Belief Update and Mispricing
Theory and Experiment

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1. Abstract

We theoretically examine the mechanism of asset mispricing using a simple sequential trading model.

We find that investors overly consider the existence of informed traders in stock markets, which causes stock mispricing. However, we also show that the mispricing finally disappears.

This study also contains numerical and experimental examinations.

2. Previous Studies

Sequential trading model under asymmetric information

- Glosten and Milgrom (1985)
- Avey and Zemsky (1993)
- Easley and O’ Hara (1992, 1997)

Asymmetric information and mispricing

- Wang (1992)
- Brunnermeier (2005)
- Barberis, Mukherjee, and Wang (2016)

3. Model Setting

We use the sequential trading model (Glosten and Milgrom, 1985).

- There is one risky asset.
- Value of risky asset θ is either 0 or 1.
- The probability is $P(\theta = 0) = P(\theta = 1) = \dfrac{1}{2}$
  (Price of asset $\in [0, 1]$.)

- We assume $\theta = 1$ without loss of generality.

- There is one market maker, and informed traders and noise traders.

- Market maker posts bid and ask prices at any time.

- Informed traders have private information which gives true value $\theta = 1$.

- Informed traders always buy risky asset.

- Noise traders randomly buy or sell with an equal probability of $\frac{1}{2}$.

- Each trader trades one unit of risky asset with the market maker.

- The market maker does not know who is an informed trader and who is a noise trader.

- ϕ represents the probability that an informed trader comes.

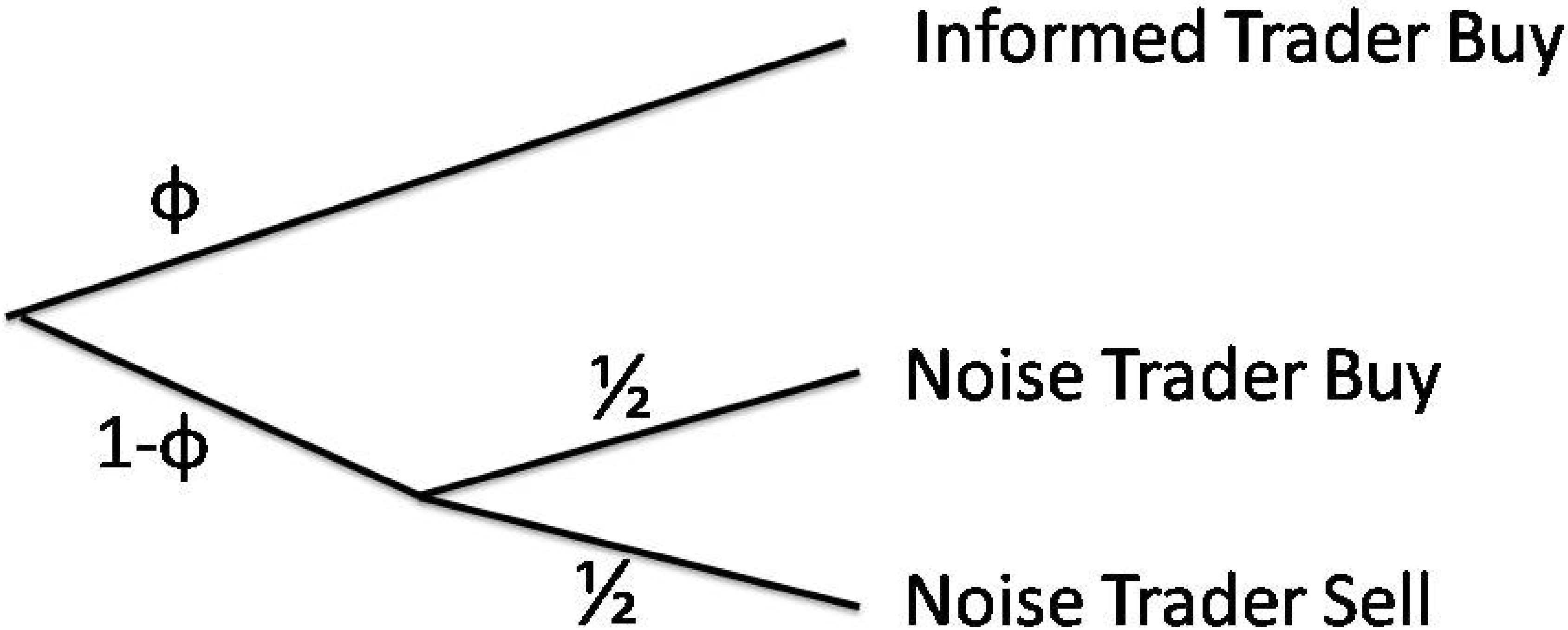
- The market maker knows the probability ϕ.

·ϕ There are two types of market;
informed market (I) and uninformed market (U).

·In an informed market, one or more informed traders exist; $\phi > 0$.

·In an uninformed market, no informed trader exists; $\phi = 0$.

