Optimal Contracts with Randomly Arriving Tasks

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Motivation and Outline

- Long-term principal-agent relationship where the environment changes over time (random opportunities, demand shocks,...)
- Study the effect of fluctuations in the environment

- This paper: A stylized contacting problem:
  - Unique optimal contract:
    - Promotion based dynamics: Wage increases over time while effort decreases over time
    - Wage stickiness
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Model

- Principal-agent with infinite horizon; discount factor $\delta$
- Every period:
  - Nature draws available task $i \in \mathcal{I} = \{1, \ldots, I\}$ with prob $q_i$
  - Agent observes $i$ and exerts effort $e \in [0, \infty)$ on task
  - Principal observes $i, e$ and pays wage $w \in [0, \infty)$

- Payoffs for $(i, e, w)$:
  - Principal: $\pi_i(e) - w$
  - Agent: $g(w) - e$
  - $\pi'_i(\cdot), g'(\cdot) > 0 > \pi''_i(\cdot), g''(\cdot)$ and satisfy $\pi_i(0) = g(0) = 0$

- Additional assumptions:
  - Tasks are ordered: $\pi'_{i+1}(e) > \pi'_i(e)$ for all $e$
  - Interior solutions: $\pi'_i(0) > \frac{1}{g''(0)}$, $\lim_{w \to \infty} \frac{1}{g'(w)} > \lim_{e \to \infty} \pi'_1(e)$
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Contracts

- History at beginning of period $t$: $h_t = \{i_s, e_s, w_s\}_{s<t}$
- A contract specifies:
  1. $work(h_t, i_t) \to [0, \infty)$ – (Job description)
  2. $pay(h_t, i_t, e_t) \to [0, \infty)$ – (Compensation plan)

- Principal’s problem:
  Choose a contract to maximize expected discounted value at time zero subject to agent’s (dynamic) incentive constraints
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Constructing the Auxiliary Problems

- **Auxiliary problem** $P^{(i)}$
  1. First period: task $i$ is available
  2. Future: Tasks arrive as in the original problem
  3. Principal is restricted to contracts of the form
     - Vector of required efforts $(e_1^{(i)}, \ldots, e_i^{(i)})$
     - Fixed periodic compensation $w^{(i)}$

- **Auxiliary problem** $P^{(i-1)}$
  1. First period: task $i - 1$ is available
  2. Future:
     - Tasks arrive as in the original problem
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- Auxiliary problem $P^{(l)}$
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Auxiliary Problems

Incentive Compatibility Constraints and Solution

Let $\lambda_i = 1 - \sum_{j>i} q_j$

The IC constraint when task $j$ is available in $P^{(i)}$ is

$$e_j \leq g(w) + \sum_{s=1}^{\infty} (\lambda_i \delta)^s \left( g(w) - \frac{1}{\lambda_i} \sum_{k \leq i} q_k e_k \right),$$

Each auxiliary problem is a convex optimization problem and so it has a unique solution.
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Properties of the Solution(s)

Lemma

The only binding constraint in the solution to $P(i)$ is $IC_i^{(i)}$.

Lemma

In the solution to $P(i)$, $\pi'_j(e_j^{(i)}) \leq \frac{1}{g'(w(i))}$ with equality if $e_j^{(i)} > 0$.

Lemma

The sequence $(w^{(1)}, w^{(2)}, \ldots, w^{(l)})$ is strictly increasing. (proof)

Corollary

Let $j \leq i$.

1. For $j > 1$, $e_j^{(i)} \geq e_{j-1}^{(i)}$, with a strict inequality if $e_j^{(i)} > 0$, and
2. For $i < l$, $e_j^{(i)} \geq e_j^{(i+1)}$, with a strict inequality if $e_j^{(i)} > 0$.
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In the solution to $P^{(i)}$ $\pi'_j(e^{(i)}_j) \leq \frac{1}{g'(w^{(i)})}$ with equality if $e^{(i)}_j > 0$.

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The sequence $(w^{(1)}, w^{(2)}, \ldots, w^{(I)})$ is strictly increasing. (proof)

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Phase mechanism

- Define $\mathcal{I}(h_t; i_t) = \max\{i_s : s \leq t\}$
- The Phase Mechanism is defined by:

$$\begin{align*}
\text{work}(h_t, i_t) &= e_i(\mathcal{I}(h_t; i_t)) \\
\text{pay}(h_t, i_t, e_t) &= \begin{cases} 
  w(\mathcal{I}(h_t; i_t)) & \text{if } e_s = \text{work}(h_s, i_s) \text{ for all } s \leq t \\
  0 & \text{otherwise.}
\end{cases}
\end{align*}$$

- In each period, contract is given by solution to $P(\mathcal{I}(h_t; i_t))$
- Contract exhibits downward wage rigidity and upward effort rigidity
Main Result

Proposition

*Phase Mechanism is the (essentially) unique optimal contract.*

- Concavity is what connects between periods
- Can be supported as a SGPE for some parameters
- Robustness: companion paper
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- **Comments:**
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  - Can be supported as a SGPE for some parameters
  - Robustness: companion paper
General contracting environment with symmetric info

- Dynamic contracting environment:
  - Principal and agent interact over time
  - Time is discrete, common discount factor
  - Periodic game in \( t \) is drawn from \( f(h_t) \)
  - \( h_t \) specifies past periodic games and actions
  - Principal can commit to a long term strategy, agent cannot

- The environment accommodates
  - “Incentivizing Randomly Arriving Tasks”
  - Labor Contracts (Harris and Holmstrom 1982, Holmstrom 1983, Postal-Vinay and Robin 2002); Dynamic Risk Sharing (Marcet and Marimon 1992, Kruger and Uhlig 2006); Foreign Investment and Entrepreneur Financing (Thomas and Worrall 1994, Albuquerque and Hopenhayn 2004); Dynamic Project Selection (Forand and Zapal 2018)
  - Other potential models with seasonal demand, R&D investments, long term projects etc.
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Results:

- Define a class of components – “convex separable activities”
- Tight condition guaranteeing that, as time goes by, these components change only in the direction that favors the agent
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Implications:
- Generalize and unify downward wage rigidity results
- Establish a general upward effort rigidity result
  - Monotonicity results in previous model are “detail free”
- New insights on foreign investment/entrepreneur financing
Proofs

- If $w^{(i+1)} \leq w^{(i)}$ then $e_j^{(i)} \leq e_j^{(i+1)}$

- Consider the continuation of $P^{(i+1)}$ when task $i$ is available. Until the arrival of a task $l > i$:
  - The worker exerts weakly more effort than under the solution of $P^{(i)}$
  - None of the IC constraints are binding

- Compensation of strictly less than $w^{(i)}$ can incentivize weakly more effort than $\{e_j^{(i)}\}$ in auxiliary problem $i$. 

(return)