Dynamic Equilibrium with Costly Short-Selling and Lending Market*

Adem Atmaz
Purdue University

Suleyman Basak
London Business School
and CEPR

Fangcheng Ruan
Purdue University

This Version: December 2020

Abstract

We develop a tractable model of costly stock short-selling and lending market within a familiar dynamic asset pricing framework. The model addresses the vast empirical literature in this market and generates implications that support many of the empirical regularities. In the model, investors' belief disagreement leads to the presence of stock lenders and short-sellers, who in turn pay shorting fees to lenders. We find that the equilibrium stock price increases in shorting fee, and both the stock price and shorting fee decrease in lenders' size. We additionally show that the stock risk premium decreases in shorting fee, while the stock volatility is increased due to costly short-selling. Notably, we demonstrate that the equilibrium short interest increases in shorting fee and predicts future stock returns negatively. Furthermore, we find that short-selling risk matters in equilibrium, and show that a higher short-selling risk can lead to lower stock returns and less short-selling activity.

JEL Classifications: G11, G12.
Keywords: Short-selling, stock lending, belief disagreement, shorting fee, short interest, stock price, stock risk premium, volatility, short-selling risk, short-selling activity.

*Email addresses: aatmaz@purdue.edu, sbasak@london.edu, and ruan13@purdue.edu. We thank Snehal Banerjee, Adam Reed, Pedro Saffi, Edward Van Wesep, and our discussant Paymon Khorrami, as well as the seminar participants at the 2020 SFS Finance Cavalcade and Purdue University for helpful comments. All errors are our responsibility.

1 Introduction

A significant fraction of stock trades in markets nowadays emerge from the activities in the stock short-selling and lending market.[1] The key imperfection in this market is that to short a stock, an investor must pay a shorting fee to the stock lender. Corresponding to the growth in the stock short-selling and lending market, a vast empirical literature (elaborated below) has developed investigating the effects of costly short-selling on the stock price, shorting fee, stock risk premium, volatility, short interest, short-selling risk and short-selling activity. The existing theoretical studies (discussed below), on the other hand, are primarily cast in stylized static settings and look at only some aspects (e.g., stock price, risk premium only), while not being suitable to address many of the empirical regularities documented in this literature.

In this paper, we fill this void and provide a comprehensive analysis of the costly stock short-selling and lending market within a familiar dynamic asset pricing framework. Our model generates rich implications that support the extensive empirical evidence on the behavior of the shorting fee, stock price, its risk premium and volatility, short interest and its predictive power, and short-selling risk and activity. We also provide new insights and predictions, and offer simple, straightforward intuitions for all our findings. Moreover, we provide an analysis on the economic determinants of the benefits of being a stock lender, as well as endogenously determining the optimal size of lenders in the economy. Specifically, we develop a tractable dynamic equilibrium model by incorporating costly stock short-selling and lending following the usual market practices into an otherwise fairly standard economy. Investors' belief disagreement, either a constant or a mean-reverting time-varying process (main model), on the stock payoff generates demand for short-selling and supply of lendable shares in our model. In this economy, the (pessimistic) short-sellers borrow stock shares from the (optimistic) lenders by paying a *shorting fee*, which along with the stock price, are determined endogenously in equilibrium. We first demonstrate that investors' subjective stock risk premia differ since in addition to the usual capital gains/losses and dividends for non-lending investors, holding the stock long yields additional income for lenders, whereas having a short position in the stock generates additional costs for short-sellers.

[1]For instance, Diether, Lee, and Werner (2009) find that roughly 30% of the trading volume in NYSE and NASDAQ is due to short-selling, while Hanson and Sunderam (2014) report that the average short interest ratios for NYSE and AMEX stocks have more than quadrupled from 1988 to 2011. On the other hand, Saffi and Sigurdsson (2010) find that the amount of global stock lending supply in December 2008 was $15 trillion (about 20% of the total market capitalization) and $3 trillion of this amount was lent out to short-sellers.

We then determine the equilibrium shorting fee and the stock price, and show that the shorting fee is driven by the disagreement, and hence it is also a mean-reverting process in our main model. We find that shorting fee increases in belief disagreement, consistent with the empirical findings of D'Avolio (2002). This occurs because not all optimistic investors can be lenders, who also can only lend a part of their stock holdings, *partial lending*. Hence, a higher disagreement leads to a lesser increase in lending supply than the increase in shorting demand, and so for the stock lending market to clear the equilibrium shorting fee increases. We also find that the shorting fee decreases in the lenders' size, since as more investors become lenders, the stock lending supply increases, which in turn leads to a lower shorting fee in the stock lending market. This effect is supported by the empirical evidence in Prado, Saffi, and Sturgess (2016), who show that stock ownership composition matters for the stock lending market and stock prices, by also showing that stocks with more concentrated institutional ownership, i.e., stocks that are held by fewer institutions but with larger stock holdings, have lower lending supply and higher shorting fees. Likewise, D'Avolio (2002) and Nagel (2005) find that shorting fee is negatively associated with institutional ownership, a plausible proxy for our lenders' size.

Turning to the equilibrium stock price, we show that it increases in the shorting fee, a result well-supported by empirical evidence (e.g. Jones and Lamont (2002), Ofek, Richardson, and Whitelaw (2004), Blocher, Reed, and Van Wesep (2013), Prado (2015)). This result can also be viewed as confirming in a dynamic setting the classic Miller (1977) argument that with higher short-selling costs, the stock price is inflated since it reflects the views of the optimistic investors more relative to those of the pessimistic investors. In our model this occurs because the stock is ultimately held by optimistic investors some of whom are lenders, who increase their stock demand and reflect their view more, due to the additional stock lending income as the shorting fee increases, which in turn leads to a higher stock price. We also find that the stock price decreases when there are more stock lenders, since as discussed above, as more investors become lenders the shorting fee decreases, which in turn leads to a lower stock price, consistent with the empirical evidence in Prado, Saffi, and Sturgess (2016).

We next examine the extensively studied relation between the short-selling costs and stock risk premium. Our key finding here is that the stock risk premium decreases in shorting fee, consistently with vast empirical evidence (e.g. Jones and Lamont (2002), Cohen, Diether, and Malloy (2007), Blocher, Reed, and Van Wesep (2013), Drechsler and Drechsler (2016), Duong, Huszár, Tan, and Zhang (2017)). This is because a higher shorting fee leads to a higher current stock price (discussed above). Therefore, when the shorting fee is higher, the stock receives lower subsequent shocks on average relative to its current price, which in turn leads to a lower risk premium. We also find that the stock risk premium increases in lenders' size, again in line with the empirical evidence in Prado, Saffi, and Sturgess (2016) and Nagel (2005). The intuition is similar to that given for the stock price in that with more investors becoming lenders, the current stock price decreases, which leads to a higher risk premium.

We further find that the stock volatility with costly short-selling is greater than that without costly short-selling, even though both stocks are subject to the same time-varying disagreement. This result is supported by the empirical evidence in Drechsler and Drechsler (2016), who find that expensive-to-short stocks have higher stock return volatility as compared to almost-costless-to-short stocks. Saffi and Sigurdsson (2010) also find that the stocks with high shorting fees are associated with high volatility. In our model, this result arises because with costly short-selling, the stock price is additionally driven by the shorting fee and the fluctuations in the shorting fee due to disagreement shocks leads to more volatile stock price changes. This is again consistent with the classic Miller (1977) argument that in the absence of short-selling costs (and wealth transfer effects) the belief disagreement itself does not affect stock prices, leading to relatively lower stock price volatilities. To the best of our knowledge, ours is the first theory work to reconcile the evidence on the expensive-to-short stocks having higher stock price volatilities.

We next turn to the stock *short interest*, a widely used measure to infer the amount of short-selling for a stock. We first determine the short interest and show that it increases in shorting fee, consistent with the empirical evidence (D'Avolio (2002), Beneish, Lee, and Nichols (2015), Drechsler and Drechsler (2016)). This is because, a higher current shorting fee corresponds to a higher current disagreement and a higher stock price, and hence the short-sellers are relatively more pessimistic now and increase their shorting demand, leading to the positive relation between the short interest and the shorting fee of the stock.

We then obtain our key implication that the short interest predicts future stock returns negatively, implying that a higher current short interest tends to be followed by lower stock prices. This predictability arises in our model because, as discussed above, a higher short interest corresponds to a higher current shorting fee, which is now expected to be lower in the future due to mean-reversion, leading to lower future stock prices on average as compared to the relatively high current stock prices. This finding is strongly supported by the vast evidence both at the individual stock levels and at the aggregate market level (Seneca (1967), Figlewski

(1981), Senchack and Starks (1993), Desai, Ramesh, Thiagarajan, and Balachandran (2002), Asquith, Pathak, and Ritter (2005), Boehmer, Huszar, and Jordan (2010), Beneish, Lee, and Nichols (2015), Rapach, Ringgenberg, and Zhou (2016)). Among them Beneish, Lee, and Nichols (2015) show that this predictability effect is stronger for costly-to-short stocks, which lends further support to our underlying mechanism. Moreover, we also find that, in addition to the short interest (a stock variable), a positive change in short interest (a flow variable) also predicts lower future returns, consistently with the findings in Boehmer, Jones, and Zhang (2008) and Diether, Lee, and Werner (2009), as well as the current short selling being positively related to past stock performance, as shown in Diether, Lee, and Werner (2009). To our best knowledge, ours is the only theory work with costly short-selling to reconcile the extensive empirical evidence on the predictive power of short interest.

We next shed light into the recent empirical findings of Engelberg, Reed, and Ringgenberg (2018), who show that stocks with higher *short-selling risk*, as measured by the shorting fee variance, have lower returns and less *short-selling activity* (volume). In our main model, due to the time-variation in disagreement, short-sellers' demand and lenders' supply of stock shares also fluctuate, which in turn lead to a time-variation in the shorting fee, resulting with short-selling risk. We find that short-selling risk matters in equilibrium, and show that a higher short-selling risk can be associated with lower stock returns and less short-selling activity as in Engelberg, Reed, and Ringgenberg (2018) due to the differences in stock partial lending. The mainly negative relation between the short-selling risk and risk premium arises because a relatively high partial lending leads to both a lower short-selling risk and a higher risk premium. The intuition for this is that when partial lending is relatively high, the shorting fee becomes less sensitive to the disagreement shocks since the increased lending supply can now absorb the short-selling demand without affecting the shorting fee as much as before. Moreover, with relatively high partial lending, the stock price decreases, therefore the stock receives higher subsequent shocks on average relative to its current price, resulting with a higher risk premium. Similarly, the mainly negative relation between the short-selling risk and short-selling activity occurs because, a relatively high partial lending also leads to a higher short-selling activity (volume). This is because, an increase in partial lending also means that short-sellers short more for any given disagreement level, hence making the short interest more sensitive to the disagreement shocks and resulting in a higher short-selling activity.

We further investigate the benefits of being a stock lender and hence a participant in the stock lending market in equilibrium as compared to just holding the stock, and show that those benefits decrease in the lenders' size but is increasing (decreasing) for relatively low (high) levels of partial lending. This is because when more investors become lenders, the lending supply increases, and so the shorting fee decreases, leading to a lower expected future lending income. The non-monotonic relation between the benefits and partial lending arises due to two opposing effects. An increase in partial lending means a lender can earn more additional income for each share held long, leading to higher benefits. However, an increase in partial lending also leads to a lower shorting fee for each successfully lent share, and hence to a lower additional expected future income, which dominates the first effect for relatively high levels of partial lending. Finally, we endogenously determine the optimal size of lenders given a cost of setting up a lending facility. We show that the lenders' optimal size decreases in the cost of entry and in fact when the cost is too high no investor would become a lender. This is fairly intuitive since an optimistic investor is reluctant to become a lender when it is more costly to do so. We also show that in the presence of entry costs, very low levels of partial lending corresponds to no lenders, and as the partial lending, and hence the expected future income from lending increases, so does the optimal size of lenders. However, for sufficiently high levels of partial lending, the optimal lenders' size starts decreasing in partial lending, since an increase in partial lending now leads to too low of a shorting fee and a lower future lending income.

Our methodological contribution and the tractability of our model is due in large part to our ability to identify the short-sellers' and lenders' subjective risk premia in the evolution of their financial wealth (Lemma 1 of Section 2.4, and Section 3.1). This in turn allows us to employ familiar martingale methods (in the case of complete markets with riskless shorting fee) and stochastic dynamic programming techniques (in the case of incomplete markets with risky shorting fee) in the determination of equilibrium.

Our paper is related to the large theoretical literature studying the effects of short-sale constraints and restrictions in the stock market. The works in this literature typically examine the effects of some investors not being able hold short stock positions, which in static settings include Miller (1977), Diamond and Verrecchia (1987), Hong and Stein (2003), Nezafat, Schroder, and Wang (2017), and in dynamic settings include Harrison and Kreps (1978), Detemple and Murthy (1997), Scheinkman and Xiong (2003), Gallmeyer and Hollifield (2008), Chabakauri (2015). More closely related papers in this literature are those that feature a stock shorting fee, like us. However, these works are typically fairly stylized and primarily focus on the effects of the shorting fee on the stock price and risk premium only.[2] In this strand, under static settings Blocher, Reed, and Van Wesep (2013), Banerjee and Graveline (2014), and under dynamic settings in which the stock is a claim to a single payoff Duffie, Gârleanu, and Pedersen (2002) (search and bargaining-based model with risk-neutral investors) and Daniel, Klos, and Rottke (2018) (a series of one-period optimization and exogenous short interest) find that a higher stock shorting fee leads to a higher stock price and lower future expected returns, as in our paper. Our paper, however, differs from these works in terms of its methodology and results. Differently from the dynamic works above, our setting is fairly standard with risk averse investors, the stock being a claim to a dividend flow rather than a single payoff, featuring no search and bargaining considerations, and is cast in a fully dynamic economy in which the short interest is determined endogenously. Moreover, due to the stock price and the shorting fee being deterministic in Duffie, Gârleanu, and Pedersen (2002), and the short interest being exogenous in Daniel, Klos, and Rottke (2018), in these works, it is not possible to address issues related to the stock volatility, predictive power of the short interest, and short-selling risk, as we do. Similarly, the other more recent dynamic models of Evgeniou, Hugonnier, and Prieto (2019) in which the shorting fee is assumed to be proportional to stock volatility, and Nutz and Scheinkman (2019) in which the shorting fee is specified exogenously, do not have our key results on the predictive power of the short interest, short-selling risk and activity, as well as our analysis on the benefits of lending and lenders' optimal size.

Finally, this paper is also related to the vast literature on heterogeneous beliefs in financial markets (Detemple and Murthy (1994), Zapatero (1998), Basak (2000, 2005), Johnson (2004), David (2008), Yan (2008), Dumas, Kurshev, and Uppal (2009), Cvitanić and Malamud (2011), Banerjee (2011), Bhamra and Uppal (2014), Ehling, Graniero, and Heyerdahl-Larsen (2017), Atmaz and Basak (2018), Andrei, Carlin, and Hasler (2019)). However, none of these works consider the costly stock short-selling and lending market as we do, hence do not have our main mechanisms and results.

The remainder of the paper is organized as follows. Section 2 presents a simple version of our model with a constant disagreement, and Section 3 our main model with a time-varying disagreement. Section 4 discusses the benefits of lending and the optimal size of lenders. Section 5 concludes. Appendix A contains all the proofs, and Appendix B discusses the parameter values employed in our figures.

[2] Relatedly, in a partial equilibrium setting, Atmaz and Basak (2019) provide an analysis of how option prices are affected by the shorting fee of the underlying stock.

2 Equilibrium with Riskless Shorting Fee

We develop a tractable model of costly stock short-selling and lending market within a familiar dynamic asset pricing framework. Investors' belief disagreement leads to the presence of stock lenders and short-sellers in the economy. Although our main model (Section 3) features a time-varying disagreement, in this Section, to demonstrate some of our key underlying economic mechanisms in the simplest way possible, we take the investors' disagreement to be a constant, which also enables us to obtain fully closed-form solutions for the equilibrium stock price and riskless shorting fee. We find that the equilibrium stock price increases in the shorting fee, and both the stock price and shorting fee decrease in the lenders' size, consistently with empirical evidence.

2.1 Securities Market

We consider an economy with an infinite horizon evolving in continuous time. There are two securities available for trading, a risky stock and a riskless asset. The risky stock is in fixed supply of Q units and is a claim to the dividend flow D with dynamics

$$dD_t = \mu dt + \sigma d\omega_t, \tag{1}$$

where the constants μ and σ are the mean and volatility of the dividend (changes), respectively, and ω is a standard Brownian motion under the objective probability measure, $\mathbb{P}$. The stock price S is to be determined endogenously in equilibrium, with posited dynamics

$$dS_t + D_t dt = \mu_{St} dt + \sigma_{St} d\omega_t, \tag{2}$$

where μ_S and σ_S are the mean and volatility of the stock price (changes), respectively. The riskless asset is in perfectly elastic supply and pays a riskless interest rate r.

2.2 Investors' Beliefs

The economy is populated by optimistic and pessimistic investors. All investors commonly observe the dividend process D and agree on its volatility but have different beliefs about its mean. The optimistic investors, with a population mass of 0.5, perceive the mean of the dividend to be $\mu+\theta$, while the pessimistic investors, with a population mass of 0.5, perceive it to be $\mu-\theta$. The quantity θ captures the *disagreement* among investors since it is the (population) weighted-difference between the optimistic and pessimistic investors' expectations on the stock payoff, $0.5\,(\mu+\theta) - 0.5\,(\mu-\theta) = \theta$. Moreover, due to the identical masses and the symmetry in beliefs, the weighted-average expectation is unbiased, $0.5\,(\mu+\theta) + 0.5\,(\mu-\theta) = \mu$.[3] In our analysis, we first construct an equilibrium by assuming the disagreement θ is a constant (Section 2.5), but later on we develop a richer equilibrium with a time-varying disagreement process (Section 3).[4]

2.3 Stock Short-Selling and Lending Market

In reality, short-sellers sell stocks that they do not own with the anticipation of making a profit from future stock price declines. To do so, they (effectively) pay a shorting fee to borrow shares from lenders who are long in the stock, and sell those borrowed shares to other willing buyers, who are not necessarily lenders of the stock.[5] Accordingly, we incorporate costly short-selling and stock lending market into our economy by capturing the standard market practices briefly discussed above. Towards that, and also in anticipation of the equilibrium outcomes of

[3]Neither the equal mass nor the belief symmetry is necessary for our main mechanism and results to hold, and they can be generalized in a straightforward manner. We consider this specification because it provides a clear benchmark economy in which without short-selling costs, the belief disagreement alone does not affect the ensuing equilibrium stock price and yields the single-investor rational beliefs economy price (as highlighted in Section 2.5). We note that in this economy, our disagreement measure θ is also the standard deviation of investors' expectations on the stock payoff, a commonly employed measure of belief disagreement.

[4]This type of constant beliefs formulation is typically referred to as "dogmatic beliefs" in the differences of opinion literature and is also adopted by numerous works (Kogan, Ross, Wang, and Westerfield (2006), Cvitanić and Malamud (2011), among others) to illustrate the main insights in simpler disagreement settings by abstracting away from the learning mechanisms when these are not crucial for the main results. That being said, in Appendix A, we also provide an analysis in which investors learn in a Bayesian way and hence provide the microfoundations of the time-varying disagreement process we consider in Section 3.

