}{x(\tilde{s})}}^{1} (x(\tilde{s})q - c)\mathrm{d}H(q).$$

Recall that $s_0 = \frac{c}{H^{-1}(1-\bar{n})}$ is the exhaustion level. Recall also that $m(s_0) = \bar{n}$ by Lemma A.1, implying $x(s_0) = s_0$ and $\tilde{V}(s_0) = \hat{V}(s_0)$. The following lemma explains why $\tilde{V}$ and $\hat{V}$ functions are important for the following analysis. In fact, we can obtain V by pasting together $\tilde{V}$ over $[0, s_0]$ and $\hat{V}$ over $[s_0, \bar{s}]$.

Lemma C.1. *The function V satisfies*

$$V(\tilde{s}) = \begin{cases} \tilde{V}(\tilde{s}), & \text{if } \tilde{s} \in [0, s_0], \\ \hat{V}(\tilde{s}), & \text{if } \tilde{s} \in (s_0, \bar{s}]. \end{cases}$$

Proof. Recall that $V(\tilde{s}) = \tilde{s}\min\{n(\tilde{s}), \bar{n}\}\mathbb{E}(q|q \geq \beta(\tilde{s})) - cn(\tilde{s})$. When $\tilde{s} \leq c$, $n(\tilde{s}) = 0$ by Proposition 1. Hence, $V(\tilde{s}) = 0 = \tilde{V}(\tilde{s})$. When $c < \tilde{s} \leq s_0$, $n(\tilde{s}) = 1 - H(\frac{c}{\tilde{s}}) \leq \bar{n}$ by Proposition 1 again. Hence,

$$V(\tilde{s}) = \tilde{s}n(\tilde{s})\mathbb{E}(q|q \geq \beta(\tilde{s})) - cn(\tilde{s}) = \int_{\frac{c}{\tilde{s}}}^{1} (\tilde{s}q - c)\mathrm{d}H(q) = \tilde{V}(\tilde{s}),$$

where the second equality comes from the fact $\beta(\tilde{s}) = H^{-1}(1 - n(\tilde{s})) = \frac{c}{\tilde{s}}$ when $c < \tilde{s} \leq s_0$ by Proposition 1. When $\tilde{s} > s_0$, $n(\tilde{s}) = m(\tilde{s}) > \bar{n}$ by Proposition 1 and Lemma A.2. Hence,

$$V(\tilde{s}) = \frac{\tilde{s}\bar{n}}{n(\tilde{s})}n(\tilde{s})\mathbb{E}(q|q \geq \beta(\tilde{s})) - cn(\tilde{s}) = \int_{\frac{cn(\tilde{s})}{\tilde{s}\bar{n}}}^{1} \left(\frac{\tilde{s}\bar{n}}{n(\tilde{s})}q - c\right)\mathrm{d}H(q) = \tilde{V}(x(\tilde{s})) = \hat{V}(\tilde{s}),$$

where the second equality comes from the fact that $\beta(\tilde{s}) = H^{-1}(1 - n(\tilde{s})) = \frac{cn(\tilde{s})}{\tilde{s}\bar{n}}$ when $\tilde{s} > s_0$ by Proposition 1. This completes the proof. ■

C.1 Properties of $\tilde{V}$ and $\hat{V}$

In this subsection, we establish some important properties of $\tilde{V}$ and $\hat{V}$. These properties will translate into the properties of V by Lemma C.1. Among all the properties, (strict) convexity of $\tilde{V}$ and $\hat{V}$ is the most important one.

The first two lemmas, Lemmas C.2 and C.3, show some properties of function $\tilde{V}$.

Lemma C.2 (Properties of $\tilde{V}$, part I)**.** *The function $\tilde{V}$ restricted on (c,∞) is continuous, strictly increasing, strictly convex and $\lim_{\tilde{s}\downarrow c}\tilde{V}(\tilde{s})=0$. Moreover, for every $\tilde{s}>c$, its left derivative satisfies*

$$\tilde{V}'_{-}(\tilde{s}) \geq \int_{\frac{c}{\tilde{s}}}^{1} q\mathrm{d}H(q).$$

Proof. It is clearly continuous, strictly increasing and $\lim_{\tilde{s}\downarrow c}\tilde{V}(\tilde{s})=0$. For the strict convexity, pick $c\leq \tilde{s}_1<\tilde{s}_2$ and $\lambda\in(0,1)$. Let $\tilde{s}^\lambda\equiv\lambda\tilde{s}_1+(1-\lambda)\tilde{s}_2$. Because $\tilde{s}_1<\tilde{s}^\lambda$, we have $\frac{c}{\tilde{s}^\lambda}<\frac{c}{\tilde{s}_1}$ and thus

$$\int_{\frac{c}{\tilde{s}^\lambda}}^{1}(\tilde{s}_1q-c)\mathrm{d}H(q)=\int_{\frac{c}{\tilde{s}^\lambda}}^{\frac{c}{\tilde{s}_1}}(\tilde{s}_1q-c)\mathrm{d}H(q)+\int_{\frac{c}{\tilde{s}_1}}^{1}(\tilde{s}_1q-c)\mathrm{d}H(q)<\int_{\frac{c}{\tilde{s}_1}}^{1}(\tilde{s}_1q-c)\mathrm{d}H(q).$$

Because $\tilde{s}^\lambda<\tilde{s}_2$, we have $\frac{c}{\tilde{s}^\lambda}>\frac{c}{\tilde{s}_2}$ and thus

$$\int_{\frac{c}{\tilde{s}^\lambda}}^{1}(\tilde{s}_2q-c)\mathrm{d}H(q)=\int_{\frac{c}{\tilde{s}_2}}^{1}(\tilde{s}_2q-c)\mathrm{d}H(q)-\int_{\frac{c}{\tilde{s}_2}}^{\frac{c}{\tilde{s}^\lambda}}(\tilde{s}_2q-c)\mathrm{d}H(q)<\int_{\frac{c}{\tilde{s}_2}}^{1}(\tilde{s}_2q-c)\mathrm{d}H(q).$$

Therefore,

$$\begin{aligned}\tilde{V}(\tilde{s}^\lambda)&=\lambda\int_{\frac{c}{s^\lambda}}^{1}(s_1q-c)\mathrm{d}H(q)+(1-\lambda)\int_{\frac{c}{s^\lambda}}^{1}(s_2q-c)\mathrm{d}H(q)\\&<\lambda\int_{\frac{c}{\tilde{s}_1}}^{1}(\tilde{s}_1q-c)\mathrm{d}H(q)+(1-\lambda)\int_{\frac{c}{\tilde{s}_2}}^{1}(s_2q-c)\mathrm{d}H(q)\\&=\lambda\tilde{V}(\tilde{s}_1)+(1-\lambda)\tilde{V}(\tilde{s}_2),\end{aligned}$$

proving the strict convexity of $\tilde{V}$.

