Dissecting Spurious Factors with Cross-Sectional Regressions

Valentina Raponi       Paolo Zaffaroni

Imperial College London       Imperial College London

December 30, 2018
Outline of the talk

1. Motivation and contribution
2. Methodology
3. Generalizations
4. Simulation evidence
5. Conclusion
Motivation

- Two-pass CSR methodology the most popular in empirical finance
Motivation

- Two-pass CSR methodology the most popular in empirical finance
  - Risk-premia estimation and inference
Motivation

- Two-pass CSR methodology the most popular in empirical finance
  - Risk-premia estimation and inference
  - Tests of asset-pricing restriction
Motivation

- Two-pass CSR methodology the most popular in empirical finance
  - Risk-premia estimation and inference
  - Tests of asset-pricing restriction
  - Risk versus characteristics
Motivation

- Two-pass CSR methodology the most popular in empirical finance
  - Risk-premia estimation and inference
  - Tests of asset-pricing restriction
  - Risk versus characteristics
  - Mutual funds/hedge funds applications
Motivation

- Two-pass CSR methodology the most popular in empirical finance
  - Risk-premia estimation and inference
  - Tests of asset-pricing restriction
  - Risk versus characteristics
  - Mutual funds/hedge funds applications
  - Corporate finance (cost of capital)

Raponi and Zaffaroni (2018) Dissecting Spurious Factors with Cross-Sect
The two-pass methodology (Fama and MacBeth (1973))

- Let asset returns follow a latent factor structure:

\[ R_{it} = E R_{it} + \beta_i' z_t + \epsilon_{it}, \]

unobserved innovation

and assume no-arbitrage and well-diversification of MV frontier.
The two-pass methodology (Fama and MacBeth (1973))

- Let asset returns follow a latent factor structure:

\[ R_{it} = ER_{it} + \beta_i' z_t + \epsilon_{it}, \]

unobserved innovation

and assume no-arbitrage and well-diversification of MV frontier.

- Then exact-pricing:

\[ ER_{it} = \gamma_0 + \gamma_1' \beta_i. \]
The two-pass methodology (Fama and MacBeth (1973))

- Let asset returns follow a latent factor structure:

\[ R_{it} = ER_{it} + \beta_i' z_t + \epsilon_{it}, \]

unobserved innovation

and assume no-arbitrage and well-diversification of MV frontier.

- Then exact-pricing:

\[ ER_{it} = \gamma_0 + \gamma'_1 \beta_i. \]

- To estimate the risk-premium \( \Gamma = (\gamma_0, \gamma'_1)' \), two-pass methodology:
The two-pass methodology (Fama and MacBeth (1973))

- Let asset returns follow a latent factor structure:
  \[ R_{it} = ER_{it} + \beta'_{i}z_{t} + \epsilon_{it}, \]
  \( \text{unobserved innovation} \)

  and assume no-arbitrage and well-diversification of MV frontier.

- Then exact-pricing:
  \[ ER_{it} = \gamma_{0} + \gamma'_{1}\beta_{i}. \]

- To estimate the risk-premium \( \Gamma = (\gamma_{0}, \gamma'_{1})' \), two-pass methodology:
  - First-pass: run \( N \) regressions with OLS, one for every asset \( i \),
    \[ R_{it} = \alpha_{i} + \beta'_{i}f_{t} + \epsilon_{it}, \ 1 \leq t \leq T. \]
The two-pass methodology (Fama and MacBeth (1973))

- Let asset returns follow a latent factor structure:

\[ R_{it} = ER_{it} + \beta_i' z_t + \epsilon_{it}, \]

unobserved innovation

and assume no-arbitrage and well-diversification of MV frontier.

- Then exact-pricing:

\[ ER_{it} = \gamma_0 + \gamma_1' \beta_i. \]

- To estimate the risk-premium \( \Gamma = (\gamma_0, \gamma_1)' \), two-pass methodology:
  - First-pass: run \( N \) regressions with OLS, one for every asset \( i \),

\[ R_{it} = \alpha_i + \beta_i' f_t + \epsilon_{it}, \quad 1 \leq t \leq T. \]

- Second-pass: run one cross-sectional regression by OLS/GLS

\[ \bar{R}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \eta_i, \quad 1 \leq i \leq N. \]

where \( \bar{R}_i = \sum_{t=1}^{T} R_{it} / T \) and \( \hat{\beta}_i \) is OLS estimator of \( \beta_i \) from first-pass.
The two-pass methodology (Fama and MacBeth (1973))

- Let asset returns follow a latent factor structure:
  \[
  R_{it} = ER_{it} + \beta_i' z_t + \epsilon_{it},
  \]
  unobserved innovation

  and assume no-arbitrage and well-diversification of MV frontier.

- Then exact-pricing:
  \[
  ER_{it} = \gamma_0 + \gamma'_1 \beta_i.
  \]

- To estimate the risk-premium \( \Gamma = (\gamma_0, \gamma'_1)' \), two-pass methodology:

  - First-pass: run \( N \) regressions with OLS, one for every asset \( i \),
    \[
    R_{it} = \alpha_i + \beta'_i f_t + \epsilon_{it}, \quad 1 \leq t \leq T.
    \]

  - Second-pass: run one cross-sectional regression by OLS/GLS
    \[
    \bar{R}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \eta_i, \quad 1 \leq i \leq N.
    \]
    where \( \bar{R}_i = \sum_{t=1}^{T} R_{it} / T \) and \( \hat{\beta}_i \) is OLS estimator of \( \beta_i \) from first-pass.

  - This gives the estimated risk-premium \( \hat{\Gamma} = (\hat{\gamma}_0, \hat{\gamma}'_1)' \).
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).

However...

...one typically assumes that the model is correctly-specified. Several ways in which the exact-pricing restriction could be wrong, that is:

\[ E^i = \gamma_0 + \gamma'_1 \beta_i + e_i \]

for some pricing-errors \( e_i \neq 0 \) for many reasons:

- missing (pervasive) factors.
- mis-measured factors (Roll’s critique).
- deviations unrelated to common factors (sentiment/behavioural).

However... it might be that \( e_i = 0 \) and yet the model is wrong: useless factors.

Raponi and Zaffaroni (2018)  
Dissecting Spurious Factors with Cross-Sectional Analysis
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).
- However...

\[
ER_t = \gamma_0 + \gamma_1 \beta_i + e_i
\]

for some pricing-errors \(e_i\).

However...it might be that \(e_i = 0\) and yet the model is wrong: useless factors.

Raponi and Zaffaroni (2018) Dissecting Spurious Factors with Cross-Sect...
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).
- However...
- ...one typically assumes that the model is correctly-specified.

\[ \text{ER}_i = \gamma_0 + \gamma_1 \beta_i + e_i \]

\( e_i \neq 0 \) for many reasons:
- missing (pervasive) factors.
- mis-measured factors (Roll’s critique).
- deviations unrelated to common factors (sentiment/behavioural).

However...it might be that \( e_i = 0 \) and yet the model is useless: useless factors.
Two-pass great results: simple, understood and works well (more on this later).

However...

...one typically assumes that the model is correctly-specified.

Several ways in which the exact-pricing restriction could be wrong, that is:

\[ ER_{it} = \gamma_0 + \gamma_1 \beta_i + e_i \]

for some pricing-errors \( e_i \).
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).
- However...
- ...one typically assumes that the model is correctly-specified.
- Several ways in which the exact-pricing restriction could be wrong, that is:

\[ ER_{it} = \gamma_0 + \gamma_1' \beta_i + e_i \]

for some pricing-errors \( e_i \).
- The \( e_i \neq 0 \) for many reasons:
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).
- However...
- ...one typically assumes that the model is correctly-specified.
- Several ways in which the exact-pricing restriction could be wrong, that is:

\[ ER_{it} = \gamma_0 + \gamma_1' \beta_i + e_i \]

for some pricing-errors \( e_i \).
- The \( e_i \neq 0 \) for many reasons:
  - missing (pervasive) factors.
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).
- However...
- ...one typically assumes that the model is correctly-specified.
- Several ways in which the exact-pricing restriction could be wrong, that is:
  \[ ER_{it} = \gamma_0 + \gamma_1' \beta_i + e_i \]
  for some pricing-errors \( e_i \).
- The \( e_i \neq 0 \) for many reasons:
  - missing (pervasive) factors.
  - mis-measured factors (Roll’s critique).
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).
- However...
- ...one typically assumes that the model is correctly-specified.
- Several ways in which the exact-pricing restriction could be wrong, that is:
  \[ ER_{it} = \gamma_0 + \gamma_1' \beta_i + e_i \]
  for some pricing-errors \( e_i \).
- The \( e_i \neq 0 \) for many reasons:
  - missing (pervasive) factors.
  - mis-measured factors (Roll's critique).
  - deviations unrelated to common factors (sentiment/behavioural).
The two-pass methodology: forms of misspecification

- Two-pass great results: simple, understood and works well (more on this later).
- However...
- ...one typically assumes that the model is correctly-specified.
- Several ways in which the exact-pricing restriction could be wrong, that is:

\[ ER_{it} = \gamma_0 + \gamma_1' \beta_i + e_i \]

for some pricing-errors \( e_i \).
- The \( e_i \neq 0 \) for many reasons:
  - missing (pervasive) factors.
  - mis-measured factors (Roll's critique).
  - deviations unrelated to common factors (sentiment/behavioural).
- However...it might be that \( e_i = 0 \) and yet the model is wrong: useless factors.
Consider special case when presumed beta-pricing models has two factors A and B. Then we think that:

\[ ER_{it} = \gamma_0 + \gamma_1 \beta_{iA} + \gamma_1 \beta_{iB}, \]

but in reality only factor A is priced:

\[ ER_{it} = \gamma_0 + \gamma_1 \beta_{iA} + \gamma_1 \beta_{iB} = \gamma_0 + \gamma_1 \beta_{iA}. \]
Consider special case when presumed beta-pricing models has two factors A and B. Then we think that:

\[ ER_{it} = \gamma_0 + \gamma_1 A \beta_i + \gamma_1 B \beta_i, \]

but in reality only factor A is priced:

\[ ER_{it} = \gamma_0 + \gamma_1 A \beta_i + \gamma_1 B \beta_i = \gamma_0 + \gamma_1 A \beta_i. \]

Both equations holds in population!
The two-pass methodology: useless factors

Consider special case when presumed beta-pricing models has two factors A and B. Then we think that:

$$ER_{it} = \gamma_0 + \gamma_1 A \beta_i A + \gamma_1 B \beta_i B,$$

but in reality only factor A is priced:

$$ER_{it} = \gamma_0 + \gamma_1 A \beta_i A + \gamma_1 B \beta_i B = \gamma_0 + \gamma_1 A \beta_i A.$$

Both equations holds in population!

So where is the problem?
The two-pass methodology: useless factors

- Problems arise when we try to estimate the risk premia, imposing factor $f_{Bt}$ when in fact $\beta_{iB} = 0$!
The two-pass methodology: useless factors

- Problems arise when we try to estimate the risk premia, imposing factor $f_{Bt}$ when in fact $\beta_{iB} = 0$!
- ...second-pass CSR:

\[
\begin{pmatrix}
\hat{\gamma}_0 \\
\hat{\gamma}_A \\
\hat{\gamma}_B
\end{pmatrix} = \hat{\Gamma} = (\hat{X}'\hat{X})^{-1}\hat{X}'\bar{R},
\]

where

\[
\hat{X} = (1_N, \hat{\beta}_A, \hat{\beta}_B) \approx (1_N, \hat{\beta}_A, 0_N) \text{ when } T \text{ large}.
\]
The two-pass methodology: useless factors

- Problems arise when we try to estimate the risk premia, imposing factor \( f_{Bt} \) when in fact \( \beta_{iB} = 0! \)

- ...second-pass CSR:

\[
\begin{pmatrix}
\hat{\gamma}_0 \\
\hat{\gamma}_A \\
\hat{\gamma}_B
\end{pmatrix} = \hat{\Gamma} = (\hat{X}'\hat{X})^{-1}\hat{X}'\bar{R},
\]

where

\[
\hat{X} = (1_N, \hat{\beta}_A, \hat{\beta}_B) \approx (1_N, \hat{\beta}_A, 0_N) \text{ when } T \text{ large}.
\]

- ‘denominator’ of \( \hat{\Gamma} \) arbitrarily close to “zero” (that is, \( (\hat{X}'\hat{X}) \) becomes singular)!
The two-pass methodology: useless factors

- Problems arise when we try to estimate the risk premia, imposing factor $f_{Bt}$ when in fact $\beta_{iB} = 0$!

- ...second-pass CSR:

  $$
  \begin{pmatrix}
  \hat{\gamma}_0 \\
  \hat{\gamma}_A \\
  \hat{\gamma}_B 
  \end{pmatrix}
  = \hat{\Gamma} = \left(\hat{X}'\hat{X}\right)^{-1}\hat{X}'\bar{R},
  $$

  where

  $$
  \hat{X} = (1_N, \hat{\beta}_A, \hat{\beta}_B) \approx (1_N, \hat{\beta}_A, 0_N) \text{ when } T \text{ large}.
  $$

- ‘denominator’ of $\hat{\Gamma}$ arbitrarily close to “zero” (that is, $(\hat{X}'\hat{X})$ becomes singular)!

- Similar problem when say the $\beta_{iB} \approx \text{constant cross-sectionally}$ (documented when $B$ is market factor).
The two-pass methodology: useless factors

- What happens to $\hat{\Gamma}$ as $T \to \infty$?
The two-pass methodology: useless factors

- What happens to $\hat{\Gamma}$ as $T \to \infty$?
- Non-standard behaviour arises.
The two-pass methodology: useless factors

- What happens to $\hat{\Gamma}$ as $T \to \infty$?
- Non-standard behaviour arises.
- Complicated: it depends on whether (netting out $f_{tB}$) the model is correctly specified (that is, whether $\beta_{iA}$ describes entirely the cross-section of expected returns).
The two-pass methodology: useless factors

- What happens to $\hat{\Gamma}$ as $T \to \infty$?
- Non-standard behaviour arises.
- Complicated: it depends on whether (netting out $f_{tB}$) the model is correctly specified (that is, whether $\beta_{iA}$ describes entirely the cross-section of expected returns).
- Complicated: it depends on the fraction of assets for which $\beta_{iB} = 0$. 

