Belief Update and Mispricing

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Abstract

This paper explores asset pricing bubbles in an asymmetric information environment. We consider two uncertainties in our model, uncertainty of the asset value and the existence of informed traders. Our model has three types of market participants: a market maker, informed traders, and uninformed traders. Assuming the market maker uses Bayesian learning, he updates his asset value belief through transactions. There are two types of markets, one with informed traders and one without. The market maker does not know whether informed traders exist or not. The market maker also updates his belief about the existence of informed traders through transactions. We find that (1) when the market maker does not know of the existence of informed traders and informed traders exist, the asset price monotonically converges to its fair value on average, and (2) when a market maker does not know of the existence of informed traders and informed traders do not exist, the asset price systematically deviates from its fair value causing asset price bubbles. This situation represents the information mirage that Camerer and Weigelt (1991) found in their asset market experiments. However, after the market maker sufficiently updates his belief, he adequately finds the non-existence of informed traders. Asset price bubbles then shrink, and the stock price returns to its fair value. The result indicates a complete bubble process:, which is not common in the literature. We also confirm this process using numerical simulations.

Keywords: security mispricing, asymmetric information, Bayesian update, sequential trading model

JEL Classification: D82, G12, G14

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1. Introduction

The purpose of this paper is to theoretically demonstrate that bubbles occur as a result of rational investor behavior. If rational investors without information believe that informed traders exist who possess private information, they attempt to infer the private information by observing transaction price. In this paper, a market maker observes trading behaviors and learns private information that other traders may have using Bayesian inference. The market maker revises his expectation of asset value based on his transaction involvement. If the presence of informed traders is uncertain, the market maker also learns the likelihood of informed trader presence. The two types of learning, of asset value and the existence of private information, simultaneously occur in the market. The market maker's expectations of both asset value and the existence of information changing dynamically affect each other. We theoretically show that this dynamic learning process causes bubbles.

There are many previous studies on the occurrence of bubbles. For example, Blanchard and Watson (1982) found rational bubbles where asset price can deviate from its fundamental value. If the expectation of future cash flows satisfies the certain condition, the market consists of rational traders and the markets satisfies the no-arbitrage conditions. This theory shows the existence of bubbles by simple explanation. However, bubbles must continue to occur from the past to the future. The theory does not explain the occurrence and collapse of bubbles.

Allen, Morris, and Postlewaite (1993) found that bubbles occur if there is information asymmetry, market transaction does not sufficiently eliminate its asymmetry, and the short sale is restricted. However, this explanation conflicts with the efficient market hypothesis and does not explain the existence of long-lived bubbles. Bubbles can be explained by the existence of irrational investors. For example, Haruvy, Laov, and Noussair (2007) found that bubbles occur when investors mistake the application of stock price for too long a period.

Hirota and Sunder (2007) experimentally examined the relation between mispricing and the investment time horizon and found that mispricing occurred when the investment time horizon is short, and investors mistakenly extrapolate the stock price that has a positive trend. Although the behavior of irrational investors causes bubbles, rational investors can arbitrage the misprices, and stock price should return to its fundamental value. Therefore, the existence of irrational investors does not explain long-lived bubbles.

Bikhchandani, Hirshleifer, and Welch (1992, 1998) showed that information cascades occur if information asymmetry exists and rational investors try to infer the information that others may have. In this situation, transactions in the markets tend to be biased towards the order of selling or buying. The occurrence of bubbles may be explained by this concept. However, a simple information cascade is difficult to create in the actual market because information is revealed through the stock price indirectly.

Camerer and Weigelt (1991) found that the information mirage occurs when private information does not exist, but traders mistakenly believe that other traders have private information. The information mirage is considered to be one type of information cascade. The information mirage may explain bubbles. However, there are few studies that explain why the information mirage occurs in the market.

We developed a theoretical model of market maker behavior. Our model is an extension of Glosten and Milgrom (1985) and Avery and Zemsky (1988). We assume that all investors are rational, and there are no irrational investors. Information is asymmetrically distributed. The market maker and noise traders do not know the true value of an asset. On the contrary, informed traders, if they exist, know the true value of an asset. The market maker, using Bayesian learning, updates his asset value belief and the existence of informed traders simultaneously. We find that although private information does not exist, the market maker must temporarily increase his estimation of the likelihood of the existence of informed traders. This implies that the information mirage occurs and causes the asset price to deviate from the fair value. In many cases, these deviations continue to increase. However, when the market maker learns sufficiently, the market maker's estimation of the likelihood of private information converges to a true value, which is zero. This causes that asset price to eventually converge to its fair value. This dynamic learning process of the market maker reveals the complete bubble process, its rise, growth, collapse.

This paper is organized as follows. Chapter 2 provides the theoretical model. Chapter 3 presents a numerical simulation of the bubble process and a discussion. Chapter 4 presents the conclusions.

2. Model

2.1. Overview

A financial asset takes the value $\theta = \{0, 1\}$. The natural probability of both $\theta = 0$ and $\theta = 1$ is $1/2$. There is one market maker and two types of traders; informed traders and noise traders. Each informed trader has private information that indicates the true value of θ. In each period, one trader buys from or sells to the market maker who continues to make a bid and ask offer to the market. Each informed trader decides to buy or sell using his private information. If a trader has information $\theta = 0$, that trader sells one unit of asset to the market maker at any positive price. On the other hand, if a trader has information $\theta = 1$, that trader buys one unit of asset from the market maker at any price less than 1. Each noise trader buys or sells randomly with a fifty percent chance of buying. However, the market maker does not know who is an informed trader and supposes that the probability of an informed trader is ϕ. Let $\mathcal{H}_t$ represent the history of the traders' actions, which is a sequence of buy or sell orders in previous periods. $\mathcal{H}_t$ is public information.

2.2. Market Maker

Let $\mu_t = \text{Prob}(\theta = 1 \mid \mathcal{H}_t)$ represent the market maker's belief. μ_t is considered to be the price of the asset because it is the expected value of the asset for the market maker.

