A more powerful subvector Anderson and Rubin test
in linear instrumental variables regression

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Overview

- Consider **subvector inference in the linear IV model**, allowing for **weak instruments** but assuming **conditional homoskedasticity**

- **Background:**
  - Projection of Anderson and Rubin (AR) test (Dufour and Taamouti, Ecta 2005).
  
  - Guggenberger, Kleibergen, Mavroeidis, and Chen (Ecta 2012, GKMC) provide power improvement:
    
    – Using $\chi^2_{k-m_W,1-\alpha}$ as critical value, rather than $\chi^2_{k,1-\alpha}$ still controls asymptotic size.
    
    – “Worst case” occurs under strong identification.
• **HERE:** consider a **data-dependent critical value** that adapts to strength of identification.

• One main objective: computational ease.

• Show: conditional subvector AR test controls finite sample/asymptotic size & has higher power than method in GKMC.

• Test in GKMC is inadmissible.

• Proposed test has a near optimality property when $m_W = 1$. 
Outline

1. Finite sample analysis
   (a) Motivation for conditional subvector AR test
   (b) Size of test when $m_W = 1$
   (c) Power analysis when $m_W = 1$
   (d) Size of test when $m_W > 1$

2. Asymptotics
Model and Objective (finite sample case)

\[ y = Y\beta + W\gamma + \varepsilon, \]
\[ Y = Z\Pi_Y + V_Y, \]
\[ W = Z\Pi_W + V_W, \]

\[ y \in \mathbb{R}^n, Y \in \mathbb{R}^{n \times m_Y}, W \in \mathbb{R}^{n \times m_W}, \text{ and } Z \in \mathbb{R}^{n \times k}. \]

- **Reduced form:**

  \[ (y : Y : W) = Z (\Pi_Y : \Pi_W) \begin{pmatrix} \beta & I_{m_Y} & 0 \\ \gamma & 0 & I_{m_W} \end{pmatrix} + (v_y : V_Y : V_W). \]

- **Objective:** test

  \[ H_0 : \beta = \beta_0 \text{ versus } H_1 : \beta \neq \beta_0 \]

s.t. size bounded by nominal size & “good” power.
Parameter space:

1. The error term is distributed as

\[ V_i \sim \text{i.i.d. } N(0, \Omega), \ i = 1, \ldots, n, \]

where \( \Omega \in R^{(m+1) \times (m+1)} \) is assumed to be known and positive definite.

2. \( Z \in R^{n \times k} \) fixed, and \( Z'Z > 0 \) \( k \times k \) matrix.

• **Note:** no restrictions on reduced form parameters \( \rightarrow \) allow for weak IV.
• Many tests available for **full vector inference**

\[ H_0 : \beta = \beta_0, \gamma = \gamma_0 \text{ vs } H_1 : \text{not } H_0 \]


Derived subvector procedures

- **Projection**: "inf" over parameter not under test, same critical value → "computationally hard" and "uninformative".


- **Plug-in approach**: Kleibergen (2004), Guggenberger and Smith (2005)...Requires strong ID of parameters not under test.

- Kleibergen (2015): subvector CLR test with correct size under weak IV and asymptotically efficient under strong IV.
• Power ranking under weak IV is unclear:

  – In just-identified case \( k = m_Y + m_W \), subvector LR statistic is equal to the subvector AR statistic, and CLR cv is \( \chi^2_{m_Y,1-\alpha} \).

  – Hence, less powerful than the test proposed here.
The Anderson and Rubin (1949) test

- **AR test stat** for full vector hypothesis

  \[ H_0 : \beta = \beta_0, \gamma = \gamma_0 \ vs \ H_1 : \text{not } H_0 \]

- AR statistic exploits \( EZ_i \varepsilon_i = 0. \)

- **AR test stat:**

  \[
  AR_n(\beta_0, \gamma_0) = \frac{(y - Y\beta_0 - W\gamma_0)'P_Z(y - Y\beta_0 - W\gamma_0)}{(1 : -\beta_0' : -\gamma_0)'\Omega (1 : -\beta_0' : -\gamma_0)'}
  \]

- AR stat is \( \chi^2_k \) under null hypothesis; critical value \( \chi^2_{k,1-\alpha} \).
• **Subvector AR statistic** for testing $H_0$ is given by

$$AR_n(\beta_0) = \min_{\gamma \in \mathbb{R}^{mW}} \frac{(Y_0 - W\gamma)'P_Z(Y_0 - W\gamma)}{(1 : -\beta'_0 : -\gamma')'\Omega(1 : -\beta'_0 : -\gamma')},$$

where $\overline{Y}_0 = y - Y\beta_0$.

