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Lobbying in Networks *

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Abstract

Evidence suggests that in the United States, organizations make significant campaign contributions to 'well-connected' legislators of all political parties to influence their voting decisions towards the company's preferred agenda. We take this empirical fact as the starting point to develop a theoretical framework to investigate whether and when the social network among legislators aids a lobby group in influencing voting decisions of legislators. Each legislator can vote for the status quo or an alternative policy. The lobbyist prefers the alternative policy and promises payments to legislators in return for their votes. A legislator is affiliated to one of the two political parties. All the legislators in one party are assumed to be biased towards the status quo, while all the legislators in the other party are biased towards the alternative policy. Legislators value the payment they receive from the lobbyist. A legislator derives additional utility from voting in line with those legislators within her party with whom she is directly connected in her social network because of the strategic complementarity in action space. Furthermore, a legislator suffers some disutility from failing to distinguish her vote from a legislator of the other party because of the strategic substitutability in actions of the opposing party legislators. Equilibrium payment to a legislator depends on her Bonacich centrality within her party and the total resources allocated to her party. The lobbyist can offset a small status quo bias with monetary transfers. The party with a relatively denser network typically receives more funds from the lobbyist. The party opposing the lobbyist may obtain relatively more funds. The model leads to a variety of additional comparative statics results that help clarify what factors affect the payments to legislators, and the resulting likelihood of the lobbyist successfully influencing their voting.

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Introduction

In United States, lobbying is authorized by the Lobbying Disclosure Act, 1995 which mandates registration of lobby firms. We have used the term 'lobbying' very loosely to refer to the interaction between a lobby and a legislator where the former actively persuades the latter to favour an alternative policy in return for campaign contributions. The role of an interest group is to promote the common interest of its members by influencing policy outcomes. In US, an interest group can fund electoral campaigns of legislators who support the group's common interests. The legislators can use the campaign contributions for funding their campaign expenses.

Lobbying by interest groups is a common phenomenon and has been extensively studied in economics and political science Snyder Jr. (1991). A typical model assumes the lobbyist can ex-ante provide or can credibly promise to ex-post deliver money or some resource valued by a legislator if she supports a policy preferred by the lobbyist. The problem facing the lobbyist is how to optimally allocate its budget among different legislators. Legislators are assumed to maximize their utility which depends on these resources and an intrinsic bias towards or against the policy preferred by the lobbyist. The basic model has been extended in multiple directions (Austen-Smith and Wright (1992); Dekel et al. (2009)). The broad message of this literature, not surprisingly, is that money matters.

Battaglini and Patacchini (2018) enrich the basic model of lobbying in an interesting way. They ignore the party affiliation of legislators and instead emphasize the role of social connections between the legislators. Specifically, their model contains two equally resourceful lobby groups that prefer different policies. Each lobby can make monetary transfers to the legislators in order to influence them to vote for its preferred policy. The key feature of their model is that each legislator not only cares about money, but also derives *additional* utility from voting in line with those legislators with whom she is directly connected, i.e., her 'neighbors' in the social network of the legislators. For analytical convenience, they assume each lobby chooses transfers to legislators in order to maximize the sum of the probabilities of the legislators voting for its preferred policy.

Social interactions among legislators have been studied since Rice (1927, 1928) and the potential impact of the social network among legislators (regardless of their political party affiliation) on their voting behavior has been noted at least since Truman (1951). Some recent studies have empirically demonstrated this possibility (Arnold et al. (2000); Fowler (2006); Cohen and Malloy (2014)). To the best of our knowledge, Battaglini and Patacchini (2018) is the first and only paper that examines the voting behavior of legislators who care about how their 'neighbours' vote in the presence of lobbying.

The issue of Net Neutrality in the United States which has been a topic of contention since the early 2000s. It means that all data packets on the internet should be treated equally with no internet service provider(ISP) possessing the power to discriminate or charge differently based on platform, source, method of communication, content, user or any other characteristic. In practice, ISPs may not intentionally block, slow down, or charge money for specific web contents which only some customers could afford. The supporters of net neutrality believe that the government has neglected individual freedom and security on the internet while the opponents are of the opinion that an intervention will impede free market innovation and investment. The Democrats launched their efforts to save net neutrality and the bill was approved on 2015 by Federal Communications Commission(FCC).

The biggest challengers (AT & T, Verizon, Comcast etc.) of net neutrality have lobbied three times harder than their proponents. In 2006 only, the opponents of net neutrality have spent approximately $71 million on lobbying and campaign contributions which is 18 times the expenditure of their counterparts. The efforts of the democrats to save net neutrality is analogous to the status quo bias of the legislators favouring a policy. Moreover, the campaign contributions from the supporters of net neutrality has been negligible compared to their oppositions and on 2017 majority favoured retaining the 2015 Open Internet Order[1]. This issue builds the premise to model a reduced form framework with a single lobby group convincing the legislators to oppose a policy in return for campaign contributions. We consider the companies opposing

[1] Yet, the FCC voted in favour of repealing the Order, with effect from June 2018 despite efforts in Congress to stay the repeal

net neutrality as a single lobby group. For simplicity, we assume all the legislators have a status quo bias. Interestingly, the top three companies (AT & T, Verizon, Comcast) opposing net neutrality have spent approximately $20 million since 1989 on campaign contributions towards Democrats while the figure is $25 million for Republicans. This postulates the model discussed in the extension where a lobby group finances the campaigns of two opposing parties. Here we assume that members in each party have uniform and opposing bias towards a policy. This is a reasonable assumption, since majority of the Democrats favour net neutrality while the Republicans oppose it.

Our model is a variant of Battaglini and Patacchini (2018), we modify two key features of their model. We study an environment with asymmetric lobby groups. To do so we assume there is only one lobby group. Each legislator derives utility from voting in line with her 'neighbors' within her own party. This can be interpreted as a strategic complementarity in legislator's actions. Later we extend the model by distinguishing the legislators on the basis of their party affiliation. Here, it is assumed that a legislator derives utility from voting in line with her 'neighbours' within her own party, but derives a disutility if she votes in line with the legislators of the other party.

The model assumes a finite set of legislators who can vote for either the status quo policy or an alternative policy. The legislators are assumed to be self-interested, partisan, individual utility maximizers with diminishing returns. There exists an exogenously given social network among the legislators. A pair of legislators is either connected or disconnected in the network. Legislators have a common level of bias towards (or, against) the status quo. Each legislator cares about her own policy preferences and also takes into account of the actions of the legislators with whom she is connected with in her social network. The lobby prefers the alternative policy and chooses monetary transfers from a fixed budget to the legislators so as to maximize the sum of their probabilitiesBattaglini and Patacchini (2018) of voting for the alternative policy. Each legislator cares about money, and also cares about voting in line with her neighbours in the social network.

We find the equilibrium payment by the lobby to a legislator depends on her pattern of connections in the network. Under our assumptions, the impact of the network structure on payments to legislators operates through a key network statistic, the *Bonacich Centrality*[2]. The higher the Bonacich centrality of a legislator in the network, the larger the payment she receives from the lobbyist. The lobbyist benefits with an increase in its budget. It also benefits if the legislators' bias towards the status quo policy decreases.

The key comparative static result relates to the marginal impact of changes in the network structure on the lobby. Specifically, we consider a pair of networks where one network is 'bigger' than another, i.e. the former network has at least one additional link relative to the latter network. If the legislators are biased towards the alternative policy, then the lobby is always relatively better off under a relatively bigger network. However, if the legislators are biased towards the status quo policy then whether and when a relatively bigger network benefits the lobby depends on the size of the bias. If the bias of the legislators towards the status quo policy is sufficiently small (large), then bigger networks benefit (hurt) the lobby. The marginal impact of a relatively bigger network on the lobbyist is ambiguous at intermediate levels of the bias towards the status quo.

Model

Let's consider a finite set of $n \in \mathbb{N}$ legislators. Each legislator simultaneously chooses between two policies $\rho \in \{A, S\}$ where S is the status quo policy and A is the alternative new policy. Each legislator has an inherent bias towards or away from the status quo policy. For simplicity, consider the legislators who are biased *towards* policy S as members of party-s and the legislators who are biased *towards* policy A as members of party-a. Thus, the legislature comprises of n_τ legislators biased towards policy ρ such that $\sum\limits_{\tau} n_\tau = n$ where $\tau \in \{s, a\}$. The legislators in each party τ are socially connected by an exogenous network within the members of their own party. The lobby utilizes the social connections among the legislators and promises them monetary payments to influence their voting decisions towards the lobby's preferred policy. All legislators

[2] Bonacich centrality was first proposed in the sociology literature Bonacich (1987)). Ballester et al. (2006); Calv́o-Armengol et al. (2009) are the earliest papers to highlight its role in understanding the equilibrium outcomes of games with complementarity between agents' actions.

vote[3] and the winning policy is chosen via plurality rule.

Definitions

In our model, two legislators i and j in party-τ are *socially connected neighbours* if $g_{ij} > 0$ where g_{ij} is a measure of strategic complementarity in j's action on i. A connection g_{ij} can be loosely interpreted as the social influence of legislator j on i. The assumption of non-negative g_{ij}'s can be interpreted as the legislators are politically aligned.[4] Assume[5], $g_{ij} \in \{0,1\}$ and we rule out strategic substitutability effect[6] from neighbours' vote. We assume, any two legislators $i \in \tau$ and $j \in \tau'$ are not connected[7], i.e. $g_{ij} = 0$. Conventionally we set, g_{ii} to 0. For any two legislators $i, j \in \tau$, a connection $g_{ij} = g_{ji} = 1$ indicates that i and j are neighbours[8], otherwise $g_{ij} = g_{ji} = 0$.

The matrix $\mathbb{G} = [g_{ij}]$ is a zero-diagonal, symmetric $n \times n$ matrix where $\mathbb{G}$ is also interpreted as the adjacency matrix of any given network. $\mathbb{G} \in \mathscr{G}$ is an unweighted, undirected[9], symmetric matrix where $\mathscr{G}$ denotes the class of graphs with n nodes. The maximum number of direct links possible for any legislator $i, j \in \tau$ is $n_\tau - 1$[10]. The network structure $\mathbb{G}$ is exogenous and is common knowledge.

Legislator's Problem

The utility of the legislator is contingent on the her vote, and the network externalities from voting in line with her neighbours and exogenous factors. The

[3]We rule out abstention.

[4]We study lobbying towards a given side of the aisle.

[5]Unlike Battaglini and Patacchini (2018) we do not normalize the sum of the social influence of each legislator i to 1. Assuming $\sum_j g_{ij} = 1$ makes their model more restrictive. Assuming $g_{ij} \in \{0,1\}$ gives us more flexibility to do some network comparative statics and is less restrictive since $\sum_j g_{ij} \leq (n-1)$.

[6]Results of this model are not directly applicable for networks with some negative ties, $g_{ij} < 0$ and some positive ties $g_{ij} > 0$.

[7]Polarization: example

[8]We rule out self loops or multiple links between a pair of nodes.

[9]Modelling directed links will lead to different results and $\mathbb{G}$ will be an asymmetric square matrix.

[10]$\mathbb{G}$ is an $n \times n$ matrix and $\mathbf{1}$ is a column vector of ones. For a graph with n nodes, the maximum number of direct links possible for a legislator in any network $\mathbb{G}$ is $\mathbb{G} \cdot \mathbf{1} = \sum_j g_{ij} \leq (n-1)$.

utility of any legislator $i \in \tau$ for any policy $\rho \in \{A, S\}$ is represented by:

$$\Pi_i(\rho) = \begin{cases} u(m_i) + \delta \sum_j g_{ij} \, v_j(A) + \epsilon_{iA}, & \text{if legislator } i \text{ votes for } A \\ \sigma_\tau + \delta \sum_j g_{ij} \, v_j(S) + \epsilon_{iS}, & \text{if legislator } i \text{ votes for } S \end{cases} \tag{1}$$

where $v_j(\rho)$ is a binary indicator of j's vote,

$$v_j(\rho) = \begin{cases} 1, & \text{if legislator } j \text{ votes for } \rho \\ 0, & \text{otherwise} \end{cases}$$

The utility function is additively separable in the monetary benefits and network effects. $u(m_i)$ represents the direct utility of legislator i from the monetary contributions to i as a compensation for her vote towards lobby's preferred policy A. m_i is the monetary transfer to the legislator i for voting in favour of A. We assume $u(\cdot)$ to be a standard increasing, concave, continuous, twice differentiable utility function. On the other hand, each legislator has their own bias σ_τ.

Assumption 1. $\sigma_s + \sigma_a = 0 \quad \cdots\cdots \quad$ [Normalization]

In our model, the legislators in party-s have a uniform positive bias σ_s towards the status quo policy S and the legislators in party-a have a uniform bias σ_a. We normalize $\sigma_s + \sigma_a$ to 0 where the level and direction of the bias is identical across all legislators in any given party. In other words, the biases for legislators of two parties are perfect substitutes and are normalised[11] to zero.

The second term describes the sum of the bilateral influences of all legislators connected to i in $\mathbb{G}$. The binary indicator function equals one if legislator j votes for the same policy as i. If any legislator i directly aligns her vote with her neighbours decision, she benefits according to the network spillover term δ. Brock and Durlauf (2001) demonstrates the presence of social network externalities where the payoff Π_i improves if she conforms with her neighbours

[11] The normalization is a simplification and can be manipulated by setting $\sigma_s + \sigma_a = \sigma$, where σ is non-zero.

votes. For any $\delta > 0$, the interpersonal connections[12] among the legislators prompts them to conform with their neighbours action. The network spillover effect δ is assumed to be homogeneous across all legislators i.

The last term is a private preference shock parameter affecting player i's preference towards any policy ρ. The error term for player i for voting S is $\epsilon_{iS} = \epsilon_i$ where ϵ_i is independently uniformly distributed between $[-\frac{1}{2\theta}, \frac{1}{2\theta}]$ with mean 0 and a density of θ. The error term for player i for voting A is normalised to zero $\epsilon_{iA} = 0$. The error ϵ_i is privately observable to legislator i. The lobby cannot observe the individual shocks of the legislators but knows the distribution.

All legislators are office motivated[13] and they do not care about policy outcome. The conditions for uniqueness and existence of the pure strategy equilibrium is based on the network structure. In our analysis, we consider a single lobbyist who influences legislators to vote for policy A. The objective of the lobby group is to maximize the sum of probabilities of legislators' votes in favour of the new policy.

