Mean-Swap Variance, Portfolio Theory, and Asset Pricing

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November, 2017

Abstract

Superior to the variance, "swap variance (SwV)" summarizes the entire probability distribution of returns and is unbiased to distributional asymmetry. Retaining the same simplicity as mean-variance (MV) model, the efficiency of mean-swap variance (MSwV) is necessary and sufficient conditions for that of stochastic dominance. The SwV is composed of a quadratic volatility and a proxy of asymmetric variation ($\mathbb{A}$). The mean-variance-asymmetry (MV$\mathbb{A}$) analysis, a three-dimensional extension of the classical MV portfolio theory and the CAPM, is consistent with expected utility maximization for all risk-averse investors and those who are downside loss-averse but prefer the prospect of potential upside gains.

Keywords: Swap-Variance, Symmetry, Asymmetry, Expected Utility Maximization, Stochastic Dominance, Mean-Variance, Capital Market Equilibrium, Co-moments, CAPM, Beta Coefficient, Prospect theory

JEL classification: D81, G02, G11, G12.

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I. **Introduction**

As is well known, the portfolio models provide a solution that separates the decision process from the question of utility maximization by restricting either the individual utility function or the assets' return distribution. The classical mean-variance (MV) portfolio theory and CAPM assume investor utility functions are quadratic or the return distributions of assets are elliptically distributed. These assumptions have been subject to much controversy over the decades. Due to these restrictions, alternative risk-return measures have been proposed. The theory of stochastic dominance (SD) for ranking investment choices does not restrict the class of utility functions, but rather it derives weak conditions for separation based on probability distributions. The SD ranking rules consider the entire return distribution of assets and thus makes no assumption about the form of the underlying probability distributions. Although general, the SD approach is subject to limitations. Levy (2015) notes that SD performs well in applied economics and finance when the decision problem is the preference for a single asset or policy. But in optimal portfolio selection, stochastic dominance performs poorly in that one has to search through all possible combinations of assets to find the optimal one.

Alternative models such as the mean-Gini approach (Yitzhaki 1982, and Shalit and Yitzhaki 1984) and the mean-lower partial moment (LPM) model (Bawa and Lindenberg 1977, Price, Price and Nantell 1982, and Harlow and Rao 1989) attempt to resolve the problems of SD optimization by transforming the stochastic dominance into a simple two-parameter framework. Nevertheless, since mean-Gini and mean-LPM efficiency are necessary but insufficient to the SD efficiency, the utility separation still fail to hold without further restrictions on probability distributions.[1] To the best of our knowledge, for almost half a century, there is still no formal

[1] That is, the SD implies mean-Gin and/or mean-LPM dominance, but not vise verse.

optimization produce for both necessarily and sufficiently constructing SD efficient portfolios that allow developing the separation and asset pricing theorems.[2]

In this paper, we formally identify all expected utilities as a function of the mean and quantity, called swap-variance (SwV), without any restriction on the form of investors' utility function as well as that of assets' return distribution.[3] This affine transformation of utility function serves as a theoretical foundation for developing SD optimization and equilibrium fundamentally. Mathematically, SwV is the twice expected difference of arithmetic and logarithmic returns adjusted by the mean and is a convergence of a polynomial weighted sum of infinite return-moments:

$$\text{SwV} = 2[E(R-r) - d\mu] \;=\; \left(\frac{\sigma}{1+\mu}\right)^2 + \mathbb{A} \geq 0, \tag{1}$$

where R is the one-period rate of return, $r = \ ln(1+R)$, $R - r \geq 0$, E is the expectation operator, μ is the expected return of R, $d\mu = \mu - ln(1+\mu)$, σ^2 is the variance, $\mathbb{A} = \left[\sum_{k=3}^{\infty}(-1)^k \binom{2}{k} \frac{\mathcal{M}^k}{(1+\mu)^k}\right]$, and $\mathcal{M}^k = E(R-\mu)^k$ is the k-th central moment of the return distribution, respectively.[4] Precisely, the right-hand side of the equation (1) quantifies that SwV composes of two components. The first is

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[2] Although Post (2003) has made an important step in this direction in that he introduced a technique to find whether the market portfolio is second degree efficient relative to all diversified portfolios composed from a given set of assets, we still do not have a stochastic dominance equilibrium.

[3] It is well known in financial literature (e.g. Neuberger, 1994, and Jiang and Oomen, 2008) that the variance swap contract can be replicated by a portfolio strategy of shorting a log-contract and simultaneously longing rebalanced forward contracts of the underlying asset. The profit/loss of such replication strategy accumulates to a quantity that is proportional to the realized variance (RV), if the jump-tail of return distribution is absent. In a continuous-time limit, Jiang and Oomen (2008) show that this quantity, they call "swap variance (SwV)," can be calculated by the accumulated difference between simple returns and log returns.