Suppose that a trader buys one unit of asset from the market maker in period *t+1*, the market maker updates his belief by Bayes rule as follows.

$$\begin{aligned}
\mu_t &= Prob(\theta = 1 \mid Buy, \mathcal{H}_t) \\
&= \frac{Prob(Buy \mid \theta = 1)\, Prob(\theta = 1)}{Prob(Buy \mid \theta = 1)\, Prob(\theta = 1) + Prob(Buy \mid \theta = 0)\, Prob(\theta = 0)} \\
&= \frac{\{\phi + (1-\phi)\, 1/2\} \cdot \mu_t}{\{\phi + (1-\phi)\, 1/2\} \cdot \mu_t + (1-\phi)\, 1/2 \cdot (1-\mu_t)} \\
&= \frac{(1+\phi)\mu_t}{1-\phi+2\phi\mu_t} \qquad (1)
\end{aligned}$$

Similarly, in the case where a trader sells one unit of asset to the market maker in period *t+1*, the market maker updates his belief as follows.

$$\begin{aligned}
\mu_t &= Prob(\theta = 1 \mid Sell, \mathcal{H}_t) \\
&= \frac{Prob(Sell \mid \theta = 1)\, Prob(\theta = 1)}{Prob(Sell \mid \theta = 1)\, Prob(\theta = 1) + Prob(Sell \mid \theta = 0)\, Prob(\theta = 0)} \\
&= \frac{(1-\phi)\, 1/2 \cdot \mu_t}{(1-\phi)\, 1/2 \cdot \mu_t + \{\phi + (1-\phi)\, 1/2\} \cdot (1-\mu_t)} \\
&= \frac{(1-\phi)\mu_t}{1+\phi-2\phi\mu_t} \qquad (2)
\end{aligned}$$

2.3. Market Information Structure

We assume that there are two types of information structure market; an informed market and an uninformed market. The informed market has one or more informed trader while the uninformed market does not have any informed traders. Table 1 shows the relation between information structure and the ratio of informed traders (ϕ).

Table 1 Market Information Structure

Market type	Ratio of informed traders	Ratio of noise traders
Uniformed market (*Uninfo*)	0	1
Informed market (*Info*)	$\phi(>0)$	$1-\phi$

We define market efficiency as follows.

Definition 1: The uninformed market is *efficient* if the price of the asset is $1/2$.

Definition 2: The informed market is *efficient* if the price of the asset is θ.

Let $\mu_t^U = \mathrm{Prob}(\theta = 1 \mid \mathcal{H}_t, Uninfo)$ represent the market maker's belief in period t under the condition that the market is uninformed. Because $\mathcal{H}_0 = null$, then $\mu_0^U = 1/2$. By Eq. (1) and Eq. (2), we obtain $\mu_t^U = 1/2$ for all t.

Let $\mu_t^I = \mathrm{Prob}(\theta = 1 \mid \mathcal{H}_t, Info)$ represent the market maker's belief in period t under the condition that the market is informed. Because $\mathcal{H}_0 = null$, then, $\mu_0^I = 1/2$. According to the action of traders; buy or sell, the market maker's belief is updated in the following two ways from Eq.(1) and Eq. (2);

$$\mu_{t+1}^I = \begin{cases} \dfrac{(1+\phi)\mu_t^I}{1-\phi+2\phi\mu_t^I} & \text{if a trader buys in t+1} \\[2ex] \dfrac{(1-\phi)\mu_t^I}{1+\phi-2\phi\mu_t^I} & \text{if a trader sells in t+1.} \end{cases} \tag{3}$$

The increment of belief is as follows:

$$\begin{aligned} \Delta\mu_{t+1}^I &\equiv \mu_{t+1}^I - \mu_t^I \\ &= \begin{cases} \dfrac{2\phi\mu_t^I(1-\mu_t^I)}{1-\phi+2\phi\mu_t^I} > 0 & \text{if a trader buys in t+1} \\[2ex] -\dfrac{2\phi\mu_t^I(1-\mu_t^I)}{1+\phi-2\phi\mu_t^I} < 0 & \text{if a trader sells in t+1.} \end{cases} \end{aligned} \tag{4}$$

We confirm that the belief increases when a trader buys, and the belief decreases when a trader sells.

PROPOSITION 1: Denote the number of buys in $\mathcal{H}_t$ as B_t and the number of sells in $\mathcal{H}_t$ as S_t . Let $G_t \equiv B_t - \mathrm{S}$. Then, μ_t^I is determined only by G_t independent of the order of buys and sells in $\mathcal{H}_t$.

ROOF: See Appendix 1.

PROPOSITION 2: If the market is informed, then $\mu_t^I \xrightarrow{p} \theta$.

PROOF: See Appendix 2.

Proposition 2 states that, eventually, the market maker finds the true value of an asset if the market is informed and the market maker recognizes the informed market. This implies that the existence of informed traders and the knowledge of their existence by the market maker is sufficient to create an efficient market.

PROPOSITION 3: If the market is uninformed, then for all $\varepsilon \in (0,1)$, $\mathrm{Prob}(\mu_t^I \leq \varepsilon) \to 1/2$ and $\mathrm{Prob}(\mu_t^I \geq 1-\varepsilon) \to 1/2$.

PROOF: See Appendix 3.

Proposition 3 states that when there is no informed trader but the market maker mistakes that the market is informed, the price of an asset eventually bifurcates into 0 and 1.

2.4. Bayesian Learning of Market Information Structure

Let $w_t = \mathrm{Prob}(\mathit{Info} \mid \mathcal{H}_t)$ represent the market maker's belief concerning market information structure. Without loss of generality, we set $\theta = 1$. Only informed traders know this information; noise traders and the market maker do not know this information.

