A Theory of Dissimilarity Between Stochastic Discount Factors*

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Abstract

This paper proposes a measure of dissimilarity between stochastic discount factors (SDFs) in different economies. The SDFs are made comparable using the respective bond prices as the numeraire. The measure is based on a probability distance metric, is dimensionless, synthesizes features of the risk-neutral distribution of currency returns, and can be extracted from currency option prices. Linking theory to data, our empirical implementation reveals a salient geographical pattern in dissimilarity across 45 pairs of industrialized economies. We compare the dissimilarity between SDFs derived from several international asset pricing models to the empirical analog, offering a dimension to gauge models.

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1. Introduction

The idea of this paper is to present a measure of dissimilarity between stochastic discount factors (SDFs) in different economies, and we use the discount bond prices as the numeraire to ensure comparability. This measure reflects the distinction between economy-specific Radon-Nikodym derivatives (change of probability), which is also a SDF with unit expectation and enables the correct pricing of gross returns scaled by the discount bond price.

In economic terms, the measure distills the heterogeneity among investors in different economies with respect to their dislike of unfavorable outcomes. The measure is not denominated in any currency unit and is dimensionless. The notion of dissimilarity may appear opaque and eclectic, but, as we show, can be rendered tractable with the help of the proposed measure.

1.1. What is our motivation for pushing the angle of dissimilarity between SDFs?

There are *a priori* reasons to think that SDFs are not homogeneous across international borders. Economies are separated by physical distance and inhabited by consumers with different consumption baskets, while investors have different wealth levels, tastes, preferences for resolution of uncertainty and risk aversions, and differential capacities to absorb adverse economic shocks (e.g., Borovička, Hansen, and Scheinkman (2016)). Still, a formal measure of dissimilarity between SDFs is lacking and not yet assimilated in international finance research.

We formalize a measure of dissimilarity with three properties: bounded between zero and one, symmetric, and satisfies triangle inequality. It is based on the Hellinger *probability distance metric* and, to our knowledge, has not been introduced in the context of comparing SDFs.

The measure can be employed to rank-order dissimilarity across various pairs of SDFs, and can address questions as: Is the Japanese SDF more dissimilar from that of Australia than that of

Canada? How does dissimilarity vary over time and with economic conditions? Do economies get closer together or farther apart during periods with adverse economic shocks?

1.2. *What is our approach to compute dissimilarity between SDFs?*

One novelty in our paper is to show how the Hellinger measure of dissimilarity between SDFs, or simply "Hellinger measure," as we will denote it henceforth for brevity, can be extracted from currency option prices under the specification of complete markets (the Radon-Nikodym derivative is unique). Our Proposition 1 establishes that the Hellinger measure can be inferred from a portfolio of options and can be updated from one period to the next.

Focusing on the economic interpretation, we further show that the Hellinger measure synthesizes risk-neutral moments of (excess) currency returns, and thus goes conceptually beyond reconciling their expectation (the currency risk premium). This interpretation is important in light of the fact that international pricing models are often calibrated and simulated, and thus a case could be made that the Hellinger measure should be an essential criterion for any parameterized international economy driven by a system of SDFs. A new question that needs to be addressed is whether an international pricing model is consistent with Hellinger measures extracted from the prices of traded currency options.[1] It is customary to characterize an international macro-finance model by its specification of the SDFs.

1.3. *What are the lessons of depicting dissimilarity for international macro-finance?*

The empirical work in this paper employs forward and spot exchange rates, as well as a panel of currency option prices to generate time series of Hellinger measures for 45 pairs of economies. Among several insights, we detect geographical patterns across these measures, which implies

[1]While we also explore using the Chi-squared distance as an alternative probability distance metric, we highlight the advantage of the Hellinger metric for interpretational and modeling reasons, and focus on it throughout.

differences in the Radon-Nikodym derivatives. For example, Japan stands apart from the remaining economies. In addition, we discern a factor structure in the Hellinger measures.

Besides computing and studying empirically Hellinger measures, we also investigate whether some extant SDF parameterizations respect the unconditional estimates of these measures. From this perspective, the Hellinger measure can be considered, in the spirit of Hansen and Jagannathan (1991), but in a two-economy context, as a diagnostic seeking consistency of an international asset pricing model with the risk-neutral distribution of currency returns. We further consider a minimum discrepancy problem in an international setting, revealing that our dissimilarity measures are consistent with the data on bond and equity index returns. Going beyond formal models, we additionally show that physical distance and cultural differences describe cross-sectional variations in dissimilarity, as embedded in the risk-neutral distribution of currency returns.

1.4. What is our value-added and how does it tie with extant research agenda's?

Our work is related to a strand of literature aiming to develop model specifications that are aligned with observed properties of interest rates, currency risk premiums, currency volatilities, and equity returns, as well as the evidence on risk sharing.[2]

Previous work has also explored various characteristics along which economies are dissimilar, such as magnitude of risk reversals, size, or resilience to external shocks. For example, Carr and Wu (2007) show that risk reversals capture skewness that changes signs, whereas Farhi and Gabaix (2016) associate reversals with country risk. Hassan (2013) argues that size is useful for explaining interest rate differentials across economies. On the other hand, Farhi and Gabaix

[2] These include Hollifield and Uppal (1997), Backus, Foresi, and Telmer (2001), Brandt, Cochrane, and Santa-Clara (2006), Pavlova and Rigobon (2007), Bakshi, Carr, and Wu (2008), Brunnermeier, Nagel, and Pedersen (2009), Verdelhan (2010), Colacito and Croce (2011, 2013), Bansal and Shaliastovich (2013), Heyerdahl-Larsen (2014), Jurek (2014), Lustig, Roussanov, and Verdelhan (2014), Favilukis, Garlappi, and Neamati (2015), Farhi and Gabaix (2016), Bakshi, Cerrato, and Crosby (2017), Maurer and Tran (2016, 2017), Colacito, Croce, Gavazzoni, and Ready (2017), Ready, Roussanov, and Ward (2017), and Zviadadze (2017), among many others.

(2016) focus on the time variation of an economy's exposure to global shocks, which is reflected in exchange rate fluctuations. The analysis of Brandt, Cochrane, and Santa-Clara (2006) implies that SDFs must be highly correlated pairwise. In addition, Bakshi, Cerrato, and Crosby (2017) offer a model to show that the unspanned components of the SDFs are distinct across economies.

Our measure of dissimilarity — which underscores the distinction between the economy-specific change of probability measures — differs from the many facets of dissimilarity inherent in economic concepts of information, gravity, network centrality, and cultural distinctions. In the two-country model of Van Nieuwerburgh and Veldkamp (2009), investors choose to have different information sets, allowing to address puzzles related to investment choices. In the vein of gravity models (e.g., Chaney (2017)), Lustig and Richmond (2015) consider a model of SDFs that can reproduce a distance-dependent factor structure, and find that distance matters for co-variation in exchange rate growth. The study of Richmond (2015) shows that countries that are more central in the global trade network are associated with lower interest rates and currency risk premiums. Karolyi (2016) examines the role of cultural disparities for reconciling international finance phenomena, including foreign/home bias in international portfolio holdings. Diebold and Yilmaz (2015) adopt a network approach to explore financial and macroeconomic connectedness.

However, our paper takes a novel angle and in contrast to previous work, we focus on formalizing and gauging the dissimilarity between SDFs. It aims to contribute by proposing and implementing a measure of dissimilarity that can be calculated from currency option prices. Through our measure, differences between SDFs (whether attributable to risk aversion, taste, technologies, culture, or other economic primitives) can be quantified and encapsulated into a single number. This number is comparable across economies and over time, and offers a way to distinguish between international pricing models. In essence, our work puts the notion of dissimilarity between SDFs on a theoretical footing and, thus, renders it a less amorphous concept.

2. A measure of dissimilarity between SDFs

This section formalizes a measure of dissimilarity between the SDFs of two economies, which is based on the Hellinger *probability distance metric*. The use of bond prices as the numeraire enables comparability. We justify our choice of the measure of dissimilarity, which we call the Hellinger measure, on theoretical and practical grounds.[3]

Additionally, we provide examples illustrating the economic nature of the Hellinger measure and identify a link to the risk-neutral distribution of currency returns when markets are complete. We show how the proposed measure can be extracted from currency option prices. The developed theory and formulations are instrumental to our exploration of international macro-finance models.

2.1. Formalizing dissimilarity

Different from Hansen and Jagannathan (1991) who develop a theory of SDFs in a single-country setting, Lucas (1982), Saá-Requejo (1994), and Backus, Foresi, and Telmer (2001) identify a relation between SDFs in a two-country setting, based on exchange rate growth. If M_{t+1} and M_{t+1}^* are the *nominal* pricing kernels in the domestic and foreign country, then let $m_{t+1} \equiv \frac{M_{t+1}}{M_t}$ ($m_{t+1}^* \equiv \frac{M_{t+1}^*}{M_t^*}$) denote the strictly positive domestic (foreign) SDF over t to $t+1$.

Let S_t be the level of the spot exchange rate, defined as the number of units of domestic currency per one unit of foreign currency. The foreign currency is the reference.

Assume also that $\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^2) < \infty$ and $\mathbb{E}_t^{\mathbb{P}}((m_{t+1}^*)^2) < \infty$, where $\mathbb{E}_t^{\mathbb{P}}(.)$ is time t conditional expectation under the physical probability measure $\mathbb{P}$. We consider m_{t+1} (m_{t+1}^*) that enforce correct pricing of domestic (foreign) returns with $\mathbb{E}_t^{\mathbb{P}}(m_{t+1}\mathbf{R}_{t+1}) = \mathbf{1}$ ($\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*\mathbf{R}_{t+1}^*) = \mathbf{1}$), where

[3] While the analysis is focused on characterizing the dissimilarity between *nominal* SDFs, we show that the framework can be adapted to also characterize the dissimilarity between *real* SDFs.

the domestic and foreign gross return vectors are linked as $\mathbf{R}_{t+1} = (\frac{S_{t+1}}{S_t})\mathbf{R}^*_{t+1}$. The gross return $R_{f,t+1}$ and $R^*_{f,t+1}$ of a domestic and foreign risk-free bond, respectively, satisfy $\frac{1}{R_{f,t+1}} = \mathbb{E}^{\mathbb{P}}_t(m_{t+1})$ and $\frac{1}{R^*_{f,t+1}} = \mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})$. By Cauchy-Schwarz, $\left|\mathbb{E}^{\mathbb{P}}_t(\sqrt{m_{t+1}}\sqrt{m^*_{t+1}})\right| < \sqrt{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})\,\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})} < \infty$.

Under no-arbitrage, it is shown in Backus, Foresi, and Telmer (2001, Proposition 1) that[4]

$$m_{t+1}\left(\frac{S_{t+1}}{S_t}\right) \;=\; m^*_{t+1}. \tag{1}$$

Associated with the exchange rate S_t in equation (1) is the one-period forward exchange rate, denoted by F_t, which satisfies $\frac{F_t}{S_t} = \frac{\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})}{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})} = \frac{R_{f,t+1}}{R^*_{f,t+1}}$, from the perspective of the domestic investor. Likewise, $\frac{F_t^{-1}}{S_t^{-1}} = \frac{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})}{\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})} = \frac{R^*_{f,t+1}}{R_{f,t+1}}$, from the perspective of the foreign investor.

Although we focus here on a discrete-time environment, our framework can be adapted to a continuous-time environment (e.g., Brandt, Cochrane, and Santa-Clara (2006)). The work of Maurer and Tran (2016, 2017) introduces the notion of risk entanglement in a setting that incorporates return jump diffusions and continuous trading, and shows that the exchange rate growth is equal to the ratio of economy-specific SDFs projected onto their respective asset return spaces, if and only if risks, which affect the conditional exchange rate growth, are completely disentangled.

To formalize the notion of m_{t+1} and m^*_{t+1} being dissimilar, we consider the transformed variables:

$$\widetilde{n}_{t+1} \equiv \frac{m_{t+1}}{\underbrace{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})}_{\text{domestic bond price is the numeraire}}} \qquad \text{and} \qquad \widetilde{n}^*_{t+1} \equiv \frac{m^*_{t+1}}{\underbrace{\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})}_{\text{foreign bond price is the numeraire}}}, \tag{2}$$

[4]Equation (1) underpins much of the work on characterizing exchange rates and their dynamics, including Hollifield and Uppal (1997, equation (7)), Brandt, Cochrane, and Santa-Clara (2006, equation (1)), Brennan and Xia (2006, Section 1), Pavlova and Rigobon (2007), Bakshi, Carr, and Wu (2008, equation (3)), Alvarez, Atkeson, and Kehoe (2009, equation (10)), Verdelhan (2010, equation (3)), Burnside, Eichenbaum, Kleshchelski, and Rebelo (2011), Lustig, Roussanov, and Verdelhan (2011, 2014), Burnside and Graveline (2014, equation (5)), Colacito and Croce (2011, 2013), Maggiori (2013), Hassan (2013), Farhi and Gabaix (2016), and Daniel, Hodrick, and Lu (2017).

which are positive random variables with unit expectation and finite second moment. Since $\mathbb{E}_t^{\mathbb{P}}(m_{t+1})$ (respectively, $\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)$) represents the zero-coupon bond price in the domestic (respectively, foreign) economy, we have used the bond price as the numeraire to scale each SDF.

To see the economic interpretation of $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$, suppose further that the density under the physical probability measure is given by $p[\nu]$, where $\nu \in \mathbb{R}^{\mathbb{N}}$ summarizes the uncertainty to be resolved at date $t+1$. Let also $\mathbb{Q}$ and $\mathbb{Q}^*$ be risk-neutral probability (pricing) measures in the domestic and foreign economy, respectively, with underlying densities $q[\nu]$ and $q^*[\nu]$. Both $\mathbb{Q}$ and $\mathbb{Q}^*$ are absolutely continuous with respect to the measure $\mathbb{P}$, with change of probability Radon-Nikodym derivatives $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and $\frac{d\mathbb{Q}^*}{d\mathbb{P}}$ (e.g., Harrison and Kreps (1979)).

Recognize that $q[\nu] = \frac{p[\nu]\, m_{t+1}[\nu]}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}$ and $q^*[\nu] = \frac{p[\nu]\, m_{t+1}^*[\nu]}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)}$. Rearranging, $\widetilde{n}_{t+1} = \frac{m_{t+1}}{\mathbb{E}_t^{\mathbb{P}}(m_{t,t+1})} = \frac{q[\nu]}{p[\nu]}$ and $\widetilde{n}_{t+1}^* = \frac{m_{t+1}^*}{\mathbb{E}_t^{\mathbb{P}}(m_{t,t+1}^*)} = \frac{q^*[\nu]}{p[\nu]}$, and thus $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$ represent change of probability (i.e., $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and $\frac{d\mathbb{Q}^*}{d\mathbb{P}}$). The transformations in (2) impart a probability function interpretation of $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$.

Remark 1 *Equation (2) decouples the discounting and Radon-Nikodym derivative components of the SDF. The Radon-Nikodym derivatives, $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$, are unique when markets are complete.*

Remark 2 *Even though m_{t+1} or m_{t+1}^* are denominated in their own currency units, say, the U.S. dollar or Japanese yen, the $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$ are rendered dimensionless (probabilities are unitless objects) by scaling m_{t+1} and m_{t+1}^* with the respective discount bond prices.*

Remark 3 *$\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$ can each be viewed as an SDF, with two features. First, $\mathbb{E}_t^{\mathbb{P}}(\widetilde{n}_{t+1}) = 1$ and $\mathbb{E}_t^{\mathbb{P}}(\widetilde{n}_{t+1}^*) = 1$, that is, $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$ are martingales. Second, $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$ enforce correct pricing with $\mathbb{E}_t^{\mathbb{P}}(\widetilde{n}_{t+1}\mathbf{A}_{t+1}) = \mathbf{1}$ and $\mathbb{E}_t^{\mathbb{P}}(\widetilde{n}_{t+1}^*\mathbf{A}_{t+1}^*) = \mathbf{1}$, where $\mathbf{A}_{t+1} \equiv \mathbb{E}_t^{\mathbb{P}}(m_{t+1})\mathbf{R}_{t+1}$ and $\mathbf{A}_{t+1}^* \equiv \mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)\mathbf{R}_{t+1}^*$ are scaled gross return vectors (e.g., Cochrane (2005, Chapter 8.1)). Section 3.3 formulates and implements a minimum discrepancy problem in an international setting to study the properties of the extracted $\widetilde{n}_{t+1}$ and $\widetilde{n}_{t+1}^*$.*

These interpretations are crucial for our purposes, and allow us to build on concepts that quantify the distance between probability measures. Among many such approaches, we focus on the Hellinger distance metric (e.g., Pollard (2002, Chapter 3)), and exploit it in our context with respect to $\widetilde{n}_{t+1}$ and $\widetilde{n}^*_{t+1}$. Section 2.5 elaborates further on the rationale for this choice.