[4] We assume asset returns are bounded with a finite range, $R \in [-1,1]$. Based on a Taylor's series of the log-return around μ, we have $r = \ln(1+R) = \ln(1+\mu) + \frac{R-\mu}{1+\mu} - \frac{(R-\mu)^2}{2(1+\mu)^2} + \sum_{k=3}^{\infty}(-1)^{k-1}\left(\frac{1}{k}\right)\frac{(R-\mu)^k}{(1+\mu)^k}$. Then, $2E(R-r) = 2[\mu - ln(1+\mu)] + \frac{E(R-\mu)^2}{(1+\mu)^2} + \sum_{k=3}^{\infty}(-1)^k \binom{2}{k} \frac{E(R-\mu)^k}{(1+\mu)^k} = 2d\mu + \frac{\sigma^2}{(1+\mu)^2} + \mathbb{A}$. Moreover, since SwV $= 2[E(R-r) - d\mu] = 2[E(\ln(1+R) - \ln(1+E(R))]$, and $E[\ln(\cdot)] \geq \ln[E(\cdot)]$ due to the concavity of logarithmic function, SwV must be non-negative.

symmetric (quadratic) variation of returns measured by the mean-adjusted variance, and the second, denoted $\mathbb{A}$ and calculated merely by the difference between SwV and $\left(\frac{\sigma}{1+\mu}\right)^2$, characterizes the asymmetric (polynomial) variation of returns on a risky asset. Intuitively, the fundamental difference between SwV and variance can be graphically observed from the distinctions of the two random variables, $2(R-r)$ and R^2, accordingly (See Figure 1). Both variables are non-negative, but $2(R-r)$ is asymmetric in nature, and R^2 is less (higher) than $2(R-r)$ for negative (positive) R. This indicates that the variance understates (overstates) the downside (upside) variation if returns are asymmetrically distributed. Figure 1 also helps to acknowledge that the distinction between *SwV* and variance is a summary statistic of the asymmetries in returns ($\mathbb{A}$).

[Insert Figure 1 here]

Noticeably, the alternating signs in the weighted sequence of third and higher order moments in the polynomial formulation of $\mathbb{A}$ suggest that larger the positive (negative) odd moments produces smaller (larger) the SwV. Since the odd moments distinguish the prospect of potential gain/loss, a significantly negative (positive) $\mathbb{A}$ is associated with a prospect of substantial gain (loss) or a possibility of profoundly positive (negative) returns. That indicates that a volatile distribution may not be necessarily risky, if the value of $\mathbb{A}$ is significant negative. In short, upside (downside) asymmetries in returns lead to relatively low (high) risk exposure so that SwV is small (larger) than the variance. We explicitly apply this notion of asymmetry embedded in SwV to the theory of expected utility maximization and develop a model for preference of choice that is robust to risk-averse investors who dislike downside-losses but prefer potential upside-gains.

We first show that the mean-SwV transformation of the expected utility function allows the derivation of both the necessary and sufficient condition for stochastic dominance, enabling risk-averse investors to discard from the efficient set of prospects that are stochastically dominated by others. Therefore, the mean-SwV (MSwV) approach separates both the knowledge of all prospects' probability distribution and that of investors' preference functions from the decision process on utility maximization. In the application of MSwV to portfolio theory, we demonstrate that similar to the formulation of portfolio variance, the SwV of a portfolio is also a weighted sum of the co-swap-variance (CoSwV) between each component asset and the portfolio. Structurally, the CoSwV contains not only the covariance but a summary of all higher order co-moments of return; thus, larger the odd co-moments (e.g., co-skewness), smaller the CoSwV, better the risk-diversification, and higher the portfolio efficiency.[5] Consequently, the SwV can replace the variance and the CoSwV can substitute covariance needed in portfolio theory whenever the MV model fails to provide consistent results of the utility maximization. Precisely, we prove that the mean and CoSwV dominance is a necessary and sufficient condition for the second-degree marginal conditional stochastic dominance (MCSD) of Shalit and Yitzhaki (1994) for all concaved utility functions. As a result, analogous to the MV analysis, the efficient set of second-degree stochastic dominance (SSD) portfolios can be determined by minimizing the portfolio's SwV for each given mean-return without searching through all possible combinations of assets as the traditional SSD algorithm requires.

[5] Mounting empirical evidence suggests that higher-order market co-moments associated with distributional variations in addition to the market volatility do explain the expected returns on financial assets. Notably, Harvey and Siddique (2000) demonstrate that under a quadratic pricing kernel, conditional skewness explains the cross-sectional variation in expected returns across assets. Dittmar (2002) extends the pricing kernel to be a cubic in the market return and show that asset returns are affected by covariance, co-skewness, and co-kurtosis with return on aggregate wealth. From an aspect of asset pricing, Vendrame, Tucker and Guermat (2016) find that covariance is associated with a positive factor premium, co-skewness demands a negative premium, and co-kurtosis has a positive premium, respectively. Furthermore, Chung, Johnson and Schill (2006) argue that although higher moment measure such as skewness and kurtosis individually provides some information about the tail of the investment return distribution, they fall far short of specifying the tail precisely. Therefore, the likelihood of extreme outcomes of an investment must be measured jointly by the entire set of all possible moments and co-moments. To specify the distributional tails of returns (or the sensitivity to the market tails) ideally, it requires information of an infinite number of moments and co-moments.

Based on the decomposition of SwV shown in (1), we further quantify the expected utility as a function of three-parameters: mean, variance, and asymmetry (MV$\mathbb{A}$), respectively. Serving as an extension of the SSD approaches, we show that MV$\mathbb{A}$ efficiency is robust to risk-averse investors and to those who prefer a prospect of upside-skewed payoffs but are averse to that of downside losses, where both MV and SSD efficient sets of assets are subsets of the MV$\mathbb{A}$ efficiency.[6] Notably, the MV$\mathbb{A}$ efficient set includes lottery-type securities (Kumar, 2009) that are SSD inefficient and commonly viewed as highly risky assets. The MV$\mathbb{A}$ optimal portfolios can then be identified by minimizing $\mathbb{A}$ of assets for every level of the mean and variance.