• Alternative representation: Let $\hat{\kappa}_i$ for $i = 1, \ldots, p = 1 + m_W$ be roots of characteristic polynomial in $\kappa$

$$\left| \kappa \Omega(\beta_0) - (\overline{Y}_0 : W)'P_Z(\overline{Y}_0 : W) \right| = 0,$$

ordered non-increasingly, where we define

$$\Omega(\beta_0) = \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \\ 0 & I_{m_W} \end{pmatrix}' \Omega \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \\ 0 & I_{m_W} \end{pmatrix}. $$
Then

\[ AR_n(\beta_0) = \hat{\kappa}_p. \]

- As discussed: When using \( \chi^2_{k,1-\alpha} \) critical values, trivially, test has correct size;

- GKMC show that this is also true for \( \chi^2_{k-m_W,1-\alpha} \) critical values.
• **Next:** AR statistic is the minimum eigenvalue of a non-central Wishart matrix.

• The roots $\hat{\kappa}_i$ solve

$$0 = \left| \hat{\kappa}_i I_{1+m_W} - \Xi'\Xi \right|, \quad i = 1, \ldots, p = 1 + m_W,$$

where $\Xi \sim N (\mathcal{M}, I_k \otimes I_p)$, and $\mathcal{M}$ is a $k \times p$.

• Under $H_0$, the noncentrality matrix becomes $\mathcal{M} = \left(0^k, \Theta_W\right)$, where

$$\Theta_W = \left(Z'Z\right)^{1/2} \Pi_W \Sigma_W^{-1/2} \Sigma_{VW} V_{W.\varepsilon},$$

$$\Sigma_{VW} V_{W.\varepsilon} = \Sigma_{VW} V_{W} - \Sigma'_{W} V_{W} \sigma^{-1}_{\varepsilon} \Sigma_{\varepsilon} V_{W}$$
and

\[
\begin{pmatrix}
\sigma_{\varepsilon \varepsilon} & \sum_{\varepsilon V_W} \\
\sum'_{\varepsilon V_W} & \sum V_W V_W
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \end{pmatrix}' \Omega \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \end{pmatrix}
\]

- **Summarizing**, under \( H_0 \)

\[
\Xi'\Xi \sim \mathcal{W}_p \left( k, I_p, \mathcal{M}'\mathcal{M} \right),
\]

non-central Wishart, with noncentrality matrix

\[
\mathcal{M}'\mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & \Theta_W'\Theta_W \end{pmatrix}
\]

and

\[
AR_n(\beta_0) = \kappa_{\text{min}}(\Xi'\Xi)
\]
• The distribution of the eigenvalues of a noncentral Wishart matrix only depends on the eigenvalues of the noncentrality matrix $\mathcal{M}'\mathcal{M}$.

• Hence, distribution of $\hat{\kappa}_i$ only depends on the eigenvalues of $\Theta'_W\Theta_W$, $\kappa_i$ say, $i = 1, \ldots, m_W$ and $\kappa = (\kappa_1, \ldots, \kappa_{m_W})'$

• When $m_W = 1$, $\kappa_1 = \Theta'_W\Theta_W$ is scalar (concentration parameter for $\gamma$ under Null).
**Theorem:** Suppose $m_W = 1$. Then, under the null hypothesis $H_0 : \beta = \beta_0$, the distribution function of the subvector AR statistic, $AR_n (\beta_0)$, is monotonically decreasing in the parameter $\kappa_1$.

![Figure 1](image-url)

**Figure 1:** The cdf of the subset AR statistic with $k = 3$ instruments, for different values of $\kappa_1 = 5, 10, 15, 100$, shown in the legend on the right.
New critical value for subvector Anderson and Rubin test

- **Relevance:** If we knew $\kappa_1$ we could implement the subvector AR test with a smaller critical value than $\chi^2_{k-m_W,1-\alpha}$ which is the critical value in the case when $\kappa_1$ is “large”.

- **Intuition for new critical value.** Let’s assume $m_W = 1$ for simplicity.