Lobby's Problem

The role of the lobby group is to influence the legislators to vote for the new policy A in exchange of monetary contributions. The lobby will allocate funds optimally to maximize the aggregate probability of votes in favour of A. The lobby's problem is given as:

$$\max_{\mathbf{m}} \sum_j p_j(\mathbf{m}) \qquad s.t. \qquad \sum_j m_j \leq M$$

The feasible vector of payments is given by $\mathbf{m} = (m_1, \cdots, m_n)$ such that $\sum_i m_i \leq M$ and $m_i \geq 0$, for all $i \in N$. The total money available to the lobby is M. The ex-ante probability of any legislator j for voting in favour of policy A is $p_j = E(v_j(A))$.

[12] In our baseline model, legislators are not negatively influenced by the action of neighbours i.e each legislator benefits by conforming with her neighbour's action.

[13] Allowing for policy motivated legislators increases the complexity of the analysis and we may lose the closed-form expressions for our equilibrium analysis. Our results hold qualitatively if bias is replaced by outcome-contingent utility. (see Battaglini and Patacchini (2018)).

Timing. In the initial influence stage, the lobby group observes the network connections among the legislators and the bias and announces a payment vector of transfers. The lobby does not know the preferences of the legislators with full certainty because of the exogenous shock parameter ϵ_i which a legislator observes privately after $\mathbf{m}$ is announced. Before making her voting decision each rational legislator i compares the expected pay-off from voting for A over S. In the final voting stage, each legislator votes based on the announced transfer and the shock. She will vote for the new policy A iff

$$E\Pi_i(S) - E\Pi_i(A) \leq 0 \tag{2}$$

Since, legislator i observes her own exogenous preference shock so, $E(\epsilon_i) = \epsilon_i$. Thus the above equation 2 leads to:

$$\begin{aligned} 0 &\leq u(m_i) - \sigma_\tau - \delta \sum_j g_{ij}\,(1-p_j) + \delta \sum_j g_{ij}(p_j) - \epsilon_i \\ \epsilon_i &\leq u(m_i) - \sigma_\tau + \delta \sum_j g_{ij}\,(2p_j - 1) \end{aligned} \tag{3}$$

Thus the legislator votes for policy A if the above condition holds. Further calculations on the derivation of the individual probability for voting in favour of A is available in the Appendix A.1. We assume θ is sufficiently small to ensure adequate uncertainty for uniqueness of solutions. In the next section, we discuss the the responsiveness of each legislator's equilibrium strategy to depends on their promised transfer, position in the network and their neighbours' votes. We solve this problem by using backward induction and the stages are discussed in the next section

Equilibrium

Voting Stage

We begin by solving the voting stage of the legislator given the monetary transfers announced by the lobby group. The probability for any player i to vote for

A is denoted by $p_i \in [0,1]$ where $\mathbf{p} = (p_1, p_2, \cdots, p_n)^\top$ is a vector of the probabilities[14] of all the players such that $\mathbf{p} : \mathbf{m}_1 \times \cdots \times \mathbf{m}_n \to [0,1]^n$. For any $\mathbf{m}$ the probability vector is a linear mapping from the probability vector $\mathbf{p}(\mathbf{m})$ to itself where $F(\mathbf{m}, \mathbf{p}(\mathbf{m}))$ is a linear transformation of $\mathbf{p}(\mathbf{m})$. For a small θ, the set F is closed, convex and continuous in $\mathbf{p}$ as it is a contraction mapping from $[0,1]^n$ to itself. So, a unique equilibrium exists. For a larger value of θ the equation may not be well behaved or unique.

We obtain a linear system of simultaneous equations of legislators' individual probabilities of voting for A. The probability of legislators choosing the new policy A is represented as:

$$\mathbf{p}(\mathbf{m}) = \begin{pmatrix} p_1(\mathbf{m}) \\ p_2(\mathbf{m}) \\ \vdots \\ p_n(\mathbf{m}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \theta \left[u(m_1) - \sigma_s + \delta \sum_j g_{1j} \left(2p_j(\mathbf{m}) - 1\right) \right] \\ \vdots \\ \frac{1}{2} + \theta \left[u(m_n) - \sigma_a + \delta \sum_j g_{nj} \left(2p_j(\mathbf{m}) - 1\right) \right] \end{pmatrix} \tag{4}$$

Solving for the equilibrium probabilities from equation 4 we get,

$$\begin{aligned} \mathbf{p}(\mathbf{m}) &= \left(\frac{1}{2} \cdot \mathbf{1} - \theta \boldsymbol{\sigma}\right) + \theta \cdot u(\mathbf{m}) + 2\theta\delta \cdot \mathbb{G} \cdot \mathbf{p}(\mathbf{m}) - \theta\delta \cdot \mathbb{G} \cdot \mathbf{1} \\ (\mathbb{I} - \delta^* \mathbb{G}) \cdot \mathbf{p} &= \frac{1}{2}(\mathbb{I} - \delta^* \mathbb{G}) \cdot \mathbf{1} + \theta \mathbf{u} - \theta \boldsymbol{\sigma} \qquad [\because 2\theta\delta = \delta^*] \end{aligned}$$

where $\mathbf{1}$ is an $n \times 1$ column vector of 1's, $u(\mathbf{m})$ is an $n \times 1$ vector of the direct utility from the transfers promised to the legislators, $\mathbf{u} = (u(m_1), u(m_2), \cdots, u(m_n))^\top$ and $\boldsymbol{\sigma} = (\sigma_s, \cdots, \sigma_a)^\top$ is a vector of bias of n legislators where n_s legislators have a bias σ_s and n_a legislators have a bias $-\sigma_s$.

Pre-multiplying both sides by $(\mathbb{I} - \delta^* \mathbb{G})^{-1}$,

$$\mathbf{p} = \frac{1}{2} \cdot \mathbf{1} + \theta (\mathbb{I} - \delta^* \mathbb{G})^{-1} \cdot \mathbf{u} - \theta (\mathbb{I} - \delta^* \mathbb{G})^{-1} \boldsymbol{\sigma} \tag{5}$$

Assumption 2. The matrix $(\mathbb{I} - \delta^* \mathbb{G}^\top)^{-1}$ is invertible.

Invertability is guaranteed if δ^* is sufficiently small. We assume the matrix

[14] $\mathbf{p}$ is a mapping from a set of possible transfers to a set of probabilities.

$(\mathbb{I}-\delta^*\mathbb{G}^\top)^{-1}$ is invertible. $\mathbb{G}$ is symmetric. The sufficient condition for invertability of the matrix $(\mathbb{I}-\delta^*\mathbb{G})^{-1}$ is similar to the assumption in Ballester et al. (2006). The following condition guarantees the inverse of the matrix exists for $\delta < \frac{1}{2\theta\zeta(\mathbb{G})}$ where $\zeta(\mathbb{G})$ is the largest eigenvalue of $\mathbb{G}$. The matrix $(\mathbb{I}-\delta^*\mathbb{G})^{-1}$ has non-negative elements, so invertibility is sufficient to ensure positive Bonacich centrality. Therefore we can infer that $\left(\mathbb{I}-\delta^*\mathbb{G}\right)^{-1}\cdot\mathbf{1}$ is positive.

Bonacich Centrality. The Bonacich centrality of a given network $\mathbb{G}$ measures the importance of a given node based on its location and all possible walks between i and j. While the k-th power of $\mathbb{G}$ tracks all the indirect connections of walk length k in $\mathbb{G}$ and is noted as $\mathbb{G}^k$. δ is a non-negative scalar which is used as a discount(decay) factor of the indirect links with successive walks. For a given network $\mathbb{G}$, the generic form of Bonacich centrality vector of legislator i is:

$$\mathbf{b}(\delta,\mathbb{G}) \;=\; \delta^0\mathbb{G}^0\mathbf{1}+\delta^1\mathbb{G}^1\mathbf{1}+\delta^2\mathbb{G}^2\mathbf{1}+\cdots \;=\; \sum_{k=0}^{+\infty}\delta^k\mathbb{G}^k\cdot\mathbf{1}$$

For any $\delta<1$, the matrix $(\mathbb{I}-\delta\mathbb{G})^{-1}$ is invertible and the sum converges to a finite value. The largest eigenvalue of matrix $\mathbb{G}$ is $\zeta(\mathbb{G})=\max\{\zeta_i(\mathbb{G})\}$. The centrality vector $\mathbf{b}(\delta,\mathbb{G})$ exists. The Bonacich centrality of a legislator i is $b_i(\delta,\mathbb{G})=\sum_{j=1}^{n}x_{ij}(\delta,\mathbb{G})$ where the term $x_{ij}(\delta,\mathbb{G})\geq 1$ gives the sum of all possible walks from i to j. By definition, for any non-negative symmetric matrix $\mathbb{G}$ if all entries in $\mathbb{G}$ are real, then any increase in an entry in $\mathbb{G}$ will not decrease the Bonacich centrality of each individuals. The Bonacich centrality vector in our analysis is written as $\mathbf{b}(\delta^*,\mathbb{G})=(\mathbb{I}-\delta^*\mathbb{G}^\top)^{-1}\mathbf{1}$.

Using equation 5, we get an unique equilibrium probability vector $\mathbf{p}(\mathbf{m})$ for a fixed $\theta<\theta^*$ such that the sum of the probabilities $\sum_j p_j(\mathbf{m})$ is as follows:

$$\mathbf{p}^\top\cdot\mathbf{1}=\frac{1}{2}\cdot\mathbf{1}^\top\cdot\mathbf{1}\;+\;\theta\cdot\mathbf{u}^\top\cdot\mathbf{b}(\delta^*,\mathbb{G})\;-\;\theta\,\boldsymbol{\sigma}^\top\cdot\mathbf{b}(\delta^*,\mathbb{G})$$

Alternative form,

$$\begin{aligned}\sum_j^n p_j(\mathbf{m}) &= \frac{n}{2} + \theta\,\Big[\sum_j u(m_j)b_j - \sigma_s \sum_{j\le n_s} b_j + \sigma_s \sum_{j>n_s} b_j\Big] \\ &= \frac{n}{2} + \theta\,\Big[\sum_j u(m_j)b_j - \sigma_s B_s + \sigma_s B_a\Big] \qquad (6)\end{aligned}$$

where $B_\tau = \sum_{j\le n_\tau} b_j$ is the sum of the Bonacich centralities of all legislators in party-τ. We address the equilibrium probabilities from solving the linear system of equations. The sum of the probabilities $\sum_j^n p_j$ in the above equation is continuous, increasing and differentiable with respect to the monetary transfer m_i.

Initial Stage

The objective of the lobbyist is to optimize the allocation of resources among the legislators such that the sum of probabilities is maximized. There exists a $\theta < \theta^*$, such that the solution is unique. Additionally, we assume the budget M to the lobbyist below some critical level[15] M^*. The budget $M < M^*_\oplus$ such that the network effects are not overwhelmed by the lobby's transfer. The first order condition is as follows:

$$\sum_j \frac{\partial p_j}{\partial m_i} = J_i[\mathbf{p}]^\top \cdot \mathbf{1} = \lambda \qquad (7)$$

for all j. Using the first order conditions we solve for the equilibrium level of monetary transfer.

We define $J_i[\mathbf{p}] = [\frac{\partial p_1}{\partial m_i}, \cdots, \frac{\partial p_n}{\partial m_i}]^\top$ as the Jacobian matrix or the first order derivative of the vector of probabilities with respect to the transfer made to the legislator i. Differentiating the optimal probability distribution vector in equation 5 with respect to the monetary transfer made to individual legislator m_i, we get $J_i[\mathbf{p}] = \theta(\mathbb{I} - \delta^*\mathbb{G})^{-1} J_i[\mathbf{u}]$. The marginal effect of any change in m_i on the

[15] For a network with no connections i.e. $\mathbb{G} = [\mathbf{0}]_{n\times n}$ the level of money which ensures the legislators vote for policy A with certainty is $M^*_\odot = \sum_\tau n_\tau\ u^{-1}(\frac{1}{2\theta} + \sigma_\tau)$. Additionally, when the legislators in each party-τ is fully connected within the members of their own party then the level of money which ensures the legislators vote for policy A with certainty is $M^*_\oplus = \sum_\tau n_\tau\ u^{-1}\big(\frac{1-\delta^*(n_\tau-2)-\delta^{*\,2}(n_\tau-1)}{2\theta} + \sigma_\tau\big)$. Note $M^*_\oplus < M^*_\odot$.

direct utility of any legislator is $u'(m_j)$ if $j = i$ and 0 otherwise. $J_i[\mathbf{u}]$ is a vector of zero's except for the i-th term which is $\frac{\partial u_i}{\partial m_i}$.[16] The sum of probabilities $\mathbb{P}$ is differentiable and increasing[17] in m_i .

From the definition of Bonacich centrality and differentiating equation 6 (see appendix A.2a) we get the following

$$J_i[\mathbf{p}]^\top \cdot \mathbf{1} \;=\; \theta \cdot J_i[\mathbf{u}]^\top \cdot \mathbf{b}(\delta^*, \mathbb{G})$$

Using the above equation, we show the Lagrangian multiplier is proportional to the equilibrium transfer m_i^* (appendix A.2b) to the legislator. The marginal cost of resources in equilibrium depends on the Bonacich centrality and marginal utility of direct transfer $\lambda^* = \frac{\lambda}{\theta} = u'(m_i) \cdot b_i(\delta^*, \mathbb{G}^\top)$. For any given parameter values of $\Omega = (\sigma_s, \delta, \theta, M, \mathbb{G})$ where $\theta < \theta^*$ and $\delta < \frac{1}{2\theta\zeta(\mathbb{G})}$ the indirect sum of probability function (analogous to indirect utility function) for the equilibrium vector of transfers $\mathbf{m}^*$ is

$$\mathscr{P}(\mathbf{m} \mid \Omega) \;=\; \frac{n}{2} + \theta\big[\sum_j u(m_j^*) \cdot b_j(\delta^*, \mathbb{G}) - \sigma_s B_s + \sigma_s B_a\big] \tag{7}$$

Since $p_i \in [0,1]$, if $M \geq M^*$, then $\mathscr{P} \to n$. If the lobby has enough money to influence the legislators then policy A wins with certainty. The objective of the lobby group is to maximize the sum of probabilities of votes in favour of policy A. So, we can verify the equilibrium sum of probabilities of voting for A converges to n for increasing m_i.

Results

In this section, we characterize the Nash equilibrium payments to the legislators and other comparative statics. In the following proposition we show the Nash equilibrium payment to each legislator depends on the Bonacich Centrality vector.

[16] Based on initial assumptions on utility, $J_i[\mathbf{u}] = (0\ 0 \cdots \frac{\partial u_i}{\partial m_i} \cdots 0)^\top$.