Finally, for any $c\leq\tilde{s}'<\tilde{s}$, we have

$$\frac{\tilde{V}(\tilde{s}')-\tilde{V}(\tilde{s})}{\tilde{s}'-\tilde{s}}=\frac{\int_{\frac{c}{\tilde{s}'}}^{1}(\tilde{s}-\tilde{s}')q\mathrm{d}H(q)}{\tilde{s}-\tilde{s}'}+\frac{\int_{\frac{c}{\tilde{s}}}^{\frac{c}{\tilde{s}'}}(\tilde{s}q-c)\mathrm{d}H(q)}{\tilde{s}-\tilde{s}'}\geq\int_{\frac{c}{\tilde{s}'}}^{1}q\mathrm{d}H(q).$$

Therefore,

$$\tilde{V}'_-(\tilde{s}) = \lim_{\tilde{s}'\uparrow\tilde{s}} \frac{\tilde{V}(\tilde{s}') - \tilde{V}(\tilde{s})}{\tilde{s}' - \tilde{s}} \geq \lim_{\tilde{s}'\uparrow\tilde{s}} \int_{\frac{c}{\tilde{s}'}}^{1} q\mathrm{d}H(q) = \int_{\frac{c}{\tilde{s}}}^{1} q\mathrm{d}H(q),$$

where the first equality, i.e., the existence of $\tilde{V}'_-$, is guaranteed since $\tilde{V}$ is convex over (c, ∞) as we have shown. ■

Lemma C.3 (Properties of $\tilde{V}$, part II)**.** *The function $\tilde{V} : [0, \infty) \to \mathbb{R}$ is continuous and increasing. For any $s > 0$, the mapping*

$$\tilde{s} \mapsto \frac{\tilde{V}(\tilde{s}) - \tilde{V}(s)}{\tilde{s} - s}$$

is increasing over $[0, s)$ and strictly so if $s > c$. As a result, $\tilde{V}$ is convex over $[0, \infty)$.

Proof. By Lemma C.2 and the fact that $\tilde{V}$ is zero over $[0, c]$, we know $\tilde{V}$ is continuous and increasing. If $s \in (0, c]$, $\frac{\tilde{V}(\tilde{s})-\tilde{V}(s)}{\tilde{s}-s} = 0$ for all $\tilde{s} < s$. Consider $s > c$. When $\tilde{s} \leq c$, $\frac{\tilde{V}(\tilde{s})-\tilde{V}(s)}{\tilde{s}-s} = -\frac{\tilde{V}(s)}{\tilde{s}-s}$. Since $\tilde{V}(s) > 0$, we know $\frac{\tilde{V}(\tilde{s})-\tilde{V}(s)}{\tilde{s}-s}$ is strictly increasing over $[0, c]$. Over $[c, s)$, $\frac{\tilde{V}(\tilde{s})-\tilde{V}(s)}{\tilde{s}-s}$ is strictly increasing since $\tilde{V}$ is strictly convex over $(c, +\infty)$ by Lemma C.2. ■

Recall the function x defined in (C.2). The next lemma lists some properties of function x, which are essentially for understanding the properties of $\hat{V}$. Notice that by its construction in (C.2), for any $\tilde{s} > 0$,

$$\frac{\tilde{s}\bar{n}}{x(\tilde{s})} = 1 - H\left(\frac{c}{x(\tilde{s})}\right). \tag{C.4}$$

Lemma C.4 (Properties of x)**.** *The function $x : (0, \infty) \to (c, \infty)$ has the following properties:*

i). $x(\tilde{s})$ is continuous, strictly increasing, $\lim_{\tilde{s}\downarrow 0} x(\tilde{s}) = c$ and $\lim_{\tilde{s}\to\infty} x(\tilde{s}) = \infty$;

ii). $x(\tilde{s})$ is convex;

iii). $x(\tilde{s}) - \tilde{s}\bar{n}$ is decreasing.

Proof. Part i): The continuity of $x(\tilde{s})$ comes from the continuity of $m(\tilde{s})$ by Lemma A.1. Consider any $0 < \tilde{s}_1 < \tilde{s}_2$. By (C.4), we know

$$\frac{\tilde{s}_1\bar{n}}{x(\tilde{s}_1)} + H\left(\frac{c}{x(\tilde{s}_1)}\right) = \frac{\tilde{s}_2\bar{n}}{x(\tilde{s}_2)} + H\left(\frac{c}{x(\tilde{s}_2)}\right) > \frac{\tilde{s}_1\bar{n}}{x(\tilde{s}_2)} + H\left(\frac{c}{x(\tilde{s}_2)}\right),$$

implying $x(\tilde{s}_1) < x(\tilde{s}_2)$. Because $m(\tilde{s}) = 1 - H\left(\frac{c}{x(\tilde{s})}\right)$ and $\lim_{\tilde{s}\downarrow 0} m(\tilde{s}) = 0$ by Lemma A.1, we know $\lim_{\tilde{s}\downarrow 0} x(\tilde{s}) = c$. Because $m(\tilde{s}) \in (0,1)$ for all $\tilde{s} > 0$, obviously we have $\lim_{\tilde{s}\to\infty} x(\tilde{s}) = \infty$.

Part ii): pick $\tilde{s}_1, \tilde{s}_2 > 0$ and $\lambda \in (0,1)$. Let $\tilde{s}^\lambda = \lambda\tilde{s}_1 + (1-\lambda)\tilde{s}_2$. Suppose, by contradiction, that $x(\tilde{s}^\lambda) > \lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)$. We have

$$\frac{1}{x(\tilde{s}^\lambda)} < \frac{1}{\lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)} = \sum_{i=1}^{2} \frac{\lambda x(\tilde{s}_i)}{\lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)} \frac{1}{x(\tilde{s}_i)}$$

Because H is strictly increasing and convex, we have

$$H\left(\frac{c}{x(\tilde{s}^\lambda)}\right) < \frac{\lambda x(\tilde{s}_1)}{\lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)} H\left(\frac{c}{x(\tilde{s}_1)}\right) + \frac{(1-\lambda)x(\tilde{s}_2)}{\lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)} H\left(\frac{c}{x(\tilde{s}_2)}\right).$$

By (C.4), we can obtain

$$1 - \frac{\tilde{s}^\lambda \bar{n}}{x(\tilde{s}^\lambda)} < \frac{\lambda x(\tilde{s}_1)}{\lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)} \left[1 - \frac{\tilde{s}_1 \bar{n}}{x(\tilde{s}_1)}\right] + \frac{(1-\lambda)x(\tilde{s}_2)}{\lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)} \left[1 - \frac{\tilde{s}_2 \bar{n}}{x(\tilde{s}_2)}\right].$$

Simple algebra yields

$$\frac{1}{x(\tilde{s}^\lambda)} > \frac{1}{\lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)},$$

which contradicts our assumption that $x(\tilde{x}^\lambda) > \lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)$. Therefore, we must have $x(\tilde{x}^\lambda) \leq \lambda x(\tilde{s}_1) + (1-\lambda)x(\tilde{s}_2)$, proving the convexity of $x(\tilde{s})$.

Part iii): by (C.4),

$$x(\tilde{s}) - \tilde{s}\bar{n} = x(\tilde{s}) H\left(\frac{c}{x(\tilde{s})}\right) = c \frac{H\left(\frac{c}{x(\tilde{s})}\right) - H(0)}{\frac{c}{x(\tilde{s})} - 0},$$

where the second equality comes from $H(0) = 0$. Because H is convex and $x(\tilde{s})$ is increasing by part i), we know $x(\tilde{s}) - \tilde{s}\bar{n}$ is decreasing. ■

Recall that $\hat{V}(\tilde{s}) = \tilde{V}(x(\tilde{s}))$ for $\tilde{s} > 0$. The next lemma gives some properties of function $\hat{V}$, wihch is built on the properties of function $\tilde{V}$ (Lemma C.2) and the properties of function x (Lemma C.4).

Lemma C.5 (Properties of $\hat{V}$)**.** *The function $\hat{V} : [0,\infty) \to \mathbb{R}_+$ is continuous, strictly increasing and strictly convex. Moreover, for every $\tilde{s} > s_0$, we have*

$$\frac{\hat{V}(\tilde{s}) - \hat{V}(s_0)}{\tilde{s} - s_0} < \tilde{V}'_-(s_0). \tag{C.5}$$

Proof. The continuity and strict monotonicity come directly from Lemma C.2 and part i) of Lemma C.4. Following Lemma C.2 and part ii) of Lemma C.4, $\hat{V}$ is strictly convex because $x(\tilde{s}) > c$ for all $\tilde{s} > 0$.

