Backward Discounting

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1 Motivation

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Motivation

- Propose a new theory of time-inconsistent preferences based on two central ingredients
  - Agents explicitly consider past outcomes in current lifetime utility
  - Agents explicitly consider utility of future selves when making current decisions
- Novel predictions with empirical support
- Use the model to analyze standard consumption savings problem, as well as other applications
  - Addictive behaviour, evolutionary fitness, elections, social discounting
Why consider \textit{backward discounting}? 

- Backward discounting + weight on future selves $\Rightarrow$ sharp form of time inconsistency...
  
  - U-shaped profile of rates of impatience
  
  - Hyperbolic models yield monotone profile

- \textbf{Key point} - Can’t be reduced to model with purely geometric discounting
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Model

Standard setting

- Time \( \in [0, T] \)
- Consumption stream for agent, \( \{c_t\}_{t=0}^T \)
- \( u(c) \) instantaneous utility function

Time-0 value

\[
\int_0^T d(s)u(c_s)ds
\]

- \( d(s) \) - effective discount factor
Backward Discounting

Postulate 1 - agents discount future streams, as well as past streams, in current utility

- Date $t$ lifetime utility

$$
\int_0^t e^{-\rho_b(t-s)}u(c_s)ds + \int_t^N e^{-\rho_f(s-t)}u(c_s)ds
$$

- $\rho_f, \rho_b$ forward and backward discount rates resp.
Different Selves

Postulate 2 - agents explicitly place weight on lifetime utility of future selves

- Today, focus on simple two-weight version, as well as $\rho_b = \rho_f$
- Place weight $\alpha$ on current self $t$, $1 - \alpha$ on some future self $T < N$
  - $T$ will interpreted as shadow parent, or retirement self
  - In paper, allow for very general weighting schemes - weight placed on all selves, past selves, allowing weights to be time-varying, etc
- Adjusted $t$-self lifetime utility:

$$\alpha \int_0^N e^{-\rho |t-s|} u(c_s) ds + (1 - \alpha) \int_0^N e^{-\rho |T-s|} u(c_s) ds$$
Rates of Impatience

- Formally, define

\[
i(t, s) = \lim_{\epsilon \to 0} \ln \left[ \frac{d(t, s)}{d(t, s + \epsilon)} \right] = -\frac{d_s(t, s)}{d(t, s)}
\]

- \( i(t, s) \) - local rate of impatience at \( s \) from the date \( t \) viewpoint
  - Standard model - \( i(t, s) = \rho \)
  - Hyperbolic discounting - \( i(t, s) \) decreasing in \( s \)
Rates of Impatience

At all pre-retirement ages $t < T$, and for $s \in [t, T)$,

$$i(t, s) = \left[ \frac{\rho_f \alpha e^{-\rho_f(s-t)} - \rho_b (1 - \alpha) e^{-\rho_b(T-s)}}{\alpha e^{-\rho_f(s-t)} + (1 - \alpha) e^{-\rho_b(T-s)}} \right]$$

For $s \geq T$,

$$i(t, s) = \left[ \frac{\rho_f \alpha e^{-\rho_f(s-t)} + \rho_f (1 - \alpha) e^{-\rho_f(T-s)}}{\alpha e^{-\rho_f(s-t)} + (1 - \alpha) e^{-\rho_f(T-s)}} \right]$$

- For $s \in [t, T)$, conflict between $t$ and $T$ selves
- $T$ self values dates increasingly in $s$, converse for $t$ self
Figure 1: Local and Instantaneous Rates of Impatience for $t = 30$, $\rho_f = \rho_b = 0.02$, $\beta = 0.3$, $\omega = 0.001$ and Various Values of $\alpha$. 

