Overview

Develop estimation methodology for an additive nonparametric panel model suitable for capturing the pricing of coupon-paying government bonds over many periods.

The novelty lies in the combination of (1) cross-sectional nonparametric methods and (2) kernel estimation for time varying dynamics.

Estimate the yield curve and its dynamics and predict individual bond prices given the full payment schedule.

Asymptotic results are provided and simulations and US bonds application show strong performance of the proposed method.

Estimation: Local constant smoothing

For \( s_i \) in the neighborhood of \( t_j \),

\[
\hat{Q}(s_i) = \sum_{j=1}^{m} \sum_{k=1}^{n} c(t_j; b_i) \sum_{x=1}^{m} \left[ \int f(s_i-t_k) \mathbb{1}(x-X(x)) dx \right] \sum_{k=1}^{n} \left[ \int f(s_i-t_k) \mathbb{1}(x-X(x)) dx \right]
\]

where \( \mathbb{1}(x) \) is the indicator function with a bandwidth parameter \( h_i \) and \( K(\cdot) \) is the kernel.

The estimator of the discount function is the minimizer of \( \tilde{Q}(t) \) such that

\[
\hat{d}(s_i) = \arg \min \tilde{Q}(s_i)
\]

where \( \tilde{Q}(t) \) is the class of all functions for which \( Q(t) \) is well defined.

Obtaining F.O.C.

- Let \( \delta d(s_i; \cdot) \) be the \( (i+1) \) dimensional Dirac delta function at \( (s_i, t) \) such that

\[
\int \delta d(s_i; t) \delta d(s_i; t) = \mathbb{1}(s_i, t)^2
\]

- Let \( d(s_i, \cdot) = \delta d(s_i, \cdot) + \phi d(s_i, \cdot) \), and differentiate \( Q(t) \) with respect to \( s_i \) at the point of \( x = 0 \).

- \[
\hat{d}(s_i, t) = \hat{d}(s_i, t) + \hat{d}(s_i, t)
\]

where:

\[
\hat{d}(s_i, t) = \sum_{i=1}^{m} \sum_{k=1}^{n} c(t_j; b_i) \sum_{x=1}^{m} \left[ \int f(s_i-t_k) \mathbb{1}(x-X(x)) dx \right] \sum_{k=1}^{n} \left[ \int f(s_i-t_k) \mathbb{1}(x-X(x)) dx \right]
\]

- \[
\hat{d}(s_i, t) = \hat{d}(s_i, t) + \hat{d}(s_i, t)
\]

where:

\[
\hat{d}(s_i, t) = \sum_{i=1}^{m} \sum_{k=1}^{n} c(t_j; b_i) \sum_{x=1}^{m} \left[ \int f(s_i-t_k) \mathbb{1}(x-X(x)) dx \right] \sum_{k=1}^{n} \left[ \int f(s_i-t_k) \mathbb{1}(x-X(x)) dx \right]
\]

- \[
\hat{d}(s_i, t) = \hat{d}(s_i, t) + \hat{d}(s_i, t)
\]

Large sample properties

\[
\frac{\hat{d}(s_i, t)}{\sqrt{nT}} \xrightarrow{d} \mathcal{N}(0, V(s_i, t))
\]

\[
\frac{\hat{d}(s_i, t)}{\sqrt{nT}} \xrightarrow{d} \mathcal{N}(0, \hat{V}(s_i, t))
\]

Theorem

Suppose that assumptions (A1)-(A4) and (B1)-(B2) hold. Then, \( \sqrt{nT} \hat{d}(s_i, t) \xrightarrow{d} \mathcal{N}(0, V(s_i, t)) \) as \( n \to \infty \).

Corollary

Suppose that all assumptions for Theorem 1 hold. Then,

\[
\sqrt{nT} \hat{d}(s_i, t) \xrightarrow{d} \mathcal{N}(0, V(s_i, t))
\]

where \( \hat{d}(s_i, t) = (nT)^{-1} \sum_{i=1}^{nT} \hat{d}(s_i, t) \) and \( \hat{V}(s_i, t) = (nT)^{-1} \sum_{i=1}^{nT} \hat{V}(s_i, t) \) with \( \hat{V}(s_i, t) \) specified in (10).

\[
\hat{d}(s_i, t) = \frac{1}{nT} \sum_{i=1}^{nT} \hat{d}(s_i, t)
\]

\[
\hat{V}(s_i, t) = \frac{1}{nT} \sum_{i=1}^{nT} \hat{V}(s_i, t)
\]

Theorem

Suppose that assumptions (A1)-(A4), (B1)-(B2) and (C1)-(C2) hold. Then, as \( n \to \infty \),

\[
\hat{d}(s_i, t) \xrightarrow{d} \mathcal{N}(0, V(s_i, t))
\]

Monte Carlo Study

Simulation design

- Time horizon of 10 years of bi-weekly data
- Number of bonds to \( n = 24 \) daily on average
- Face value: 100
- Zero coupon bonds (1-12 months), Coupon bonds (1-10 years, annual payment)
- Replace each expiring bond with a new one in the same data structure but with a different identification number
- Bond prices are generated by \( \cdot \) and for a given \( \cdot, \) \( \hat{d} \) has a ARMA(0,0) structure with AR coefficient \( -0.1 \) and MA coefficient \( 0.2 \) to allow for temporal dependence
- Variance is set to increase over duration across bond types
- Discount function is generated from (2) with the parameter vector \( \beta_0 = 0, \beta_1 = 0.05, \beta_2 = 2, \tau_1 = 0.75 \) and \( \tau_2 = 125 \).

Simulation outputs

- Figure: Cubic time-dynamics of the discount curve
- Figure: Three-month Treasury Bill Rates
- Figure: Estimates for the discount function \( \hat{d}(s_i, t) \) with pointwise confidence intervals
- Figure: Estimates for the yield curve \( \hat{d}(s_i, t) \) with pointwise confidence intervals
- Figure: 3-dimensional shapes of \( \hat{d}(s_i, t) \) and \( \hat{d}(s_i, t) \)

Codes and Package

r-package yeevo has been developed in line with this paper and is available at https://github.com/bonsook/yeevo