Because $\tilde{V}'_-(s_0) \geq \int_{\frac{c}{s_0}}^1 q\mathrm{d}H(q)$ by Lemma C.2 and because $\bar{n} = 1 - H\left(\frac{c}{s_0}\right)$ by construction, we know $\tilde{V}'_-(s_0) > \bar{n}\mathbb{E}q$. Hence, to show (C.5), it suffices to show

$$\frac{\hat{V}(\tilde{s}) - \hat{V}(s_0)}{\tilde{s} - s_0} \leq \bar{n}\mathbb{E}q, \ \forall \tilde{s} > s_0. \tag{C.6}$$

Because $\hat{V}$ is strictly convex, $\tilde{s} \mapsto (\hat{V}(\tilde{s}) - \hat{V}(s_0))/(\tilde{s} - s_0)$ is strictly increasing over (s_0, ∞). Thus, to show (C.6), it suffices to show

$$\lim_{\tilde{s}\to\infty} \frac{\hat{V}(\tilde{s}) - \hat{V}(s_0)}{\tilde{s} - s_0} \leq \bar{n}\mathbb{E}q.$$

To see this, note that for $\tilde{s} > s_0$,

$$\begin{aligned}
\frac{\hat{V}(\tilde{s}) - \hat{V}(s_0)}{\tilde{s} - s_0} &= \frac{x(\tilde{s}) - x(s_0)}{\tilde{s} - s_0} \frac{\tilde{V}(x(\tilde{s})) - \tilde{V}(x(s_0))}{x(\tilde{s}) - x(s_0)} \\
&\leq \bar{n}\left[\frac{\int_{\frac{c}{x(s_0)}}^1 (x(\tilde{s}) - x(s_0))q\mathrm{d}H(q)}{x(\tilde{s}) - x(s_0)} + \frac{\int_{\frac{c}{x(\tilde{s})}}^{\frac{c}{x(s_0)}} (x(\tilde{s})q - c)\mathrm{d}H(q)}{x(\tilde{s}) - x(s_0)}\right] \\
&= \bar{n}\left[\int_{\frac{c}{x(s_0)}}^1 q\mathrm{d}H(q) + \frac{x(\tilde{s})}{x(\tilde{s}) - x(s_0)} \int_{\frac{c}{x(\tilde{s})}}^{\frac{c}{x(s_0)}} q\mathrm{d}H(q) - \frac{c\int_{\frac{c}{x(\tilde{s})}}^{\frac{c}{x(s_0)}} \mathrm{d}H(q)}{x(\tilde{s}) - x(s_0)}\right],
\end{aligned}$$

where the inequality comes from $x(\tilde{s}) - \bar{n}\tilde{s} \leq x(s_0) - \bar{n}s_0$ by part iii) of Lemma C.4. Because $x(\tilde{s}) \to \infty$ as $\tilde{s} \to \infty$ by part i) of Lemma C.4, we immediately see

$$\lim_{\tilde{s}\to\infty} \frac{\hat{V}(\tilde{s}) - \hat{V}(s_0)}{\tilde{s} - s_0} \leq \bar{n}\left[\int_{\frac{c}{x(s_0)}}^1 q\mathrm{d}H(q) + \int_0^{\frac{c}{x(s_0)}} q\mathrm{d}H(q)\right] = \bar{n}\mathbb{E}q,$$

completing the proof. ■

C.2 Common secant lines of $\tilde{V}$ and $\hat{V}$

Based on the analysis in the previous subsection, we construct a special kind of common secant lines of $\tilde{V}$ and $\hat{V}$ in this subsection. These common secant lines will play an important role in the proof of Proposition 3.

The next lemma is a standard mathematical result regarding a convex function and its secant line. Since it will be used repeatedly in what follows, we include it here for completeness. Its proof is omitted.

Lemma C.6. *Suppose $f : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ is a (strict) convex function. Let ℓ be a secant line of f that passes through $(\theta_1, f(\theta_1))$ and $(\theta_2, f(\theta_2))$, where $\underline{\theta} < \theta_1 < \theta_2 < \overline{\theta}$. Then, $f(\theta) \geq (>) \ell(\theta)$ if $\theta \in [\underline{\theta}, \theta_1) \cup (\theta_2, \overline{\theta}]$ and $f(\theta) \leq (<) \ell(\theta)$ if $\theta \in (\theta_1, \theta_2)$. Moreover, both of the following two functions are convex over $[\underline{\theta}, \overline{\theta}]$:*

$$f_1(\theta) = \begin{cases} f(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_1], \\ \ell(\theta), & \text{if } \theta \in (\theta_1, \overline{\theta}), \end{cases} \quad \text{and } f_2(\theta) = \begin{cases} \ell(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_2], \\ f(\theta), & \text{if } \theta \in (\theta_2, \overline{\theta}]. \end{cases}$$

We now construct some secant lines of $\hat{V}$ and proceed to show that they are also secant lines of $\tilde{V}$.

For every $s > s_0$, let $\ell_s : \mathbb{R} \to \mathbb{R}$ be the secant line of $\hat{V}$ that passes through $(s_0, \hat{V}(s_0))$ and $(s, \hat{V}(s))$. That is,

$$\ell_s(\tilde{s}) = \frac{\hat{V}(s) - \hat{V}(s_0)}{s - s_0}(\tilde{s} - s_0) + \hat{V}(s_0), \ \forall \tilde{s}.$$

Because $\hat{V}$ is strictly increasing by Lemma C.5, the slope of ℓ_s is positive and thus ℓ_s is strictly increasing in $\tilde{s}$.

The next two lemmas, Lemmas C.7 and C.8, show that ℓ_s is also a secant line of $\tilde{V}$.

Lemma C.7. *For every $s > s_0$, there exists $\tilde{s} \in (0, s_0)$ such that $l_s(\tilde{s}) > V(\tilde{s})$.*

Proof. Consider any $s > s_0$. Pick $\tilde{s} \in (c, s_0)$ such that

$$\frac{\hat{V}(s) - \hat{V}(s_0)}{s - s_0} < \frac{\tilde{V}(\tilde{s}) - \tilde{V}(s_0)}{\tilde{s} - s_0}.$$

By (C.5) in Lemma C.5, such $\tilde{s}$ exists. Therefore,

$$\frac{\ell_s(\tilde{s}) - \hat{V}(s_0)}{\tilde{s} - s_0} = \frac{\hat{V}(s) - \hat{V}(s_0)}{s - s_0} < \frac{\tilde{V}(\tilde{s}) - \tilde{V}(s_0)}{\tilde{s} - s_0}.$$

Because $\hat{V}(s_0) = \tilde{V}(s_0)$, we immediately obtain $\ell_s(\tilde{s}) > \tilde{V}(\tilde{s})$. ■

Lemma C.8. *For every $s > s_0$, there exists a unique $\alpha(s) \in (0, s_0)$ such that $\ell_s(\alpha(s)) = \tilde{V}(\alpha(s))$. Therefore, for every $s > s_0$, ℓ_s is a secant line of $\tilde{V}$ that passes through $(\alpha(s), \tilde{V}(\alpha(s)))$ and $(s_0, \tilde{V}(s_0))$.*