Let' simplify even further and assume true model is zero-factor model but we insist and use $f_{tB}$.

Case I (Kan and Zhang (1999)): $\beta_{iB} = 0$ when model correctly specified ($E(R_{it}) = \gamma_0$), $\hat{\gamma}_B \rightarrow d Z'1MZ_1Z_2'$ for zero-mean normal r.v.s $Z_1$, $Z_2$.

when model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$), $\hat{\gamma}_B \rightarrow p \pm \infty$.

In particular $\hat{\gamma}_B \approx \sqrt{T}Z'1McZ_1MZ_1$ where $c = \gamma_0 1N + e_i$.
The two-pass methodology: useless factors

- Let' simplify even further and assume true model is zero-factor model but we insist and use $f_{tB}$.
- Case I (Kan and Zhang (1999)): $\beta_{iB} = 0$
Let' simplify even further and assume true model is zero-factor model but we insist and use $f_{tB}$.

Case I (Kan and Zhang (1999)): $\beta_{iB} = 0$

when model correctly specified ($E(R_{it}) = \gamma_0$)

$$\hat{\gamma}_B \rightarrow_d \frac{Z'_1MZ_2}{Z'_1MZ_1}$$

for zero-mean normal r.v.s $Z_1, Z_2$. 
The two-pass methodology: useless factors

- Let's simplify even further and assume true model is zero-factor model but we insist and use $f_{tB}$.
- Case I (Kan and Zhang (1999)): $\beta_iB = 0$
- when model correctly specified ($E(R_{it}) = \gamma_0$)

$$\hat{\gamma}_B \xrightarrow{d} \frac{Z_1'MZ_2}{Z_1'MZ_1} \text{ for zero-mean normal r.v.s } Z_1, Z_2.$$

- when model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$)

$$\hat{\gamma}_B \xrightarrow{p} \pm \infty.$$
The two-pass methodology: useless factors

- Let's simplify even further and assume true model is zero-factor model but we insist and use $f_{tB}$.
- Case I (Kan and Zhang (1999)): $\beta_{iB} = 0$
- when model correctly specified ($E(R_{it}) = \gamma_0$)
  \[ \hat{\gamma}_B \to_d \frac{Z_1'MZ_2}{Z_1'MZ_1} \]  for zero-mean normal r.v.s $Z_1, Z_2$.
- when model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$)
  \[ \hat{\gamma}_B \to_p \pm \infty. \]
- In particular $\hat{\gamma}_B \approx \sqrt{T} \frac{Z_1'Mc}{Z_1'MZ_1}$ where $c = \gamma_01_N + e$. 
Case II (Kleibergen (2009)): $\beta_{iB} = \beta / \sqrt{T}$ for some $\beta \neq 0$. 

When model correctly specified ($E(R_{it}) = \gamma_0$), 

\[ \hat{\gamma}_B \to^d (\beta + Z_1)' M (\beta + Z_1) (\beta + Z_1)' M (Z_1 + \beta) \] 

for zero-mean normal r.v.s $Z_1, Z_2$. 

When model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$), 

\[ \hat{\gamma}_B \to^p \pm \infty. \] 

In particular, 

\[ \hat{\gamma}_B \approx \sqrt{T} (\beta + Z_1)' M (\beta + Z_1) (\beta + Z_1)' M (Z_1 + \beta). \]
The two-pass methodology: useless factors

- Case II (Kleibergen (2009)): $\beta_{iB} = \beta / \sqrt{T}$ for some $\beta \neq 0$.
- when model correctly specified ($E(R_{it}) = \gamma_0$)

$$\hat{\gamma}_B \overset{d}{\rightarrow} \frac{(\beta + Z_1)'M(\beta + Z_1)}{(\beta + Z_1)'M(Z_1 + \beta)}$$

for zero-mean normal r.v.s $Z_1, Z_2$. 
The two-pass methodology: useless factors

- Case II (Kleibergen (2009)): \( \beta_{iB} = \beta / \sqrt{T} \) for some \( \beta \neq 0 \).
- when model correctly specified \( (E(R_{it}) = \gamma_0) \)

\[
\hat{\gamma}_B \xrightarrow{d} \frac{(\beta + Z_1)'M(\beta + Z_1)}{(\beta + Z_1)'M(Z_1 + \beta)}
\]

for zero-mean normal r.v.s \( Z_1, Z_2 \).

- when model is misspecified \( (E(R_{it}) = \gamma_0 + e_i \text{ with non-zero } e_i) \)

\[
\hat{\gamma}_B \xrightarrow{p} \pm \infty.
\]
Case II (Kleibergen (2009)): $\beta_{iB} = \beta / \sqrt{T}$ for some $\beta \neq 0$.

when model correctly specified ($E(R_{it}) = \gamma_0$)

$$\hat{\gamma}_B \xrightarrow{d} \frac{(\beta + Z_1)' M (\beta + Z_1)}{(\beta + Z_1)' M (Z_1 + \beta)}$$

for zero-mean normal r.v.s $Z_1, Z_2$.

when model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$)

$$\hat{\gamma}_B \xrightarrow{p} \pm \infty.$$ 

In particular $\hat{\gamma}_B \approx \sqrt{T} \frac{(\beta+Z_1)' Mc}{(\beta+Z_1)' M (\beta+Z_1)}$. 
Case $\beta_iB = 0$ and correctly specified model
Case $\beta_{iB} = 0$ and misspecified model
Useless factors (Fig 1.3 from Kleibergen (2009) JOE)

Case $\beta_{1B} \neq 0, \beta_{iB} = 0, 2 \leq i \leq N$ and correctly specified model
The two-pass methodology: useless factors

- Some more (peculiar) results.
The two-pass methodology: useless factors

- Some more (peculiar) results.
- Case I (Kan and Zhang 1999): $\beta_{iB} = 0$. 

IN SUMMARY: due useless factors inference on beta-pricing models is corrupted using standard CSRs methods valid for large-$T$. Gospodinov et al. (2017): GMM-tests of asset pricing restriction on SDF parameters have power equal to size when useless factors.
The two-pass methodology: useless factors

- Some more (peculiar) results.
- Case I (Kan and Zhang 1999): $\beta_{iB} = 0$.
- when model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$)

\[
R^2 \xrightarrow{d} \zeta \text{ (some random variable)}, \\
t_{\beta_B} \xrightarrow{p} \pm \infty.
\]
The two-pass methodology: useless factors

- Some more (peculiar) results.
- Case I (Kan and Zhang 1999): $\beta_{iB} = 0$.
- when model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$)

$$R^2 \xrightarrow{d} \zeta \text{ (some random variable)},$$

$$t_{\beta_B} \xrightarrow{p} \pm \infty.$$

- IN SUMMARY: due useless factors inference on beta-pricing models is corrupted using standard CSRs methods valid for large-$T$. 

Gospodinov et al. (2017): GMM-tests of asset pricing restriction on SDF parameters have power equal to size when useless factors.

Raponi and Zaffaroni (2018) Dissecting Spurious Factors with Cross-Sect

December 30, 2018 14 / 75
The two-pass methodology: useless factors

- Some more (peculiar) results.
- Case I (Kan and Zhang 1999): $\beta_{iB} = 0$.
- When model is misspecified ($E(R_{it}) = \gamma_0 + e_i$ with non-zero $e_i$)

\[
\begin{align*}
R^2 & \rightarrow_d \zeta \text{ (some random variable)}, \\
T_{\beta_B} & \rightarrow_p \pm \infty.
\end{align*}
\]

IN SUMMARY: due useless factors inference on beta-pricing models is corrupted using standard CSRs methods valid for large-$T$.

- Gospodinov et al. (2017): GMM-tests of asset pricing restriction on SDF parameters have power equal to size when useless factors.
The two-pass methodology: useless factors

- Existing methodologies to tackle the effect of useless factors (all designed for large- $T$ except double-asymptotics procedure of Anatolyev and Mikusheva (2018)): ingenious yet sophisticated approaches (non-standard).
The two-pass methodology: useless factors

- Existing methodologies to tackle the effect of useless factors (all designed for large-$T$ except double-asymptotics procedure of Anatolyev and Mikusheva (2018)): ingenious yet sophisticated approaches (non-standard).

The two-pass methodology: useless factors

- Existing methodologies to tackle the effect of useless factors (all designed for large- $T$ except double-asymptotics procedure of Anatolyev and Mikusheva (2018)): ingenious yet sophisticated approaches (non-standard).
- Burnside (2016): rank-tests on parameters of factor-SDF.
The two-pass methodology: useless factors

- Existing methodologies to tackle the effect of useless factors (all designed for large- \( T \) except double-asymptotics procedure of Anatolyev and Mikusheva (2018)): ingenious yet sophisticated approaches (non-standard).
- Burnside (2016): rank-tests on parameters of factor-SDF.
The two-pass methodology: useless factors

- Existing methodologies to tackle the effect of useless factors (all designed for large-$T$ except double-asymptotics procedure of Anatolyev and Mikusheva (2018)): ingenious yet sophisticated approaches (non-standard).


- Burnside (2016): rank-tests on parameters of factor-SDF.


- Anatolyev and Mikusheva (2018): estimation procedure based on sample-splitting instrumental variables regression robust to weak identification (near-zero betas) and omitted weak factors.
This paper: methodology to test for useless factors valid when $T$ is fixed, possibly very small, and number of assets $N$ large.
This paper: **methodology** to test for **useless factors** valid when $T$ is fixed, possibly very small, and number of assets $N$ large.

This paper: **completely standard** as it just uses the plain OLS CSR estimator.
Useless factors: motivation

- This paper: **methodology** to test for useless factors valid when \( T \) is fixed, possibly very small, and number of assets \( N \) large.

- This paper: completely **standard** as it just uses the plain OLS CSR estimator.

- This paper: traditional asymptotics (normal and chi-square limiting distributions immune of nuisance-parameters).
This paper: methodology to test for useless factors valid when $T$ is fixed, possibly very small, and number of assets $N$ large.

This paper: completely standard as it just uses the plain OLS CSR estimator.

This paper: traditional asymptotics (normal and chi-square limiting distributions immune of nuisance-parameters).

This paper: distinction between lack of identification (zero betas) and weak identification (quasi-zero betas) irrelevant.
Useless factors: motivation

- Idea: troubles with OLS CSR due to the fact it is a ‘good’ estimator for large $T$ and model correctly specified.
Useless factors: motivation

- Idea: troubles with OLS CSR due to the fact it is a ‘good’ estimator for large $T$ and model correctly specified.
- ....but OLS CSR is not a good estimator when $T$ is fixed (first-order bias!).
Useless factors: motivation

- Idea: troubles with OLS CSR due to the fact it is a ‘good’ estimator for large $T$ and model correctly specified.
- ....but OLS CSR is not a good estimator when $T$ is fixed (first-order bias!).
- How does it behave when $T$ is fixed but one only takes $N$ large?
Useless factors: motivation

- Idea: troubles with OLS CSR due to the fact it is a ‘good’ estimator for large $T$ and model correctly specified.
- ....but OLS CSR is not a good estimator when $T$ is fixed (first-order bias!).
- How does it behave when $T$ is fixed but one only takes $N$ large?
- This sampling scheme empirically motivated as tens of thousands of assets traded every day (individual assets) but only short time-series used in practice (for data availability; for structural breaks; for time-variation of parameters, etc.)
Useless factors: motivation

- Idea: troubles with OLS CSR due to the fact it is a ‘good’ estimator for large $T$ and model correctly specified.
- ....but OLS CSR is not a good estimator when $T$ is fixed (first-order bias!).
- How does it behave when $T$ is fixed but one only takes $N$ large?
- This sampling scheme empirically motivated as tens of thousands of assets traded every day (individual assets) but only short time-series used in practice (for data availability; for structural breaks; for time-variation of parameters, etc.)
- Our result: OLS CSR is a powerful tool to dissect useless factors in a large-$N$ environment!
Useless factors: base case

- From now on $g_t$ denotes $K_g \times 1$ useless factor: $\text{cov}(g_t, R_{it}) = 0$ all $i$. 
Useless factors: base case

- From now on $g_t$ denotes $K_g \times 1$ useless factor: $\text{cov}(g_t, R_{it}) = 0$ all $i$.
- Assume correctly-specified zero-factor model:

$$ER_{it} = \gamma_0.$$
Useless factors: base case

- From now on $g_t$ denotes $K_g \times 1$ useless factor: $\text{cov}(g_t, R_{it}) = 0$ all $i$.
- Assume correctly-specified zero-factor model:
  \[ ER_{it} = \gamma_0. \]
- ...but we estimate one-factor model:
  \[ R_{it} = \alpha_i + \beta_{ig} g_t + \epsilon_{it}. \]
Useless factors: base case

- From now on $g_t$ denotes $K_g \times 1$ useless factor: $\text{cov}(g_t, R_{it}) = 0$ all $i$.
- Assume correctly-specified zero-factor model:
  \[ ER_{it} = \gamma_0. \]
- ...but we estimate one-factor model:
  \[ R_{it} = \alpha_i + \beta_{ig} g_t + \epsilon_{it}. \]
- Risk premia OLS CSR estimator:
  \[ \hat{\Gamma}_g = (\hat{X}_g' \hat{X}_g)^{-1} \hat{X}_g' \bar{R} \text{ with } \hat{X}_g = [1_N, \hat{B}_g]. \]
Useless factors: base case

