THE CUTOFF STRUCTURE OF TOP TRADING CYCLES IN SCHOOL CHOICE

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TOP TRADING CYCLES FOR SCHOOL CHOICE

- **School Choice**: Assigning students to schools
  - Allow students to choose schools
  - Account for siblings, neighborhood status

- **Top Trading Cycles (TTC)** is an attractive mechanism
  - Pareto efficient and strategy-proof for students
  - Policy lever: school priorities can guide the allocation

- **But TTC is rarely used**
  - Difficult to assess how changes in input (priorities and preferences) affect the TTC allocation
THE CUTOFF STRUCTURE OF TTC

- Characterizing the TTC assignment
  - TTC assignment given by \( n^2 \) admissions cutoffs

- Calculating the TTC cutoffs
  - Solve for sequential trade by looking at trade balance equations
  - TTC cutoffs are solutions to a differential equation

- Structure of the TTC assignment
  - Comparative statics
  - Welfare comparisons with other school choice mechanisms
  - Designing TTC priorities
RELATED LITERATURE

▷ **School choice – theory and practice**

▷ **Cutoff representations of school choice mechanisms**

▷ **Characterizations of TTC mechanism**
  - Shapley & Scarf (1973), attributed to David Gale
THE TTC ALGORITHM

### School Priorities

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**Step 1:**
- Schools point to their favorite student
- Students point to their favorite school
- Choose a cycle, assign included students to their favorite school.
### THE TTC ALGORITHM

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## The TTC Algorithm

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THE TTC ALGORITHM

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Step $k$:
- Schools point to their favorite remaining student
- Students point to their favorite remaining school
- Choose a cycle, assign included students to their favorite school.
THE TTC ALGORITHM

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\[ \vdots \]

\[ s_7 \quad s_1 \quad s_1 \]

Step \( k \):

- Schools point to their favorite remaining student
- Students point to their favorite remaining school
- Choose a cycle, assign included students to their favorite school.
CHARACTERIZING THE TTC ASSIGNMENT
SCHOOL CHOICE MODEL

- Finite number of students $\theta = (\succ^\theta, r^\theta)$
  - Student $\theta$ has preferences $\succ^\theta$ over schools
  - $r^\theta_c \in [0,1]$ is the rank of student $\theta$ at school $c$ (percentile in $c$’s priority list)

- Finite number of schools $c$
  - School $c$ can admit $q_c$ students
  - $\succ^c$ a strict ranking over students
Student $\theta_1$
- prefers 1 to 2
- highly ranked at 1
- highly ranked at 2

Student $\theta_2$
- prefers 2 to 1
- highly ranked at 1
- poorly ranked at 2
EXAMPLE

- 2/3 students prefer school 1
- Ranks are uniformly i.i.d. across schools
- $q_1 = q_2$
EXAMPLE – TTC ASSIGNMENT

- Assigned to school 1
- Assigned to school 2
- Unassigned
EXAMPLE – TTC ASSIGNMENT

- Assigned to school 1
- Assigned to school 2
- Unassigned
Theorem.
The TTC assignment is given by cutoffs \( \{p_b^c\} \) where:

- Each student \( \theta \) has a budget set

\[
B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}
\]

- Students assigned to their favorite school in their budget set

\[
\mu(\theta) = \max_{\succ \theta}(B(p, \theta))
\]

**Interpretation:** \( p_b^c \) is the minimal priority at school \( b \) that allows trading a seat at school \( b \) for a seat at school \( c \)
EXAMPLE – ASSIGNMENT VIA CUTOFFS

\[ B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c \} \]

Budget set
\{1,2\}

Budget set
\{2\}
EXAMPLE – ASSIGNMENT VIA CUTOFFS

$B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$

- Budget set \{1,2\}
- Budget set \{2\}

$B_1(\theta, p)$: Budget set from rank at school 1

$B_2(\theta, p)$: Budget set from rank at school 2

$p_1^1 = p_1^2$
EXAMPLE – ASSIGNMENT VIA CUTOFFS

\[ \mu(\theta) = \max_{\geq \theta} B(p, \theta) \]

- Assigned to school 1
- Assigned to school 2
- Unassigned
GENERAL STRUCTURE OF CUTOFFS

There is a renaming of the schools such that

- Each student’s budget set is

\[ C^{(\ell)} = \{\ell, \ldots, n\} \]

- The cutoffs are ordered

\[ p_c^1 \geq p_c^2 \geq \cdots \geq p_c^c = p_c^d \]

for all \( c < d \)
CALCULATING TTC CUTOFFS
CONTINUUM MODEL

- **Finite** number of schools $c \in C = \{1, \ldots, n\}$
  - School $c$ can admit a mass $q_c$ of students

- **Measure $\eta$** specifying a distribution of a continuous mass of students
  - A student $\theta \in \Theta$ is given by $\theta = (\succ^\theta, r^\theta)$
  - Student $\theta$ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the student’s rank at school $c$
    (percentile in $c$ priority list)
TTC ASSIGNMENT VIA CUTOFFS