Informed Market（θ＝1）



The probability of buy is $\dfrac{1+\phi}{2}$

The probability of sell is $\dfrac{1-\phi}{2}$

Let ω_k be trade event at time k:

$\omega_k = B$, if trader buys at k.

$\omega_k = S$, if trader sells at k.

Let $\xi_n(h_n) = P(M = I|h_n)$: market maker's belief that the market is informed under trading history $h_n = \omega_1\omega_2\cdots\omega_n$.

We focus on this conditional probability.

4. Trader's belief based on Bayes theory

Using Bayesian theory, we have

$$\xi_n = P(M = I|h_n) = \frac{P(h_n, M = I)}{P(h_n, M = I) + P(h_n, M = U)}.$$

The first term of denominator is

$$P(h_n, M = I)$$

$$= P(h_n, M = I, \theta = 1) + P(h_n, M = I, \theta = 0)$$

$$= P(h_n|M = I, \theta = 1)P(M = I, \theta = 1)$$
$$+P(h_n|M = I, \theta = 0)P(M = I, \theta = 0).$$

Under M=I and θ =1,we have

$$\begin{cases} P(\omega_k = B|M = I, \theta = 1) = \dfrac{1+\phi}{2} \\ \\ P(\omega_k = S|M = I, \theta = 1) = \dfrac{1-\phi}{2} \end{cases} .$$

Then, we obtain

$$P(h_n|M = I, \theta = 1) = {}_n C_{B_n} \left(\frac{1+\phi}{2}\right)^{B_n} \left(\frac{1-\phi}{2}\right)^{S_n} ,$$

where B_n is the total number of buys and S_n is the total number of sells in $h_n = \omega_1 \omega_2 \cdots \omega_n$.

Then, we obtain

$$\xi_n = P(M = I|h_n)$$

$$= \frac{(1+\phi)^{B_n}(1-\phi)^{S_n} \cdot \frac{1}{2}\xi_0 + (1-\phi)^{B_n}(1+\phi)^{S_n} \cdot \frac{1}{2}\xi_0}{(1+\phi)^{B_n}(1-\phi)^{S_n} \cdot \frac{1}{2}\xi_0 + (1-\phi)^{B_n}(1+\phi)^{S_n} \cdot \frac{1}{2}\xi_0 + (1-\xi_0)}.$$

So, ξ_n depends only on the number of buys and sells and does not depend on the order of them.

5. Bayesian update of trader's belief

We have

$$\xi_{n+1} = P(M = I|h_{n+1})$$

$$= \frac{P(\omega_{n+1}, M = I|h_n)}{P(\omega_{n+1}, M = I|h_n) + P(\omega_{n+1}, M = U|h_n)}.$$

In the case of $\omega_{n+1} = B$, the first term of denominator is

$$P(\omega_{n+1}, M = I|h_n) = P(\omega_{n+1} = B|M = I, \theta = 1, h_n)$$

$$\times P(\theta = 1|M = I, h_n) P(M = I|h_n)$$

$$+ P(\omega_{n+1} = B|M = I, \theta = 0, h_n)$$

$$\times P(\theta = 0|M = I, h_n) P(M = I|h_n).$$

Under $M = I$ and $\theta = 1$, the probability of $\omega_{n+1} = B$ is $P(\omega_{n+1} = B|M = I, \theta = 1, h_n) = \dfrac{1+\phi}{2}$.

In order to update $\{P(M = I \mid h_n)\}$, we need $P(\theta = 1|M = I, h_n) =: \mu_n(h_n) = \mu_n(\omega_1 \omega_2 \cdots \omega_n)$.

Using μ_n, the first term of the denominator of $\xi_{n+1} = P(M = I \mid h_{n+1})$ is

$$P(\omega_{n+1}, M = I|h_n) = \frac{1+\phi}{2}\mu_n \xi_n + \frac{1-\phi}{2}(1-\mu_n)\xi_n.$$

Under $M = I$, we have the Bayesian update for $\{\mu_n\}$ when $\omega_{n+1} = B$ as follows:

$$\mu_{n+1}(\omega_1\omega_2\cdots\omega_n B) = P(\theta = 1|\omega_{n+1} = B,\ h_n)$$

$$= \frac{P(\omega_{n+1} = B|\theta = 1)P(\theta = 1|h_n)}{P(\omega_{n+1} = B|\theta = 1)P(\theta = 1|h_n) + P(\omega_{n+1} = B|\theta = 0)P(\theta = 0|h_n)}$$

$$= \frac{\frac{1+\phi}{2}\mu_n(h_n)}{\frac{1+\phi}{2}\mu_n(h_n) + \frac{1-\phi}{2}(1-\mu_n(h_n))} = \frac{(1+\phi)\mu_n(h_n)}{1-\phi+2\phi\cdot\mu_n(h_n)}.$$

Similarly, we can have a Bayesian update for μ_n when $\omega_{n+1} = S$. Then, we obtain

$$\mu_{n+1}(h_{n+1}) = \begin{cases} \dfrac{(1+\phi)\mu_n}{1-\phi+2\phi\mu_n} & \omega_{n+1} = B \\ \dfrac{(1-\phi)\mu_n}{1+\phi-2\phi\mu_n} & \omega_{n+1} = S \end{cases},$$

where $\mu_0(\omega) = \dfrac{1}{2}$.