Proof. Consider any $s > s_0$. Because ℓ_s is a secant line of $\hat{V}$ that passes through $(s_0, \hat{V}(s_0))$ and $(s, \hat{V}(s))$ and because $\hat{V}$ is strictly convex by Lemma C.5, we know $\ell_s(0) < \hat{V}(0) = 0 = \tilde{V}(0)$ by Lemma C.6. By Lemma C.7, we also know that $\ell_s(\tilde{s}) > \tilde{V}(\tilde{s})$ for some $\tilde{s} \in (0, s_0)$. Since $\tilde{V}$ is continuous by Lemma C.2, by the intermediate value theorem, we know that there exists $\hat{s} \in (0, \tilde{s}) \subset (0, s_0)$ such that $\ell_s(\hat{s}) = \tilde{V}(\hat{s})$. As a result, we have

$$\frac{\tilde{V}(\hat{s}) - \hat{V}(s_0)}{\hat{s} - s_0} = \frac{\hat{V}(s) - \hat{V}(s_0)}{s - s_0},$$

or equivalently

$$\frac{\tilde{V}(\hat{s}) - \tilde{V}(s_0)}{\hat{s} - s_0} = \frac{\hat{V}(s) - \hat{V}(s_0)}{s - s_0}$$

since $\hat{V}(s_0) = \tilde{V}(s_0)$. Notice that, by Lemma C.3, the left hand side in the above equation is strictly increasing in $\hat{s}$, thereby proving the uniqueness of $\hat{s}$. Letting $\alpha(s)$ be this $\hat{s}$ completes the proof. ■

The last two lemmas in this subsection, Lemmas C.9 and C.10, establish some properties of α.

Lemma C.9. *The function $\alpha : (s_0, \infty) \to (0, s_0)$ is continuous and strictly increasing.*

Proof. By construction, we have

$$\frac{\tilde{V}(\alpha(s)) - \tilde{V}(s_0)}{\alpha(s) - s_0} = \frac{\hat{V}(s) - \hat{V}(s_0)}{s - s_0}, \ \forall s > s_0.$$

Because $\hat{V}$ is strictly convex by Lemma C.5, we know the right hand side of the above equality is strictly increasing. By Lemma C.3, $\alpha(s)$ must be strictly increasing too.

For continuity, consider a monotone sequence $\{s_k\}$ such that either $s_k \uparrow s$ or $s_k \downarrow s$ for some $s > s_0$. Since α is monotone, we know $\lim_k \alpha(s_k)$ exists. Because

$$\frac{\tilde{V}(\alpha(s_k)) - \tilde{V}(s_0)}{\alpha(s_k) - s_0} = \frac{\hat{V}(s_k) - \hat{V}(s_0)}{s_k - s_0}, \ \forall k,$$

and because both $\tilde{V}$ and $\hat{V}$ are continuous by Lemmas C.2 and C.5, we know

$$\frac{\tilde{V}(\lim_k \alpha(s_k)) - \tilde{V}(s_0)}{\lim_k \alpha(s_k) - s_0} = \frac{\hat{V}(s) - \hat{V}(s_0)}{s - s_0}$$

implying that $\lim_k \alpha(s_k) = \alpha(s)$ by Lemma C.3 again. This proves the continuity of $\alpha(s)$. ■

Lemma C.10. *The function* $s \mapsto \mathbb{E}_G(\tilde{s} \,|\, \alpha(s) \leq \tilde{s} \leq s)$ *is continuous and strictly increasing over* $(s_0, \bar{s}]$.

Proof. Continuity comes from the continuity of $\alpha(s)$ by Lemma C.9 and the assumption that G is continuous. Strict monotonicity comes from the strict monotonicity of α by Lemma C.9 and the assumption that G is strictly increasing. ∎

With all the previous preparation, we are now ready to prove Propositions 3 and 4. The proof has to deal with two cases separately, depending on the comparison between $\mathbb{E}_G(\tilde{s} \,|\, \alpha(\bar{s}) \leq \tilde{s} \leq \bar{s})$ and s_0. Subsection C.3 deals with the case where $\mathbb{E}_G(\tilde{s} \,|\, \alpha(\bar{s}) \leq \tilde{s} \leq \bar{s}) \geq s_0$ (Case I). Subsection C.4 deals with the case where $\mathbb{E}_G(\tilde{s} \,|\, \alpha(\bar{s}) \leq \tilde{s} \leq \bar{s}) < s_0$ (Case II). The proofs of both these two cases are built on Theorem 1 in Dworczak and Martini (2019), despite of some differences in details.

C.3 Proof of Propositions 3 and 4, Case I

In this subsection, we maintain the assumption that $\mathbb{E}_G(\tilde{s} \,|\, \alpha(\bar{s}) \leq \tilde{s} \leq \bar{s}) \geq s_0$.

Because $\mathbb{E}_G(\tilde{s} \,|\, \alpha(s) \leq \tilde{s} \leq s)$ is continuous and strictly increasing in s over $(s_0, \bar{s})$ by Lemma C.10 and because $\lim_{s \downarrow s_0} \mathbb{E}_G(\tilde{s} \,|\, \alpha(s) \leq \tilde{s} \leq s) < s_0$, we know that there exists a unique $s_+ \in (s_0, \bar{s}]$ such that $\mathbb{E}_G(\tilde{s} \,|\, \alpha(s_+) \leq \tilde{s} \leq s_+) = s_0$. Let $s_- = \alpha(s_+)$. Define

$$W(\tilde{s}) \equiv \begin{cases} \tilde{V}(\tilde{s}), & \text{if } \tilde{s} \in [0, s_-], \\ \ell_{s_+}(\tilde{s}), & \text{if } \tilde{s} \in (s_-, s_+), \\ \hat{V}(\tilde{s}), & \text{if } \tilde{s} \in [s_+, \bar{s}]. \end{cases} \tag{C.7}$$

The following lemma states two basic properties of function W.

Lemma C.11. *The function* W *in (C.7) satisfies*

i). $W(\tilde{s}) \geq V(\tilde{s})$ for $\tilde{s} \in [0, \bar{s}]$, with strict inequality if and only if $\tilde{s} \in (s_-, s_0) \cup (s_0, s_+)$;

ii). W is convex.

Proof. Part i): if $\tilde{s} \in [0, s_-] \cup [s_+, \bar{s}]$, $W(\tilde{s}) = V(\tilde{s})$ by construction. Since $\ell_{s+}(s_0) = \hat{V}(s_0) = \tilde{V}(s_0) = V(s_0)$, $W(s_0) = V(s_0)$ too. Because ℓ_{s_+} is a secant line of $\hat{V}$ that passes through $(s_0, \hat{V}(s_0))$ and $(s_+, \hat{V}(s_+))$ and because $\hat{V}$ is strictly convex by Lemma C.5, we know $W(\tilde{s}) = \ell_{s_+}(\tilde{s}) > \hat{V}(\tilde{s}) = V(\tilde{s})$ for $\tilde{s} \in (s_0, s_+)$ by Lemma C.6.

Consider the case where $s_- \geq c$. Similarly as above, because ℓ_{s_+} is a secant line of $\tilde{V}$ that passes through $(s_-, \tilde{V}(s_-))$ and $(s_0, \tilde{V}(s_0))$ and because $\tilde{V}$ is strictly convex over $[c, \infty)$ by Lemma C.2, if $s_- \geq c$, we know $W(\tilde{s}) = \ell_{s_+}(\tilde{s}) > \tilde{V}(\tilde{s}) = V(\tilde{s})$ for $\tilde{s} \in (s_-, s_0)$.

Finally, consider the case where $s_- < c$. We know $\ell_{s+}(\tilde{s}) > 0 = \tilde{V}(\tilde{s}) = V(\tilde{s})$ for $\tilde{s} \in (s_-, c]$, for ℓ_{s_+} is strictly increasing and $\ell_{s_+}(s_-) = \tilde{V}(s_-) = 0$. For $\tilde{s} \in (c, s_0)$, we have

$$\ell_{s_+}(\tilde{s}) = \frac{\tilde{V}(s_-) - \tilde{V}(s_0)}{s_- - s_0}(\tilde{s} - s_0) + \tilde{V}(s_0) > \frac{\tilde{V}(c) - \tilde{V}(s_0)}{c - s_0}(\tilde{s} - s_0) + \tilde{V}(s_0) \geq \tilde{V}(\tilde{s}),$$

where the first inequality comes from Lemma C.3. The second inequality comes from convexity of $\tilde{V}$. Hence, we also have $W(\tilde{s}) = \ell_{s_+}(\tilde{s}) > V(\tilde{s})$ for $\tilde{s} \in (s_-, s_0)$.