[5]The exact mechanics of stock short-selling are somewhat more involved but its essentials are captured by our description above (see D'Avolio (2002) and Reed (2013) for an extensive discussion of short-selling). Briefly, to protect lenders, short-sellers leave a collateral typically equals to 102% of the market value of the borrowed amount in an account, which earns the interest rate. This interest income is shared between the short-seller and the lender. The short-seller's account earns the rebate amount, while the lender's account earns the shorting fee, which is effectively what short-sellers pay to lenders for each share they short. Existing research shows that other considerations such as search costs and bargaining may also play a role in this market (Duffie, Gârleanu, and Pedersen (2002)). Kolasinski, Reed, and Ringgenberg (2013) provide evidence on the dispersion of shorting fees and how search frictions can affect those dispersion and short-selling costs, similar findings are also reported in Chague, De-Losso, De Genaro, and Giovannetti (2017). In our model there is one type of lender leading to a unique shorting fee, which can be viewed as the average shorting fee.

Sections 2–4, we further categorize the optimists and pessimists populations into two groups with respect to their ability to participate in the stock short-selling and lending market. The *lenders*, ℓ, with a population mass λ, are the optimistic investors who can only successfully lend a fraction $0 < \alpha \leq 1$ of their long position to short-sellers, where henceforth we refer to α as *partial lending.* The remaining optimistic investors are *holders*, h, with a population mass $0.5 - \lambda$, and they do not participate in the stock short-selling and lending market. As D'Avolio (2002) discusses, we need such investors to help clear the stock market as "the outstanding securities must come to rest with non-lending investors willing to hold these securities despite forgoing the loan fees capitalized into the equilibrium price".[6] While the quantity λ is taken as given in Sections 2–3, it is endogenously determined in the equilibrium of Section 4.

The *short-sellers*, s, with a population mass λ_s, are the pessimistic investors who can borrow from lenders by paying for each share the *shorting fee* ϕ_t, which is to be determined endogenously in equilibrium. Finally, the remaining pessimistic investors are *non-participants*, n, with a population mass $0.5 - \lambda_s$, and they do not participate in the stock short-selling and lending market. By being pessimistic and not participating in the short-selling market, in equilibrium, non-participants do not hold any stock positions, and therefore their presence is not crucial in the determination of the endogenous stock price or the shorting fee, and hence for our main mechanism and results. Therefore, without loss of generality, we set their population mass to 0, i.e., $\lambda_s = 0.5$, and consider an economy with three types of investors ℓ, h, and s.

2.4 Investors' Preferences and Optimization

Each type of investor $i = \ell, h, s$, is endowed at time zero with the same initial wealth W_0. The investor then chooses a consumption process c_i, and an admissible portfolio strategy ψ_i, the number of shares in the stock, to maximize her subjective expected constant absolute risk

[6] The partial lending feature of ℓ-type investors can also be justified on the grounds that in reality investors may not be able to lend and earn additional income from all their long stock holdings. This is evident from the presence of the excess supply of lendable shares for most stocks. For example, Saffi and Sigurdsson (2010) find that out of \$15 trillion of stocks that are available globally for short-sellers to borrow, only \$3 trillion was actually lent out in December 2008. Moreover, the partial lending feature in our model also enables us to make a distinction between short interest (fraction of outstanding shares held by short-sellers) and lending supply (fraction of outstanding shares available for lending), as in the data. On the other hand, the presence of h-type investors in our model guarantees that in equilibrium, the short interest is determined endogenously rather than exogenously implied by the market clearing conditions.

aversion (CARA) preferences over her life-time consumption

$$\mathrm{E}_i \left[\int_0^\infty e^{-\rho t} \frac{e^{-\gamma c_{it}}}{-\gamma} dt \right],$$

where $\gamma > 0$ is the absolute risk aversion and $\rho > 0$ is the time discount factor. Here, E_i denotes the expectation under each i-type investor's subjective beliefs $\mathbb{P}_i$, on which is defined her perceived Brownian motion ω_i, given by $\omega_{it} = \omega_t - \theta t/\sigma$, for $i = \ell, h$, and $\omega_{it} = \omega_t + \theta t/\sigma$, for $i = s$, due to their beliefs as described above. The financial wealth of each i-type investor, W_i, evolves over time in a distinct manner, since in addition to the usual capital gains/losses and dividends, holding the stock long yields an additional income for lenders, whereas having a short position in the stock implies an additional cost for short-sellers. Lemma 1 presents the dynamics of each i-type investor's financial wealth, and identifies each investor's (shorting fee incorporated) subjective risk premium.

Lemma 1 (**Financial wealth dynamics and investor type-specific risk premia**). *In the economy with costly stock short-selling and lending market, the financial wealth of each i-type investor, $i = \ell, h, s$, evolves according to*

$$dW_{it} = W_{it} r dt + \psi_{it} \pi_{it} dt + \psi_{it} \sigma_{St} d\omega_{it} - c_{it} dt, \tag{3}$$

where π_{it} is the each i-type investor's subjective risk premium and is given by

$$\pi_{it} = \begin{cases} \pi_{St} + \frac{\sigma_{St}}{\sigma}\theta + \alpha\phi_t & \text{for } i = \ell, \\ \pi_{St} + \frac{\sigma_{St}}{\sigma}\theta & \text{for } i = h, \\ \pi_{St} - \frac{\sigma_{St}}{\sigma}\theta + \phi_t & \text{for } i = s, \end{cases} \tag{4}$$

where $\pi_{St} = \mu_{St} - rS_t$ is the objective risk premium.

The first term in the investors' subjective risk premium (4) is the standard objective risk premium. The second term is due to their subjective beliefs and adding it gives the subjective stock risk premium if there is no additional shorting fee cost or income for the investor. For the optimists this term is positive, while for the pessimists is negative. The third term, if exists, adjusts for the additional shorting fee cost or income from being a short-seller or a lender in equilibrium, respectively. For example, if an investor turns out to be a lender in equilibrium, she must be an optimist and earning a shorting fee ϕ_t for each share she successfully lends out of her long stock position, thereby increasing her subjective stock risk premium by $\sigma_{St}\theta/\sigma+\alpha\phi_t$ over and above the objective risk premium. The subjective and objective risk premia will be determined endogenously in equilibrium.

Our explicit identification of the investor type-specific risk premia above facilitates much tractability in our subsequent general equilibrium analysis. In particular, it enables us to further identify investor-specific risk-neutral measures in this setting with investors facing different market imperfections. This in turn allows us to employ familiar martingale methods and stochastic dynamic programming techniques in the determination of equilibrium.

2.5 Equilibrium

The costly stock short-selling and lending market economy is said to be in *equilibrium* if the stock price S, the shorting fee ϕ, and the investors' consumption and portfolio strategies (c_i, ψ_i), $i = \ell, h, s$, are such that (i) all investors choose their optimal consumption and portfolio strategies given the stock price, the shorting fee, and their beliefs, (ii) the stock market and the stock short-selling and lending market clear, i.e., $\lambda\psi_{\ell t} + (\frac{1}{2} - \lambda)\psi_{ht} + \frac{1}{2}\psi_{st} = Q$, and $\lambda\alpha\psi_{\ell t}\phi_t + \frac{1}{2}\psi_{st}\phi_t = 0$, at all times t, respectively.

We will often make comparisons with the costless short-selling ($\phi_t = 0$) benchmark economy denoted with an upper bar ($\bar{\ }$). Since there is no shorting fee in the benchmark economy, only the stock market needs to clear in equilibrium. With costly short-selling, the stock short-selling and lending market needs to additionally clear. In particular, the equilibrium shorting fee must be such that the total amount shorted equals to the total amount lent. To ensure that the model is well-behaved, the shorting fee is positive and only s-type pessimistic investors are short-sellers in equilibrium, we assume the following parameter restriction throughout our analysis with constant disagreement in this Section

$$\frac{1/2 + \lambda\alpha}{1/2 - \lambda\alpha}\gamma\sigma^2 Q < \theta < 2\gamma\sigma^2 Q. \tag{5}$$

This essentially states that for a non-trivial short-selling activity to arise in equilibrium, we need to have at the minimum some level of disagreement among investors and this disagreement cannot be too large so that we prevent h-type optimistic investors becoming short-sellers.

In this costly stock short-selling economy with a constant disagreement, there is one source of uncertainty and two securities available for trading for all investors, and hence markets are dynamically complete, which implies the existence of a unique (type-specific) risk-neutral measure for each investor type. We employ standard martingale methods (Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989)) under the risk-neutral measures to solve for each investor's optimal consumption and portfolio strategies and apply market clearing conditions above to obtain the equilibria. Proposition 1 reports the equilibrium in this economy and presents the equilibrium shorting fee ϕ and the stock price S, along with their key properties.

Proposition 1 (**Equilibrium shorting fee and stock price**). *In the costly stock short-selling economy with a constant disagreement, the equilibrium exists with three types of investors, ℓ, h, s, and the equilibrium shorting fee and the stock price are given by*

$$\phi = -\frac{1}{(1/2-\lambda\alpha)-\lambda\alpha\,(1-\alpha)\,/(1/2+\lambda\alpha)}\frac{1}{r}\gamma\sigma^2 Q+\frac{1}{(1/2+\lambda\alpha)-\lambda\alpha\,(1-\alpha)\,/(1/2-\lambda\alpha)}\frac{1}{r}\theta, \quad (6)$$

$$S_t = \bar{S}_t + \Big(\frac{1}{2} + \lambda\alpha\Big)\frac{1}{r}\phi. \quad (7)$$

The equilibrium stock price in the benchmark economy with costless short-selling is given by $\bar{S}_t = \frac{1}{r}D_t + \frac{\mu}{r^2} - \frac{\gamma\sigma^2 Q}{r^2}$.

Consequently, in the costly stock short-selling economy with a constant disagreement,

i) the shorting fee is increasing in disagreement θ, and is decreasing in lenders' size λ and partial lending α,

ii) the stock price is increasing in shorting fee ϕ, and is decreasing in lenders' size λ and partial lending α.

We see that the equilibrium shorting fee (6) is driven by the disagreement θ, the size of the stock lenders λ, the partial lending α, as well as the risk adjustment term $\gamma\sigma^2 Q$. More notably, we find that the shorting fee increases in belief disagreement (property (i)). This is because not all optimistic investors can be lenders, who also can only lend part of their stock holdings, and so a higher disagreement leads to a lesser increase in lending supply than the increase in shorting demand. Hence, for the stock lending market to clear the equilibrium shorting fee must increase. This result is consistent with the empirical evidence in D'Avolio (2002), who shows that the shorting fee of a stock is high when the investor disagreement is high.

Moreover, we show that the shorting fee decreases in the lenders' size (property (i)). This is because as more investors become lenders, the stock lending supply increases, leading to a lower shorting fee in the stock lending market. The empirical support for this effect is provided by Prado, Saffi, and Sturgess (2016), who show that stock ownership composition matters for the stock lending market and stock prices. In particular, they find that stocks with more concentrated institutional ownership, i.e., stocks that are held by fewer institutions but with larger stock holdings, corresponding to lower lenders' size λ in our model, have lower lending supply and higher shorting fees. Likewise, D'Avolio (2002) and Nagel (2005) find that shorting fee is negatively associated with institutional ownership, a plausible proxy for the lenders' size in our model. Similarly, we find that the shorting fee decreases in partial lending. This is because as partial lending increases, lenders increase their stock holdings and their actual lending, since they can now earn more additional income from lending, which increases the stock lending supply and leads to a lower shorting fee in the stock lending market.

Turning to the equilibrium stock price, we first see that the stock price in the benchmark economy with costless short-selling (and disagreement) is as in the standard economy with a single investor with unbiased beliefs. With costly short-selling, the equilibrium stock price (7) has a simple structure and is additionally driven by the shorting fee ϕ, the size of the lenders λ and partial lending α. The sensitivity of the stock price to shorting fee is captured by the positive term $(1/2 + \lambda\alpha)$, which economically is the total size of investors present in the stock short-selling market relative to the total size of investors in the stock market, and hence is always less than 1. A notable implication here is that the stock price increases in the shorting fee (property (ii)). This result can be viewed as the dynamic version of the classic Miller (1977) argument that with higher short-selling costs, the stock price is inflated since it reflects the views of the optimistic investors more relative to those of the pessimistic investors, who, due to the increased cost of short-selling, reflect their views on the stock less. In our model this occurs because the stock is ultimately held by optimistic investors and among them there are lenders, who increase their stock demand due to the additional stock lending income as the shorting fee increases, which in turn leads to a higher stock price.[7] This result is well supported by empirical evidence (e.g. Jones and Lamont (2002), Ofek, Richardson, and Whitelaw (2004), Blocher, Reed, and Van Wesep (2013), Prado (2015)).

We also see that the stock price decreases in the lenders' size (property (ii)). This is because as we discuss above as more investors become lenders the shorting fee decreases, which in turn

[7]As we show in Appendix A, the investors' stock demands are given by $\psi_{it} = \pi_{it}/\gamma\sigma_S^2 r$, where the subjective risk premium π_{it} is as in (4) of Lemma 1. We see that a higher shorting fee leads to an increase in the lenders' long position.

leads to a lower stock price. The empirical support for this effect is provided by Prado, Saffi, and Sturgess (2016), who show that stocks with more concentrated institutional ownership, corresponding to lower lenders' size λ in our model, have higher stock prices and lower future returns on average.[8] Similarly, we find that the stock price decreases in partial lending because a higher partial lending leads to a lower shorting fee as discussed above.

We note that the effects of partial lending on the stock price and the shorting fee are due to mechanisms similar to those for the lenders' size. In our subsequent analysis when both quantities have the same effect, to avoid repetition we only highlight the results with respect to lenders' size, which also have clear empirical counterparts. However, when the effects of partial lending are different from those for the lenders' size, we highlight them as we do in Sections 3.3 and 4.

As discussed in the Introduction, consistent with the classic Miller (1977) argument, a number of existing theoretical works find a similar result that a higher shorting fee leads to a higher stock price, as in our Proposition 1. Moreover, Duffie, Gârleanu, and Pedersen (2002), Blocher, Reed, and Van Wesep (2013), and Daniel, Klos, and Rottke (2018) also find that the shorting fee increases in disagreement. Our findings complement these works by demonstrating that these results also arise in our fairly standard simple dynamic framework. However, to the best of our knowledge, our findings on lenders' size and partial lending are new. Moreover, we carry out our main analysis with a time-varying disagreement in the next section, which supports the results of Proposition 1 and also generates various other novel implications.

3 Equilibrium with Risky Shorting Fee

In this Section, we present our main model with a time-varying disagreement, which leads to a richer equilibrium and to a risky shorting fee as in the data. We show that our earlier implications with a riskless shorting fee on the equilibrium stock price and shorting fee remain equally valid with a risky shorting fee. We additionally find that the stock risk premium is decreasing in the risky shorting fee, while the stock volatility is increased in the presence of costly short-selling, all consistently with empirical evidence. Notably, we demonstrate that the

[8] On the other hand, the empirical evidence on the effects of overall lending supply on stock prices is somewhat mixed. For example, Blocher, Reed, and Van Wesep (2013) show that a reduction in the lending supply of stock shares increases the prices of expensive-to-short stocks. In contrast, Kaplan, Moskowitz, and Sensoy (2013) find that an increase in the lending supply of expensive-to-short stocks has no significant effect on the stock price.

equilibrium short interest is increasing in the shorting fee and predicts future stock returns negatively, supporting the vast empirical evidence. Furthermore, we find that short-selling risk matters in equilibrium, and show that a lower short-selling risk can be associated with higher stock returns and more short-selling activity, also consistently with the recent empirical evidence.

3.1 Economy with Time-Varying Disagreement

We retain all the primitives of Section 2, the securities market, the investors' types and preferences, apart from investors' beliefs, so as to have a time-varying disagreement in our model. In particular, rather than assuming investors perceive the mean of the dividend as $\mu \pm \theta$, where θ is a constant, we assume that they perceive it as $\mu \pm \theta_t$, where θ_t follows an Ornstein-Uhlenbeck process with dynamics

$$d\theta_t = \kappa \left(\mu_\theta - \theta_t\right) dt + \sigma_\theta d\omega_{\theta t}, \tag{8}$$

where the constants κ, μ_θ, and σ_θ are the speed of mean reversion, long-run mean, and volatility of the disagreement, respectively, and ω_θ is a standard Brownian motion, independent from ω, under the objective probability measure, $\mathbb{P}$.[9] In Appendix A, we discuss the microfoundations of this type of mean-reverting, Gaussian disagreement process by considering a Bayesian learning environment, in which investors are symmetrically informed but have different interpretations of signals as often considered in the dynamic differences in beliefs models with stationary disagreement (e.g., Scheinkman and Xiong (2003), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2010)). With this disagreement process, the stock price S has the posited dynamics

$$dS_t + D_t dt = \mu_{St} dt + \sigma_{1t} d\omega_t + \sigma_{2t} d\omega_{\theta t}, \tag{9}$$

[9]It is well-known that in a framework like ours, i.e., CARA preferences with Gaussian payoff, the stock price can take negative values with positive probability. Due to being a Gaussian process, now additionally the disagreement θ_t, as well as the equilibrium shorting fee that will be driven by the disagreement, can take negative values with positive probabilities. However, our model allows us to control these probabilities and obtain almost always positive shorting fee and disagreement for suitable choices of parameter values. For example, for the parameters values in Table 1 of Appendix B, the probabilities of the shorting fee and the disagreement being positive at the steady-state are $\mathbb{P}(\phi_t > 0) = 0.999$ and $\mathbb{P}(\theta_t > 0) = 1 - \Phi(-20.35) \simeq 1$, where Φ is the cumulative distribution function of standard normal distribution. Therefore, in our subsequent analysis we treat these quantities as if they were positive almost surely.

where the diffusion terms σ_1 and σ_2 now determine the volatility of the stock price (changes) through the relation $\sigma_{St}^2 = \sigma_{1t}^2 + \sigma_{2t}^2$. Moreover, under this specification, the financial wealth of each i-type investor, $i = \ell, h, s$, evolves according to

$$dW_{it} = W_{it} r dt + \psi_{it}\pi_{it} dt + \psi_{it}\sigma_{1t} d\omega_{it} + \psi_{it}\sigma_{2t} d\omega_{\theta t} - c_{it} dt,$$

where each i-type investor's subjective risk premium is now given by

$$\pi_{it} = \begin{cases} \pi_{St} + \frac{\sigma_{1t}}{\sigma}\theta_t + \alpha\phi_t & \text{for } i = \ell, \\ \pi_{St} + \frac{\sigma_{1t}}{\sigma}\theta_t & \text{for } i = h, \\ \pi_{St} - \frac{\sigma_{1t}}{\sigma}\theta_t + \phi_t & \text{for } i = s. \end{cases} \tag{10}$$

In this costly stock short-selling economy with a time-varying disagreement, there are two sources of uncertainty and two securities available for trading for all investors, and hence markets are dynamically incomplete. We employ the standard stochastic dynamic programming method (Merton (1971)) to solve for each investor's optimal consumption and portfolio strategies and apply market clearing conditions to obtain the linear equilibrium.