We define the measure of dissimilarity between SDFs as

$$
\begin{aligned}
H_t &\equiv \frac{1}{2}\mathbb{E}_t^{\mathbb{P}}\left(\left[\sqrt{\widetilde{n}_{t+1}} - \sqrt{\widetilde{n}^*_{t+1}}\,\right]^2\right), && (3)\\
&= \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}\left[\sqrt{\underbrace{\frac{q[\nu]}{p[\nu]}}_{\text{Ratio-of-densities}}} - \sqrt{\underbrace{\frac{q^*[\nu]}{p[\nu]}}_{\text{Ratio-of-densities}}}\right]^2 p(\nu)d\nu, && (4)
\end{aligned}
$$

and denote it throughout as the "Hellinger measure." If $q(v)$ dominates $q^*(v)$ in certain states (say, in the tails of the distribution due to higher risk aversion), then it will amplify H_t, irrespective of whether the markets *are complete or incomplete*.

The Hellinger measure in equation (3) satisfies the axioms of distance: (i) $H_t[a,b] > 0$, (ii) $H_t[a,b] = H_t[b,a]$, and (iii) $H_t[a,b] + H_t[a,c] \geq H_t[b,c]$ (the triangle inequality). In addition

$$H_t \in [0,1], \quad \text{whereby } \; H_t = 0 \; \text{ for } m_{t+1} = m^*_{t+1}. \tag{5}$$

We employ H_t to quantify the dissimilarity between the Radon-Nikodym derivatives of different economies.

An innovation to be highlighted shortly is that one can extract H_t (as in (3)) from currency option prices under a (minimal) set of assumptions.

Expanding in equation (3) as in $(\sqrt{\widetilde{n}_{t+1}} - \sqrt{\widetilde{n}^*_{t+1}}\,)^2 = \widetilde{n}_{t+1} + \widetilde{n}^*_{t+1} - 2\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}$, and noting that $\mathbb{E}_t^{\mathbb{P}}(\widetilde{n}_{t+1}) = \mathbb{E}_t^{\mathbb{P}}(\widetilde{n}^*_{t+1}) = 1$, we can further write

$$H_t = 1 - \mathbb{E}_t^{\mathbb{P}}\left(\sqrt{\widetilde{n}_{t+1}\,\widetilde{n}^*_{t+1}}\right) = 1 - \frac{1}{\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})\,\mathbb{E}_t^{\mathbb{P}}(m^*_{t+1})}}\mathbb{E}_t^{\mathbb{P}}\left(\sqrt{m_{t+1}\,m^*_{t+1}}\right). \tag{6}$$

Returning to equation (5), we clarify the null hypothesis of *zero* Hellinger measure. Recall that in complete markets $m_{t+1}\frac{S_{t+1}}{S_t} = m^*_{t+1}$, hence if $m_{t+1} = m^*_{t+1}$, then $\mathbb{E}_t^{\mathbb{P}}(m_{t+1}) = \mathbb{E}_t^{\mathbb{P}}(m^*_{t+1})$ and $H_t = 0$, implying $\frac{S_{t+1}}{S_t} = 1$. On the other hand, if $\widetilde{n}_{t+1} = \widetilde{n}^*_{t+1}$ then $\frac{m_{t+1}}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})} = \frac{m^*_{t+1}}{\mathbb{E}_t^{\mathbb{P}}(m^*_{t+1})}$ and hence $\frac{S_{t+1}}{S_t} - \frac{\mathbb{E}_t^{\mathbb{P}}(m^*_{t+1})}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})} = 0$ (point by point). Both of these versions of exchange rate determination do not appear to align with data realities, therefore we expect to empirically reject the hypothesis of zero Hellinger measure.

The Hellinger measure in equation (6) can be seen as distinguishing between SDFs via the moments and cross-moments of $\log(m_{t+1})$ and $\log(m^*_{t+1})$, of all orders:

$$\begin{aligned} H_t &= 1 - \mathbb{E}_t^{\mathbb{P}}\left(\exp\left(\frac{1}{2}\log(\widetilde{n}_{t+1}) + \frac{1}{2}\log(\widetilde{n}^*_{t+1})\right)\right), & (7)\\ &= -\mathbb{E}_t^{\mathbb{P}}\left(\hbar + \frac{\hbar^2}{2!} + \frac{\hbar^3}{3!} + \frac{\hbar^4}{4!} + \dots\right), \quad \text{and} & (8)\\ &\qquad \text{where } \hbar \equiv \frac{1}{2}\log(\widetilde{n}_{t+1}) + \frac{1}{2}\log(\widetilde{n}^*_{t+1}). & (9)\end{aligned}$$

To gain some insights on the information encoded in H_t, we offer an example.

Example 1 Consider a model of $\log(m_{t+1})$ and $\log(m^*_{t+1})$, in which the parameters Λ and Λ_* reflect potentially different sensitivity to common disasters:

$$\log\left(\frac{M_{t+1}}{M_t}\right) = -\mu - \sigma\varepsilon_{t+1} - \Lambda\sum_{i=N_t}^{N_{t+1}} z_i, \qquad \log\left(\frac{M^*_{t+1}}{M^*_t}\right) = -\mu_* - \sigma_*\varepsilon^*_{t+1} - \Lambda_*\sum_{i=N_t}^{N_{t+1}} z_i, \tag{10}$$

where μ, μ_*, σ, and σ_* are constants. The shocks ε_{t+1} and ε^*_{t+1} are each $\mathcal{N}(0,1)$ with covariance ρ,

N_t is a Poisson process with parameter λ, and $\{z_i\}$ is a sequence of independent random variables each distributed $\mathcal{N}(\mu_z, \sigma_z^2)$. Thus, $\sum_{i=N_t}^{N_{t+1}} z_i$ is a compound Poisson process (analogous to the one also considered in Bakshi, Carr, and Wu (2008, equation (8)) and Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan (2015, Section 2.1)). It can be shown that (see Internet Appendix I),

$$H_t = 1 - \exp\left(-\frac{1}{8}(\sigma^2 + \sigma_*^2) + \frac{1}{4}\sigma\sigma_*\rho + \lambda\{e^{d_0} - \frac{1}{2}e^{-\Lambda\mu_z + \frac{1}{2}\Lambda^2\sigma_z^2} - \frac{1}{2}e^{-\Lambda_*\mu_z + \frac{1}{2}\Lambda_*^2\sigma_z^2}\}\right), \qquad (11)$$

where $d_0 \equiv -\frac{1}{2}(\Lambda + \Lambda_*)\mu_z + \frac{1}{8}(\Lambda + \Lambda_*)^2\sigma_z^2$.

The Hellinger measure in this disaster model is not time-varying, yet can be useful for understanding cross-sectional variation across economies, in terms of σ, σ_*, ρ, λ, Λ, and Λ_*. ♣

2.2. *The Hellinger measure in a simple model with tail exposures*

Next, consider a state-space model in a two-date world with non-homogeneous SDFs. There are four states, indexed by $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$. The associated physical probabilities, denoted by $p[\omega]$, and SDFs are

$$p[\omega] = \begin{pmatrix} \frac{1 - \frac{(1-\upsilon)(1+\varsigma)}{2}}{2} \\ \frac{(1-\upsilon)\varsigma}{2} \\ \frac{(1-\upsilon)\varsigma}{2} \\ \frac{1 - \frac{(1-\upsilon)(1+\varsigma)}{2}}{2} \end{pmatrix}, \qquad m[\omega] = \begin{pmatrix} 1+\Lambda \\ 1 \\ 1 \\ 1-\Lambda \end{pmatrix}, \quad \text{and} \quad m^*[\omega] = \begin{pmatrix} 1+\Lambda_* \\ 1 \\ 1 \\ 1-\Lambda_* \end{pmatrix}, \qquad (12)$$

where $0 \leq \upsilon \leq 1$, $0 \leq \varsigma \leq 1$, $0 \leq \Lambda < 1$, $0 \leq \Lambda_* < 1$, and $\sum_\omega p[\omega_j] = 1$.

The parameter υ calibrates the exposure to the extremes, ω_1 represents the most unpleasant extreme state, and ω_4 the most desirable one, and they are of equal probability.

Our parametrization of $m[\omega]$ and $m^*[\omega]$ reflects the intuition that marginal utility is high (low) in the bad (good) state. By inflating (deflating) marginal utilities in the bad (good) state relative to the middle states, we keep the means of m and m^* equal to unity and their skewness to zero, which allows to isolate the effects of SDF volatility and kurtosis on the Hellinger measure.

Based on the assumed model structure, the variances of the domestic and foreign SDFs are $\Lambda^2\left(1-\frac{1}{2}(\varsigma+1)(1-\upsilon)\right)$ and $\Lambda_*^2\left(1-\frac{1}{2}(\varsigma+1)(1-\upsilon)\right)$, respectively, and are both increasing in the tail exposure υ. Besides, the kurtosis of the two SDFs is the same and equals $\frac{1}{1-\frac{1}{2}(\varsigma+1)(1-\upsilon)}$, which increases in ς and decreases in υ, but does not depend on Λ and Λ_*.

Under the additional assumption that $\Lambda > \Lambda_*$, namely, the domestic country (say, the US) is more heavily exposed to the common shock than the foreign country (say, Australia). The foreign currency depreciates (appreciates) in the bad (good) state. The mean exchange rate growth is $\sum_\omega p[\omega_j]\left(m^*[\omega]/m[\omega]-1\right)=\frac{\Lambda(\Lambda-\Lambda_*)(1+\upsilon-\varsigma(1-\upsilon))}{2\left(1-\Lambda^2\right)}$, which is positive when $\Lambda > \Lambda_*$. Thus, the foreign currency (with positive expected return) is relatively riskier, a trait that is featured in the treatments of Hassan (2013), Farhi and Gabaix (2016), and Ready, Roussanov, and Ward (2017).

In this model, the Hellinger measure is

$$H=\frac{1}{4}\left(2-\sqrt{(1-\Lambda)(1-\Lambda_*)}-\sqrt{(1+\Lambda)(1+\Lambda_*)}\right)(1+\upsilon-\varsigma(1-\upsilon)). \tag{13}$$

Therefore, H increases with the difference between Λ and Λ_*, which can be seen by setting $\Lambda=\Lambda_*+\hbar_0$ and taking a derivative with respect to $\hbar_0$. Moreover, increasing the probability of the extremes tends to increase the Hellinger measure, which is captured by the positive derivative of H with respect to υ. The means of the SDFs do not alter the Hellinger measure. This model can produce cross-sectional variation in H, depending on the absolute and relative tail exposures.

Searching for a simple insight, the Hellinger measure captures key sources of heterogeneity to which international macro-finance models can be calibrated across a system of economies.

2.3. *Extracting the measure of dissimilarity from currency option prices*

This subsection shows how to compute the Hellinger measure from the prices of traded options on foreign currency. Because the SDFs m_{t+1} and m^*_{t+1} are positive random variables and $\frac{m^*_{t+1}}{m_{t+1}}$ exists, it follows from the definition of the Hellinger measure in equation (6) that

$$H_t = 1 - \frac{1}{\sqrt{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})\,\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})}}\,\mathbb{E}^{\mathbb{P}}_t\left(m_{t+1}\sqrt{\frac{m^*_{t+1}}{m_{t+1}}}\right), \tag{14}$$

$$= 1 - \frac{1}{\sqrt{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})\,\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})}}\,\mathbb{E}^{\mathbb{P}}_t\left(m_{t+1}\sqrt{\frac{S_{t+1}}{S_t}}\right), \quad \text{(via equation (1))} \tag{15}$$

$$= 1 - \sqrt{\frac{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})}{\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})}}\,\mathbb{E}^{\mathbb{Q}}_t\left(\sqrt{\frac{S_{t+1}}{S_t}}\right), \tag{16}$$

where, for any suitable random variable x_{t+1}, $\mathbb{E}^{\mathbb{Q}}_t(x_{t+1}) = \frac{1}{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})}\mathbb{E}^{\mathbb{P}}_t(m_{t+1}\,x_{t+1})$.

The tractability of the Hellinger measure in equations (14)–(15) stems from the fact that both $\widetilde{n}_{t+1}$ and $\widetilde{n}^*_{t+1}$ represent change of probability measures, and are unique in complete markets. Equation (16) shows that the calculation of H_t involves determining the value of the payoff $\sqrt{\frac{S_{t+1}}{S_t}}$. It also indicates that H_t is convex in S_{t+1}, implying that greater risk-neutral dispersion of $\frac{S_{t+1}}{S_t}$ will increase the dissimilarity between the two SDFs.[5]

Next, suppose that European options with strike price K and underlier S are traded in the domestic country (the foreign currency is the reference).

The following result shows how to calculate the Hellinger measure in equation (16) using option prices, with the foreign currency as the reference.

[5]Clarifying the development of the Hellinger measure, we note that $\frac{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})}{\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})} = \frac{1}{\frac{F_t}{S_t}} = \frac{1}{\mathbb{E}^{\mathbb{Q}}_t\left(\frac{S_{t+1}}{S_t}\right)}$. Hence, we can express equation (16) as $H_t = 1 - \mathbb{E}^{\mathbb{Q}}_t\left(\sqrt{\frac{S_{t+1}}{S_t}}\right) / \sqrt{\mathbb{E}^{\mathbb{Q}}_t\left(\frac{S_{t+1}}{S_t}\right)}$. For any concave function $g[x]$, Jensen's inequality implies that $\mathbb{E}^{\mathbb{Q}}_t(g[x]) < g[\mathbb{E}^{\mathbb{Q}}_t(x)]$. Because $\sqrt{\frac{S_{t+1}}{S_t}}$ is concave in $\frac{S_{t+1}}{S_t}$, it holds that $H_t > 0$.

Proposition 1 *The Hellinger measure can be computed at each time t as*

$$H_t \;=\; \sqrt{\frac{R_{f,t+1}^2}{16 F_t}} \left(\int_{\{K>F_t\}} \frac{C_t[K]}{K^{3/2}} \, dK \;+\; \int_{\{K<F_t\}} \frac{P_t[K]}{K^{3/2}} \, dK \right), \tag{17}$$

where $C_t[K]$ ($P_t[K]$) is the time t price of a call (put) on the foreign exchange with strike price K, denominated in domestic currency units.

Proof: See Internet Appendix II. ■

Ensuring the internal consistency of the approach, we emphasize that the Hellinger measure can be computed from the perspective of the foreign investor as well. Specifically, $H_t = 1 - \frac{1}{\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})\,\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)}} \mathbb{E}_t^{\mathbb{P}} \left(m_{t+1}^* \sqrt{\frac{m_{t+1}}{m_{t+1}^*}} \right)$. Thus, it follows that

$$H_t \;=\; 1 \;-\; \frac{1}{\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})\,\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)}} \mathbb{E}_t^{\mathbb{P}} \left(m_{t+1}^* \sqrt{\frac{1/S_{t+1}}{1/S_t}} \right), \tag{18}$$

$$\;=\; 1 \;-\; \sqrt{\frac{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}} \, \mathbb{E}_t^{\mathbb{Q}*} \left(\sqrt{\frac{1/S_{t+1}}{1/S_t}} \right). \tag{19}$$

Suppose there are European options traded in the foreign country with strike price $\frac{1}{K}$ and underlier $\frac{1}{S}$ (the domestic currency is the reference). Then, the analog to Proposition 1 is Corollary 1.

Corollary 1 *Let $C_t^*[K^*]$ ($P_t^*[K^*]$) be the time t price of a call (put) with strike price $K^* \equiv \frac{1}{K}$, denominated in foreign currency units. Then*

$$H_t = \sqrt{\frac{(R_{f,t+1}^*)^2}{16 F_t^{-1}}} \left(\int_{\{K^*>F_t^{-1}\}} \frac{C_t^*[K^*]}{(K^*)^{3/2}} \, dK^* \;+\; \int_{\{K^*<F_t^{-1}\}} \frac{P_t^*[K^*]}{(K^*)^{3/2}} \, dK^* \right), \tag{20}$$

where F_t^{-1} is the forward exchange rate with the domestic currency as the reference.

Proposition 1 and Corollary 1 highlight a novelty in that the Hellinger measure is amenable to computation from the prices of out-of-the-money calls and puts from the perspective of either country. Importantly, our method allows to obtain *time-varying* estimates of H_t without specifying the functional form of the SDFs or invoking distributional assumptions.

2.4. Interpretation in terms of currency returns

We argue that the Hellinger measure can be an important quantity to consider when parameterizing international asset pricing models, as it embeds information about the risk-neutral distribution of currency returns when markets are complete.