[17] The first order condition from equation 6 yields, $\sum_j \frac{\partial p_j}{\partial m_i} = \theta \sum_j x_{ij} u'(m_j)$, is positive as $u'(.)$ is increasing. The second order sufficiency condition gives, $\sum_j \frac{\partial^2 p_j}{\partial m_i^2} = \theta \sum_j x_{ij} u''(m_j)$ the sum of probabilities to be concave because of the diminishing marginal returns from money

Comparative Statics

The objective of the lobby group is to influence the legislators to choose policy A in exchange of some monetary payments. In other words, the lobby maximizes the sum of probabilities of votes in favour of policy A. In the following proposition, we analyze the effect of role of money and legislator's bias on the lobby's objective.

PROPOSITION 1. *For any given network* $\mathbb{G}$ *,* **(a)** *the Nash equilibrium transfer* m_i^* *to each legislator* $i \in \tau$ *depends on the Bonacich centrality vector,* **(b)** *the equilibrium sum of probabilities* $\mathscr{P}$ *is increasing in* M*, and* **(c)** *if* $B_s > B_a$ *, then* $\frac{\partial \mathscr{P}}{\partial \sigma_s} < 0$.

Part(a) in proposition entails that in equilibrium each legislator i receives monetary contribution according to the Bonacich centrality vector. The proof of the proposition is available in the Appendix A.2b where the equilibrium transfer to the legislator j is $m_j^* = m_j^*(M, \mathbf{b})$. If we assume a logarithmic[18] utility function i.e. $u(m_i) = log(m_i)$ then each legislator receives transfer proportional to their individual weighted Bonacich centrality i.e.

$$m_i^* = \frac{b_i}{B_\tau} \cdot M\Big(\frac{B_\tau}{B_\tau + B_{\tau'}}\Big) = \frac{b_i}{B} \cdot M$$

where B is the sum of the individual Bonacich centralities of all legislators. Note, the equilibrium transfer to each legislator $i \in \tau$ is proportional to their relative centrality within the party and the total money to the party.

Part(b) explains that more money never disadvantages the legislator. If the budget allocation of the lobbyist improves then they will always use it in their favour to improve upon the sum of probabilities of votes for A. Any increase in the budget allocation for lobbying activities will lead to an increase in

[18]This class of result will also hold for iso-elastic or CRRA utility functions:

$$u(m_i) = \begin{cases} \dfrac{(m_i)^{1-\gamma} - 1}{1-\gamma} & if \quad \gamma \neq 1 \\ \log(m_i) & if \quad \gamma = 1 \end{cases}$$

where $u'(m_i) = (m_i)^{-\gamma}$. So, the equilibrium transfer to each legislator is $m_i^* = \frac{b_i^{\frac{1}{\gamma}}}{\sum_j b_j^{\frac{1}{\gamma}}} M$

vote share. Differentiating the sum of probabilities $\mathscr{P}$ with respect to M we get, $\frac{d\mathscr{P}}{dM} > 0$.

Part(c) demonstrates any increase in the legislators biased towards the status quo policy disadvantages the lobby if legislators in party-s are 'better connected' than legislators in party-a. We define, legislators in party-s are 'better-connected' than those in party-a if $B_s > B_a$. Any increase in the legislator's bias towards status quo policy gives $\frac{d\mathscr{P}}{d\sigma_s} < 0$. If legislators in party-s are relatively better connected than that of party-a, then an increase in σ_s increase the likelihood of legislators choosing policy S over A. Hence, any increase in σ_s will disadvantage the lobby.

If none of the legislators in either party are connected with other legislators then the Bonacich centrality of i is 1. Thus, the sum of Bonacich centrality of each party equals the number of members in the party i.e. $B_s = n_s$ and $B_a = n_a$. In a fully disconnected network, if the number of legislators in party-s exceeds the number of legislators in party-a then any increase in σ_s will disadvantage the lobby.

Example 1. (Logarithmic Utility) For any given graph $\mathbb{G}$ and a logarithmic utility function $u(m_i) = \log m_i$ in equation 7, the first order condition yields the marginal cost of resources. We get,

$$\lambda^* = \frac{b_i}{m_i} \qquad \text{and} \qquad \sum_i m_i = M$$

for all i. By algebraic manipulation, the equilibrium transfer is $m_i^* = \frac{b_i}{B} \cdot M$. Notice that the ratio of equilibrium transfers between two legislators is proportional to their Bonacich centrality i.e. $\frac{m_i^*}{m_j^*} = \frac{b_i}{b_j}$. Ceteris paribus, any improvement in the centrality of any two legislators i and j will increase their centrality and their equilibrium transfer. From the indirect sum of probability function, we get

$$\mathscr{P}(\mathbf{m}^*) \ = \ \frac{n}{2} \ + \ \theta\big[\sum_j b_j \cdot \log\big(\frac{b_j}{B} \cdot M\big) \ - \ \sigma_s B_s \ + \ \sigma_s B_a\big] \tag{7a}$$

We have already shown that the optimal transfer to the legislator is proportional to their Bonacich centrality. Differentiating the sum of probabilities of the lobbyist for A with respect to the available budget M and the status quo bias conforms our proposition if party-s is relatively better connected.

Network Effects

In this subsection, we consider the marginal impact of changes in the network structure on the lobby. Specifically, we consider a pair of networks where one network is 'bigger' than another, i.e. the bigger network has at least one additional link relative to the smaller network.

Consider any incomplete network $\mathbb{G}_\tau$ where $\mathbb{G} = \bigcup_\tau \mathbb{G}_\tau$ and $\mathbb{G}^\oplus = \mathbb{G}^\oplus_\tau \cup \mathbb{G}_{\tau'}$ with $\mathbb{G}_\tau \subset \mathbb{G}^\oplus_\tau$. Thus $\mathbb{G}, \mathbb{G}^\oplus \in \mathscr{G}$ where $\mathbb{G} \subset \mathbb{G}^\oplus$ and $\mathscr{G}$ is the set of all possible networks. We know, network $\mathbb{G}^\oplus{}_\tau$ has atleast one additional connection compared to $\mathbb{G}_\tau$. If $\mathbb{G} \subset \mathbb{G}^\oplus$, then by definition $\zeta(\mathbb{G}^\oplus) > \zeta(\mathbb{G})$. A bigger network implies higher maximum eigenvalues $\max\{\zeta_i(\mathbb{G}^\oplus)\} > \max\{\zeta_i(\mathbb{G})\}$ and thus the Bonacich centrality vectors of the two networks are such that $\mathbf{b}^\oplus > \mathbf{b}$ where $b_i^\oplus > b_i$ for all $i \in \tau$ and $b_i^\oplus = b_i$ for all $i \in \tau'$.

Without any loss of generality, we assume any incomplete network $\mathbb{G}_s$ where $\mathbb{G} = \mathbb{G}_s \cup \mathbb{G}_a$ and $\mathbb{G}^\oplus = \mathbb{G}^\oplus_s \cup \mathbb{G}_a$ with $\mathbb{G}_s \subset \mathbb{G}^\oplus_s$. The Bonacich centrality vector of $\mathbb{G}^\oplus$ is greater than that of $\mathbb{G}$ i.e. $\mathbf{b}^\oplus > \mathbf{b}$ where $b_i^\oplus > b_i$ for all $i \in s$ and $b_i^\oplus = b_i$ for all $i \in a$. Assume that legislators in party-s have a positive status quo bias σ_s which implies legislators in party-a have a bias $-\sigma_s$ (Assumption 1). The equilibrium the optimal transfer vectors are $\mathbf{m}^*$ and $\mathbf{m}^*_\oplus$. Plugging the equilibrium transfer vector in equation 7 we get the equilibrium sum of probabilities under both networks $\mathbb{G}$ and $\mathbb{G}^\oplus$:

$$\begin{aligned} \mathscr{P}(\mathbf{m}^*, \mathbb{G}) &= \frac{n}{2} + \theta\big[\sum_j u(m_j^*) b_j - \sigma_s B_s + \sigma_s B_a\big] \qquad (7b) \\ \mathscr{P}(\mathbf{m}^*_\oplus, \mathbb{G}^\oplus) &= \frac{n}{2} + \theta\big[\sum_j u(m_{j\oplus}^*) b_j^\oplus - \sigma_s B_s^\oplus + \sigma_s B_a\big] \end{aligned}$$

Our next proposition demonstrates that if the status quo bias is below a critical level then the lobby benefits from a bigger network.

PROPOSITION 2. *For any incomplete network* $\mathbb{G}_\tau$ *where* $\mathbb{G}_\tau \subset \mathbb{G}_\tau^\oplus$ *with* $\mathbb{G} = \bigcup_\tau \mathbb{G}_\tau$ *and* $\mathbb{G}^\oplus = \mathbb{G}_\tau^\oplus \cup \mathbb{G}_{\tau'}$, *the equilibrium sum of probabilities* $\mathscr{P}(\mathbb{G}^\oplus \,|\sigma_s) \geq \mathscr{P}(\mathbb{G} \,|\, \sigma_s)$ *if* $\sigma_s \leq \widehat{\sigma}_s(\mathbb{G}, \mathbb{G}^\oplus)$.

Proposition 2 shows that a bigger network benefits the lobby if the status quo bias of the legislator is below a critical threshold. The lobby prefers the alternative policy A and maximises the sum of probabilities for A. We measure the impact of a bigger network on $\mathscr{P}$ when the status quo bias of the legislators are positive. The optimal transfer vectors to legislators in $\mathbb{G}$ and $\mathbb{G}^\oplus$ are $\mathbf{m}^*$ and $\mathbf{m}_\oplus^*$ respectively. Comparing the equilibrium sum of probabilities in equation 7b, we get the critical status quo bias $\widehat{\sigma}_s(\mathbb{G}, \mathbb{G}^\oplus)$ below which the bigger network benefits the lobby.

If the legislators do not have a status quo bias i.e. if $\sigma_s = 0$ than a bigger network never disadvantages the lobby since $\mathscr{P}^\oplus \geq \mathscr{P}$ where $\mathscr{P}^\oplus = \mathscr{P}(\mathbf{m}_\oplus^*, \mathbb{G}^\oplus)$ and $\mathscr{P} = \mathscr{P}(\mathbf{m}^*, \mathbb{G})$. A bigger network in τ benefits lobby if the status quo bias is reasonably small. In other words, additional connections increase strategic complementarity in actions among legislators in a given party. For example, a bigger network in party-s increase the likelihood of legislators in s benefiting from policy S if $\sigma_s > \widehat{\sigma}_s(\mathbb{G}, \mathbb{G}^\oplus)$ because of the increased positive spillover from conforming with neighbours. This is due to the improved spillover effect from complementarity in the legislators voting decisions. If the bias σ_s is below the critical threshold $\widehat{\sigma}_s$ the lobby utilize the network spillover in their favour to overcome the status quo bias among s with money. If the budget available to the lobbyist is inadequate to overcome the effect of the status quo bias, a bigger network in s increases the likelihood of policy S being chosen.

We show the marginal impact of a bigger network on the lobby. If the status quo bias is below $\widehat{\sigma}_s(\mathbb{G}, \mathbb{G}^\oplus)$, the effect of money to legislators overwhelms σ_s and the lobby benefits from the complementarity of the bigger network. On the other hand, if the status quo bias exceeds $\widehat{\sigma}_s(\mathbb{G}, \mathbb{G}^\oplus)$ the budget available to the lobby is inadequate to overcome the effect of bias and the lobby benefits from the smaller network.

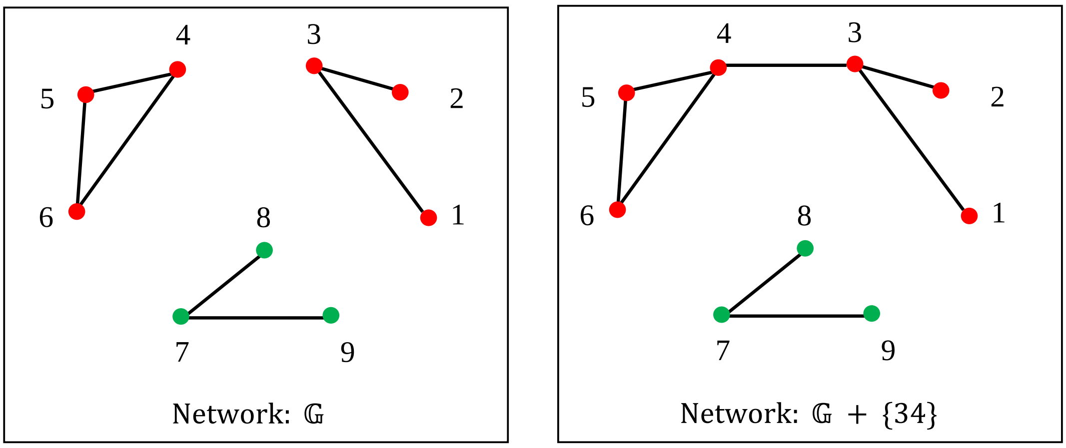

Figure 1: A Simple Network

We assume an incomplete network $\mathbb{G}_s$ where $\mathbb{G} = \mathbb{G}_s \cup \mathbb{G}_a$ and $\mathbb{G}^{\oplus} = \mathbb{G}_s^{\oplus} \cup \mathbb{G}_a$ with $\mathbb{G}_s \subset \mathbb{G}_s^{\oplus}$. We know, the equilibrium sum of probabilities of legislators voting for A under both networks $\mathbb{G}$ and $\mathbb{G}^{\oplus}$ are $\mathscr{P}(\mathbf{m}^*)$ and $\mathscr{P}(\mathbf{m}_{\oplus}^*)$ respectively. From proposition 2, the bigger network $\mathbb{G}^{\oplus}$ benefits lobby if the critical value of the status quo bias is below the critical level of bias $\hat{\sigma}_s(\mathbb{G}, \mathbb{G}^{\oplus})$. The critical value of the bias depends on the initial and the final networks $\mathbb{G}$ and $\mathbb{G}^{\oplus}$. There may exist many networks which are bigger than $\mathbb{G}$. Let $\mathscr{Z}(\mathbb{G})$ is the set of all possible networks bigger than $\mathbb{G}$. For every $\mathbb{G}^{\oplus} \in \mathscr{Z}(\mathbb{G})$, there exists a critical bias $\hat{\sigma}_s^{\mathscr{Z}}(\mathbb{G}, \mathbb{G}^{\oplus})$ for all networks bigger than $\mathbb{G}$. Hence, we rank all the critical values of status quo bias where $\underline{\hat{\sigma}_s} = \min\left\{\hat{\sigma}_s^{\mathscr{Z}}(\mathbb{G}, \mathbb{G}^{\oplus})\right\}$ and $\overline{\hat{\sigma}_s} = \max\left\{\hat{\sigma}_s^{\mathscr{Z}}(\mathbb{G}, \mathbb{G}^{\oplus})\right\}$. In the following proposition we show the minimum critical value of status quo bias below which any network bigger than $\mathbb{G}$ never disadvantages the lobby and the maximum critical value of bias above which any network bigger than $\mathbb{G}$ always disadvantages the lobby.