3.2 Shorting Fee, Stock Price and Its Dynamics

Proposition 2 reports the equilibrium in this economy with a time-varying disagreement, and presents the equilibrium shorting fee ϕ, the stock price S, the investors' indirect utility functions $J_i(W_{it}, \theta_t, t) = \max_{(c_i,\psi_i)} \mathrm{E}_{it}\left[\int_t^\infty e^{-\rho u}\frac{e^{-\gamma c_{iu}}}{-\gamma} du\right]$, $i = \ell, h, s$, along with their key properties.

Proposition 2 (Equilibrium risky shorting fee and stock price). *In the costly stock short-selling economy with a time-varying disagreement, the linear equilibrium exists with three types of investors, ℓ, h, s, and the equilibrium shorting fee and the stock price are given by*

$$\phi_t = \phi_0 + \phi_1\theta_t, \tag{11}$$

$$S_t = \bar{S}_t - \frac{\mu}{r^2} + \frac{\gamma\sigma^2 Q}{r^2} + A + B\phi_t, \tag{12}$$

where the positive constants ϕ_1 and B solve the equations

$$\Big[\left(2\lambda\alpha K_{\ell 2}+K_{s2}\right)\sigma_\theta^2-\Big(\frac{1}{2}+\lambda\alpha\Big)\left(\kappa+r\right)\Big]r\phi_1 B-\Big(\frac{1}{2}-\lambda\alpha\Big)+\Big(\frac{1}{2}+\lambda\alpha^2\Big)r\phi_1=0, \quad (13)$$

$$\Big[\left(2\lambda\left(1-\alpha\right)K_{\ell 2}+\left(1-2\lambda\right)K_{h2}\right)\sigma_\theta^2-\Big(\frac{1}{2}-\lambda\alpha\Big)\left(\kappa+r\right)\Big]r\phi_1 B+\Big(\frac{1}{2}-\lambda\alpha\Big)+\lambda\alpha\left(1-\alpha\right)r\phi_1=0, \quad (14)$$

and the constants ϕ_0 and A solve

$$\Big(\lambda\alpha K_{\ell 1}+\frac{1}{2}K_{s1}\Big)\sigma_\theta^2\phi_1 B+\Big(\frac{1}{2}+\lambda\alpha\Big)\Big(\frac{\mu}{r}+\kappa\bar{\theta}\phi_1 B-r\left(A+\phi_0 B\right)\Big)+\Big(\frac{1}{2}+\lambda\alpha^2\Big)\phi_0=0, \quad (15)$$

$$\frac{\Big[\lambda(1-\alpha)K_{\ell 1}+\Big(1/2-\lambda\Big)K_{h1}\Big]\sigma_\theta^2\phi_1 B+\Big(1/2-\lambda\alpha\Big)\Big(\mu/r+\kappa\bar{\theta}\phi_1 B-r(A+\phi_0 B)\Big)+\lambda\alpha(1-\alpha)\,\phi_0}{r\gamma\left(\sigma_\theta^2\phi_1^2B^2+\sigma^2/r^2\right)Q}=1, \quad (16)$$

and the i-type investor's, $i=\ell,h,s$, indirect utility function is given by

$$J_i\left(W_{it},\theta_t,t\right)=-e^{-\rho t}e^{-\gamma r W_{it}}e^{K_{i0}+K_{i1}\theta_t+K_{i2}\theta_t^2}, \quad (17)$$

where the constants K_{i0}, K_{i1}, K_{i2}, and each i-type investor's optimal consumption and portfolio strategies (c_i,ψ_i) are provided in the Appendix A. The equilibrium stock price in the benchmark economy with costless short-selling is given by $\bar{S}_t=\frac{1}{r}D_t+\frac{\mu}{r^2}-\frac{\gamma\sigma^2 Q}{r^2}$, and the i-type investor's, $i=\ell,h,s$, indirect utility function is given by $\bar{J}_i\left(W_{it},\theta_t,t\right)=-e^{-\rho t}e^{-\gamma r W_{it}}e^{\bar{K}_{i0}+\bar{K}_{i1}\theta_t+\bar{K}_{i2}\theta_t^2}$, where the constants $\bar{K}_{i0}$, $\bar{K}_{i1}$, $\bar{K}_{i2}$, and each i-type investor's optimal consumption and portfolio strategies $(\bar{c}_i,\bar{\psi}_i)$ are provided in the Appendix A.

Consequently, in the costly stock short-selling economy with a time-varying disagreement,

i) the shorting fee is increasing in disagreement θ_t,

ii) the stock price is increasing in shorting fee ϕ_t.

Proposition 2 reveals that in the costly stock short-selling economy with a time-varying disagreement, the shorting fee and the stock price now have a richer structure, and are additionally driven by the speed of mean reversion, long-run mean, and volatility of the disagreement, κ, μ_θ, and σ_θ, respectively. Since it is driven by the mean-reverting disagreement process, the stock shorting fee also becomes a mean-reverting process. Notably, the shorting fee (11) is now risky since the fluctuations in the disagreement lead to time-variation in the shorting fee, which in turn introduces additional fluctuations in the stock price (12). On the other hand, investors'

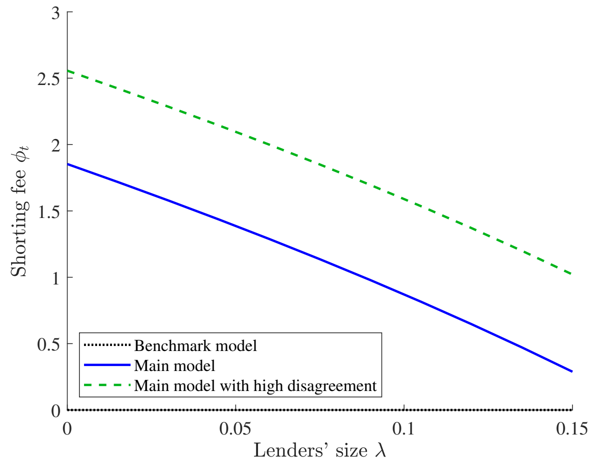
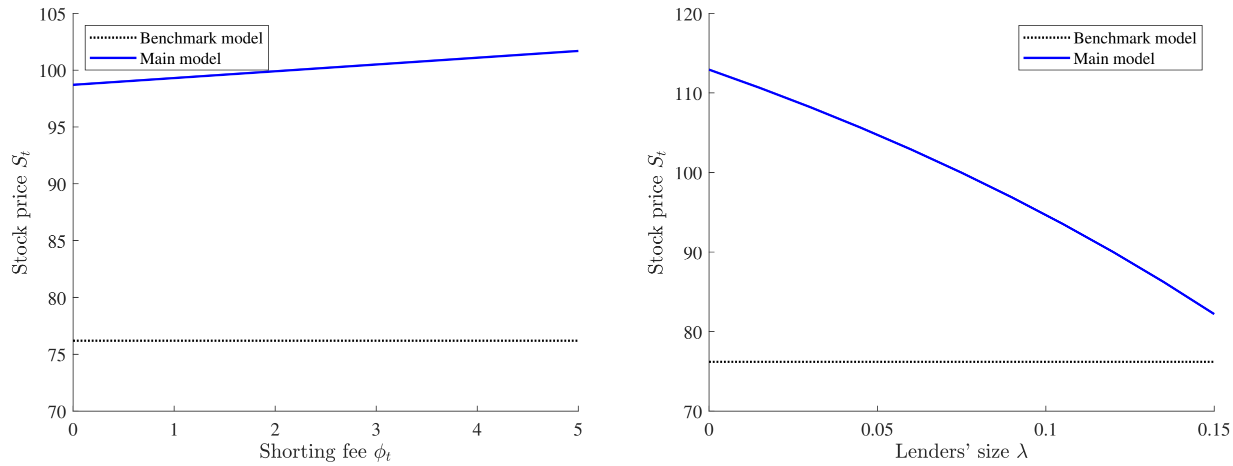


Figure 1: **Shorting fee behavior.** This figure plots the equilibrium shorting fee ϕ_t against the lenders' size λ. The solid line corresponds to our main economy with costly short-selling, the dashed line corresponds to the main economy with higher long-run mean of disagreement, and the dotted line corresponds to the benchmark economy with costless short-selling. The parameter values follow from Table 1 of Appendix B.

value function (17) has a familiar structure that typically arises in dynamic linear equilibrium asset pricing models with Gaussian processes.

Proposition 2 also confirms that our main implications for the shorting fee and the stock price discussed in the previous section remain valid with a time-varying disagreement. The economic mechanism and intuition for these results are similar to those of Proposition 1. We first see that the shorting fee increases in belief disagreement (property (i)). Increasing disagreement leads to a larger increase in shorting demand than that in lending supply given the presence of non-lending optimists and lenders being able to lend only a fraction of their stock holdings, thereby increasing the shorting fee. Figure 1, plots the equilibrium shorting fee against the lenders' size and illustrates that our analytical negative relation result between these two quantities of Section 2 continues to hold in this economy. This is because when more investors become lenders, the increased lending supply leads to a lower shorting fee in the stock lending market.

We also find that the stock price increases in shorting fee (property (ii)), as also illustrated in Figure 2, Panel (a), which plots the equilibrium stock price against the shorting fee. This is because the stock is ultimately held by optimistic investors and among them there are lenders, who increase their stock demand due to the additional stock lending income as the shorting



(a) Effects of shorting fee

(b) Effects of lenders' size

Figure 2: **Stock price behavior.** These panels plot the equilibrium stock price S_t against the shorting fee ϕ_t and lenders' size λ. The solid lines correspond to our main economy with costly short-selling, while the dotted lines correspond to the benchmark economy with costless short-selling. The parameter values follow from Table 1 of Appendix B.

fee increases, which in turn leads to a higher stock price.[10] Moreover, as illustrated in Figure 2, Panel (b), when more investors become lenders, the stock price goes down in equilibrium (property (ii)). This occurs because, as discussed above, when more investors become lenders the shorting fee decreases, which in turn leads to a lower stock price.

There is much empirical evidence documenting that stocks with higher shorting fee earn lower future returns, as well as their returns being more volatile. We now investigate our model implication for the stock risk premium and volatility. As discussed earlier, in this economy, the risk premium perceived by each i-type investor, $i = \ell, h, s$, differs from the (observed) objective risk premium, $\pi_{St} = \mu_{St} - rS_t$, with the relation between them being given by (10). To make our results comparable to empirical studies, we present our results in terms of the objective risk premium as observed in the data. Proposition 3 presents the equilibrium stock risk premium and volatility and their key properties.

Proposition 3 (**Equilibrium risk premium and volatility**). *In the costly stock short-selling*

[10]We note that, even with a time-varying disagreement, in the absence of short-selling costs, the stock price in the benchmark economy with costless short-selling is still as in the standard economy with a single investor with unbiased beliefs. Therefore, the dashed line in Figure 2, Panel (a), serves only to illustrate the level of the stock price in that economy and should not be interpreted as the shorting fee varying in our benchmark economy. This consideration also applies to all the subsequent figures.

economy with a time-varying disagreement, the equilibrium stock risk premium and volatility are given by

$$\pi_{St} = \bar{\pi}_S + \left(\frac{\mu}{r} - \frac{\gamma\sigma^2 Q}{r} - rA + \kappa B(\phi_0 + \phi_1\bar{\theta})\right) - (\kappa + r)\, B\phi_t, \tag{18}$$

$$\sigma_S = \sqrt{\bar{\sigma}_S^2 + \sigma_\theta^2 \phi_1^2 B^2}, \tag{19}$$

where the shorting fee ϕ_t, and the constants ϕ_0, ϕ_1, A, and B are as in Proposition 2. The equilibrium stock risk premium and volatility in the benchmark economy with costless short-selling are given by $\bar{\pi}_S = \gamma\sigma^2 Q/r$ and $\bar{\sigma}_S = \sigma/r$, respectively.

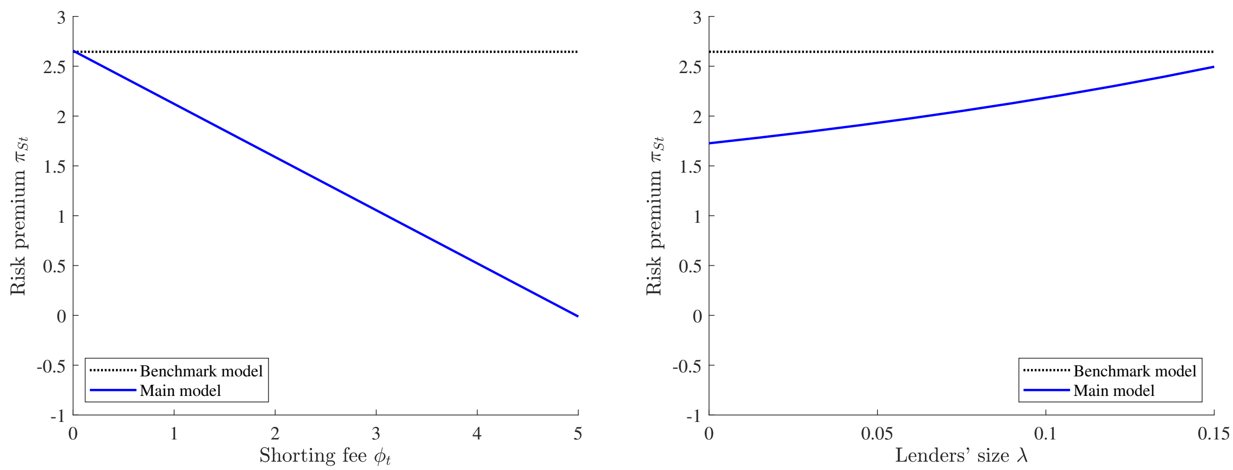
Consequently, in the costly stock short-selling economy with a time-varying disagreement,

i) the stock risk premium is decreasing in shorting fee ϕ_t,

ii) the stock volatility is higher than that in the benchmark economy with costless short-selling.

Proposition 3 shows that, like the stock price, the stock risk premium has a rich structure, and is driven by the time-varying shorting fee and hence the disagreement parameters, in addition to the usual risk adjustment term. The key implication here is that stock risk premium decreases in shorting fee (property (i)), as illustrated in Figure 3, Panel (a), consistently with vast empirical evidence (e.g. Jones and Lamont (2002), Cohen, Diether, and Malloy (2007), Blocher, Reed, and Van Wesep (2013), Drechsler and Drechsler (2016), Duong, Huszár, Tan, and Zhang (2017)).[11] This is because a higher shorting fee leads to a higher current stock price (Proposition 2). Therefore, when the shorting fee is higher, the stock receives lower subsequent shocks on average relative to its current price, which in turn leads to a lower risk premium. Figure 3, Panel (b), plots the equilibrium risk premium against the lenders' size and reveals that the higher lenders' size leads to an increase in the stock risk premium. This is because as more investors become lenders, the current stock price decreases (Proposition 2), therefore, the stock receives higher subsequent shocks on average relative to its current price, which in turn leads to a higher risk premium. The empirical support for this result is provided by Prado, Saffi, and Sturgess (2016), who show that stocks with more concentrated institutional ownership, e.g. stocks that are held by fewer institutions but with larger stock holdings, corresponding to lower lenders' size λ in our model, have lower future returns on average. Similarly, Nagel (2005)

[11] We note that even though we focus on the effects of the shorting fee on the risk premium in Proposition 3, we find that the stock mean return μ_{St} and the Sharpe ratio π_{St}/σ_S are also decreasing in the shorting fee.



(a) Effects of shorting fee

(b) Effects of lenders' size

Figure 3: **Risk premium behavior.** These panels plot the equilibrium stock risk premium π_{St} against the shorting fee ϕ_t and lenders' size λ. The solid lines correspond to our main economy with costly short-selling, while the dotted lines correspond to the benchmark economy with costless short-selling. The parameter values follow from Table 1 of Appendix B.

shows that high institutional ownership (a plausible proxy for our lenders' size) is associated with higher future stock returns on average.

We also find that in the costly stock short-selling economy, the stock volatility is higher than that in the benchmark economy with costless short-selling (property (ii)) even though both economies have the same time-varying disagreement θ_t. This is again consistent with a classic Miller (1977) argument that in the absence of short-selling costs (and wealth transfer effects) the belief disagreement itself does not affect stock prices, leading to relatively lower stock price (change) volatilities. In contrast, with costly short-selling, stock prices are additionally driven by the shorting fee (Proposition 2), and the fluctuations in the shorting fee due to disagreement shocks leads to more volatile stock price changes (Figure 4). This result is consistent with empirical evidence in Drechsler and Drechsler (2016) who find that expensive-to-short stocks have higher stock return volatility compared to almost-costless-to-short stocks. Similarly, Saffi and Sigurdsson (2010) find a positive association between high shorting fees and high volatility.[12]

[12]As Figure 4 illustrates, the stock volatility does not vary much with lenders' size λ, thereby implying that two stocks with the same level of high shorting fee but differing in lenders' size, and hence differing in lending supply, may have similar volatility. In our model this occurs because the fluctuations in the disagreement is now

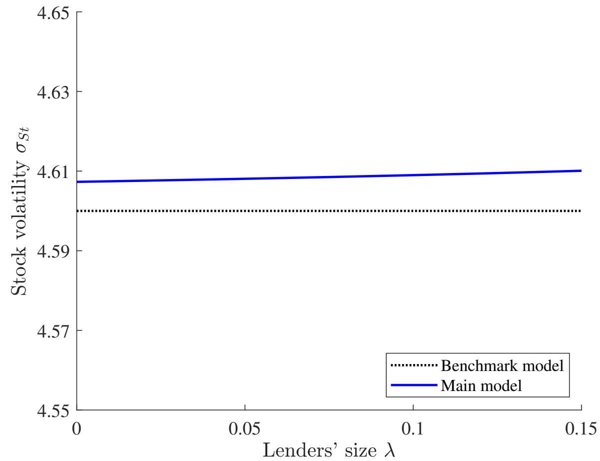


Figure 4: **Stock volatility behavior.** This figure plots the equilibrium stock volatility against the lenders' size λ. The solid line corresponds to our main economy with costly short-selling, while the dotted lines correspond to the benchmark economy with costless short-selling. The parameter values follow from Table 1 of Appendix B.

As discussed in the Introduction, a number of existing theoretical works in various settings find results similar to our risk premium findings of Proposition 3. As in the case of the stock price, our findings complement these works by demonstrating that these results also arise in our fairly standard dynamic asset-pricing framework. However, our volatility findings are novel in the sense that, to the best of our knowledge, ours is the first theory work to reconcile the evidence on the expensive-to-short stocks having higher stock price volatilities.