To develop this interpretation, we appeal to a Taylor expansion and write $\sqrt{x} = \sqrt{\mu_x} + \frac{x-\mu_x}{2\sqrt{\mu_x}} - \frac{(x-\mu_x)^2}{8\mu_x^{3/2}} + \frac{(x-\mu_x)^3}{16\mu_x^{5/2}} - \frac{5(x-\mu_x)^4}{128\mu_x^{7/2}} + O\left((x-\mu_x)^5\right)$. If $x \equiv \frac{S_{t+1}}{S_t}$ and $\mu_x \equiv \mathbb{E}_t^{\mathbb{Q}}\left(\frac{S_{t+1}}{S_t}\right) = \frac{F_t}{S_t} = \frac{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}$, then one can define

$$\mathrm{rx}_{t+1} \equiv \frac{S_{t+1}}{S_t} - \frac{F_t}{S_t} \text{ as the excess currency return.} \tag{21}$$

Its expectation under the $\mathbb{P}$ measure, i.e., $\mathbb{E}_t^{\mathbb{P}}(\mathrm{rx}_{t+1})$, defines the currency risk premium.

Substituting the Taylor expansion of $\sqrt{x}$ into equation (16) gives the following representation:

$$H_t = \frac{1}{8(\frac{F_t}{S_t})^2}\mathbb{E}_t^{\mathbb{Q}}\left(\mathrm{rx}_{t+1}^2\right) - \frac{1}{16(\frac{F_t}{S_t})^3}\mathbb{E}_t^{\mathbb{Q}}\left(\mathrm{rx}_{t+1}^3\right) + \frac{5}{128(\frac{F_t}{S_t})^4}\mathbb{E}_t^{\mathbb{Q}}\left(\mathrm{rx}_{t+1}^4\right) + \dots \tag{22}$$

The Hellinger measure can be viewed as a weighted sum of risk-neutral moments of rx_{t+1}.

2.5. Comparison with other possible measures of dissimilarity and approaches

The Hellinger measure in equation (4) is $\frac{1}{2}\mathbb{E}_t^{\mathbb{P}}(\{\sqrt{\frac{q[\mathrm{v}]}{p[\mathrm{v}]}} - \sqrt{\frac{q^*[\mathrm{v}]}{p[\mathrm{v}]}}\}^2)$, which is the second moment of the difference in the square-root of the change of probability Radon-Nikodym derivatives. We recognize that a number of other measures of dissimilarity can in principle be considered, for example, based on total variation or the Kullback-Leibler, or the Kolmogorov metrics (e.g., Gibbs and Su (2002, Table 1)).

However, among these, and many others, the Hellinger measure has the appealing features of being a proper distance metric, independent of the choice of the reference measure (the $\mathbb{P}$ measure in our case), and symmetric. Moreover, the payoff underlying the Hellinger measure meets a differentiability requirement, facilitating the development in Proposition 1.

Integral to our empirical characterizations, the expression for the Hellinger measure can be calculated from data on currency options in the currency units of either economy. Being option-based and forward-looking, the Hellinger measure can reflect time variation in dissimilarities between SDFs, and this time-series dimension possibly sets it apart from alternatives.

Still, we consider the (symmetric) Chi-squared distance metric (e.g., Lindsay, Markatou, Ray, Yang, and Chen (2008, Section 2.3)), which can be extracted from currency option prices:

$$\begin{aligned}
\mathcal{C}_t &\equiv \mathrm{E}_t^{\mathbb{P}}\left(\frac{(\widetilde{n}_{t+1}-\widetilde{n}^*_{t+1})^2}{(\widetilde{n}_{t+1}+\widetilde{n}^*_{t+1})/2}\right), \qquad \text{(Chi-squared distance)} && (23)\\
&= \int_{\mathbb{R}^{\mathbb{N}}} \frac{(\frac{q[\nu]}{p[\nu]}-\frac{q^*[\nu]}{p[\nu]})^2}{(\frac{q[\nu]}{p[\nu]}+\frac{q^*[\nu]}{p[\nu]})/2}\, p(\nu)d\nu, && (24)\\
&= \mathrm{E}_t^{\mathbb{P}}\left(\frac{m_{t+1}}{\mathrm{E}_t^{\mathbb{P}}(m_{t+1})}\frac{(1-\frac{\mathrm{E}_t^{\mathbb{P}}(m_{t+1})}{\mathrm{E}_t^{\mathbb{P}}(m^*_{t+1})}\frac{m^*_{t+1}}{m_{t+1}})^2}{(1+\frac{\mathrm{E}_t^{\mathbb{P}}(m_{t+1})}{\mathrm{E}_t^{\mathbb{P}}(m^*_{t+1})}\frac{m^*_{t+1}}{m_{t+1}})/2}\right), && (25)\\
&= \mathrm{E}_t^{\mathbb{P}}\left(\frac{m_{t+1}}{\mathrm{E}_t^{\mathbb{P}}(m_{t+1})}\frac{(1-\frac{S_{t+1}}{F_t})^2}{(1+\frac{S_{t+1}}{F_t})/2}\right), \qquad \left(\text{since } \frac{F_t}{S_t}=\frac{\mathbb{E}_t^{\mathbb{P}}(m^*_{t+1})}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}\right) && (26)\\
&= \mathrm{E}_t^{\mathbb{Q}}\left(\frac{(1-\frac{S_{t+1}}{F_t})^2}{(1+\frac{S_{t+1}}{F_t})/2}\right), && (27)\\
&= R_{f,t+1}\int_{\{K>F_t\}} \mho[K]\, C_t[K]\, dK \;+\; R_{f,t+1}\int_{\{K<F_t\}} \mho[K]\, P_t[K]\, dK, && (28)
\end{aligned}$$

where $\mho[K] \equiv \frac{4}{F_t^2(\frac{K}{F_t}+1)}+\frac{8(1-\frac{K}{F_t})}{F_t^2(\frac{K}{F_t}+1)^2}+\frac{4(1-\frac{K}{F_t})^2}{F_t^2(\frac{K}{F_t}+1)^3}$. Our implementation of the Chi-squared measure in equation (28) shows that the Hellinger and Chi-squared measures are in agreement (this is intuitively so since equation (27) can also be expressed as $\frac{1}{(\frac{F_t}{S_t})^2}\mathbb{E}_t^{\mathbb{Q}}(\text{rx}_{t+1}^2) - \frac{1}{2(\frac{F_t}{S_t})^3}\mathbb{E}_t^{\mathbb{Q}}(\text{rx}_{t+1}^3) + \frac{1}{4(\frac{F_t}{S_t})^4}\mathbb{E}_t^{\mathbb{Q}}(\text{rx}_{t+1}^4) + \ldots$). However, equation (23) involves the conditional expectation of the ratio of polynomials in m_{t+1} and m_{t+1}^*, precluding analytical solutions of the Chi-squared measure for some of the international asset pricing models in Section 3.4.1. For these reasons, we feature the Hellinger measure in the remainder of the paper.[6]

While the Hellinger measure involves computing the risk-neutral expectation $\mathbb{E}_t^{\mathbb{Q}}\left(\sqrt{\frac{S_{t+1}}{S_t}}\right)$, the Chi-squared measure involves computing $\mathrm{E}_t^{\mathbb{Q}}\left(\frac{(1-\frac{S_{t+1}}{F_t})^2}{(1+\frac{S_{t+1}}{F_t})/2}\right)$. The form of the payoff under the risk-neutral measure can be traced to the respective probability distance metric. The common thread is that both measures are related to the risk-neutral distribution of currency returns.

The Hellinger measure is distinct from codependence. If $L_t[x_{t+1}] \equiv -\mathbb{E}_t^{\mathbb{P}}(\log(x_{t+1})) + \log(\mathbb{E}_t^{\mathbb{P}}(x_{t+1}))$, then $H_t = 1 - \exp\left(L_t[\sqrt{\widetilde{n}_{t+1}\widetilde{n}_{t+1}^*}] - \frac{1}{2}L_t[\widetilde{n}_{t+1}] - \frac{1}{2}L_t[\widetilde{n}_{t+1}^*]\right)$; therefore H_t differs from codependence between SDFs in the manner of Hansen (2012, page 930) (see Internet Appendix III). This is more directly seen when (m_{t+1}, m_{t+1}^*) is bivariate lognormal (as per the expression of the Hellinger measure in equation (29)).

In incomplete international markets, some SDF pairs (m_{t+1}, m_{t+1}^*) satisfy $m_{t+1}(\frac{S_{t+1}}{S_t}) - m_{t+1}^* = 0$, while others do not (e.g., Bakshi, Cerrato, and Crosby (2017, Definitions 1 and 2)). In this incomplete markets setting, there is multiplicity of (m_{t+1}, m_{t+1}^*), and, thus, an infinite number of possible values of such a distance measure, prompting us to take infimums over m_{t+1} and m_{t+1}^*, of $\mathbb{E}_t^{\mathbb{P}}\left(\sqrt{m_{t+1}m_{t+1}^*}\right)$. Thus, one can, at most, strive to obtain an upper bound on the Hellinger mea-

[6] Is it possible to characterize the Hellinger measure specifically for the martingale component of the SDFs, and extract it from option data? To entertain such a possibility, consider SDFs that admit multiplicative decomposition (unique in the sense of Alvarez and Jermann (2005) and Hansen and Scheinkman (2009)), implying $m_{t+1}^P = m_{t+1}R_{t+1,\infty}$ and $m_{t+1}^{P*} = m_{t+1}^*R_{t+1,\infty}^*$, where m_{t+1}^P (m_{t+1}^{P*}) is the martingale component of the domestic (foreign) SDF, and $R_{t+1,\infty}$ ($R_{t+1,\infty}^*$) is the return of the domestic (foreign) discount bond with infinite maturity. Since $\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^P) = 1$ and $\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^{P*}) = 1$, it holds that $H_t \equiv 1 - \mathbb{E}_t^{\mathbb{P}}\left(\sqrt{m_{t+1}^P m_{t+1}^{P*}}\right) = 1 - \mathbb{E}_t^{\mathbb{Q}}\left(\sqrt{m_{t+1}m_{t+1}^*R_{t+1,\infty}R_{t+1,\infty}^*}\right)$. Analytical tractability in this case is hampered by the fact that m_{t+1}, m_{t+1}^*, $R_{t+1,\infty}$, and $R_{t+1,\infty}^*$ are correlated (e.g., Bakshi, Chabi-Yo, and Gao (2017) and Christensen (2017)).

sure as $1 - \inf_{(m_{t+1}, m^*_{t+1})} \frac{1}{\sqrt{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})\,\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})}} \mathbb{E}^{\mathbb{P}}_t \left(\sqrt{m_{t+1} m^*_{t+1}}\right)$. This optimization problem must be solved subject to pricing restrictions and subject to ruling out extremely lucrative investment opportunities across incomplete international markets (in the flavor of Bakshi, Cerrato, and Crosby (2017, Problem 1)). In the case of incomplete markets, even an unconditional upper bound on the Hellinger measure may be untractable, and, in contrast to our Proposition 1, extracting a measure of dissimilarity from currency option prices may be infeasible.

Finally, is it possible to quantify dissimilarity using risk-neutral densities extracted from equity options markets (see equation (4))? This requires parametric assumptions and a richer market that trades *product options* of all strike pairs in domestic and foreign equities. Instead, our innovation is to tap into an approach that infers the price of $\sqrt{\frac{m^*_{t+1}}{m_{t+1}}}$.

3. Linking the theory of dissimilarity to data and models

Next we explore the stylized features of the Hellinger measure and its variation through time. While international macro-finance models often strive to reproduce empirical characteristics like interest rates, currency risk premiums, and SDF volatilities, an additional important dimension that can be captured with the help of the Hellinger measure is consistency with the risk-neutral distribution of currency returns.

3.1. Panel of currency option prices and data construction

The main data source for this study is J. P. Morgan's DataQuery database.[7] We employ end-of-month observations of spot and forward exchange rates, option-implied volatility quotes, and

[7] Similar data has been used in currency market research by Bates (1996), Carr and Wu (2007), Bakshi, Carr, and Wu (2008), Burnside, Eichenbaum, Kleshchelski, and Rebelo (2011), Caballero and Doyle (2012), Gavazzoni, Sambalaibat, and Telmer (2013), Branger, Herold, and Mucky (2014), Jurek (2014), Jurek and Xu (2014), Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan (2015), Mueller, Stathopoulos, and Vedolin (2016), Della Corte, Sarno, and Ramadorai (2016), Della Corte, Kozhan, and Neuberger (2016), Londono and Zhou (2017), Zviadadze, Chernov, and Graveline (2018), and Daniel, Hodrick, and Lu (2017) among others.

LIBORs over the 222-month sample period from 1/1996 to 6/2014 (from 12/1998 for the Euro and Norwegian krone). Our focus is on the most actively traded G-10 currencies and their respective economies: New Zealand (NZ), Australia (AU), United Kingdom (UK), Norway (NO), Sweden (SD), Canada (CA), US, Euro-zone (EU), Switzerland (SW), and Japan (JP).

Quoted implied volatilities, the building blocks for the option prices used in our study, are available for all pairs of G-10 currencies (45 pairs in total). They correspond to European options of constant maturity (30 days in our work) and five levels of constant delta: 10 and 25 delta puts, 10 and 25 delta calls, and at-the-money. For details, see Internet Appendix IV.

While standard, this format of option data requires some further manipulation in order to render it suitable for our study. In particular, we need to transform the five volatilities provided on each date into prices of options (both puts and calls) with a wide range of strikes, since our approach requires numerical integration involving such option prices.

For this purpose, we interpolate the volatilities on a grid of deltas and extrapolate conservatively beyond the minimum and maximum strikes, keeping constant the volatilities given at these strikes, following Jiang and Tian (2005), Carr and Wu (2009), and Reiswich and Wystup (2009). Next, we transform these deltas into strike prices as in Jurek (2014, equations (9a), (9b) and (10)), and as noted also in Wystup (2006). With the volatilities and strikes thus obtained, we calculate the corresponding call and put prices using the Garman and Kohlhagen (1983) formula. All inputs for the calculation of the option prices come from the same database to ensure consistency.[8] Having access to all 45 cross-rates among the G-10 currencies and the corresponding option volatility quotes allows for direction calculation of the pairwise Hellinger measures.

[8]We note that, in principle, further manipulation may be needed in cases when certain currency appears as a reference currency in some underlying exchange rates, but as a non-reference one in others, for example, the USD is reference currency against CAD or JPY, but not against AUD or EUR. The issue can be addressed by symmetry relations as in Jurek (2014, equations (11) and (12)), and via our Corollary 1. However, the Hellinger measure is by design symmetric, and hence, the issue of the reference currency is inessential.

3.2. Empirical attributes of the Hellinger measure

While the theory bounds the possible values of the Hellinger measure between zero and one, any *a priori* intuition of what values of the measure should be considered "large" or "small" is still lacking. To provide a perspective, we consider the following baseline model.

Example 2 Suppose that $\log(m_{t+1})$ and $\log(m^*_{t+1})$ are bivariate normally distributed. If $\mathrm{var}^{\mathbb{P}}_t(.)$ and $\mathrm{cov}^{\mathbb{P}}_t(.,.)$ denote conditional variance and covariance under the physical probability measure, then (see Internet Appendix V),

$$\begin{aligned} H^{\mathrm{LN}}_t \;&\equiv\; 1-\exp\left(-\frac{1}{8}\{\mathrm{var}^{\mathbb{P}}_t(\log(m^*_{t+1}))+\mathrm{var}^{\mathbb{P}}_t(\log(m_{t+1}))\}\right.\\ &\qquad \left.+\;\frac{1}{4}\mathrm{cov}^{\mathbb{P}}_t(\log(m^*_{t+1}),\log(m_{t+1}))\right), &(29)\\ &=\; 1-\exp\left(-\frac{1}{8}\mathrm{var}^{\mathbb{P}}_t(\log(\frac{S_{t+1}}{S_t}))\right). &(30)\end{aligned}$$

When (m_{t+1},m^*_{t+1}) is lognormal, the dissimilarity between m_{t+1} and m^*_{t+1} can be inferred from the conditional variance of $\log(\frac{S_{t+1}}{S_t})$ under the physical probability measure. ♣

Isolating the impact of distributional non-normalities in $(\log(m_{t+1}),\log(m^*_{t+1}))$**:** Equation (30) of Example 2 allows us to address two questions: How important are non-normalities in $(\log(m_{t+1}),\log(m^*_{t+1}))$ to the Hellinger measures extracted in a model-free manner from currency options? What are the anchoring values of Hellinger measures?