(a) $i(t, s)$, various $s$  
(b) $i(t, t)$, various $t$
Testable Implications

Theorem 1

1. For $t < T$, $i(t,s)$ is decreasing in $s$ for $s \in (t,T]$

2. For each $t < T$, $i(t,s)$ jumps up as $s$ crosses $T$

3. $i(t,t)$ is decreasing in $t$, and jumps up as $t$ crosses $T$

4. For $t > T$, $s > t$, $i(t,s) = \rho$

- (1) - standard present-bias time-inconsistency

- (3), (4), (5) - past retirement age, conflict between different selves disappear, return to standard geometric discounting

- Plan to make sacrifices in middle age, enjoy post-retirement
Testable implications - Evidence

Novel model predictions

- Increased patience across immediate choices into middle age, decreases post-retirement
  - Harrison et al 2002, Read et al 2004

- Younger people discount hyperbolically, older discount geometrically
  - Read et al 2004, Green et al 1994
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Consumption-Savings

Embed model into standard consumption-savings problem

- $u(c) = \ln c$
- Flow income $y_s$ per period (no uncertainty)
- Constant interest rate $r$ on borrowing/lending
- $A_s$ denotes total wealth in period $s$
  - If $F_s = \text{financial wealth}$, and $M_s = \int_s^N e^{-r(\tau-s)} y_\tau d\tau$ the present value of future income earnings, then $A_s = F_s + M_s$
  - $A_s$ evolves according to $\dot{A}_s = rA_s - c_s$
Planned Consumption

Naive agent

- At each date $t$, agent solves date $t$ problem, assuming future selves will honor current plan
  - Commitment versus equilibrium solutions. Look for solution to time 0 problem (details)
Proposition 2

Theorem 2

The optimal consumption profile at date 0 satisfies

\[ c_t(A) = \left[ \frac{\alpha e^{-\rho t} + (1 - \alpha) e^{-\rho |T-t|}}{\alpha e^{-\rho t} a_t + (1 - \alpha) p_t} \right] A \equiv \lambda_t A \]  

\[ a_t = \rho^{-1} \left[ (\rho - 1)e^{-\rho(N-t)} + 1 \right] \]  

\[ p_t = \rho^{-1} e^{-\rho (t-T)} \left[ (\rho - 1)e^{-\rho(N-t)} + 1 \right] \]  

\[ = \rho^{-1} \left\{ [(\rho - 1)e^{-\rho(N-T)} + 1] + [1 - e^{-\rho(T-t)}] \right\} \]  

for \( t > T \)  

for \( t < T \)
Benchmarking

- To compare solution to standard model, set $\alpha = 1$
  - $\bar{\lambda}_t = \frac{1}{a_t}$

- Now form the ratio $\theta_t = \frac{\lambda_t}{\bar{\lambda}_t}$
  - If $\theta_t < 1$, then planned saving greater than standard

**Theorem 3**

For $t < T$, $\theta_t < 1$. For $t \geq T$, $\theta = 1$. Furthermore, there exists $\hat{\alpha} \in (0, 1]$ such that if $\alpha \leq \hat{\alpha}$, $\theta_t$ always increases in $t$; while if $\alpha > \hat{\alpha}$, $\theta_t$ first decreases and then increases in $t$.

- Before retirement, agent saves more than in standard model. Afterwards, same.

- For high enough weight on shadow parent, agent does bulk of saving in middle age.
Sophisticated agent

- Solution takes time-inconsistency into account, i.e. time $t$ agent takes into account decisions of future agents

- Standard approach in discrete time - model problem as a game, in which separate agents at each instant $t$ make consumption choices, solve via backward induction

- But how to model game in continuous time?
  