Again, the main advantage of MSwV and MV$\mathbb{A}$ optimization is that the expected utility maximization of investors can be distinguished entirely by a function of finite summary statistics independent of individual preference function or the knowledge of probability or decision weight distribution. Consequently, the MSwV and MV$\mathbb{A}$ optimal portfolios allow us to derive the SD equilibrium return on assets as well as the SD systematic risk measures. Specifically, we show that similar to the MV beta coefficient in form, the MSwV-beta (or the SSD-beta) is a ratio between CoSwV and the market SwV. Since CoSwV accommodates all possible higher order co-moments of returns between an asset and the SSD optimal (market) portfolio, the MSwV-beta is sensitive to the asymmetries in market returns. It makes MSwV-beta becomes a more general proxy of systematic risk than the traditional MV-beta. Finally, incorporating with upside-skewness preference and downside-asymmetry aversion, we develop a two-factor linear model as a result of the MV$\mathbb{A}$ equilibrium. This model is an extension of the MSwV approach for quantifying the systematic impacts of symmetry and asymmetry separately on the required return on risky assets.

[6] The MVA inefficient assets must also be dominated by SSD or MV rules. Note that the neither the MV efficient set is a subset of SSD, nor vice versa.

Empirically, the MV𝔸 two-factor pricing model is statistically valid as compared with the conventional multi-factor models.

The rest of the paper is organized as follows: Section II derives the expected utility as a function of mean and SwV. The stochastic dominance rules based on distributional moments and SwV are then defined and proved. We also illustrate that the MV𝔸 model serves as an extension of the MSwV and MV approaches for incorporating with asymmetry-preference. Section III shows the derivation of the co-SwV as well as its application to MCSD orderings. Section IV demonstrates the application of MSwV and MV𝔸 to the determination of stochastic dominance optimal portfolios. Based on the MSwV and MV𝔸 optimization, the stochastic dominance oriented asset pricing models are then developed in Section IV. Section V illustrates the empirical analysis, and section VI contains brief concluding remarks.

II. Swap-Variance, Asymmetry, and Stochastic Dominance

Stochastic dominance provides a way of analyzing risky investment decisions when an investor's utility function U is not fully known but is presumed to be in a class of real-valued functions. An asset i unconditionally and stochastically dominates an asset j, if and only if $EU(R_i) \geq EU(R_j)$, where $EU(R_i)$ and $EU(R_j)$ are expected utilities of returns on assets i and j, respectively. Without loss of generality, we apply the Taylor-series of the utility function $U(R)$ about mean return and with some mathematical arrangements, the utility function can be expressed by the following equation:

$$U(R) = U(\mu) + U'(\mu)(R - \mu) + \mathbb{U}(U, \mu, R)[2(R - r) - 2d\mu], \quad (2)$$

where $U' \geq 0,\ \mathbb{U}(U,\mu,R) = \sum_{k=2}^{\infty} w_k A_k,$ is a weighted sum of all derivatives of the utility functions**.** Specifically, $A_k = \left[\frac{[-(1+\mu)]^k}{2(k-1)!} U^{(k)}(\mu)\right]$, where $U^{(k)}$ is the *k*-th derivative of the utility function, and $w_k = \left[\frac{2(-1)^k}{k}\left(\frac{R-\mu}{1+\mu}\right)^k\right] \Big/ [2(R-r) - 2d\mu]$.[7] Next, based on the assumption that utility function is non-decreasing, continuous and differentiable on R, we apply the *mean value theorem* for integrals to the expected utility of return such that:

$$EU(R) = \quad U(\mu) + \ \mathbb{U}(U,\mu,R^o) \cdot \text{SwV} \tag{3}$$

where $= E(R),\ \text{SwV} = 2E(R-r) - 2d\mu$, as shown in (1), and $\mathbb{U}(U,\mu,R^o)$ is the function of $\mathbb{U}(\cdot)$ at some point of return, $R^o \in (-1, 1)$ so that it equals the probability weighted average of $\mathbb{U}(U,\mu,R)$, where the weights could be either objective-probability or subjective-decision oriented. It is important to note that equation (3) holds for all forms of the utility functions in which $\mathbb{U}(\mathrm{U}, \mu, \mathrm{R^o})$ is negative, zero and positive for concave, linear, and convex function, respectively.[8]

A. MSwV Stochastic Dominance Rules

The implication of Equation (3) is crucial. First, with no assumption on the form of either utility function or that of return distribution, the expected utility can be characterized as a function

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[7] $U(R) = U(\mu) + \ U'(\mu)(R-\mu) + \sum_{k=2}^{\infty} \frac{1}{k!} U^{(k)}(\mu)(R-\mu)^k = U(\mu) + \ U'(\mu)(R-\mu) + \sum_{k=2}^{\infty} \left[\frac{(-1)^k(1+\mu)^k}{2(k-1)!} U^{(k)}(\mu)\right]\left[\frac{2(-1)^k}{k}\frac{(R-\mu)^k}{(1+\mu)^k}\right]$. Now, let $A_k = \left[\frac{(-1)^k(1+\mu)^k}{2(k-1)!} U^{(k)}(\mu)\right],\ B_k = \left[\frac{2(-1)^k}{k}\frac{(R-\mu)^k}{(1+\mu)^k}\right],\ B = \sum_{k=2}^{\infty} B_k = 2(R-r) - 2d\mu$, and $w_k = \frac{B_k}{B}$. Define $\mathbb{U}(U,\mu,R) = \sum_{k=2}^{\infty} w_k A_k = \frac{1}{B}\sum_{k=2}^{\infty} A_k B_k$. We have $\sum_{k=2}^{\infty} A_k B_k \ = \mathbb{U}(U,\mu,R)[2(R-r) - 2d\mu]$, and consequently, $U(R) = U(\mu) + \ U'(\mu)(R-\mu) + \mathbb{U}(U,\mu,R)[2(R-r) - 2d\mu]$.