- From now on $g_t$ denotes $K_g \times 1$ useless factor: $\text{cov}(g_t, R_{it}) = 0$ all $i$.
- Assume correctly-specified zero-factor model:
  \[
  ER_{it} = \gamma_0.
  \]
- ...but we estimate one-factor model:
  \[
  R_{it} = \alpha_i + \beta_{ig} g_t + \epsilon_{it}.
  \]
- Risk premia OLS CSR estimator:
  \[
  \hat{\Gamma}_g = (\hat{X}_g' \hat{X}_g)^{-1} \hat{X}_g' \bar{R} \text{ with } \hat{X}_g = [1_N, \hat{B}_g].
  \]
- In particular
  \[
  \hat{\beta}_{ig} = 0_{K_g} + (\bar{G}' \bar{G})^{-1} \bar{G}' \epsilon_i \text{ where } \bar{G} = G - 1_T \bar{g}'.
  \]
Theorem

Under Assumptions 1-5 and correct specification:

(i)
\[ \hat{\Gamma}_g - \left( \begin{array}{c} \gamma_0 \\ 0_{Kg} \end{array} \right) = O_p \left( \frac{1}{\sqrt{N}} \right). \]

(ii)
\[ \sqrt{N} \left( \hat{\Gamma}_g - \left( \begin{array}{c} \gamma_0 \\ 0_K \end{array} \right) \right) \xrightarrow{d} N(0_{K+1}, V) \]

where

\[ V = \begin{pmatrix} \frac{\sigma^2}{T} & 0'_{K} \\ 0_{K} & \frac{1}{\sigma^4} C' U_\varepsilon C \end{pmatrix}, \quad \text{with} \quad C = \left( \frac{1_T}{T} \otimes \tilde{G} \right). \]
Remark: the risk premia associated with useless factors go to zero.
Useless factors: base case

- Remark: the risk premia associated with useless factors go to zero.
- Remark: consistent estimation of zero-beta rate.
Useless factors: base case

- Remark: the risk premia associated with useless factors go to zero.
- Remark: consistent estimation of zero-beta rate.
- Remark: asymptotic covariance matrix can be consistently estimated.
Remark: the risk premia associated with useless factors go to zero.
Remark: consistent estimation of zero-beta rate.
Remark: asymptotic covariance matrix can be consistently estimated.
Remark: correctly-sized Wald test for $H_0 : \gamma_g = 0_{K_g}$. 
Useless factors: base case

Let $\hat{e}_g = \bar{R} - \hat{X}_g \hat{\Gamma}_g$ denote the vector of pricing errors (OLS CSR residuals).
Useless factors: base case

- Let $\hat{e}_g = \bar{R} - \hat{X}_g \hat{\Gamma}_g$ denote the vector of pricing errors (OLS CSR residuals).
- The $t$ statistic for the $k$-th regression coefficient (where $c_{g, kk}$ denotes the $(k, k)$-th element of the matrix $(\hat{X}_g' \hat{X}_g)^{-1}$) is:

$$t_{g, k} = \frac{\hat{\gamma}_{g, k}}{s_g \cdot \sqrt{c_{g, kk}}}, \quad 2 \leq k \leq K + 1 \text{ with } s_g^2 = \frac{\hat{e}_g' \hat{e}_g}{N - K - 1}. \quad (1)$$
Useless factors: base case

- Let $\hat{e}_g = \bar{R} - \hat{X}_g \hat{\Gamma}_g$ denote the vector of pricing errors (OLS CSR residuals).
- The $t$ statistic for the $k$-th regression coefficient (where $c_{g,kk}$ denotes the $(k, k)$-th element of the matrix $(\hat{X}_g' \hat{X}_g)^{-1}$) is:

$$t_{g,k} = \frac{\hat{\gamma}_{g,k}}{s_g \cdot \sqrt{c_{g,kk}}}, \quad 2 \leq k \leq K + 1 \text{ with } s_g^2 = \frac{\hat{e}_g' \hat{e}_g}{N - K - 1}. \quad (1)$$

- The R-squared (where we define $\mathcal{M} = I_N - \frac{1_N 1_N'}{N}$) is:

$$R_{CRSg}^2 = 1 - \frac{\hat{e}_g' \hat{e}_g}{\bar{R}' \mathcal{M} \bar{R}}. \quad (2)$$
Useless factors: base case

- Let $\hat{e}_g = \bar{R} - \hat{X}_g \hat{\Gamma}_g$ denote the vector of pricing errors (OLS CSR residuals).
- The $t$ statistic for the $k$-th regression coefficient (where $c_{g, kk}$ denotes the $(k, k)$-th element of the matrix $(\hat{X}_g' \hat{X}_g)^{-1}$) is:

$$
t_{g, k} = \frac{\hat{\gamma}_{g, k}}{s_g \cdot \sqrt{c_{g, kk}}}, \quad 2 \leq k \leq K + 1 \text{ with } s_g^2 = \frac{\hat{e}_g' \hat{e}_g}{N - K - 1}. \quad (1)
$$

- The R-squared (where we define $\mathcal{M} = I_N - \frac{1_{N1_N'}}{N}$) is:

$$
R^2_{\text{CRSg}} = 1 - \frac{\hat{e}_g' \hat{e}_g}{\bar{R}' \mathcal{M} \bar{R}} \quad (2)
$$

- The $F$-statistic to test whether all the $K$ coefficients except for the intercept are zero is:

$$
F_{\text{CSRg}} = \frac{R^2_{\text{CSR}} / K}{(1 - R^2_{\text{CSR}}) / (N - K - 1)} \quad (3)
$$
Theorem

Under Assumptions 1-5 and correct specification:

(i) \( t_{g,k} \rightarrow N(0, T_{k}^{4} + \sigma_{4}^{4}) \)

(ii) \( R_{2}^{CRS_{g}} \rightarrow 0 \)

(iii) \( F_{CSR_{g}} \rightarrow \chi_{2}^{2}(k_{4} + \sigma_{4}^{4}) / K \)

Raponi and Zaffaroni (2018)
Dissecting Spurious Factors with Cross-Sect
Theorem

Under Assumptions 1-5 and correct specification:
Useless factors: base case

Theorem

Under Assumptions 1-5 and correct specification:

(i)

\[ t_{g,k} \xrightarrow{d} \mathcal{N} \left( 0, \frac{\frac{1}{T} k_4 + \sigma_4}{\sigma^4} \right) \]
Theorem

Under Assumptions 1-5 and correct specification:

(i)

\[ \begin{align*} 
t_{g,k} & \overset{d}{\rightarrow} \mathcal{N} \left( 0, \frac{1}{T} k_4 + \sigma_4 \right) 
\end{align*} \]

(ii)

\[ R_{CRSg}^2 \rightarrow 0 \]
Theorem

Under Assumptions 1-5 and correct specification:

(i)
\[ t_{g,k} \xrightarrow{d} \mathcal{N} \left(0, \frac{1}{T} k_4 + \sigma_4 \right) \]

(ii)
\[ R^2_{CRSg} \rightarrow 0 \]

(iii)
\[ F_{CSRg} \xrightarrow{d} \chi^2_K \left( \frac{k_4}{T} + \sigma_4 \right) / K \]
**Useless factors: base case**

- Inference can be carried out with t and F tests.
Useless factors: base case

- Inference can be carried out with t and F tests.
- When $\sigma_4 = \sigma^4$ and $k_4 = 0$, then $t_{g,k} \overset{d}{\to} \mathcal{N}(0,1)$ and $F_{CSR_g} \overset{d}{\to} \chi^2_{K/K}$.
Useless factors: base case

- Inference can be carried out with t and F tests.
- When $\sigma_4 = \sigma^4$ and $k_4 = 0$, then $t_{g,k} \xrightarrow{d} \mathcal{N}(0, 1)$ and $F_{CSRg} \xrightarrow{d} \chi^2_{K/K}$
- When $\sigma_4 \neq \sigma^4$, limiting distributions not conventional but can be made so by estimating nuisance parameters: there exists $\hat{\sigma}^2, \hat{\sigma}_4$ such that

  $$\hat{\sigma}_4 \xrightarrow{p} \sigma_4, \hat{\sigma}^2 \xrightarrow{p} \sigma^2.$$
Useless factors: base case

- Inference can be carried out with t and F tests.
- When $\sigma_4 = \sigma^4$ and $k_4 = 0$, then $t_{g,k} \xrightarrow{d} \mathcal{N}(0,1)$ and $F_{CSRg} \xrightarrow{d} \chi^2_{K/K}$.
- When $\sigma_4 \neq \sigma^4$, limiting distributions not conventional but can be made so by estimating nuisance parameters: there exists $\hat{\sigma}^2, \hat{\sigma}_4$ such that
  \[ \hat{\sigma}_4 \xrightarrow{p} \sigma_4, \hat{\sigma}^2 \xrightarrow{p} \sigma^2. \]
- $R^2$ is not inflated (goes to zero, as it should).
Useless factors (base case): misspecified case

- Let $1'_N c / N \rightarrow \mu_c$ and $c'M_{1_N} c / N \rightarrow \nu_c$ where $ER_{it} = c_i = \gamma_0 + e_i$. 
Useless factors (base case): misspecified case

- Let $1'Nc / N \to \mu_c$ and $c'M_1Nc / N \to \nu_c$ where $ER_{it} = c_i = \gamma_0 + e_i$.

**Theorem**

*Under Assumptions 1-5 and misspecification:*

\[ \hat{\Gamma} - (\mu_0K) = O_p(1/\sqrt{N}) \]

\[ \sqrt{N}(\hat{\Gamma}_g - (\mu_0K)) \overset{d}{\to} N(0_K + V + W) \]

where $V$ as for correctly-specified case and $W = \begin{bmatrix} 0 & 0' \\ 0_K & \nu_c \sigma^2 \tilde{G}' \tilde{G} \end{bmatrix}$.
Useless factors (base case): misspecified case

- Let $1_N' c / N \rightarrow \mu_c$ and $c' M_1 c / N \rightarrow \nu_c$ where $ER_{it} = c_i = \gamma_0 + e_i$.

Theorem

**Under Assumptions 1-5 and misspecification:**

(i) \[
\hat{\Gamma} - \begin{pmatrix} \mu_c \\ 0_K \end{pmatrix} = O_p \left( \frac{1}{\sqrt{N}} \right).
\]
Useless factors (base case): misspecified case

- Let $1_N'c/N \to \mu_c$ and $c'M_{1_N}c/N \to \nu_c$ where $ER_{it} = c_i = \gamma_0 + e_i$.

**Theorem**

*Under Assumptions 1-5 and misspecification:*

(i)  
\[ \hat{\Gamma} - \begin{pmatrix} \mu_c \\ 0_K \end{pmatrix} = O_p \left( \frac{1}{\sqrt{N}} \right). \]

(ii)  
\[ \sqrt{N} \left( \hat{\Gamma}_g - \begin{pmatrix} \mu_c \\ 0_K \end{pmatrix} \right) \xrightarrow{d} \mathcal{N} \left( 0_{K+1}, V + W \right), \]

where $V$ as for correctly-specified case and $W = \begin{pmatrix} 0 & 0'_{K} \\ 0_K & \frac{\nu_c}{\sigma^2} \tilde{G}' \tilde{G} \end{pmatrix}$. 

Raponi and Zaffaroni (2018)
Useless factors: base case with misspecification

Theorem

Under Assumptions 1-5 and misspecification:

(i) \( t \rightarrow N(0, \nu_c + \kappa_4 + \tau \sigma_4 + \tau^2 \sigma_2 \nu_c + \sigma_2 \tau) \)

(ii) \( R^2 \rightarrow 0 \)

(iii) \( F \rightarrow \chi^2_K(\nu_c + \kappa_4 + \tau \sigma_4 + \tau^2 \sigma_2 \nu_c + \sigma_2 \tau) \) / \( K \)
Useless factors: base case with misspecification

**Theorem**

*Under Assumptions 1-5 and misspecification:*

\[ t^g, k^d \to N \left( 0, \nu_c + \kappa^4 + T \sigma^4 T^2 \sigma^2 \nu_c + \sigma^2 T \right) \]

\[ R^2_{CSg} \to 0 \]

\[ F_{CSRg} \to \chi^2_K \left( \nu_c + \kappa^4 + T \sigma^4 T^2 \sigma^2 \nu_c + \sigma^2 T \right) / K \]

Raponi and Zaffaroni (2018)
Theorem

Under Assumptions 1-5 and misspecification:

(i)

\[ t_{g,k} \xrightarrow{d} \mathcal{N} \left( 0, \frac{\nu_c + \frac{\kappa_4 + T \sigma_4}{T^2 \sigma^2}}{\nu^c + \frac{\sigma^2}{T}} \right) \]
Useless factors: base case with misspecification

Theorem

Under Assumptions 1-5 and misspecification:

(i)

\[ t_{g,k} \overset{d}{\rightarrow} \mathcal{N} \left( 0, \frac{\nu_c + \frac{\kappa_4 + T \sigma_4}{T^2 \sigma^2}}{\nu_c + \frac{\sigma^2}{T}} \right) \]

(ii)

\[ R_{CRSg}^2 \rightarrow 0 \]
Useless factors: base case with misspecification

Theorem

Under Assumptions 1-5 and misspecification:

(i) \( t_{g,k} \overset{d}{\rightarrow} \mathcal{N} \left( 0, \frac{\nu_c + \frac{\kappa_4 + T \sigma_4}{T^2 \sigma^2}}{\nu_c + \frac{\sigma^2}{T}} \right) \)

(ii) \( R_{CRSg}^2 \rightarrow 0 \)

(iii) \( F_{CSRg} \overset{d}{\rightarrow} \chi^2_K \left( \frac{\nu_c + \frac{\kappa_4 + T \sigma_4}{T^2 \sigma^2}}{\nu_c + \frac{\sigma^2}{T}} \right) / K \)
Useless factors (base case): base case with misspecification

- Correctly-specified case obtained for $\nu_c = 0$. 

\[ \hat{\mu}_c = \frac{1}{N} \sum \bar{R}, \quad \hat{\nu}_c = \frac{(1/N) \sum \bar{R}}{N} - \hat{\mu}_c^2. \]
Correctly-specified case obtained for $\nu_c = 0$.