Before proceeding to answers, we define

$$H^{[i,j]}_t \;\equiv\; \sqrt{H} \text{ (based on monthly data, in \%) for economy pair } (i,j) \text{ in month } t, \qquad (31)$$

$$\overline{H}^{[45]}_t \;\equiv\; \text{Cross-sectional average of } H^{[i,j]}_t \text{ for all 45 pairs of economies in month } t \qquad (32)$$

$$\overline{H}^{\mathrm{US},[9]}_t \;\equiv\; \text{Cross-sectional average of } H^{[\mathrm{US},j]}_t \text{ for nine other pairs of economies in month } t. \qquad (33)$$

Table 1 (Panel A) reports the mean, standard deviation, minimum, maximum, and percentiles of $\overline{H}_t^{[45]}$ and $\overline{H}_t^{\mathrm{US},[9]}$. Panel B reports the counterparts constructed from the empirical analog to equation (30). Specifically, we proxy the *monthly* $\mathrm{var}_t^{\mathbb{P}}(\log(\frac{S_{t+1}}{S_t}))$ as the quadratic return variation, utilizing *daily* currency returns (e.g., Andersen, Bollerslev, Diebold, and Labys (2003)):

$$\mathcal{H}_t \;\equiv\; 1 \;-\; \exp(-\frac{1}{8}\sum_{n=1}^{22}\{\log(\frac{S_{t+n\Delta t}}{S_{t+(n-1)\Delta t}})\}^2). \tag{34}$$

For comparison, we compute and report (see Panel C of Table 1) the following deviations

$$\log(\frac{\overline{H}_t^{[45]}}{\overline{\mathcal{H}}_t^{[45]}}) \qquad \text{and} \qquad \log(\frac{\overline{H}_t^{\mathrm{US},[9]}}{\overline{\mathcal{H}}_t^{\mathrm{US},[9]}}). \tag{35}$$

The central observation from Table 1 is that the deviations $\log(\frac{\overline{H}_t^{[45]}}{\overline{\mathcal{H}}_t^{[45]}})$ ($\log(\frac{\overline{H}_t^{\mathrm{US},[9]}}{\overline{\mathcal{H}}_t^{\mathrm{US},[9]}})$), are sizable, time-varying, and switch sign, with a minimum of -19.4% (-19.7%) and a maximum of 47% (53%). We interpret the time-varying nature of the deviations (seen from Figure 1) as the contribution of stochastically-varying risk-neutral moments of currency returns (some evidence on risk-neutral stochastic skewness is provided in Carr and Wu (2007)).

[Figure 1 about here.]

The takeaway is that the Hellinger measure based on $\mathrm{var}_t^{\mathbb{P}}(\log(\frac{S_{t+1}}{S_t}))$ appears to omit a sizable portion of the variation in the dissimilarity between SDFs, computed using Proposition 1.

Distinct geographical pattern: Here we construct, for each economy, the time series of cross-sectional average measures with respect to the remaining nine economies:

$$\overline{H}_t^{i,[9]} \;\equiv\; \text{Cross-sectional average: } \tfrac{1}{9}\textstyle\sum_{j=1}^{9} H_t^{[i,j]} \text{ for economy } i \text{ in month } t. \tag{36}$$

Using $\overline{H}_t^{[45]}$ as a benchmark against which to compare other Hellinger measures, we consider *differences* between $\overline{H}_t^{i,[9]}$ and $\overline{H}_t^{[45]}$ and, importantly, estimate confidence intervals for these differences. Measures that are found to be significantly different from $\overline{H}_t^{[45]}$ can be considered to be large or small in our sample.

These averages aggregate information for the individual economies and can identify geographical patterns, if present in the Hellinger measures. Table 2 shows the *mean differences* between $\overline{H}_t^{i,[9]}$ and $\overline{H}_t^{[45]}$. We also show 95% confidence intervals for these mean differences, obtained from 10,000 bootstrapped series of Hellinger measures, using the stationary bootstrap as in Politis and Romano (1994), with optimal block size determined following Politis and White (2004). This procedure has been shown to account for conditional heteroskedasticity (e.g., Goncalves and White (2002)), as well as non-normality. In addition, we show the minimum and maximum values of the mean differences obtained in the respective 10,000 bootstrap samples.

The Euro-zone has the lowest Hellinger measures, implying that its SDF is on average the closest to all SDFs in the sample. Its mean difference with $\overline{H}_t^{[45]}$ is negative and significant at the 5% level, and, in fact, *all* bootstrapped mean differences are also negative. The mean differences with $\overline{H}_t^{[45]}$ are also significantly negative for the UK, Switzerland, Norway, and Sweden, while they are statistically insignificant for the US and Canada, and positive and significant for Australia, New Zealand, and Japan. For the latter two economies, the minimum bootstrapped differences are also positive.

While Table 2 hints at a pattern with low Hellinger measures for the European economies, intermediate measures for the North American economies, and higher measures for the three Asia-Pacific economies, any such attempt at geographical classification brings further questions such as: Are the measures involving the European economies uniformly low, or only with respect to each other? Are the measures for the Asia-Pacific economies uniformly large, or only with respect to the other regions?

To elaborate, Table 3 shows mean differences between each Hellinger measure $H_t^{[i,j]}$ and $\overline{H}_t^{[45]}$, and identifies in bold those that are statistically significant. It is seen that (i) most of these differences are significant, (ii) the measures among the European economies tend to be small, but are intermediate (large) for these economies with respect to the US and Canada (the three Asia-Pacific economies), and (iii) measures for the US and Canada are overall intermediate, while those of Japan, New Zealand, and Australia (to some extent) are the largest.

Importantly, the measure for the US with respect to Canada, and that for Australia with respect to New Zealand are small, consistent with a regional distinction among Hellinger measures. However, Japan exhibits large measures with respect to most European countries, and even larger ones with respect to Australia and New Zealand, which points to a four-region geographic classification based on Hellinger measures.

Strong common component: We also ask how different (or similar) is the time series behavior of the Hellinger measures. The first principal component of the 45 time series of measures explains more than 97% of the variation, thus a strong single-factor structure appears to be another essential feature of the Hellinger measure. The first principal component is closely associated with $\overline{H}_t^{[45]}$, justifying the use of this average as a benchmarking device.

[Figure 2 about here.]

For an illustration, Figure 2 plots $\overline{H}_t^{[45]}$ together with the average measures for the Eurozone, US, UK, New Zealand, and Japan. The Hellinger measures appear to vary with economic conditions. All six plots exhibit an increase around the financial crisis at the end of 2008, and a smaller, but again common, increase around 1998–2000. The series have sizable amplitude, with the maximum value of $\overline{H}_t^{[45]}$ being about four times higher than its minimum value. Furthermore, they are persistent, and the first-order autocorrelation coefficient of $\overline{H}_t^{[45]}$ equals 0.87.

Summary: Table 1 and Figure 1 indicate that deviations from the normality of $(\log(m_{t+1}),\log(m^*_{t+1}))$ are economically important. Specifically, the Hellinger measure extracted from daily currency returns differs substantially from our model-free construction based on currency option prices.

Moreover, Table 2 shows that the Euro-zone and UK have the lowest Hellinger measures $\overline{H}_t^{i,[9]}$, indicating that their SDFs on average are the closest to those of all remaining economies in the sample. In contrast, the SDFs of Japan and New Zealand on average differ the most from all the remaining ones. In most cases these distinctions are statistically significant, as evidenced by the reported bootstrap results.

Complementing these findings, Table 3 shows the differences between the 45 individual Hellinger measures $H_t^{[i,j]}$ and the average Hellinger measure $\overline{H}_t^{[45]}$. The lowest measure is observed for the Euro-zone and Switzerland, which hence have the most similar SDFs, while Japan and New Zealand exhibit the highest measure, and hence the most distinct SDFs.

3.3. Economic interpretation and consistency with returns of bonds and equities

Here we go beyond the currency options data and construct the Hellinger (and Chi-squared) measures using international equity and bond market data. The measures are consistent with Radon-Nikodym derivatives $\widetilde{n}$ and $\widetilde{n}^*$ based on a minimum discrepancy problem (e.g., Borovička, Hansen, and Scheinkman (2016, Section VIII)), which entails minimizing the expectation of convex functions $\mathbf{d}[\widetilde{n}] = \widetilde{n}\log\widetilde{n}$ and $\mathbf{d}[\widetilde{n}^*] = \widetilde{n}^*\log\widetilde{n}^*$, as formalized below:

$$\inf_{\widetilde{n}} \; \mathbb{E}^{\mathbb{P}}(\mathbf{d}[\widetilde{n}]) \;=\; \sup_{\boldsymbol{\zeta}\in\mathbb{R}^6,\, \varpi\in\mathbb{R}} \; \inf_{\widetilde{n}} \mathbb{E}^{\mathbb{P}}(\widetilde{n}\log\widetilde{n} - \boldsymbol{\zeta}'(\widetilde{n}\mathbf{A} - \mathbf{1}) - \varpi(\widetilde{n}-1)), \tag{37}$$

$$\inf_{\widetilde{n}^*} \; \mathbb{E}^{\mathbb{P}}(\mathbf{d}[\widetilde{n}^*]) \;=\; \sup_{\boldsymbol{\zeta}_*\in\mathbb{R}^6,\, \varpi_*\in\mathbb{R}} \; \inf_{\widetilde{n}^*} \mathbb{E}^{\mathbb{P}}(\widetilde{n}^*\log\widetilde{n}^* - \boldsymbol{\zeta}_*'(\widetilde{n}^*\mathbf{A}^* - \mathbf{1}) - \varpi_*(\widetilde{n}^*-1)), \tag{38}$$

where $\boldsymbol{\zeta}$ ($\boldsymbol{\zeta}_*$) are the Lagrange multipliers associated with $\mathbb{E}^{\mathbb{P}}(\widetilde{n}\mathbf{A}) = \mathbf{1}$ ($\mathbb{E}^{\mathbb{P}}(\widetilde{n}^*\mathbf{A}^*) = \mathbf{1}$) and $\boldsymbol{\varpi}$ ($\boldsymbol{\varpi}_*$) are the Lagrangian multipliers associated with $\mathbb{E}^{\mathbb{P}}(\widetilde{n}) = 1$ ($\mathbb{E}^{\mathbb{P}}(\widetilde{n}^*) = 1$). These restrictions are imposed unconditionally.

The featured choice of $\mathbf{d}[.]$ guarantees the positivity of the Radon-Nikodym derivatives $\widetilde{n}$ and $\widetilde{n}^*$. Our goal is to examine whether such extracted measures of dissimilarity agree with what is reported in Tables 1, 2, and 3 and illustrate the economic relevance of our dissimilarity measure.

Our empirical exercises involve the scaled return vectors $\mathbf{A}_{t+1}$ and $\mathbf{A}^*_{t+1}$, given as:

$$\underbrace{\mathbf{A}_{t+1}}_{6\times 1} = \frac{1}{R_{f,t+1}} \underbrace{\begin{pmatrix} R_{f,t+1} \\ R_{\text{bond},t+1} \\ R_{\text{equity},t+1} \\ (\frac{S_{t+1}}{S_t})R^*_{f,t+1} \\ (\frac{S_{t+1}}{S_t})R^*_{\text{bond},t+1} \\ (\frac{S_{t+1}}{S_t})R^*_{\text{equity},t+1} \end{pmatrix}}_{\mathbf{R}_{t+1}} \quad \text{and} \quad \underbrace{\mathbf{A}^*_{t+1}}_{6\times 1} = \frac{1}{R^*_{f,t+1}} \underbrace{\begin{pmatrix} (\frac{S_t}{S_{t+1}})R_{f,t+1} \\ (\frac{S_t}{S_{t+1}})R_{\text{bond},t+1} \\ (\frac{S_t}{S_{t+1}})R_{\text{equity},t+1} \\ R^*_{f,t+1} \\ R^*_{\text{bond},t+1} \\ R^*_{\text{equity},t+1} \end{pmatrix}}_{\mathbf{R}^*_{t+1}}. \tag{39}$$

Further, the U.S. is the home economy and the foreign economy is each of the economies with the remaining G-10 currencies. Germane to the calculations in equation (39) is the return vector $\mathbf{R}_{t+1}$, which contains the gross returns of the U.S. risk-free bond ($R_{f,t+1}$), the U.S. ten-year Treasury bond ($R_{\text{bond},t+1}$), and the U.S. MSCI equity index ($R_{\text{equity},t+1}$) in U.S. dollars, together with the returns of the foreign counterparts (denoted by * superscripts) converted into U.S. dollars, and an analogous vector of returns $\mathbf{R}^*_{t+1}$ expressed in the foreign currency. The vector of domestic and foreign gross returns are related as $\mathbf{R}_{t+1} = (\frac{S_{t+1}}{S_t})\mathbf{R}^*_{t+1}$.

As in Bakshi, Chabi-Yo, and Gao (2017, Case 2), the problem in (37) and (38) has the following explicit solution:

$$\widetilde{n} = \exp(\boldsymbol{\zeta}'\mathbf{A} + \varpi - 1), \ \text{ where } (\boldsymbol{\zeta}, \varpi) \text{ solves } \inf_{(\boldsymbol{\zeta},\varpi)} -\varpi - \mathbf{1}'\boldsymbol{\zeta} + \mathbb{E}^{\mathbb{P}}(\exp(\mathbf{A}'\boldsymbol{\zeta} + \varpi - 1)), \quad (40)$$

$$\widetilde{n}^* = \exp(\boldsymbol{\zeta}_*'\mathbf{A}^* + \varpi_* - 1), \ \text{ where } (\boldsymbol{\zeta}_*, \varpi_*) \text{ solves } \inf_{(\boldsymbol{\zeta}_*,\varpi_*)} -\varpi_* - \mathbf{1}'\boldsymbol{\zeta}_* + \mathbb{E}^{\mathbb{P}}(\exp((\mathbf{A}^*)'\boldsymbol{\zeta}_* + \varpi_* - 1)). \quad (41)$$

We then compute the unconditional Hellinger measure (equation (3)) as $\frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}(\sqrt{\widetilde{n}_t} - \sqrt{\widetilde{n}_t^*})^2$, and likewise for the Chi-squared measure (equation (23)) as $\frac{1}{T}\sum_{t=1}^{T} 2(\widetilde{n}_t - \widetilde{n}_t^*)^2/(\widetilde{n}_t + \widetilde{n}_t^*)$.

Table 4 reports the extracted Hellinger and Chi-squared measures. It is seen that the average Hellinger measure is close to those derived from currency options. Furthermore, the relative ranking between the Hellinger measures is also preserved, and is also mirrored by the ranking between the Chi-squared measures. These results imply that the structure of risk differs across economies, and our measures capture in a consistent way this difference.

Additionally, we report the annualized volatility of the Radon-Nikodym derivatives, given by $(\frac{12}{T}\sum_{t=1}^{T}(\widetilde{n}_t - 1)^2)^{1/2}$ and $(\frac{12}{T}\sum_{t=1}^{T}(\widetilde{n}_t^* - 1)^2)^{1/2}$. The volatility of the U.S. (foreign) Radon-Nikodym derivatives that correctly prices the considered set of assets range between 74% and 105% (70% and 104%).

3.4. *Implications for international asset pricing models*

This section examines Hellinger measures obtained from several models of international economies, including those in Verdelhan (2010), driven by the surplus consumption ratio, and in Lustig, Roussanov, and Verdelhan (2014), which features global and economy-specific state variables, as well as a long-run risk model from Colacito and Croce (2011), and a model with time-varying probability of disasters.

Our focus is on studying the consistency of these models with the risk-neutral-distribution of currency returns (which is available here for a large cross-section of economies). Since the models were not intended to match this data feature, we employ the Hellinger measure as a tool for differentiation.

Our null hypothesis is whether the 95% confidence intervals of model-based H_t bracket the options-market-based unconditional values based on Proposition 1.

3.4.1. Models

A. Verdelhan (2010, Section I.A): In this model, the SDF in each economy satisfies the following:

$$\log(m_{t+1}) = \log(\beta) - \gamma(g + (\phi - 1)(s_t - \bar{s})) - \gamma(1 + \lambda[s_t])(\log(C_{t+1}/C_t) - g), \quad (42)$$

$$\log(C_{t+1}/C_t) = g + u_{t+1} \quad \text{for} \quad u_{t+1} \sim \text{ i.i.d. } \mathcal{N}(0, \sigma^2), \text{ and} \quad (43)$$

$$s_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda[s_t]\,(\log(C_{t+1}/C_t) - g), \quad (44)$$

where s_t is the log surplus consumption ratio, and $\lambda[s_t] = \frac{1}{\bar{S}}\sqrt{1 - 2(s_t - \bar{s})} - 1$ when $s \leq s_{\max}$ and zero otherwise.