We define $a_n = \frac{1}{\mu_n} - 1$, then we have

$\mu_n \to 0 \Leftrightarrow a_n \to \infty,\ \mu_n \to \frac{1}{2} \Leftrightarrow a_n \to 1,\ \mu_n \to 1 \Leftrightarrow a_n \to 0.$

So, we have the following proposition 1 and corollary 2.

Proposition 1:

$$a_n = \left(\frac{1+\phi}{1-\phi}\right)^{S_n - B_n}.$$

Corollary 2:

μ_n depends on the number of buys and sells and not on the order in which they are executed.

In the case of informed markets, μ_n converges to 1 a.s. when $\theta = 1$, and 0 a.s. when $\theta = 0$.

In the case of uninformed markets, μ_n converges to either 0 or 1 a.s. The probability that μ_n converges to 0 is $\frac{1}{2}$ and vice versa.

Remember

$$\xi_{n+1} = P(M = I|h_{n+1})$$

$$= \frac{P(\omega_{n+1}, M = I|h_n)}{P(\omega_{n+1}, M = I|h_n) + P(\omega_{n+1}, M = U|h_n)}.$$

In the case of $\omega_{n+1} = B$, the first term of the denominator is

$$P(\omega_{n+1} = B, M = I|h_n) = \frac{1+\phi}{2}\mu_n\xi_n + \frac{1-\phi}{2}(1-\mu_n)\xi_n.$$

When $\omega_{n+1} = B$, the second term of the denominator is

$$P(\omega_{n+1} = B, M = U|h_n) = \frac{1}{2}(1-\xi_n).$$

Therefore, we have the following Bayesian update for $\xi_n = \xi_n(h_n) = P(M = I|h_n)$.

$$\xi_{n+1} = \begin{cases} \dfrac{\frac{1}{2} + \phi(\mu_n - \frac{1}{2})}{\frac{1}{2} + \phi(\mu_n - \frac{1}{2})\xi_n} \cdot \xi_n & \omega_{n+1} = B \\ \\ \dfrac{\frac{1}{2} - \phi(\mu_n - \frac{1}{2})}{\frac{1}{2} - \phi(\mu_n - \frac{1}{2})\xi_n} \cdot \xi_n & \omega_{n+1} = S \end{cases}.$$

Note: $\mu_n = P(\theta = 1|M = I, h_n)$ is easier to consider than $\xi_n = P(M = I|h_n)$.

Example:

Let us consider the case of $h_5 = BS\,BBB$,

$\xi_5 = P(M = I \mid h_5)$ is not close to one because
$\mu_5 = P(S_5 = 1 \; or \; 4 \mid M = U) \fallingdotseq \dfrac{1}{3}$,

but $P(\theta = 1 \mid M = I, h_5)$ is close to one.

We have an explicit form of ξ_n as follows.

Proposition 3:

$$\xi_n = P(M = I|h_n)$$

$$= \frac{(1+\phi)^{B_n}(1-\phi)^{S_n} \cdot \frac{1}{2}\xi_0 + (1-\phi)^{B_n}(1+\phi)^{S_n} \cdot \frac{1}{2}\xi_0}{(1+\phi)^{B_n}(1-\phi)^{S_n} \cdot \frac{1}{2}\xi_0 + (1-\phi)^{B_n}(1+\phi)^{S_n} \cdot \frac{1}{2}\xi_0 + (1-\xi_0)}$$

The above equation is the same as the solution based on the Bayesian theory in section 4.

From the proposition 3, we have the following corollaries 4 and 5.

Corollary 4: ξ_n *does not depend on the order of buys and sells and only depends on their number.*

In the case of informed markets, ξ_n *converges to* 1 *a.s.*

In the case of uninformed markets, ξ_n *convergesto* 0 *a.s.*

Corollary 5: *In the case of uninformed markets,* $E[\xi_n]$ *monotonically decreases and converges to* 0 *.*

On the contrary, in the case of informed markets, $E[\xi_n]$

Let us define asset price p_n:

$$p_n = \xi_n \mu_n + \frac{1}{2}(1 - \xi_n).$$

We also define market efficiency in Definition 6.

Definition 6:

In the case of informed markets, the market is efficient with respect to price p if $p = \theta$.

Similarly, in the case of uninformed markets, the market is efficient if $p = \frac{1}{2}$.

From corollary 2 and corollary 4, we have the theorem7.