Qualitatively, the results do not differ from correctly-specified case.
Useless factors (base case): base case with misspecification

- Correctly-specified case obtained for $\nu_c = 0$.
- Qualitatively, the results do not differ from correctly-specified case.
- All quantities can be consistently estimated for $N \to \infty$:

$$
\hat{\mu}_c = 1'_N \bar{R} / N, \quad \hat{\nu}_c = 1'_N \bar{R}^2 / N - \hat{\mu}_c^2.
$$
How are the traditional FM t-ratios behaving? Before we have seen non-traditional t-ratios.
Useless factors (base case): FM t-ratios

- How are the traditional FM t-ratios behaving? Before we have seen non-traditional t-ratios.
- Let $Z = (Z_0 \cdots Z_K)' \sim N(0_{K+1}, V + W)$ as defined before.
Useless factors (base case): FM t-ratios

- How are the traditional FM t-ratios behaving? Before we have seen non-traditional t-ratios.
- Let $Z = (Z_0 \cdots Z_K)' \sim N(0_{K+1}, V + W)$ as defined before.
- Let $\frac{e'1_N}{\sqrt{N}} \rightarrow_d \xi \sim N(0, \sigma^2 I_T)$, $\frac{(e'e - N\sigma^2 I_T)}{\sqrt{N}} \rightarrow_d \Xi$ with $\text{vec}(\Xi) \sim N(0, U_\epsilon)$.
Useless factors (base case): FM t-ratios

- How are the traditional FM t-ratios behaving? Before we have seen non-traditional t-ratios.
- Let \( Z = (Z_0 \cdots Z_K)' \sim N(0_{K+1}, V + W) \) as defined before.
- Let \( \frac{\epsilon' \epsilon}{\sqrt{N}} \rightarrow_d \xi \sim N(0, \sigma^2 I_T) \), \( \frac{(\epsilon' \epsilon - N\sigma^2 I_T)}{\sqrt{N}} \rightarrow_d \Xi \) with \( \text{vec}(\Xi) \sim N(0, U_\epsilon) \).
- Let

\[
\Phi_k = \frac{1}{(T - 1)^{1/2}} \left( \left( \begin{array}{c} 1'_{k+1,K+1} \left( \Sigma_X + \Lambda \right)^{-1} \left( \begin{array}{ccc} \zeta' A \xi & \zeta' A \Xi P \\ P' \Xi A \Xi P & P' \Xi A \Xi P \end{array} \right) \left( \Sigma_X + \Lambda \right)^{-1} 1_{k+1,K+1} \right)^{1/2} \right), \quad k = 1, \ldots, K.
\]

for \( A = I_T - \frac{1_T 1_T'}{T} - \tilde{G} (\tilde{G}' \tilde{G})^{-1} \tilde{G}' \).

Useless factors (base case): FM t-ratios

- How are the traditional FM t-ratios behaving? Before we have seen non-traditional t-ratios.

- Let $Z = (Z_0 \cdots Z_K)' \sim N(0_{K+1}, V + W)$ as defined before.

- Let $\frac{\epsilon'1_N}{\sqrt{N}} \rightarrow_d \zeta \sim N(0, \sigma^2 I_T)$, $\frac{(\epsilon'\epsilon - N\sigma^2 I_T)}{\sqrt{N}} \rightarrow_d \Xi$ with $\text{vec}(\Xi) \sim N(0, U_\epsilon)$.

- Let

$$\Phi_k \equiv \left( \frac{1}{(T-1)} \right)^{1/2} \left( i'_{k+1,K+1} (\Sigma X + \Lambda)^{-1} \begin{pmatrix} \zeta' A \zeta & \zeta' A \Xi P \\ P' \Xi A \zeta & P' \Xi A \Xi P \end{pmatrix} \right) (\Sigma X + \Lambda)^{-1} i_{k+1,K+1} \right)^{1/2}, \quad k = 1, \ldots, K.$$  

for $A = I_T - \frac{1_T 1_T'}{T} - \tilde{G} (\tilde{G}' \tilde{G})^{-1} \tilde{G}'$.

- These non-conventional quantities characterize the FM t-ratios when $N$ is large.
Theorem

Under Assumptions 1-5:

(i) for the ex-ante risk premia

\[ |t_{FM}(\hat{\gamma}_0)| = \frac{|\hat{\gamma}_0 - \mu_c|}{SE_{FM}^0} \rightarrow_p Z_0 \]  

and  

\[ \sqrt{N}|t_{FM}(\hat{\gamma}_1)| = \sqrt{N} \frac{|\hat{\gamma}_1|}{SE_{FM}^k} \rightarrow_d Z_k. \]

(ii) for the ex-post risk premia

\[ |t_{FM}(\hat{\gamma}_0)| = \frac{|\hat{\gamma}_0 - \mu_c|}{SE_{FM,P}^0} \rightarrow_d Z_0 \]  

and  

\[ |t_{FM}(\hat{\gamma}_1)| = \frac{|\hat{\gamma}_1|}{SE_{FM,P}^k} \rightarrow_d Z_k. \]
Non-standard distributions arise for all cases.
Useless factors (base case): FM t-ratios

- Non-standard distributions arise for all cases.
- Shanken’s correction vanishes as $N$ diverges. The same applies for correctly-specified models without useless factors.
Useless factors (base case): FM t-ratios

- Non-standard distributions arise for all cases.
- Shanken’s correction vanishes as $N$ diverges. The same applies for correctly-specified models without useless factors.
- In fixed-$T$ ex post risk premia $\Gamma^P = \Gamma + \bar{f} -Ef_t$ should be considered: however ex ante FM t-ratios goes to zero unlike ex post.
Non-standard distributions arise for all cases.

Shanken’s correction vanishes as $N$ diverges. The same applies for correctly-specified models without useless factors.

In fixed-$T$ ex post risk premia $\Gamma^P = \Gamma + \bar{f} - Ef_t$ should be considered: however ex ante FM t-ratios goes to zero unlike ex post.

Same results for correctly-specified (except that $\mu_c = \gamma_0$) and misspecified cases.
Useless factors (base case): FM t-ratios

- Non-standard distributions arise for all cases.
- Shanken’s correction vanishes as $N$ diverges. The same applies for correctly-specified models without useless factors.
- In fixed-$T$ ex post risk premia $\Gamma^P = \Gamma + \bar{f} - Ef_t$ should be considered: however ex ante FM t-ratios goes to zero unlike ex post.
- Same results for correctly-specified (except that $\mu_c = \gamma_0$) and misspecified cases.
- Same results (obviously) for zero-beta rate t-ratios.
The true model is

\[ R_t = \alpha + B_f f_t + 0_{N,H} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t. \]
The true model is

\[ R_t = \alpha + B_f f_t + 0_{N,K} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t. \]

Estimated risk premia (setting \( \tilde{D} = (\tilde{F}, \tilde{G}) \))

\[ \hat{\Gamma}_{0fg} = (\hat{X}_f' \hat{X}_f)^{-1} \hat{X}_f' \tilde{R} \text{ where } (\hat{B}_f, \hat{B}_g) = R' \tilde{D}(\tilde{D}' \tilde{D})^{-1}. \]
Useless factors: useful with useless

- The true model is

\[ R_t = \alpha + B_f f_t + 0_{N,K} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t. \]

- Estimated risk premia (setting \( \tilde{D} = (\tilde{F}, \tilde{G}) \))

\[ \hat{\Gamma}_{0fg} = (\hat{X}_{fg}' \hat{X}_{fg})^{-1} \hat{X}_{fg}' \hat{R} \text{ where } (\hat{B}_f, \hat{B}_g) = R' \tilde{D} (\tilde{D}' \tilde{D})^{-1}. \]

- Under correct specification

\[ ER_{it} = \gamma_0 + \gamma'_{1f} \beta_{if}. \]
Useless factors: useful with useless

- The true model is

\[ R_t = \alpha + B_f f_t + 0_{N,K} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t. \]

- Estimated risk premia (setting \( \tilde{D} = (\tilde{F}, \tilde{G}) \))

\[ \hat{\Gamma}_{0fg} = (\hat{X}_{fg}' \hat{X}_{fg})^{-1} \hat{X}_{fg}' \bar{R} \text{ where } (\hat{B}_f, \hat{B}_g) = R' \tilde{D} (\tilde{D}' \tilde{D})^{-1}. \]

- Under correct specification

\[ ER_{it} = \gamma_0 + \gamma'_1 f \beta_{if}. \]

- Under misspecification

\[ ER_{it} = \gamma_0 + \gamma'_1 f \beta_{if} + e_i. \]
Useless factors: useful with useless ($G$ and $F$ orthogonal)

**Theorem**

When $\tilde{G}'\tilde{F} = 0$ ($G$ and $F$ orthogonal):

(i) 

$$
\hat{\Gamma}_{fg} - \begin{pmatrix}
\gamma_0 + d_0 \\
\gamma_{Pf} + d_1 \\
0_{K_g}
\end{pmatrix} = O_p \left( \frac{1}{\sqrt{N}} \right).
$$

(ii) 

$$
\left( \hat{\Gamma}_{fg} - \begin{pmatrix}
\gamma_0 + d_0 \\
\gamma_{Pf} + d_1 \\
0_{K_g}
\end{pmatrix} \right) \xrightarrow{d} N \left( 0, \left( \Sigma_{X_{fg}} + \Lambda_{fg} \right)^{-1} \left( V_{fg} + W_{fg} \right) \left( \Sigma_{X_{fg}} + \Lambda_{fg} \right)^{-1} \right).
$$
Theorem

When $\tilde{G}'\tilde{F} = 0$ ($G$ and $F$ orthogonal):

(i) 

$$t_{g,k_g} \overset{d}{\to} \mathcal{N} \left(0, \frac{d_1'\tilde{\Sigma}_f d_1 + \sigma^{-2} W_{[k_g,k_g]}}{\sigma^2 T + \gamma_{1f}'\sigma^2 (\tilde{F}'\tilde{F})^{-1} D^{-1}\tilde{\Sigma}_f \gamma_{1f}} \right).$$

(ii) 

$$R_{CRS_{fg}}^2 = 1 - \frac{\hat{e}'_{fg} \hat{e}_{fg}}{\bar{R}' \mathcal{M}_N \bar{R}} \to 1 - \frac{\sigma^2}{T} + \gamma_{1f}'\sigma^2 (\tilde{F}'\tilde{F})^{-1} D^{-1}\tilde{\Sigma}_f \gamma_{1f}$$
Theorem

When $\tilde{G}' \tilde{F} = 0$ ($G$ and $F$ orthogonal):

(iii) Let

$$F_{CSR_{fg}} = \frac{(\hat{e}_f' \hat{e}_f' - \hat{e}_{fg} \hat{e}_{fg}) / K_g}{\hat{e}_{fg}' \hat{e}_{fg} / (N - (K_f + K_g + 1))}$$

be the $F$-statistic to test the null hypothesis $\gamma_{1g}^P = 0_{K_g}$.

Then

$$F_{CSR_{fg}} \xrightarrow{d} (Z_1', Z_2') \frac{\mathcal{W}_{fg} / K_g}{\sigma^2 \tilde{T} - d_1' \tilde{\Sigma}_{\beta_f} \gamma_{1f}^P} \left( \begin{array}{c} Z_1 \\ Z_2 \end{array} \right)$$

where $Z_1 \equiv \mathcal{N} \left( 0_{T^2}, U_{\epsilon} \right)$ and $Z_2 \equiv \mathcal{N} \left( 0_T, \sigma^2 d_1' \tilde{\Sigma}_{\beta_f} d_1 I_T \right)$ are two normally distributed vectors of dimension $T^2 \times 1$ and $T \times 1$, respectively, and where $\mathcal{W}_{fg}$ suitable matrix.
Risk premia estimates for F (useful) are first-order biased.
Useless factors: useful with useless ($G$ and $F$ orthogonal)

- Risk premia estimates for $F$ (useful) are first-order biased.
- Instead, risk premia for $G$ (useless) converges to zero.
Useless factors: useful with useless ($G$ and $F$ orthogonal)

- Risk premia estimates for $F$ (useful) are first-order biased.
- Instead, risk premia for $G$ (useless) converges to zero.
- Results more complicated than previous case but similar spirit: all quantities can be consistently estimated and test with correct size and power be derived.
Useless factors: useful with useless ($G$ and $F$ not orthogonal)

- When $G$ and $F$ not orthogonal:

\[
\hat{\Gamma}_P \rightarrow \begin{pmatrix}
\gamma_0 - \mu' \beta_f (I_K f - E^{-1} \Sigma \beta_f) \\
\gamma_P 1_f E^{-1} \Sigma \beta_f \end{pmatrix}
\] (5)

Set $\theta_f = E^{-1} \Sigma \beta_f$ and $\theta_g = -\frac{\sigma}{2} Q' f (E^{-1} \Sigma \beta_f).$

Under the null of useless factors, the following linear restriction holds:

$H_0: \theta_g = A \theta_f.$

for an observed $A = (\tilde{G}' \tilde{G} - \tilde{G}' \tilde{F} (\tilde{F}' \tilde{F})^{-1} \tilde{F}' \tilde{G}) (\tilde{F}' \tilde{F})^{-1} - \tilde{G}' \tilde{G} D^{-1}.$

Using the distribution of part of our theorem, derive the test.