- Under null, when $\kappa_1$ “is large”, the larger root $\hat{\kappa}_1$ is a sufficient statistic for $\kappa_1$, see Muirhead (1978).

- Muirhead provides approximate, nuisance parameter free, density of $AR_n (\beta_0) = \hat{\kappa}_2$ given $\hat{\kappa}_1$ (which measures strength of identification).
• The **new critical value** for the subvector AR-test at significance level $1 - \alpha$ is given by

\[ 1 - \alpha \text{ quantile of (approximation of } AR_n \text{ given } \hat{\kappa}_1) \]

• Denote cv by

\[ c_{1-\alpha}(\hat{\kappa}_1, k - m_W) \]

Depends only on $\alpha$, $k - m_W$, and $\hat{\kappa}_1$.

• We find, by simulations over fine grid of values of $\kappa_1$, that test controls size.

• It improves on the GKMC procedure in terms of power.
• **Theorem:** Suppose $m_W = 1$. The subvector Anderson Rubin test that uses the new conditional critical value $c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ has correct size under the assumptions above.

**Details**

• Again: $\kappa_1 \geq 0$ is nonzero latent root of $\mathcal{M}'\mathcal{M}$ (nuisance parameter).

• When the root is “large”, the conditional density of $AR_n(\beta_0) = \hat{\kappa}_2$ given $\hat{\kappa}_1$ can be approximated by

$$f_{\hat{\kappa}_2|\hat{\kappa}_1}(x) \sim f_{\chi^2_{k-1}}(x) (\hat{\kappa}_1 - x)^{1/2} g(\hat{\kappa}_1),$$

where $f_{\chi^2_{k-1}}$ is the density of a $\chi^2_{k-1}$ and $g$ is a function that does not depend on $\kappa_1$. (Muirhead, 1978 due to Leach, 1969).
• Analytical formula for $g$.

• Conditional quantiles can be computed by numerical integration.

• Conditional critical values can be tabulated → implementation of new test is trivial and fast.

• They are increasing in $\hat{\kappa}_1$ and converging to quantiles of $\chi^2_{k-1}$.
Critical value function $c_{1-\alpha}(\hat{\kappa}_1, k - 1)$ for $\alpha = 0.05$. 
Table of conditional critical values

\[ \alpha = 5\%, \quad k - m_W = 4 \]

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Null rejection frequency of subset AR test based on conditional (red) and $\chi^2_{k-1}$ (blue) critical values, as function of $\kappa_1$. 10000 MC simulations with importance sampling over a grid of 42 points.
Power

- The subvector AR statistic is the LR statistic for testing \( H'_0 : \rho(A) \leq m_W \) against \( H'_1 : \rho(A) = m_W + 1 \) for \( A = E[Z'(y - Y\beta_0 : W)] \), where the data is \( Z'(y - Y\beta_0 : W) \).

- \( H_0 : \beta = \beta_0 \) implies \( H'_0 \) but the converse is not true:
  
  \(- \quad H'_0 \) holds iff \( \rho(\Pi_Y (\beta - \beta_0) : \Pi_W) \leq m_W \), which includes \( H_1 : \beta \neq \beta_0 \) when \( H'_0 \setminus H_0 \) holds, i.e., if \( \Pi_W \) is rank deficient or \( \Pi_Y(\beta - \beta_0) \in \text{span}(\Pi_W) \).

- Under \( H'_0 \), \( (\hat{\kappa}_1, \ldots, \hat{\kappa}_p) \) are distributed as eigenvalues of \( \mathcal{W}_p (k, I_p, \mathcal{M}'\mathcal{M}) \) with rank deficient noncentrality.
• Thus, every test $\varphi(\hat{k}_1, \ldots, \hat{k}_p) \in [0, 1]$ that has size $\alpha$ under $H_0$ must also have size $\alpha$ under $H'_0$, so cannot have power exceeding size under alternatives $H'_0 \setminus H_0$.

• In other words, size $\alpha$ tests $\varphi(\hat{k}_1, \ldots, \hat{k}_p)$ can only have nontrivial power under alternatives $\rho(A) = m_W + 1$.

• We use this insight to derive a power envelope for tests of the form $\varphi(\hat{k}_1, \ldots, \hat{k}_p)$.

• Consider only the case $m_W = 1$. 
Testing $\rho (M) \leq 1$ against $\rho (M) = 2$, where $\Xi \sim N (M, I)$.