Theorem 7:

The probability that the market is efficient under p_n *converges to* 1 *a.s.*

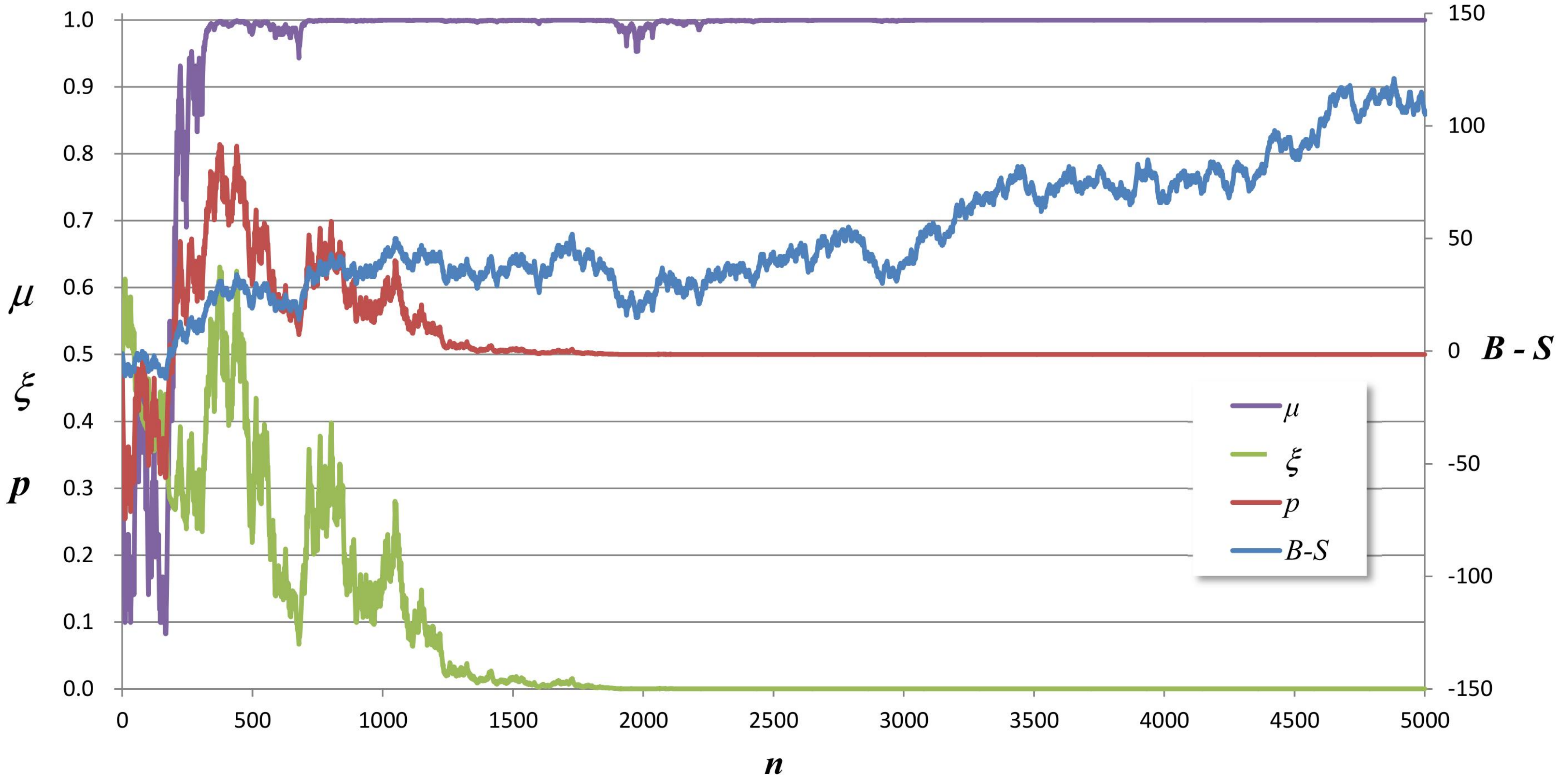


Figure 1 The transition of price by Bayesian updates when the market is uninformed.