If $F$ and $G$ orthogonal in sample, then $H_0: \theta_g = 0.$

Bias-adjusted estimator for $\gamma_P$ can be obtained (not the focus here).
Useless factors: useful with useless ($G$ and $F$ not orthogonal)

When $G$ and $F$ not orthogonal:

\[
\hat{\Gamma}^P_{fg} \xrightarrow{P} \left( \gamma_0 - \mu'_\beta_f (I_{K_f} - E^{-1} \Sigma_{\beta_f}) \gamma^P_{1_f} \right) \begin{pmatrix}
\gamma_0 \\
E^{-1} \Sigma_{\beta_f} \gamma^P_{1_f} \\
A E^{-1} \Sigma_{\beta_f} \gamma^P_{1_f}
\end{pmatrix}
\]
Useless factors: useful with useless ($G$ and $F$ not orthogonal)

- When $G$ and $F$ not orthogonal:
  
  \[
  \hat{\Gamma}_{fg}^P \xrightarrow{p} \begin{pmatrix}
  \gamma_0 - \mu'_{\beta_f} (I_{K_f} - E^{-1}\Sigma_{\beta_f}) \gamma_1^P \\
  E^{-1}\Sigma_{\beta_f} \gamma_1^P \\
  A E^{-1}\Sigma_{\beta_f} \gamma_1^P
  \end{pmatrix}
  \] (5)

- Set

  \[
  \theta_f = E^{-1}\Sigma_{\beta_f} \gamma_1^P \text{ and } \theta_g = -\frac{D}{\sigma^2} Q_{fg} E^{-1}\Sigma_{\beta_f} \gamma_1^P.
  \]
Useless factors: useful with useless ($G$ and $F$ not orthogonal)

- When $G$ and $F$ not orthogonal:

\[
\hat{\Gamma}_{fg}^P \xrightarrow{P} \begin{pmatrix}
\gamma_0 - \mu'_\beta_f (I_{K_f} - E^{-1} \Sigma_{\beta_f}) \gamma_{1f}^P \\
E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P \\
AE^{-1} \Sigma_{\beta_f} \gamma_{1f}^P
\end{pmatrix}
\]  

(5)

- Set

\[
\theta_f = E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P \quad \text{and} \quad \theta_g = \frac{D}{\sigma^2} Q_{fg} E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P.
\]

- Under the null of useless factors, the following linear restriction holds:

\[
H_0 : \theta_g = A \theta_f.
\]

for an observed $A = (\tilde{G}' \tilde{G} - \tilde{G}' \tilde{F} (\tilde{F}' \tilde{F})^{-1} \tilde{F}' \tilde{G}) (\tilde{F}' \tilde{F})^{-1} \tilde{F}' \tilde{G} D^{-1}$. 

Useless factors: useful with useless ($G$ and $F$ not orthogonal)

- When $G$ and $F$ not orthogonal:

$$\hat{\Gamma}_{fg}^P \overset{P}{\to} \begin{pmatrix} \gamma_0 - \mu'_\beta_f (I_{K_f} - E^{-1}\Sigma_{\beta_f}) \gamma_{1f}^P \\ E^{-1}\Sigma_{\beta_f} \gamma_{1f}^P \\ AE^{-1}\Sigma_{\beta_f} \gamma_{1f}^P \end{pmatrix}$$ (5)

- Set

$$\theta_f = E^{-1}\Sigma_{\beta_f} \gamma_{1f}^P$$ and $$\theta_g = -\frac{D}{\sigma^2} Q_{fg} E^{-1}\Sigma_{\beta_f} \gamma_{1f}^P.$$  

- Under the null of useless factors, the following linear restriction holds:

$$H_0 : \theta_g = A\theta_f.$$  

for an observed $$A = (\tilde{G}'\tilde{G} - \tilde{G}'\tilde{F}(\tilde{F}'\tilde{F})^{-1}\tilde{F}'\tilde{G})(\tilde{F}'\tilde{F})^{-1}\tilde{F}'\tilde{G}D^{-1}.$$  

- Using the distribution of part of our theorem, derive the test.
Useless factors: useful with useless ($G$ and $F$ not orthogonal)

- When $G$ and $F$ not orthogonal:
  
  \[ \hat{\Gamma}^{P}_{fg} \xrightarrow{P} \left( \gamma_0 - \mu'_f (I_{K_f} - E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P) \gamma_{1f}^P \right) \]

  \[ \left( \begin{array}{c}
  E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P \\
  A E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P
  \end{array} \right) \]  
  (5)

- Set 
  \[ \theta_f = E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P \]  
  and \[ \theta_g = -\frac{D}{\sigma^2} Q_{fg} E^{-1} \Sigma_{\beta_f} \gamma_{1f}^P. \]

- Under the null of useless factors, the following linear restriction holds:
  \[ H_0 : \theta_g = A \theta_f. \]

  for an observed \[ A = (\tilde{G}' \tilde{G} - \tilde{G}' \tilde{F}(\tilde{F}' \tilde{F})^{-1} \tilde{F}' \tilde{G})(\tilde{F}' \tilde{F})^{-1} \tilde{F}' \tilde{G} D^{-1}. \]

- Using the distribution of part of our theorem, derive the test.

- If $F$ and $G$ orthogonal in sample, then $H_0 : \theta_g = 0.$
Useless factors: useful with useless (\(G\) and \(F\) not orthogonal)

- When \(G\) and \(F\) not orthogonal:
  
  \[
  \hat{\Gamma}_{fg}^P \rightarrow \begin{pmatrix}
  \gamma_0 - \mu'_{\beta_f} (I_{K_f} - E^{-1} \Sigma_{\beta_f}) \gamma_{1_f}^P \\
  E^{-1} \Sigma_{\beta_f} \gamma_{1_f}^P \\
  AE^{-1} \Sigma_{\beta_f} \gamma_{1_f}^P
  \end{pmatrix}
  \]  
  (5)

- Set

  \[
  \theta_f = E^{-1} \Sigma_{\beta_f} \gamma_{1_f}^P \text{ and } \theta_g = -\frac{D}{\sigma^2} Q_{fg} E^{-1} \Sigma_{\beta_f} \gamma_{1_f}^P.
  \]

- Under the null of useless factors, the following linear restriction holds:

  \[
  H_0 : \theta_g = A \theta_f.
  \]

  for an observed \(A = (\tilde{G}' \tilde{G} - \tilde{G}' \tilde{F} (\tilde{F}' \tilde{F})^{-1} \tilde{F}' \tilde{G}) (\tilde{F}' \tilde{F})^{-1} \tilde{F}' \tilde{G} D^{-1}.
  \]

- Using the distribution of part of our theorem, derive the test.

- If \(F\) and \(G\) orthogonal in sample, then \(H_0 : \theta_g = 0\).

- Bias-adjusted estimator for \(\gamma_{1_f}^P\) can be obtained (not the focus here).
The true model is still
\[ R_t = \alpha + B_f f_t + 0_{N,K} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t. \]

but we estimate
\[ R_t = \alpha + B_{f1} f_{1t} + B_g g_t + \text{residual setting } F = (F_1, F_2)(\text{misspecification:} \]
The true model is still

\[ R_t = \alpha + B_f f_t + 0_{N, K_g} g_t + \epsilon_t = \alpha + B_f f_t + \epsilon_t. \]

but we estimate

\[ R_t = \alpha + B_{f_1} f_{1t} + B_g g_t + \text{residual} \text{ setting } F = (F_1, F_2) \text{(misspecification:} \]

As a special case, we could miss out F entirely, so estimated model:

\[ R_t = \alpha + B_g g_t + \text{residual}. \]
Theorem (i)

\[
\hat{\Gamma}_{f_{1g}} - \left( \begin{array}{c}
\gamma_0 + \tilde{d}_0 \\
\tilde{d}_{11} \gamma_{1f}^{P[1]} + \tilde{d}_{12} \gamma_{1f}^{P[2]} \\
0_{K_g}
\end{array} \right) = O_p \left( \frac{1}{\sqrt{N}} \right)
\]
Useless factors: useful with useless and misspecification (G and F orthogonal)

**Theorem**

\[ \sqrt{N} \left( \hat{\Gamma}_{fg} - \begin{pmatrix} \gamma_0 + \tilde{d}_0 \\ \tilde{d}_{11} \gamma_{1f} + \tilde{d}_{12} \gamma_{1f} \\ 0 \end{pmatrix} \right) \]

\[ \overset{d}{\rightarrow} \mathcal{N} \left( 0, \left( \Sigma^{[1]}_{X_{fg}} + \Lambda^{[1]}_{fg} \right)^{-1} \left( V_{fg} + W_{fg} \right) \left( \Sigma^{[1]}_{X_{fg}} + \Lambda^{[1]}_{fg} \right)^{-1} \right) \]

*Problem is that acm is function of both \( F_1 \) and \( F_2 \) so not feasible!*
Useless factors: useful with useless and misspecification ($G$ and $F$ orthogonal)

- In particular $V_{fg}$ equal

$$
\sigma^2 \left( \frac{1}{T} + (\bar{d}_{11} \gamma_{1_f}^{P[1]} + \bar{d}_{12} \gamma_{1_f}^{P[2]})' (\bar{F}' \bar{F})^{-1} (\bar{d}_{11} \gamma_{1_f}^{P[1]} + \bar{d}_{12} \gamma_{1_f}^{P[2]}) \right) \Sigma_{X_{fg}}^{[1]} \\
+ \sigma^2 \Omega_{fg},
$$

with $\Omega_{fg}$ equal to

$$
\begin{pmatrix}
0 & 0'_{K_{f1}} & 0'_{K_g} \\
0_{K_{f1}} & \vartheta(\bar{F}'[1] \bar{F}[1])^{-1} - (\Sigma_{[1]}^{[1]} \bar{d}_{1} - \Sigma_{[1,2]}^{[1]} \gamma_{1_f}^{P[2]}) - (\bar{d}_{1}' \Sigma_{[1]}^{[1]} - \gamma_{1_f}^{P[2]}' \Sigma_{[2,1]}^{[1]}) & 0_{K_{f1} \times K_g} \\
0_{K_g} & 0_{K_g \times K_{f1}} & \vartheta(\bar{G}' \bar{G})^{-1}
\end{pmatrix}
$$
Useless factors: useful with useless and misspecification ($G$ and $F$ orthogonal)

- $W_{fg}$ equal

\[
\begin{bmatrix}
0'_{K_f} \\
(Q_f^{[1,2]'} \otimes P_f^{[1]'}) U_\varepsilon (Q_f^{[1,2]} \otimes P_f^{[1]}) \\
(Q_f^{[1,2]'} \otimes P_g') U_\varepsilon (Q_f^{[1,2]} \otimes P_f^{[1]}) \\
(Q_f^{[1,2]'} \otimes P_g') U_\varepsilon (Q_f^{[1,2]} \otimes P_g)
\end{bmatrix}
\begin{bmatrix}
0'_{K_g} \\
(Q_f^{[1,2]'} \otimes P_f^{[1]'}) U_\varepsilon (Q_f^{[1,2]} \otimes P_g)
\end{bmatrix}
\]

with

\[
Q_f^{[1,2]} = \left(\frac{1}{T} - P_f^{[1]} \tilde{d}_{11} \gamma_{1f} P^{[1]} - P_f^{[1]} \tilde{d}_{12} \gamma_{1f} P^{[2]}\right).
\]
Useless factors: useful with useless and misspecification (G and F orthogonal)

\[ t_{f_{1g}, k_g} = \frac{\hat{\gamma}_{1g, k_g}}{s_{f_{1g}} \cdot \sqrt{c_{g, k_g} k_g}} \]

is the \( t \)-statistic for the \( k_g \)-th regression coefficient (\( k_g = 1, ..., K_g \)) and \( c_{g, k_g} k_g \) is the \((k_g, k_g)\)-th element of the matrix \((\hat{X}_{f_{1g}}' \hat{X}_{f_{1g}})^{-1}\).
Useless factors: useful with useless and misspecification ($G$ and $F$ orthogonal)

\[ t_{f_{1g},k_g} = \frac{\hat{\gamma}_{1g,k_g}}{s_{f_{1g}} \cdot \sqrt{c_{g,k_g,k_g}}} \]

is the $t$-statistic for the $k_g$-th regression coefficient ($k_g = 1, ..., K_g$) and $c_{g,k_g,k_g}$ is the $(k_g, k_g)$-th element of the matrix $(\hat{X}_{f_{1g}}' \hat{X}_{f_{1g}})^{-1}$.

\[ R^2_{CSR_{f_{1g}}} = 1 - \frac{\hat{e}_{f_{1g}}' \hat{e}_{f_{1g}}}{\bar{R}' \bar{M}_N \bar{R}}. \]
Useless factors: useful with useless and misspecification (\(G\) and \(F\) orthogonal)

\[
t_{f_1g,k_g} = \frac{\hat{\gamma}_{1g,k_g}}{s_{f_1g} \cdot \sqrt{c_{g,k_g} k_g}}
\]

is the \(t\)-statistic for the \(k_g\)-th regression coefficient (\(k_g = 1, \ldots, K_g\)) and \(c_{g,k_g} k_g\) is the \((k_g, k_g)\)-th element of the matrix \((\hat{X}_{f_1g} \hat{X}_{f_1g})^{-1}\).

\[
R^2_{CSR_{f_1g}} = 1 - \frac{\hat{e}_{f_1g}^l \hat{e}_{f_1g}^f}{\bar{R}' \bar{M}_N \bar{R}}.
\]

\[
F_{CSR_{f_1g}} = \frac{(\hat{e}_{f_1g}^l \hat{e}_{f_1g}^f - \hat{e}_{f_1g}^l \hat{e}_{f_1g}^f) / K_g}{\hat{e}_{f_1g}^l \hat{e}_{f_1g}^f / (N - K_{f_1} - K_g - 1)}
\]

is the \(F\)-statistic to test \(\gamma_{1g}^P = 0_{K_g}\).
Useless factors: useful with useless and misspecification ($G$ and $F$ orthogonal)