Suppose that a trader buys one unit of asset from the market maker in period *t+1*, the market maker updates his belief concerning the information structure by Bayes rule as follows.

$$
\begin{aligned}
w_t &= Prob(\mathit{Info} \mid \mathit{Buy}, \mathcal{H}_t) \\
&= \frac{Prob(\mathit{Buy} \mid \mathit{Info})\, Prob(\mathit{Info})}{Prob(\mathit{Buy} \mid \mathit{Info})\, Prob(\mathit{Info}) + Prob(\mathit{Buy} \mid \mathit{Unifo})\, Prob(\mathit{Uninfo})} \\
&= \frac{[\mu_{t+1}^I\{\phi + (1-\phi)\,1/2\} + (1-\mu_{t+1}^I)(1-\phi)\,1/2]w_t}{[\mu_{t+1}^I\{\phi + (1-\phi)\,1/2\} + (1-\mu_{t+1}^I)(1-\phi)\,1/2]w_t + 1/2\,(1-w_t)} \\
&= \frac{(2\mu_{t+1}^I - 1)\phi w_t + w_t}{(2\mu_{t+1}^I - 1)\phi w_t + 1}\ .
\end{aligned} \tag{5}
$$

From Eq. (3), we have $\mu_{t+1}^{I} = \dfrac{(1+\phi)\mu_t^I}{1-\phi+2\phi\mu_t^I}$. Substituting this into Eq. (5), we obtain

$$w_{t+1} = \frac{2(2\mu_t^I - 1)\phi + \phi^2 + 1}{(2\mu_t^I - 1)\phi(1+w_t) + \phi^2 w_t + 1} w_t \ . \tag{6}$$

The increment of the information structure belief is as follows:

$$\begin{aligned} \Delta w_{t+1} &\equiv w_{t+1} - w_t \\ &= \frac{2(2\mu_t^I - 1 + \phi)\phi(1-w_t)w_t}{(2\mu_t^I - 1)\phi(1+w_t) + \phi^2 w_t + 1} \ . \end{aligned} \tag{7}$$

Because the dominator of Eq. (7) is positive, the sign of increment Δw_{t+1} is determined by the following three cases when a trader buys one asset and
$0 < w_t < 1$.

$$\begin{cases} \mu_t^I > \dfrac{1-\phi}{2} & \rightarrow \quad \Delta w_{t+1} > 0 \\[2ex] \mu_t^I = \dfrac{1-\phi}{2} & \rightarrow \quad \Delta w_{t+1} = 0 \\[2ex] \mu_t^I < \dfrac{1-\phi}{2} & \rightarrow \quad \Delta w_{t+1} < 0 \ . \end{cases} \tag{8}$$

Similarly, when a trader sells one unit of asset to the market maker in period *t+1*, the market maker updates the information structure belief as follows.

$$\begin{aligned} w_t &= Prob(Info \mid Sell, \mathcal{H}_t) \\ &= \frac{Prob(Sell \mid Info)\ Prob(Info)}{Prob(Sell \mid Info)\ Prob(Info) + Prob(Sell \mid Unifo)\ Prob(Uninfo)} \\ &= \frac{[\mu_{t+1}^I(1-\phi)\ 1/2 + (1-\mu_{t+1}^I)\{\phi + (1-\phi)\ 1/2\}]w_t}{[\mu_{t+1}^I(1-\phi)\ 1/2 + (1-\mu_{t+1}^I)\{\phi + (1-\phi)\ 1/2\}]w_t + 1/2\,(1-w_t)} \\ &= \frac{(2\mu_{t+1}^I - 1)\phi w_t - w_t}{(2\mu_{t+1}^I - 1)\phi w_t - 1} \ . \end{aligned} \tag{9}$$

From Eq. (3), we have $\mu_{t+1}^{I} = \dfrac{(1-\phi)\mu_t^I}{1+\phi-2\phi\mu_t^I}$. Then, using Eq. (9) we obtain

$$w_{t+1} = \frac{2(2\mu_t^I - 1)\phi + \phi^2 - 1}{(2\mu_t^I - 1)\phi(1+w_t) + \phi^2 w_t - 1} w_t \ . \tag{10}$$

The increment of the information structure belief is as follows;

$$\Delta w_{t+1} = \frac{2(2\mu_t^I - 1 - \phi)\phi(1 - w_t)w_t}{(2\mu_t^I - 1)\phi(1 + w_t) - \phi^2 w_t - 1} \ . \tag{11}$$

Because the dominator of Eq. (11) is negative, when a trader sells one asset and $0 < w_t < 1$, we have

$$\begin{cases} \mu_t^I > \dfrac{1+\phi}{2} & \rightarrow \quad \Delta w_{t+1} < 0 \\ \mu_t^I = \dfrac{1+\phi}{2} & \rightarrow \quad \Delta w_{t+1} = 0 \\ \mu_t^I < \dfrac{1+\phi}{2} & \rightarrow \quad \Delta w_{t+1} > 0\ . \end{cases} \tag{12}$$

PROPOSITION 4: If the market is informed and $0 < w_0 < 1$, then $w_t \overset{p}{\rightarrow} 1$.

PROOF: See Appendix.

Proposition 4 states that the market maker finally finds that the market is informed. From Propositions 2 and 4, we know that the informed market becomes efficient after the market maker has sufficiently learned μ_t^I and w_t.

PROPOSITION 5: If the market is uninformed and $0 < w_0 < 1$, then, $w_t \overset{p}{\rightarrow} 0$.

PROOF: See Appendix.

Proposition 5 states that the market maker finally finds that the market is uninformed. Recalling $\mu_t^U = 1/2$ for all t, we know that the uninformed market also becomes efficient after the market maker sufficiently learns μ_t^I and w_t.