Part ii): Consider the following two functions

$$W_1(\tilde{s}) \equiv \begin{cases} \tilde{V}(\tilde{s}), & \text{if } \tilde{s} \in [0, s_-], \\ \ell_{s_+}(\tilde{s}), & \text{if } \tilde{s} \in (s_-, \bar{s}], \end{cases} \text{ and } W_2(\tilde{s}) \equiv \begin{cases} \ell_{s_+}(\tilde{s}), & \text{if } \tilde{s} \in [0, s_+], \\ \hat{V}(\tilde{s}), & \text{if } \tilde{s} \in (s_+, \bar{s}]. \end{cases}$$

Because ℓ_{s_+} is a secant line of $\tilde{V}$ that passes through $(s_-, \tilde{V}(s_-))$ and $(s_0, \tilde{V}(s_0))$ and because $\tilde{V}$ is convex by Lemma C.2, we know $\tilde{V}(\tilde{s}) \geq \ell_{s_+}(\tilde{s})$ for $\tilde{s} \in [0, s_-)$ by Lemma C.6. Hence $W_1(\tilde{s}) \geq W_2(\tilde{s})$ for $\tilde{s} \in [0, s_-)$. Because ℓ_{s_+} is also a secant line of $\hat{V}$ that passes through $(s_0, \hat{V}(s_0))$ and $(s_+, \hat{V}(s_+))$ by construction and because $\hat{V}$ is strictly convex by Lemma C.2, we know $\ell_{s_+}(\tilde{s}) < \hat{V}(\tilde{s})$ for $\tilde{s} \in (s_+, \bar{s}]$ by Lemma C.6 again. Hence $W_1(\tilde{s}) \leq W_2(\tilde{s})$ for $\tilde{s} \in (s_+, \bar{s}]$. Therefore, we have $W = \max\{W_1, W_2\}$. Because both W_1 and W_2 are convex by Lemma C.6, so is W. ■

We are now ready to prove Proposition 3.

Proof of Proposition 3, Case I. Consider $\tilde{G}^*$ in (12). It is the c.d.f. that puts all the mass over $[s_-, s_+]$ under G to the atom s_0. Since $\mathbb{E}_G(\tilde{s} \,|\, s_- \leq \tilde{s} \leq s_+) = s_0$ by construction, we immediately know that $\tilde{G}^*$ is a mean-preserving contraction of G. Because G and $\tilde{G}^*$ coincide over $[0, s_-]$ and $[s_+, \bar{s}]$, we have

$$\begin{aligned} &\int_0^{\bar{s}} W(\tilde{s}) \mathrm{d}G(\tilde{s}) - \int_0^{\bar{s}} W(\tilde{s}) \mathrm{d}\tilde{G}^*(\tilde{s}) \\ =&\int_{s_-}^{s_+} W(\tilde{s}) \mathrm{d}G(\tilde{s}) - \int_{s_-}^{s_+} W(\tilde{s}) \mathrm{d}\tilde{G}^*(\tilde{s}) \\ =&(G(s_+) - G(s_-))W(s_0) - (\tilde{G}^*(s_+) - \tilde{G}^*(s_-))W(s_0) \\ =&0, \end{aligned} \tag{C.8}$$

where the second equality comes from linearity of W over $[s_-, s_+]$. The last equality comes from $G(s_-) = \tilde{G}^*(s_-)$ and $G(s_+) = \tilde{G}^*(s_+)$.

Therefore, for any mean-preserving contraction $\tilde{G}$ of G, we have[27]

$$\begin{aligned}\int_0^{\bar{s}} V(\tilde{s})\mathrm{d}\tilde{G}^*(\tilde{s}) &= \int_0^{\bar{s}} W(\tilde{s})\mathrm{d}\tilde{G}^*(\tilde{s}) = \int_0^{\bar{s}} W(\tilde{s})\mathrm{d}G(\tilde{s}) \\ &\geq \int_0^{\bar{s}} W(\tilde{s})\mathrm{d}\tilde{G}(\tilde{s}) \geq \int_0^{\bar{s}} V(\tilde{s})\mathrm{d}\tilde{G}(\tilde{s}),\end{aligned} \tag{C.9}$$

where the first equality comes from the fact that $\mathrm{supp}\tilde{G}^* = [0, s_-] \cup \{s_0\} \cup [s_+, \bar{s}] = \{\tilde{s} \in [0, \bar{s}] \,|\, W(\tilde{s}) = V(\tilde{s})\}$ by Lemma C.11. The second equality comes from (C.8). The first inequality comes from the facts that $\tilde{G}$ is a mean-preserving contraction of G and that W is convex by Lemma C.11. The second inequality comes from $W \geq V$ by Lemma C.11 again. Since $\tilde{G}$ is arbitrary, we know $\tilde{G}^*$ is an optimal disclosure rule. ■

To prove Proposition 4, we need some further properties of W, which is stated in the following lemma. It simply comes from the strict convexity of $\tilde{V}$ and $\hat{V}$ developed above. Thus, its proof is omitted. Let $W'_-(\tilde{s})$ be the left derivative of W at $\tilde{s} \in [0, \bar{s}]$ with the convention that $W'_-(0) = 0$.

Lemma C.12. *Every pair $(\tilde{s}, s) \in [0, \bar{s}]^2$ satisfies*

$$W(s) \geq W'_-(\tilde{s})(s - \tilde{s}) + W(\tilde{s}), \tag{C.10}$$

with equality if and only if

$$(\tilde{s}, s) \in A \equiv \begin{cases} [0, s_-]^2 \cup (s_-, s_+] \times [s_-, s_+] \cup \{(\hat{s}, \hat{s}) \,|\, \hat{s} \in (s_+, \bar{s}]\}, & \text{if } s_- \leq c, \\ [0, c]^2 \cup (s_-, s_+] \times [s_-, s_+] \cup \{(\hat{s}, \hat{s}) \,|\, \hat{s} \in (c, s_-] \cup (s_+, \bar{s}]\}, & \text{if } s_- > c. \end{cases} \tag{C.11}$$

We are now ready to prove Proposition 4.

Proof of Proposition 4, Case I. Because $\tilde{G}$ is a mean-preserving contraction of G, by Theorem 6 in Blackwell (1951), there exist a probability space $(\Omega, \mathcal{F}, \widehat{P})$ and two random variables S and $\tilde{S}$ such that (i) the distribution of S is G, (ii) the distribution

[27] The sequence of (in)equalities in (C.9) comes from the proof of Theorem 1 in Dworczak and Martini (2019). We repeat it here, rather than directly applying the result, because we need this sequence of (in)equalities in the following proof of Proposition 4.

of $\tilde{S}$ is $\tilde{G}$, and (iii) $\widehat{E}(S|\tilde{S}) = \tilde{S}$ where the expectation $\widehat{E}$ is with respect to $\widehat{P}$.[28] Note that because $\tilde{G}$ is optimal by assumption, the inequalities in (C.9) all become equalities. Thus, we have

$$\widehat{E}W(S) = \widehat{E}W(\tilde{S}) = \widehat{E}V(\tilde{S}). \tag{C.12}$$

In what follows, we proceed to prove Proposition 4. The proof is divided into a series of small steps for clarity.