Denoting by ρ the correlation between the shocks to domestic and foreign consumption growth u_{t+1} and u^*_{t+1}, the Hellinger measure can be derived as

$$H_t = 1 - \exp\left(-\frac{\gamma^2(1 + \lambda[s_t])^2\sigma^2}{8} - \frac{\gamma^2(1 + \lambda[s^*_t])^2\sigma^2}{8} + \frac{\gamma^2(1 + \lambda[s_t])(1 + \lambda[s^*_t])\sigma^2\rho}{4}\right). \quad (45)$$

Since s_t and s^*_t vary over time, so does the measure of dissimilarity.[9]

B. Lustig, Roussanov, and Verdelhan (2014, Section 4.2): In this model, there are $i = 1, \ldots, N$

[9]We emphasize that this international asset pricing model (and some other models considered here) does not admit closed-form solution for the Chi-squared distance measure in equation (23). The reason is that the conditional expectation of the ratio of two polynomials (i.e., the numerator (denominator) consists of three (two) exponential terms) in (23) has no analytical solution to the best of our knowledge when uncertainty u_{t+1} is distributed normal.

economies, where $(u^i_{t+1}, u^w_{t+1}, u^g_{t+1}) \sim$ i.i.d. $\mathcal{N}(0,1)$, and

$$\log(m^i_{t+1}) = -\alpha - \chi z^i_t - \sqrt{\gamma z^i_t}\, u^i_{t+1} - \tau z^w_t - \sqrt{\delta^i z^w_t}\, u^w_{t+1} - \sqrt{\kappa z^i_t}\, u^g_{t+1}, \quad (46)$$

$$z^i_{t+1} = (1-\phi)\theta + \phi z^i_t - \sigma\sqrt{z^i_t}\, u^i_{t+1}, \text{ and} \quad (47)$$

$$z^w_{t+1} = (1-\phi^w)\,\theta^w + \phi^w z^w_t - \sigma^w \sqrt{z^w_t}\, u^w_{t+1}. \quad (48)$$

The model accommodates heterogeneity in the exposure to global shocks u^w and u^g. For any two economies (domestic and foreign), the Hellinger measure can be derived as

$$\begin{aligned} H_t = 1 - \exp\Bigg(& -\frac{1}{8}\gamma z^*_t - \frac{1}{8}\gamma z_t - \frac{1}{8}\delta^* z^w_t - \frac{1}{8}\delta z^w_t + \frac{1}{4}\sqrt{\delta^*\delta}\, z^w_t \\ & -\frac{1}{8}\kappa z_t - \frac{1}{8}\kappa z^*_t + \frac{1}{4}\sqrt{\kappa^2 z_t z^*_t}\Bigg). \end{aligned} \quad (49)$$

The time variation in the Hellinger measures is due to the state variables z_t, z^*_t, and z^w_t.

C. Colacito and Croce (2011, Section II.B): This long-run risk model exploits a first-order linear approximation of the SDFs (see their equation (4); see also Bansal and Shaliastovich (2013), Lewis and Liu (2015), and Zviadadze (2017)) as:

$$\log(m_{t+1}) = \log\delta - \frac{1}{\psi}x_t + \kappa_c \frac{1-\gamma\psi}{\psi(1-\rho_x\kappa_c)}\varepsilon_{x,t+1} - \gamma\varepsilon_{c,t+1}, \quad (50)$$

$$\log(m^*_{t+1}) = \log\delta - \frac{1}{\psi}x^*_t + \kappa_c \frac{1-\gamma\psi}{\psi(1-\rho_x\kappa_c)}\varepsilon_{x*,t+1} - \gamma\varepsilon_{c*,t+1}, \quad (51)$$

$$x_{t+1} = \rho_x x_t + \varepsilon_{x,t+1} \quad \text{and} \quad x^*_{t+1} = \rho_x x^*_t + \varepsilon_{x^*,t+1}. \quad (52)$$

The two economies are assumed to be symmetric with same preference and transition laws parameters, and the shocks are independent, identically, and normally distributed: $\varepsilon_{c,t+1} \sim \mathcal{N}(0,\sigma^2)$, $\varepsilon_{x,t+1} \sim \mathcal{N}(0,\varphi_e^2\sigma^2)$, $\varepsilon_{c^*,t+1} \sim \mathcal{N}(0,\sigma^2)$, and $\varepsilon_{x^*,t+1} \sim \mathcal{N}(0,\varphi_e^2\sigma^2)$. The correlation between

$\varepsilon_{c,t+1}$ and $\varepsilon_{c^*,t+1}$ is ρ_c^{hf}, and the correlation between $\varepsilon_{x,t+1}$ and $\varepsilon_{x^*,t+1}$ is ρ_x^{hf}. We can show that

$$H_t = 1 - \exp\left(-\frac{c_0^2}{4}\varphi_e^2\sigma^2 + \frac{c_0^2}{4}\rho_x^{hf}\varphi_e^2\sigma^2 - \frac{1}{4}\gamma^2\sigma^2 + \frac{1}{4}\gamma^2\rho_c^{hf}\sigma^2\right), \tag{53}$$

where $c_0 \equiv \kappa_c \frac{1-\gamma\psi}{\psi(1-\rho_x\kappa_c)}$. In this model, the Hellinger measure is determined by risk aversion (γ), elasticity of intertemporal substitution (ψ), and parameters of transition laws.

D. Model with time-varying probability of common disasters: The Hellinger measures in our sample exhibit a pronounced single-factor structure, and the model we introduce next is tailored to reflect such a framework. Drawing on Santa-Clara and Yan (2010), Gabaix (2012), and Wachter (2013), it offers a multi-economy refinement and allows for asymmetric effects of common disasters. The model also relates to Aït-Sahalia, Cacho-Diaz, and Laeven (2015) and Farhi and Gabaix (2016), and admits a closed-form Hellinger measure.

Generalizing Example 1, this model incorporates time-varying probability of disasters λ_t, and the dynamics of the pricing kernels (M_t, M_t^*) of two economies is given by

$$\frac{dM_t}{M_{t^-}} = -\mu\,dt - \sigma\,dB_t + (e^{-\Lambda z} - 1)dN_t, \tag{54}$$

$$\frac{dM_t^*}{M_{t^-}^*} = -\mu_*\,dt - \sigma_*\,dB_t^* + (e^{-\Lambda_* z} - 1)dN_t, \tag{55}$$

$$dN_t = \begin{cases} 1 & \text{with probability } \lambda_t\,dt \\ 0 & \text{with probability } 1-\lambda_t\,dt, \end{cases} \tag{56}$$

$$d\lambda_t = \kappa(\theta - \lambda_t)\,dt + \eta\sqrt{\lambda_t}\,dB_t^\bullet, \tag{57}$$

$$z \sim \mathcal{N}\left(\mu_z, \sigma_z^2\right). \tag{58}$$

When a disaster occurs, it can affect the economies asymmetrically, and in this case $\Lambda \neq \Lambda_*$. The Brownian motions (B_t, B_t^*) have correlation ρ, and are uncorrelated with the Brownian motion $B_t^\bullet$.

The (percentage) jump size z is normally distributed, but the framework is amenable to alternative jump size distributions.

In this model of economies exposed to a common disaster, the τ-period Hellinger measure is

$$H_t = 1 - \exp\left(a_1(\tau) - \frac{1}{2}a_2(\tau) - \frac{1}{2}a_3(\tau) + \{b_1(\tau) - \frac{1}{2}b_2(\tau) - \frac{1}{2}b_3(\tau)\}\lambda_t\right). \tag{59}$$

The expressions for $a_j(\tau)$ and $b_j(\tau)$, for $j = 1,\ldots,3$, are displayed in Appendix A.

3.4.2. Model-based versus actual Hellinger measures

Table 5 shows results for the Hellinger measures computed from the four models, as well as four actual average Hellinger measures inferred from the data: $\overline{H}_t^{[45]}$, $\overline{H}_t^{\text{US},[9]}$ (the cross-sectional average of US versus all other nine economies), the smallest (i.e., EU|SW, the EU and Switzerland economy pair), and the largest (i.e., NZ|JP, the New Zealand and Japan economy pair).

We also provide bootstrap confidence intervals, obtained by first estimating all ARMA(p,q) models, with $p \leq 2$ and $q \leq 2$, to the log of the Hellinger measures, and then simulating from the best model according to the Bayesian Information Criterion (BIC).

We use parameter as provided in Verdelhan (2010, Table II), Lustig, Roussanov, and Verdelhan (2014, Table 5) and Colacito and Croce (2011, Table I), and displayed in the note to Table 5. For the disaster model, we take κ, θ, and η from Wachter (2013, equation (2) and Table 1). We also set μ_z to -0.15 and σ_z to 0.15 (but we do vary them over a wide range to assess robustness, as clarified in footnote 10), whereas σ and ρ parameters are determined jointly by targeting SDFs' volatilities of 50%, and a $\rho = 0.95$, in line with Brandt, Cochrane, and Santa-Clara (2006).

The Hellinger measures extracted from currency option prices are in nominal terms. We define the real domestic (foreign) pricing kernels as $\frac{M_t}{\Pi_t}$ ($\frac{M_t^*}{\Pi_t^*}$), where the domestic (foreign) price level is Π_t (Π_t^*) (i.e., Constantinides (1992, page 534)). Then the Hellinger measure in real terms, that is $1 - \frac{1}{\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}\frac{\Pi_t}{\Pi_{t+1}})\,\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*\frac{\Pi_t^*}{\Pi_{t+1}^*})}}\mathbb{E}_t^{\mathbb{P}}\left(\sqrt{m_{t+1}\, m_{t+1}^*\frac{\Pi_t}{\Pi_{t+1}}\frac{\Pi_t^*}{\Pi_{t+1}^*}}\right)$, will coincide with the nominal counterpart, if the inflation processes are independent of each other and from the SDFs. For a related clarification on the role of inflation in models of real and nominal economies, see Lustig, Roussanov, and Verdelhan (2014, page 536).

To obtain time series of Hellinger measures, we resort (when needed) to simulations that yield the dynamics of the latent state variables. For example, for the model in Verdelhan (2010), we first generate $\{u_t, u_t^*\}_{t=1,\ldots,222}$ and construct the time series of $(\lambda[s_t], \lambda[s_t^*])$ using the evolution of the log consumption surplus ratio as per equation (44), and then obtain the series of the Hellinger measure according to equation (45).

The main observation is that the models differ with respect to the Hellinger measures that they generate. For example, those from the habit-based model are about twice higher than the actual measures. We recognize, however, that the assumed model parameters have been calibrated over a very different period, which is a likely explanation for the discrepancy.

On the other hand, the model of Lustig, Roussanov, and Verdelhan (2014) is able to mimic the actual Hellinger measures, except the smallest ones. We also note that although not time-varying, the Hellinger measure generated by the long-run risk model matches reasonably well the average measure observed in the data. Finally, the disaster model also generates Hellinger measures with the correct magnitude on average. However, it is unable to reproduce the variation in Hellinger measures, both in time series, and across various simulations.[10]

Our exercise shows that although these models were not designed from the vantage point of risk-neutral distribution of currency returns, they still could come close to achieving certain

[10] We also consider an exercise where we varied $-0.25 \leq \mu_z \leq -0.05$, and $0.05 \leq \sigma_z \leq 0.25$. These combinations of jump size distribution parameters (μ_z, σ_z) still generate Hellinger measures consistent with the distribution of the actual measures in our sample.

consistency with the options data.

3.4.3. Other variables capturing dissimilarity

Having considered several formal pricing models, we recognize that the international finance literature has explored a variety of further approaches, allowing for various other determinants of dissimilarity among economies like distance, cultural differences, and informational asymmetries. In this section we focus on some of these variables, specifically on their relation with the Hellinger measure, which we examine in a regression framework.

We rely on an observation from equation (29) of Example 2, where the Hellinger measure is, to first-order, linear in the variance of the two log SDFs' and the covariance between them. Guided by Hansen and Jagannathan (1991, Section V) and Brandt, Cochrane, and Santa-Clara (2006), we associate the log SDF with equity market returns, and employ in our regressions the equity return variances and covariances corresponding to each pair of economies.

We explore physical distance as a possible determinant of dissimilarity, prompted by the geographical pattern in Hellinger measures, as revealed in Tables 2 and 3. In this choice, we possibly relate to studies of the explanatory power of distance for international trade (e.g., the gravity equation of Tinbergen (1962), or more recently Chaney (2017)), financial flows (e.g., Okawa and van Wincoop (2012)), and also currency risk premia (Lustig and Richmond (2015)), among others.

Besides, following a voluminous literature that aims to link various distinctions between economies to their cultural differences (for the financial perspective see, e.g., Grinblatt and Keloharju (2001), Stulz and Williamson (2003), and Guiso, Sapienza, and Zingales (2008)), we construct a variable that combines the six Hofstede cultural measures (available at https://geert-hofstede.com/). In line with Karolyi (2016), this variable is calculated as the root mean squared difference between the respective cultural measures for each pair of economies (for the Euro-zone we first average the measures for its largest members).

Each column in Table 6 refers to a regression of the 45 individual average Hellinger measures $H_t^{[i,j]}$ on the corresponding values of some combination of the above four explanatory variables, and shows estimated intercept and slope(s), corresponding p-values based on White's standard errors (in curly brackets), and adjusted R^2. The regressions indicate that these variables describe a large part of the variation in average Hellinger measures. Besides, the four variables capture distinct dimensions of the differences between economies, since the significance of the slope estimates is largely preserved in the regressions combining several of these variables.

Additionally, we note that the variance (covariance) proxies are positively (negatively) related to the Hellinger measures, consistent with the implications of Examples 1 and 2, and with the illustrative model in Section 2.2. While we use the distance between capitals to show that the slope coefficients on physical distance are positive, the results are practically identical for all four versions of the distance measure in Mayer and Zignago (2011). We further find that the cultural measure proxy is positively and significantly related to the Hellinger measures. Finally, the intercept in the full regression specification (Column I) is statistically insignificant, affirming the explanatory power of the variables for the Hellinger measures.

These results show that the information contained in physical distance and cultural differences is essential for understanding the dissimilarities between economies, as also reflected in the risk-neutral distribution of currency returns.

4. Conclusion

In this paper, we have developed and operationalized a measure of dissimilarity for SDFs in different economies. By using the bond prices as the numeraire, we ensure comparability. While the SDF is a fundamental object embedding information about discounting and risk, such a formal measure has not been proposed previously, even though the international finance literature has explored multiple other dimensions of distinction among economies.

The proposed measure follows from the Hellinger metric of distance between probability measures, and inherits from it features like boundedness and symmetry. The measure allows to gauge how different or similar the SDFs of various pairs of economies are. The measure is not denominated in any currency unit and is hence dimensionless. Intuitively, the measure of dissimilarity reflects how economy-specific Radon-Nikodym derivatives (its square-root) are distinct from one another and can be extracted from a portfolio of currency options when international markets are complete.

Unlike certain static indicators of difference, the Hellinger measure is time-varying and can reflect the dynamics of dissimilarity. We characterize the economic nature of the Hellinger measure and its mapping to the risk-neutral distribution of currency returns.

The empirical implementation relies on data for 10 industrialized economies and finds a significant variation in the Hellinger measures across them, as well as a geographical pattern, which, in particular, sets Japan apart from the remaining economies. The empirical exercise also reveals a pronounced factor structure in the Hellinger measures.

We argue that the Hellinger measure can be used as a criterion for judging whether a parameterized international economy driven by a system of SDFs agrees with the risk-neutral distribution of currency returns, and is in the spirit of Hansen and Jagannathan (1991), but in a two-economy context. We generate model-based Hellinger measures from several international asset pricing models, and outline certain hurdles that some of them face in reproducing the values obtained from the data.

A. Appendix A: Model with time-varying probability of disasters

Given the setting in equations (54)–(58), we need to solve three partial integro-differential equations, for (i) $\mathbb{E}_t^{\mathbb{P}}(\sqrt{m_{t+\tau} m^*_{t+\tau}})$, (ii) $\mathbb{E}_t^{\mathbb{P}}(m_{t+1})$, and (iii) $\mathbb{E}_t^{\mathbb{P}}(m^*_{t+1})$. We conjecture solutions and solve the resulting ordinary differential equations, details for which are provided in Internet Appendix VI.