3. Asset Price Bubbles

This section shows how bubbles arise in an uninformed market using numerical simulations. We set $\phi = 0.1$. A typical simulated asset price path is shown in Figure 1. μ_t is the asset price that is calculated as $\mu_t \equiv w_t\mu_t^I + (1 - w_t)\mu_t^U$. Proposition 3 shows that μ_t^I bifurcates into 0 and 1 when the market is uninformed. Figure 1 is a case where μ_t converges to 1. Although there is no informed trader, the market maker's information structure belief w_t increases in the beginning period. This increase of w_t causes a deviation of μ_t from the fair value of 1/2. Around period 500, the market price μ_t approaches the upper bound 1, which shows that an asset price bubble occurs. This bubble continues for over 500 periods. When the market maker sufficiently learns that there are no informed traders and w_t begins to decrease. Finally w_t converges to 0 around the period 1,400, the market maker recognizes the true state, and the market price returns to the fair value 1/2.

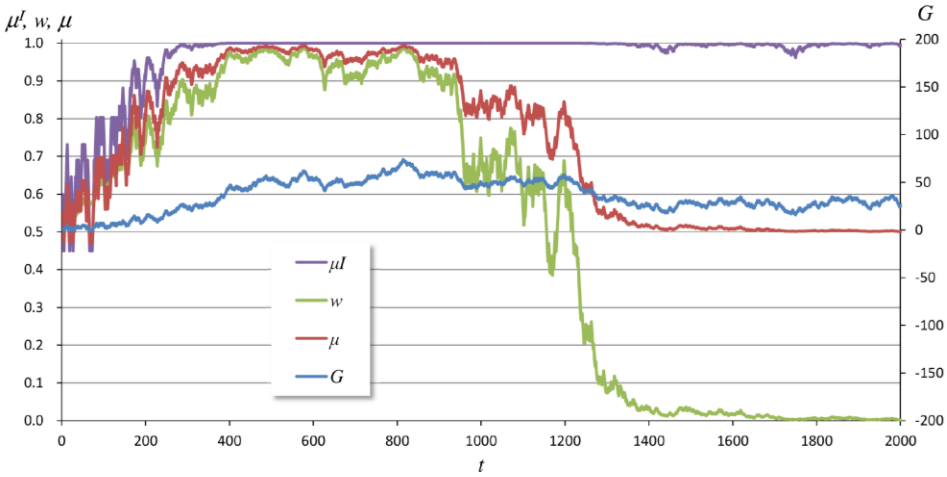

Figure 1　An example of a bubble in an uninformed market

The increase of w_t in Figure 1 is not a special case, but this always happens. Figure 2 shows the dynamics of the distribution of w_t that is calculated by 5,000 simulated paths. We find that w_t increases for the beginning periods without exception. w_t for the 50th percentile continues to increase for 500 or more periods. This implies that the market maker has the wrong belief concerning the existence of informed traders at the initial learning stage. The market maker mistakenly believes that informed traders exist in the market although this is not the case. This mistaken belief represents the phenomenon of the information mirage that Camerer and Weigelt (1991) found in their asset market experiments. The information mirage causes market price deviations from the fair value and creates asset price bubbles as shown in Figure 1. Figure 2 shows that the information mirage disappears after sufficient learning. The 50th percentile drops to 0 after the period 3,000.

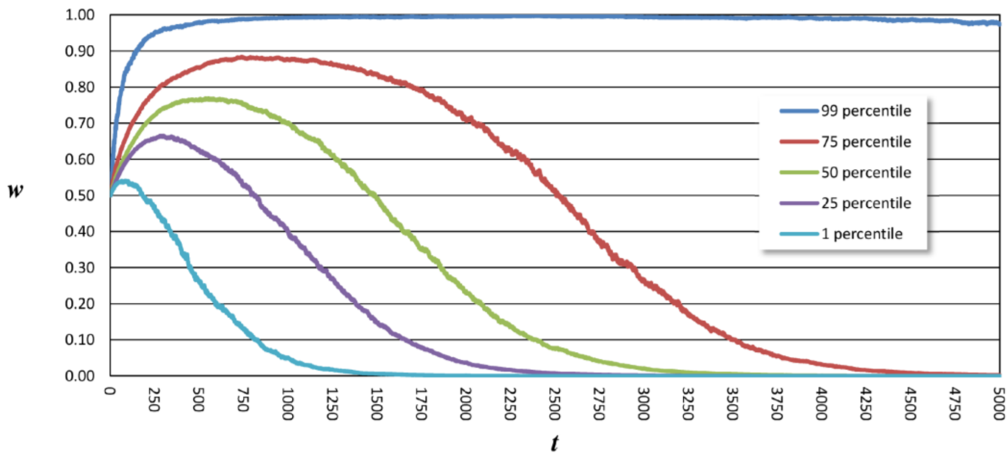

Figure 2　The dynamics of the distribution of w_t in an uninformed market.

Figure 3 shows the dynamics of the distribution of μ_t. We find that the distribution of μ_t rapidly widens for the beginning periods, then shrinks and gradually converges to its true value 1/2. Figure 4 shows the distribution of μ_t in the period 500. μ_t has a bimodal distribution, which implies that positive or negative bubbles occur in almost all cases.