**Theorem**

\[ tf_{1g,kg} \xrightarrow{d} \mathcal{N} \left( 0, \frac{\vartheta + \sigma^{-2} W_{kg,kg}}{\sigma^2 / T + \Gamma_{1f}' \tilde{\Sigma} \Sigma_{1f} \Gamma_{1f}'} \right), \]

where $W_{kg,kg}$ denotes the $(kg, kg)$-th element of the matrix $(Q_f^{[1,2]}' \times \tilde{G}') U\varepsilon (Q_f^{[1,2]} \times P_g)$, $\Gamma_{1f}' = \left[ \gamma_{1f}'^{[1]}, \gamma_{1f}'^{[2]} \right]'$ and

\[
\tilde{\Sigma} \chi_f = \begin{pmatrix}
\tilde{\Sigma}^{[1]}_{\beta_f} - \Sigma^{[1]}_{\beta_f} D^{-1} \tilde{\Sigma}^{[1]}_{\beta_f} & \sigma^2 (\tilde{F}^{[1]}' \tilde{F}^{[1]})^{-1} D^{-1} \tilde{\Sigma}^{[1,2]}_{\beta_f} \\
\sigma^2 \tilde{\Sigma}^{[1,2]}'_{\beta_f} D^{-1} (\tilde{F}^{[1]}' \tilde{F}^{[1]})^{-1} & \Sigma^{[2]}_{\beta_f} - \tilde{\Sigma}^{[1,2]}'_{\beta_f} D^{-1} \tilde{\Sigma}^{[1,2]}_{\beta_f}
\end{pmatrix}.
\]
Useless factors: useful with useless and misspecification (G and F orthogonal)

\[ R_{CSR_{f_1g}}^2 \xrightarrow{p} 1 - \frac{\sigma^2}{T} + \frac{\Gamma^P_1\tilde{\Sigma}X_f\Gamma^P_1}{\sigma^2 + \Gamma^P_1\tilde{\Sigma}\beta_f\Gamma^P_1} \]

where \( \Gamma^P_1 = \begin{bmatrix} \gamma^P[1]'_1, \gamma^P[2]'_1 \end{bmatrix}' \) and \( \tilde{\Sigma}_f = \begin{pmatrix} \tilde{\Sigma}_f^{[1]} & \tilde{\Sigma}_f^{[1,2]} \\ \tilde{\Sigma}_f^{[2,1]} & \tilde{\Sigma}_f^{[2]} \end{pmatrix} \).

\[ F_{CSR_{f_1g}} \xrightarrow{d} (Z_1', Z_2') \frac{\mathcal{W}_{fg}/K_g}{\sigma^2 / T + \Gamma^P_1\tilde{\Sigma}X_f\Gamma^P_1} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \]

where \( Z_1 \equiv \mathcal{N}(0_{T^2}, \theta \mathcal{I}_T) \) and \( Z_2 \equiv \mathcal{N}(0_T, \theta \sigma^2 I_T) \) are two normally distributed vectors of dimension \( T^2 \times 1 \) and \( T \times 1 \).
Results extend to $G$ and $F$ not orthogonal.
Results extend to $G$ and $F$ not orthogonal.

Problem: asymptotic distributions depend on $F_2$ which is not observed. Bounds can be derived but inaccurate for large $N$. 

Solution: estimate the useful factors by PCA and derive asymptotics for useless factors based on the PCA distribution (along the idea of Giglio and Xiu (2017)).
Useless factors: useful with useless and misspecification

- Results extend to $G$ and $F$ not orthogonal.
- Problem: asymptotic distributions depend on $F_2$ which is not observed. Bounds can be derived but inaccurate for large $N$.
- Solution: estimate the useful factors by PCA and derive asymptotics for useless factors based on the PCA distribution (along the idea of Giglio and Xiu (2017)).
Simulation results: base case

- The table reports the percentage bias (Bias) and root mean squared error (RMSE), all in percent, over 10,000 simulated data sets.
Simulation results: base case

- The table reports the percentage bias (Bias) and root mean squared error (RMSE), all in percent, over 10,000 simulated data sets.

- DGP

\[ R_t = \gamma_0 1_N + \epsilon_t, \]

where \( \epsilon_t \sim \mathcal{N}(0, \Sigma) \) and where we calibrate \( \gamma_0 \) as

\[ \gamma_0 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} R_{it}. \]
Simulation results: base case

- The table reports the percentage bias (Bias) and root mean squared error (RMSE), all in percent, over 10,000 simulated data sets.
- DGP

\[ R_t = \gamma_0 1_N + \epsilon_t, \]

where \( \epsilon_t \sim \mathcal{N}(0, \Sigma) \) and where we calibrate \( \gamma_0 \) as
\[ \gamma_0 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} R_{it}. \]

- Fitted Model is a One-Factor Model \( R_{it} = a_i + b_i g_t + u_{it} \), where \( g_t \) is the excess market return (from Kenneth French’s website) from January 2008 to December 2010 for \( T=36 \), and the excess market return from January 2008 to December 2013 for \( T=72 \).
Simulation results: base case

- The table reports the percentage bias (Bias) and root mean squared error (RMSE), all in percent, over 10,000 simulated data sets.
- DGP

\[ R_t = \gamma_0 1_N + \epsilon_t, \]

where \( \epsilon_t \sim \mathcal{N}(0, \Sigma) \) and where we calibrate \( \gamma_0 \) as
\[ \gamma_0 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} R_{it}. \]
- Fitted Model is a One-Factor Model \( R_{it} = a_i + b_i g_t + u_{it} \), where \( g_t \) is the excess market return (from Kenneth French’s website) from January 2008 to December 2010 for \( T=36 \), and the excess market return from January 2008 to December 2013 for \( T=72 \).
- The table also reports the \( R \)-squared \( (R^2) \) of the fitted model for different cross-sections of \( N = 100, 500, 1000, 3000 \) stocks.
Simulation results: base case (Bias and RMSE) - Scalar $\Sigma$

Table I
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor ($\Sigma$ scalar)

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias ($\hat{\gamma}_0$)</td>
<td>0.32%</td>
<td>0.18%</td>
<td>0.12%</td>
<td>0.11%</td>
</tr>
<tr>
<td>RMSE ($\hat{\gamma}_0$)</td>
<td>0.184</td>
<td>0.083</td>
<td>0.058</td>
<td>0.035</td>
</tr>
<tr>
<td>Bias ($\hat{\gamma}_1$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE ($\hat{\gamma}_1$)</td>
<td>0.429</td>
<td>0.191</td>
<td>0.134</td>
<td>0.082</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.006</td>
<td>0.002</td>
<td>0.001</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Panel A: $T = 36$

Raponi and Zaffaroni (2018) Dissecting Spurious Factors with Cross-Sect
December 30, 2018 46 / 75
Table I  
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor ($\Sigma$ scalar) 

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>0.05%</td>
<td>0.04%</td>
<td>0.03%</td>
<td>0.04%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.146</td>
<td>0.066</td>
<td>0.046</td>
<td>0.028</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_1$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_1$)</td>
<td>0.379</td>
<td>0.166</td>
<td>0.119</td>
<td>0.072</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.002</td>
<td>0.002</td>
<td>0.001</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Panel B: $T = 72$
The table presents the size properties of t-tests of statistical significance.
The table presents the size properties of $t$-tests of statistical significance. The null hypothesis is that the parameter of interest is equal to its true value.
The table presents the size properties of $t$-tests of statistical significance.

The null hypothesis is that the parameter of interest is equal to its true value.

$t_{FM}(\cdot)$ denotes the $t$-statistic associated with the OLS estimator that uses the traditional Fama-MacBeth standard error.
The table presents the size properties of $t$-tests of statistical significance.

The null hypothesis is that the parameter of interest is equal to its true value.

$t_{FM}(\cdot)$ denotes the $t$-statistic associated with the OLS estimator that uses the traditional Fama-MacBeth standard error.

$t(\cdot)$ denotes the $t$-statistic associated with the OLS estimator.
The table presents the size properties of t-tests of statistical significance.

The null hypothesis is that the parameter of interest is equal to its true value.

\( t_{FM}(\cdot) \) denotes the t-statistic associated with the OLS estimator that uses the traditional Fama-MacBeth standard error.

\( t(\cdot) \) denotes the t-statistic associated with the OLS estimator.

The t-statistics are compared with the critical values from a standard normal distribution.
Simulation results: base case (t-test) - Scalar $\Sigma$

Table II
Empirical size of t-tests in a One-Factor Model with a useless factor ($\Sigma$ Scalar)

Panel A: $T = 36$

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_{FM}(\hat{\gamma}_0)$</td>
<td>$t_{FM}(\hat{\gamma}_1)$</td>
<td>$t(\hat{\gamma}_0)$</td>
<td>$t(\hat{\gamma}_1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.105</td>
<td>0.053</td>
<td>0.010</td>
<td>0.105</td>
<td>0.053</td>
<td>0.013</td>
</tr>
<tr>
<td>500</td>
<td>0.108</td>
<td>0.053</td>
<td>0.011</td>
<td>0.108</td>
<td>0.054</td>
<td>0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.105</td>
<td>0.051</td>
<td>0.011</td>
<td>0.103</td>
<td>0.053</td>
<td>0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.101</td>
<td>0.050</td>
<td>0.010</td>
<td>0.101</td>
<td>0.053</td>
<td>0.011</td>
</tr>
<tr>
<td>100</td>
<td>0.105</td>
<td>0.053</td>
<td>0.010</td>
<td>0.105</td>
<td>0.055</td>
<td>0.013</td>
</tr>
<tr>
<td>500</td>
<td>0.107</td>
<td>0.052</td>
<td>0.011</td>
<td>0.108</td>
<td>0.054</td>
<td>0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.106</td>
<td>0.051</td>
<td>0.011</td>
<td>0.103</td>
<td>0.053</td>
<td>0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.102</td>
<td>0.050</td>
<td>0.010</td>
<td>0.102</td>
<td>0.052</td>
<td>0.011</td>
</tr>
</tbody>
</table>
### Table II
Empirical size of t-tests in a One-Factor Model with a useless factor ($\Sigma$ Scalar)

Panel A: $T = 72$

<table>
<thead>
<tr>
<th>N</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_{FM}(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.105</td>
<td>0.054</td>
<td>0.011</td>
<td>0.101</td>
<td>0.052</td>
<td>0.011</td>
</tr>
<tr>
<td>500</td>
<td>0.105</td>
<td>0.052</td>
<td>0.011</td>
<td>0.098</td>
<td>0.049</td>
<td>0.009</td>
</tr>
<tr>
<td>1000</td>
<td>0.102</td>
<td>0.052</td>
<td>0.009</td>
<td>0.097</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>3000</td>
<td>0.102</td>
<td>0.051</td>
<td>0.010</td>
<td>0.099</td>
<td>0.050</td>
<td>0.009</td>
</tr>
<tr>
<td>$t(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.103</td>
<td>0.053</td>
<td>0.009</td>
<td>0.098</td>
<td>0.055</td>
<td>0.011</td>
</tr>
<tr>
<td>500</td>
<td>0.103</td>
<td>0.051</td>
<td>0.010</td>
<td>0.098</td>
<td>0.046</td>
<td>0.009</td>
</tr>
<tr>
<td>1000</td>
<td>0.101</td>
<td>0.051</td>
<td>0.009</td>
<td>0.096</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>3000</td>
<td>0.101</td>
<td>0.051</td>
<td>0.010</td>
<td>0.099</td>
<td>0.050</td>
<td>0.009</td>
</tr>
</tbody>
</table>
Simulation results: base case (F-test) - Scalar $\Sigma$

**Table III**  
**Empirical size of $F$-tests in a One-Factor Model with a useless factor ($\Sigma$ scalar)**

The table presents the size properties of $F$-tests of statistical significance. The $F$-statistics are compared with the critical values from a $\chi^2_K \left( \frac{\sigma_4}{\sigma_4^2} / K \right)$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Panel A: $T = 36$</th>
<th>Panel A: $T = 72$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.107 0.056 0.012</td>
<td>0.108 0.056 0.012</td>
</tr>
<tr>
<td>500</td>
<td>0.101 0.052 0.011</td>
<td>0.104 0.053 0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.101 0.051 0.011</td>
<td>0.101 0.052 0.010</td>
</tr>
<tr>
<td>3000</td>
<td>0.100 0.049 0.010</td>
<td>0.101 0.051 0.010</td>
</tr>
</tbody>
</table>
## Table IV

Bias and RMSE of the OLS Estimator One-Factor Model with a useless factor ($\Sigma$ Diagonal).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>-0.15%</td>
<td>0.15%</td>
<td>0.08%</td>
<td>0.02%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.923</td>
<td>0.425</td>
<td>0.308</td>
<td>0.190</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_1$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_1$)</td>
<td>0.764</td>
<td>0.330</td>
<td>0.227</td>
<td>0.135</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.030</td>
<td>0.006</td>
<td>0.003</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Panel A: $T = 36$
Simulation results: base case (Bias and RMSE) - Diagonal $\Sigma$

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>0.07%</td>
<td>0.03%</td>
<td>0.03%</td>
<td>0.02%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.400</td>
<td>0.160</td>
<td>0.127</td>
<td>0.075</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_1$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_1$)</td>
<td>1.070</td>
<td>0.521</td>
<td>0.332</td>
<td>0.208</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.069</td>
<td>0.018</td>
<td>0.008</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Panel B: $T = 72$
**Table V**

Empirical size of $t$-tests in a One-Factor Model with a useless factor ($\Sigma$ Diagonal)