[8] For any risk-averse investor, his/her concave utility must be not greater than his/her equivalently risk-neutral utility (i.e., graphically, the tangency line at any return level) due to the utility-discount of risk. That is, $EU(R) - U(\mu) \leq 0$. Since $E(R-r) - d\mu = \ln(1+\mu) - E[\ln(1+R)] \geq 0$ because of the concavity of logarithmic function, $\mathbb{U}(U,\mu,R^o) = \frac{EU(R)-U(\mu)}{(SwV-2d\mu)} \leq 0$. On the other hand, for convex utilities, $EU(R) - U(\mu) \geq 0$, and $\mathbb{U}(U,\mu,R^o) > 0$.

of the mean (μ) and SwV. Second, the mean-SwV transformation makes the consistency between the expected utility maximization and the prospect theory, where the probability distribution is not necessarily required in decision making.

Theorem 1 (First Degree Stochastic Dominance)
Let R_i and R_j be two uncertain prospects. Based on equation (3), the conditions, $\mu_i \geq \mu_j$ and $SwV_i = SwV_j$, are necessary and sufficient to have $EU(R_i) \geq EU(R_j)$ for all utility functions.

Theorem 2 (Second Degree Stochastic Dominance: The Mean-Swap Variance Dominance)
Let R_i and R_j be two uncertain prospects. Based on equation (3), the conditions, $\mu_i \geq \mu_j$ and $SwV_i \leq SwV_j$ are necessary and sufficient conditions to ensure that $EU(R_i) \geq EU(R_j)$ for all concave utility functions.

The proof of Theorems 1 and 2 is given in the Appendix. The main contribution of the above theorems is the convergence of the SD rules from the comparison of entire probability distributions to the two-statistic (MSwV) analysis that retains the same simplicity as the MV model. The theorems indicate that the swap-variance is more generalized risk proxy than the variance is for all risk-averse expected utility maximizers. Notably, there are risk characters, embedded in the *SwV*, other than return volatility that risk-averse investors care about in which the variance fails to measure them.

The following numerical example shows that the MSwV approach correctly discriminate SSD inefficient assets, but the MV model fails to do so. Suppose returns on two securities, R_1 and R_2, that are lognormally distributed, where $r_1 \sim N(0.1, 0.22)$ and $r_2 \sim N(-0.15, 0.24)$, respectively. Apparently, R_2 is stochastically dominated by R_1, and a profitable investment can be formed by a long/short position between R_1 and R_2, if investors know the form of return distributions. Now, without the knowledge of distributional forms, suppose investors employ the three methods, MV, SSD and M*SwV*, for security selection, we have the following results:

	Security 1		Security 2
μ	0.132	>	-0.114
σ^2	0.0636	>	0.0465
SwV	0.0484	<	0.0576
SSD	$\int_{-1}^{R} \frac{1}{2} + \frac{1}{2} erf[\frac{\ln(1+t)-0.1}{0.22\sqrt{2}}]\, dt$[9]	<[10]	$\int_{-1}^{R} \frac{1}{2} + \frac{1}{2} erf[\frac{\ln(1+t)+0.15}{0.24\sqrt{2}}]\, dt$

According to the MV trade-off ($\sigma_1^2 > \sigma_2^2$ and $\mu_1 > \mu_2$), incorrectly, no dominance between the two securities makes them as efficient as to each other. Nevertheless, consistent with SSD ordering, the MSwV rule ($\mu_1 > \mu_2$, but $\mathrm{SwV}_1 < \mathrm{SwV}_2$) enables one to correctly discriminate the dominated security R_2 from the dominating one, R_1. This example highlights a significant bias of the traditional MV model and the superiority of the MSwV analysis under the conventional assumption of log-normality. The implication of this example is that if returns on risky assets are asymmetrically distributed, then many efficient assets or portfolios determined by the MV analysis are in fact inefficient. The MSwV, on the other hand, used as convenient as the MV, provides unbiased results.

B. *$MV\mathbb{A}$ and Stochastic Dominance Efficiency*

To further examine how return-variation other than the volatility affects expected utility of investors, we separate the second order of return-moment from the swap-variance and have a focus on the asymmetries in returns. Again, without loss of generality, the utility function (2) can be further decomposed as follows:

$$U(R) = U(\mu) + U'(\mu)(R-\mu) + \frac{1}{2}U''(\mu)(R-\mu)^2 + \mathbb{U}^{(3)}(U,\mu,R) \cdot \mathcal{A} \qquad (4)$$

[9] $erf(\cdot)$ is error function, and it is monotonically increasing in its whole define of domain

[10] The "<" holds when the upper bound of holding period return is 16.28, so we consider asset 1 second-degree stochastically dominates asset 2.