Theorem.
The TTC assignment is given by cutoffs \( \{p_b^c\} \) where:

- Each student \( \theta \) has a budget set
  \[
  B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}
  \]

- Students assigned to their favorite school in their budget set
  \[
  \mu(\theta) = \max_{\succ \theta} (B(p, \theta))
  \]

Cutoffs \( p_b^c \) are the solutions to a differential equation.
Theorem.
The TTC cutoffs \( \{p_b^c\} \) are given by

\[ p_b^c = \gamma_b(t^{(c)}) \]

where \( \gamma \) satisfies the marginal trade balance equations

\[
\sum_{a \in C} \gamma_a'(t) H_a^c(\gamma(t)) = \sum_{a \in C} \gamma_c'(t) H_c^a(\gamma(t)) \quad \forall t, c.
\]

\( H_b^c(x) \) is the marginal density of students who have rank \( \leq x \), are top ranked at school \( b \) and most prefer school \( c \).
TRADE BALANCE EQUATIONS

\[
\# \left\{ \begin{array}{c}
\text{Students assigned to } c \\
\text{by time } t
\end{array} \right\} = \# \left\{ \begin{array}{c}
\text{Students who traded } c \\
\text{by time } t
\end{array} \right\}
\]

for all times \( t \).

▶ Necessary condition for aggregate trade

▶ Equivalent to the differential equation \( \gamma'(t) = d(\gamma(t)) \), where \( \gamma_c(t) \) is the rank of students pointed to by school \( c \) at time \( t \).

▶ \( \gamma \) is the TTC path
**TRADE BALANCE – VISUALIZATION**

$\gamma_c(t)$: Rank of students pointed to by school $c$ at time $t$

\[ 1 > \theta 2 \quad \text{and} \quad 2 > \theta 1 \]

- $\gamma_2(t)$: Assigned students
- $\gamma_1(t)$: Unassigned students
- $\gamma(t)$: Offered students
Trade Balance – Visualization

$\gamma_c(t)$: Rank of students pointed to by school $c$ at time $t$

Diagram showing the ranks and assignments of students.
TRADE BALANCE – VISUALIZATION

\( \gamma_c(t) \): Rank of students pointed to by school \( c \) at time \( t \)

\[ \gamma'_2(t) \text{(density of } 1 > 2) = \gamma'_1(t) \text{(density of } 2 > 1) \]

\( \gamma_2(t) \):

\[ 1 > \theta \ 2 \]

\( \gamma_1(t) \):

\[ 2 > \theta \ 1 \]
TRADE BALANCE – VISUALIZATION

\( \gamma_c(t) \): Rank of students pointed to by school \( c \) at time \( t \)

\[ \gamma_2(t) (\text{density of } 1 > 2) = \gamma_1'(t) (\text{density of } 2 > 1) \]
**CAPACITY EQUATIONS**

**Stopping times** $t^{(c)}$

$$ t^{(c)} = \min \left\{ t : \# \left\{ \text{Students assigned to } c \text{ by time } t \right\} \geq q_c \right\} $$

- Necessary condition for market clearing
- Equivalent to equations involving $\gamma(t^{(c)})$
Theorem.

The TTC assignment is given by computing cutoffs \( \{p_b^c\} \)

\[ p_b^c = \gamma_b(t^{(c)}) \]

where \( \gamma \) satisfies the marginal trade balance equations, and assigning students to their favorite school in their budget set

\[ B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\} \]

\[ \mu(\theta) = \max_{\triangleright_\theta} B(p, \theta). \]

- Closed form solutions, comparative statics
- Admissions probabilities
EXAMPLE: CALCULATING TTC CUTOFFS

2/3 of students prefer school 1, ranks are uniformly i.i.d. across schools, $q_1 = q_2$
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- Marginal trade balance equations given valid gradient:
  $$\gamma'(t) = d(\gamma(t))$$
EXAMPLE: CALCULATING TTC CUTOFFS

2/3 of students prefer school 1, ranks are uniformly i.i.d. across schools, \( q_1 = q_2 \)

- TTC path \( \gamma \) with initial condition \( \gamma(0) = 1 \) and satisfying
  \[
  \sum_{a \in C} \gamma_a'(t) H_a^c(\gamma(t)) = \sum_{a \in C} \gamma_c'(t) H_c^a(\gamma(t))
  \]
EXAMPLE: CALCULATING TTC CUTOFFS

1 > \theta 2

2 > \theta 1

2/3 of students prefer school 1, ranks are uniformly i.i.d. across schools, q_1 = q_2

- TTC path \gamma indicates the run of TTC
- Cutoffs \( p \) are the points at which schools reach capacity
EXAMPLE: CALCULATING TTC CUTOFFS

- Valid gradient
  \[ d(x) = -\begin{bmatrix} \frac{x_1}{x_1 + 2x_2} & \frac{2x_2}{x_1 + 2x_2} \end{bmatrix} \]  
  \((d(\cdot) \text{ balances marginal densities})\)