3.3 Short Interest and Short-Selling Risk

A widely used and closely watched measure to infer the amount of short-selling for a stock is its *short interest*, which is the fraction of its outstanding shares held by short-sellers. There is vast empirical evidence documenting that a higher current short interest predicts lower future stock returns, both for individual stocks and the stock market itself. We now examine our model implication for the short interest, $\mathcal{SI}$, defined as $\mathcal{SI}_t = -\frac{1}{2}\psi_{st}/Q$, and its predictive ability for

mostly absorbed by the fluctuations in shorting fee, whose marginal effects on the stock price is smaller compared to fundamental dividend shocks. This finding may help us understand the somewhat surprising evidence in Saffi and Sigurdsson (2010) and Kaplan, Moskowitz, and Sensoy (2013), who do not find a significant relation between the lending supply and stock volatility.

future stock returns in the regression

$$S_{t+h} - S_t = \alpha_{\mathcal{SI}} + \beta_{\mathcal{SI}} \mathcal{SI}_t + \epsilon_{t+h}, \tag{20}$$

where $\alpha_{\mathcal{SI}}$ is some constant, $\beta_{\mathcal{SI}}$ is the slope coefficient that we are interested in, and ϵ_{t+h} are the error terms.[13] Towards that, Proposition 4 reports the equilibrium short interest $\mathcal{SI}_t$ and the slope coefficient $\beta_{\mathcal{SI}}$ in the predictive regression, along with their key properties.

Proposition 4 (**Equilibrium short interest and predictability**). *In the costly stock short-selling economy with a time-varying disagreement, the equilibrium short interest and its slope coefficient in the predictive regression* (20) *are given by*

$$\begin{aligned}\mathcal{SI}_t = \overline{\mathcal{SI}}_t + \frac{1}{2} + \frac{1}{2}\frac{1}{\gamma\sigma^2 Q}\frac{\phi_0}{\phi_1} + \frac{1}{2r}\left(\frac{(\kappa + r - 2K_{s2}\sigma_\theta^2)\,\phi_1 B + 1/r - \phi_1}{\gamma\sigma_S^2 Q} - \frac{r}{\gamma\sigma^2 Q}\right)\frac{1}{\phi_1}\phi_t \\ - \frac{1}{2r}\frac{\Big(\mu/r + \kappa\bar{\theta}\phi_1 B - r\,(A + \phi_0 B) + \phi_0/(r\phi_1)\Big) + \Big((\kappa + r - 2K_{s2}\sigma_\theta^2)\,\phi_0 + K_{s1}\sigma_\theta^2\phi_1\Big)B}{\gamma\sigma_S^2 Q},\end{aligned} \tag{21}$$

$$\beta_{\mathcal{SI}} = -2r\phi_1 B \frac{\gamma\sigma_S^2 Q}{(\kappa + r - 2K_{s2}\sigma_\theta^2)\,\phi_1 B + 1/r - \phi_1}\left(1 - e^{-\kappa h}\right), \tag{22}$$

where the stock volatility σ_S is as in Proposition 3, and the shorting fee ϕ_t, and the constants ϕ_0, ϕ_1, A, B, K_{s1}, and K_{s2} are as in Proposition 2. The equilibrium short interest and its slope coefficient in the predictive regression (20) *in the benchmark economy with costless short-selling are given by $\overline{\mathcal{SI}}_t = -1/2 + \theta_t/(2\gamma\sigma^2 Q)$, and $\bar{\beta}_{\mathcal{SI}} = 0$, respectively.*

Consequently, in the costly stock short-selling economy with a time-varying disagreement,

i) the short interest is increasing in shorting fee ϕ_t,

ii) higher short interest predicts lower future stock returns.

Proposition 4 shows that in our economy with a time-varying disagreement, the short interest increases in shorting fee (property (i)), as also illustrated in Figure 5. This is because, a higher current shorting fee corresponds to a higher current disagreement and a higher stock price, and hence the short-sellers are relatively more pessimistic now and increase their shorting demand,

[13] We do not find it necessary to impose a non-negativity restriction on our short interest measure, since the reasonable parameter values that ensure the almost always positivity of disagreement, also ensure the almost always positivity of the short interest in our model. For example, for the parameters values in Table 1 of Appendix B, the probability of $\mathcal{SI}_t$ being positive at the steady-state is $\mathbb{P}(\mathcal{SI}_t > 0) = 1 - \Phi(-44.04) \simeq 1$, where Φ is the cumulative distribution function of standard normal distribution.

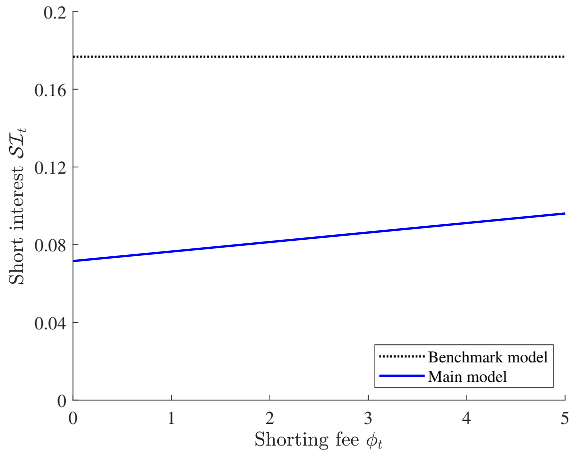


Figure 5: **Short interest behavior.** This figure plots the equilibrium short interest $\mathcal{SI}_t$ against the shorting fee ϕ_t. The solid line corresponds to our main economy with costly short-selling, while the dotted line corresponds to the benchmark economy with costless short-selling. The parameter values follow from Table 1 of Appendix B.

leading to the positive relation between the short interest and the shorting fee of the stock. This result is consistent with the empirical evidence in Drechsler and Drechsler (2016), who find that expensive-to-short stocks have higher short interest compared to cheap-to-short stocks. The typical positive relation between the short interest and shorting fee is also documented by many other studies, including D'Avolio (2002) and Beneish, Lee, and Nichols (2015).

The key implication of Proposition 4 is that the short interest predicts future stock returns negatively, implying a higher current short interest tends to be followed by lower stock prices (property (ii)). This predictability arises in our model because, as discussed above, a higher short interest corresponds to a higher current shorting fee, which is now expected to be lower in the future due to mean-reversion, leading to lower future stock prices on average as compared to the relatively high current stock prices. We would like to highlight here that the predictive ability of short interest is solely due to the presence of shorting fee. This is because in the benchmark economy, the short interest is still time-varying (as it is driven by disagreement), but it does not predict future returns since the stock price does not depend on disagreement when the short-selling is costless. Therefore, our model shows that the current short interest is an "informative" signal for future returns for costly-to-short stocks but not for stocks that are costless to short.

In the literature, empirical investigations of whether the short interest predicts future returns or not date back to Seneca (1967), who finds that high short interest predicts lower future returns for the S&P500. Subsequent research also confirmed these findings for both at the individual stock levels and the aggregate market level. For the individual stock levels, for example, Desai, Ramesh, Thiagarajan, and Balachandran (2002) show that stocks with a high short interest experience lower returns and this predictability persists up to 12 months. Similarly, Boehmer, Huszar, and Jordan (2010) show that stocks with a low short interest in the previous month experience higher returns in the next month. Similar findings are also in Figlewski (1981), Senchack and Starks (1993), Asquith, Pathak, and Ritter (2005), and Beneish, Lee, and Nichols (2015). Among them Beneish, Lee, and Nichols (2015) show that the effect is stronger for costly-to-short stocks, which is in line with our finding that in the benchmark economy short interest is still time-varying but it does not predict future returns. For the aggregate market, Rapach, Ringgenberg, and Zhou (2016) find that a higher aggregate short interest predicts lower future stock market returns both at the monthly and at the yearly horizon, and in fact argue that the short interest is the strongest known predictor of aggregate stock market returns. On the theory side however, to our knowledge, our analysis is the first to demonstrate that a higher short interest predicts lower future stock returns for various horizons, as well as obtaining an endogenous dynamic short interest with a stationary steady-state distribution.

Moreover, several empirical studies additionally show that, in addition to the short interest (a stock variable), a positive change in short interest (a flow variable) also predicts lower future returns (Boehmer, Jones, and Zhang (2008), Diether, Lee, and Werner (2009)). Our model is also consistent with these findings since the slope coefficient in the regression of future stock returns on past changes in short interest $S_{t+h} - S_t = \alpha_{\Delta\mathcal{SI}} + \beta_{\Delta\mathcal{SI}} \left(\mathcal{SI}_t - \mathcal{SI}_{t-h}\right) + \epsilon_{t+h}$, is negative. Furthermore, Diether, Lee, and Werner (2009) also show that the current short selling is positively related to past stock performance. Our model is also consistent with this finding since the slope coefficient in the regression of the current short interest on past stock price changes $\mathcal{SI}_t = \alpha_{\Delta S} + \beta_{\Delta S} \left(S_t - S_{t-h}\right) + \epsilon_t$, is positive.[14]

In recent empirical work, Engelberg, Reed, and Ringgenberg (2018) find that stocks with higher short-selling risk, as measured by the shorting fee variance, have lower returns and less short-selling activity (volume), as measured by the number of shares shorted each day. Since our model generates a time-varying shorting fee with a stationary steady-state distribution,

[14]We omit the analysis of these two results for brevity, but they can be shown in a straightforward manner following the steps of the proof Proposition 4.

our model is well-suited to shed light into these findings.[15] We take our measure of *short-selling risk*, V_ϕ, as in Engelberg, Reed, and Ringgenberg (2018), and define it to be the variance of the shorting fee (at the steady-state), $V_\phi = \lim_{t\to\infty} \mathrm{Var}[\phi_t]$. For our measure of *short-selling activity* (or volume), $\sigma_{\mathcal{SI}}$, we consider the volatility of the short interest changes, $\sigma_{\mathcal{SI}} = \sqrt{\mathrm{Var}_t [d\mathcal{SI}_t]/dt}$, consistently with the trading volume proxies employed by works in continuous-time settings (e.g., Xiong and Yan (2010), Longstaff and Wang (2012)).[16] Proposition 5 presents the equilibrium short-selling risk (shorting fee variance) and the short-selling activity (volatility of the short interest changes) in our settings.

Proposition 5 (**Equilibrium short-selling risk and short-selling activity**). *In the costly stock short-selling economy with a time-varying disagreement, the equilibrium short-selling risk and short-selling activity are given by*

$$V_\phi = \phi_1^2 \frac{\sigma_\theta^2}{2\kappa}, \tag{23}$$

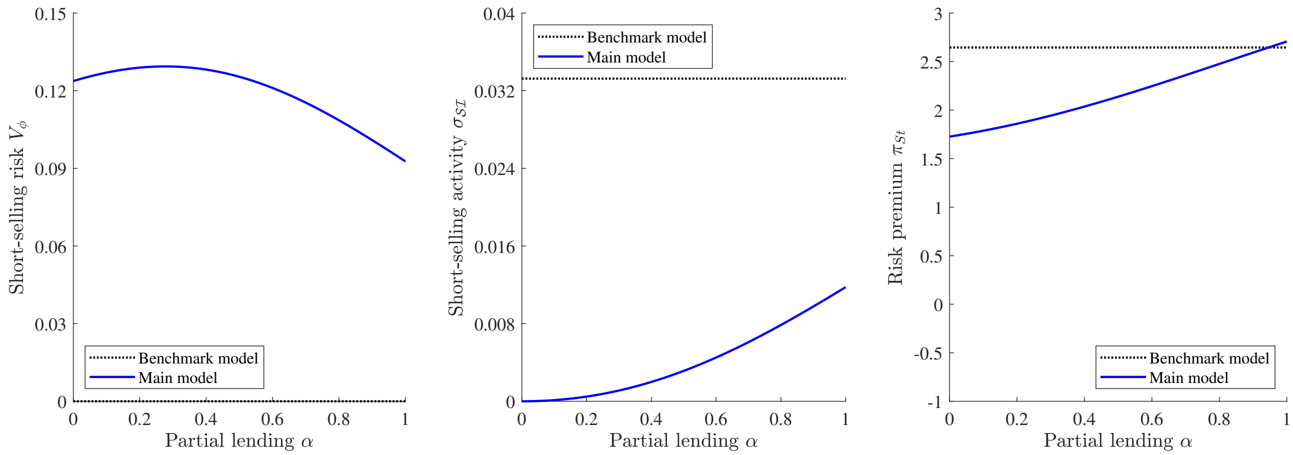
$$\sigma_{\mathcal{SI}} = \bar{\sigma}_{\mathcal{SI}} - \frac{\sigma_\theta}{2r}\left(\frac{r}{\gamma\sigma^2 Q} - \frac{(\kappa + r - 2K_{s2}\sigma_\theta^2)\,\phi_1 B + 1/r - \phi_1}{\gamma\sigma_S^2 Q}\right), \tag{24}$$

where the stock volatility σ_S is as in Proposition 3, and the constants ϕ_1, B, and K_{s2} are as in Proposition 2. The equilibrium short-selling risk and activity in the benchmark economy with costless short-selling are given by $\bar{V}_\phi = 0$ and $\bar{\sigma}_{\mathcal{SI}} = \sigma_\theta/(2\gamma\sigma^2 Q)$, respectively.

In this economy, due to the time-variation in disagreement, short-sellers' demand and the lenders' supply of stock shares in the stock lending market also fluctuate, which in turn lead to a time-variation in the shorting fee, resulting in the presence of short-selling risk (a non-zero shorting fee variance) in equilibrium. Figure 6, Panel (a), plots the equilibrium short-selling risk against the partial lending α, and illustrates that a higher partial lending leads to a lower short-selling risk for relatively high partial lending levels. This is because, in that region, the increased

[15] Engelberg, Reed, and Ringgenberg (2018) also find that stocks with higher short-selling risk have less price efficiency, where their price efficiency measure is computed by regressing the weekly stock returns on the current and the lagged (value-weighted) market returns, which is something we cannot capture meaningfully in our framework.

[16] As is well recognized in the case of trading volume, employing the standard definition of short-selling volume, by taking the absolute value of the short interest changes, $|d\mathcal{SI}_t|$, in a continuous-time setting is problematic since the local variation of the driving uncertainty (Brownian motion) is unbounded. The measure $\sigma_{\mathcal{SI}}$ does not suffer from this issue and captures the unexpected short-selling volume by not taking into account of the expected changes in the short interest.

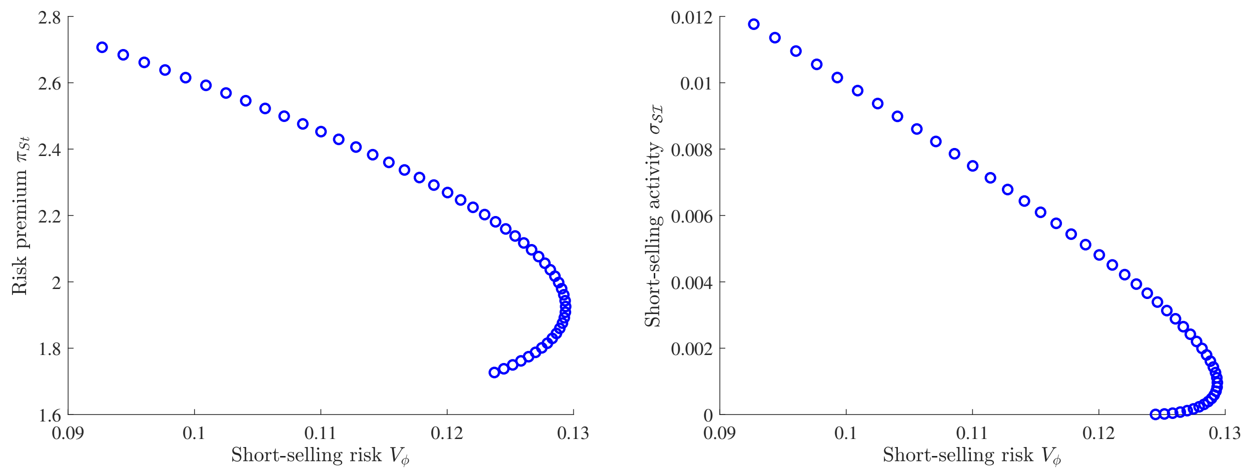


(a) Short-selling risk (b) Short-selling activity (c) Risk premium

Figure 6: **Effects of partial lending.** These panels plot the equilibrium short-selling risk V_ϕ, short-selling activity $\sigma_{S\mathcal{I}}$, and risk premium π_{St} against the partial lending α. The solid lines correspond to our main economy with costly short-selling, while the dotted lines correspond to the benchmark economy with costless short-selling. The parameter values follow from Table 1 of Appendix B.

lending supply, due to an increase in partial lending, now can absorb the short-selling demand without affecting the shorting fee as much as before. As the shorting fee is less sensitive to the disagreement shocks (lower ϕ_1 in (11)), its long-run variance is also lower. On the other hand, Figure 6, Panel (b), plots the equilibrium short-selling activity against the partial lending, and illustrates that a higher partial lending leads to a higher short-selling activity. This is because, an increase in partial lending also means that short-sellers short more for any given disagreement level, hence making the short interest more sensitive to the disagreement shocks and resulting in a higher short-selling activity. Finally, Figure 6 Panel (c) plots the equilibrium risk premium against the partial lending and illustrates that a higher partial lending also leads to a higher risk premium. This is because as partial lending increases, the current stock price decreases (Proposition 2), and therefore the stock receives higher subsequent shocks on average relative to its current price, which in turn leads to a higher risk premium.

Having illustrated the effects of the partial lending on the short-selling risk, short-selling activity, and stock risk premium, we here argue that the differences in stock partial lending could very well be behind the evidence in Engelberg, Reed, and Ringgenberg (2018). To illustrate this, Figure 7 presents scatter plots of the equilibrium risk premium (Panel (a)) and the short-



(a) Risk premium (b) Short-selling activity

Figure 7: **Effects of short-selling risk.** These panels scatter plot the equilibrium stock risk premium π_{St} and short-selling activity $\sigma_{S\mathcal{I}}$ against the short-selling risk V_ϕ with varying levels of partial lending α. The parameter values follow from Table 1 of Appendix B.

selling activity (Panel (b)) against the short-selling risk, where each point is obtained by varying the partial lending. We see that when the stock has higher short-selling risk, it tends to have lower risk premium and lower short-selling activity, as documented in Engelberg, Reed, and Ringgenberg (2018).[17]

The mainly negative risk premium short-selling risk relation occurs because, a relatively high partial lending leads to both a lower short-selling risk and a higher risk premium (as discussed above). The intuition for this, again, is that when partial lending is relatively high, both the shorting fee becomes less sensitive to the disagreement shocks leading to lower short-selling risk, and at same time the current stock price decreases (Proposition 2). Therefore, the stock receives higher subsequent shocks on average relative to its current price, which in turn leads to a higher risk premium. Similarly, the mainly negative short-selling activity short-selling risk relation occurs because, a relatively high partial lending leads to both a lower short-selling risk and a higher short-selling activity (as discussed above). Therefore, when partial lending is relatively high, in addition to the short-selling risk decreasing, short-selling activity increases since short-sellers short more for any given disagreement level. Moreover, as we discuss in our

[17]We also find a similar result if instead of short interest volatility (a flow measure) we use the short-interest itself (a stock measure) for the short-selling activity. However, to be consistent with Engelberg, Reed, and Ringgenberg (2018), who employ a flow measure, we demonstrate our results using the short interest volatility.

determination of parameter values in Appendix B, the empirical evidence in Aggarwal, Saffi, and Sturgess (2015) suggests a sufficiently high level, 43%, as a proxy for the partial lending for a typical stock in special, which implies that for empirically plausible values, the negative relation regions in Figure 7 are more likely to arise. In sum, our model offers a plausible explanation for the evidence in Engelberg, Reed, and Ringgenberg (2018) by demonstrating that the differences in partial lending across stocks may very well be behind their findings.[18]

4 Benefits of Lending and Lenders' Optimal Size

As clearly evident from our analysis so far and the related empirical evidence, the lenders' size and partial lending are important driving factors in the stock short-selling and lending market. In this Section, we first investigate the benefits of being a lender and hence a participant in the stock lending market in equilibrium as compared to just holding the stock, and illustrate that those benefits decrease in the lenders' size but are non-monotonically related to partial lending. We then endogenously determine the optimal size of lenders in our economies given a cost of setting up a lending facility, and show that the optimal size decreases in the cost of entry, and is non-monotonic, first increasing then decreasing, in partial lending, while the equilibrium stock price and shorting fee increase in the cost of entry.