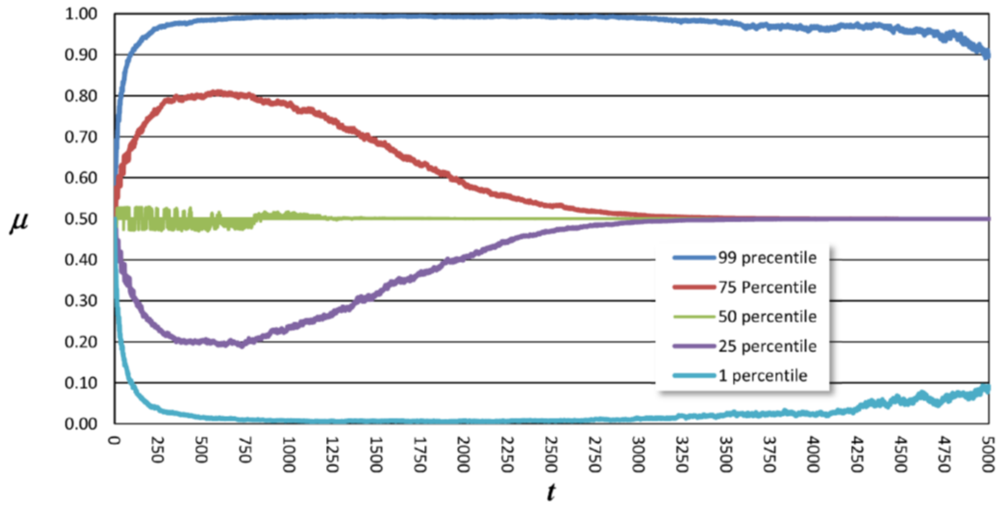

Figure 3 The dynamics of the distribution of μ_t in an uninformed market

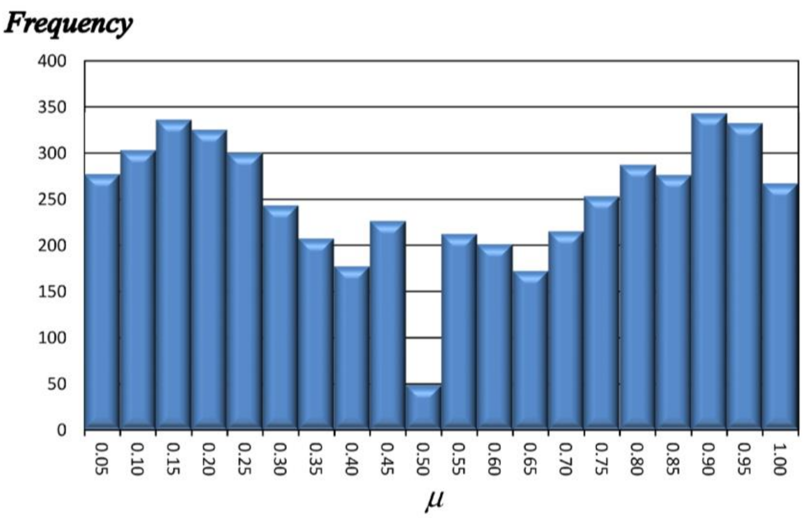

Figure 4 The distribution of μ_t in the period 500

There are two reasons why the asset bubbles occur in this model. First, Bayesian learning of the market maker causes w_t to rise in the early periods. Second, the learning speed of w_t is slower than that of μ_t^I. Figure 1 shows that μ_t^I rapidly converges to 1; however, w_t slowly converges to its true value 0. This time, lag amplifies the deviation of asset price from its fair value. The learning concerning market information structure is more difficult than that of asset value in an uninformed market.

For the informed market, the market rapidly converges to an efficient market as expected. Figure 5 shows a typical price path and other variables. In this case, the true asset value is 1. μ_t^I and w_t quickly close to each true value, and μ_t also converges to its true value 1. The informed market becomes efficient within a short period.

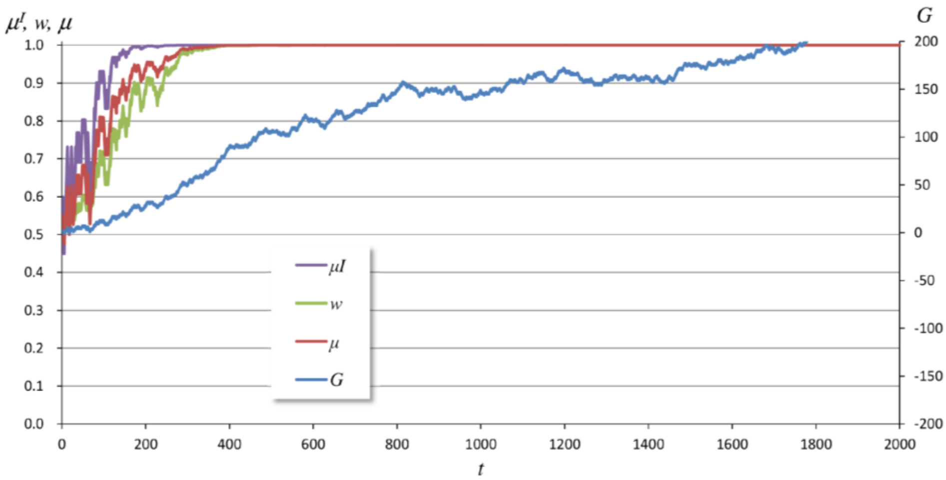

Figure 5　An example in an informed market

4. Conclusion

This paper theoretically explored asset pricing dynamics in an asymmetric information environment. We also showed numerical simulations demonstrating that asset bubbles arise, grow, and burst.

In our model, a market maker inferred two uncertainties, the uncertainty of the existence of private information and the uncertainty of asset value. The market maker updated his beliefs concerning these two uncertainties simultaneously using Bayesian learning. We found that when the market maker does not know whether informed traders exist, and informed traders do exist, the asset price rapidly converges to its fair value. However, when the market maker does not know whether the informed traders exist, and informed traders do not exist, the market maker inferred the existence of informed traders incorrectly and made buying and selling offers at different prices from the fair value. The asset price systematically deviated from the fair values causing asset price bubbles. The bubble grew and continued for a long period. However, after sufficient learning, the market maker realized the true situation that there was no informed trader, and the bubble began to shrink and finally disappeared. This model explains the complete bubble process: its rise, growth, and burst.