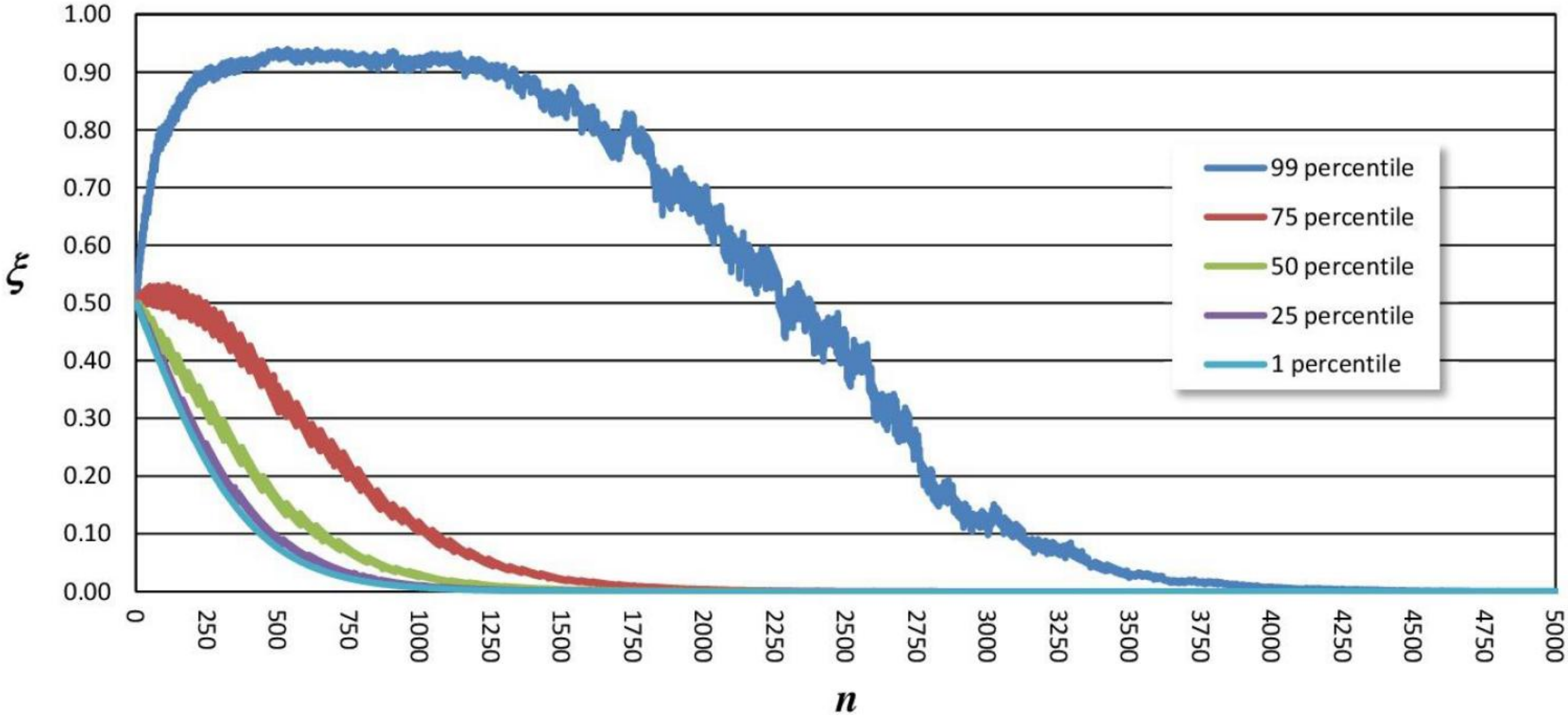


Figure 2 The transition of ξ_n by Bayesian updates when the market is uninformed.

6.Quasi-Bayesian update of market maker's belief

Remember, the update of $\{\xi_n\}$.

$$\xi_{n+1} = \begin{cases} \dfrac{\frac{1}{2} + \phi(\mu_n - \frac{1}{2})}{\frac{1}{2} + \phi(\mu_n - \frac{1}{2})\xi_n} \cdot \xi_n & \omega_{n+1} = B \\ \\ \dfrac{\frac{1}{2} - \phi(\mu_n - \frac{1}{2})}{\frac{1}{2} - \phi(\mu_n - \frac{1}{2})\xi_n} \cdot \xi_n & \omega_{n+1} = S \end{cases}.$$

	Bayesian update		Quasi-Bayesian
$P(\theta = 1 \vert M = I, h_n)$	: $\mu_n \rightarrow \mu_{n+1}$	$\Longrightarrow$	$\mu_n \rightarrow \mu_{n+1}$
	$\searrow$		$\downarrow$
$P(M = I \vert h_n)$	: $\xi_n \rightarrow \xi_{n+1}$		$\widetilde{\xi}_n \rightarrow \widetilde{\xi}_{n+1}$

We define Quasi-Bayesian update as follows by replacing μ_n to μ_{n+1}:

$$\widetilde{\xi}_{n+1} = \begin{cases} \dfrac{\frac{1}{2} + \phi(\mu_{n+1} - \frac{1}{2})}{\frac{1}{2} + \phi(\mu_{n+1} - \frac{1}{2})\widetilde{\xi}_n} \cdot \widetilde{\xi}_n & \omega_{n+1} = B \\ \\ \dfrac{\frac{1}{2} - \phi(\mu_{n+1} - \frac{1}{2})}{\frac{1}{2} - \phi(\mu_{n+1} - \frac{1}{2})\widetilde{\xi}_n} \cdot \widetilde{\xi}_n & \omega_{n+1} = S \end{cases},$$

where $\widetilde{\xi}_0 = \xi_0$.

We have the following propositions and theorems.

Proposition 9:

$\xi_n < \widetilde{\xi}_n$. for any $\omega, n \geq 1$.

This proposition states that the probability of $M = I$ is always higher for Quasi-Bayesian updates than for Bayesian updates.