- TTC path
  \[ \gamma(t) = \left(t^{1/3}, t^{2/3}\right) \]  
  \((\gamma'(t) = d(\gamma(t)))\)

- TTC cutoffs
  \[ p^1 = \left((1 - 3q_1)^{1/3}, (1 - 3q_1)^{2/3}\right) \]  
  \((p^c_b = \gamma_b(t^{(c)})\))
TRADE BALANCE IS SUFFICIENT

- Trade balance of gradient is mathematically equivalent to stationarity of a Markov chain
  - schools $\Leftrightarrow$ states
  - transition probability $p_{bc} \Leftrightarrow$ mass of students $b$ points to, who want $c$
  - trade balance $\Leftrightarrow$ stationarity

- Unique solution within each communicating class

- Different solutions yield the same allocation
  - Multiplicity only because of disjoint trade cycles
  - Different paths clear the same cycles at different rates
CONTINUUM TTC GENERALIZES DISCRETE TTC

- **Trade Balance Uniquely Determines the Allocation**
  - Differential equation and TTC path may not be unique, but all give the same allocation

- **Consistent with Discrete TTC**
  - Can naturally embed discrete TTC in the continuum model
  - The continuum embedding gives the same allocation as TTC in the discrete model

- **Convergence**
  - If two distributions of students have full support and total variation distance $\varepsilon$, then the TTC allocations differ on a set of students of measure $O(\varepsilon |C|^2)$. 
APPLICATIONS
Effect of marginal increase in desirability of school 2

\[ p_1^1 = p_1^2 \]

\[ r_2^0 \]

\[ p_2^1 \]

\[ p_2^2 \]

\[ \emptyset \]
COMPARATIVE STATICS - WELFARE

\( n \) schools, **MNL utility model** (McFadden 1973):

- Student preferences given by **MNL** utility model:
  \[
  u_{S}(c) = \delta_{c} + \varepsilon_{sc}
  \]
  - \( \delta_{c} \) is invested quality, \( \varepsilon_{\theta c} \) is mean 0 random EV iid
  - Random priority, independent for each school

- Constraints on total quality
- What are the welfare maximizing quality levels \( \sum_{c} \delta_{c} \leq N \)?
COMPARATIVE STATICS - WELFARE

Effects of increasing school quality on student welfare:
(under MNL model, for $n = 2$ and $\delta_1/q_1 > \delta_2/q_2$)

$$\frac{dSW}{d\delta_1} = q_1 - q_1 e^{\delta_2-\delta_1} \ln(1 + e^{\delta_1-\delta_2})$$

- Directly improves welfare of those who stay at the school
- Indirectly affects welfare through changing the allocation
TTC WELFARE GIVEN $n = 2, \delta_1 + \delta_2 = 2$

Maximizing efficient investment vs maximizing choice

Optimal investment maximizes choice

Maximizing efficient investment vs maximizing choice
COMPARING TTC & DA, \( q_1 = q_2 = \frac{3}{8} \)

<table>
<thead>
<tr>
<th>( \delta_1 = \delta_2 = 1, OPT )</th>
<th>( \delta_1 = 2, \delta_2 = 0 )</th>
<th>( \delta_1 - \delta_2 )</th>
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<td>( \frac{1}{1+\ln(2)} \approx 1.69 )</td>
<td>( \frac{1}{1+\ln(2)} \approx 1.20 )</td>
<td>( \frac{1}{1+\ln(2)} \approx 1.11 )</td>
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**TTC**

1+\ln(2) \approx 1.69

**DA**

1 + \left(\frac{1}{3}\right)\ln(2) \approx 1.23
COMPARING TTC & DA, \( q_1 = \frac{1}{2}, \ q_2 = \frac{1}{4} \)

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<tr>
<th>( \delta_1 = \delta_2 = 1 )</th>
<th>( OPT = 1.75 )</th>
<th>( \delta_1 - \delta_2 = 1.40 )</th>
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<tr>
<td><strong>TTC</strong></td>
<td>( 1 + \frac{2}{3} \ln(2) \approx 1.46 )</td>
<td>( {1,2} )</td>
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\[\text{Assigned Student Welfare} \]

\[\text{Assigned Student Welfare} \]
DESIGNING TTC PRIORITIES

- Symmetric economy with two schools
  - Equal capacities
  - Student equally likely to prefer either
  - Priorities are uniformly random iid

- Consider changing the ranking of students with
  \( r_c^\theta \geq m \) for both \( c = 1,2 \)
TTC PRIORITIES ARE “BOSSY”

- The change affects the allocation of other students
- Changed students have the same assignment
CONCLUSIONS

▷ Cutoff description of TTC
  ▷ $n^2$ admissions cutoffs

▷ Tractable framework for analyzing TTC
  ▷ Trade balance equations
  ▷ TTC cutoffs are a solution to a differential equation
  ▷ Can give closed form expressions

▷ Structure of the TTC assignment
  ▷ Equalizing school popularity leads to more efficient sorting on horizontal preferences
  ▷ Welfare comparisons
  ▷ TTC priorities are “bossy”
Thank you!