PROPOSITION 3. *For any incomplete $\mathbb{G}_\tau$ where $\mathbb{G}_\tau \subset \mathbb{G}_\tau^{\oplus}$ with $\mathbb{G} = \mathbb{G}_\tau \cup \mathbb{G}_{\tau'}$ and $\mathbb{G}^{\oplus} = \mathbb{G}_\tau^{\oplus} \cup \mathbb{G}_{\tau'}$, there exist $\underline{\hat{\sigma}_s}$ and $\overline{\hat{\sigma}_s}$ where any bigger $\mathbb{G}^{\oplus}$, **(a)** never disadvantages lobby if $\sigma_s \leq \underline{\hat{\sigma}_s}$, **(b)** always disadvantages lobby if $\sigma_s > \overline{\hat{\sigma}_s}$, and **(c)** the effect is ambiguous if $\sigma_s \in (\underline{\hat{\sigma}_s}, \overline{\hat{\sigma}_s}]$.*

In Proposition 3, we characterize the critical levels of status quo bias and the

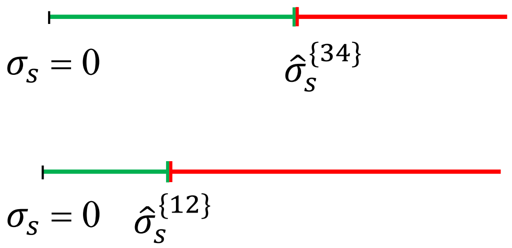

Figure 2: Critical Status Quo Biases

marginal impact on any bigger network on the lobby. This directly follows from Proposition 2 which holds for a specific $\mathbb{G}^{\oplus}$. For Proposition 3, the proof is quite straightforward. For any given network $\mathbb{G}$, there exists a minimum critical value of the status quo bias $\underline{\hat{\sigma}_s}$ which is calculated by comparing the sum of probabilities of legislators voting for A under $\mathbb{G}$ and $\mathbb{G}^{\oplus} \in \mathcal{Z}(\mathbb{G})$. If the status quo bias doesn't exceed the minimum critical value, then any network bigger than $\mathbb{G}$ never disadvantages lobby. In other words, if $\sigma_s \leq \underline{\hat{\sigma}_s}$ then a bigger network never disadvantage the lobby.

Using similar arguments, for any given network $\mathbb{G}$, there exists a maximum critical value of the status quo bias $\overline{\hat{\sigma}_s}$. If the status quo bias exceeds the maximum critical value, then any network bigger than $\mathbb{G}$ never advantages lobby. In other words, if $\sigma_s > \overline{\hat{\sigma}_s}$ then a bigger network never advantages the lobby. The result is ambiguous for $\sigma_s \in (\underline{\hat{\sigma}_s}, \overline{\hat{\sigma}_s}]$ and depends on the specific $\mathbb{G}^{\oplus}$. We demonstrate this result with a simple example where the bigger network has only one additional connection in party-s and keep the number of connections in party-a constant.

If the lobby has adequate resources and the status quo bias is below the minimum critical value, then the lobby advantages from the bigger network by using money to overcome the status quo bias. A bigger network increases the spillover effect from voting for a given policy, substantial resources helps lobby overcome the status quo bias and use the spillover in their advantage by increasing the likelihood of policy A being chosen. Similarly, if the lobby has insufficient resources to overcome σ_s an increased spillover from a bigger network reduces the likelihood of policy A being chosen.

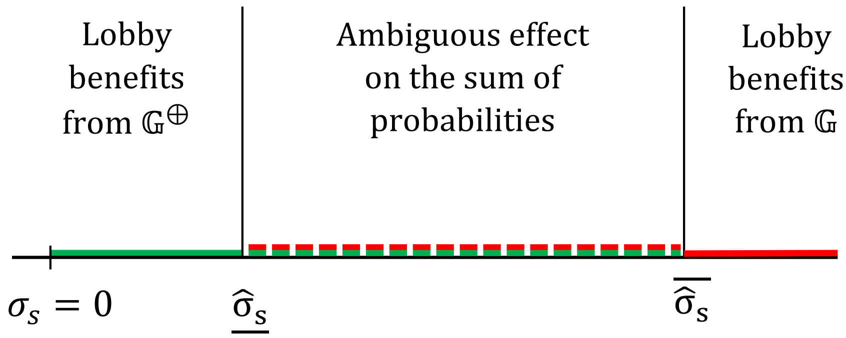

Figure 3: Impact of a Bigger Network

Example 2. Assume an incomplete network $\mathbb{G}_s$ where $\mathbb{G} = \mathbb{G}_s \cup \mathbb{G}_a$ and $\mathbb{G}^{\oplus} = \mathbb{G}_s^{\oplus} \cup \mathbb{G}_a$ with $\mathbb{G}_s \subset \mathbb{G}_s^{\oplus}$. Consider $\mathbb{G}_s^{\oplus} = \mathbb{G} + \{jk\}$ where $\{jk\} \notin \mathbb{G}$ but $\{jk\} \in \mathbb{G}^{\oplus}$. The two agents $j \in \tau$ and $k \in \tau$ are not connected in $\mathbb{G}_s$ but are connected in $\mathbb{G}_s^{\oplus}$. An incomplete graph $\mathbb{G}_s$ implies there exist at least one unconnected edge between any two agents i and j such that $g_{ij} = 0$. Let $\mathcal{Z}(\mathbb{G})$ be the set of all unconnected links in graph $\mathbb{G}$. Proposition 2 shows a bigger network $\mathbb{G}^{\oplus}\{ik\}$ is beneficial to the lobbyist i.e. $\mathscr{P}(\mathbb{G}^{\oplus}\{ik\}) \geq \mathscr{P}(\mathbb{G})$ if $\sigma_s \leq \hat{\sigma}_s(\mathbb{G}, \mathbb{G}^{\oplus}\{ik\})$ for all $\{ik\} \in \mathcal{Z}(\mathbb{G})$. Hence we find the critical values of status quo biases for all possible elements in $\mathcal{Z}(\mathbb{G})$. The minimum critical status quo bias is $\underline{\hat{\sigma}_s} = \min\left\{\hat{\sigma}_s^{\mathcal{Z}}(\mathbb{G}, \mathbb{G}^{\oplus})\right\}$ and the maximum critical status quo bias is $\overline{\hat{\sigma}_s} = \max\left\{\hat{\sigma}_s^{\mathcal{Z}}(\mathbb{G}, \mathbb{G}^{\oplus})\right\}$.

The network $\mathbb{G}$ has 9 legislators where $n_s = 6$ and $n_a = 3$. The nodes in red represent the legislators of party-s and the nodes in green are members of party-a. The bigger network $\mathbb{G}_s^{\oplus}$ has one additional connection between the legislators 3 and 4 i.e. $\mathbb{G}_s + \{34\}$. The network $\mathbb{G}$ in Figure 1 illustrates that legislators $\{34\}$, $\{12\}$, $\{26\}$, $\cdots$ are not connected in network $\mathbb{G}$. Note, legislators in s and a are not connected. The additional connection $\{34\}$ benefits the lobby if $\mathscr{P}(\mathbb{G}^{\oplus}\{34\}) > \mathscr{P}(\mathbb{G})$ if $\sigma_s \leq \hat{\sigma}_s(\mathbb{G}, \mathbb{G}^{\oplus}\{34\})$. The critical value of the bias is $\hat{\sigma}_s^{\{34\}}$. Similarly, instead of $\{34\}$ if we add $\{12\}$ in $\mathbb{G}$ we get $\hat{\sigma}_s^{\{12\}}$. Using this argument we can get critical value of status quo bias for every unconnected node in $\mathcal{Z}$. Hence, we find the critical values $\underline{\hat{\sigma}_s}(\mathbb{G})$ and $\overline{\hat{\sigma}_s}(\mathbb{G})$.

For the remaining analysis, we assume a logarithmic[19] utility function of the legislator. In the following proposition we see the impact of a bigger net-

[19]This class of result may also hold for iso-elastic utility functions.

work on the equilibrium transfer to the legislator.

PROPOSITION 4. *For any incomplete* $\mathbb{G}_\tau$ *where* $\mathbb{G} = \mathbb{G}_\tau \cup \mathbb{G}_{\tau'}$ *and* $\mathbb{G}^\oplus = \mathbb{G}_\tau^\oplus \cup \mathbb{G}_{\tau'}$, ***(a)*** m_i^* *increases in* B_τ *for at least one* $i \in \tau$, ***(b)*** m_i^* *is monotonically decreasing in* $B_{\tau'}$ *for all* $i \in \tau'$, *and* ***(c)*** *the total transfer to party* τ *is increasing in* B_τ.

Proposition 4 highlights the effect of a bigger network on the legislators. Let's assume a logarithmic utility function and an incomplete network $\mathbb{G}_s$ where $\mathbb{G} = \mathbb{G}_s \cup \mathbb{G}_a$ and $\mathbb{G}^\oplus = \mathbb{G}_s^\oplus \cup \mathbb{G}_a$. Part(a) of the proposition entails that in a bigger network atleast one legislator in party-s receives a higher transfer. Given $\mathbb{G}_a$, a bigger $\mathbb{G}_s$ implies increased complementarity effect in party-s. The lobby utilizes the increased complementarity effect in their favour by reallocating resources from legislators of party-a to some legislators in party-s. In a bigger network $\mathbb{G}_s$, the sum of centralities of legislators increases i.e. $B_s^\oplus > B_s$. Since, legislators in party-a are not connected with legislators in party-s (Assumption 1), B_a is constant. Thus, the relative centrality of legislators in party-a is $\frac{b_i}{(B_s+B_a)}$ decreases. The Nash equilibrium transfer to legislators in a goes down. This is demonstrated in part(b) and part(c) of the proposition. Naturally, the sum of equilibrium transfers M_a to legislators in party-a decreases. So, atleast one of the legislators in party-s receives a higher equilibrium transfer.

Conflict

In the preceding analysis, our model provides a prediction of the equilibrium transfers to legislators when the legislators in each party are connected with only positive externalities within themselves. In this subsection, the legislators in each party benefits from the positive externalities of the legislators they are friends with and suffer from negative externalities of the oppositions decision.

For simplicity, we assume $g_{ij} \in \{-1, 0, 1\}$ where the link between legislators are unweighted. Here we study both strategic substitutability and complementarity effect on the legislators behaviour. g_{ij}^τ represents social connections between two legislators i and j where both legislators are members of the same party. For any two legislators of the same party τ, a connection $g_{ij}^\tau = g_{ji}^\tau = 1$

where $i, j \in \tau$ indicates that two legislators i and j are connected. Legislators i and j are defined as *compatible* neighbours if $g_{ij}^{\tau} = 1$ which captures the strategic complementarity effect of j's action on i's vote. For example, if legislator i of party s is connected to another member j of the same party, i.e. $g_{ij}^{s} = 1$, then i and j are compatible neighbours.

We also assume each member of party τ is negatively linked to every member of party τ' i.e. $g_{ij}^{\tau'} = \{-1\}$ where i is a member of party τ and j is from party τ'. In such cases legislators i and j are defined as *conflicting* neighbours. $g_{ij}^{\tau\tau'} < 0$ is the strategic substitutability effect of the action of j's action on i's vote. Again, if a legislator i of party s is connected to another legislator j of party a. The payoff of a legislator $i \in \tau$ for choosing policy A policy is

$$\Pi_i(A) = u(m_i) + \delta \sum_{j=1}^{n_\tau} g_{ij}^{\tau} v_j(A) + \sum_{j'=(n_\tau+1)}^{n} \kappa_\tau g_{ij'}^{\tau'} v_{j'}(A)$$

where j, j' votes for same policy A. The utility of any legislator i if she votes for policy S is:

$$\Pi_i(S) = \sigma_\tau + \delta \sum_{j=1}^{n_\tau} g_{ij}^{\tau} v_j(S) + \sum_{j'=(n_\tau+1)}^{n} \kappa_\tau g_{ij'}^{\tau\tau'} v_{j'}(S) + \epsilon_i$$

where j, j' votes for same policy S with legislator $j \in \tau$ and $j' \in \tau'$ where $\tau \in \{s, a\}$.

The first two terms in the above equations are similar to the legislators payoff in the baseline model. If legislator i is directly linked with j from her own party τ and her vote is similar to j's then she benefits from the peer-effects of social interactions through a positive spillover effect. The positive network spillover effect δ is assumed to be homogeneous across all legislators i.

The negative spillover effect $\kappa_\tau \in \{\kappa_s, \kappa_a\}$ can be interpreted as the degree of conflict. If a member $j' \in \tau'$ of the opposing party votes in line with $i \in \tau$, she receives a disutility κ_τ for conforming with an opposing party. κ_τ is interpreted as the degree of conflict or aversion agent i gets for voting in line with a member of the opposing party. κ_τ is common knowledge. Every member in party s loathes voting in line with legislators of the opposing party a by a magnitude of κ_s. Similarly, legislators in a receives disutility of κ_a for conforming votes with any member in s. The net effect of spillovers on a legislator depends on her party affiliation and the action of her neighbours. If we assume the conflict $\kappa_\tau = 0$, we get our baseline model.

The utility from money is logarithmic $u(m_i) = \log m_i$ and $m_i > 0$. From the first order condition we get,

$$\omega_s \cdot \frac{b_i}{m_i} = \omega_a \cdot \frac{b_j}{m_j}$$

for all $i \in \tau$ and $j \in \tau'$. The Lagrangian multiplier is proportional to the equilibrium transfer m_i^* to the legislator. The equilibrium transfer to any individual $i \in \tau$ is

$$m_i^* \;=\; \left(\frac{b_i}{B_\tau}\right) \cdot M\phi_\tau$$

where $\phi_s = \left[\frac{1}{1+\left(\frac{(1/B_s)-\kappa_s^*}{(1/B_a)-\kappa_a^*}\right)}\right]$ and $\phi_a = \left[\frac{1}{1+\left(\frac{(1/B_a)-\kappa_a^*}{(1/B_s)-\kappa_s^*}\right)}\right]$ are the proportions that determines the total monetary allocation to each party. Using the optimal transfer $m_i^*(\boldsymbol{b}, M)$, we get the total probability of the legislators to vote in favour of the new policy A:

$$\begin{aligned}
\mathscr{P}(\mathbf{m}^*) \;=\; & \frac{n}{2} \;+\; \frac{\theta(1-\kappa_a^* B_a)}{(1-\kappa_a^*\kappa_s^* B_a B_s)} \sum_{j\in s} u(m_j^*) b_j \;+\; \frac{\theta(1-\kappa_s^* B_s)}{(1-\kappa_a^*\kappa_s^* B_a B_s)} \sum_{j'\in a} u(m_{j'}^*) b_{j'} \\
& + \; \frac{\theta\sigma_s}{(1-\kappa_a^*\kappa_s^* B_a B_s)} (B_a - B_s) \;+\; \frac{\theta\sigma_s B_a B_s}{(1-\kappa_a^*\kappa_s^* B_a B_s)} (\kappa_a^* - \kappa_s^*)
\end{aligned}$$

The equilibrium transfer to each legislator i in party $\tau \in \{s, a\}$ is a function of their relative position in the party and the total budget allocated to the party. This is demonstrated in the following proposition.