Proposition 10:

$\widetilde{\xi}_n$ *depends on the order of buys and sells.*

We have the following inequalities of paths where the number of buys are the same as that of h_n.

$$\begin{cases} \widetilde{\xi}_n(BB\cdots BSS\cdots S) \le \widetilde{\xi}_n(h_n) \le \widetilde{\xi}_n(BSBS\cdots BSB\cdots B) & B_n \ge S_n \\ \widetilde{\xi}_n(SS\cdots SBB\cdots B) \le \widetilde{\xi}_n(h_n) \le \widetilde{\xi}_n(SBSB\cdots SBS\cdots S) & B_n \le S_n \end{cases}.$$

For example: $\widetilde{\xi}_5(BSBSB) > \widetilde{\xi}_5(BBSSB) > \widetilde{\xi}_5(BBSBS) > \widetilde{\xi}_5(BBBSS)$

Theorem 11: *When the market is uninformed, there exists a function $m(n), n \in N$ that has the following three properties:*

1. $\mathbf{E}[\widetilde{\xi}_n] > m(n) \quad for\ all\ n \in \mathbf{N}$,

2. $m(0) = \xi_0, \ \lim_{n \to \infty} m(n) = 0$,

3. $m(n)$ *has a local maximum in* $0 < n < \infty$.

This theorem states that we have a local maximum for $E[\widetilde{\xi}_n]$, indicating that the probability of $M = I$ increases erroneously initially even though $M = U$.

On the contrary, $E[\xi_n]$ monotonically converges to 0.

Theorem 12:

If the market is informed, $\widetilde{\xi}_n$ *diverges to* 1 *a.s., and if the market is uninformed,* $\widetilde{\xi}_n$ *converges to* 0 *a.s.*

This theorem means that $\widetilde{\xi}_n$ finally converges to the true value.

Let us define asset price $\widetilde{p}_n$:

$$\widetilde{p}_n = \widetilde{\xi}_n \mu_n + \frac{1}{2}(1 - \widetilde{\xi}_n).$$

Then, we have the following theorem 13.

Theorem 13:

The probability that the market is efficient with respect to price $\widetilde{p}$ converges to 1 a.s.

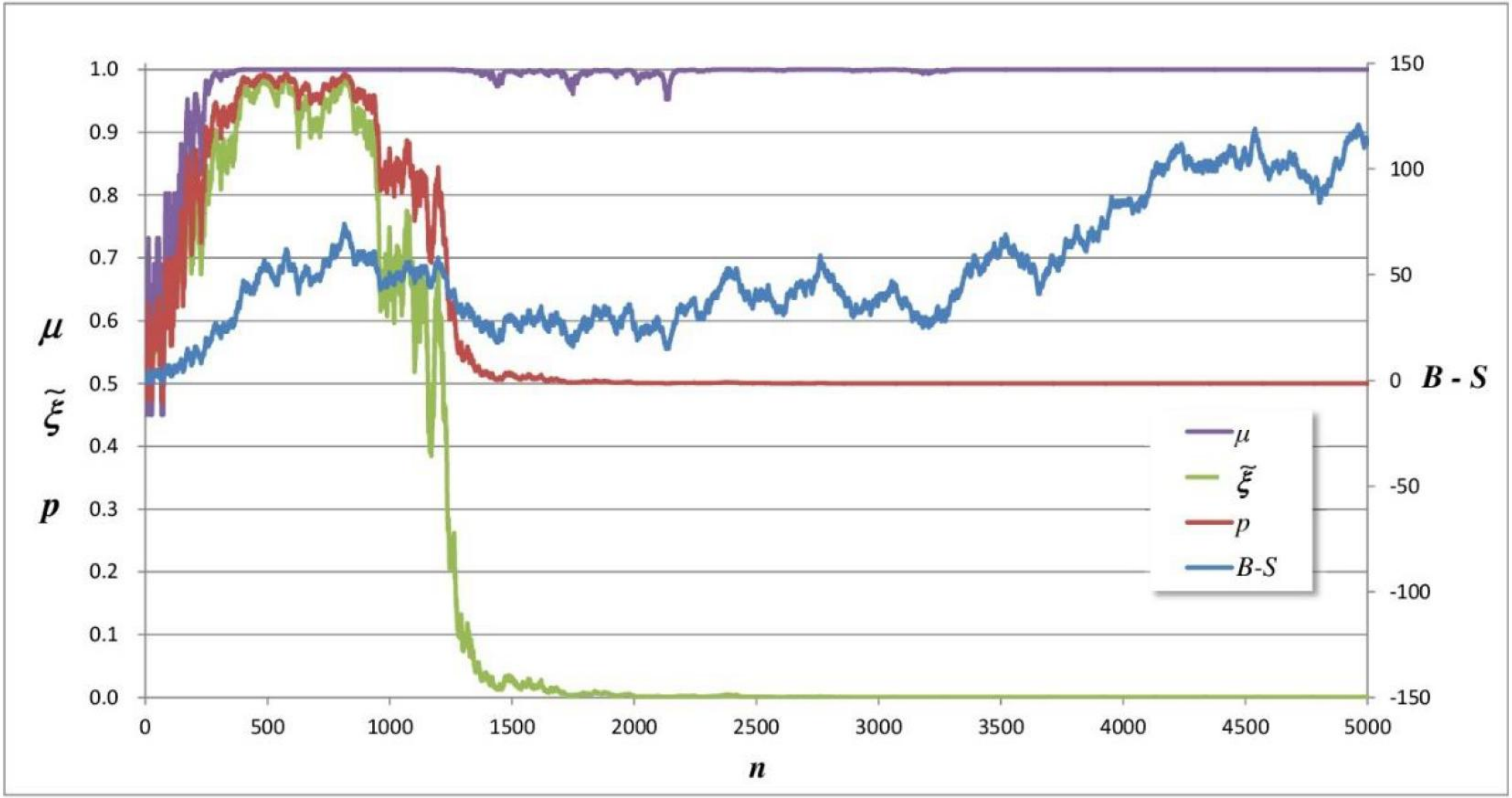


Figure 3 The transition of Price by the Quasi-Bayesian updates when the market is uninformed.