Step 1: $\widehat{P}((\tilde{S}, S) \in A) = 1$, where A is defined in (C.11).

Because

$$W(S) \geq W'_{-}(\tilde{S})(S - \tilde{S}) + W(\tilde{S})$$

by Lemma C.12, we have

$$\begin{aligned} \widehat{E}W(S) \geq \widehat{E}\Big(W'_{-}(\tilde{S})(S - \tilde{S}) + W(\tilde{S})\Big) &= \widehat{E}\Big(\widehat{E}\Big(W'_{-}(\tilde{S})(S - \tilde{S}) + W(\tilde{S}) \,\Big|\, \tilde{S}\Big)\Big) \\ = \widehat{E}\Big(W'_{-}(\tilde{S})(\widehat{E}(S|\tilde{S}) - \tilde{S}) + W(\tilde{S})\Big) &= \widehat{E}W(\tilde{S}) = \widehat{E}W(S), \end{aligned}$$

where the last equality comes from the first equality in (C.12). Therefore, we have

$$W(S) = W'_{-}(\tilde{S})(S - \tilde{S}) + W(\tilde{S}) \quad \widehat{P}\text{-}a.s.$$

By Lemma C.12, we know $\widehat{P}((\tilde{S}, S) \in A) = 1$.

Step 2: $\widehat{P}(\tilde{S} \in (s_-, s_0) \cup (s_0, s_+)) = 0$.

This directly comes from the second equality in (C.12) and Lemma C.11.

Step 3: $\widehat{P}(\tilde{S} = s_+) = 0$.

By the construction of A in (C.11), we have

$$\begin{aligned} &(\tilde{S} \notin (s_-, s_0) \cup (s_0, s_+)) \cap (S \in (s_-, s_+)) \cap ((\tilde{S}, S) \in A) \\ =&(\tilde{S} \in \{s_0, s_+\}) \cap (S \in (s_-, s_+)) \\ =&(\tilde{S} \in \{s_0, s_+\}) \cap ((\tilde{S}, S) \in A) - (\tilde{S} \in \{s_0, s_+\}) \cap (S \in \{s_-, s_+\}). \end{aligned} \tag{C.13}$$

[28]See also

Thus,

$$\begin{aligned}\widehat{E}(S;\ S\in(s_-,s_+)) &= \widehat{E}(S;\ \tilde{S}\notin(s_-,s_0)\cup(s_0,s_+), S\in(s_-,s_+), (\tilde{S},S)\in A)\\ &= \widehat{E}(S;\ \tilde{S}\in\{s_0,s_+\}, (\tilde{S},S)\in A)\\ &= \widehat{E}(S;\ \tilde{S}\in\{s_0,s_+\})\\ &= \widehat{E}(\tilde{S};\ \tilde{S}\in\{s_0,s_+\}),\end{aligned}$$

where the first and third equality comes from Steps 1 and 2. The second equality comes from (C.13) and the fact that $\widehat{P}(\tilde{S}\in\{s_0,s_+\}, S\in\{s_-,s_+\})=0$ since G is continuous. The last equality comes from $\widehat{E}(S|\tilde{S})=\tilde{S}$. Since $s_0=\widehat{E}(S|s_-\le S\le s_+)$ by construction and G is continuous, the above equality implies

$$s_0\widehat{P}(S\in(s_-,s_+)) = s_0\widehat{P}(\tilde{S}=s_0)+s_+\widehat{P}(\tilde{S}=s_+).$$

But (C.13) also implies $\widehat{P}(S\in(s_-,s_+))=\widehat{P}(\tilde{S}=s_0)+\widehat{P}(\tilde{S}=s_+)$. Hence $\widehat{P}(\tilde{S}=s_+)=0$.

Step 4: $\tilde{G}(\tilde{s})=\tilde{G}^*(\tilde{s})$ for $\tilde{s}\ge\min\{c,s_-\}$.

Define

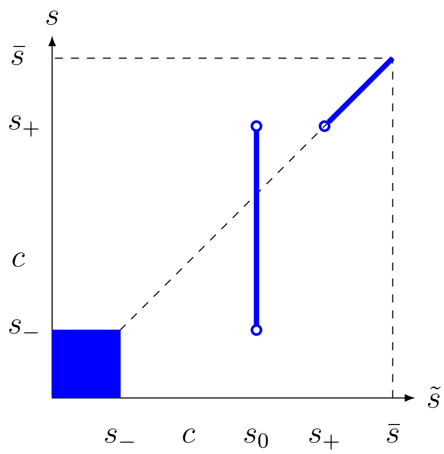
$$\begin{aligned}\widehat{A} &\equiv A-\{(\tilde{s},s)\in[0,\bar{s}]^2\,|\,\tilde{s}\in(s_-,s_0)\cup(s_0,s_+]\}-\{s_0\}\times\{s_-,s_+\}\\ &= \begin{cases}[0,s_-]^2\cup\{s_0\}\times(s_-,s_+)\cup\{(\hat{s},\hat{s})\,|\,\hat{s}\in(s_+,\bar{s}]\}, & \text{if } s_-\le c,\\ [0,c]^2\cup\{s_0\}\times(s_-,s_+)\cup\{(\hat{s},\hat{s})\,|\,\hat{s}\in(c,s_-]\cup(s_+,\bar{s}]\}, & \text{if } s_->c.\end{cases}\end{aligned} \tag{C.14}$$

By Steps 1 - 3, we know $\widehat{P}((\tilde{S},S)\in\widehat{A})=1$.

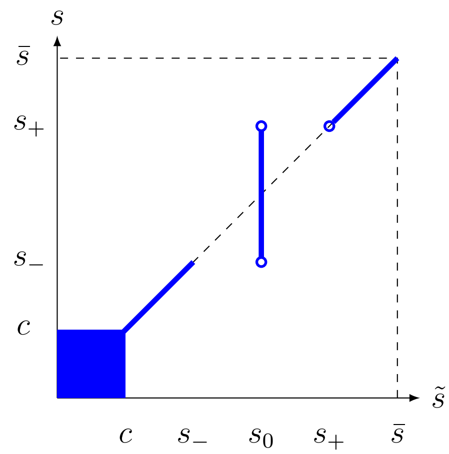
Figure 4 gives an illustration of set $\widehat{A}$ for the two cases, $s_-\le c$ and $s_->c$, respectively. The desired result should then be obvious. For example, consider the case $c<s_-$. We have

$$(\tilde{S}\le\tilde{s})\cap((\tilde{S},S)\in\widehat{A}) = \begin{cases}(S\le\tilde{s})\cap((\tilde{S},S)\in\widehat{A}), & \text{if } \tilde{s}\in[c,s_-),\\ (S\le s_-)\cap((\tilde{S},S)\in\widehat{A}), & \text{if } \tilde{s}\in[s_-,s_0),\\ (S\le s_+)\cap((\tilde{S},S)\in\widehat{A}), & \text{if } \tilde{s}\in[s_0,s_+),\\ (S\le\tilde{s})\cap((\tilde{S},S)\in\widehat{A}), & \text{if } \tilde{s}\in[s_+,\bar{s}].\end{cases}$$

Therefore, $\tilde{G}(\tilde{s})=\tilde{G}^*(\tilde{s})$ for $\tilde{s}\ge c$. The other case, $s_-\le c$, is similar.



(a) $s_- \leq c$



(b) $s_- > c$

Figure 4: The set $\widehat{A}$ in (C.14)

Step 5: $\tilde{G}$ is a mean-preserving contraction of $\tilde{G}^*$.