PROPOSITION 5. *For any given network* $\mathbb{G}$, ***(a)*** *the Nash equilibrium transfer* m_i^* *to each legislator* $i \in \tau$ *depends on the conflict-weighted Bonacich centrality vec-*

*tor, and **(b)** the equilibrium $\mathscr{P}$ is decreasing in the status quo bias σ_s if $\kappa_s \geq \kappa_s + \left(\frac{B_a - B_s}{2\theta B_a B_s}\right)$.*

Part(a) of the proposition 5 entails that in equilibrium each legislator $i \in \tau$ receives monetary contribution according to their Bonacich centrality vector within their party. For a logarithmic utility function each legislator receives transfer proportional to their weighted individual Bonacich centrality $\left(\frac{b_i}{B_s}\right) M\phi_s$ (see appendix). The equilibrium transfer of each legislator is a function of their relative centrality in the party and total share of funds available to the party. In equilibrium, members in party s gets $\left(\frac{b_i}{B_s}\right) M_s$ and legislators in a receives $\left(\frac{b_i}{B_a}\right) M_a$.

Part(b) of the proposition demonstrates the sum of probabilities $\mathscr{P}$ decreases with increase in status quo bias i.e. $\frac{d\mathscr{P}}{d\sigma_s} < 0$ if $\left(\frac{B_a - B_s}{B_a B_s}\right) < 2\theta(\kappa_s - \kappa_s)$. If the degree of conflict κ_s is sufficiently higher than κ_a i.e. $\kappa_s > \kappa_a + \frac{B_a - B_s}{2\theta B_a B_s}$. If the legislators in party-a is better connected (higher Bonacich Centrality) than party-s, a sufficiently high conflict of legislators in party-s will disadvantage the lobby. A sufficiently higher conflict of the legislators in s can offset the benefits to the lobby from a well connected party-a. The legislators in party-s have a status quo bias $\sigma_s > 0$ while the members in party-a members have equal and opposite bias towards policy A. An increase in status quo bias σ_s increases the likelihood of the legislators of s choosing policy S as it increases expected utility from the positive spillover effect within the party because of the complementarity in actions. Simultaneously, the expected utility of the legislators of a voting for policy A also increases. If the conflict κ_s is sufficiently high then the legislators in party s are likely to choose policy S to benefit from the complementarity effect and reduce the substitutability effect from opposition members voting for S. In equilibrium, the lobby uses a considerable portion of her budget efficiently among the legislators in party-s to influence their decisions towards policy A.

For a given budget M, any increase in conflict of party-s decreases the expected payoff of the legislators in s because of the negative spillover effects from similar votes by conflicting neighbours. The lobby's equilibrium transfer to legislators in s increase to influence them towards the alternative policy A. This result is discussed in the following proposition.

PROPOSITION 6. *Assume a logarithmic utility function, (a) the equilibrium transfers made to any individual $i \in s$ is monotonically decreasing in κ_a and monotonically increasing in κ_s, and (b) the total contribution to party-s is monotonically decreasing in κ_a and increasing in κ_s.*

For logarithmic utility function any increase in the degree of conflict parameter κ_s will always improve the equilibrium transfer to $i \in s$ and any increase in the conflict parameter of a will be reduce the equilibrium transfer of i since $\frac{dm_i^*}{d\kappa_s} > 0$ and $\frac{dm_i^*}{d\kappa_a} < 0$. The first half of the proposition describes that any increase in the the degree of conflict κ_s improves the centrality of the legislators of s and their individual budget allocation increases. Legislators in s are connected via negative links with their conflicting neighbours and gets disutility for conforming with oppositions' votes. If the degree of conflict of the opposition party κ_a increases, legislators in s are negatively affected which is reflected in their equilibrium payoff.

This proposition shows any increase in κ_s will improve the total budget allocated to the members of party-s. As κ_s increases, the expected utility of a legislator from a policy reduces. Because of the substitutability in actions, the transfer to each legislator in s goes down. Thus, the lobby announces more transfer to legislators in s to maintain the equilibrium sum of probabilities. So any increase in κ_s the total share of funds contributed to party-s increases and hence individual share for all members in s increases. Similarly if κ_a increases the proportion of funds going to party-a decreases. As the degree of conflict of given party increases, the Centrality of the opposition legislators are worsened because of negative linkage between parties. Any increase in κ_s impacts the utility of any legislator $i \in s$ as they want to distinguish their votes from the oppositions'. This increase in κ_s has a negative impact on the centrality of opposing party legislators and positively affects their own centrality because of the strategic substitutability effect.

Additionally, the effect of a bigger network (higher sum of Bonacich centrality)of legislators of party-a on the equilibrium monetary transfer to any individual $i \in s$. We notice that $\frac{dm^*_{i\in a}}{d\sigma_s} < 0$ i.e. any increase in the density of graph among legislators of s will be worsen the equilibrium transfers to all the legislators of a. In equilibrium, the transfers to every individual $i \in a$ unambiguously decrease. The total fund allocated to party-a falls. Hence, any bigger network in party-s will improve the total fund allocation to s i.e. $\frac{dM_s}{d\sigma_s} > 0$ and any increase in connections among legislators in L will reduce the equilibrium fund allocation to a, $\frac{dM_a}{d\sigma_s} < 0$.

Single Party Framework

In the previous subsection we assume that there are two parties where n_s legislators are biased towards policy S and n_a legislators are biased towards policy A. In this subsection we provide an analysis where the number of legislators in a given party converges to 0. For simplicity, assume $n_a \to 0,\ n_s \to n$ and $\sigma_s = \sigma$. This is an extreme case where all the legislators are only biased towards the status quo policy. The role of the lobby is to influence the legislators towards the lobby's preferred policy A. The equilibrium sum of probabilities of the legislators voting for A is given by

$$\mathscr{P}(\mathbf{m}^*) = \frac{n}{2} + \theta\big[\sum_j u(m^*_j)\cdot b_j(\delta^*,\mathbb{G}) - \sigma B\big]$$

The equilibrium predictions of our analysis, remains identical to the baseline model: the equilibrium transfers to legislator i by lobby is proportional to her relative Bonacich centrality.

PROPOSITION 7. *For any given network* $\mathbb{G}$ *,* **(a)** *the Nash equilibrium transfer* m^*_i *to each legislator* $i \in \tau$ *depends on the Bonacich centrality vector, and* **(b)** *the equilibrium sum of probabilities* $\mathscr{P}$ *is decreasing in* σ*.*

Proposition 7 demonstrates that in equilibrium each legislator i receives monetary transfer according to the Bonacich centrality vector. The proof of the proposition is available in the appendix. For a logarithmic utility function each legislator receives transfer proportional to their individual Bonacich centrality i.e. $m_i^* = \frac{b_i}{B}$. More money never disadvantages the lobby. If the budget allocation of the lobby improves, they will always use it in their favour to improve upon the sum of probabilities of votes for A. Secondly, any increase in the status quo bias σ always disadvantages the lobby. An increase in the legislator's bias disadvantages the lobby. An increase in positive status quo bias increase the likelihood of policy S being chosen. Any increase in σ increases the expected payoff of legislators voting in favour of policy S. Hence, any increase in bias disadvantages the lobby.

PROPOSITION 8. *For any incomplete* $\mathbb{G}$ *where* $\mathbb{G} \subset \mathbb{G}^{\oplus}$ *there exist* $\underline{\widehat{\sigma}}$ *and* $\overline{\widehat{\sigma}}$ *such that any bigger network* $\mathbb{G}^{\oplus}$, ***(a)*** *never disadvantages lobby if* $\sigma \leq \underline{\widehat{\sigma}}$, ***(b)*** *always disadvantages lobby if* $\sigma > \overline{\widehat{\sigma}}$, *and* ***(c)*** *the effect is ambiguous if* $\sigma \in (\underline{\widehat{\sigma}}, \overline{\widehat{\sigma}}\,]$.

If the legislators have a positive bias towards the alternative policy A or a negative bias away from the status quo S then a bigger network always benefits the lobby. A negative status quo bias favours the lobby. A relatively well connected graph improves the sum of probabilities of A when the legislators are biased towards A (see Appendix). In this case, all the legislators are biased towards A and the lobby announces transfers to legislators for voting in favour of A, hence a bigger network has increased complementarity effects which always favours lobby. The rest is analogous to our Proposition 3. If the status quo bias is less than the critical $\underline{\widehat{\sigma}} = \min\{\sigma^{\mathcal{I}}(\mathbb{G}, \mathbb{G}^{\oplus})\}$, the lobby utilizes her resources and increased network spillover effect to offset the status quo bias. Similarly, if the status quo bias is larger than the critical $\overline{\widehat{\sigma}} = \max\{\sigma^{\mathcal{I}}(\mathbb{G}, \mathbb{G}^{\oplus})\}$, the lobby benefit from a sparser network as she lacks adequate resources to offset the status quo bias.

Conclusion

In this paper, we present a theory on the role of transfers on voting decisions of connected legislators. We exploit the idea of the lobby using money as a tool to influence connected legislators by paying them according to their Bonacich centrality vector. We measure the sum of voting probabilities for the new policy and its responsiveness to the changes in the budget constraint and the legislator's bias. This approach provides us with a premise to do a comparative study between voting outcomes under different networks. If a party biased towards status quo policy is relatively well connected, any increase in status quo bias disadvantages lobby. We also provide a positive sharp cut-off for the critical value of status bias beyond which the lobbyist cannot gain from a bigger network. We show that a bigger network is beneficial to the lobby if the bias is smaller than a minimum critical value and hurts her when the actual value of the bias is beyond the maximum critical value. Additionally, any bigger network in the party biased towards the status quo reduces the transfers to legislators biased towards the alternative policy. Moreover, we provide some analysis of a model with conflict and a limiting case with legislators biased towards only one policy.

In the initial part of our analysis we have focused on equilibrium transfers and impacts of different networks under complementarities among the legislators actions. Later, we enrich our understanding of the role of monetary transfer on voting decisions of legislators connected by both positive and negative links. In other words, each legislator benefits of the positive links by conforming with neighbours vote within her own party and also gets disutility from voting in line with members from the opposition party. We infer that the lobbyist can influence connected legislators by paying them according to the Bonacich centrality vector. We have considered strategic substitutability and complementarity among legislators' ties. Any increase in the conflict of a given party improves the individual and the overall equilibrium payments of the party. The increase in conflict will worsen the position (centrality) of the opposition legislators because of the substitutability effect. We find that any additional links in a given party improves the overall fund allocation to that party. We see that for a relatively well connected graph, the legislators' bias towards new policy hurts the lobbyist if the status quo bias is above a critical threshold. One may also examine a model with strategic rather than probabilistic voting, incomplete or asymmetric information, and endogenize the degree of conflict between the parties.

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Appendix A

[A.1] The probability $p_j(\mathbf{m})$ of each legislator j for voting in favour of the new policy is derived from the cumulative distribution function of the uniform error distribution $\epsilon_i \sim U[-\frac{1}{2\theta}, \frac{1}{2\theta}]$:

$$-\frac{1}{2\theta} \leq \epsilon_i \leq u(m_i) - \sigma_\tau + \delta \sum_j g_{ij}(2p_j - 1)$$

Thus, the cumulative distribution function is as follows:

$$\begin{aligned} p_i(\mathbf{m}) &= \theta\big[u(m_j) - \sigma_\tau + \delta \sum_j g_{ij}(2p_j(\mathbf{m}) - 1) - (-\frac{1}{2\theta})\big] \\ p_i(\mathbf{m}) &= \frac{1}{2} + \theta\big[u(m_j) - \sigma_\tau + \delta \sum_j g_{ij}(2p_j(\mathbf{m}) - 1)\big] \end{aligned} \tag{A.1}$$

[A.2] **Proof of Proposition 1(a):** The sum of probabilities is differentiable and the inverse $(\mathbb{I} - \delta^* \mathbb{G}^\top)^{-1}$ exists(by), we get:

$$\begin{aligned} & J_i[\mathbf{p}] = \theta.(\mathbb{I} - \delta^* \mathbb{G})^{-1}.J_i[\mathbf{u}] \\ \Leftrightarrow\quad & J_i[\mathbf{p}]^\top = \theta.J_i[\mathbf{u}]^\top.(\mathbb{I} - \delta^* \cdot \mathbb{G}^\top)^{-1} \\ \Leftrightarrow\quad & J_i[\mathbf{p}]^\top.\mathbf{1} = \theta.J_i[\mathbf{u}]^\top.(\mathbb{I} - \delta^* \cdot \mathbb{G}^\top)^{-1}.\mathbf{1} \\ \Leftrightarrow\quad & J_i[\mathbf{p}]^\top.\mathbf{1} = \theta.J_i[\mathbf{u}]^\top.\mathbf{b}(\delta^*, \mathbb{G}^\top) \end{aligned} \tag{A.2a}$$

Solving the optimal value of transfer m_i^* from equation 7a and appendix A.2a we get,

$$\begin{aligned} & \lambda = \theta \cdot J_i[\mathbf{u}]^\top \cdot \mathbf{b}(\delta^*, \mathbb{G}^\top) = J_i[\mathbf{p}]^\top \cdot \mathbf{1} \\ \Leftrightarrow\quad & \lambda^* = J_i[\mathbf{u}]^\top \cdot \mathbf{b}(\delta^*, \mathbb{G}^\top) = u'(m_j) \cdot b_j; \qquad \forall\, j \in n \\ \Leftrightarrow\quad & \lambda^* = u'(m_1) \cdot b_1 = \cdots = u'(m_n) \cdot b_n \end{aligned}$$

Without any loss of generality we can include the constant θ in the Lagrangian parameter λ^*. Incorporating the above result $m_j = u'^{-1}\big(u'(m_i).\frac{b_i}{b_j}\big)$

in the budget equation of the lobbyist. We get,

$$\begin{aligned} M &= \sum_j m_j = \sum_j u'^{-1}\big(u'(m_i).\frac{b_i}{b_j}\big) \\ \Leftrightarrow \quad m_j^* &= m_j^*(M, \mathbf{b}) \end{aligned} \tag{A.2b}$$

[A.3] Proof of Proposition 2: Consider any incomplete network $\mathbb{G}_\tau$ where $\mathbb{G} = \mathbb{G}_s \cup \mathbb{G}_a$ and $\mathbb{G}^\oplus = \mathbb{G}_s^\oplus \cup \mathbb{G}_a$ with $\mathbb{G}_s \subset \mathbb{G}_s^\oplus$. Thus $\mathbb{G}, \mathbb{G}^\oplus \in \mathscr{G}$ where $\mathbb{G} \subset \mathbb{G}^\oplus$. We know, network ${\mathbb{G}^\oplus}_\tau$ has atleast one additional connection compared to $\mathbb{G}_\tau$. If $\mathbb{G} \subset \mathbb{G}^\oplus$, then by definition $\zeta(\mathbb{G}^\oplus) > \zeta(\mathbb{G})$. A bigger network implies higher maximum eigenvalues $\max\{\zeta_i(\mathbb{G}^\oplus)\} > \max\{\zeta_i(\mathbb{G})\}$ and thus the Bonacich centrality vectors of the two networks are such that $\mathbf{b}^\oplus > \mathbf{b}$ where $b_i^\oplus > b_i$ for all $i \in \tau$ and $b_i^\oplus = b_i$ for all $i \in \tau'$.