Panel A: $T = 36$

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_{FM}(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td>$t_{FM}(\hat{\gamma}_1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.103</td>
<td>0.052</td>
<td>0.010</td>
<td>0.113</td>
<td>-0.060</td>
<td>0.015</td>
</tr>
<tr>
<td>500</td>
<td>0.101</td>
<td>0.050</td>
<td>0.010</td>
<td>0.101</td>
<td>0.053</td>
<td>0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.101</td>
<td>0.050</td>
<td>0.011</td>
<td>0.103</td>
<td>0.054</td>
<td>0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.100</td>
<td>0.050</td>
<td>0.009</td>
<td>0.102</td>
<td>0.050</td>
<td>0.011</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$t(\hat{\gamma}_0)$</th>
<th></th>
<th></th>
<th>$t(\hat{\gamma}_1)$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.095</td>
<td>0.048</td>
<td>0.010</td>
<td>0.113</td>
<td>0.059</td>
<td>0.014</td>
</tr>
<tr>
<td>500</td>
<td>0.098</td>
<td>0.048</td>
<td>0.009</td>
<td>0.102</td>
<td>0.050</td>
<td>0.011</td>
</tr>
<tr>
<td>1000</td>
<td>-0.099</td>
<td>0.050</td>
<td>0.011</td>
<td>0.103</td>
<td>0.052</td>
<td>0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.102</td>
<td>0.050</td>
<td>0.009</td>
<td>0.100</td>
<td>0.050</td>
<td>0.012</td>
</tr>
</tbody>
</table>
Table V

Empirical size of $t$-tests in a One-Factor Model with a useless factor ($\Sigma$ Diagonal)

<table>
<thead>
<tr>
<th>Panel B: $T = 72$</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>0.10</td>
<td>0.05</td>
<td>0.01</td>
<td>0.10</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>$t_{FM}(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.101</td>
<td>0.047</td>
<td>0.007</td>
<td>0.134</td>
<td>0.077</td>
<td>0.022</td>
</tr>
<tr>
<td>500</td>
<td>0.100</td>
<td>0.050</td>
<td>0.010</td>
<td>0.108</td>
<td>0.057</td>
<td>0.013</td>
</tr>
<tr>
<td>1000</td>
<td>0.098</td>
<td>0.049</td>
<td>0.010</td>
<td>0.103</td>
<td>0.053</td>
<td>0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.051</td>
<td>0.010</td>
<td>0.101</td>
<td>0.052</td>
<td>0.011</td>
</tr>
<tr>
<td>$t(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.087</td>
<td>0.041</td>
<td>0.006</td>
<td>0.122</td>
<td>0.072</td>
<td>0.023</td>
</tr>
<tr>
<td>500</td>
<td>0.096</td>
<td>0.050</td>
<td>0.009</td>
<td>0.104</td>
<td>0.057</td>
<td>0.012</td>
</tr>
<tr>
<td>1000</td>
<td>0.097</td>
<td>0.047</td>
<td>0.010</td>
<td>0.108</td>
<td>0.053</td>
<td>0.013</td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.051</td>
<td>0.010</td>
<td>0.101</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>$t_{FM}(\hat{\gamma}_1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Simulation results: base case (F-test) - Diagonal $\Sigma$**

**Table VI**

**Empirical size of $F$-tests in a One-Factor Model with a useless factor ($\Sigma$ Diagonal)**

The table presents the size properties of $F$-tests of statistical significance. The $F$-statistics are compared with the critical values from a $\chi^2_K \left( \frac{\sigma_4^4}{\sigma_4^4} / K \right)$.

<table>
<thead>
<tr>
<th>Panel A: $T = 36$</th>
<th>Panel A: $T = 72$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>0.10   0.05   0.01</td>
</tr>
<tr>
<td>100</td>
<td>0.113  0.060  0.015</td>
</tr>
<tr>
<td>500</td>
<td>0.101  0.053  0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.102  0.052  0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.102  0.050  0.011</td>
</tr>
</tbody>
</table>
Simulation results: base case (Bias and RMSE) - Full $\Sigma$ ($\delta = 0.5$)

Table VII
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor ($\Sigma$ Full - $\delta = 0.5$).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>-0.16%</td>
<td>0.13%</td>
<td>0.06%</td>
<td>0.05%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.923</td>
<td>0.425</td>
<td>0.305</td>
<td>0.189</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_1$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_1$)</td>
<td>1.253</td>
<td>0.474</td>
<td>0.349</td>
<td>0.196</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.031</td>
<td>0.006</td>
<td>0.003</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Panel A: $T = 36$
Simulation results: base case (Bias and RMSE) - Full $\Sigma$ ($\delta = 0.5$)

Table VII
Bias and RMSE of the OLS Estimator in a One-Factor Model with a useless factor ($\Sigma$ Full - $\delta = 0.5$).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>-0.03%</td>
<td>0.02%</td>
<td>0.05%</td>
<td>0.03%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.353</td>
<td>0.178</td>
<td>0.118</td>
<td>0.078</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_1$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_1$)</td>
<td>0.764</td>
<td>0.329</td>
<td>0.230</td>
<td>0.138</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.090</td>
<td>0.015</td>
<td>0.009</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Panel B: $T = 72$
Simulation results: base case (t-test) - Full $\Sigma (\delta = 0.5)$

Table VIII
Empirical size of $t$-tests in a One-Factor Model with a useless factor($\Sigma$ Full - $\delta = 0.5$)

Panel A: $T = 36$

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_{FM}(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td>$t_{FM}(\hat{\gamma}_1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.103</td>
<td>0.053</td>
<td>0.010</td>
<td>0.113</td>
<td>0.060</td>
<td>0.015</td>
</tr>
<tr>
<td>500</td>
<td>0.102</td>
<td>0.050</td>
<td>0.010</td>
<td>0.101</td>
<td>0.053</td>
<td>0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.102</td>
<td>0.049</td>
<td>0.010</td>
<td>0.103</td>
<td>0.053</td>
<td>0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.050</td>
<td>0.010</td>
<td>0.101</td>
<td>0.052</td>
<td>0.011</td>
</tr>
</tbody>
</table>

|     | $t(\hat{\gamma}_0)$ |      |      | $t(\hat{\gamma}_1)$ |      |      |
| 100 | 0.096 | 0.048 | 0.010 | 0.113 | 0.059 | 0.014 |
| 500 | 0.097 | 0.048 | 0.010 | 0.101 | 0.051 | 0.011 |
| 1000| 0.102 | 0.048 | 0.010 | 0.104 | 0.052 | 0.011 |
| 3000| 0.099 | 0.050 | 0.010 | 0.103 | 0.051 | 0.010 |
Simulation results: base case (t-test) - Full $\Sigma$ ($\delta = 0.5$)

Table VIII
Empirical size of $t$-tests in a One-Factor Model with a useless factor ($\Sigma$ Full - $\delta = 0.5$)

Panel B: $T = 72$

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_{FM}(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td>$t_{FM}(\hat{\gamma}_1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.102</td>
<td>0.046</td>
<td>0.005</td>
<td>0.138</td>
<td>0.080</td>
<td>0.030</td>
</tr>
<tr>
<td>500</td>
<td>0.107</td>
<td>0.053</td>
<td>0.009</td>
<td>0.106</td>
<td>0.055</td>
<td>0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.099</td>
<td>0.045</td>
<td>0.009</td>
<td>0.101</td>
<td>0.052</td>
<td>0.014</td>
</tr>
<tr>
<td>3000</td>
<td>0.101</td>
<td>0.049</td>
<td>0.011</td>
<td>0.097</td>
<td>0.049</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>$t(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td>$t(\hat{\gamma}_1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.087</td>
<td>0.044</td>
<td>0.008</td>
<td>0.129</td>
<td>0.073</td>
<td>0.022</td>
</tr>
<tr>
<td>500</td>
<td>0.102</td>
<td>0.050</td>
<td>0.009</td>
<td>0.105</td>
<td>0.052</td>
<td>0.011</td>
</tr>
<tr>
<td>1000</td>
<td>0.096</td>
<td>0.047</td>
<td>0.009</td>
<td>0.100</td>
<td>0.050</td>
<td>0.013</td>
</tr>
<tr>
<td>3000</td>
<td>0.100</td>
<td>0.049</td>
<td>0.010</td>
<td>0.097</td>
<td>0.049</td>
<td>0.011</td>
</tr>
</tbody>
</table>
Simulation results: base case (F-test) - Full $\Sigma$ ($\delta = 0.5$)

Table IX
Empirical size of $F$-tests in a One-Factor Model with a useless factor ($\Sigma$ Full - $\delta = 0.5$)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Panel A: $T = 36$</th>
<th>Panel A: $T = 72$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.10</td>
<td>0.05</td>
</tr>
<tr>
<td>100</td>
<td>0.113</td>
<td>0.060</td>
</tr>
<tr>
<td>500</td>
<td>0.101</td>
<td>0.053</td>
</tr>
<tr>
<td>1000</td>
<td>0.103</td>
<td>0.053</td>
</tr>
<tr>
<td>3000</td>
<td>0.101</td>
<td>0.051</td>
</tr>
</tbody>
</table>
• DGP is

\[ R_t = \gamma_0 1_N + B_f (\gamma_1 + f_t - E[f]) + \epsilon_t, \]

where \( \epsilon_t \sim \mathcal{N}(0, \sigma^2 I_T) \) and where we calibrate \( \gamma_0 \) and \( \gamma_1 \) as the OLS estimates from the one factor model (CAPM).

• Fitted Model is a Two-Factor Model”

\[ R_t = \alpha + B_f f_t + B_g g_t + \epsilon_t, \]

where \( g_t \) is an orthogonal (useless) factor to \( f_t \).

• All factors are orthogonalized to each other such that \( \tilde{F}' \tilde{G} = 0_{K_f \times K_g} \)
Simulation results: useful plus useless (Bias and RMSE) - scalar $\Sigma$

### Table XIII
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors ($\Sigma$ scalar).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>0.78%</td>
<td>0.06%</td>
<td>-0.15%</td>
<td>0.10%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.291</td>
<td>0.132</td>
<td>0.071</td>
<td>0.047</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_{1f}$)</td>
<td>0.43%</td>
<td>0.07%</td>
<td>0.08%</td>
<td>-0.03%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_{1f}$)</td>
<td>0.211</td>
<td>0.102</td>
<td>0.053</td>
<td>0.039</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_{1g}$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_{1g}$)</td>
<td>1.769</td>
<td>0.766</td>
<td>0.543</td>
<td>0.326</td>
</tr>
<tr>
<td>Bias($R^2$)</td>
<td>4.66%</td>
<td>1.62%</td>
<td>0.38%</td>
<td>0.22%</td>
</tr>
</tbody>
</table>
Simulation results: useful plus useless (Bias and RMSE) - scalar $\Sigma$

**Table XIII**

Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors ($\Sigma$ scalar).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Bias}(\hat{\gamma}_0)$</td>
<td>0.10%</td>
<td>0.05%</td>
<td>0.06%</td>
<td>0.07%</td>
</tr>
<tr>
<td>$\text{RMSE}(\hat{\gamma}_0)$</td>
<td>0.582</td>
<td>0.195</td>
<td>0.079</td>
<td>0.055</td>
</tr>
<tr>
<td>$\text{Bias}(\hat{\gamma}_{1f})$</td>
<td>0.27%</td>
<td>0.08%</td>
<td>0.11%</td>
<td>0.03%</td>
</tr>
<tr>
<td>$\text{RMSE}(\hat{\gamma}_{1f})$</td>
<td>0.278</td>
<td>0.125</td>
<td>0.052</td>
<td>0.033</td>
</tr>
<tr>
<td>$\text{Bias}(\hat{\gamma}_{1g})$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$\text{RMSE}(\hat{\gamma}_{1g})$</td>
<td>1.215</td>
<td>0.540</td>
<td>0.376</td>
<td>0.227</td>
</tr>
<tr>
<td>$\text{Bias}(R^2)$</td>
<td>4.00%</td>
<td>1.57%</td>
<td>0.65%</td>
<td>0.39%</td>
</tr>
</tbody>
</table>

Panel B: $T = 72$
Table XIV
Empirical Size of \( t \)-tests in a correctly specified model with useful and useless factors (\( \Sigma \) Scalar)

Panel A: \( T = 36 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( t(\hat{\gamma}_0) )</td>
<td>( t(\hat{\gamma}_{1f}) )</td>
<td>( t(\hat{\gamma}_{1g}) )</td>
<td>( t(\hat{\gamma}_0) )</td>
<td>( t(\hat{\gamma}_{1f}) )</td>
<td>( t(\hat{\gamma}_{1g}) )</td>
<td>( t(\hat{\gamma}_0) )</td>
<td>( t(\hat{\gamma}_{1f}) )</td>
<td>( t(\hat{\gamma}_{1g}) )</td>
</tr>
<tr>
<td>100</td>
<td>0.102</td>
<td>0.049</td>
<td>0.011</td>
<td>0.099</td>
<td>0.053</td>
<td>0.009</td>
<td>0.100</td>
<td>0.053</td>
<td>0.012</td>
</tr>
<tr>
<td>500</td>
<td>0.102</td>
<td>0.052</td>
<td>0.009</td>
<td>0.098</td>
<td>0.048</td>
<td>0.008</td>
<td>0.099</td>
<td>0.051</td>
<td>0.012</td>
</tr>
<tr>
<td>1000</td>
<td>0.100</td>
<td>0.051</td>
<td>0.011</td>
<td>0.098</td>
<td>0.048</td>
<td>0.009</td>
<td>0.100</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.050</td>
<td>0.010</td>
<td>0.101</td>
<td>0.052</td>
<td>0.010</td>
<td>0.099</td>
<td>0.049</td>
<td>0.010</td>
</tr>
</tbody>
</table>
Simulation results: useful plus useless (t-test) - scalar $\sum$