4.1 Benefits of Lending

Absent any other frictions, we can show that among optimistic investors, stock lenders have a higher time-0 indirect utility than the non-lending holders do. This is simply because the lenders partially lend their long stock positions and claim shorting fees, while non-lenders do not. In this Subsection, we study the benefits of such participation in the stock lending market in equilibrium.

To quantify such benefits of lending, we consider a *benefit measure* η that represents the fraction of initial wealth a lender would be willing to give up in equilibrium to become indifferent to just being a holder of the stock and not participate in the stock lending market. That is, η

[18]We would like to highlight here that our numerical analysis shows that the changes in short-selling risk due to variations in other quantities such as κ, μ_θ, σ_θ, σ, and λ in our model would not generate the observed relation between the short-selling risk and the risk premium and the short-selling activity simultaneously.

solves

$$J_\ell\left((1-\eta)\,W_0,\theta_0,0\right) = J_h\left(W_0,\theta_0,0\right), \tag{25}$$

where $J_i(\cdot)$ is the indirect utility of i-type investor, $i=\ell,h$, defined at time t as $J_i\left(W_{it},\theta_t,t\right) = \max_{(c_i,\psi_i)} \mathrm{E}_{it}\left[\int_t^\infty e^{-\rho u}\frac{e^{-\gamma c_{iu}}}{-\gamma}du\right]$. Proposition 6 presents the equilibrium benefits of lending with both constant and time-varying disagreements.

Proposition 6 (**Benefits of lending**)**.** *In the costly stock short-selling economy with a constant disagreement, the equilibrium benefits of lending are given by*

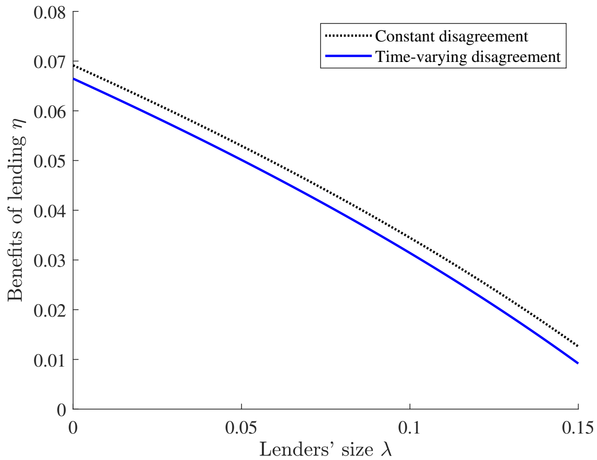
$$\begin{aligned}\eta = {} & \frac{1}{\gamma r W_0}\frac{-\left(1/\alpha-1/2+\lambda\alpha\right)\left(1/2+\lambda\alpha\right)\gamma^2\sigma^2Q^2+2\left(1-\alpha\right)\left(1/(4\alpha)+\lambda^2\alpha\right)\gamma Q\theta}{2r(1/(4\alpha)-\lambda^2\alpha-\lambda+\lambda\alpha)^2} \\ & +\frac{1}{\gamma r W_0}\frac{\left(1/2+\lambda\alpha-2\lambda\right)\left(1/2-\lambda\alpha\right)\sigma^{-2}\theta^2}{2r(1/(4\alpha)-\lambda^2\alpha-\lambda+\lambda\alpha)^2},\end{aligned} \tag{26}$$

and with a time-varying disagreement by

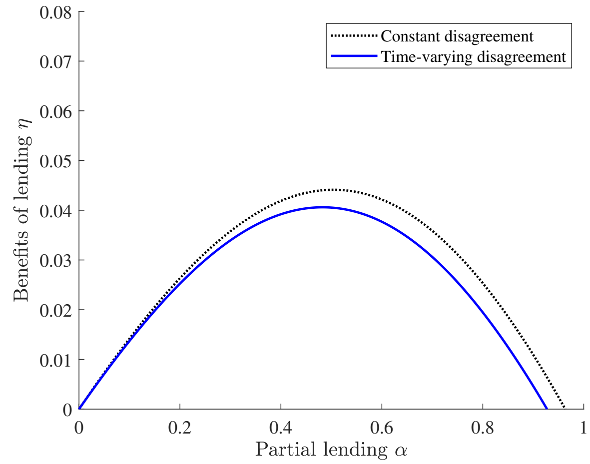
$$\eta = \frac{1}{\gamma r W_0}\left[\left(K_{h0}+K_{h1}\theta_0+K_{h2}\theta_0^2\right)-\left(K_{\ell 0}+K_{\ell 1}\theta_0+K_{\ell 2}\theta_0^2\right)\right], \tag{27}$$

where the constants K_{i0}, K_{i1}, and K_{i2}, $i=\ell,h$, are as in Proposition 2.

Proposition 6 gives the equilibrium benefits of lending as a fraction of initial wealth in closed-form for the constant disagreement economy, and in analytical form for the time-varying disagreement economy. Figure 8, Panel (a), reveals the negative relation between the benefits of lending and the lenders' size in both these economies. This is because when more investors become lenders, the lending supply increases, and so the shorting fee decreases. Consequently, a lender expects a lower future lending income, and hence would need to give up less of her wealth to become indifferent to being a non-lender. On the other hand, Figure 8, Panel (b), reveals that the benefits of lending is increasing (decreasing) for relatively low (high) levels of partial lending. This non-monotonic relation arises due to two opposing effects of partial lending. First, an increase in partial lending means a lender can earn more additional income for each share held long, leading to higher benefits of lending. Second, an increase in partial lending leads to a lower shorting fee, and hence to a lower additional income for each successfully lent share, leading to lower benefits of lending. The second effect dominates the first one only when the partial lending is sufficiently high, since in this case the shorting fee and hence the lending income is too low, generating the downward sloping relation depicted in Figure 8, Panel (b).



(a) Effects of lenders' size



(b) Effects of partial lending

Figure 8: **Benefits of lending.** These panels plot the equilibrium benefits of lending η against the lenders' size λ and partial lending α. The solid lines correspond to our main economy with time-varying disagreement and risky shorting fee, while the dotted lines correspond to the economy with constant disagreement and riskless shorting fee. The parameter values follow from Table 1 of Appendix B.

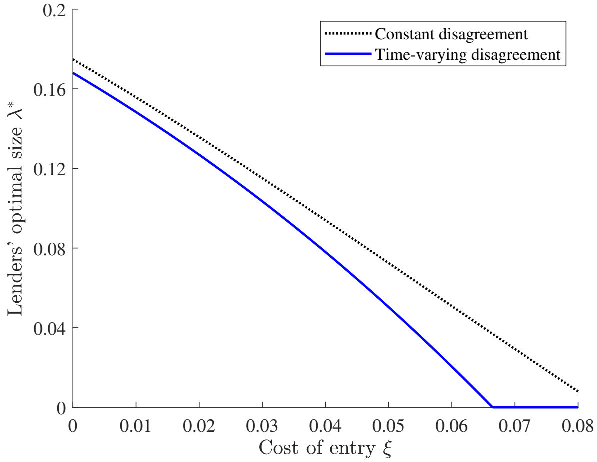
4.2 Lenders' Optimal Size

In this Subsection, we determine endogenously the optimal size of stock lenders in the economy, given a cost of entry into the lending market. In particular, we introduce an *entry cost*, ξ, of setting up a lending facility, which is expressed as a fraction of investors' initial wealth W_0. The optimists then decide whether to pay the cost and become lenders, or not pay the cost and remain as non-lenders.[19] In equilibrium, the lenders' optimal size λ^* is determined endogenously such that the time-0 indirect utility of both types of optimistic investors are equated. That is, the lenders' optimal size λ^* solves

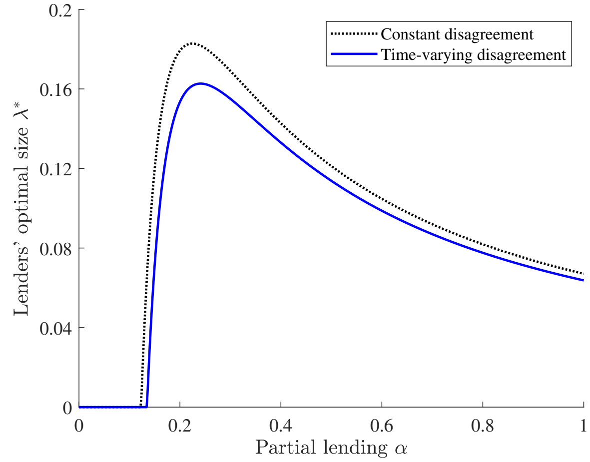
$$J_\ell\left((1-\xi)\, W_0, \theta_0, 0; \lambda^*\right) = J_h\left(W_0, \theta_0, 0; \lambda^*\right). \tag{28}$$

Proposition 7 presents the lenders' optimal sizes with both constant and time-varying disagreements.

[19]We note that even though the cost of entry of this Subsection and the benefits of lending of Subsection 4.1 are defined similarly, since they have different economic roles and interpretations we use distinct notation for each of them.



(a) Effects of cost of entry



(b) Effects of partial lending

Figure 9: **Lenders' optimal size in equilibrium.** These panels plot the lenders' optimal size λ^* against the cost of entry ξ and partial lending α. The solid lines correspond to our main economy with time-varying disagreement and risky shorting fee, while the dotted lines correspond to the economy with constant disagreement and riskless shorting fee. The parameter values follow from Table 1 of Appendix B.

Proposition 7 (**Lenders' optimal size in equilibrium**). *In the costly stock short-selling economy with a constant disagreement and the cost of entry ξ, the lenders' optimal size λ^* solves*

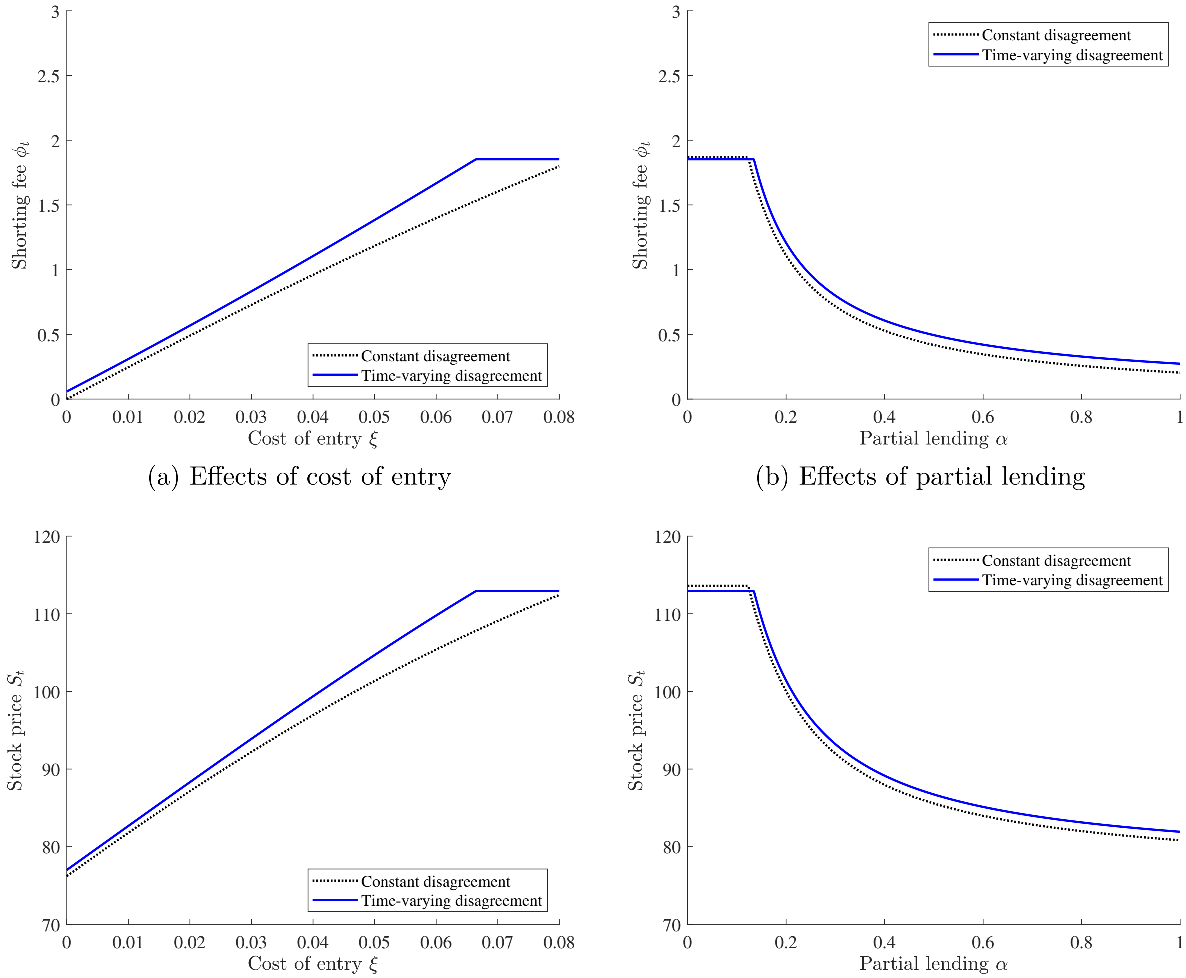
$$
\begin{aligned}
&-(1/\alpha - 1/2 + \lambda^*\alpha)(1/2 + \lambda^*\alpha)\gamma^2\sigma^2 Q^2 + 2(1-\alpha)\left(1/(4\alpha) + \lambda^{*2}\alpha\right)\gamma Q\theta \\
&+ (1/2 + \lambda^*\alpha - 2\lambda^*)(1/2 - \lambda^*\alpha)\sigma^{-2}\theta^2 - 2\xi\gamma r^2 W_0 (1/(4\alpha) - \lambda^{*2}\alpha - \lambda^* + \lambda^*\alpha)^2 = 0, \qquad (29)
\end{aligned}
$$

and with a time-varying disagreement and the cost of entry ξ, the lenders' optimal size λ^* solves

$$
\left(K_{h0} + K_{h1}\theta_0 + K_{h2}\theta_0^2\right) - \left(K_{\ell 0} + K_{\ell 1}\theta_0 + K_{\ell 2}\theta_0^2\right) - \xi\gamma r W_0 = 0, \qquad (30)
$$

where the constants K_{i0}, K_{i1}, K_{i2}, $i = \ell, h$, are functions of λ^* and are as in Proposition 2.

Proposition 7 gives the analytical expressions that the lenders' optimal size solves for both the constant and time-varying disagreement economies. We illustrate in Figure 9, Panel (a), that the lenders' optimal size decreases in the cost of entry and when the cost is too high no



(a) Effects of cost of entry (b) Effects of partial lending

(c) Effects of cost of entry (d) Effects of partial lending

Figure 10: **Effects of cost of entry and partial lending on shorting fee and stock price.** These panels plot the equilibrium shorting fee ϕ_t and stock price S_t against the cost of entry ξ and partial lending α under the optimal lenders' size. The solid lines correspond to our main economy with time-varying disagreement and risky shorting fee, while the dotted lines correspond to the economy with constant disagreement and riskless shorting fee. The parameter values follow from Table 1 of Appendix B.

investor becomes a lender. This is fairly intuitive since an optimistic investor is reluctant to become a lender when it is more costly to do so. On the other hand, from Figure 9, Panel (b), we see that in the presence of entry costs, very low levels of partial lending corresponds to no lenders, since the low potential future income from lending does not outweigh the entry cost

ξ. As the lenders can lend a higher fraction of each share held long, the future income from lending increases and so does the optimal size of lenders. However, for sufficiently high levels of partial lending, the optimal lenders' size is decreasing in partial lending, since an increase in partial lending now leads to too low of a shorting fee and a lower future lending income.

We now turn to investigating the effects of cost of entry and partial lending on the equilibrium shorting fee and stock price when the stock lenders' size is endogenously determined as above. Figure 10, Panels (a) and (c), show that the shorting fee and the stock price are both increasing in cost of entry. This is because a higher cost of entry leads to a lower lenders' optimal size, which leads to a higher shorting fee and a higher stock price for the reasons as discussed in Propositions 1 and 2. Similarly, Figure 10, Panels (b) and (d), show that the shorting fee and the stock price are both decreasing in partial lending. These results are a bit more subtle and due to two effects. First effect is due to the fact that a higher partial lending itself leads to a lower shorting fee and lower stock price (Propositions 1 and 2). Second effect is due to the impact of partial lending on lenders' optimal size. As discussed above, for low levels of partial lending, lenders' size increases in partial lending, which reinforces the first effect (Propositions 1 and 2), and hence leading to a steep decrease in the shorting fee and the stock price. For sufficiently high levels of partial lending, lenders' size decreases in partial lending, which partially counters the first effect, leading to a less steep decrease in the shorting fee and the stock price.

In the existing theoretical literature, in a dynamic setting Duffie, Gârleanu, and Pedersen (2002) endogenize the short-selling capital by introducing a fixed cost of entry to being a short-seller, whereas in a static setting Banerjee and Graveline (2014) endogenize the fraction of lenders' long stock position that they can lend out by introducing a convex cost function. Our findings in this section complement these works primarily by identifying the economic determinants of the benefits of being a stock lender, as well as endogenizing the lenders' size by introducing a fixed cost of entry to being a lender, and show how this cost of entry along with partial lending affect the equilibrium stock price and shorting fee.

5 Conclusion

In this paper, we provide a comprehensive analysis of the costly stock short-selling and lending market within a familiar dynamic asset pricing framework. Our model generates rich implications that support the extensive empirical evidence on the behavior of the shorting fee, stock price, its risk premium and volatility, short interest and its predictive power, and short-selling risk and activity, and also offer simple straightforward intuitions for them. Moreover, we provide an analysis on the economic determinants of the benefits of being a stock lender, as well as endogenously determining the optimal size of lenders.