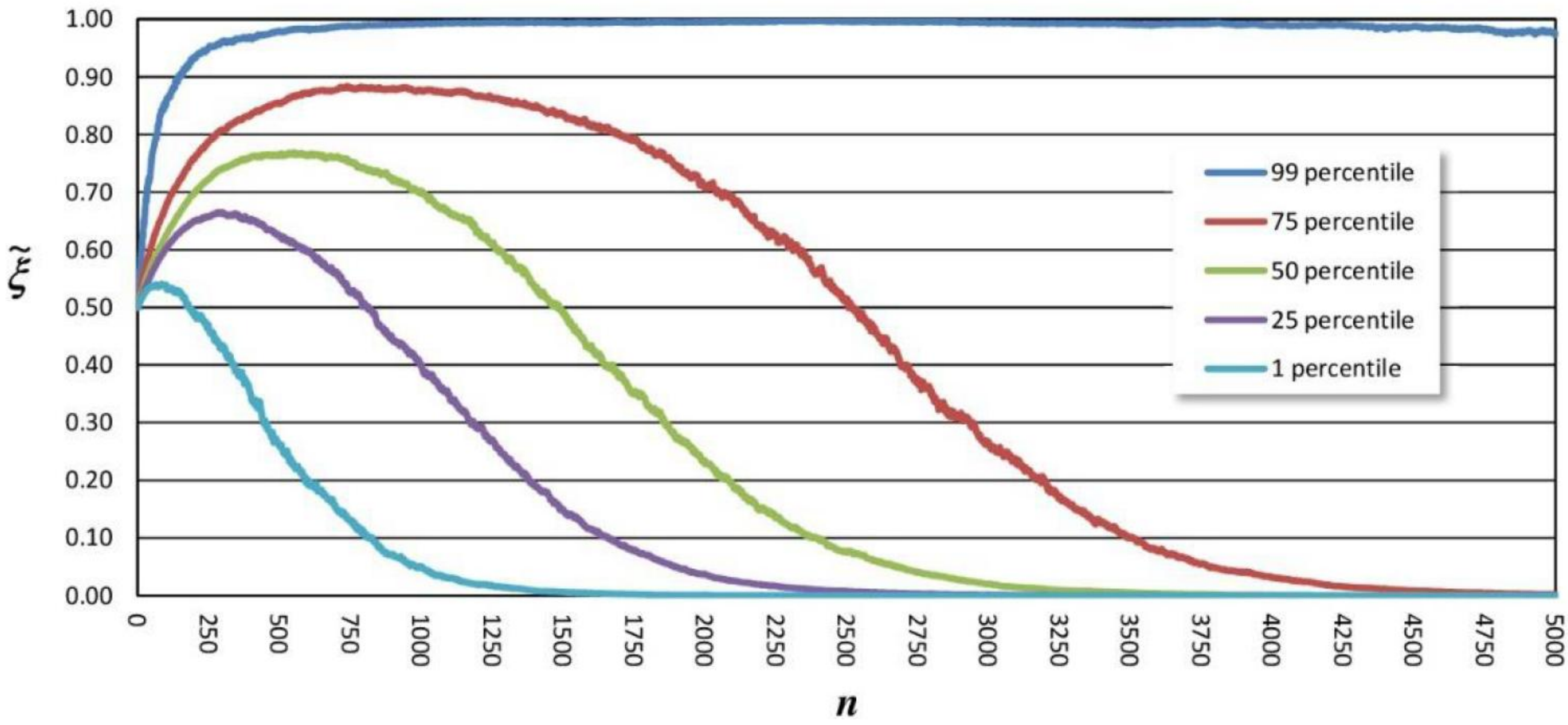


Figure 4 The transition of $\widetilde{\xi}_n$ by the Quasi-Bayesian updates when the market is uninformed.