where $\mathcal{A} = [2(R-r) - 2d\mu] - \left(\frac{R-\mu}{1+\mu}\right)^2$, $\mathbb{U}^{(3)}(U,\mu,R) = \sum_{k=3}^{\infty} \omega_k A_k < 0$, $A_k = \left[\frac{(-1)^k (1+\mu)^k}{2(k-1)!} U^{(k)}(\mu)\right]$ and $\omega_k = \left[\frac{2(-1)^k}{k}\left(\frac{R-\mu}{1+\mu}\right)^k\right] \Big/ \mathcal{A}$.[11] We note that $\mathcal{A}$ depicts the asymmetries in return (see Figure 1 for the case that $\mu = 0$). The quantity $\mathcal{A}$ is negative (positive) if R is greater (less) than μ. Further, the inequality, $\mathbb{U}^{(3)}(U,\mu,R) < 0$, holds if $U''' > 0$.[12] Now, from (1), (3), and (4), we can rewrite the expected utility as a function of mean, variance and the asymmetry measure $\mathbb{A}$ as follows:

$$EU(R) = U(\mu) + \frac{1}{2}U''(\mu) \cdot \sigma^2 + \mathbb{U}^{(3)}(U,\mu,R^o) \cdot \mathbb{A} \tag{5}$$

where $\mathbb{A} = \text{SwV} - \left(\frac{\sigma}{1+\mu}\right)^2$. Since $\mathbb{U}^{(3)}(U,\mu,R^o)$ is non-positive, larger (smaller) the asymmetric risk, and more positive (negative) value of $\mathbb{A}$, lower (higher) the expected utility. Also, since the odd higher moments embedded in $\mathbb{A}$ distinguish the prospect of potential gain/loss, a high (low) $\mathbb{A}$ results from either a significant chance of loss (gain) or a probability of substantially negative (positive) returns. In the next theorem, we show that for all risk-averse investors who also prefer

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[11] $U(R) = U(\mu) + U'(\mu)(R-\mu) + \frac{1}{2}U''(\mu)(R-\mu)^2 + \sum_{k=3}^{\infty}\left[\frac{(-1)^k(1+\mu)^k}{2(k-1)!}U^{(k)}(\mu)\right]\left[\frac{2(-1)^k}{k}\frac{(R-\mu)^k}{(1+\mu)^k}\right]$. Now, let $A_k = \left[\frac{(-1)^k(1+\mu)^k}{2(k-1)!}U^{(k)}(\mu)\right]$, $B_k = \left[\frac{2(-1)^k}{k}\frac{(R-\mu)^k}{(1+\mu)^k}\right]$, $\mathcal{B} = \sum_{k=3}^{\infty} B_k = [2(R-r) - 2d\mu\,] - \left(\frac{R-\mu}{1+\mu}\right)^2$, and $\omega_k = \frac{B_k}{\mathcal{B}}$. Define $\mathbb{U}^{(3)}(U,\mu,R) = \sum_{k=3}^{\infty} w_k A_k = \frac{1}{\mathcal{B}}\sum_{k=3}^{\infty} A_k B_k$. We have $\sum_{k=3}^{\infty} A_k B_k = \mathbb{U}^{(3)}(U,\mu,R)\left\{[2(R-r) - 2d\mu] - \left(\frac{R-\mu}{1+\mu}\right)^2\right\}$, and consequently, $U(R) = U(\mu) + U'(\mu)(R-\mu) + \mathbb{U}^{(3)}(U,\mu,R)\left\{[2(R-r) - 2d\mu] - \left(\frac{R-\mu}{1+\mu}\right)^2\right\}$.

[12] This is to prove that if $U'''(R) > 0$, then $\mathbb{U}^{(3)}(U,\mu,R) < 0$. Take the Taylor Expansion on $U'''(R)$ around μ, $U'''(R) = U'''(\mu) + U^{(4)}(\mu)(R-\mu) + \sum_{k=5}^{\infty}\frac{U^{(k)}(\mu)}{(k-3)!}(R-\mu)^{k-3}$. Define $q(R) = \int_{\mu}^{R} U'''(s)ds$, $q(R) = U'''(\mu)(R-\mu) + \frac{U^{(4)}(\mu)}{2}(R-\mu)^2 + \sum_{k=5}^{\infty}\frac{U^{(k)}(\mu)}{(k-2)!}(R-\mu)^{k-2}$. Since$U'''(R) > 0$, $q(R) > 0$ as $R > \mu$ while $q(R) < 0$ as $R < \mu$. Now consider $Q(R) = \int_{\mu}^{R} q(s)ds$. Then $Q(R) = \frac{U'''(\mu)}{2!}(R-\mu)^2 + \frac{U^{(4)}(\mu)}{3!}(R-\mu)^3 + \sum_{k=5}^{\infty}\frac{U^{(k)}(\mu)}{(k-1)!}(R-\mu)^{k-1}$. Because of the property of $q(R)$, $Q(R)$ reaches its minimum as $R = \mu$. Thus, $Q(R) > 0$. Now, consider $\mathbb{Q}(R) = \int_{\mu}^{R} Q(s)ds$. Then $\mathbb{Q}(R) = \frac{U'''(\mu)}{3!}(R-\mu)^3 + \frac{U^{(4)}(\mu)}{4!}(R-\mu)^4 + \sum_{k=5}^{\infty}\frac{U^{(k)}(\mu)}{k!}(R-\mu)^k$. Since $Q(R) > 0$ for all R. $\mathbb{Q}(R)$ is a monotonic increasing function with $\mathbb{Q}(\mu) = 0$, and thus $\mathbb{Q}(R) > 0$ as $R > \mu$ while $\mathbb{Q}(R) < 0$ as R, μ. Compared to equation (4), $\mathbb{Q}(R) = \mathbb{U}^{(3)}(U,\mu,R) \cdot \mathcal{A}$. So $\mathbb{U}^{(3)}(U,\mu,R) < 0$ for all R.