The final expression for the Hellinger measure is displayed in equation (59). We present $a_j(\tau)$ and $b_j(\tau)$, for $j = 1, \ldots, 3$ below:

$$\begin{aligned}
a_1(\tau) &= \left(-\frac{1}{8}(\sigma^2+\sigma_*^2)+\frac{1}{4}\sigma\sigma_*\rho\right)\tau \\
&\quad -\kappa\theta\left(\frac{v_1-\kappa}{\eta^2}\right)\tau-\frac{2\kappa\theta}{\eta^2}\log\left(1-\frac{(v_1-\kappa)\left(1-e^{-\tau v_1}\right)}{2v_1}\right), && (60)\\
a_2(\tau) &= -\kappa\theta\left(\frac{v_2-\kappa}{\eta^2}\right)\tau-\frac{2\kappa\theta}{\eta^2}\log\left(1-\frac{(v_2-\kappa)\left(1-e^{-\tau v_2}\right)}{2v_2}\right), \text{ and} && (61)\\
a_3(\tau) &= -\kappa\theta\left(\frac{v_3-\kappa}{\eta^2}\right)\tau-\frac{2\kappa\theta}{\eta^2}\log\left(1-\frac{(v_3-\kappa)\left(1-e^{-\tau v_3}\right)}{2v_3}\right), && (62)
\end{aligned}$$

with

$$\begin{aligned}
c_1 &= \exp\left(-\frac{1}{2}(\Lambda+\Lambda_*)\mu_z+\frac{1}{8}\sigma_z^2(\Lambda+\Lambda_*)^2\right)-1, & v_1 &= \sqrt{\kappa^2-2\eta^2 c_1}, && (63)\\
c_2 &= \exp\left(-\Lambda\mu_z+\frac{1}{2}\sigma_z^2\Lambda^2\right)-1, & v_2 &= \sqrt{\kappa^2-2\eta^2 c_2}, \text{ and} && (64)\\
c_3 &= \exp\left(-\Lambda_*\mu_z+\frac{1}{2}\sigma_z^2\Lambda_*^2\right)-1. & v_3 &= \sqrt{\kappa^2-2\eta^2 c_3}, && (65)
\end{aligned}$$

and

$$b_j(\tau)=\frac{2c_j\left(1-e^{-\tau v_j}\right)}{2v_j-(v_j-\kappa)\left(1-e^{-\tau v_j}\right)}, \qquad \text{for } j=1,\ldots,3. \tag{66}$$

■

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Table 1

Contribution of non-normalities in $(\log(m_{t+1}), \log(m^*_{t+1}))$ to the Hellinger measure

Each Hellinger measure is computed as in equation (17):

$$H_t \;=\; \sqrt{\frac{R^2_{f,t+1}}{16F_t}\left(\int_{\{K>F_t\}} \frac{C_t[K]}{K^{3/2}}\,dK \;+\; \int_{\{K<F_t\}} \frac{P_t[K]}{K^{3/2}}\,dK\right)},$$

where $C_t[K]$ ($P_t[K]$) is the price of a call (put) on the foreign exchange with strike price K and a constant maturity of 30 days, F_t is the (30 day) forward exchange rate, and $R_{f,t+1}$ is the gross return on the domestic risk-free bond. We compute the empirical analog to equation (30) using *daily* currency returns within a *month* as (Δt is one day)

$$\mathcal{H}_t \;\equiv\; 1-\exp\Big(-\frac{1}{8}\sum_{n=1}^{22}\{\log(\frac{S_{t+n\Delta t}}{S_{t+(n-1)\Delta t}})\}^2\Big).$$

We emphasize that a square-root transformation is applied to each Hellinger measure and multiplied by 100 (for ease of reporting). Thus, we define the following:

$$\begin{aligned}
H_t^{[i,j]} &\equiv \sqrt{H} \text{ (based on monthly data, in \%) for economy pair } (i,j) \text{ in month } t, \\
\overline{H}_t^{[45]} &\equiv \text{Cross-sectional average of } H_t^{[i,j]} \text{ for all 45 pairs of economies in month } t, \text{ and} \\
\overline{H}_t^{\mathrm{US},[9]} &\equiv \text{Cross-sectional average: } \tfrac{1}{9}\textstyle\sum_{j=1}^{9} H_t^{[\mathrm{US},j]} \text{ in month } t.
\end{aligned}$$

Likewise

$$\begin{aligned}
\mathcal{H}_t^{[i,j]} &\equiv \sqrt{\mathcal{H}} \text{ (based on monthly data, in \%) for economy pair } (i,j) \text{ in month } t, \\
\overline{\mathcal{H}}_t^{[45]} &\equiv \text{Cross-sectional average of } \mathcal{H}_t^{[i,j]} \text{ for all 45 pairs of economies in month } t, \text{ and} \\
\overline{\mathcal{H}}_t^{\mathrm{US},[9]} &\equiv \text{Cross-sectional average: } \tfrac{1}{9}\textstyle\sum_{j=1}^{9} \mathcal{H}_t^{[\mathrm{US},j]} \text{ in month } t.
\end{aligned}$$

The deviations $\log(\overline{H}_t^{[45]}/\overline{\mathcal{H}}_t^{[45]})$ and $\log(\overline{H}_t^{\mathrm{US},[9]}/\overline{\mathcal{H}}_t^{\mathrm{US},[9]})$ represent the contribution of non-normalities in $(\log(m_{t+1}), \log(m^*_{t+1}))$ to the Hellinger measure inferred from options market. The sample period is from 1/1996 to 6/2014 (and from 1/1999 for pairs including NO or EU).

	Properties				Percentiles				
	Mean	Std.	Min.	Max.	5th	25th	50th	75th	95th
A. Hellinger measures based on currency option prices (constant maturity of 30 days)									
$\overline{H}_t^{[45]}$	1.14	0.33	0.64	3.39	0.80	0.92	1.09	1.26	1.65
$\overline{H}_t^{\mathrm{US},[9]}$	1.12	0.34	0.59	3.34	0.73	0.94	1.08	1.24	1.64
B. Hellinger measures based on variance of currency returns (monthly, from daily data)									
$\overline{\mathcal{H}}_t^{[45]}$	1.05	0.36	0.55	3.90	0.69	0.84	0.99	1.13	1.66
$\overline{\mathcal{H}}_t^{\mathrm{US},[9]}$	1.03	0.37	0.48	3.53	0.61	0.83	0.97	1.15	1.77
C. Deviations									
$\log(\overline{H}_t^{[45]}/\overline{\mathcal{H}}_t^{[45]})$, %	10.3	12.8	-19.4	47.0	-10.5	1.2	10.0	18.2	31.9
$\log(\overline{H}_t^{\mathrm{US},[9]}/\overline{\mathcal{H}}_t^{\mathrm{US},[9]})$, %	11.1	14.4	-19.7	53.0	-10.0	1.1	10.5	19.9	38.4

Table 2

Economy-specific Hellinger measures versus average measure across all economies

The Hellinger measure is computed as in equation (17):

$$H_t \;=\; \sqrt{\frac{R_{f,t+1}^2}{16F_t}\left(\int_{\{K>F_t\}}\frac{C_t[K]}{K^{3/2}}\,dK \;+\; \int_{\{K<F_t\}}\frac{P_t[K]}{K^{3/2}}\,dK\right)},$$

where $C_t[K]$ ($P_t[K]$) is the price of a call (put) on the foreign exchange with strike price K and a constant maturity of 30 days, F_t is the (30 day) forward exchange rate, and $R_{f,t+1}$ is the gross return on the domestic risk-free bond. We define the following:

$$\begin{aligned}
H_t^{[i,j]} &\equiv \sqrt{H} \text{ (based on monthly data, in \%) for economy pair } (i,j) \text{ in month } t,\\
\overline{H}_t^{i,[9]} &\equiv \text{Cross-sectional average: } \tfrac{1}{9}\textstyle\sum_{j=1}^{9} H_t^{[i,j]} \text{ for economy } i \text{ in month } t, \text{ and}\\
\overline{H}_t^{[45]} &\equiv \text{Cross-sectional average of } H_t^{[i,j]} \text{ for all 45 pairs of economies in month } t.
\end{aligned}$$

The first column in the table shows the mean *difference* between each of the economy-specific average measures $\overline{H}_t^{i,[9]}$ and $\overline{H}_t^{[45]}$. The remaining columns show 95% confidence intervals for these mean differences, obtained with 10,000 stationary bootstrap samples, as well as the respective minimums and maximums of the bootstrapped mean differences. The sample period is from 1/1996 to 6/2014 (and from 1/1999 for pairs including NO or EU). All numbers are reported in percent.

$\overline{H}_t^{i,[9]} - \overline{H}_t^{[45]}$	Mean	[95% CI]	Minimum	Maximum
Euro-zone (EU)	-0.17	[-0.21 -0.13]	-0.26	-0.12
United Kingdom (UK)	-0.09	[-0.10 -0.07]	-0.11	-0.05
Switzerland (SW)	-0.07	[-0.11 -0.03]	-0.14	0.02
Norway (NO)	-0.05	[-0.08 -0.03]	-0.11	0.00
Sweden (SD)	-0.03	[-0.06 -0.01]	-0.08	0.01
United States (US)	-0.01	[-0.08 0.05]	-0.15	0.09
Canada (CA)	0.00	[-0.04 0.04]	-0.07	0.08
Australia (AU)	0.04	[0.01 0.07]	-0.01	0.11
New Zealand (NZ)	0.12	[0.08 0.17]	0.05	0.22
Japan (JP)	0.23	[0.16 0.32]	0.09	0.43

Table 3

Individual Hellinger measures versus $\overline{H}_t^{[45]}$

For each pair of G-10 economies, we calculate the Hellinger measure as per equation (17). For our purposes, we compute the following:

$$\begin{aligned} H_t^{[i,j]} &\equiv \sqrt{H} \text{ (based on monthly data, in \%) for economy pair } (i,j) \text{ in month } t, \quad \text{and} \\ \overline{H}_t^{[45]} &\equiv \text{Cross-sectional average of } H_t^{[i,j]} \text{ for all 45 pairs of economies in month } t. \end{aligned}$$

We show the mean *difference* between each individual Hellinger measure $H_t^{[i,j]}$ and $\overline{H}_t^{[45]}$. The differences shown in bold are significant at the 95% confidence level, using stationary bootstrap. The sample period is from 1/1996 to 6/2014 (from 1/1999 for pairs including NO or EU), and the numbers are in percent. The economies are denoted as in Table 2.

	UK	SW	NO	SD	US	CA	AU	NZ	JP
EU	**-0.30**	**-0.60**	**-0.38**	**-0.40**	-0.04	-0.04	-0.02	**0.10**	**0.17**
UK		**-0.20**	**-0.14**	**-0.14**	**-0.20**	**-0.11**	-0.01	**0.09**	**0.16**
SW			**-0.24**	**-0.24**	-0.02	**0.05**	**0.14**	**0.23**	**0.08**
NO				**-0.37**	**0.11**	0.03	0.06	**0.15**	**0.30**
SD					0.08	**0.08**	**0.07**	**0.18**	**0.32**
US						**-0.27**	0.08	**0.16**	0.04
CA							-0.05	0.06	**0.25**
AU								**-0.29**	**0.37**
NZ									**0.43**

Table 4
Hellinger and Chi-squared measures from minimum discrepancy problems

First we solve the optimization (minimum discrepancy) problems in equations (40) and (41), with the U.S. as the domestic economy and each of seven other economies as the foreign one. Comparable data on bond returns for Norway (NO) and Sweden (SD) is not available. Then we use the time series of the extracted Radon-Nikodym derivatives $\widetilde{n}_{t+1}$ and $\widetilde{n}^*_{t+1}$ to compute the Hellinger measure in equation (3) and the Chi-squared measure in equation (23), whereby:

- $R_{f,t+1}$ and $R^*_{f,t+1}$ are gross returns of the domestic and foreign risk-free bonds (in local currency);
- $R_{\text{bond},t+1}$ and $R^*_{\text{bond},t+1}$ are gross returns of the domestic and foreign bonds with constant maturity of ten years (in local currency);
- $R_{\text{equity},t+1}$ and $R^*_{\text{equity},t+1}$ are gross returns of the domestic and foreign equity (MSCI, total return, in local currency).

The vectors of (scaled) asset returns employed in our calculations are:

$$\underbrace{\mathbf{A}_{t+1}}_{6\times 1} = \frac{1}{R_{f,t+1}} \underbrace{\begin{pmatrix} R_{f,t+1} \\ R_{\text{bond},t+1} \\ R_{\text{equity},t+1} \\ (\frac{S_{t+1}}{S_t})R^*_{f,t+1} \\ (\frac{S_{t+1}}{S_t})R^*_{\text{bond},t+1} \\ (\frac{S_{t+1}}{S_t})R^*_{\text{equity},t+1} \end{pmatrix}}_{\mathbf{R}_{t+1}} \qquad \text{and} \qquad \underbrace{\mathbf{A}^*_{t+1}}_{6\times 1} = \frac{1}{R^*_{f,t+1}} \underbrace{\begin{pmatrix} (\frac{S_t}{S_{t+1}})R_{f,t+1} \\ (\frac{S_t}{S_{t+1}})R_{\text{bond},t+1} \\ (\frac{S_t}{S_{t+1}})R_{\text{equity},t+1} \\ R^*_{f,t+1} \\ R^*_{\text{bond},t+1} \\ R^*_{\text{equity},t+1} \end{pmatrix}}_{\mathbf{R}^*_{t+1}},$$

where S_t is the level of the exchange rate with the foreign currency as the reference. We report the square root (in %) of the unconditional Hellinger measure given by $\frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}(\sqrt{\widetilde{n}_t}-\sqrt{\widetilde{n}^*_t})^2$, and the Chi-squared measure given by $\frac{1}{T}\sum_{t=1}^{T}2(\widetilde{n}_t-\widetilde{n}^*_t)^2/(\widetilde{n}_t+\widetilde{n}^*_t)$. The annualized volatility (shown in %) of the (unit mean) Radon-Nikodym derivatives is $(\frac{12}{T}\sum_{t=1}^{T}(\widetilde{n}_t-1)^2)^{1/2}$ and $(\frac{12}{T}\sum_{t=1}^{T}(\widetilde{n}^*_t-1)^2)^{1/2}$, respectively. T is the sample length in months, and all samples end in 12/2016.

	T	Hellinger measure ($\sqrt{H}\times 100$)	Chi-squared measure	Volatility $\widetilde{n}$ (%)	$\widetilde{n}^*$ (%)
EU	191	0.81	0.096	105	104
UK	336	0.74	0.081	77	76
SW	336	0.89	0.107	84	84
CA	319	0.62	0.056	85	84
AU	336	0.97	0.127	74	70
NZ	309	0.98	0.138	76	70
JP	336	0.91	0.113	92	95

Table 5

Hellinger measures in international macro-finance models

We present the Hellinger measures computed from four international models: (i) Model A: Verdelhan (2010) (equation (45)), (ii) Model B: Lustig, Roussanov, and Verdelhan (2014) (equation (49)), (iii) Model C: Colacito and Croce (2011) (equation (53)), and (iv) Model D: model with time-varying probability of disasters (equation (59)). The parameters in Models A, B, and C are as specified in the respective studies.

Model A.	g (%)	σ (%)	γ	ϕ	ρ	β	$\bar{s}$	s_{max}
	0.53	0.51	2.00	0.995	0.15	1.00	log(0.07)	log(0.12)

Model B.	α (%)	χ	τ	γ	κ	ϕ	θ (%)	σ (%)	ϕ^w	θ^w (%)	σ^w (%)
	0.76	0.89	0.06	0.04	2.78	0.91	0.77	0.68	0.99	2.09	0.28

Model C.	γ	ψ	σ	κ_c	κ_x	φ_e	ρ_c^{hf}	ρ_x^{hf}
	4.25	2.0	0.0068	0.997	0.987	0.048	0.30	1.00

Model D.	σ	σ^*	ρ	Λ	Λ^*	κ	θ	η	μ_z	σ_z
	0.30	0.30	0.95	1.2	0.8	0.08	0.0355	0.067	-0.15	0.15

From each model, we simulate 10,000 series of measures, each of length 222 as in our data, and calculate the respective time-series averages. Displayed are the mean of these statistics, as well as the 2.5 and 97.5 percentiles of the simulated distribution of the time-series averages. We employ a square-root transformation to each Hellinger measure and then multiplied it by 100 (for ease of reporting). We consider three different parameterizations of δ and δ^* for Model B. The last four columns refer to the average measure $\overline{H}_t^{[45]}$, $\overline{H}_t^{\text{US},[9]}$, and the smallest (i.e., EU|SW), and largest (i.e., NZ|JP) Hellinger measure as per Table 3. In our calculations, $\overline{H}_t^{\text{US},[9]}$ is the cross-sectional average $\frac{1}{9}\sum_{j=1}^{9} H_t^{[\text{US},j]}$ in month t. To obtain the bootstrap confidence intervals for the reported average measures obtained in the data, we first fit all ARMA(p,q) models – with $p \leq 2$ and $q \leq 2$ – to the log of the Hellinger measures. Next, we simulate from the best model according to BIC (which turns out in each case to be AR(1)).

	Models						Data			
	A	B			C	D				
		(i)	(ii)	(iii)						
δ		0.22	0.22	0.35			$\overline{H}_t^{[45]}$	$\overline{H}_t^{\text{US},[9]}$	Smallest	Largest
δ^*		0.49	0.36	0.36					EU|SW	NZ|JP
Mean H	2.93	1.60	1.26	1.07	1.21	1.31	1.14	1.12	0.54	1.56
							Bootstrap			
2.5 perc.	1.96	1.53	1.18	0.98	-	1.30	0.96	0.95	0.39	1.27
97.5 perc.	4.37	1.69	1.36	1.18	-	1.34	1.30	1.30	0.69	1.84

Table 6

Hellinger measure and other dissimilarity variables

We regress the averages of the 45 individual Hellinger measures $H_t^{[i,j]}$ (as defined in equation (31)) on the corresponding values of several explanatory variables: $\overline{H}^{[i,j]} = \Psi_0 + \sum_k \Psi^k X_k^{[i,j]} + e^{[i,j]}$. The explanatory variables $X_k^{[i,j]}$ are (i) average variances of the monthly equity returns of the respective two economies (times 100), (ii) covariance between the monthly equity returns of these economies (times 100), (iii) log distance between the corresponding pair of capital cities (divided by 100), (iv) square root of the sum of squared differences between six cultural measures of the respective two economies (divided by 100). The regression models I to VII include various combinations of the explanatory variables. Equity returns are calculated from MSCI indexes, and the cultural measures are from Hofstede, Hofstede, and Minkov (2010) (available at https://geert-hofstede.com/). The p-values, based on White's standard errors that allow for heteroscedasticity, are reported in curly brackets, and the adjusted R^2s are in percent.