If we compare two given graphs $\mathbb{G}$ and $\mathbb{G}^\oplus$, the equilibrium transfer vectors are $\mathbf{m}^*$ and $\mathbf{m}_\oplus^*$ respectively where $m_{i\oplus} \geq m_i$ for some i and $m_{i\oplus} < m_i$ for others. We compare the the sum of probabilities for A for both networks $\mathscr{P}(\mathbb{G})$ and $\mathscr{P}(\mathbb{G}^\oplus)$ in equilibrium to find the critical level of status quo bias. The bigger network benefits lobby if

$$\begin{aligned} &\mathscr{P}(\mathbf{m}^*, \mathbb{G}) \;\leq\; \mathscr{P}(\mathbf{m}_\oplus^*, \mathbb{G}^\oplus) \\ &\sum_j u(m_j^*)b_j - \sigma_s B_s + \sigma_s B_a \;\leq\; \sum_j u(m_{j\oplus}^*)b_j^\oplus - \sigma_s B_s^\oplus + \sigma_s B_a \\ &\sigma_s \;\leq\; \frac{(\sum_j u(m_{j\oplus}^*)b_j^\oplus - \sum_j u(m_j^*)b_j)}{(B_s^\oplus - B_s)} \;=\; \widehat{\sigma}_s(\mathbb{G}, \mathbb{G}^\oplus) \end{aligned}$$

[A.4] Proof of Proposition 3: Consider any incomplete network $\mathbb{G}_\tau$ where $\mathbb{G} = \mathbb{G}_s \cup \mathbb{G}_a$ and $\mathbb{G}^\oplus = \mathbb{G}_s^\oplus \cup \mathbb{G}_a$ with $\mathbb{G}_s \subset \mathbb{G}_s^\oplus$. Thus $\mathbb{G}, \mathbb{G}^\oplus \in \mathscr{G}$ where $\mathbb{G} \subset \mathbb{G}^\oplus$. We know, network $\mathbb{G}_\tau^\oplus$ has atleast one additional connection compared to $\mathbb{G}_\tau$. $\mathscr{Z}(\mathbb{G})$ be the set of all networks bigger than $\mathbb{G}$ where $g_{ij} = 0$ for all $i \in \tau$ and $j \in \tau'$. The network $\mathbb{G}_{z(\mathbb{G})}^\oplus$ is bigger than $\mathbb{G}$ where $z(\mathbb{G}) \subset \mathscr{Z}(\mathbb{G})$. Hence, we can calculate the critical values of all networks bigger than $\mathbb{G}$. The maximum critical value is $\overline{\widehat{\sigma}_s} = \max_{z \subset \mathscr{Z}} \left\{\widehat{\sigma}_s^{z(\mathbb{G})}\right\}$ and the minimum critical value is $\underline{\widehat{\sigma}_s} = \min_{z \subset \mathscr{Z}} \left\{\widehat{\sigma}_s^{z(\mathbb{G})}\right\}$.

Any bigger network is beneficial to the lobby i.e. $\mathscr{P}(\mathbb{G}^{\oplus}_{z(\mathbb{G})}) > \mathscr{P}(\mathbb{G})$ when $\sigma_s < \underline{\widehat{\sigma}_s}$. This implies $\mathbb{G}^{\oplus}_{z(\mathbb{G})}$ benefits the lobby, if the actual value of the status quo bias is relatively small i.e. $\sigma_s \leq \underline{\widehat{\sigma}_s}$. Similarly, any bigger network disadvantages the lobby i.e. $\mathscr{P}(\mathbb{G}^{\oplus}_{z(\mathbb{G})}) \leq \mathscr{P}(\mathbb{G})$ when $\sigma_s > \overline{\widehat{\sigma}_s}$. This implies $\mathbb{G}^{\oplus}_{z(\mathbb{G})}$ definitely hurts the lobby, if the actual value of the status quo bias is relatively high i.e. $\sigma_s > \overline{\widehat{\sigma}_s}$.

[A.5] Proof of Proposition 4: Assume a logarithmic utility function $u(m_i) = log m_i$. The Nash equilibrium payments to each legislator is $m_i^* = \frac{b_i}{B} M$ where $B = B_a + B_s$. Consider any incomplete network $\mathbb{G}_s$ where $\mathbb{G} = \mathbb{G}_s \cup \mathbb{G}_a$ and $\mathbb{G}^{\oplus} = \mathbb{G}^{\oplus}_s \cup \mathbb{G}_a$ with $\mathbb{G}_s \subset \mathbb{G}^{\oplus}_s$. By definition $\zeta(\mathbb{G}^{\oplus}) > \zeta(\mathbb{G})$. A bigger network implies higher maximum eigenvalues $\max\{\zeta_i(\mathbb{G}^{\oplus})\} > \max\{\zeta_i(\mathbb{G})\}$ and the Bonacich centrality vectors of the two networks are such that $\mathbf{b}^{\oplus} > \mathbf{b}$ where $b^{\oplus}_i \geq b_i$ for all $i \in \tau$ and $b^{\oplus}_i = b_i$ for all $i \in \tau'$.

The equilibrium transfer to any legislator in the bigger network is given by $m_i^{*\oplus} = \frac{b_i}{B^{\oplus}} M$ where $B^{\oplus} = B_a + B^{\oplus}_s$. Any increase in the sum of Bonacich centralities of the legislators from B_s to $B^{\oplus}_s$ reduces the equilibrium payoff of the legislators in party-a. Without any loss of generality, holding $B_{\tau'}$ constant, any increase in B_τ reduces the equilibrium transfer from $\frac{b_i}{B} \cdot M$ to $\frac{b_i}{B^{\oplus}} \cdot M$ for all $i \in \tau'$. Thus, the total transfer to party-τ' goes down. Additionally, the transfers to the legislators in party-τ goes up or down depending on the relative centrality of the legislators i.e. the ratio $\frac{b^{\oplus}_i}{B^{\oplus}}$.

[A.7] Proof of Proposition 8(a): By definition, when $\mathbb{G} \subset \mathbb{G}^{\oplus}$ then, $b_j < b^{\oplus}_j$ for all j. Here we compare the sum of probabilities of votes in favour of A for different graphs. Take any two graph $\mathbb{G}$ and $\mathbb{G}^{\oplus}$ where one is a subset of the other $\mathbb{G} \subset \mathbb{G}^{\oplus} \in \mathscr{G}$. Let, the equilibrium the optimal transfer vectors are $\mathbf{m}^*$ and $\mathbf{m}^*_{\oplus}$. If the all legislators are unbiased or have a bias towards the alternative policy, then bigger network always benefits the lobby's objective.

For a graph $\mathbb{G}$ with $\mathbf{m}^*$ as the optimal transfer, the sum of probabilities is given by $\mathscr{P}(\mathbf{m}^*, \mathbb{G}) = \max_{\mathbf{m}} \{\mathscr{P}(\mathbf{m}, \mathbb{G})\} \geq \mathscr{P}(\mathbf{m}^*_{\oplus}, \mathbb{G})$ (by definition). Similarly, for graph $\mathbb{G}^{\oplus}$ with $\mathbf{m}^*_{\oplus}$ optimal transfer, the sum of probabilities of voting for A is given by $\mathscr{P}(\mathbf{m}^*_{\oplus}, \mathbb{G}^{\oplus}) = \max_{\mathbf{m}} \{\mathscr{P}(\mathbf{m}, \mathbb{G}^{\oplus})\} \geq \mathscr{P}(\mathbf{m}^*, \mathbb{G}^{\oplus})$ (by definition). We already mentioned, $\mathscr{P}(\mathbf{m}^*, \mathbb{G}) \geq \mathscr{P}(\mathbf{m}^*_{\oplus}, \mathbb{G})$ and using $\mathbf{m}^*$ as transfer vector in graph $\mathbb{G}^{\oplus}$, we get the following $\mathscr{P}(\mathbf{m}^*, \mathbb{G}) < \mathscr{P}(\mathbf{m}^*, \mathbb{G}^{\oplus})$, since $b_j < b^{\oplus}_j$ for all j. But by optimization, $\mathbf{m}^*_{\oplus}$ is the optimal transfer vector for graph $\mathbb{G}^{\oplus}$. Hence, $\mathscr{P}(\mathbf{m}^*, \mathbb{G}^{\oplus}) \leq \mathscr{P}(\mathbf{m}^*_{\oplus}, \mathbb{G}^{\oplus})$. Therefore the sum of probabilities for the new policy under the denser graph always improves.

[A.5a] For any given M, when $\sigma = 0$ if we use the same optimal transfer $\mathbf{m}^*(\mathbf{b}, \mathbb{G})$ for both graphs {$\mathbb{G}$ and $\mathbb{G}^{\oplus}$} where $\mathbb{G} \subset \mathbb{G}^{\oplus}$. We verify whether a bigger graph is beneficial for the lobby group in the absence of bias. The relation between the sum of probabilities $\mathscr{P}(\mathbb{G}, m^*)$ and $\mathscr{P}(\mathbb{G}^{\oplus}, m^*)$ is as follows:

$$\begin{aligned}
& \mathscr{P}(\mathbb{G}, \mathbf{m}^*) \; < \; \mathscr{P}(\mathbb{G}^{\oplus}, \mathbf{m}^*) \\
\Leftrightarrow \quad & \frac{n}{2} + \theta \sum_j b_j \cdot u(m^*_j) \; < \; \frac{n}{2} + \theta \sum_j b^{\oplus}_j \cdot u(m^*_j) \\
\Leftrightarrow \quad & \sum_j b_j \cdot u(m^*_j) \; < \; \sum_j b^{\oplus}_j \cdot u(m^*_j) \qquad [\because b_j \; < \; b^{\oplus}_j \qquad \forall j \in n] \\
\Leftrightarrow \quad & 0 < \sum_j (b^{\oplus}_j - b_j).u(m^*_j) \qquad [\because u(\cdot) > 0]
\end{aligned} \tag{A.3}$$

The aggregate Bonacich centrality in a denser network is greater, i.e. $\sigma^{\oplus} > \sigma$ where $\sigma^{\oplus}$ and σ are the sum of the Bonacich centralities of $\mathbb{G}^{\oplus}$ and $\mathbb{G}$ respectively.

[A.5b] When the legislators have a positive bias $\sigma > 0$ for the status quo policy S, if separate optimal transfer $\mathbf{m}^*$ and $\mathbf{m}^*_{\oplus}$ maximizes the sum of probabilities of votes in favour of A for the graphs $\{\mathbb{G}, \mathbb{G}^{\oplus}\} \in \mathscr{G}$ where $\mathbb{G} \subset \mathbb{G}^{\oplus}$. The relation between the sum of probabilities $\mathscr{P}(\mathbf{m}^*, \mathbf{G})$ and $\mathscr{P}(\mathbb{G}^{\oplus}, \mathbf{m}^*_{\oplus})$ is as follows:

$$
\begin{aligned}
& \mathscr{P}(\mathbb{G},\mathbf{m}^*) \;<\; \mathscr{P}(\mathbb{G}^{\oplus},\mathbf{m}^*_{\oplus}) \\
\Leftrightarrow \quad & \sum_j b_j \cdot u(m^*_j) \;-\; \sigma\sum_j b_j \;<\; \sum_j b^{\oplus}_j.u(m^*_{j\oplus}) \;-\; \sigma\sum_j b^{\oplus}_j \\
\Leftrightarrow \quad & \sigma\sum_j (b^{\oplus}_j - b_j) \;<\; \sum_j \left(b^{\oplus}_j u(m^*_{j\oplus}) - b_j.u(m^*_j)\right) \qquad \text{(A.5b)} \\
\Leftrightarrow \quad & \sigma \;<\; \frac{\sum_j \left(b^{\oplus}_j u(m^*_{j\oplus}) \;-\; b_j.u(m^*_j)\right)}{\sum_j (b^{\oplus}_j - b_j)}\Bigg]
\end{aligned}
$$

Since $\sigma > 0$, both the R.H.S and L.H.S in the above equation is positive. Thus the relation between the sum of probabilities, $\mathscr{P}(\mathbb{G},\mathbf{m}^*) < \mathscr{P}(\mathbb{G}^{\oplus},\mathbf{m}^*_{\oplus})$ holds true if $\sigma < \widehat{\sigma}(\mathbb{G},\mathbb{G}^{\oplus})$.

[A.5c] Let's compare two networks $\mathbb{G}$ and $\mathbb{G}^{\oplus}_{\{rs\}}$ where an additional link $\{rs\}$ is beneficial to the lobbyist i.e. $\mathscr{P}(\mathbb{G}) < \mathscr{P}(\mathbb{G}^{\oplus}_{rs})$ if $\sigma < \widehat{\sigma}^{\{rs\}}(\mathbb{G}\mathbb{G}^{\oplus}_{\{rs\}})$. The critical value of the bias is given by $\widehat{\sigma}^{\{rs\}}$. With slight abuse of notation, if instead we join two other nodes $\{st\} \in \left\{\mathcal{Z}\middle|\{rs\}\right\}$ in $\mathbb{G}$ where $r \neq t$. In this case, we compare the networks $\mathbb{G}$ and $\mathbb{G}^{\oplus}_{\{st\}}$. Adding this new link is beneficial to the lobbyist i.e. $\mathscr{P}(\mathbb{G}) < \mathscr{P}(\mathbb{G}^{\oplus}_{st})$ if $\sigma < \widehat{\sigma}^{\{st\}}(\mathbb{G}\mathbb{G}^{\oplus}_{\{st\}})$. Using this argument we can get critical value of status quo bias for every unconnected node in $\mathcal{Z}$.