Equivalently, $H_0' : \kappa_2 = 0, \kappa_1 \geq \kappa_2$ against $H_1' : \kappa_2 > 0, \kappa_1 \geq \kappa_2$.

Maximal invariant is $\hat{\kappa}_1, \hat{\kappa}_2$ (Muirhead, 2009, Section 10.2).

Likelihood (James, 1964)

$$lik (\kappa | \hat{\kappa}) = \exp \left( - \frac{\kappa_1 + \kappa_2}{2} \right) \, _0F_1^{(2)} \left( \frac{k}{2}; \frac{1}{4} \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \begin{pmatrix} \hat{\kappa}_1 & 0 \\ 0 & \hat{\kappa}_2 \end{pmatrix} \right)$$

Computed using the algorithms developed by Koev and Edelman (2008), available in C and Matlab.
Power bounds

- Point-optimal power bounds for reduced rank testing problem using least favourable distribution $\Lambda^{LF}$ over nuisance parameter $\kappa_1$.

- Two methods: Andrews Moreira and Stock (JoE, 2008, Sec 4.2) – AMS.
  - assumes one-point $\Lambda^{LF}$, gives lower and upper bounds on envelope.


- Implementation: 42 points evenly spaced in log-scale between 0 and 99.
Power of conditional subvector AR test $\varphi_c (\hat{\kappa}) = 1\{\hat{\kappa}_2 > c_{1-\alpha}(\hat{\kappa}_1, k-1)\}$ relative to power bound (left) and power of $\varphi_c$, $\varphi_{GKMC} (\hat{\kappa}) = 1\{\hat{\kappa}_2 > \chi_{k-1, 1-\alpha}^2\} = 1\{\hat{\kappa}_2 > c_{1-\alpha}(\infty, k-1)\}$ and bound at $\kappa_1 = \kappa_2$ (right) for $k = 5$. Computed using 10000 MC replications.
• Little scope for power improvement over proposed test.
Size for $m_W > 1$

When $m_W = 1$ the new subvector AR test has correct size and uniformly improves the power of the test in GKMC.

→ Generalize this result to any $m_W$.

We define a new subvector AR test that rejects when

$$AR_n (\beta_0) > c_{1-\alpha}(\kappa_{\text{max}} (\Xi'\Xi), k - m_W).$$

Note: We condition on the LARGEST eigenvalue of the Wishart matrix.

Show now that this test has correct size and has uniformly larger power than the test in GKMC.
**Theorem:** Under the null $H_0 : \beta = \beta_0$, there exists a random orthogonal matrix $O$, such that for

$$\tilde{\Xi} = \Xi O \in \mathbb{R}^{k \times p}, \text{ and its upper left submatrix } \tilde{\Xi}_{11} \in \mathbb{R}^{k-m_W+1 \times 2}$$

$\tilde{\Xi}_{11} \tilde{\Xi}_{11}$ is a non-central Wishart $2 \times 2$ matrix of order $k - m_W + 1$ (cond’l on $O$), whose noncentrality matrix, $\tilde{\mathcal{M}}_1^T \tilde{\mathcal{M}}_1$ say, is of reduced rank.

It then follows that

(i) $AR_n (\beta_0) = \kappa_{\min} (\Xi' \Xi) = \kappa_{\min} (\tilde{\Xi}' \tilde{\Xi})$

$$\leq \kappa_{\min} (\tilde{\Xi}_{11}' \tilde{\Xi}_{11}) \leq \kappa_{\max} (\tilde{\Xi}_{11}' \tilde{\Xi}_{11})$$

$$\leq \kappa_{\max} (\tilde{\Xi}' \tilde{\Xi}) = \kappa_{\max} (\Xi' \Xi)$$
and thus

\[
P(AR_n(\beta_0) > c_{1-\alpha}(\kappa_{\max}(\Xi', \Xi), k - m_W)) \\
\leq P(\kappa_{\min}(\tilde{\Xi}'_{11}, \tilde{\Xi}_{11}) > c_{1-\alpha}(\kappa_{\max}(\tilde{\Xi}'_{11}, \tilde{\Xi}_{11}), k - m_W)) \\
\leq \alpha,
\]

where the last inequality follows from the case \(m_W = 1\) (by conditionning on \(O\)).