(dislike) positively (negatively) asymmetric payoffs, the preference of choice can be made by the orders of the three parameters: mean, variance, and asymmetry, respectively.

Theorem 3 (The Mean-Volatility-Asymmetry (MV$\mathbb{A}$) Dominance and Efficiency)
Let R_i and R_j be two uncertain prospects. Based on equation (5), these inequalities, $\mu_i \geq \mu_j$, $\sigma_i \leq \sigma_j$, and $\mathbb{A}_i \leq \mathbb{A}_j$ are the sufficient condition for the expected utility inequality, $EU(R_i) \geq EU(R_j)$, and for all utility functions with $U' > 0$, $U'' < 0$, and $U''' > 0$. Since $MV\mathbb{A}$ dominance must be the SSD dominance but not vice versa, the SSD efficient set is thus a subset of the $MV\mathbb{A}$ efficiency.

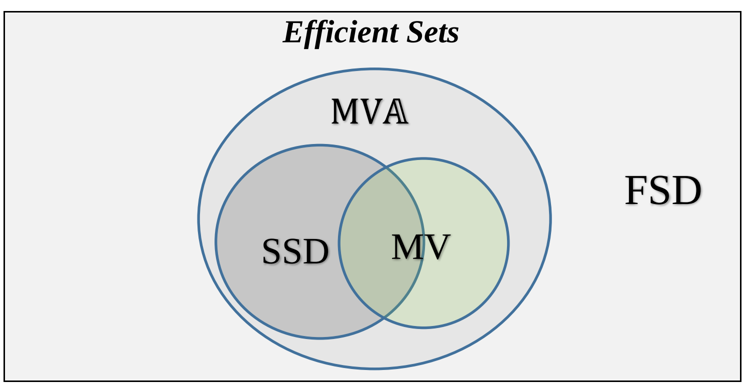
The proof of Theorem 3 is similar to that of Theorems 1 and 2, except that the dominance of MV$\mathbb{A}$ is not a necessary condition to that of expected utility.[13] Intuitively, the MV$\mathbb{A}$ model based on (4) and (5) takes both aversion of symmetric volatility ($U'' < 0$) and that of asymmetric variation ($\mathbb{U}^{(3)} < 0$) into consideration. Also, from (4), the inequality of $U''' > 0$ ensures that of $\mathbb{U}^{(3)} < 0$. That indicates the efficient assets in the MV$\mathbb{A}$ set include those that are chosen by investor who prefer upside skewed outcomes as well as those who are downside asymmetry averse. Noticeably, the MV$\mathbb{A}$ approach reduces to the MV model, if either U''' or $\mathbb{A}$ is zero.

To illustrate that some SSD (or MSwV) inefficient assets for risk-averse investors may not be viewed as the inferior assets for those who also prefer positively asymmetric outcomes, we use a simple counterexample. Consider two random prospects X and Y with discrete distributions of returns. Both X and Y have two possible investment outcomes: -0.20 with a probability of 90% and 0.65 with a probability of 10%, as well as -0.20 with a probability of 40% and -0.05 with a probability of 60%, respectively. We summarize the key statistics are follows:

[13] Specifically, the non-necessity of MV$\mathbb{A}$ to expected utility dominance is because it is impossible that both $\frac{U''(\cdot)}{\mathbb{U}^{(3)}(\cdot)}$ and $\frac{U'(\cdot)}{\mathbb{U}^{(3)}(\cdot)}$ simultaneously approach zero for non-decreasing concave utility functions with a non-negative U'''.

	Upside Gain Preference				
	X	*Prob.*		*Y*	*Prob.*
Outcome 1	-0.20	90%		-0.20	40%
Outcome 2	0.65	10%		-0.05	60%
μ	-0.115		<	-0.110	
σ^2	0.065		>	0.005	
SwV	0.057		>	0.007	
$\mathbb{A}$	-0.026		<	0.000	

Based on the MSwV and MV ranking rules, asset *Y* is superior to *X* in that the mean (SwV or variance) of *Y* is higher (lower) than that of *X*. Also, from the aspect of the probability distribution, *X* is also dominated by *Y* under the SSD rule. Nevertheless, investors may be unwilling to discriminate *X* as a dominated choice in that a prospect of dramatic positive-payoff (0.65) could be attractive as compared with the alternative *Y* that has all negative investment outcomes. Consequently, if we take those investors who care about the upside potential into consideration, *X* may then be as efficient as *Y* in the risk-return tradeoff. The MV$\mathbb{A}$ model can detect this efficiency; correctly, the negative $\mathbb{A}$ shows that asset *X* has positively asymmetric (skewed) payoff and thus low asymmetric risk, which increases the expected utility as shown in (5). For a graphical illustration, we depict the relationship of efficiency among FSD, MV$\mathbb{A}$, SSD, and MV in the following simple chart:



Apparently, the investment alternative *X* in the previous example can be viewed as a lottery-type security, defined by Kumar (2009), that has low negative expected returns, high variance, and a small probability of a substantial payoff (i.e., a significantly negative $\mathbb{A}$). Our analysis above indicates that the lottery-type securities, although they are dominated assets under the SSD and MV framework, could still be expected utility efficient for investors have preference of upside potential even if they are risk-averse in general. Therefore, the traditional methodologies of security selection and portfolio efficiency analysis under the classical risk-aversion assumption may be too restrictive.