Since $\tilde{G}$ is a mean-preserving contraction of G and $\tilde{G}^*$ coincides with G over $[0, s_-]$, we have $\int_0^s \tilde{G}(\tilde{s})\mathrm{d}\tilde{s} \leq \int_0^s G(\tilde{s})\mathrm{d}\tilde{s} = \int_0^s \tilde{G}^*(\tilde{s})\mathrm{d}\tilde{s}$, for all $s \in [0, s_-]$. Because $\tilde{G}$ and $\tilde{G}^*$ coincide over $[s_-, \bar{s}]$, we then immediately know $\int_0^s \tilde{G}(\tilde{s})\mathrm{d}\tilde{s} \leq \int_0^s \tilde{G}^*(\tilde{s})\mathrm{d}\tilde{s}$ for all $s \in [s_-, \bar{s}]$. ■

C.4 Proof of Propositions 3 and 4, Case II

In this subsection, we deal with the case where $\mathbb{E}_G(\tilde{s} \mid \alpha(\bar{s}) \leq \tilde{s} \leq \bar{s}) < s_0$. The proofs of Propositions 3 and 4 for this case are very similar to those of the previous case. We only outline the main idea.

The major difference between the proofs of the current case and those of the previous case is the construction of W function. Because $\mathbb{E}_G(\tilde{s} \mid s_0 \leq \tilde{s} \leq \bar{s}) < s_0$ and because $s \mapsto \mathbb{E}_G(\tilde{s} \mid s \leq \tilde{s} \leq \bar{s})$ is continuous and strictly increasing over $[0, \bar{s}]$, there exists a unique $s_- \in (\alpha(\bar{s}), s_0)$ such that $\mathbb{E}_G(\tilde{s} \mid s_- \leq \tilde{s} \leq \bar{s}) = s_0$. Let $s_+ = \bar{s}$ and define

$$W(\tilde{s}) = \begin{cases} \tilde{V}(\tilde{s}), & \text{if } \tilde{s} \in [0, s_-], \\ \ell(\tilde{s}), & \text{if } \tilde{s} \in (s_-, s_+], \end{cases} \tag{C.15}$$

where $\ell : \mathbb{R} \to \mathbb{R}$ is the secant line of $\tilde{V}$ that passes through $(s_-, \tilde{V}(s_-))$ and $(s_0, \tilde{V}(s_0))$. The following lemma is an analogue of Lemma C.11.

Lemma C.13. *The function W in (C.15) satisfies*

i). $W(\tilde{s}) \geq V(\tilde{s})$ for $\tilde{s} \in [0, \bar{s}]$, with strict inequality if and only if $\tilde{s} \in (s_-, s_0) \cup (s_0, s_+]$;

ii). W is convex.

Notice that, because $s_+ = \bar{s}$, $\tilde{G}^*$ in (12) becomes

$$\tilde{G}^*(\tilde{s}) = \begin{cases} G(\tilde{s}), & \text{if } \tilde{s} \in [0, s_-), \\ G(s_-), & \text{if } \tilde{s} \in [s_-, s_0), \\ 1, & \text{if } \tilde{s} \in [s_0, \bar{s}]. \end{cases}$$

Then, we can apply a similar argument as before, i.e., (in)equalities (C.9), to prove Proposition 3 for this case.

For Proposition 4, we need the following lemma, which is an analogue of Lemma C.12.

Lemma C.14. *Every pair $(\tilde{s}, s) \in [0, \bar{s}]^2$ satisfies*

$$W(s) \geq W'_-(\tilde{s})(s - \tilde{s}) + W(\tilde{s}),$$

with equality if and only if

$$(\tilde{s}, s) \in A \equiv \begin{cases} [0, s_-]^2 \cup (s_-, s_+] \times [s_-, s_+], & \text{if } s_- \leq c, \\ [0, c]^2 \cup (s_-, s_+] \times [s_-, s_+] \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in (c, s_-]\}, & \text{if } s_- > c. \end{cases}$$

Then, we can apply a similar argument as before to prove Proposition 4 for this case.

Appendix D Proofs for Section 4

D.1 Proof of Proposition 5

Proof. Because H has a continuous density, $\tilde{V}(\tilde{s})$, $\hat{V}(\tilde{s})$ and $x(\tilde{s})$, defined in Appendix C are differentiable at $\tilde{s} = s_0$. By (C.4), we know

$$x'(\tilde{s}) = \frac{\bar{n}}{1 - H(\frac{c}{x(\tilde{s})}) + \frac{c}{x(\tilde{s})} h(\frac{c}{x(\tilde{s})})}.$$

Since $x(s_0) = s_0 = \frac{c}{H^{-1}(1-\bar{n})}$, we have

$$x'(s_0) = \frac{\bar{n}}{\bar{n} + \frac{c}{s_0} h(\frac{c}{s_0})} < 1.$$

Because $\hat{V}'(s_0) = \tilde{V}'(s_0)x'(s_0)$, we know $\hat{V}'(s_0) < \tilde{V}'(s_0)$. Pick $\gamma \in (\hat{V}'(s_0), \tilde{V}'(s_0))$. Because $\hat{V}(s_0) = \tilde{V}(s_0)$, there exists an interval $[\tilde{s}_-, \tilde{s}_+]$ around s_0 such that the straight line $\ell(\tilde{s}) = \gamma(\tilde{s} - s_0) + \tilde{V}(s_0) > V(\tilde{s})$ for all $\tilde{s} \in [\tilde{s}_-, \tilde{s}_+]$ except $\tilde{s} = s_0$ in which case $\ell(\tilde{s}) = V(\tilde{s})$. Pick an interval $[s_-, s_+] \subset [\tilde{s}_-, \tilde{s}_+]$ such that $\mathbb{E}_G(\tilde{s} \,|\, s_- \leq \tilde{s} \leq s_+) = s_0$.

Consider the disclosure rule $\tilde{G}$ that censors $s \in [s_-, s_+]$ and fully discloses otherwise. Then

$$\int_0^{\bar{s}} V(\tilde{s}) \mathrm{d}\tilde{G}(\tilde{s}) - \int_0^{\bar{s}} V(\tilde{s}) \mathrm{d}G(\tilde{s}) = \int_{s_-}^{s_+} (\ell(\tilde{s}) - V(\tilde{s})) \mathrm{d}G(\tilde{s}) > 0,$$

showing that $\tilde{G}$ is better than the full transparency policy G. ■

D.2 Proof of Proposition 6

Proof. By the proof of Proposition 3, we know

$$\mathbb{E}_{G_1}(\tilde{s} \,|\, s_-^1 \leq s_0 \leq s_+^1) = s_0 = \mathbb{E}_{G_2}(\tilde{s} \,|\, s_-^2 \leq s_0 \leq s_+^2).$$

By SMLRP, we know $\mathbb{E}_{G_1}(\tilde{s} \,|\, s_-^1 \leq \tilde{s} \leq s_+^1) < \mathbb{E}_{G_2}(\tilde{s} \,|\, s_-^1 \leq \tilde{s} \leq s_+^1)$. Thus, we have

$$\mathbb{E}_{G_2}(\tilde{s} \,|\, s_-^2 \leq s_0 \leq s_+^2) < \mathbb{E}_{G_2}(\tilde{s} \,|\, s_-^1 \leq \tilde{s} \leq s_+^1). \tag{D.16}$$

In what follows, we discuss two cases. First, consider the case $s_+^2 = \bar{s}$. If $s_+^1 < \bar{s}$, by the proof of Proposition 3, we know $s_-^1 = \alpha(s_+^1) < \alpha(\bar{s}) \leq s_-^2$, where α is defined in Lemma C.8. This contradicts (D.16). Thus, we must have $s_+^1 = \bar{s}$. Inequality (D.16) then implies $s_-^2 < s_-^1$.