**Table XIV**

Empirical Size of $t$-tests in a correctly specified model with useful and useless factors ($\sum$ Scalar)

Panel B: $T = 72$

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t(\hat{\gamma}_0)$</td>
<td></td>
<td></td>
<td>$t(\hat{\gamma}_{1f})$</td>
<td></td>
<td></td>
<td>$t(\hat{\gamma}_{1g})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.097</td>
<td>0.051</td>
<td>0.009</td>
<td>0.101</td>
<td>0.048</td>
<td>0.010</td>
<td>0.110</td>
<td>0.054</td>
<td>0.012</td>
</tr>
<tr>
<td>500</td>
<td>0.103</td>
<td>0.051</td>
<td>0.013</td>
<td>0.096</td>
<td>0.048</td>
<td>0.009</td>
<td>0.098</td>
<td>0.048</td>
<td>0.010</td>
</tr>
<tr>
<td>1000</td>
<td>0.099</td>
<td>0.052</td>
<td>0.010</td>
<td>0.095</td>
<td>0.049</td>
<td>0.010</td>
<td>0.098</td>
<td>0.048</td>
<td>0.011</td>
</tr>
<tr>
<td>3000</td>
<td>0.101</td>
<td>0.052</td>
<td>0.010</td>
<td>0.102</td>
<td>0.052</td>
<td>0.010</td>
<td>0.100</td>
<td>0.049</td>
<td>0.009</td>
</tr>
</tbody>
</table>
Simulation results: useful plus useless (Bias and RMSE) - diagonal $\Sigma$

### Table XV

Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors ($\Sigma$ Diagonal).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>N = 100</th>
<th>N = 500</th>
<th>N = 1000</th>
<th>N = 3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>0.09%</td>
<td>0.06%</td>
<td>0.02%</td>
<td>0.04%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.052</td>
<td>0.034</td>
<td>0.029</td>
<td>0.021</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_{1_f}$)</td>
<td>0.11%</td>
<td>0.05%</td>
<td>0.03%</td>
<td>0.02%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_{1_f}$)</td>
<td>0.038</td>
<td>0.029</td>
<td>0.025</td>
<td>0.017</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_{1_g}$)</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_{1_g}$)</td>
<td>1.820</td>
<td>1.203</td>
<td>0.958</td>
<td>0.543</td>
</tr>
<tr>
<td>Bias($R^2$)</td>
<td>2.30%</td>
<td>0.90%</td>
<td>0.23%</td>
<td>0.18%</td>
</tr>
</tbody>
</table>

Panel A: $T = 36$
Simulation results: useful plus useless (Bias and RMSE) - diagonal $\Sigma$

Table XV
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors ($\Sigma$ Diagonal).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Bias}(\hat{\gamma}_0)$</td>
<td>0.09%</td>
<td>0.03%</td>
<td>0.02%</td>
<td>0.02%</td>
</tr>
<tr>
<td>$\text{RMSE}(\hat{\gamma}_0)$</td>
<td>0.036</td>
<td>0.032</td>
<td>0.029</td>
<td>0.023</td>
</tr>
<tr>
<td>$\text{Bias}(\hat{\gamma}_{1f})$</td>
<td>0.13%</td>
<td>0.03%</td>
<td>0.03%</td>
<td>0.03%</td>
</tr>
<tr>
<td>$\text{RMSE}(\hat{\gamma}_{1f})$</td>
<td>0.032</td>
<td>0.023</td>
<td>0.019</td>
<td>0.017</td>
</tr>
<tr>
<td>$\text{Bias}(\hat{\gamma}_{1g})$</td>
<td>0.001</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$\text{RMSE}(\hat{\gamma}_{1g})$</td>
<td>1.807</td>
<td>0.922</td>
<td>0.653</td>
<td>0.392</td>
</tr>
<tr>
<td>$\text{Bias}(R^2)$</td>
<td>3.13%</td>
<td>1.07%</td>
<td>0.19%</td>
<td>0.17%</td>
</tr>
</tbody>
</table>

Panel B: $T = 72$
Simulation results: useful plus useless (t-test) - diagonal $\Sigma$

### Table XVI

Empirical Size of $t$-tests in a correctly specified model with useful and useless factors ($\Sigma$ Diagonal)

Panel A: $T = 36$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$t(\hat{\gamma}_0)$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>$t(\hat{\gamma}_{1f})$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>$t(\hat{\gamma}_{1g})$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.101</td>
<td>0.051</td>
<td>0.011</td>
<td></td>
<td>0.109</td>
<td>0.068</td>
<td>0.016</td>
<td></td>
<td>0.112</td>
<td>0.056</td>
<td>0.016</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>0.101</td>
<td>0.051</td>
<td>0.011</td>
<td></td>
<td>0.084</td>
<td>0.045</td>
<td>0.009</td>
<td></td>
<td>0.106</td>
<td>0.049</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.102</td>
<td>0.052</td>
<td>0.010</td>
<td></td>
<td>0.099</td>
<td>0.051</td>
<td>0.010</td>
<td></td>
<td>0.103</td>
<td>0.054</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.049</td>
<td>0.010</td>
<td></td>
<td>0.099</td>
<td>0.051</td>
<td>0.010</td>
<td></td>
<td>0.099</td>
<td>0.049</td>
<td>0.009</td>
<td></td>
</tr>
</tbody>
</table>

Raponi and Zaffaroni (2018) Dissecting Spurious Factors with Cross-Sect regression
Simulation results: useful plus useless (t-test) - diagonal $\Sigma$

Table XVI
Empirical Size of $t$-tests in a correctly specified model with useful and useless factors ($\Sigma$ Diagonal)

Panel B: $T = 72$

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t(\hat{\gamma}_0)$</td>
<td>$t(\hat{\gamma}_{1f})$</td>
<td>$t(\hat{\gamma}_{1g})$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.079</td>
<td>0.045</td>
<td>0.007</td>
<td>0.073</td>
<td>0.042</td>
<td>0.003</td>
<td>0.106</td>
<td>0.056</td>
<td>0.014</td>
</tr>
<tr>
<td>500</td>
<td>0.105</td>
<td>0.054</td>
<td>0.010</td>
<td>0.089</td>
<td>0.045</td>
<td>0.008</td>
<td>0.098</td>
<td>0.053</td>
<td>0.010</td>
</tr>
<tr>
<td>1000</td>
<td>0.097</td>
<td>0.049</td>
<td>0.009</td>
<td>0.097</td>
<td>0.048</td>
<td>0.010</td>
<td>0.098</td>
<td>0.049</td>
<td>0.010</td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.049</td>
<td>0.010</td>
<td>0.098</td>
<td>0.049</td>
<td>0.010</td>
<td>0.100</td>
<td>0.049</td>
<td>0.010</td>
</tr>
</tbody>
</table>
Table XVII

Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors ($\Sigma$ Full, $\delta = 0.5$).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$ Bias</td>
<td>0.42%</td>
<td>0.38%</td>
<td>0.09%</td>
<td>0.06%</td>
</tr>
<tr>
<td>$\gamma_0$ RMSE</td>
<td>0.086</td>
<td>0.042</td>
<td>0.035</td>
<td>0.021</td>
</tr>
<tr>
<td>$\gamma_1f$ Bias</td>
<td>0.06%</td>
<td>0.03%</td>
<td>0.04%</td>
<td>0.01%</td>
</tr>
<tr>
<td>$\gamma_1f$ RMSE</td>
<td>0.066</td>
<td>0.023</td>
<td>0.020</td>
<td>0.017</td>
</tr>
<tr>
<td>$\gamma_1g$ Bias</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$\gamma_1g$ RMSE</td>
<td>1.211</td>
<td>0.916</td>
<td>0.903</td>
<td>0.543</td>
</tr>
<tr>
<td>$R^2$</td>
<td>2.90%</td>
<td>1.42%</td>
<td>0.49%</td>
<td>0.38%</td>
</tr>
</tbody>
</table>
Simulation results: useful plus useless (Bias and RMSE) - Full $\Sigma$ ($\delta = 0.5$)

Table XVII
Bias and RMSE of the OLS Estimator in a correctly specified model with useful and useless factors ($\Sigma$ Full, $\delta = 0.5$).

<table>
<thead>
<tr>
<th>Statistics</th>
<th>$N = 100$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 3000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias($\hat{\gamma}_0$)</td>
<td>0.02%</td>
<td>0.04%</td>
<td>0.01%</td>
<td>0.02%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_0$)</td>
<td>0.052</td>
<td>0.044</td>
<td>0.030</td>
<td>0.023</td>
</tr>
<tr>
<td>Bias($\hat{\gamma}_{1f}$)</td>
<td>0.04%</td>
<td>0.07%</td>
<td>0.02%</td>
<td>0.02%</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_{1f}$)</td>
<td>0.056</td>
<td>0.036</td>
<td>0.028</td>
<td>0.021</td>
</tr>
<tr>
<td>$\hat{\gamma}_{1g}$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>RMSE($\hat{\gamma}_{1g}$)</td>
<td>1.872</td>
<td>0.864</td>
<td>0.653</td>
<td>0.393</td>
</tr>
<tr>
<td>$R^2$</td>
<td>2.19%</td>
<td>1.29%</td>
<td>0.46%</td>
<td>0.29%</td>
</tr>
</tbody>
</table>
Simulation results: useful plus useless (t-test) - Full $\Sigma$ ($\delta = 0.5$)

Table XVIII
Empirical Size of $t$-tests in a correctly specified model with useful and useless factors ($\Sigma$ Full - $\delta = 0.5$)

Panel A: $T = 36$

<table>
<thead>
<tr>
<th>N</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t(\hat{\gamma}_0)$</td>
<td>$t(\hat{\gamma}_1f)$</td>
<td>$t(\hat{\gamma}_1g)$</td>
<td>$t(\hat{\gamma}_0)$</td>
<td>$t(\hat{\gamma}_1f)$</td>
<td>$t(\hat{\gamma}_1g)$</td>
<td>$t(\hat{\gamma}_0)$</td>
<td>$t(\hat{\gamma}_1f)$</td>
<td>$t(\hat{\gamma}_1g)$</td>
</tr>
<tr>
<td>100</td>
<td>0.127</td>
<td>0.069</td>
<td>0.016</td>
<td>0.126</td>
<td>0.072</td>
<td>0.019</td>
<td>0.102</td>
<td>0.053</td>
<td>0.010</td>
</tr>
<tr>
<td>500</td>
<td>0.107</td>
<td>0.055</td>
<td>0.014</td>
<td>0.110</td>
<td>0.054</td>
<td>0.013</td>
<td>0.102</td>
<td>0.052</td>
<td>0.009</td>
</tr>
<tr>
<td>1000</td>
<td>0.104</td>
<td>0.052</td>
<td>0.012</td>
<td>0.103</td>
<td>0.049</td>
<td>0.008</td>
<td>0.100</td>
<td>0.050</td>
<td>0.010</td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.048</td>
<td>0.010</td>
<td>0.099</td>
<td>0.051</td>
<td>0.010</td>
<td>0.100</td>
<td>0.050</td>
<td>0.010</td>
</tr>
</tbody>
</table>

Panel B: $T = 72$

<table>
<thead>
<tr>
<th></th>
<th>$t(\hat{\gamma}_0)$</th>
<th>$t(\hat{\gamma}_1f)$</th>
<th>$t(\hat{\gamma}_1g)$</th>
<th>$t(\hat{\gamma}_0)$</th>
<th>$t(\hat{\gamma}_1f)$</th>
<th>$t(\hat{\gamma}_1g)$</th>
<th>$t(\hat{\gamma}_0)$</th>
<th>$t(\hat{\gamma}_1f)$</th>
<th>$t(\hat{\gamma}_1g)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.076</td>
<td>0.035</td>
<td>0.007</td>
<td>0.065</td>
<td>0.036</td>
<td>0.006</td>
<td>0.104</td>
<td>0.056</td>
<td>0.013</td>
</tr>
<tr>
<td>500</td>
<td>0.088</td>
<td>0.040</td>
<td>0.009</td>
<td>0.088</td>
<td>0.044</td>
<td>0.009</td>
<td>0.102</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>1000</td>
<td>0.095</td>
<td>0.047</td>
<td>0.009</td>
<td>0.096</td>
<td>0.046</td>
<td>0.008</td>
<td>0.102</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>3000</td>
<td>0.099</td>
<td>0.048</td>
<td>0.010</td>
<td>0.099</td>
<td>0.049</td>
<td>0.010</td>
<td>0.099</td>
<td>0.048</td>
<td>0.010</td>
</tr>
</tbody>
</table>
Conclusion

- Framework for testing useless factors within the context of beta-pricing models.
Framework for testing useless factors within the context of beta-pricing models.

- Designed for when $N$ is large and $T$ is fixed, possibly very small ($T > K$ is enough).
Conclusion

- Framework for testing useless factors within the context of beta-pricing models.
- Designed for when $N$ is large and $T$ is fixed, possibly very small ($T > K$ is enough).
- Unlike the large-$T$ methods, our approach is simple (based simply on the OLS CSR).
Conclusion

- Framework for testing useless factors within the context of beta-pricing models.
- Designed for when \( N \) is large and \( T \) is fixed, possibly very small (\( T > K \) is enough).
- Unlike the large-\( T \) methods, our approach is simple (based simply on the OLS CSR).
- Unlike the large-\( T \) methods, our results do NOT depend on degree of misspecification.
Conclusion

- Framework for testing useless factors within the context of beta-pricing models.
- Designed for when $N$ is large and $T$ is fixed, possibly very small ($T > K$ is enough).
- Unlike the large-$T$ methods, our approach is simple (based simply on the OLS CSR).
- Unlike the large-$T$ methods, our results do NOT depend on degree of misspecification.
- Our results lead to conventional asymptotic distributions of OLS CSR estimator and test statistics.
THANKS