In addition to the upside skewness preference, the MV$\mathbb{A}$ approach is also useful in detecting downside asymmetry (disappointment) aversion that the MV model fails to do so.[14] Consider two mutually exclusive investment projects, *G* and *H*, have almost identical means. As shown in the following table, *G* dominates *H* in the mean-variance tradeoff in that the variance of project *G* (0.014) is lower than that of project *H* (0.016):

	Downside Loss Aversion				
	G	*Prob.*		*H*	*Prob.*
Outcome 1	0.175	90%		0.01	50%
Outcome 2	-0.225	10%		0.26	50%
μ	0.135		≈	0.135	
σ^2	0.014		<	0.016	
SwV	0.014		>	0.012	
$\mathbb{A}$	0.003		>	0.000	

Nevertheless, from the prospect payouts shown in the above table, risk-averse investors may not view *G* is a superior alternative to *H* for risk-averse investors. In fact, *H* stochastically dominates *G* for that *G* has a larger SwV than *H*, and the downside asymmetry-risk ($\mathbb{A}$) of project *G* is greater than that of project *H*. In short, unlike the variance, the sign of $\mathbb{A}$ derived from the SwV provides

[14] The downside asymmetry-aversion is closely related to the notion of disappointment aversion in Gul (1991).

valuable indications for the reference of choice between the upside gain-preference and the downside loss-aversion.

Although the simplicity of MSwV and MVA rules shown in the above theorems demonstrates their superiority to the conventional stochastic dominance approach, the goal of this paper is applying them to the development of SD portfolio efficiency and equilibrium, consisting with the notion of expected utility maximization. In the next section, we extend the MSwV and MVA to the marginal and conditional ordering conditions of assets within a portfolio for serving as an essential step toward the development of stochastic dominance optimization and capital market equilibrium.

III. Co-*SwV*, Co-Asymmetry and MCSD

Let $R_p\left(=\sum_{i=1}^{N} w_i R_i\right)$ be the return on a core portfolio of N risky assets, where w_i is the share of wealth invested in asset *i*, and $\sum_{i=1}^{N} w_i = 1$. We assume investors are maximizing their expected utility of R_p. Shalit and Yitzhaki (1994) showed this inequality,

$$E\left[U'(R_p)(R_i - R_j)\right] \geq 0, \tag{6}$$

is the necessary and sufficient condition for all risk-averse investors to prefer marginally increase the share of one asset over another in the core portfolio. In other words, asset *i* is said to marginally and conditionally stochastically dominate (*MCSD*) asset *j*, if and only if the inequality (6) holds.

A. The Co-Swap Variance and MCSD

Analogical to (3), we transform the difference of expected marginal utility between returns on component assets *i* and *j* of a portfolio *p* as:

$$E\left[U'\left(R_p\right)\left(R_i - R_j\right)\right] = U'\left(R_p\right)\left(\mu_i - \mu_j\right) + \mathbb{U}\left(U, \mu_p, R_p^o\right)\left[CoSwV\left(R_i, R_p\right) - CoSwV\left(R_j, R_p\right)\right] \qquad (7)$$

where $CoSwV\left(R_i, R_p\right)$ is the co-swap variance between returns on asset *i* and those on the portfolio, and $\mathbb{U}\left(U, \mu_p, R_p^o\right)$, previously defined in (3), is non-positive in value.[15] The formulation of the co-swap variance is shown in the following equation:

$$\begin{aligned} CoSwV\left(R_i, R_p\right) &= 2E\left[\left(\frac{R_i - \mu_i}{R_p - \mu_p}\right)(R_p - r_p)\right] - 2d\mu_p \\ &= \frac{CoV\left(R_i, R_p\right)}{\left(1+\mu_p\right)^2} + \sum_{k=3}^{\infty}(-1)^k \left(\frac{2}{k}\right) \frac{Co\mathcal{M}^k\left(R_i, R_p\right)}{\left(1+\mu_p\right)^k}, \end{aligned} \qquad (8)$$

where $r_p = ln\left(1 + R_p\right)$, $d\mu_p = \mu_p - ln\left(1 + \mu_p\right)$, $CoV\left(R_i, R_p\right) = E\left[(R_i - \mu_i)(R_p - \mu_i)\right]$, and $Co\mathcal{M}^k\left(R_i, R_p\right) = E\left[(R_i - \mu_i)(R_p - \mu_p)^{k-1}\right]$ is the *k*-th order co-moment between R_i and R_p, accordingly.[16]

Equation (7) shows that the marginal expected utility of return can be characterized by the mean and CoSwV without any restriction on the form of the utility functions and that of the return distributions. Although the covariance plays the key role for risk-diversification, the higher orders of co-moments between assets and the portfolio, from equations (7) and (8), are crucial for the determination of portfolio efficiency. We show, in the following theorems, that the mean and CoSwV can be employed to determine the necessary and sufficient condition for the MCSD.