Appendix 1 Proof of Proposition 1

Suppose that a trader buys one asset from the market maker in period *t*. The market maker updates his asset value belief using Eq.(3), then

$$\mu_{t+1}^{I} = \frac{(1+\phi)\mu_t^I}{1-\phi+2\phi\mu_t^I} \ . \tag{A.1}$$

Suppose that the next trader sells one asset to the market maker in period *t+1*. The market maker updates his asset value belief using Eq.(3), then

$$\mu_{t+2}^{I} = \frac{(1-\phi)\mu_{t+1}^I}{1+\phi-2\phi\mu_{t+1}^I} \tag{A.2}$$

From Eq. (A.1) and Eq. (A.2), we obtain

$$\mu_{t+2}^I = \mu_t^I \ .$$

Similarly, in the case where a trader sells one asset in period *t* and the next trader buys one asset in period *t+1*, we have

$$\mu_{t+1}^{I} = \frac{(1-\phi)\mu_t^I}{1+\phi-2\phi\mu_t^I} \tag{A.3}$$

and

$$\mu_{t+2}^{I} = \frac{(1+\phi)\mu_{t+1}^I}{1-\phi+2\phi\mu_{t+1}^I} \ . \tag{A.4}$$

From Eq. (A.3) and Eq. (A.4), we obtain

$$\mu_{t+2}^I = \mu_t^I \ .$$

Hence, a buy and sell pair does not change the market maker's asset value belief, and we can remove such pairs from $\mathcal{H}_t$ without changing μ_t^I. After all pairs are removed from $\mathcal{H}_t$, $\mathcal{H}_t$ constitutes buys only or sells only. Therefore, μ_t^I is determined by G_t independent of orders of buys and sells in $\mathcal{H}_t$. ■

Appendix 2 Proof of Proposition 2

From Eq. (4), μ_t^I is a monotonically increasing function of G_t. Because $0 \le \mu_t^I \le 1$, as $G_t \to \infty$, μ_t^I has a finite limit. Similarly, as $G_t \to -\infty$, μ_t^I also has a finite limit. When $\mu_{t+1}^I = \mu_t^I$, we obtain $\mu_t^I = 0 \ or \ 1$ from Eq. (3). Because $\mu_0^I = 1/2$, as $G_t \to \infty$, $\mu_t^I \to 1$ and as $G_t \to -\infty$, $\mu_t^I \to 0$. When $\theta = 1$, $\mathrm{Prob}(Buy) > \mathrm{Prob}(Sell)$, then $G_t \xrightarrow{p} \infty$; therefore, $\mu_t^I \xrightarrow{p} 1$. When $\theta = 0$, $\mathrm{Prob}(Sell) > \mathrm{Prob}(Buy)$, then, $G_t \xrightarrow{p} -\infty$; therefore, $\mu_t^I \xrightarrow{p} 0$. ■

Appendix 3 Proof of Proposition 3

From Proposition 1, $\forall \varepsilon \in (0,1)$, $\exists \alpha$, $(\mu_t^I \leq \varepsilon) \Leftrightarrow (G_t \leq \alpha)$. We define $\beta \equiv \frac{\alpha}{t}$. Then, $(\mu_t^I \leq \varepsilon) \Leftrightarrow \left(\frac{G_t}{t} \leq \beta\right)$. When the market is uninformed, G_t is a one dimension simple random walk. So, we have $\mathrm{E}[G_t]=0$ and $\sigma[G_t] = \sqrt{t}$. From the central limit theorem, as $t \to \infty$, $\mathrm{Prob}\left(\frac{G_t}{t} \leq \beta\right) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\beta} e^{-\frac{x^2}{2}}dx$. Hence, $\mathrm{Prob}(\mu_t^I \leq \varepsilon) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\beta} e^{-\frac{x^2}{2}}dx$. Because $t \to \infty$, $\beta \to 0$, we obtain $\mathrm{Prob}(\mu_t^I \leq \varepsilon) \to 1/2$. Similarly, we obtain $\mathrm{Prob}(\mu_t^I \geq 1-\varepsilon) \to 1/2$. ■

Appendix 4 Proof of Proposition 4

When a trader buys one asset from the market maker, from Eq. (6) we have

$$\frac{1}{w_{t+1}} - 1 = \frac{1+\phi(2\mu_t^I - 1)}{1+2\phi(2\mu_t^I - 1)+\phi^2}\left(\frac{1}{w_t} - 1\right). \tag{A.5}$$

When a trader sell one asset to the market maker, from Eq. (10) we have

$$\frac{1}{w_{t+1}} - 1 = \frac{1-\phi(2\mu_t^I - 1)}{1-2\phi(2\mu_t^I - 1)+\phi^2}\left(\frac{1}{w_t} - 1\right). \tag{A.6}$$

Assume that $\forall \varepsilon > 0$, $\exists T > 0$, for all $t > T$,

$$\frac{1+\phi(2\mu_t^I - 1)}{1+2\phi(2\mu_t^I - 1)+\phi^2} < \frac{1}{1+\phi} + \varepsilon \quad and \quad \frac{1-\phi(2\mu_t^I - 1)}{1-2\phi(2\mu_t^I - 1)+\phi^2} < \frac{1}{1-\phi} + \varepsilon.$$

When a trader buys one asset from the market maker in period $t > T$, using Eq. (A.5) we have

$$\frac{1}{w_{t+1}} - 1 < \left(\frac{1}{1+\phi} + \varepsilon\right)\left(\frac{1}{w_t} - 1\right). \tag{A.7}$$

Whereas when a trader sells one asset to the market maker in period $t > T$, using Eq. (A.6) we have

$$\frac{1}{w_{t+1}} - 1 < \left(\frac{1}{1-\phi} + \varepsilon\right)\left(\frac{1}{w_t} - 1\right). \tag{A.8}$$

Consider that traders buy B times and sell S times after the period T. In period $t = T + B + S$, we have

$$\frac{1}{w_t} - 1 < \left(\frac{1}{1+\phi} + \varepsilon\right)^B \left(\frac{1}{1-\phi} + \varepsilon\right)^S \left(\frac{1}{w_T} - 1\right). \tag{A.9}$$