Without any loss of generality, we rank all the critical values of status quo bias in $\mathcal{Z}$. The minimum critical value of the bias is $\underline{\widehat{\sigma}} = \min_{rs\in\mathcal{Z}}\left\{\widehat{\sigma}^{\{rs\}}\right\}$ where $\{ij\} \in \mathcal{Z}$. Thus adding the link $\{ij\}$ in $\mathbb{G}$ will be beneficial to the lobbyist i.e. $\mathscr{P}(\mathbb{G}^{\oplus}_{ij}) > \mathscr{P}(\mathbb{G})$ when $\sigma < \underline{\widehat{\sigma}}$. This implies adding any link $\{rs\} \in \mathcal{Z}$ benefits the lobbyist, if the actual value of the status quo bias is relatively small i.e. $\sigma \leq \underline{\widehat{\sigma}}$.

Similarly, there must exist a link $\{kl\} \in \mathcal{Z}$ which yields the maximum critical value of bias $\overline{\widehat{\sigma}} = \max_{rs\in\mathcal{Z}}\left\{\widehat{\sigma}^{\{rs\}}\right\}$. Thus, adding the new link $\{kl\}$ will hurt the lobby group i.e. $\mathscr{P}(\mathbb{G}^{\oplus}_{kl}) < \mathscr{P}(\mathbb{G})$ when $\sigma > \overline{\widehat{\sigma}}$. Using a similar argument as above, we get a network $\mathbb{G}^{\oplus}_{\{kl\}}$ that has the maximum critical value of the bias $\overline{\widehat{\sigma}}$.

[B.1] The probability $p_j(\mathbf{m})$ of any legislator $j \in \tau$ for voting in favour of the new policy is derived from the cumulative distribution function of the uniform error distribution $\epsilon_i \sim U[-\frac{1}{2\theta}, \frac{1}{2\theta}]$:

$$
\begin{aligned}
-\frac{1}{2\theta} \;&\leq\; \epsilon_i \;\leq\; u(m_i) - \sigma_\tau + \delta \sum_j g^{\tau}_{ij}(2p_j - 1) + \kappa_{\tau'} \sum_{j'} g^{\tau\tau'}_{ij'}(2p_{j'} - 1) \\
\Leftrightarrow\; p_j(\mathbf{m}) \;&=\; \theta\big[u(m_j) - \sigma_\tau + \delta \sum_j g_{ij}(2p_j(\mathbf{m}) - 1) + \kappa_\tau \sum_{j'} g^{\tau}_{ij'}(2p_{j'}(\mathbf{m}) - 1) + (\frac{1}{2\theta})\big] \\
\Leftrightarrow\; p_j(\mathbf{m}) \;&=\; \frac{1}{2} + \theta\big[u(m_j) - \sigma_\tau + \delta \sum_j g_{ij}(2p_j(\mathbf{m}) - 1) + \kappa_\tau \sum_{j'} g^{\tau}_{ij'}(2p_{j'}(\mathbf{m}) - 1)\big]
\end{aligned}
\tag{B.1}
$$

[B.2] The adjacency matrix between agents is given by $\mathbb{G}$ where $\mathbb{G} = 2\theta\mathbb{G}'$. For the rest of the analysis we partition the matrix $\mathbb{G}$ into the following:

$$
\hat{\mathbb{G}} = \left[\begin{array}{c|c} \mathbb{G}_s & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{G}_a \end{array}\right]; \quad \mathbb{K} = \left[\begin{array}{c|c} \mathbf{0} & -\kappa_s \mathbb{1} \\ \hline -\kappa_a \mathbb{1} & \mathbf{0} \end{array}\right]; \quad \mathbb{G} = \left[\begin{array}{c|c} \delta^* \mathbb{G}_s & -\kappa^*_s \mathbb{1} \\ \hline -\kappa^*_a \mathbb{1} & \delta^* \mathbb{G}_a \end{array}\right]
\tag{B.2}
$$

where $\delta^* = 2\delta\theta$ and $\kappa^*_\tau = 2\kappa_\tau\theta$ and $2\theta = \theta^*$. The matrix $\mathbb{G}$ is symmetric if $\mathbb{G}_s = \mathbb{G}_a$, $\kappa_s = \kappa_a$ and $\mathbb{1}^\top = \mathbb{1}$ where $\mathbb{1}$ is a symmetric matrix of ones.

[B.3]By manipulating the equation B.1 we represent the above equation in matrix form:

$$
\begin{aligned}
\mathbf{p} &= \frac{1}{2} \cdot \mathbf{1} + \theta\big(u(\mathbf{m}) - \boldsymbol{\sigma}\big) + 2\theta \cdot \mathbb{G}'\mathbf{p} - \theta \cdot \mathbb{G}'\mathbf{1} \\
\mathbf{p} &= \frac{1}{2} \cdot \mathbb{I}\mathbf{1} + \theta\big(u(\mathbf{m}) - \boldsymbol{\sigma}\big) + 2\theta \cdot \mathbb{G}'\mathbf{p} - \frac{1}{2} 2\theta \cdot \mathbb{G}'\mathbf{1} \\
(\mathbb{I} - \mathbb{G})\mathbf{p} &= \frac{1}{2}(\mathbb{I} - \mathbb{G})\mathbf{1} + \theta\big(u(\mathbf{m}) - \boldsymbol{\sigma}\big) \\
\mathbf{p} &= \frac{1}{2} \cdot \mathbf{1} + \theta(\mathbb{I} - \mathbb{G})^{-1}\big(u(\mathbf{m}) - \boldsymbol{\sigma}\big)
\end{aligned}
\tag{B.3}
$$

where $\mathbf{p}$ is the probability vector of voting in favour of policy A.

[B.4] Using the result from Appendix C.2 we can calculate the sum of probabilities of voting in favour of policy A

$$
\begin{aligned}
\mathbb{P}_A &= \mathbf{p}^\top \cdot \mathbf{1} \\
&= \frac{1}{2} \cdot \mathbf{1}^\top \mathbf{1} + \theta(\mathbf{u}^\top - \boldsymbol{\sigma}^\top)(\mathbb{I} - \mathbb{G}^\top)^{-1} \cdot \mathbf{1} \\
&= \frac{1}{2} \cdot \mathbf{1}^\top \mathbf{1} + \theta(\mathbf{u}^\top - \boldsymbol{\sigma}^\top) \cdot \boldsymbol{b} \qquad [\because \sigma_s + \sigma_a = 0] \\
&\text{In Equilibrium,} \\
&= \frac{n}{2} + \frac{\theta(1-\kappa_a^* B_a)}{(1-\kappa_a^* B_a \kappa_s^* B_s)}\Big(\sum_{j\in L} u(m_j^*) b_j - \sigma_s \sum_{j\in L} b_j\Big) + \\
&\frac{\theta(1-\kappa_s^* B_s)}{(1-\kappa_a^* B_a \kappa_s^* B_s)}\Big(\sum_{j'\in R} u(m_{j'}^*) b_{j'} - \sigma_a \sum_{j'\in R} b_{j'}\Big) \\
&= \frac{n}{2} + \frac{\theta(1-\kappa_a^* B_a)}{(1-\kappa_a^* B_a \kappa_s^* B_s)} \sum_{j\in L} u(m_j^*) b_j + \frac{\theta(1-\kappa_s^* B_s)}{(1-\kappa_a^* B_a \kappa_s^* B_s)} \sum_{j'\in R} u(m_{j'}^*) b_{j'} \\
&+ \frac{\theta\sigma_s}{(1-\kappa_a^* B_a \kappa_s^* B_s)}(B_a - B_s) + \frac{\theta\sigma_s B_a B_s}{(1-\kappa_a^* B_a \kappa_s^* B_s)}(\kappa_a^* - \kappa_s^*)
\end{aligned}
\tag{B.4}
$$

[B.5] Proof of Proposition 5: Assuming that the sum of probabilities are differentiable and the inverse $(\mathbb{I} - \mathbb{G}^\top)^{-1}$ exists, we get:

$$
\begin{aligned}
& J_i[\mathbf{p}] = \theta \cdot (\mathbb{I} - \mathbb{G})^{-1} \cdot J_i[\mathbf{u}] \qquad [\because J_i[\cdot] = \frac{d}{dm_i}] \\
\Leftrightarrow\ & J_i[\mathbf{p}]^\top = \theta \cdot J_i[\mathbf{u}]^\top \cdot (\mathbb{I} - \mathbb{G}^\top)^{-1} \\
\Leftrightarrow\ & J_i[\mathbf{p}]^\top \cdot \mathbf{1} = \theta \cdot J_i[\mathbf{u}]^\top \cdot (\mathbb{I} - \mathbb{G}^\top)^{-1} \cdot \mathbf{1} \\
\Leftrightarrow\ & J_i[\mathbf{p}]^\top \cdot \mathbf{1} = \theta \cdot J_i[\mathbf{u}]^\top \boldsymbol{b} \qquad [\text{using Equation C.2}]
\end{aligned}
\tag{B.5}
$$

Solving the optimal value of transfer we get,

$$
\begin{aligned}
& \lambda \ = \ \theta \cdot J_i[\mathbf{u}]^\top\, \boldsymbol{b}(\delta^*, \kappa_s^*, \kappa_a^*, \mathbb{G}^\top) \ = \ J_i[\mathbf{p}]^\top \cdot \mathbf{1} \\
\Leftrightarrow\ & \lambda^* = J_i[\mathbf{u}]^\top \boldsymbol{b} = u'(m_i).b_i; \qquad \forall i \in \{s, a\} \\
\Leftrightarrow\ & \lambda^* = \frac{(1-\kappa_a^* B_a)}{(1-\kappa_a^* \sigma_a \kappa_s^* \sigma_s)} u'(m_1) \cdot b_1 \ = \ \cdots \ = \ \frac{(1-\kappa_s^* \sigma_s)}{(1-\kappa_a^* B_a \kappa_s^* B_s)} u'(m_n) \cdot b_n
\end{aligned}
$$

We assume the utility from money to be logarithmic $u(m_i) = \log m_i$ and $m_i > 0$, then for any two individuals i and j, we solve the first order condition which yields the marginal cost of resources(see Appendix B.5). We get, $\omega_s \cdot \frac{b_i}{m_i} = \omega_a \cdot \frac{b_j}{m_j}$ for all $i \in s$ and $j \in a$. By algebraic manipulation, we acquire the equilibrium transfers:

$$m^*_{i\in s} = \frac{(1-\kappa^*_a\sigma_a)b_i \cdot M}{(1-\kappa^*_a\sigma_a)\sigma_s + (1-\kappa^*_s\sigma_s)\sigma_a} = \left(\frac{\omega_s}{\omega_s\sigma_s + \omega_a\sigma_a}\right)b_i M = \left(\frac{b_i}{\sigma_s}\right) \cdot M\phi_s$$

$$m^*_{i'\in a} = \frac{(1-\kappa^*_s\sigma_s)b_{i'} \cdot M}{(1-\kappa^*_a\sigma_a)\sigma_s + (1-\kappa^*_s\sigma_s)\sigma_a} = \left(\frac{\omega_a}{\omega_s\sigma_s + \omega_a\sigma_a}\right)b_j M = \left(\frac{b_{i'}}{\sigma_a}\right) \cdot M\phi_a$$

where $\phi_s = \left[\frac{1}{1+\left(\frac{(1/\sigma_s)-\kappa^*_s}{(1/\sigma_a)-\kappa^*_a}\right)}\right]$ and $\phi_a = \left[\frac{1}{1+\left(\frac{(1/\sigma_a)-\kappa^*_a}{(1/\sigma_s)-\kappa^*_s}\right)}\right]$ are the proportions that determines the total monetary allocation to each party.

[B.6] **Lemma ??**: For party L, given any two graphs $\mathbb{G}_s, \mathbb{G}^\oplus_s \in \mathcal{G}$, where $\mathbb{G}_s \subset \mathbb{G}^\oplus_s$, the unweighted Bonacich centrality vector of the graphs are $\mathbf{b}_s$ and $\mathbf{b}^\oplus_s$ where $\mathbf{b}^\oplus_s > \mathbf{b}^T_s$ and $\sigma^\oplus_s > \sigma_s$. If $\mathbb{G}_a$ is unchanged, then the unweighted centrality of the graph $\mathbb{G}_a$ is given by $\mathbf{b}_a$. We know that $\mathbb{G} \subset \mathbb{G}^\oplus$, thus using Appendix C.2 we can infer that the weights of the graph will be affected for any change in $\mathbb{G}$ to $\mathbb{G}^\oplus$. We compare the change in weights of the centrality of the members of party L when the sum of the centrality of L increase from σ_s to $\sigma^\oplus_s$. Ceteris paribus, if σ_s increase ω_s increases. In other words as connections within the members of party L becomes bigger, the weights of their centrality increases from $\omega_s(\mathbf{b}_s, \mathbf{b}_a|\kappa_s, \kappa_a)$ to $\omega^\oplus_s(\mathbf{b}^\oplus_s, \mathbf{b}_a|\kappa_s, \kappa_a)$. Since σ_s and σ_a are large, our result will holds for a sufficiently small θ, κ_a and κ_s where $(1-\kappa^*_a\sigma_a) \geq 0$ and $(1-\kappa^*_s\sigma_s) \geq 0$.

$$\begin{gathered}\sigma_s < \sigma^\oplus_s \implies \left(\frac{1-\kappa^*_a\sigma_a}{1-\kappa^*_a\sigma_a\kappa^*_s\sigma_s}\right) < \left(\frac{1-\kappa^*_a\sigma_a}{1-\kappa^*_a\sigma_a\kappa^*_s\sigma^\oplus_s}\right) \implies \omega_s < \omega^\oplus_s \\ \text{So,} \quad \boldsymbol{\omega}^\top_s \cdot \mathbf{b}_s \leq \boldsymbol{\omega}^{\oplus T}_s \cdot \mathbf{b}_s \implies \boldsymbol{\omega}^\top_s \cdot \mathbf{b}_s \leq \boldsymbol{\omega}^{\oplus T}_s \cdot \mathbf{b}^\oplus_s \quad [\because \mathbf{b}^\oplus_s > \mathbf{b}_s]\end{gathered} \tag{B.6}$$

where $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}^\oplus_s$ are $n_s \times 1$ vectors of weights of the left party. Similarly if $\mathbb{G}_s$ is fixed and $\mathbb{G}_a \subset \mathbb{G}^\oplus_a$, then $\omega_a < \omega^\oplus_a$.