	I	II	III	IV	V	VI	VII
Ψ_0	0.01	0.00	0.04	-0.01	0.04	-0.01	0.03
	{0.38}	{0.68}	{0.00}	{0.09}	{0.00}	{0.25}	{0.00}
Ψ^{var}	0.02	0.06	0.04		0.08		
	{0.10}	{0.00}	{0.00}		{0.00}		
Ψ^{cov}	-0.06	-0.08	-0.13		-0.17		
	{0.00}	{0.00}	{0.00}		{0.00}		
Ψ^{distance}	0.36	0.40		0.49		0.52	
	{0.00}	{0.00}		{0.00}		{0.00}	
Ψ^{culture}	0.01		0.01	0.01			0.01
	{0.00}		{0.00}	{0.00}			{0.00}
$\overline{R}^2$ (%)	78	68	63	75	49	63	19

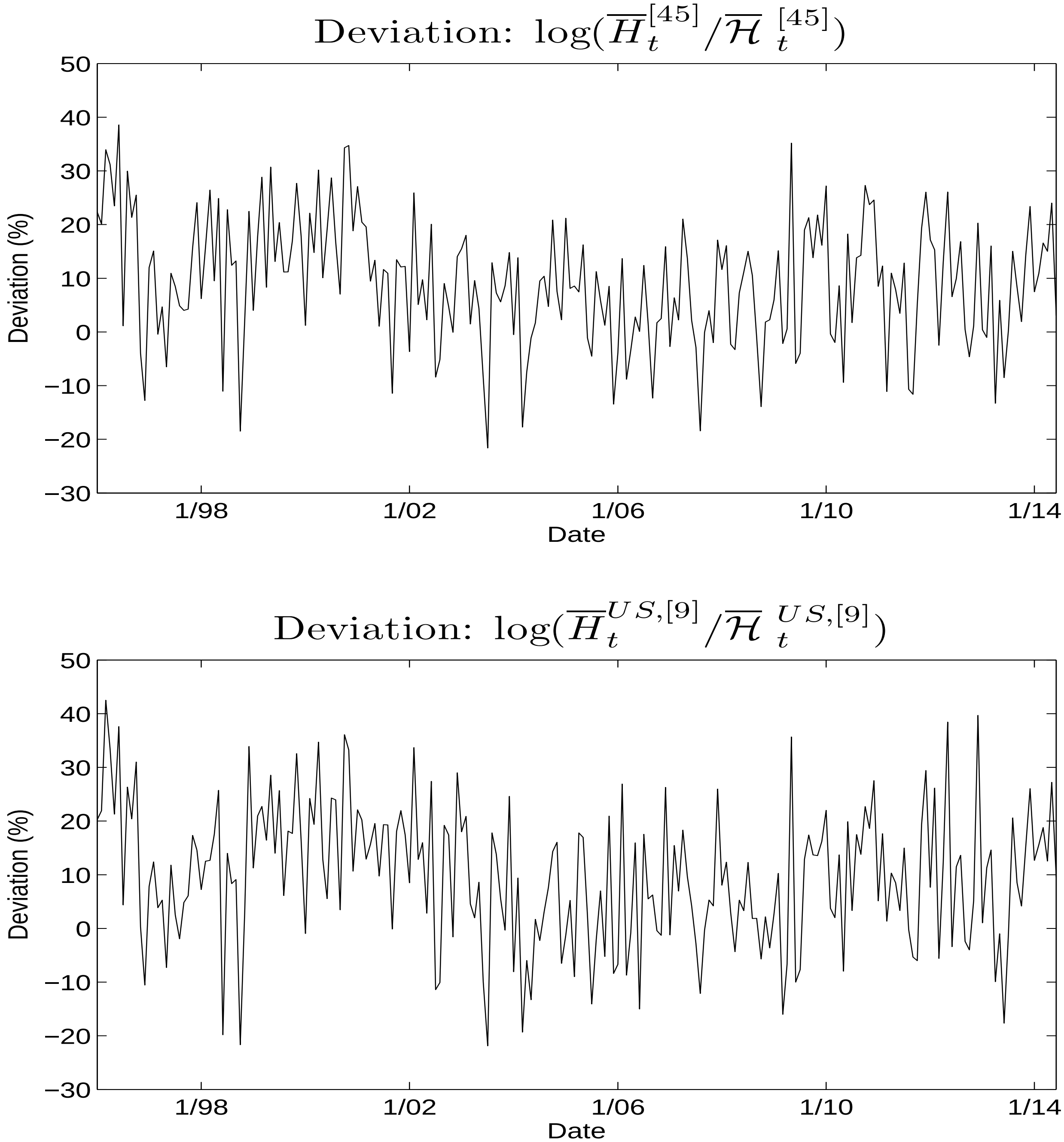

Figure 1. **Depicting the impact of distributional non-normalities in** $(\log(m_{t+1}), \log(m^*_{t+1}))$ **on the Hellinger measure**

The figure plots the time series of the deviations $\log(\overline{H}_t^{[45]}/\overline{\mathcal{H}}_t^{[45]})$ and $\log(\overline{H}_t^{\mathrm{US},[9]}/\overline{\mathcal{H}}_t^{\mathrm{US},[9]})$, over the sample of 1/1996 to 6/2014. We compute H_t using equation (17) of Proposition 1, and as described in equations (31)–(33). In contrast, we compute $\mathcal{H}_t$ using the expression in equation (30), utilizing daily (log) currency returns to calculate the conditional variance $\mathrm{var}_t^{\mathbb{P}}(\log(\frac{S_{t+1}}{S_t}))$. As reported in the tables, a square-root transformation is applied to each Hellinger measure and multiplied by 100.

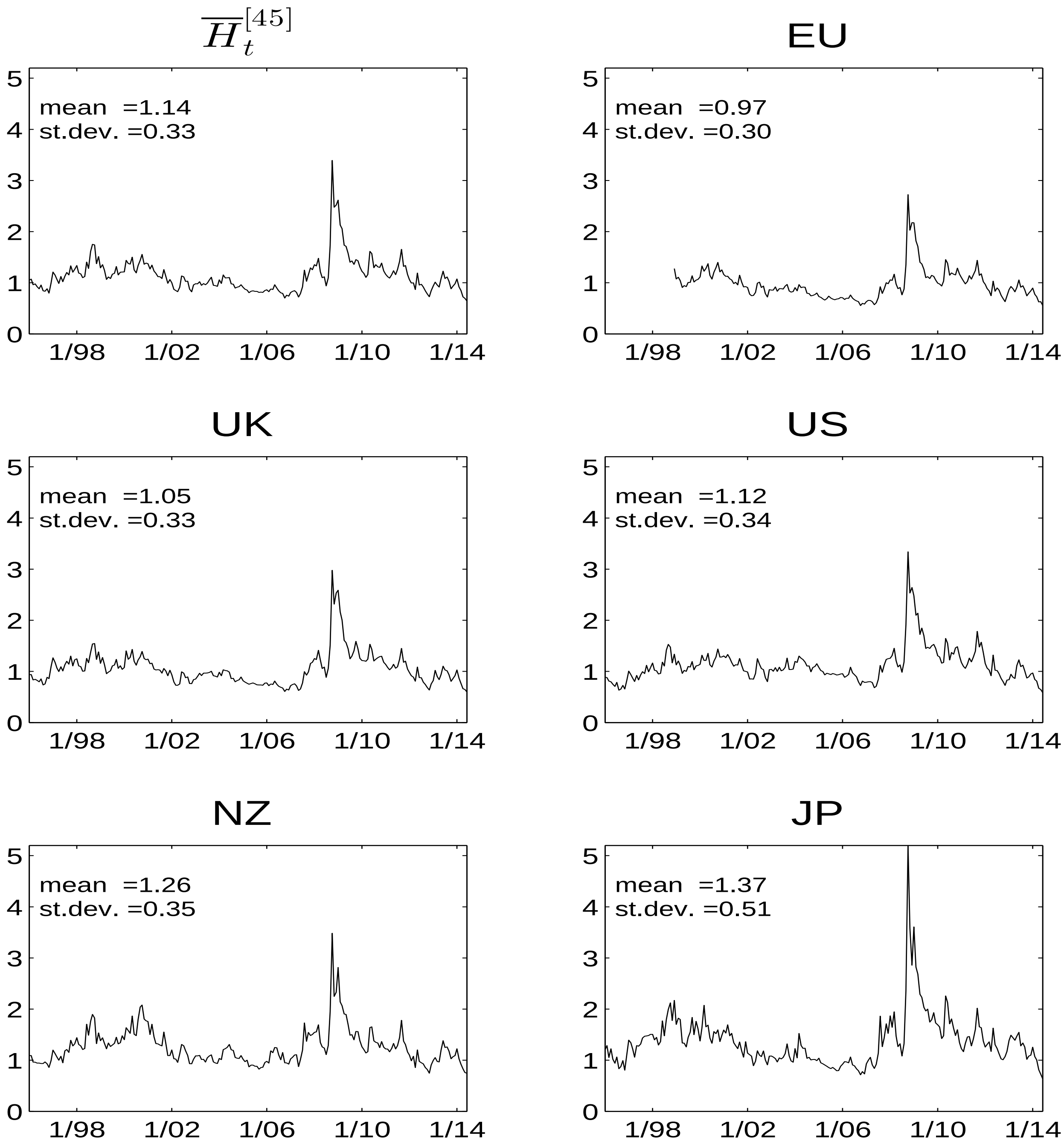

Figure 2. **Averages of Hellinger measures**

The figure plots the time series of Hellinger measures (i.e., $\sqrt{H} \times 100$). That is, as reported in the tables, a square-root transformation is applied to each Hellinger measure and multiplied by 100. The first plot shows the cross-sectional average $\overline{H}_t^{[45]}$, as defined in equation (33), and the remaining plots show economy-specific measures $\overline{H}_t^{i,[9]}$, as in equation (36) for several economies. The sample period is from 1/1996 to 6/2014 (from 1/1999 for EU).

A Theory of Dissimilarity Between Stochastic Discount Factors

Gurdip Bakshi, Xiaohui Gao, and George Panayotov

Internet Appendix: Not for Publication

Abstract

This Internet Appendix provides the proofs of results presented in the main text. We derive the Hellinger measure for a model with constant probability of disasters in Section I. The proof of expression (17) in Proposition 1 is in Section II, while Section III shows that the Hellinger measure is distinct from codependence. The details of the data construction are in Section IV. Section V derives the Hellinger measure displayed in equation (30), when (m_{t+1}, m^*_{t+1}) is distributed log-normal. The steps leading to the Hellinger measure in the model with time-varying probability of disasters are in Section VI.

I. Hellinger measure in a model with disasters

Based on the dynamics of $\log(\frac{M_{t+1}}{M_t})$ and $\log(\frac{M^*_{t+1}}{M^*_t})$ in equation (10), and the moment generating function of $\sum_{i=N_t}^{N_{t+1}} z_i$ (e.g., Privault (2016, Proposition 15.6)),

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}_t(\sqrt{m_{t+1}\, m^*_{t+1}}) &= \mathbb{E}^{\mathbb{P}}_t\left(\exp(-\frac{1}{2}(\mu+\mu^*) - \frac{1}{2}\sigma\varepsilon_{t+1} - \frac{1}{2}\sigma_*\varepsilon^*_{t+1})\right)\mathbb{E}^{\mathbb{P}}_t\left(\exp(\{-\frac{\Lambda}{2} - \frac{\Lambda_*}{2}\}\sum_{i=N_t}^{N_{t+1}} z_i)\right), \\
&= \exp\left(-\frac{1}{2}\mu + \frac{1}{2}(\frac{\sigma}{2})^2 - \frac{1}{2}\mu_* + \frac{1}{2}(\frac{\sigma_*}{2})^2 + \frac{1}{4}\sigma_m\sigma_*\rho + \lambda(e^{d_0}-1)\right), \qquad \text{(A1)}
\end{aligned}$$

where $d_0 \equiv \{-\frac{1}{2}\Lambda - \frac{1}{2}\Lambda_*\}\mu_z + \frac{1}{2}\{-\frac{1}{2}\Lambda - \frac{1}{2}\Lambda_*\}^2\sigma_z^2$. Additionally,

$$\begin{aligned}
\sqrt{\mathbb{E}^{\mathbb{P}}_t(m_{t+1})} &= \exp\left(-\frac{1}{2}\mu + \frac{1}{4}\sigma^2 + \frac{1}{2}\lambda(e^{-\Lambda\mu_z + \frac{1}{2}\Lambda^2\sigma_z^2} - 1)\right), \qquad \text{(A2)} \\
\sqrt{\mathbb{E}^{\mathbb{P}}_t(m^*_{t+1})} &= \exp\left(-\frac{1}{2}\mu_* + \frac{1}{4}\sigma_*^2 + \frac{1}{2}\lambda(e^{-\Lambda_*\mu_z + \frac{1}{2}\Lambda_*^2\sigma_z^2} - 1)\right). \qquad \text{(A3)}
\end{aligned}$$

Therefore, we have established the expression for H_t presented in equation (11). ■

II. Proof of Proposition 1

The proof involves computing the intrinsic value of a claim with a payoff equal to $\sqrt{\frac{S_{t+1}}{S_t}}$, which is determined using a positioning in currency calls and puts.

Any twice-continuously differentiable payoff function $G[S_{t+1}]$ with bounded expectation can be synthesized as per Bakshi and Madan (2000, Appendix A.3) and Carr and Madan (2001, equation (1)):

$$\begin{aligned}
G[S_{t+1}] &= G[F_t] + G_S[F_t](S_{t+1} - F_t) \\
&\quad + \int_{\{K>F_t\}} G_{SS}[K]\,(S_{t+1}-K)^+\,dK + \int_{\{K<F_t\}} G_{SS}[K]\,(K - S_{t+1})^+\,dK, \qquad \text{(B1)}
\end{aligned}$$

where $a^+ \equiv \max(a,0)$. $G_S[F_t]$ is the first-order derivative of the payoff $G[S_{t+1}]$ with respect to S_{t+1} evaluated at F_t, and $G_{SS}[K]$ is the second-order derivative, with respect to S_{t+1} evaluated at K. The terms under the integrals in equation (B1) are weighted payoffs of European put or call options. In our context, consider $G[S_{t+1}] = (\frac{S_{t+1}}{S_t})^{1/2}$. Then

$$G[F_t] = \left(\frac{S_{t+1}}{S_t}\right)^{1/2}\Big|_{S_{t+1}=F_t} = \sqrt{\frac{F_t}{S_t}}, \tag{B2}$$

$$G_S[F_t] = \frac{dG[S_{t+1}]}{dS_{t+1}}\Big|_{S_{t+1}=F_t} = \frac{(1/2)}{S_t^{1/2}}(S_{t+1})^{-1/2}\Big|_{S_{t+1}=F_t} = \frac{1}{2\sqrt{S_t F_t}}, \text{ and} \tag{B3}$$

$$G_{SS}[K] = \frac{d^2G[S_{t+1}]}{dS_{t+1}^2}\Big|_{S_{t+1}=K} = \frac{-1}{4S_t^{1/2}}S_{t+1}^{-3/2}\Big|_{S_{t+1}=K} = \frac{-1}{4S_t^{1/2}}K^{-3/2}. \tag{B4}$$

The value of $\mathbb{E}_t^{\mathbb{Q}}\left(\sqrt{\frac{S_{t+1}}{S_t}}\right)$ in equation (16) is then a sum of level, slope, and curvature terms:

$$\text{Level term}: \quad \sqrt{\frac{F_t}{S_t}} = \sqrt{\frac{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}} = \sqrt{\frac{R_{f,t+1}}{R_{f,t+1}^*}}. \tag{B5}$$

$$\text{Slope term}: \quad G_S[F_t]\,\mathbb{E}_t^{\mathbb{Q}}(S_{t+1}-F_t) = \frac{G_S[F_t]}{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}\underbrace{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}(S_{t+1}-F_t))}_{\text{zero cost of forward}} = 0. \tag{B6}$$

$$\text{Curvature term for call}: \quad \frac{-K^{-3/2}}{4S_t^{1/2}\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}\underbrace{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}(S_{t+1}-K)^+)}_{\equiv C_t[K]}. \tag{B7}$$

$$\text{Curvature term for put}: \quad \frac{-K^{-3/2}}{4S_t^{1/2}\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}\underbrace{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}(K-S_{t+1})^+)}_{\equiv P_t[K]}. \tag{B8}$$