Then,

$$\frac{1}{w_t}-1<\left\{\left(\frac{1+2\varepsilon}{1-\phi^2}+\varepsilon^2\right)\left(\frac{(1-\phi)\{1+\varepsilon(1+\phi)\}}{(1+\phi)\{1+\varepsilon(1-\phi)\}}\right)^{\frac{B-S}{B+S}}\right\}^{\frac{B+S}{2}}\left(\frac{1}{w_T}-1\right)\ . \qquad \text{(A.10)}$$

Because the market is informed, from Proposition 2, $\forall\varepsilon'>0$, $\forall\delta'>0$, $\exists T'>0$, for all $t>T'$, $\text{Prob}(1-\mu_t^I<\varepsilon')>1-\delta'$. Hence, $\forall\varepsilon>0$, $\forall\delta>0$, $\exists T>0$, for all $t>T$,

$$Prob\left(\frac{1+\phi(2\mu_t^I-1)}{1+2\phi(2\mu_t^I-1)+\phi^2}<\frac{1}{1+\phi}+\varepsilon\ \right)>1-\delta$$

and

$$Prob\left(\frac{1-\phi(2\mu_t^I-1)}{1-2\phi(2\mu_t^I-1)+\phi^2}<\frac{1}{1-\phi}+\varepsilon\right)>1-\delta\ .$$

From Eq. (A.10), $\forall\varepsilon>0$, $\forall\delta>0$, $\exists T>0$, for all $t>T+B+S$, we have

$$Prob\left(\frac{1}{w_t}-1<\left\{\left(\frac{1+2\varepsilon}{1-\phi^2}+\varepsilon^2\right)\left(\frac{(1-\phi)\{1+\varepsilon(1+\phi)\}}{(1+\phi)\{1+\varepsilon(1-\phi)\}}\right)^{\frac{B-S}{B+S}}\right\}^{\frac{B+S}{2}}\left(\frac{1}{w_T}-1\right)\right)$$

$$>1-\delta\ . \qquad \text{(A.11)}$$

Because the market is informed, using the law of large numbers, $t\to\infty$, $\dfrac{B-S}{B+S}\to\phi$. When $0<\phi<1$, we have

$$\left(\frac{1}{1-\phi^2}\right)\left(\frac{1-\phi}{1+\phi}\right)^{\phi}=\left(\frac{1}{1-\phi}\right)^{1-\phi}\left(\frac{1}{1+\phi}\right)^{1+\phi}<1\ .$$

Denote that ε (> 0) is sufficiently small, then, $\exists T>0$, for all $t>T$,

$$\left(\frac{1+2\varepsilon}{1-\phi^2}+\varepsilon^2\right)\left(\frac{(1-\phi)\{1+\varepsilon(1+\phi)\}}{(1+\phi)\{1+\varepsilon(1-\phi)\}}\right)^{\frac{B-S}{B+S}}<1\ .$$

Because $t\to\infty$, $\dfrac{B+S}{2}\to\infty$, from Eq. (A.11), we obtain that as $t\to\infty$, $\dfrac{1}{w_t}-1\xrightarrow{p}0$..

Therefore, $w_t\xrightarrow{p}1$. ■

Appendix 5 Proof of Proposition 5

When a trader buys one asset from the market maker in period $t+1$, from Eq. (6) we have

$$\frac{1}{w_{t+1}} - 1 = \frac{1 + \phi(2\mu_t^I - 1)}{1 + 2\phi(2\mu_t^I - 1) + \phi^2}\left(\frac{1}{w_t} - 1\right) . \tag{A.12}$$

When a trader sells one asset to the market maker in period $t+1$, from Eq. (10) we have

$$\frac{1}{w_{t+1}} - 1 = \frac{1 - \phi(2\mu_t^I - 1)}{1 - 2\phi(2\mu_t^I - 1) + \phi^2}\left(\frac{1}{w_t} - 1\right) . \tag{A.13}$$

Assume that $\forall\varepsilon > 0,\ \exists T > 0$, for all $t > T$,

$$\frac{1 + \phi(2\mu_t^I - 1)}{1 + 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1+\phi} - \varepsilon \quad and \quad \frac{1 - \phi(2\mu_t^I - 1)}{1 - 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1-\phi} - \varepsilon\, .$$

When a trader buys one asset from the market maker in period $t > T$, using Eq. (A.12) we have

$$\frac{1}{w_{t+1}} - 1 > \left(\frac{1}{1+\phi} - \varepsilon\right)\left(\frac{1}{w_t} - 1\right) . \tag{A.14}$$

Whereas when a trader sells one asset to the market maker in period $t > T$, using Eq. (A.13) we have

$$\frac{1}{w_{t+1}} - 1 > \left(\frac{1}{1-\phi} - \varepsilon\right)\left(\frac{1}{w_t} - 1\right) . \tag{A.15}$$

Consider that traders buy B times and sell S times after the period T. In period $t = T + B + S$, we have

$$\frac{1}{w_t} - 1 > \left(\frac{1}{1+\phi} - \varepsilon\right)^B \left(\frac{1}{1-\phi} - \varepsilon\right)^S \left(\frac{1}{w_T} - 1\right) . \tag{A.16}$$

Then,

$$\frac{1}{w_t} - 1 > \left\{\left(\frac{1+2\varepsilon}{1-\phi^2} + \varepsilon^2\right)\left(\frac{(1-\phi)\{1-\varepsilon(1+\phi)\}}{(1+\phi)\{1-\varepsilon(1-\phi)\}}\right)^{\frac{B-S}{B+S}}\right\}^{\frac{B+S}{2}} \left(\frac{1}{w_T} - 1\right) . \tag{A.17}$$

Next, assume that $\forall\varepsilon > 0,\ \exists T > 0$, for all $t > T$,

$$\frac{1 + \phi(2\mu_t^I - 1)}{1 + 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1-\phi} - \varepsilon \quad and \quad \frac{1 - \phi(2\mu_t^I - 1)}{1 - 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1+\phi} - \varepsilon\, .$$