Given the above situation, as connections become bigger for party L, we compare the effects on the weights of the centrality of the legislators of party R. Since the connections in $\mathbb{G}_a$ is fixed, $\mathbf{b}_s$ and σ_a are unchanged. Let's consider the weights $\omega_a(\mathbf{b}_s, \mathbf{b}_a|\kappa_s, \kappa_a)$ of the centrality of the other party R improves, then

$$\begin{aligned}\omega_a < \omega_a^{\oplus} \implies & \Big(\frac{1-\kappa_s^*\sigma_s}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}\Big) < \Big(\frac{1-\kappa_s^*\sigma_s^{\oplus}}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s^{\oplus}}\Big) \\ \implies & (1-\kappa_a^*\sigma_a) \not< 0 \qquad [\because (1-\kappa_a^*\sigma_a) \geq 0] \\ \implies & \boldsymbol{\omega}_a^{\top} \cdot \mathbf{b}_a \geq \boldsymbol{\omega}_a^{\oplus T} \cdot \mathbf{b}_a\end{aligned} \tag{B.7}$$

where $\boldsymbol{\omega}_a$ and $\boldsymbol{\omega}_a^{\oplus}$ are $(n-n_s)\times 1$ vectors of weights of the right party. Hence, $\omega_a \geq \omega_a^{\oplus}$. Analogously if $\mathbb{G}_s$ is fixed and $\mathbb{G}_a \subset \mathbb{G}_a^{\oplus}$, then $\omega_s \geq \omega_s^{\oplus}$.

[B.8] Proof of Proposition (c): Using equation B.4, for any given graph $\mathbb{G}$, the sum of probabilities of A with equilibrium transfer $\mathbf{m}^*$ is given by:

$$\mathbb{P}_A(\mathbb{G}, \mathbf{m}^*) = \frac{n}{2} + \theta\omega_s \sum_{j\in L} u(m_j^*) b_j + \theta\omega_a \sum_{j'\in R} u(m_{j'}^*) b_{j'} - \theta\sigma_s[\omega_s\sigma_s - \omega_a\sigma_a] \tag{B.8a}$$

where m_j^* is the equilibrium transfer to an individual $j \in L$ and $m_{j'}^*$ is the equilibrium transfer to an individual $j' \in R$. Similarly, for any given graph $\mathbb{G} \subset \mathbb{G}^{\oplus}$ where $\mathbb{G}_s \subset \mathbb{G}_s^{\oplus}$ and $\mathbb{G}_a$ is fixed the weighted Bonacich centrality vector changes from $\boldsymbol{b}$ to $\boldsymbol{b}^{\oplus}$. From equations B.6 and B.7 we know the following results, $\mathbf{b}_a$ is unchanged, $\mathbf{b}_s < \mathbf{b}_s^{\oplus}$, $\omega_s < \omega_s^{\oplus}$ and $\omega_a > \omega_a^{\oplus}$. The sum of probabilities of A with equilibrium transfer $\mathbf{m}_{\oplus}^*$ is given by:

$$\mathbb{P}_A(\mathbb{G}^{\oplus}, \mathbf{m}_{\oplus}^*) = \frac{n}{2} + \theta\omega_s^{\oplus} \sum_{j\in L} u(m_{j\oplus}^*) b_j^{\oplus} + \theta\omega_a^{\oplus} \sum_{j'\in R} u(m_{j'\oplus}^*) b_{j'} - \theta\sigma_s[\omega_s^{\oplus}\sigma_s^{\oplus} - \omega_a^{\oplus}\sigma_a] \tag{B.8b}$$

where $m_{j\oplus}^*$ is the equilibrium transfer to an individual $j \in L$ and $m_{j'\oplus}^*$ is the equilibrium transfer to an individual $j' \in R$. Comparing equations B.8a and B.8b, we show that the sum of probabilities increases with an increase in density if the following condition holds:

$$
\begin{aligned}
& \mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}^*_{\oplus}) \geq \mathbb{P}_A(\mathbb{G},\mathbf{m}^*) \\
\implies \quad & \omega_s^{\oplus}\sum_{j\in L} u(m^*_{j\oplus})b_j^{\oplus} + \omega_a^{\oplus}\sum_{j'\in R} u(m^*_{j'\oplus})b_{j'} - \sigma_s[\omega_s^{\oplus}\sigma_s^{\oplus} - \omega_a^{\oplus}\sigma_a] \\
\geq \quad & \omega_s\sum_{j\in L} u(m^*_j)b_j + \omega_a\sum_{j'\in R} u(m^*_{j'})b_{j'} - \sigma_s[\omega_s\sigma_s - \omega_a\sigma_a] \\
\sigma_s \leq & \frac{\left(\omega_s^{\oplus}\sum_{j\in L} u(m^*_{j\oplus})b_j^{\oplus} - \omega_s\sum_{j\in L} u(m^*_j)b_j\right) + \left(\omega_a^{\oplus}\sum_{j'\in R} u(m^*_{j'\oplus})b_{j'} - \omega_a\sum_{j'\in R} u(m^*_{j'})b_{j'}\right)}{(\omega_s^{\oplus}\sigma_s^{\oplus} - \omega_s\sigma_s) + (\omega_a\sigma_a - \omega_a^{\oplus}\sigma_a)} \\
& = \sigma_s^c(\mathbf{m}^*,\mathbf{m}^*_{\oplus},\mathbb{G},\mathbb{G}^{\oplus}) = \sigma_s^{\mathbf{cmm}_{\oplus}}
\end{aligned}
\tag{B.8}
$$

We know that $\mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}^*_{\oplus}) = \max\{\mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}_{\oplus})\}$. But the new optimal transfer vector $\mathbf{m}^*_{\oplus}$ might lead to a $\mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}^*_{\oplus})$ lower than $\mathbb{P}_A(\mathbb{G})$ based on the network. The equilibrium transfers $\mathbf{m}^*(\sigma_s,\sigma_a)$ and $\mathbf{m}^*_{\oplus}(\sigma_s^{\oplus},\sigma_a)$ are functions of the sum of Bonacich Centrality of all the legislators. If the lobbyist sticks to their previous transfer vector $\mathbf{m}^*$, the sum of probabilities for A is given by:

$$
\mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}^*) = \frac{n}{2} + \theta\omega_s^{\oplus}\sum_{j\in L} u(m^*_j)b_j^{\oplus} + \theta\omega_a^{\oplus}\sum_{j'\in R} u(m^*_{j'})b_{j'} - \theta\sigma_s[\omega_s^{\oplus}\sigma_s^{\oplus} - \omega_a^{\oplus}\sigma_a]
\tag{B.8c}
$$

Lobbyist will prefer the transfer vector $\mathbf{m}^*$ only if $\mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}^*) \geq \mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}^*_{\oplus})$. Comparing equations B.8a and B.8c, we show that the sum of probabilities increases with an increase in density if the following condition holds:

$$
\begin{aligned}
& \mathbb{P}_A(\mathbb{G}^{\oplus},\mathbf{m}^*) \geq \mathbb{P}_A(\mathbb{G},\mathbf{m}^*) \\
\implies \quad & \omega_s^{\oplus}\sum_{j\in L} u(m^*_j)b_j^{\oplus} + \omega_a^{\oplus}\sum_{j'\in R} u(m^*_{j'})b_{j'} - \sigma_s[\omega_s^{\oplus}\sigma_s^{\oplus} - \omega_a^{\oplus}\sigma_a] \\
\geq \quad & \omega_s\sum_{j\in L} u(m^*_j)b_j + \omega_a\sum_{j'\in R} u(m^*_{j'})b_{j'} - \sigma_s[\omega_s\sigma_s - \omega_a\sigma_a] \\
\sigma_s \leq & \frac{\left(\omega_s^{\oplus}\sum_{j\in L} u(m^*_j)b_j^{\oplus} - \omega_s\sum_{j\in L} u(m^*_j)b_j\right) + \left(\omega_a^{\oplus}\sum_{j'\in R} u(m^*_{j'})b_{j'} - \omega_a\sum_{j'\in R} u(m^*_{j'})b_{j'}\right)}{(\omega_s^{\oplus}\sigma_s^{\oplus} - \omega_s\sigma_s) + (\omega_a\sigma_a - \omega_a^{\oplus}\sigma_a)} \\
& = \sigma_s^c(\mathbf{m}^*,\mathbf{m}^*,\mathbb{G},\mathbb{G}^{\oplus}) = \sigma_s^{\mathbf{cmm}}
\end{aligned}
\tag{B.9}
$$

Appendix C

[C.1] To calculate the groupwise Bonacich Centrality from the inverse $(\mathbb{I} - \mathbb{G}^{\top})^{-1} \cdot \mathbf{1}$, we use the simplified Binomial Inverse Theorem:

$$(\mathbb{A}+\mathbb{B})^{-1} = \mathbb{A}^{-1} - \mathbb{A}^{-1}(\mathbb{I}+\mathbb{B}\mathbb{A}^{-1})^{-1}\mathbb{B}\mathbb{A}^{-1}$$

where $\mathbb{A}$ and $\mathbb{B}$ are non-singular matrices. The left party Bonacich centrality is represented by $(\mathbb{I} - \delta^* \mathbb{G}_s)^{-1} \cdot \mathbf{1} = \mathbf{b}_s$ where $\mathbf{b}_s$ is an $n_s \times 1$ matrix and similarly $(\mathbb{I} - \delta^* \mathbb{G}_a)^{-1} \cdot \mathbf{1} = \mathbf{b}_a$ is the column vector of the right party Bonacich Centrality with $(n - n_s)$ elements.

[C.2] Assuming that both $(\mathbb{I} - \delta^* \mathbb{G}_s)^{-1}$ and $(\mathbb{I} - \delta^* \mathbb{G}_a)^{-1}$ are invertible we can say that:

$$(\mathbb{I}-\delta^*\mathbb{G}_s)^{-1}\cdot\mathbb{1} = \begin{bmatrix} b_1 & \dots & b_1 \\ \vdots & \ddots & \vdots \\ b_{n_s} & \dots & b_{n_s} \end{bmatrix} \quad \text{and} \quad (\mathbb{I}-\delta^*\mathbb{G}_a)^{-1}\cdot\mathbb{1} = \begin{bmatrix} b_{(n_s+1)} & \dots & b_{(n_s+1)} \\ \vdots & \ddots & \vdots \\ b_n & \dots & b_n \end{bmatrix}$$

where $\mathbb{1}$ is a symmetric matrix of ones of order n_s and $(n - n_s)$ respectively. The sum of the Bonacich centrality of any party $\tau \in \{L, R\}$ is given by $\sigma_\tau = \sum_{j=1}^{n_\tau} b_j^\tau$.

[C.3] Using the formulae for Inverse of Partitioned matrix we compute the party specific weighted Bonacich conditional on the parameter values(degree of conflict, spillover effect and the graph):

$$(\mathbb{I} - \mathbb{G}^{\top})^{-1} \cdot \mathbf{1} = \left[\begin{array}{c|c} (\mathbb{I} - \delta^* \mathbb{G}_s^{\top}) & \kappa_a^* \mathbb{1} \\ \hline \kappa_s^* \mathbb{1} & (\mathbb{I} - \delta^* \mathbb{G}_a^{\top}) \end{array}\right]^{-1} \cdot \mathbf{1} \tag{C.1}$$

Using the formulae of Inverse of Block matrices, equation C.1 is inverted and represented in terms of party-specific centrality as follows:

$$\left[\begin{array}{c|c} \left((\mathbb{I}-\delta^*\mathbb{G}_s^\top)-\kappa_a^*\kappa_s^*\mathbb{1}\cdot(\mathbb{I}-\delta^*\mathbb{G}_a^\top)^{-1}\cdot\mathbb{1}\right)^{-1} & -\kappa_a^*(\mathbb{I}-\delta^*\mathbb{G}_s^\top)^{-1}\cdot\mathbb{1}\left((\mathbb{I}-\delta^*\mathbb{G}_a^\top)-\kappa_a^*\kappa_s^*\cdot\mathbb{1}(\mathbb{I}-\delta^*\mathbb{G}_s^\top)^{-1}\cdot\mathbb{1}\right)^{-1} \\ \hline -\kappa_s^*(\mathbb{I}-\delta^*\mathbb{G}_s^\top)^{-1}\mathbb{1}\cdot\left((\mathbb{I}-\delta^*\mathbb{G}_s^\top)-\kappa_a^*\kappa_s^*\mathbb{1}\cdot(\mathbb{I}-\delta^*\mathbb{G}_a^\top)^{-1}\cdot\mathbb{1}\right)^{-1} & \left((\mathbb{I}-\delta^*\mathbb{G}_a^\top)-\kappa_a^*\kappa_s^*\mathbb{1}\cdot(\mathbb{I}-\delta^*\mathbb{G}_s^\top)^{-1}\cdot\mathbb{1}\right)^{-1} \end{array}\right]$$

$$= \left[\begin{array}{ccc} \frac{1}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}(\mathbb{I}-\delta^*\mathbb{G}_s^\top)^{-1}\cdot\mathbf{1}_{n_s} & + & \frac{-\kappa_a^*\sigma_a}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}(\mathbb{I}-\delta^*\mathbb{G}_s^\top)^{-1}\cdot\mathbf{1}_{n_s} \\ \frac{-\kappa_s^*\sigma_s}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}(\mathbb{I}-\delta^*\mathbb{G}_a^\top)^{-1}\cdot\mathbf{1}_{(n-n_s)} & + & \frac{1}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}(\mathbb{I}-\delta^*\mathbb{G}_a^\top)^{-1}\cdot\mathbf{1}_{(n-n_s)} \end{array}\right]$$

$$\Longrightarrow \left(\frac{1}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}\right)\begin{bmatrix} (1-\kappa_a^*\sigma_a)b_1 \\ \vdots \\ (1-\kappa_a^*\sigma_a)b_{n_s} \\ (1-\kappa_s^*\sigma_s)b_{n_s+1} \\ \vdots \\ (1-\kappa_s^*\sigma_s)b_n \end{bmatrix} = \boldsymbol{\omega}^\top\cdot\mathbf{b} = \boldsymbol{b} \tag{C.2}$$

The $n\times 1$ vector of weights are given by $\boldsymbol{\omega} = \{\omega_s \cdots \omega_s\ \omega_a \cdots \omega_a\}^\top$ where $\omega_s = \left(\frac{1-\kappa_a^*\sigma_a}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}\right)$, $\omega_a = \left(\frac{1-\kappa_s^*\sigma_s}{1-\kappa_a^*\sigma_a\kappa_s^*\sigma_s}\right)$ and $\mathbf{b}$ is the unweighted party specific centrality of the legislators.