We find that the equilibrium stock price increases in the shorting fee, which in turn increases in belief disagreement. We also find that both the stock price and shorting fee decrease in the lenders' size and partial lending. We additionally show that the stock risk premium decreases in the shorting fee, while the stock volatility is increased in the presence of costly short-selling. More notably, we show that the equilibrium short interest increases in the shorting fee and predicts future stock returns negatively. Furthermore, we demonstrate that higher short-selling risk can be associated with lower stock returns and less short-selling activity. These implications of our model are all consistent with empirical evidence, and in particular, to the best of our knowledge, our results on the predictive power of the short interest, as well as on the stock volatility and the short-selling risk, are all new and have not been demonstrated in the extant theoretical literature.

So as to not unnecessarily complicate our model, we do not consider some specific features of the actual stock short-selling and lending practices. For instance, in our model 100% of the short-selling proceeds are kept as collateral (see proof of Lemma 1 in Appendix A). This rate is very close that in the US for domestic stocks, for which lenders typically require 102% of the short-selling proceeds as a collateral to help protect themselves (D'Avolio (2002)). Moreover, in our model, lenders get all of the shorting fee upon lending a share. In reality, this is true for large institutions with internal lending facilities, which directly lend to short-sellers. Other lenders typically use an agent bank/brokerage and receive only a fraction of the shorting fee, with the rest paid to the agent bank/brokerage for providing the lending service. Our framework would be able to accommodate these generalizations. Furthermore, in Section 4, we only consider the benefits of lending in collecting the shorting fees, and not the potential costs of lending in the form of lenders' forgoing their voting rights (D'Avolio (2002)). While it may be challenging to consider such a setup with a dynamic trade-off between the benefits and costs of lending, it may generate significant implications for policymaking that is concerned with regulating these markets. Relatedly, our framework may also accommodate other regulatory interventions such as the short-selling bans and their welfare consequences. We leave these considerations for future research.

Appendix A: Proofs

Proof of Lemma 1. The financial wealth dynamics of each i-type investor, $i = \ell, h, s$, are obtained as follows. Since the holders, $i = h$, do not participate in the stock short-selling and lending market, their stock position yields the usual capital gains/losses and dividends, and hence the evolution of their financial wealth is the familiar

$$dW_{ht} = (W_{ht} - \psi_{ht} S_t)\, rdt + \psi_{ht}\, (dS_t + D_t dt) - c_{ht} dt. \tag{A.1}$$

The stock position of the lenders, $i = \ell$, on the other hand, in addition to the capital gains/losses and dividends, yields an additional lending income of $\alpha\phi_t$ per stock share held, and hence the evolution of their financial wealth becomes

$$dW_{\ell t} = (W_{\ell t} - \psi_{\ell t} S_t)\, rdt + \psi_{\ell t}\, (dS_t + D_t dt) + \psi_{\ell t} \alpha \phi_t dt - c_{\ell t} dt.$$

Finally, the stock position of the short-sellers, $i = s$, in addition to the capital gains/losses and dividends, effectively incurs an additional short-selling cost of ϕ_t per share, and hence the evolution of their financial wealth becomes

$$dW_{st} = (W_{st} - \psi_{st} S_t)\, rdt + \psi_{st}\, (dS_t + D_t dt) + \psi_{st} \phi_t dt - c_{st} dt. \tag{A.2}$$

This can be seen from the mechanics of the standard stock short-selling market practices as follows. Since in equilibrium, these investors are short-sellers, $\psi_{st} < 0$, their time-t wealth is given by $W_{st} = F_t + \psi_{st} S_t + M_t$, where F_t is the amount invested in the riskless asset, and M_t is the total amount of short-selling proceeds that are collateralized, and hence cannot be invested in other securities, and is given by $M_t = -\psi_{st} S_t$. The collateral earns the interest rate rM_t, which is shared between the short-seller and the lender. The lender's account earns the shorting fee, $-\phi_t \psi_{st} > 0$, while the short-seller's account earns the remainder rebate amount, $rM_t + \phi_t \psi_{st}$, which implies the short-selling cost dynamics for the s-type investor as $dM_t = (rM_t + \phi_t \psi_{st}) dt = -\psi_{st}\, (rS_t - \phi_t)\, dt$ in the evolution of her financial wealth

$$dW_{st} = dF_t + \psi_{st}\, (dS_t + D_t dt) + dM_t - c_{st} dt.$$

Substituting $dF_t = W_{st} rdt$ along with dM_t and rearranging yields (A.2).

Next, we note that the optimistic investors agreeing on the dividend levels, and hence on its changes, yields the consistency relation $dD_t = \mu dt + \sigma d\omega_t = (\mu + \theta)\, dt + \sigma d\omega_{it}$, for $i = \ell, h$, which yields the relation between the perceived and objective Brownian motions as stated in the main text, $\omega_{it} = \omega_t - \theta t/\sigma$ for $i = \ell, h$. A similar consideration yields the relation $\omega_{st} = \omega_t + \theta t/\sigma$ for the short-sellers. Substituting the posited stock price dynamics (2) into (A.1)–(A.2), and using the relations for the perceived and the objective Brownian motions above leads to (3). □

Proof of Proposition 1. To determine the equilibrium stock price in the costly short-selling economy with a constant disagreement, we first conjecture and later verify that the equilibrium shorting fee ϕ is constant, and the stock price takes the simple linear form

$$S_t = \frac{1}{r} D_t + A, \tag{A.3}$$

for some constant A, which, along with ϕ, are to be determined endogenously in equilibrium. Using the stock price form (A.3) and the posited stock price dynamics (2), we obtain the mean and volatility of the stock price (changes) as $\mu_{St} = \mu/r + D_t$ and $\sigma_S = \sigma/r$, respectively, which in turn implies each i-type investor's subjective risk premium to be a constant given by

$$\pi_i = \begin{cases} \frac{\mu+\theta}{r} - rA + \alpha\phi & \text{for } i = \ell, \\ \frac{\mu+\theta}{r} - rA & \text{for } i = h, \\ \frac{\mu-\theta}{r} - rA + \phi & \text{for } i = s. \end{cases} \tag{A.4}$$

We next take advantage of the dynamic market completeness in this economy and employ standard martingale methods (Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989)) to restate each i-type investor's, $i = \ell, h, s$, optimization problem as

$$\max_{c_i} \mathrm{E}_i^* \left[\int_0^\infty e^{-\rho t} Z_{it}^{-1} \frac{e^{-\gamma c_{it}}}{-\gamma} dt \right], \qquad \text{s.t.} \qquad \mathrm{E}_i^* \left[\int_0^\infty e^{-rt} c_{it} dt \right] = W_0, \tag{A.5}$$

where Z_{it} is the likelihood ratio between the i-type investor's subjective and risk-neutral measures, and due to the wealth dynamics in (3) is given by

$$Z_{it} = \frac{d\mathbb{P}_i^*}{d\mathbb{P}_i} = e^{-\int_0^t \frac{\pi_i}{\sigma_S} d\omega_{iu} - \frac{1}{2}\int_0^t \left(\frac{\pi_i}{\sigma_S}\right)^2 du} = e^{-\frac{\pi_i}{\sigma_S}\omega_{it} - \frac{1}{2}\left(\frac{\pi_i}{\sigma_S}\right)^2 t},$$

and E_i^* denotes the expectation taken under i-type investor's risk-neutral measure $\mathbb{P}_i^*$. The first-order condition of each i-type investor's, $i = \ell, h, s$, optimization problem (A.5) gives her optimal consumption as

$$c_{it} = \frac{1}{\gamma}\left[-\ln y_i - (\rho - r)\, t + \frac{\pi_i}{\sigma_S}\omega_{it}^* - \frac{1}{2}\left(\frac{\pi_i}{\sigma_S}\right)^2 t\right], \tag{A.6}$$

where y_i is the corresponding Lagrangian multiplier and $\omega_{it}^* = \omega_{it} + (\pi_i/\sigma_S)t$ for $i = \ell, h, s$.

We then derive the optimal wealth of each i-type investor from the relation

$$W_{it} = \mathrm{E}_{it}^*\left[\int_t^\infty e^{-r(u-t)} c_{iu} du\right]. \tag{A.7}$$

Taking the expectation of the optimal consumption (A.6), along with the fact that $\mathrm{E}_{it}^*\left[\omega_{iu}^*\right] = \omega_{it}^*$, yields

$$\mathrm{E}_{it}^*\left[c_{iu}\right] = c_{it} - \frac{1}{\gamma}(\rho - r)(u - t) - \frac{1}{2\gamma}\left(\frac{\pi_i}{\sigma_S}\right)^2 (u - t),$$

which after substituting into (A.7) and taking the integration gives

$$W_{it} = \frac{1}{r}c_{it} - \frac{1}{\gamma r^2}(\rho - r) - \frac{1}{2\gamma r^2}\left(\frac{\pi_i}{\sigma_S}\right)^2. \tag{A.8}$$

We determine each i-type investor's optimal portfolio strategy ψ_{it} by matching the volatility of her wealth dynamics in (3) with the corresponding dynamics obtained using (A.6) and (A.8)

$$dW_{it} = \frac{1}{r}dc_{it} = \ldots dt + \frac{1}{\gamma r}\frac{\pi_i}{\sigma_S}d\omega_{it}^*,$$

leading to the optimal portfolios

$$\psi_i = \frac{1}{\gamma r}\frac{\pi_i}{\sigma_S^2}. \tag{A.9}$$

Finally, to verify our conjectures, we use the portfolios (A.9) and impose the stock market clearing condition, $\lambda\psi_{\ell t} + (\frac{1}{2} - \lambda)\psi_{ht} + \frac{1}{2}\psi_{st} = Q$, to obtain

$$\lambda\frac{1}{\gamma r}\frac{\pi_\ell}{\sigma_S^2} + (\frac{1}{2} - \lambda)\frac{1}{\gamma r}\frac{\pi_h}{\sigma_S^2} + \frac{1}{2}\frac{1}{\gamma r}\frac{\pi_s}{\sigma_S^2} = Q,$$

which along with (A.4) gives the constant A and verifies the equilibrium stock price as in (7). We finally impose the stock short-selling and lending market clearing condition, $\lambda\alpha\psi_{\ell t}\phi_t + \frac{1}{2}\psi_{st}\phi_t = 0$

to obtain the equilibrium shorting fee as in (6). In the benchmark economy with costless short-selling, the stock price $\bar{S}_t$ is simply obtained by setting the shorting fee $\phi = 0$ in (7).

Property (i), which states that the shorting fee is increasing in disagreement, follows from the fact that

$$\frac{\partial \phi}{\partial \theta} = \frac{1}{r} \frac{1/2 - \lambda\alpha}{(1/2 + \lambda\alpha)(1/2 - \lambda\alpha) - \lambda\alpha\,(1 - \alpha)} > 0.$$

Property (i) also states that the shorting fee is decreasing in lenders' size, which follows from the fact that

$$\frac{\partial \phi}{\partial \lambda} = -\frac{\alpha}{2r} \frac{\theta - (2\,(1/2 + \lambda\alpha)^2 + 1 - \alpha)\gamma\sigma^2 Q/(1 - \alpha - 2\,(1/2 - \lambda\alpha)^2)}{((1/2 + \lambda\alpha)(1/2 - \lambda\alpha) - \lambda\alpha\,(1 - \alpha))^2} < 0,$$

which is negative since the numerator is positive given (5). Property (i) that shorting fee is decreasing in partial lending follows from the fact that

$$\frac{\partial \phi}{\partial \alpha} = -2\lambda \frac{[\alpha\,(1 - \lambda - \alpha\lambda) - (1/4 - \lambda^2\alpha^2)]\,\theta + [(1 - \alpha)\,(1 + \lambda\alpha) - (1/4 - \lambda^2\alpha^2)]\,\gamma\sigma^2 Q}{((1/2 + \lambda\alpha)(1/2 - \lambda\alpha) - \lambda\alpha\,(1 - \alpha))^2} < 0,$$

which is negative since the numerator is positive given (5). Property (ii), which states that the stock price is increasing in shorting fee, follows from the fact that $\partial S_t/\partial\phi = (1/2 + \alpha\lambda)/r > 0$. Property (ii) also states that the stock price is decreasing in lenders' size, which follows from the fact that

$$\frac{\partial S_t}{\partial \lambda} = -\frac{\alpha}{r^2} \frac{-(1 - \alpha)\,(1/4 + \lambda^2\alpha^2)\,\theta + (1 - \alpha\,(1/2 - \lambda\alpha))\,(1/2 + \lambda\alpha)\,\gamma\sigma^2 Q}{((1/2 + \lambda\alpha)(1/2 - \lambda\alpha) - \lambda\alpha\,(1 - \alpha))^2} < 0,$$

which is negative since the numerator is positive given (5). Property (ii) that stock price is decreasing in partial lending follows from the fact that

$$\frac{\partial S_t}{\partial \alpha} = -\frac{\lambda}{r} \frac{(-2\lambda^2\alpha^2 + \alpha - 1/2)\,\theta + 2\,(1 - \alpha)\,(1/2 + \lambda\alpha)\,\gamma\sigma^2 Q}{((1/2 + \lambda\alpha)(1/2 - \lambda\alpha) - \lambda\alpha\,(1 - \alpha))} < 0,$$

which is negative since the numerator is positive given (5). □

Further discussion on disagreement process (8)**.** In this note, we demonstrate how the dynamics for the belief disagreement (8) could arise in an economy with Bayesian investors, who are symmetrically informed but have different interpretations of signals, as in the dynamic differences in beliefs models with stationary disagreement (e.g., Scheinkman and Xiong (2003), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2010)). In these models, the quantity the investors disagree on (typically, the mean of a fundamental process) is a mean-reverting unobservable process, and investors observe signals but have different prior knowledge on the informativeness of signals, which leads to investors to react differently to the signals and therefore to have different posterior beliefs, and hence disagree.

To illustrate this disagreement in the presence of learning in our setting, we adopt the Xiong and Yan (2010) approach, and assume that the mean dividend change μ is now an unobservable process with dynamics $d\mu_t = \kappa_\mu (\bar{\mu} - \mu_t)\, dt + \sigma_\mu d\omega_{\mu t}$, where the constants κ_μ, $\bar{\mu}$, and σ_μ are the speed of mean reversion, long-run mean, and volatility of the mean dividend change process, respectively, and ω_μ is a standard Brownian motion, independent from ω. All investors observe the signal $dN_t = d\omega_{\theta t}$, where ω_θ is a standard Brownian motion independent of all other Brownian motions. Even though the signal N is "pure noise" and is not informative, each i-type investor, $i = \ell, h, s$, believes it contains useful information by having different prior knowledge of it in the form of $dN_t = \varphi_i d\omega_{\mu t} + \sqrt{1-\varphi_i^2} d\omega_{\theta t}$, where the parameter φ_i captures the each i-type investor's perceived correlation between the signal dN_t and $d\omega_{\mu t}$. As in Xiong and Yan (2010), we also assume that each i-type investor misperceives the volatility of μ as $k_i \sigma_\mu$ rather than σ_μ, which also helps us to isolate the effects of belief disagreement, as shown below.

In this setting, each i-type investor, $i = \ell, h, s$, estimates μ from the observations of the dividend and the signal, and hence their time-t information is the filtration $\mathcal{G}_t = \sigma \{D_s, N_s : 0 \leq s \leq t\}$. Assuming investors view the prior distribution of μ as normal with mean m_o and variance V_o, the application of the standard Kalman filtering (e.g., Liptser and Shiryaev (2001)) yields the posterior mean $m_{it} = \mathbb{E}[\mu_t | \mathcal{G}_t]$ and the posterior variance $V_{it} = \mathbb{E}\left[(\mu_t - m_{it})^2 | \mathcal{G}_t\right]$ as

$$
\begin{aligned}
dm_{it} &= \kappa_\mu (\overline{\mu} - m_{it})\, dt + \frac{1}{\sigma} V_t d\widehat{\omega}_{it} + \varphi_i k_i \sigma_\mu dN_t, \\
dV_{it} &= -\left[\frac{1}{\sigma^2} V_t^2 + 2\kappa_\mu V_t - \left(1 - \varphi_i^2\right) k_i^2 \sigma_\mu^2\right] dt,
\end{aligned}
$$

where each i-type investor's perceived Brownian motion is given by $d\widehat{\omega}_{it} = \frac{1}{\sigma}(dD_t - m_{it} dt)$. By choosing the constants $\varphi_\ell = \varphi_h = -\varphi_s = \varphi > 0$, and $k_\ell = k_h = k_s = 1/\sqrt{1-\varphi^2}$, similarly to Xiong and Yan (2010), we shut down the channels due to investors' overconfidence on signal precision and isolate the effects of disagreement. We also note that with this specification the ℓ-type and the h-type investors have identical beliefs, and more importantly the posterior variance of all investors become identical, and is equal to $\overline{V}_\ell = \overline{V}_h = \overline{V}_s = \overline{V} = \sigma^2\sqrt{\kappa_\mu^2 + \sigma_\mu^2/\sigma^2} - \sigma^2\kappa_\mu$, at the steady-state, which we base our analysis. Finally, by defining the belief disagreement simply as the difference between the ℓ-type and the h-type investors' posterior mean, $\theta_t = m_{\ell t} - m_{st}$, (as in our main model also), we obtain the disagreement dynamics, similarly to Xiong and Yan (2010), as

$$d\theta_t = \kappa\,(0 - \theta_t)\,dt + \sigma_\theta d\omega_{\theta t}, \tag{A.10}$$

where the constants are given by $\kappa = \kappa_\mu + \overline{V}/\sigma^2$ and $\sigma_\theta = 2\varphi\sigma_\mu/\sqrt{1-\varphi^2}$.

Hence, we see that the main features of our disagreement process (8) also arise in this Bayesian learning environment, namely the disagreement being a stationary mean-reverting process with independent shocks. We see that differently from (8), the disagreement process (A.10) has a zero long-run mean, whereas for generality we assume it to be non-zero in our main analysis. However, this is not crucial and our main analysis remains valid if we also take it to be zero, since the long-run mean of disagreement does not play an important role in our mechanisms and results, though it helps us ensure that the equilibrium shorting fee is positive with a high probability at the steady-state. In sum, we highlight that our model can easily accommodate this more richer learning environment with unobservable stochastic dividend mean, but it would unnecessarily complicate the analysis as it would introduce an additional state process μ, therefore, for clarity we carry out our analysis with our disagreement as specified in (8).