When a trader sells one asset to the market maker in period $t > T$, using Eq. (A.12) we have

$$\frac{1}{w_{t+1}} - 1 > \left(\frac{1}{1-\phi} - \varepsilon\right)\left(\frac{1}{w_t} - 1\right). \tag{A.18}$$

Whereas when a trader sells one asset to the market maker in period $t > T$, using Eq. (A.13) we have

$$\frac{1}{w_{t+1}} - 1 > \left(\frac{1}{1+\phi} - \varepsilon\right)\left(\frac{1}{w_t} - 1\right). \tag{A.19}$$

Consider that traders buy B times and sell S times after the period T. In period $t = T + B + S$, we have

$$\frac{1}{w_t} - 1 > \left(\frac{1}{1-\phi} - \varepsilon\right)^B \left(\frac{1}{1+\phi} - \varepsilon\right)^S \left(\frac{1}{w_T} - 1\right). \tag{A.20}$$

Then,

$$\frac{1}{w_t} - 1 > \left\{\left(\frac{1+2\varepsilon}{1-\phi^2} + \varepsilon^2\right)\left(\frac{(1+\phi)\{1-\varepsilon(1-\phi)\}}{(1-\phi)\{1-\varepsilon(1+\phi)\}}\right)^{\frac{B-S}{B+S}}\right\}^{\frac{B+S}{2}} \left(\frac{1}{w_T} - 1\right). \tag{A.21}$$

Assume that $\forall \varepsilon > 0,\ \exists T > 0$, for all $t > T$,

$$\frac{1+\phi(2\mu_t^I - 1)}{1 + 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1+\phi} - \varepsilon \quad and \quad \frac{1-\phi(2\mu_t^I - 1)}{1 - 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1-\phi} - \varepsilon$$

or

$$\frac{1+\phi(2\mu_t^I - 1)}{1 + 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1-\phi} - \varepsilon \quad and \quad \frac{1-\phi(2\mu_t^I - 1)}{1 - 2\phi(2\mu_t^I - 1) + \phi^2} > \frac{1}{1+\phi} - \varepsilon.$$

From Eq. (A.17) and Eq. (A.21), in period $t = T + B + S$, we have

$$\frac{1}{w_t} - 1 >$$

$$\left\{\left(\frac{1+2\varepsilon}{1-\phi^2} + \varepsilon^2\right) min\left[\left(\frac{(1-\phi)\{1-\varepsilon(1+\phi)\}}{(1+\phi)\{1-\varepsilon(1-\phi)\}}\right)^{\frac{B-S}{B+S}}, \left(\frac{(1+\phi)\{1-\varepsilon(1-\phi)\}}{(1-\phi)\{1-\varepsilon(1+\phi)\}}\right)^{\frac{B-S}{B+S}}\right]\right\}^{\frac{B+S}{2}} \left(\frac{1}{w_T} - 1\right). \tag{A.22}$$

Because the market is uninformed, from Proposition 3, $\forall \varepsilon' > 0$, $\forall \delta' > 0$, $\exists T' > 0$, for all $t > T'$, $\mathrm{Prob}(\mu_t^I < \varepsilon') + \mathrm{Prob}(1 - \mu_t^I < \varepsilon') > 1 - \delta'$. Hence, $\forall \varepsilon > 0$, $\forall \delta > 0$, $\exists T > 0$, for all $t > T$,

$$\begin{bmatrix} Prob\left(\dfrac{1+\phi(2\mu_t^I-1)}{1+2\phi(2\mu_t^I-1)+\phi^2} > \dfrac{1}{1+\phi} - \varepsilon\right) > 1-\delta \\ and \\ Prob\left(\dfrac{1-\phi(2\mu_t^I-1)}{1-2\phi(2\mu_t^I-1)+\phi^2} > \dfrac{1}{1-\phi} - \varepsilon\right) > 1-\delta \end{bmatrix}$$

or

$$\begin{bmatrix} Prob\left(\dfrac{1+\phi(2\mu_t^I-1)}{1+2\phi(2\mu_t^I-1)+\phi^2} > \dfrac{1}{1-\phi} - \varepsilon\right) > 1-\delta \\ and \\ Prob\left(\dfrac{1-\phi(2\mu_t^I-1)}{1-2\phi(2\mu_t^I-1)+\phi^2} > \dfrac{1}{1+\phi} - \varepsilon\right) > 1-\delta \end{bmatrix}.$$

From Eq. (A.22), $\forall\varepsilon > 0$, $\forall\delta > 0$, $\exists T > 0$, for all $t > T + B + S$,

$$Prob\left(\frac{1}{w_t} - 1 > \left\{\left(\frac{1+2\varepsilon}{1-\phi^2} + \varepsilon^2\right) min\left[\left(\frac{(1-\phi)\{1-\varepsilon(1+\phi)\}}{(1+\phi)\{1-\varepsilon(1-\phi)\}}\right)^{\frac{B-S}{B+S}}, \left(\frac{(1+\phi)\{1-\varepsilon(1-\phi)\}}{(1-\phi)\{1-\varepsilon(1+\phi)\}}\right)^{\frac{B-S}{B+S}}\right]\right\}^{\frac{B+S}{2}} \left(\frac{1}{w_T} - 1\right)\right)$$

$$> 1 - \delta\,. \tag{A.23}$$

Because the market is uninformed, $\text{Prob}(Buy) = \text{Prob}(Sell) = 1/2$. Using the law of large numbers, as $t \to \infty$, $\dfrac{B-S}{\mathrm{B+S}} \to 0$, $\dfrac{B+S}{2} \to \infty$. Denote $\varepsilon < \dfrac{\phi^2}{2}$, we have $\dfrac{1-2\varepsilon}{1-\phi^2} + \varepsilon^2 > 1$. From Eq. (A.23), we obtain that as $t \to \infty$, $\dfrac{1}{w_t} - 1 \xrightarrow{p} \infty$. Therefore, $w_t \xrightarrow{p} 0$. ■

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