(ii) new conditional test is uniformly more powerful than test in GKMC (because \(c_{1-\alpha}(\cdot, k - m_W)\) is increasing and converging to \(\chi^2_{k-m_W, 1-\alpha}\) as argument goes to infinity).
Asymptotic case

- **Parameter space** $\mathcal{F}$ under the null hypothesis $H_0 : \beta = \beta_0$. Let $U_i = (\varepsilon_i, V_{W,i}')$ and $F$ distribution of $(U_i, V_{Y,i}, Z_i)$

$\mathcal{F}$ is set of all $(\gamma, \Pi_W, \Pi_Y, F)$ s.t.

$\gamma \in R^{m_W}, \Pi_W \in R^{k \times m_W}, \Pi_Y \in R^{k \times m_Y},$

$E_F(\|T_i\|^{2+\delta}) \leq B, \text{ for } T_i \in \{Z_i\varepsilon_i, \text{vec}(Z_iV_{W,i}'), V_{W,i}\varepsilon_i, \varepsilon_i, V_{W,i}, Z_i\},$

$E_F(Z_i(\varepsilon_i, V_{W,i}', V_{Y,i}')) = 0,$

$E_F(\text{vec}(Z_i U_i')(\text{vec}(Z_i U_i'))') = (E_F(U_i U_i') \otimes E_F(Z_i Z_i')),$

$\kappa_{\min}(A) \geq b \text{ for } A \in \{E_F(Z_i Z_i'), E_F(U_i U_i')\}$

for some $b > 0, B < \infty$, where $\kappa_{\min}(\cdot)$ is smallest eigenvalue, “$\otimes$” Kronecker product, $\text{vec}(\cdot)$ column vectorization.
- Subvector AR stat equals

\[ AR_n(\beta_0) = \kappa_{\min} \left( \left( \frac{Y' M Z Y}{n-k} \right)^{-1/2} (\overline{Y}' P Z \overline{Y}) \left( \frac{Y' M Z Y}{n-k} \right)^{-1/2} \right) \]

where

\[ \overline{Y} := (y - Y \beta_0 : W) \in R^{n \times (1+mW)} \]

- GKMC showed \( \varphi_{GKMC} = 1 \left\{ AR_n(\beta_0) > \chi^2_{k-mW,1-\alpha} \right\} \) has correct asymptotic size for parameter space \( \mathcal{F} \).

- **Current paper:** \( \varphi_c = 1 \left\{ AR_n(\beta_0) > c_{1-\alpha}(\hat{\kappa}_{\max},k-mW) \right\} \) has correct asy size.
Asymptotic Size of conditional subvector AR test


- The complication relative to GKMC is that we need joint limiting distribution of $\hat{\kappa}_1, \ldots, \hat{\kappa}_p$, not just the minimum, $\hat{\kappa}_p$.

- Fortunately, we can use the results of Andrews and Guggenberger (2015) on limit distribution of eigenvalues of quadratic forms.

- It turns out that joint limit depends only on localization parameters corresponding to the singular values of

$$\left((E_{FZ_iZ_i'})^{1/2}(\Pi_W\gamma, \Pi_W)\Omega(\beta_0)^{-1/2},\right.$$
which correspond to singular values of $\Theta_W$ (concentration matrix) in the finite sample case.

- Hence, replicates the finite sample, normal, fixed IV, known variance matrix setup.

- Correct asymptotic size then follows from correct finite sample size.
Takeaways

- We can obtain uniform power improvement over the subvector AR test in GKMC by using data-dependent critical values.

- We propose one such test whose conditional cv’s are easy to compute and can be tabulated.

- In the case $m_W = 1$, i.e., when there is a single endogenous regressor whose coefficient is unrestricted under $H_0$, the proposed cv’s are an increasing function of a first-stage F statistic for that regressor.

- There is little scope for further power improvement when $m_W = 1$ – our proposed test is nearly optimal.
**Current work:** Drop assumption of conditional homoskedasticity → allow for heteroskedasticity.

- Lee (2014) found an example in which the subvector AR with $\chi^2_{k-m_W,1-\alpha}$ cv’s overrejects when the covariance matrix does not have Kronecker product form.

- Importantly, this does not apply to iid data.

- So far, we have found correct size of the heteroskedasticity robust subvector AR test that uses $\chi^2_{k-m_W,1-\alpha}$ cv’s when $m_W = 1$ and $k = 2$.

- We are working on generalizing this to higher dimensions.