Proof of Proposition 2. To determine the equilibrium stock price in the costly short-selling economy with a time-varying disagreement, we first conjecture, and later verify, that the equilibrium stock price and the shorting fee take the simple linear forms of

$$S_t = \frac{1}{r}D_t + \tilde{A} + \tilde{B}\theta_t, \tag{A.11}$$

$$\phi_t = \phi_0 + \phi_1\theta_t, \tag{A.12}$$

for some constants $\tilde{A}$, $\tilde{B}$, ϕ_0, and ϕ_1 to be determined endogenously in equilibrium. Using the stock price form (A.11), the disagreement dynamics (8) and the posited stock price dynamics (9), we obtain the mean and the diffusion terms of the stock price (changes) as $\mu_{St} = \mu/r + \tilde{B}\kappa\bar{\theta} - \tilde{B}\kappa\theta_t + D_t$, $\sigma_1 = \sigma/r$, and $\sigma_2 = \tilde{B}\sigma_\theta$, respectively, which in turn implies $\sigma_S = \sqrt{\sigma_1^2 + \sigma_2^2}$, and each i-type investor's subjective risk premium in a linear form as $\pi_{it} = \pi_{i0} - \pi_{i1}\theta_t$, where

$$\pi_{i0} = \begin{cases} \frac{\mu}{r} + \tilde{B}\kappa\bar{\theta} - \tilde{A}r + \alpha\phi_0 & \text{for } i = \ell, \\ \frac{\mu}{r} + \tilde{B}\kappa\bar{\theta} - \tilde{A}r & \text{for } i = h, \\ \frac{\mu}{r} + \tilde{B}\kappa\bar{\theta} - \tilde{A}r + \phi_0 & \text{for } i = s, \end{cases} \qquad \pi_{i1} = \begin{cases} (\kappa + r)\,\tilde{B} - \frac{1}{r} - \alpha\phi_1 & \text{for } i = \ell, \\ (\kappa + r)\,\tilde{B} - \frac{1}{r} & \text{for } i = h, \\ (\kappa + r)\,\tilde{B} + \frac{1}{r} - \phi_1 & \text{for } i = s. \end{cases}$$

We next solve each i-type investor's optimization problem using the standard stochastic dynamic programming method. From the theory of stochastic control, investors' optimal consumption c_i and portfolio strategy ψ_i satisfy the Hamilton–Jacobi–Bellman equation

$$\begin{aligned} 0 = \max_{(c_i, \psi_i)} \frac{e^{-\rho t - \gamma c_{it}}}{-\gamma} + \frac{\partial J_i}{\partial t} + (W_{it} r + \psi_{it}\pi_{it} - c_{it})\,\frac{\partial J_i}{\partial W} + \frac{1}{2}\psi_{it}^2\sigma_S^2 \frac{\partial^2 J_i}{\partial W^2} \\ + \kappa\left(\bar{\theta} - \theta_t\right)\frac{\partial J_i}{\partial \theta} + \frac{1}{2}\sigma_\theta^2 \frac{\partial^2 J_i}{\partial \theta^2} + \psi_{it}\tilde{B}\sigma_\theta^2 \frac{\partial^2 J_i}{\partial W \partial \theta}, \end{aligned} \tag{A.13}$$

where $J_i\left(W_{it}, \theta_t, t\right) = \max_{(c_i, \psi_i)} \mathrm{E}_{it}\left[\int_t^\infty e^{-\rho u}\frac{e^{-\gamma c_{iu}}}{-\gamma}du\right]$ is i-type investor's indirect utility function. We proceed by taking i-type investor's indirect utility $J_i(\cdot)$ as in (17). Taking the first-order conditions of (A.13) with respect to c_i and ψ_i, and substituting the indirect utility (17) gives the optimal consumption and portfolio strategy, respectively, as

$$c_{it} = rW_{it} - \frac{\ln(\gamma r)}{\gamma} - \frac{K_{i0} + K_{i1}\theta_t + K_{i2}\theta_t^2}{\gamma}, \tag{A.14}$$

$$\psi_{it} = \frac{\pi_{it} + \tilde{B}\sigma_\theta^2\left(K_{i1} + 2K_{i2}\theta_t\right)}{\gamma r \sigma_S^2}. \tag{A.15}$$

Substituting (17), (A.14), and (A.15) into (A.13) and rearranging gives the following quadratic equation in θ_t,

$$\begin{aligned} 0 = {} & (r - \rho) - r\ln(\gamma r) - rK_{i0} - rK_{i1}\theta_t - rK_{i2}\theta_t^2 \\ & - \frac{1}{2\sigma_S^2}\left[\left(\pi_{i0} + K_{i1}\tilde{B}\sigma_\theta^2\right) + \left(2K_{i2}\tilde{B}\sigma_\theta^2 - \pi_{i1}\right)\theta_t\right]^2 \\ & + \kappa\bar{\theta}K_{i1} + \kappa\left(2\bar{\theta}K_{i2} - K_{i1}\right)\theta_t - 2\kappa K_{i2}\theta_t^2 + \frac{1}{2}\sigma_\theta^2\left[2K_{i2} + \left(K_{i1} + 2K_{i2}\theta_t\right)^2\right], \end{aligned}$$

which implies

$$0 = (r - \rho) - r\ln(\gamma r) - rK_{i0} - \frac{1}{2\sigma_S^2}\left(\pi_{i0} + K_{i1}\tilde{B}\sigma_\theta^2\right)^2 + \kappa\bar{\theta}K_{i1} + \frac{1}{2}\sigma_\theta^2\left(2K_{i2} + K_{i1}^2\right), \quad \text{(A.16)}$$

$$0 = \kappa\left(2\bar{\theta}K_{i2} - K_{i1}\right) - rK_{i1} - \frac{1}{\sigma_S^2}\left(\pi_{i0} + K_{i1}\tilde{B}\sigma_\theta^2\right)\left(2K_{i2}\tilde{B}\sigma_\theta^2 - \pi_{i1}\right) + 2\sigma_\theta^2 K_{i1}K_{i2}, \quad \text{(A.17)}$$

$$0 = 2\sigma_\theta^2 K_{i2}^2 - (r + 2\kappa)K_{i2} - \frac{1}{2\sigma_S^2}\left(2K_{i2}\tilde{B}\sigma_\theta^2 - \pi_{i1}\right)^2. \quad \text{(A.18)}$$

To determine the constants $\tilde{A}$, $\tilde{B}$, ϕ_0, and ϕ_1, and hence verify our conjectured equilibrium shorting fee and the stock price, we first impose the stock market clearing condition, $\lambda\psi_{\ell t} + (\frac{1}{2} - \lambda)\psi_{ht} + \frac{1}{2}\psi_{st} = Q$. Substituting (A.15), and rearranging yields a linear equation of θ_t. By the method of undetermined coefficients, we must have

$$\left[(2\lambda(1-\alpha)K_{\ell 2} + (1-2\lambda)K_{h2})\sigma_\theta^2 - \left(\frac{1}{2} - \lambda\alpha\right)(\kappa + r)\right]r\tilde{B} + \left(\frac{1}{2} - \lambda\alpha\right) + \lambda\alpha(1-\alpha)r\phi_1 = 0, \quad \text{(A.19)}$$

$$\frac{\left[\lambda(1-\alpha)K_{\ell 1} + \left(1/2 - \lambda\right)K_{h1}\right]\sigma_\theta^2\tilde{B} + \left(1/2 - \lambda\alpha\right)\left(\mu/r + \kappa\bar{\theta}\tilde{B} - r\tilde{A}\right) + \lambda\alpha(1-\alpha)\phi_0}{r\gamma\left(\sigma_\theta^2\tilde{B}^2 + \sigma^2/r^2\right)Q} = 1. \quad \text{(A.20)}$$

We next impose the stock short-selling and lending market clearing condition, $\lambda\alpha\psi_{\ell t}\phi_t + \frac{1}{2}\psi_{st}\phi_t = 0$. Substituting (A.15) and again using the method of undetermined coefficients leads to

$$\left[(2\lambda\alpha K_{\ell 2} + K_{s2})\sigma_\theta^2 - \left(\frac{1}{2} + \lambda\alpha\right)(\kappa + r)\right]r\tilde{B} - \left(\frac{1}{2} - \lambda\alpha\right) + \left(\frac{1}{2} + \lambda\alpha^2\right)r\phi_1 = 0, \quad \text{(A.21)}$$

$$\left(\lambda\alpha K_{\ell 1} + \frac{1}{2}K_{s1}\right)\sigma_\theta^2\tilde{B} + \left(\frac{1}{2} + \lambda\alpha\right)\left(\frac{\mu}{r} + \kappa\bar{\theta}\tilde{B} - r\tilde{A}\right) + \left(\frac{1}{2} + \lambda\alpha^2\right)\phi_0 = 0. \quad \text{(A.22)}$$

We then jointly solve the constants $K_{\ell 2}$, K_{h2}, K_{s2}, $\tilde{B}$, and ϕ_1 from the five equations (A.18) for $i = \ell, h, s$, (A.19), and (A.21). For this, we first notice (A.18) is quadratic in K_{i2}, which yields two distinct roots, one is positive and the other is negative. We obtain a unique linear equilibrium by imposing the condition that if the disagreement approaches to being constant, i.e., $\sigma_\theta \to 0$, we should obtain the same economic quantities as in our constant disagreement economy. This identification rules out the positive roots of K_{i2}. Consequently, we take the negative root of the quadratic equation (A.18) and obtain

$$K_{i2} = \frac{-a_{i1} - \sqrt{a_{i1}^2 - 4a_{i2}a_{i0}}}{2a_{i2}} < 0, \quad \text{(A.23)}$$

where $a_{i2} = 2\sigma_\theta^2\sigma^2/r^2$, $a_{i1} = 2\tilde{B}\sigma_\theta^2\pi_{i1} - (r+2\kappa)\,\sigma_S^2$, and $a_{i0} = \pi_{i1}^2/2$. The constant $\tilde{B}$ and ϕ_1 are then jointly solved from the resulting non-linear equation obtained by substituting (A.23) into (A.19) and (A.21), which implies that $\tilde{B}$, and ϕ_1 must have the same sign. Again, we impose the condition that if the disagreement approaches to being constant, i.e., $\sigma_\theta \to 0$, we should obtain the same economic quantities as in our constant disagreement economy, in which $\tilde{B}$ and ϕ_1 are both positive, so we conclude that is also the case here. Moreover, this is consistent with the special case of our model with $\alpha = 1$, in which we have a closed-from solution for $\tilde{B} = 1/r(\kappa + r) > 0$, which also guarantees $\phi_1 > 0$.

We next jointly determine the constants $K_{\ell 1}$, K_{h1}, K_{s1}, $\tilde{A}$, and ϕ_0, from the five equations (A.17) for $i = \ell, h, s$, (A.20), and (A.22). For this, we rearrange (A.17), which yields

$$K_{i1} = \frac{2\kappa\bar{\theta}K_{i2}\sigma_S^2 - \left(2K_{i2}\tilde{B}\sigma_\theta^2 - \pi_{i1}\right)\pi_{i0}}{(\kappa + r)\,\sigma_S^2 + \left(2K_{i2}\tilde{B}\sigma_\theta^2 - \pi_{i1}\right)\tilde{B}\sigma_\theta^2 - 2\sigma_\theta^2\sigma_S^2 K_{i2}}. \tag{A.24}$$

The constant $\tilde{A}$ and ϕ_0 are then backed out from the resulting non-linear equation obtained by substituting (A.24), together with the solved constants, $K_{\ell 2}$, K_{h2}, K_{s2}, $\tilde{B}$, and ϕ_1, into (A.20) and (A.22). Finally, we determine the constant K_{i0} by substituting (A.23), (A.24) into (A.16) and using the solved constants, which yields

$$K_{i0} = \frac{1}{r}\left[(r - \rho) - r\ln(\gamma r) - \frac{1}{2\sigma_S^2}\left(\pi_{i0} + K_{i1}\tilde{B}\sigma_\theta^2\right)^2 + \kappa\bar{\theta}K_{i1} + \frac{1}{2}\sigma_\theta^2\left(2K_{i2} + K_{i1}^2\right)\right].$$

Thus, we have verified our conjectured equilibrium stock price (A.11) and the shorting fee (A.12) by determining the constants $\tilde{A}$, $\tilde{B}$, ϕ_0, and ϕ_1, as well as determining the constants K_{i0}, K_{i1}, and K_{i2} in the indirect utility of each i-type investor (17), $i = \ell, h, s$. Lastly, we use the monotonic relation between the shorting fee and the disagreement (A.12), and rewrite the stock price in terms of the shorting fee as in (12) by defining $A = \tilde{A} - \phi_0\tilde{B}/\phi_1$ and $B = \tilde{B}/\phi_1$. Substituting A and B into (A.19), (A.21), (A.20), and (A.22) gives (13), (14), (15) and (16), respectively.

In the benchmark economy with costless short-selling, we solve the equilibrium stock price again by conjecturing a linear form, $\bar{S}_t = \bar{A} + \bar{B}\theta_t + D_t/r$, and taking i-type investor's indirect utility $\bar{J}_i(\cdot)$ as in (17). Following the similar steps to those in our main model with costly short-selling above, taking $\phi_0 = \phi_1 = 0$, and substituting (A.15) into the same stock market clearing condition, immediately yields $\bar{A} = \mu/r^2 - \gamma\sigma^2 Q/r^2$ and $\bar{B} = 0$, verifying our conjecture.

The constants $\bar{K}_{i0}$, $\bar{K}_{i1}$, and $\bar{K}_{i2}$ in the indirect utility function are then solved through a similar system of equations of (A.16)–(A.18) by substituting $\bar{A}$, $\bar{B}$ and setting $\phi_0 = \phi_1 = 0$. Finally, by substituting $\bar{A}$, $\bar{B}$, $\bar{K}_{i0}$, $\bar{K}_{i1}$, and $\bar{K}_{i2}$ into (A.14) and (A.15), we obtain each i-type investor's optimal consumption $\bar{c}_i$ and portfolio strategy $\bar{\psi}_i$, respectively. Specifically, we have $\psi_{it} = Q + \theta_t/\gamma\sigma^2$ for $i = \ell, h$, and $\psi_{it} = Q - \theta_t/\gamma\sigma^2$ for $i = s$.

Property (i), which states that the shorting fee is increasing in disagreement follows from the fact that $\phi_1 > 0$ as discussed above. Property (ii), which states that the stock price is increasing in shorting fee, follows from the facts that $\partial S_t/\partial\phi_t = \tilde{B}/\phi_1 > 0$ since $\tilde{B} > 0$ and $\phi_1 > 0$. □

Proof of Proposition 3. In the costly short-selling economy, the equilibrium stock risk premium is given by $\pi_{St} = \mu_{St} - rS_t$, where μ_{St} is determined from the drift term in the dynamics of the stock price (A.11)

$$\mu_{St} = \frac{\mu}{r} + \tilde{B}\kappa\left(\bar{\theta} - \theta_t\right) + D_t,$$

which in turn along with the stock price (A.11) gives the stock risk premium as

$$\pi_{St} = \frac{\mu}{r} + \tilde{B}\kappa\bar{\theta} - r\tilde{A} - (\kappa + r)\,\tilde{B}\theta_t.$$

Using the monotonic relation between the shorting fee and the disagreement (11), together with $A = \tilde{A} - \phi_0\tilde{B}/\phi_1$ and $B = \tilde{B}/\phi_1$, we obtain (18). The equilibrium stock volatility (19) is determined by $\sigma_{St} = \sqrt{\sigma_{1t}^2 + \sigma_{2t}^2}$, where the diffusion terms are obtained using the dynamics of the stock price expression (A.11), $\sigma_{1t} = \sigma/r$ and $\sigma_{2t} = \sigma_\theta\tilde{B} = \sigma_\theta\phi_1 B$.

In the costless short-selling benchmark economy, given the stock price as stated in Proposition 2, the stock mean return and volatility are given by $\bar{\mu}_{St} = \mu/r + D_t$ and $\bar{\sigma}_{St} = \sigma/r$, respectively, and hence the stock risk premium is $\bar{\pi}_S = \gamma\sigma^2 Q/r$.

Property (i), which states that the stock risk premium is decreasing in shorting fee, follows from the fact that the partial derivative of stock risk premium in (18) with respect to shorting fee, $\partial\pi_{St}/\partial\phi_t = -(\kappa + r)\,B$ is negative since $B > 0$ as discussed in the proof of Proposition 2. Property (ii), which states that the stock volatility is higher than that in the benchmark economy with costless short-selling is immediate from (19). □

Proof of Proposition 4. In the costly short-selling economy, the equilibrium short interest is given by $\mathcal{SI}_t = -\frac{1}{2}\psi_{st}/Q$ and is obtained by substituting the short-sellers' optimal portfolio strategy (A.15) as

$$\mathcal{SI}_t = -\frac{1}{2r}\frac{\left(\mu/r + \tilde{B}\kappa\bar{\theta} - \tilde{A}r + \phi_0\right) + K_{s1}\tilde{B}\sigma_\theta^2}{\gamma\sigma_S^2 Q} + \frac{1}{2r}\frac{\left(\kappa + r - 2K_{s2}\sigma_\theta^2\right)\tilde{B} + 1/r - \phi_1}{\gamma r\sigma_S^2 Q}\theta_t, \quad \text{(A.25)}$$

which after using the monotonic relation between the shorting fee and disagreement (11), along with $A = \tilde{A} - \phi_0\tilde{B}/\phi_1$ and $B = \tilde{B}/\phi_1$, and rearranging becomes (21). The slope coefficient in the predictive regression (20) is given by

$$\beta_{\mathcal{SI}} = \frac{\mathrm{Cov}\left[\mathcal{SI}_t, S_{t+h} - S_t\right]}{\mathrm{Var}\left[\mathcal{SI}_t\right]}, \quad \text{(A.26)}$$

where the covariance between the short interest and stock price changes, derived from (A.11) and (A.25), is

$$\mathrm{Cov}\left[\mathcal{S}I_t, S_{t+h} - S_t\right] = -\tilde{B}\frac{\sigma_\theta^2}{4\kappa r}\frac{\left(\kappa + r - 2K_{s2}\sigma_\theta^2\right)\tilde{B} + 1/r - \phi_1}{\gamma\sigma_S^2 Q}\left(1 - e^{-\kappa h}\right), \quad \text{(A.27)}$$

and the variance of short interest in (A.25), using $\mathrm{Var}\left[\theta_t\right] = \sigma_\theta^2/(2\kappa)$, is

$$\mathrm{Var}\left[\mathcal{SI}_t\right] = \frac{\sigma_\theta^2}{8\kappa r^2}\left(\frac{\left(\kappa + r - 2K_{s2}\sigma_\theta^2\right)\tilde{B} + 1/r - \phi_1}{\gamma r\sigma_S^2 Q}\right)^2, \quad \text{(A.28)}$$

where the constants K_{s2} and ϕ_1 are defined as in Proposition 2. Substituting (A.27) and (A.28) into (A.26) and using $B = \tilde{B}/\phi_1$ yields (22).

In the costless short-selling benchmark economy, the equilibrium short interest is obtained similarly given $\bar{\psi}_{st}$ in Proposition 2, as $\overline{\mathcal{SI}}_t = -1/2 + \theta_t/(2\gamma\sigma^2 Q)$. The slope coefficient in the predictive regression (20) is zero because the stock price does not depend on disagreement.