Equation (16) then implies $H_t = 1 - \sqrt{\frac{S_t}{F_t}}(\sqrt{\frac{F_t}{S_t}} + \int_{\{K>F_t\}} \frac{-K^{-3/2}}{4S_t^{1/2}\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}C_t[K]dK + \int_{\{K<F_t\}} \frac{-K^{-3/2}}{4S_t^{1/2}\mathbb{E}_t^{\mathbb{P}}(m_{t+1})}P_t[K]dK)$. To obtain the final expression, note that

$$\sqrt{\frac{S_t}{F_t}}\,\frac{-K^{-3/2}}{4\sqrt{S_t}\,\mathbb{E}_t^{\mathbb{P}}(m_{t+1})} \text{ is the same as } -K^{-3/2}\sqrt{\frac{R_{f,t+1}^2}{16F_t}}. \tag{B9}$$

The proof of equation (17) of Proposition 1 is complete. ■

III. Hellinger measure is distinct from codependence

Consider two positive random variables x_{t+1} and y_{t+1}. Codependence, as in Hansen (2012, page 930) or Chabi-Yo and Colacito (2017, equation (1)), is denoted by $\mathbf{c}_t[x,y]$ and defined as

$$\mathbf{c}_t[x,y] \equiv L_t[x] + L_t[y] - L_t[xy], \qquad \text{where} \qquad L_t[a] \equiv -\mathbb{E}_t^{\mathbb{P}}(\log(a)) + \log(\mathbb{E}_t^{\mathbb{P}}(a)). \tag{C1}$$

Set $a = \sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}$, for $\widetilde{n}_{t+1}$ and $\widetilde{n}^*_{t+1}$ as defined in equation (2). Then

$$L_t[\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}] \;=\; -\mathbb{E}_t^{\mathbb{P}}(\log(\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}})) \;+\; \log(\mathbb{E}_t^{\mathbb{P}}\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}). \tag{C2}$$

Rearrange the above expression to consider

$$\begin{aligned}
\log(\mathbb{E}_t^{\mathbb{P}}\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}) \;&=\; \mathbb{E}_t^{\mathbb{P}}(\log(\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}})) + L[\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}], && \text{(C3)}\\
&=\; \frac{1}{2}\mathbb{E}_t^{\mathbb{P}}(\log(\widetilde{n}_{t+1})) + \frac{1}{2}\mathbb{E}_t^{\mathbb{P}}(\log(\widetilde{n}^*_{t+1})) + L[\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}], && \text{(C4)}\\
&=\; -\frac{1}{2}L[\widetilde{n}_{t+1}] \;+\; \frac{1}{2}\underbrace{\log(\mathbb{E}_t^{\mathbb{P}}(\widetilde{n}_{t+1}))}_{=0} \;-\; \frac{1}{2}L[\widetilde{n}^*_{t+1}] \\
&\qquad +\; \frac{1}{2}\underbrace{\log\mathbb{E}_t^{\mathbb{P}}((\widetilde{n}^*_{t+1}))}_{=0} \;+\; L[\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}], && \text{(C5)}\\
&=\; L[\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}] - \frac{1}{2}L[\widetilde{n}_{t+1}] - \frac{1}{2}L[\widetilde{n}^*_{t+1}]. && \text{(C6)}
\end{aligned}$$

Since $\mathbb{E}_t^{\mathbb{P}}(\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}) = 1 - H_t$ by definition, therefore,

$$H_t = 1 - \exp\left(L_t[\sqrt{\widetilde{n}_{t+1}\widetilde{n}^*_{t+1}}] - \frac{1}{2}L_t[\widetilde{n}_{t+1}] - \frac{1}{2}L_t[\widetilde{n}^*_{t+1}]\right). \tag{C7}$$

This expression is not quite of the form $\mathbf{c}_t[x,y] \equiv L_t[x] + L_t[y] - L_t[xy]$. Specifically, $\mathbf{c}_t[m,m^*] = -\text{cov}_t^{\mathbb{P}}(\log(m),\log(m^*))$ when $(m,\, m^*)$ is lognormal, whereas H_t is as displayed in (E5). ■

IV. Details of the data construction

Throughout, the remaining time to expiration of the options and forward contracts, τ, equals 30/360.

For ease of exposition, we employ the following notation (e.g., Wystup (2006) and Jurek (2014, Section 4.1)):

$S_t^{j,i}$: spot price of one unit of currency j (the foreign currency) in terms of currency i (the domestic currency).

$F_{t,t+\tau}^{j,i}$: τ period forward price of one unit of currency j (the foreign currency) in terms of currency i (the domestic currency).

r_t^i (r_t^j): τ-period matched (net) risk-free rate in the domestic (foreign) currency.

By currency market conventions, option data is quoted as 10 delta, 25 delta, and at-the-money call or put volatilities. We denote them by $\sigma[\delta_C]$ or $\sigma[\delta_P]$. The ATM volatility is derived from the price of a delta-neutral straddle, and the strike price K of the delta-neutral straddle satisfies $\delta_C[K] + \delta_P[K] = 0$.

Let K_{ATM}, K_{δ_C}, and K_{δ_P} be the strike prices for the respective call and put options and corresponding to call (respectively, put) deltas δ_C (respectively, δ_P).

We apply the following standard conversion formulas:

$$K_{\text{ATM}} = F_{t,t+\tau}^{j,i} \exp\left(\frac{1}{2}\sigma_t[\text{ATM}]^2\tau\right), \tag{D1}$$

$$K_{\delta_C} = F_{t,t+\tau}^{j,i} \exp\left(\frac{1}{2}\sigma_t^2[\delta_C]\tau - \sigma_t[\delta_C]\,\sqrt{\tau}\,N^{-1}\left[\exp(r_t^j\,\tau)\,\delta_C\right]\right), \text{ and} \tag{D2}$$

$$K_{\delta_P} = F_{t,t+\tau}^{j,i} \exp\left(\frac{1}{2}\sigma_t^2[\delta_P]\tau + \sigma_t[\delta_P]\,\sqrt{\tau}\,N^{-1}\left[-\exp(r_t^j\,\tau)\,\delta_P\right]\right). \tag{D3}$$

Via the put-call symmetry relations in Wystup (2006), we also obtain

$$
\begin{aligned}
C_t(S_t^{i,j}, \frac{1}{K}, \tau, r_t^j, r_t^i) &= \frac{1}{S_t^{j,i}K} P_t(S_t^{j,i}, K, \tau, r_t^i, r_t^j) \text{ and} && \text{(D4)} \\
P_t(S_t^{i,j}, \frac{1}{K}, \tau, r_t^j, r_t^i) &= \frac{1}{S_t^{j,i}K} C_t(S_t^{j,i}, K, \tau, r_t^i, r_t^j), && \text{(D5)}
\end{aligned}
$$

which yield the option prices for an underlying $S_t^{i,j}$, when only data for the same currency pair, but with the underlying quoted as $S_t^{j,i} = \frac{1}{S_t^{i,j}}$ is available. ■

V. Hellinger measure when (m_{t+1}, m_{t+1}^*) is distributed lognormal

Consider $H_t = 1 - \frac{1}{\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})\,\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)}} \mathbb{E}_t^{\mathbb{P}}\left(\sqrt{m_{t+1}\, m_{t+1}^*}\right)$, when (m_{t+1}, m_{t+1}^*) is jointly lognormal.

The moment generating function of the bivariate normal distribution of (x_{t+1}, y_{t+1}) is

$$
\mathbb{E}_t^{\mathbb{P}}\left(e^{\phi x_{t+1} + \varphi y_{t+1}}\right) = \exp\left(\phi \mu_x + \varphi \mu_y + \frac{1}{2}(\phi^2 \sigma_x^2 + \varphi^2 \sigma_y^2 + 2\,\phi\,\varphi\,\sigma_{x,y})\right), \qquad \text{(E1)}
$$

where $\mu_x(\mu_y)$ and σ_x^2 (σ_y^2) are, respectively, the expected value and variance, and $\sigma_{x,y}$ is the co-variance.

It then follows that

$$
\begin{aligned}
\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+1})} &= \exp\left(\frac{1}{2}\mathbb{E}_t^{\mathbb{P}}(\log(m_{t+1})) + \frac{1}{4}\mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}))\right) \text{ and} && \text{(E2)} \\
\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+1}^*)} &= \exp\left(\frac{1}{2}\mathbb{E}_t^{\mathbb{P}}(\log(m_{t+1}^*)) + \frac{1}{4}\mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}^*))\right). && \text{(E3)}
\end{aligned}
$$

The numerator of the Hellinger measure is

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}}(\sqrt{m_{t+1}m_{t+1}^*}) &= \exp\left(\frac{1}{2}\mathbb{E}_t^{\mathbb{P}}(\log(m_{t+1}))+\frac{1}{2}\mathbb{E}_t^{\mathbb{P}}(\log(m_{t+1}^*))+\frac{1}{8}\mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}))\right. \\ &+ \left.\frac{1}{8}\mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}^*))+\frac{1}{4}\mathrm{cov}_t^{\mathbb{P}}(\log(m_{t+1}),\log(m_{t+1}^*))\right). \end{aligned} \tag{E4}$$

Canceling terms and simplifying, we obtain the expression for the Hellinger measure (under the lognormality assumption for $(m_{t+1},\, m_{t+1}^*)$) presented below:

$$H_t^{\mathrm{LN}} = 1-\exp\left(-\frac{1}{8}\{\mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}^*))+\mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}))-2\,\mathrm{cov}_t^{\mathbb{P}}(\log(m_{t+1}^*),\log(m_{t+1}))\}\right). \tag{E5}$$

Since $\frac{S_{t+1}}{S_t}=\frac{m_{t+1}^*}{m_{t+1}}$, we have the identity that

$$\mathrm{var}_t^{\mathbb{P}}(\log(\frac{S_{t+1}}{S_t})) = \mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}^*))+\mathrm{var}_t^{\mathbb{P}}(\log(m_{t+1}))-2\mathrm{cov}_t^{\mathbb{P}}(\log(m_{t+1}^*),\log(m_{t+1})). \tag{E6}$$

The final expression for H_t^{LN} presented in equation (30) is verified. ■

VI. Model with time-varying probability of disasters

In light of the dynamics of (M_t, M_t^*) in equations (54)–(58) (e.g., the setup in Runggaldier (2003, Section 3, equation (30))), consider the characteristic function of the remaining uncertainty

$$\mathcal{G}(t,\tau;\phi,\phi^*) \;=\; \mathbb{E}_t^{\mathbb{P}}\left(e^{i\phi\,\log(M_{t+\tau})+i\phi^*\,\log(M_{t+\tau}^*)}\right). \tag{F1}$$

$\mathcal{G}(t,\tau;\phi,\phi^*)$ solves the partial integro-differential equation

$$\frac{1}{2}\mathcal{G}_{MM}\sigma^2M^2+\mathcal{G}_M(-\mu)M+\frac{1}{2}\mathcal{G}_{M^*M^*}\sigma_*^2(M^*)^2+\mathcal{G}_{M^*}(-\mu_*)M^*+\mathcal{G}_{MM^*}\rho\sigma\sigma_*(MM^*)$$
$$+\frac{1}{2}\mathcal{G}_{\lambda\lambda}\eta^2\lambda+\mathcal{G}_\lambda\kappa(\theta-\lambda_t)-\mathcal{G}_\tau+\lambda\mathbb{E}_t^{\mathbb{P}}\left(\mathcal{G}[Me^{-\Lambda z},M^*e^{-\Lambda_* z}]-\mathcal{G}[M,M^*]\right)\;=\;0. \qquad \text{(F2)}$$

We also have the condition that $\mathcal{G}(t+\tau,0;\phi,\phi^*)=e^{i\phi\log(M_t)+i\phi^*\log(M_t)}$. The solution is

$$\mathcal{G}(t,\tau;\phi,\phi^*)=e^{a(\tau)+b(\tau)\lambda_t+i\phi\log(M_t)+i\phi^*\log(M_t^*)}. \qquad \text{(F3)}$$

Substituting the conjecture (F3) into equation (F2), we obtain

$$\frac{1}{2}(i\phi)(i\phi-1)\sigma^2+(i\phi)(-\mu)+\frac{1}{2}(i\phi^*)(i\phi^*-1)(\sigma_*)^2+(i\phi^*)(-\mu_*)+(i\phi)(i\phi^*)(\sigma\sigma_*)\rho$$
$$+\frac{1}{2}b^2(\tau)\eta^2\lambda+b(\tau)\kappa(\theta-\lambda_t)-a'(\tau)-b'(\tau)\lambda+\lambda_t\,\mathbb{E}_t^{\mathbb{P}}\left(e^{-i\phi\Lambda z-i\phi^*\Lambda_* z}-1\right)\;=\;0, \qquad \text{(F4)}$$

and also

$$\frac{1}{2}b^2(\tau)\eta^2-b(\tau)\kappa-b'(\tau)+\mathbb{E}_t^{\mathbb{P}}\left(e^{-i\phi\Lambda z-i\phi^*\Lambda_* z}-1\right)\;=\;0. \qquad \text{(F5)}$$

The solution to equation (F5) is

$$b(\tau;\phi,\phi^*)=\frac{2c\,(1-e^{-\tau v})}{2v-(v-\kappa)\,(1-e^{-\tau v})}, \qquad \text{(F6)}$$

which is verified by direct substitution. The coefficients c and v are

$$c[\phi,\phi^*]=\exp\left(-(i\phi\xi+i\phi^*\xi_*)\mu_z+\frac{1}{2}\sigma_z^2(i\phi\xi+i\phi^*\xi_*)^2\right)-1, \qquad v[\phi,\phi^*]\equiv\sqrt{\kappa^2-2\eta^2c}. \qquad \text{(F7)}$$

Moreover

$$\int_0^\tau b(s) = -\left(\frac{v-\kappa}{\eta^2}\right)\tau - \frac{2}{\eta^2}\log\left(1-\frac{(v-\kappa)\left(1-e^{-\tau v}\right)}{2v}\right). \tag{F8}$$

Therefore, we determine the solution to $a(\tau)$ in equation (F4) as

$$\begin{aligned} a(\tau;\phi,\phi^*) &= \{\frac{1}{2}(i\phi)(i\phi-1)\sigma^2+(i\phi)(-\mu)+\frac{1}{2}(i\phi^*)(i\phi^*-1)\sigma_*^2+(i\phi^*)(-\mu_*) \\ &+ (i\phi)(i\phi^*)\sigma\sigma_*\rho\}\,\tau-\kappa\theta\left(\frac{v-\kappa}{\eta^2}\right)\tau-\frac{2\kappa\theta}{\eta^2}\log\left(1-\frac{(v-\kappa)\left(1-e^{-\tau v}\right)}{2v}\right). \end{aligned} \tag{F9}$$

Completing the solution, define

$$\mathcal{G}_1(t,\tau;\phi) = \frac{\mathcal{G}(t,\tau;\phi,0)}{(M_t)^{i\phi}} = \mathbb{E}_t^{\mathbb{P}}\left(e^{i\phi\log\left(\frac{M_{t+\tau}}{M_t}\right)}\right), \tag{F10}$$

$$\mathcal{G}_2(t,\tau;\phi^*) = \frac{\mathcal{G}(t,\tau;0,\phi^*)}{(M_t^*)^{i\phi^*}} = \mathbb{E}_t^{\mathbb{P}}\left(e^{i\phi^*\log\left(\frac{M_{t+\tau}^*}{M_t^*}\right)}\right), \text{ and} \tag{F11}$$

$$\mathcal{G}_3(t,\tau;\phi,\phi^*) = \frac{\mathcal{G}(t,\tau;\phi,\phi^*)}{M_t^{i\phi}(M_t^*)^{i\phi^*}} = \mathbb{E}_t^{\mathbb{P}}\left(e^{i\phi\log\left(\frac{M_{t+\tau}}{M_t}\right)+i\phi^*\log\left(\frac{M_{t+\tau}^*}{M_t^*}\right)}\right). \tag{F12}$$

Therefore,

$$\mathbb{E}_t^{\mathbb{P}}(m_{t+\tau}) = \mathcal{G}_1(t,\tau;\frac{1}{i}), \tag{F13}$$

$$\mathbb{E}_t^{\mathbb{P}}(m_{t+\tau}^*) = \mathcal{G}_2(t,\tau;\frac{1}{i}), \qquad \text{and} \tag{F14}$$

$$\mathbb{E}_t^{\mathbb{P}}(\sqrt{m_{t+\tau}m_{t+\tau}^*}) = \mathcal{G}_3(t,\tau;\frac{1}{2i},\frac{1}{2i}). \tag{F15}$$

Thus, we can compute $H_t = 1-\frac{1}{\sqrt{\mathbb{E}_t^{\mathbb{P}}(m_{t+\tau})\mathbb{E}_t^{\mathbb{P}}(m_{t+\tau}^*)}}\mathbb{E}_t^{\mathbb{P}}\left(\sqrt{m_{t+\tau}m_{t+\tau}^*}\right)$.

The Hellinger measure varies with λ_t, and the final expression is displayed in equation (59), where $a_j(\tau)$ and $b_j(\tau)$, for $j=1,\ldots,3$, are in equations (60)–(66). ■