  - Hard to interpret - each agent controls an instant, choice of $c$ affects nothing

- Take a novel approach...
Equilibrium Consumption

Sophisticated agent

- Break \([0, N]\) into sub-intervals length \(\Delta\). Assume one agent controls each, acts as “mini-planner”
  - Suppose agent controlling \([t, t + \Delta]\) chooses \(\{c_s\}_{s=t}^{t+\Delta}\) under constraint that \(A_{t+\Delta} = \hat{A}\)
  - Solve for optimal control \(\{c_t(A_s, s : \hat{A})\}_{s=t}^{t+\Delta}\) as above - let \(U_t(A, \hat{A})\) denote value of optimal control to this agent

- Induces a standard game with finitely many players. Solve via backward induction. Looks like \(J_t(A) = \max_{\hat{A}} U_t(A, \hat{A}) + e^{-\rho\Delta} J_{t+\Delta}(\hat{A})\)

- Combine \(\{c_t(A_s, s : \hat{A})\}_{s=t}^{t+\Delta}\) and optimal \(\hat{A}\) to solve for rates of consumption at \(t\) - \(c_t^\Delta(A)\),

- Define equilibrium of original game to be profile obtained by \(\lim_{\Delta \to 0} c_t^\Delta(A)\)
Equilibrium Consumption

**Theorem 4**

The equilibrium consumption profile satisfies

\[ c_t^*(A) = \left[ \frac{\alpha + (1 - \alpha)e^{-\rho|T-t|}}{\alpha a_t + (1 - \alpha)p_t} \right] A_t \equiv \lambda_t^* A \quad (5) \]

where \( a_t, p_t \) satisfy (2) and (3).

**Theorem 5**

1. \( \theta_t, \theta_t^* < 1 \) for each \( t < T \).
2. For dates \( t \geq T, \theta_t, \theta_t^* = 1 \).
3. In both the planning and equilibrium problems, \( \theta_t, \theta_t^* \) are increasing in \( \alpha \).
4. \( \theta_t^* > \theta_t \) for each \( t < T \). Furthermore, \( \theta_t^* \) rise monotonically over time, whereas \( \theta_t \) may be U-shaped.
(a) Planned and Equilibrium $\theta_t$, $\rho = 0.05$, $t = 30$, $N = 80$, $T = 65$, $r = 0.03$ and $\alpha = 0.5$. 

(b) Equilibrium $\theta_t$, varying $\alpha$
Predictions

Countries with greater inter-generational linkages have higher savings rates

- Eye-balling data on East-Asian countries vs other OECD seems in line
- 1980-2013 savings rates: Japan, S Korea, China around 30, whereas UK, US, France, Germany around 15
  - Interest rates much lower in former countries

Naive vs sophisticated

- Naive show U-shaped savings rates, sophisticates monotone
- Testable?
Summary

- Model of time-preferences, in which agents
  - Backward discount
  - Weight on future utilities
- Generates novel implications with empirical support
- Embed preferences into standard life-cycle model
- Going forward...
  - Infinite horizon, uncertainty
  - Policy implications (designing $\alpha$)
Planned Consumption

Naive agent

- At each date \( t \), agent solves date \( t \) problem, assuming future selves will honor current plan

  - Commitment versus equilibrium solutions. Look for solution to time 0 problem (details)

- Define value functions \( V(A, t) \), \( W(A, t) \) as

\[
V(A, t) = \int_t^N e^{-\rho(s-t)} \ln(c_s) ds, \quad W(A, t) = \int_t^N e^{-\rho|T-s|} \ln(c_s) ds
\]

where \( \{c_s\} \) is the optimal plan
Planned Consumption

- Sup value of time $t$ problem, viewed from time 0 is

\[ e^{-\rho t} \alpha V(A, t) + (1 - \alpha)W(A, t) \]

- Use this to write time $t$ problem in standard form

\[
0 = \sup_{c_t} \alpha e^{-\rho t} \left[ \ln c_t + \dot{A}_t V_A(A, t) + V_t(A, t) - \rho V(A, t) \right] \\
+ (1 - \alpha) \left[ e^{-\rho |T-t|} \ln c_t + \dot{A} W_A(A, t) + W_t(A, t) \right]
\]

where $\dot{A}_t = rA_t - c_t$