Property (i), which states that the short interest is increasing in shorting fee, follows from the fact that the partial derivative of short interest with respect to ϕ_t, $\partial\mathcal{SI}_t/\partial\phi_t = \left[\left(\kappa + r - 2K_{s2}\sigma_\theta^2\right)\tilde{B} + 1/r - \phi_1\right]/\left(2r^2\phi_1\gamma\sigma_S^2 Q\right)$ is positive. This is because $\left(\kappa + r - 2K_{s2}\sigma_\theta^2\right)\tilde{B} + 1/r - \phi_1 > 0$. To show this, we first substitute (A.21) to this inequality and get a necessary and sufficient condition $2K_{\ell 2}\sigma_\theta^2\tilde{B} - (\kappa + r)\tilde{B} + 1/r + \alpha\phi_1 > 0$. We then substitute the equilibrium $K_{\ell 2}$ from (A.23) into the left-hand-side of this condition and rearrange to obtain $\kappa(r + \kappa)\tilde{B}^2 + r(\alpha\phi_1 + 1/r)\tilde{B} - (\alpha\phi_1 + 1/r)^2 > 0$, which always holds since $0 < \tilde{B} <$

$1/\left(r\left(r+\kappa\right)\right)+(1-\alpha)\lambda\alpha\phi_1/((r+\kappa)(1/2-\lambda\alpha))$ as implied by (A.19) with $K_{\ell 2}<0$ and $K_{h2}<0$. Property (ii) that a higher short interest predicts lower future stock returns follows from the fact that $\beta_{\mathcal{SI}}$ is negative. This is again because except for the negative sign all the terms in (22) are positive including the denominator term $\left(\kappa+r-2K_{s2}\sigma_\theta^2\right)\tilde{B}+1/r-\phi_1$, as shown above. $\square$

Proof of Proposition 5. In the costly short-selling economy, the equilibrium short-selling risk is given by $V_\phi=\lim_{t\to\infty}\mathrm{Var}\left[\phi_t\right]$. Substituting the equilibrium shorting fee (11) and using the steady state variance, $\mathrm{Var}\left[\theta_t\right]=\sigma_\theta^2/(2\kappa)$, yields (23). The equilibrium short-selling activity is defined as the volatility of short interest changes, $\sigma_{\mathcal{SI}}=\sqrt{\mathrm{Var}_t\left[d\mathcal{SI}_t\right]/dt}$. Taking the derivative of (A.25), substituting (11), and using the disagreement dynamics (8) and $B=\tilde{B}/\phi_1$ gives (24).

In the costless short-selling benchmark economy, the shorting fee is zero and thus the variance of the shorting fee is zero. The volatility of short interest is calculated using $\overline{\mathcal{SI}}_t=-1/2+\theta_t/(2\gamma\sigma^2 Q)$, and is given by $\bar{\sigma}_{\mathcal{SI}}=\sigma_\theta/(2\gamma\sigma^2 Q)$. $\square$

Proof of Proposition 6. In the costly stock short-selling economy with a constant disagreement, the i-type investor's indirect utility, $i=\ell,h,s$, is defined as

$$J_i\left(W_{it},t\right)=\max_{(c_i,\psi_i)}\mathrm{E}_{it}^*\left[\int_t^\infty e^{-\rho u}Z_{iu}^{-1}\frac{e^{-\gamma c_{iu}}}{-\gamma}du\right]. \tag{A.29}$$

Substituting the optimal consumption (A.6) into the static budget constraint in (A.5) gives the Lagrangian multiplier y_i. Substituting y_i into (A.6) and then into (A.29), and taking the integral yields the indirect utility function as

$$J_i\left(W_{it},t\right)=-e^{-\rho t}e^{-\gamma r W_{it}}e^{-\ln(\gamma r)-\frac{1}{r}\left(\rho-r+\frac{1}{2}\left(\frac{\pi_i}{\sigma_S}\right)^2\right)}, \tag{A.30}$$

where π_i is the subjective risk premium as given in (4). Substituting (A.30) at $t=0$ into the equality $J_\ell\left(\left(1-\eta\right)W_0,0\right)=J_h\left(W_0,0\right)$ and rearranging gives (26). In the economy with a time-varying disagreement, the benefits of lending solves (25). Substituting (17) at $t=0$ into (25) and rearranging gives (27). $\square$

Proof of Proposition 7. In the costly short-selling economy with a constant disagreement, given the cost of entry ξ, the lenders' optimal size λ^* solves $J_\ell\left(\left(1-\xi\right)W_0,0;\lambda^*\right)=J_h\left(W_0,0;\lambda^*\right)$. Substituting (A.30) at $t=0$ into this equality and rearranging yields (29). In the economy with a time-varying disagreement, lenders' optimal size solves (28). Substituting the indirect utility (17) at $t=0$ into (28) and rearranging gives (30). $\square$

Appendix B: Parameter Values in Figures

In this Appendix, we discuss the parameter values and their choices employed in our Figures. Table 1 summarizes the parameter values used. We note that the behavior of the equilibrium quantities depicted in our Figures is typical and does not vary much with alternative plausible values of parameters.

We start by setting the interest rate to $r = 0.025$ and the time discount factor to $\rho = 0.015$, as often used in other works with similar settings (e.g., Barberis, Greenwood, Jin, and Shleifer (2015)). We also set the absolute risk aversion coefficient to $\gamma = 1$. The parameters in the dividend dynamics (1), the mean of dividend changes, μ, and the volatility of dividend changes, σ, are matched to the corresponding quantities in the data for the aggregate dividend growth rate, 0.018 and 0.115, respectively, as in Bansal and Yaron (2004). Next, we choose the initial dividend level D_0 and the supply of stock Q such that the stock price in our costly short-selling economy with time-varying disagreement is around 100. To do this, we first set the initial dividend to $D_0 = 3.83$ such that the initial price-dividend ratio S_0/D_0 matches the average price-dividend ratio of the stock market in the data, which is reported by Bansal and Yaron (2004) to be 26, after rounding. Given the level of initial dividend we then set the supply of stock to $Q = 5$ for stock price to be around 100.

For the partial lending α, the fraction of lenders' long position that is actually lent to short-sellers, we use the average ratio of total stock amount actually lent out to total amount available to borrow for stocks on special (i.e., stocks with shorting fees higher than 1%) as a proxy, which is reported by Aggarwal, Saffi, and Sturgess (2015) to be 43%. We set the lenders' size in our model so that the lending supply (fraction of outstanding shares held by lenders) in our main model with time-varying disagreement matches the corresponding quantity in the data (Engelberg, Reed, and Ringgenberg (2018) report this to be 18%), which yields $\lambda = 0.078$. We note that we endogenize this quantity in Section 4 and our analysis there yields the lenders' optimal size to be quite close to this value. For our analysis in Sections 2–3, we do not need to assign any values to the initial wealth of investors. However, in our analysis of Section 4, to obtain the benefits of lending and cost of entry, which is simply set to $\xi = 0.02$, as a fraction of investors' initial wealth, we find it useful to assign a value to it, and so simply set it to $W_0 = 5000$ as in Barberis, Greenwood, Jin, and Shleifer (2015).

For the second moment parameters of the disagreement process, we use the summary sta-

Parameter	Symbol	Value
Initial dividend	D_0	3.83
Mean of the dividend changes	μ	0.018
Volatility of the dividend changes	σ	0.115
Interest rate	r	0.025
Supply of stock shares	Q	5
Absolute risk aversion coefficient	γ	1
Time discount factor	ρ	0.015
Long-run mean of the disagreement	μ_θ	0.0895
Volatility of the disagreement changes	σ_θ	0.0058
Speed of mean reversion of the disagreement	κ	0.87
Partial lending	α	0.43
Population mass of lenders	λ	0.078
Investors' initial wealth	W_0	5000
Cost of entry	ξ	0.02

Table 1: **Parameter values.** This table reports the parameter values used in our Figures.

tistics provided by Yu (2011), who report the standard deviation and the one-month auto-correlation of disagreement as 0.0038 and 0.93, respectively. Matching these quantities to the corresponding ones in our model gives $0.0038 = \sigma_\theta/\sqrt{2\kappa}$, and $0.93 = \exp(-\kappa/12)$, which yields the volatility of the disagreement as $\sigma_\theta = 0.0058$, and speed of mean reversion as $\kappa = 0.87$. We set the mean of the disagreement changes such that in our main model, the probability of shorting fee being positive is 99.9%. With the equilibrium shorting fee (11) as in Proposition 2, this probability is given by $\lim_{t\to\infty} \mathbb{P}(\phi_t > 0) = 1 - \Phi\left(-(\mu_\theta + \phi_0/\phi_1)/\sqrt{\sigma_\theta^2/2\kappa}\right) = 99.9\%$, which yields the mean of the disagreement changes as $\mu_\theta = 0.0895$. This procedure yields the parameter values in Table 1.

We further note that with these parameter values, the probability of the disagreement θ_t and the short interest $\mathcal{SI}_t$ being positive in the steady-state become $\lim_{t\to\infty} \mathbb{P}(\theta_t > 0) = 1 - \Phi\left(-\mu_\theta/\sqrt{\sigma_\theta^2/2\kappa}\right) = 1 - \Phi(-20.35) \simeq 1$, and $\lim_{t\to\infty} \mathbb{P}(\mathcal{SI}_t > 0) = 1 - \Phi(-44.04) \simeq 1$, respectively, as reported in Section 3.

References

Aggarwal, Reena, Pedro A. C. Saffi, and Jason Sturgess, 2015, The role of institutional investors in voting: Evidence from the securities lending market, *Journal of Finance* 70, 2309–2346.

Andrei, Daniel, Bruce Carlin, and Michael Hasler, 2019, Asset pricing with disagreement and uncertainty about the length of business cycles, *Management Science* 65, 2900–2923.

Asquith, Paul, Parag A. Pathak, and Jay R. Ritter, 2005, Short interest, institutional ownership, and stock returns, *Journal of Financial Economics* 78, 243–276.

Atmaz, Adem, and Suleyman Basak, 2018, Belief dispersion in the stock market, *Journal of Finance* 73, 1225–1279.

Atmaz, Adem, and Suleyman Basak, 2019, Option prices and costly short-selling, *Journal of Financial Economics* 134, 1–28.

Banerjee, Snehal, 2011, Learning from prices and the dispersion in beliefs, *Review of Financial Studies* 24, 3025–3068.

Banerjee, Snehal, and Jeremy J. Graveline, 2014, Trading in derivatives when the underlying is scarce, *Journal of Financial Economics* 111, 589–608.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *Journal of Finance* 59, 1481–1509.

Barberis, Nicholas, Robin Greenwood, Lawrence Jin, and Andrei Shleifer, 2015, X-CAPM: An extrapolative capital asset pricing model, *Journal of Financial Economics* 115, 1–24.

Basak, Suleyman, 2000, A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk, *Journal of Economic Dynamics and Control* 24, 63–95.

Basak, Suleyman, 2005, Asset pricing with heterogeneous beliefs, *Journal of Banking & Finance* 29, 2849–2881.

Beneish, M. D., C. M. C. Lee, and D. C. Nichols, 2015, In short supply: Short-sellers and stock returns, *Journal of Accounting and Economics* 60, 33–57.

Bhamra, Harjoat S., and Raman Uppal, 2014, Asset prices with heterogeneity in preferences and beliefs, *Review of Financial Studies* 27, 519–580.

Blocher, Jesse, Adam V. Reed, and Edward D. Van Wesep, 2013, Connecting two markets: An equilibrium framework for shorts, longs, and stock loans, *Journal of Financial Economics* 108, 302–322.

Boehmer, Ekkehart, Zsuzsa R. Huszar, and Bradford D. Jordan, 2010, The good news in short interest, *Journal of Financial Economics* 96, 80–97.

Boehmer, Ekkehart, Charles M. Jones, and Xiaoyan Zhang, 2008, Which shorts are informed?, *Journal of Finance* 63, 491–527.

Chabakauri, Georgy, 2015, Asset pricing with heterogeneous preferences, beliefs, and portfolio constraints, *Journal of Monetary Economics* 75, 21–34.

Chague, Fernando, Rodrigo De-Losso, Alan De Genaro, and Bruno Giovannetti, 2017, Well-connected short-sellers pay lower loan fees: A market-wide analysis, *Journal of Financial Economics* 123, 646–670.

Cohen, Lauren, Karl B. Diether, and Christopher J. Malloy, 2007, Supply and demand shifts in the shorting market, *Journal of Finance* 62, 2061–2096.

Cox, John C., and Chi-fu Huang, 1989, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33–83.

Cvitanić, Jakša, and Semyon Malamud, 2011, Price impact and portfolio impact, *Journal of Financial Economics* 100, 201–225.

Daniel, Kent, Alexander Klos, and Simon Rottke, 2018, Overconfidence, information diffusion, and mispricing persistence, Working paper, Columbia University.

David, Alexander, 2008, Heterogeneous beliefs, speculation, and the equity premium, *Journal of Finance* 63, 41–83.

D'Avolio, Gene, 2002, The market for borrowing stock, *Journal of Financial Economics* 66, 271–306.

Desai, Hemang, K. Ramesh, S. Ramu Thiagarajan, and Bala V. Balachandran, 2002, An investigation of the informational role of short interest in the Nasdaq market, *Journal of Finance* 57, 2263–2287.

Detemple, Jérôme B., and Shashidhar Murthy, 1994, Intertemporal asset pricing with heterogeneous beliefs, *Journal of Economic Theory* 62, 294–320.

Detemple, Jérôme B., and Shashidhar Murthy, 1997, Equilibrium asset prices and no-arbitrage with portfolio constraints, *Review of Financial Studies* 10, 1133–1174.

Diamond, Douglas W., and Robert E. Verrecchia, 1987, Constraints on short-selling and asset price adjustment to private information, *Journal of Financial Economics* 18, 277–311.

Diether, Karl B., Kuan-Hui Lee, and Ingrid M. Werner, 2009, Short-sale strategies and return predictability, *Review of Financial Studies* 22, 575–607.

Drechsler, Itamar, and Qingyi Freda Drechsler, 2016, The shorting premium and asset pricing anomalies, Working Paper, University of Pennsylvania.

Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen, 2002, Securities lending, shorting, and pricing, *Journal of Financial Economics* 66, 307–339.

Dumas, Bernard, Alexander Kurshev, and Raman Uppal, 2009, Equilibrium portfolio strategies in the presence of sentiment risk and excess volatility, *Journal of Finance* 64, 579–629.

Duong, Truong X., Zsuzsa R. Huszár, Ruth S. K. Tan, and Weina Zhang, 2017, The information value of stock lending fees: Are lenders price takers?, *Review of Finance* 21, 2353–2377.

Ehling, Paul, Alessandro Graniero, and Christian Heyerdahl-Larsen, 2017, Asset prices and portfolio choice with learning from experience, *Review of Economic Studies* 85, 1752–1780.

Engelberg, Joseph E., Adam V. Reed, and Matthew C. Ringgenberg, 2018, Short-selling risk, *Journal of Finance* 73, 755–786.

Evgeniou, Theodoros, Julien Hugonnier, and Rodolfo Prieto, 2019, Costly short sales and nonlinear asset pricing, Working paper, INSEAD.

Figlewski, Stephen, 1981, The informational effects of restrictions on short sales: Some empirical evidence, *Journal of Financial and Quantitative Analysis* 16, 463–476.

Gallmeyer, Michael, and Burton Hollifield, 2008, An examination of heterogeneous beliefs with a short-sale constraint in a dynamic economy, *Review of Finance* 12, 323–364.

Hanson, Samuel G., and Adi Sunderam, 2014, The growth and limits of arbitrage: Evidence from short interest, *Review of Financial Studies* 27, 1238–1286.

Harrison, Michael J., and David M. Kreps, 1978, Speculative investor behavior in a stock market with heterogeneous expectations, *Quarterly Journal of Economics* 92, 323–336.

Hong, Harrison, and Jeremy C. Stein, 2003, Differences of opinion, short-sales constraints, and market crashes, *Review of Financial Studies* 16, 487–525.

Johnson, Timothy C., 2004, Forecast dispersion and the cross section of expected returns, *Journal of Finance* 59, 1957–1978.

Jones, Charles M., and Owen A. Lamont, 2002, Short-sale constraints and stock returns, *Journal of Financial Economics* 66, 207–239.

Kaplan, Steven N., Tobias J. Moskowitz, and Berk a. Sensoy, 2013, The effects of stock lending on security prices: An experiment, *Journal of Finance* 68, 1891–1936.

Karatzas, Ioannis, John P. Lehoczky, and Steven E. Shreve, 1987, Optimal portfolio and consumption decisions for a small investor on a finite horizon, *SIAM Journal on Control and*

Optimization 25, 1557–1586.

Kogan, Leonid, Stephen A. Ross, Jiang Wang, and Mark M. Westerfield, 2006, The price impact and survival of irrational traders, *Journal of Finance* 61, 195–229.

Kolasinski, Adam C., Adam V. Reed, and Matthew C. Ringgenberg, 2013, A multiple lender approach to understanding supply and search in the equity lending market, *Journal of Finance* 68, 559–595.

Liptser, Robert S., and Albert N. Shiryaev, 2001, *Statistics of Random Processes* (Springer).

Longstaff, Francis A., and Jiang Wang, 2012, Asset pricing and the credit market, *Review of Financial Studies* 25, 3169–3215.

Merton, Robert C., 1971, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373–413.

Miller, Edward M., 1977, Risk, uncertainty, and divergence of opinion, *Journal of Finance* 32, 1151–1168.

Nagel, Stefan, 2005, Short sales, institutional investors and the cross-section of stock returns, *Journal of Financial Economics* 78, 277–309.

Nezafat, Mahdi, Mark Schroder, and Qinghai Wang, 2017, Short-sale constraints, information acquisition, and asset prices, *Journal of Economic Theory* 172, 273–312.

Nutz, Marcel, and Jose A. Scheinkman, 2019, Shorting in speculative markets, Working paper, Columbia University.

Ofek, Eli, Matthew Richardson, and Robert F. Whitelaw, 2004, Limited arbitrage and short sales restrictions: Evidence from the options markets, *Journal of Financial Economics* 74, 305–342.

Prado, Melissa Porras, 2015, Future lending income and security value, *Journal of Financial and Quantitative Analysis* 50, 869–902.

Prado, Melissa Porras, Pedro A. C. Saffi, and Jason Sturgess, 2016, Ownership structure, limits to arbitrage, and stock returns: Evidence from equity lending markets, *Review of Financial Studies* 29, 3211–3244.

Rapach, David E., Matthew C. Ringgenberg, and Guofu Zhou, 2016, Short interest and aggregate stock returns, *Journal of Financial Economics* 20, 0–1.

Reed, Adam V., 2013, Short selling, *Annual Review of Financial Economics* 5, 245–258.

Saffi, Pedro A. C., and Kari Sigurdsson, 2010, Price efficiency and short selling, *Review of Financial Studies* 24, 821–852.

Scheinkman, José A., and Wei Xiong, 2003, Overconfidence and speculative bubbles, *Journal of Political Economy* 111, 1183–1220.

Senchack, A. J., and Laura T. Starks, 1993, Short-sale restrictions and market reaction to short-interest announcements, *Journal of Financial and Quantitative Analysis* 28, 177–194.

Seneca, Joseph J., 1967, Short interest: Bearish or bullish?, *Journal of Finance* 22, 67–70.

Xiong, Wei, and Hongjun Yan, 2010, Heterogeneous expectations and bond markets, *Review of Financial Studies* 23, 1433–1466.

Yan, Hongjun, 2008, Natural selection in financial markets: Does it work?, *Management Science* 54, 1935–1950.

Yu, Jialin, 2011, Disagreement and return predictability of stock portfolios, *Journal of Financial Economics* 99, 162–183.

Zapatero, Fernando, 1998, Effects of financial innovations on market volatility when beliefs are heterogeneous, *Journal of Economic Dynamics and Control* 22, 597–626.