Next, consider the case $s_+^2 < \bar{s}$. If $s_+^1 \leq s_+^2$, by the proof of Proposition 3, we have $s_-^1 = \alpha(s_+^1) \leq \alpha(s_+^2) = s_-^2$, contradicting to (D.16). Thus, we must have $s_+^2 < s_+^1$. This also implies $s_-^2 = \alpha(s_+^2) < \alpha(s_+^1) \leq s_-^1$. ■

D.3 Proof of Proposition 7

Proof. When H is uniform, it is easy to calculate $s_0(\bar{n}) = \frac{c}{1-\bar{n}}$ and

$$V(\tilde{s}, \bar{n}) = \begin{cases} 0, & \text{if } \tilde{s} \leq c, \\ \frac{(\tilde{s}-c)^2}{2\tilde{s}}, & \text{if } c < \tilde{s} \leq \frac{c}{1-\bar{n}}, \\ \frac{(\tilde{s}\bar{n})^2}{2(c+\tilde{s}\bar{n})}, & \text{if } \tilde{s} > \frac{c}{1-\bar{n}}. \end{cases}$$

Moreover, it is also straightforward to calculate $\alpha(s, \bar{n}) = \frac{cs}{c+s}$ as defined in Lemma C.8, which turns out to be independent of $\bar{n}$. Because $c < \mathbb{E}_G(\tilde{s}) < \mathbb{E}_G(\tilde{s} \,|\, \alpha(\bar{s}, \bar{n}) \leq \tilde{s} \leq \bar{s}) < \bar{s}$, there exists $\bar{n}^* \in (0, 1 - \frac{c}{\bar{s}})$ such that $\mathbb{E}_G(\tilde{s} \,|\, \alpha(\bar{s}, \bar{n}^*) \leq \tilde{s} \leq \bar{s}) = s_0(\bar{n}^*)$. Because α is independent of $\bar{n}$ and because $s_0(\bar{n})$ is strictly increasing in $\bar{n}$, we know $\mathbb{E}_G(\tilde{s} \,|\, \alpha(\bar{s}, \bar{n}) \leq \tilde{s} \leq \bar{s}) > s_0(\bar{n})$ if and only if $\bar{n} < \bar{n}^*$. By the proof of Proposition 3, we know that the optimal censorship rule is middle censorship if $\bar{n} < \bar{n}^*$ and upper censorship if $\bar{n} \geq \bar{n}^*$.

Consider $\bar{n}_1 < \bar{n}_2 < \bar{n}^*$. By the proof of Proposition 3, we know

$$\mathbb{E}_G(\tilde{s} \,|\, \alpha(s_+(\bar{n}_1), \bar{n}_1) \leq \tilde{s} \leq s_+(\bar{n}_1)) = s_0(\bar{n}_1)$$

and

$$\mathbb{E}_G(\tilde{s} \,|\, \alpha(s_+(\bar{n}_2), \bar{n}_2) \leq \tilde{s} \leq s_+(\bar{n}_2)) = s_0(\bar{n}_2).$$

Because $s_0(\bar{n}_1) < s_0(\bar{n}_2)$ and α is independent of $\bar{n}$, we know $s_+(\bar{n}_1) < s_+(\bar{n}_2)$ and $s_-(\bar{n}_1) = \alpha(s_+(\bar{n}_1), \bar{n}_1) < \alpha(s_+(\bar{n}_2), \bar{n}_2) = s_-(\bar{n}_2)$.

When $\bar{n} \in [\bar{n}^*, 1 - \frac{c}{\bar{s}})$, by the proof of Proposition 3 again, we know

$$\mathbb{E}_G(\tilde{s} \,|\, s_-(\bar{n}) \leq \tilde{s} \leq \bar{s}) = s_0(\bar{n}),$$

which implies that $s_-(\bar{n})$ is strictly increasing. ∎

D.4 Proof of Proposition 8

Proof. The overall proof is similar to that of Proposition 3. We can similarly define

$$\tilde{V}(\tilde{s}) = \begin{cases} 0, & \text{if } \tilde{s} \in [0, c], \\ \int_{\frac{c}{\tilde{s}}}^{1} [(1+\gamma)\tilde{s}q - c] \mathrm{d}H(q), & \text{if } \tilde{s} \in (c, \infty) \end{cases}$$

as (C.1) and

$$\hat{V}(\tilde{s}) = \tilde{V}(x(\tilde{s}))$$

as (C.3), where x is exactly the same as before, i.e., (C.2). Because H is uniform, we can directly calculate

$$\tilde{V}(\tilde{s}) = \begin{cases} 0, & \text{if } \tilde{s} \leq c, \\ \frac{[(1+\gamma)\tilde{s}-(1-\gamma)c](\tilde{s}-c)}{2\tilde{s}}, & \text{if } \tilde{s} > c, \end{cases}$$

and

$$\hat{V}(\tilde{s}) = \frac{[(1+\gamma)\tilde{s}\bar{n} + 2\gamma c]\tilde{s}\bar{n}}{2(c+\tilde{s}\bar{n})}.$$

It is easy to verify that they are both convex. Moreover, it is easy to calculate α, as defined in Lemma C.8, is

$$\alpha(\tilde{s}, \gamma) = \frac{c(1-\gamma)\tilde{s}\bar{n}}{(1+\gamma)\tilde{s}\bar{n} + c[(2-\bar{n})\gamma + \bar{n}]}.$$

Therefore, we can similarly show that censoring an interval $[s_-(\gamma), s_+(\gamma)]$ with conditional mean s_0 is optimal. Moreover, we also know that $s_-(\gamma) \geq \alpha(s_+(\gamma), \gamma)$ with equality if $s_+(\gamma) < \bar{s}$.

Assume $\gamma_2 > \gamma_1$. Suppose $s_+(\gamma_1) < \bar{s}$ first. If $s_+(\gamma_2) \leq s_+(\gamma_1)$, then we have $s_-(\gamma_2) = \alpha(s_+(\gamma_2), \gamma_2) < \alpha(s_+(\gamma_2), \gamma_1) \leq \alpha(s_+(\gamma_1), \gamma_1) = s_-(\gamma_1)$, where the first inequality comes from the fact that $\alpha(s, \gamma)$ is strictly decreasing in γ. The second inequality comes from the fact that $\alpha(s, \gamma)$ is increasing in s. But then, we have

$$\mathbb{E}[\tilde{s}|s_-(\gamma_2) \leq \tilde{s} \leq s_+(\gamma_2)] < \mathbb{E}[\tilde{s}|s_-(\gamma_1) \leq \tilde{s} \leq s_+(\gamma_1)] = s_0,$$

a contradiction. Therefore, we must have $s_+(\gamma_2) > s_+(\gamma_1)$. Because

$$\mathbb{E}[\tilde{s}|s_-(\gamma_2) \leq \tilde{s} \leq s_+(\gamma_2)] = \mathbb{E}[\tilde{s}|s_-(\gamma_1) \leq \tilde{s} \leq s_+(\gamma_1)], \tag{D.17}$$

we also must have $s_-(\gamma_2) < s_-(\gamma_1)$.

Suppose $s_+(\gamma_1) = \bar{s}$ next. If $s_+(\gamma_2) < s_+(\gamma_1)$, we know $s_-(\gamma_2) > s_-(\gamma_1)$ by (D.17). Then, $\alpha(s_+(\gamma_2), \gamma_2) = s_-(\gamma_2) > s_-(\gamma_1) \geq \alpha(s_+(\gamma_1), \gamma_1) > \alpha(s_+(\gamma_2), \gamma_2)$, a contradiction. Therefore, we must have $s_+(\gamma_2) = s_+(\gamma_1) = \bar{s}$. This in turn implies $s_-(\gamma_2) = s_-(\gamma_1)$ by (D.17). ■

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