7. Experiments

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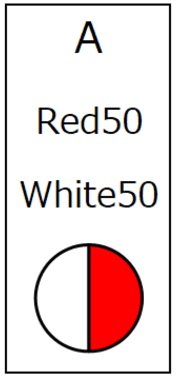

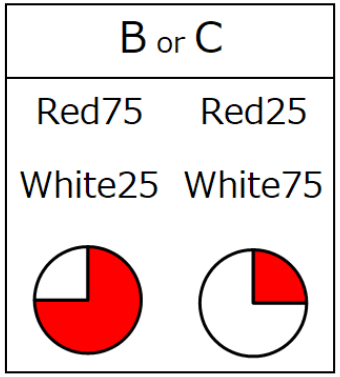

Round	Picked ball			Probability of box "B or C"												Box		
1	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
2	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
3	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
4	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
5	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
6	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
7	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
8	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
9	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C
10	Red	White		0%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%		A	B	C

Figure 5　Answer sheet used in experiments

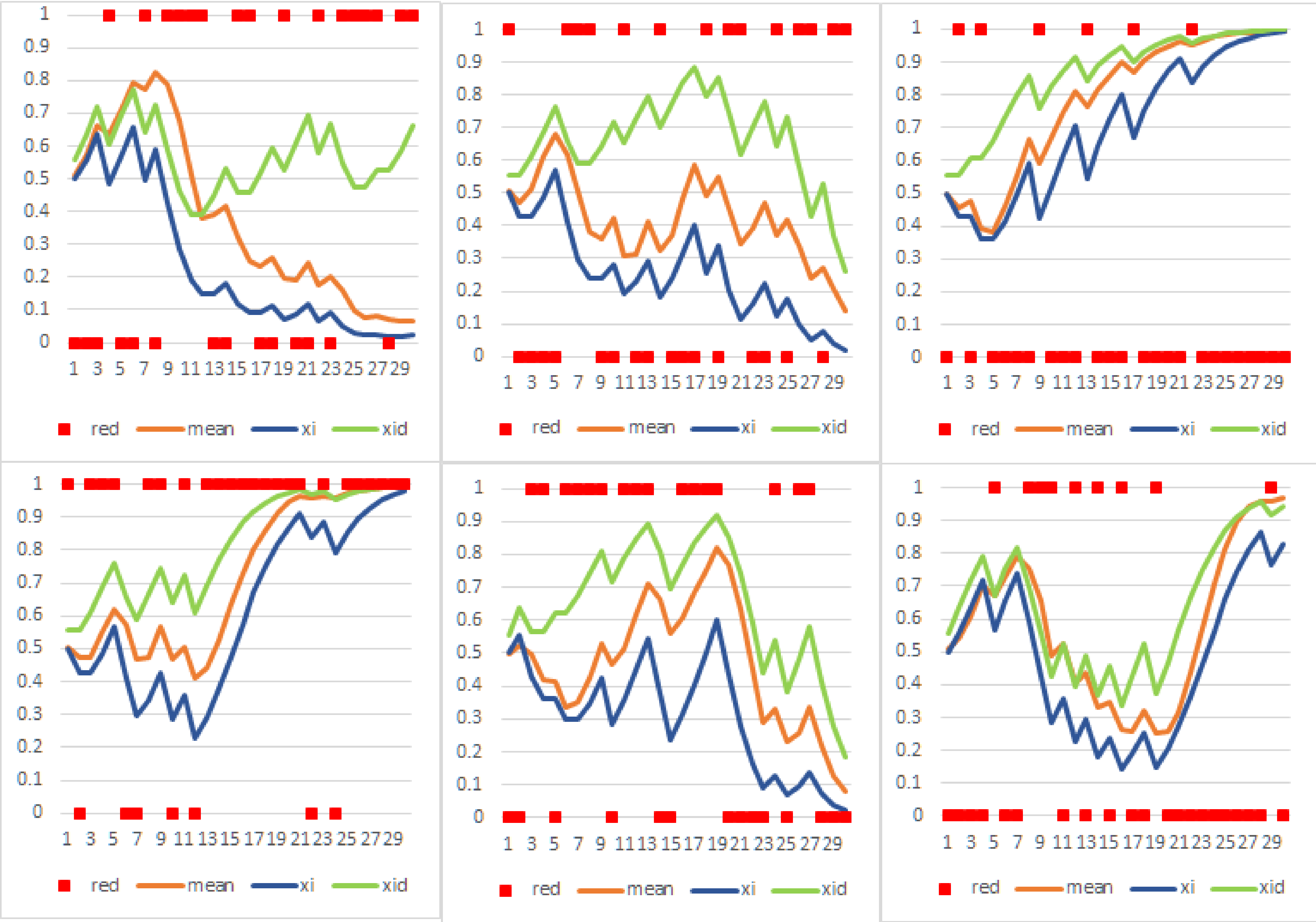

Figure 6 Results of the experiment

8. Conclusion

In this study, we investigate the belief update process of market makers regarding risky asset.

We assume two uncertainties in stock markets: uncertainty of asset value and that of existence of informed traders.

By using the sequential trading model, we first show that the updating process for the asset value μ_n and the existence of informed traders ξ_n are consistent with the Bayesian theory.

Next, we introduce the quasi-Bayesian update model which investors tend to apply to their belief updating.

We find that

(1) the probability of the existence of informed traders by the quasi-Bayesian update model $\widetilde{\xi}_n$ is always higher than that of Bayesian model,

(2) $E[\widetilde{\xi}_n]$ rises erroneously initially even though there is no informed trader in the market,

(3) $\widetilde{\xi}_n$ converges to the true value regarding the existence of informed traders and the market becomes efficient.

We experimentally investigate the belief update process of traders and find that traders’ behavior is consistent with the quasi-Bayesian update model.