[15] Equation (7) can be derived as follows: $E\left[U'\left(R_p\right)\left(R_i - R_j\right)\right] = E\left[U'\left(R_p\right)\left(\mu_i - \mu_j\right)\right] + E\{U'\left(R_p\right)\left[(R_i - \mu_i) - \left(R_j - \mu_j\right)\right]\}$, where $E\{U'\left(R_p\right)\left[(R_i - \mu_i) - \left(R_j - \mu_j\right)\right]\} = E\left\{\left[U'\left(\mu_p\right) + \sum_{k=2}^{\infty}\frac{1}{k!}U^{(k)}\left(\mu_p\right)\left(R_p - \mu_p\right)^{k-1}\right]\left[(R_i - \mu_i) - \left(R_j - \mu_j\right)\right]\right\} = E\sum_{k=2}^{\infty}\frac{\left(1+\mu_p\right)^k}{2(k-1)!}(-1)^k U^{(k)}\left(\mu_p\right)\{\frac{(-1)^k 2}{k\left(1+\mu_p\right)^k}\left(R_p - \mu_p\right)^{k-1}\left[(R_i - \mu_i) - \left(R_j - \mu_j\right)\right]\}$
$= \mathbb{U}\left(U, \mu_p, R_p^o\right)\left[CoSwV\left(R_i, R_p\right) - CoSwV\left(R_j, R_p\right)\right]$.

[16] The derivation of the closed-formed CoSwV is as follows: $CoSwV\left(R_i, R_p\right) = \sum_{k=2}^{\infty}\frac{2(-1)^k}{k\left(1+\mu_p\right)^k}\int_{-1}^{b}\left(R_i - \mu_i\right)\left(R_p - \mu_p\right)^{k-1} dF\left(R_i, R_p\right) = \int_{-1}^{b}\frac{\left(R_i - \mu_i\right)}{\left(R_p - \mu_p\right)}\left[\sum_{k=2}^{\infty}\frac{2(-1)^k}{k\left(1+\mu_p\right)^k}\left(R_p - \mu_p\right)^k\right] dF\left(R_i, R_p\right) = 2E\left[\left(\frac{R_i - \mu_i}{R_p - \mu_p}\right)(R_p - r_p)\right] - 2d\mu_p$.

Theorem 4 (Mean-Co*SwV* MCSD)

Assume R_i and R_j are two uncertain prospects in a portfolio R_p. Based on equation (7), the conditions $\mu_i \geq \mu_j$ and $CoSwV(R_i, R_p) \leq CoSwV(R_j, R_p)$ are necessary and sufficient for R_i to marginally and conditionally dominate R_j for all concave utilities.[17]

If returns on assets are symmetrically distributed, then from the above theorem, risk averse investors prefer to hold those assets with the larger expected return and lower correlations with the core portfolio.

Further, let $Co\mathbb{A}(R_i, R_p) = \left[CoSwV(R_i, R_p) - \frac{CoV(R_i,R_p)}{(1+\mu_p)^2}\right]$ be the sensitivity of *i*-th asset's return to the symmetric (asymmetric) price movement of the core portfolio. Equation (7) can then be expanded as:

$$\begin{aligned} E[U'(R_p)(R_i - R_j)] &= U'(R_p)(\mu_i - \mu_j) + U''(R_p)[CoV(R_i, R_p) - CoV(R_j, R_p)] \\ &\quad + \mathbb{U}^{(3)}(U, \mu_p, R_p^o)[Co\mathbb{A}(R_i, R_p) - Co\mathbb{A}(R_j, R_p)], \end{aligned} \tag{10}$$

where $CoV(R_i, R_p)$ is the covariance between returns on asset *i* and those on the portfolio, and $\mathbb{U}^{(3)}(U, \mu_p, R_p^o)$, previously defined in (5), is non-positive in value.[18]

B. *The Co-Asymmetry and MCSD*

In the following theorem, we demonstrate that for those investors who care about the potential gain/loss due to asymmetries in returns additional to price fluctuations, the co-asymmetry

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[17] Analogous to these of Theorems 2 and 3, we omit the proof of this theorem.

[18] The derivation of equation (10) is shown as follows: $E[U'(R_p)(R_i - R_j)] = E[U'(R_p)(\mu_i - \mu_j)] + E\{U'(R_p)[(R_i - \mu_i) - (R_j - \mu_j)]\}$, where $E\{U'(R_p)[(R_i - \mu_i) - (R_j - \mu_j)]\} = E\left\{\left[U'(\mu_p) + U''(\mu_p)(R_p - \mu_p) + \sum_{k=3}^{\infty} \frac{1}{(k-1)!} U^{(k)}(\mu_p)(R_p - \mu_p)^{k-1}\right][(R_i - \mu_i) - (R_j - \mu_j)]\right\} = U''(\mu_p)[CoV(R_i, R_p) - CoV(R_j, R_p)] + E \sum_{k=3}^{\infty} \frac{k(1+\mu_p)^k}{2(k-1)!} (-1)^k U^{(k)}(\mu_p)\{\frac{(-1)^k 2}{k(1+\mu_p)^k} (R_p - \mu_p)^{k-1}[(R_i - \mu_i) - (R_j - \mu_j)]\} = U''(\mu_p)[CoV(R_i, R_p) - CoV(R_j, R_p)] + \mathbb{U}^{(3)}(U, \mu_p, R_p^o)\left[\left(CoSwV(R_i, R_p) - \frac{CoV(R_i,R_p)}{(1+\mu_p)^2}\right) - \left(CoSwV(R_j, R_p) - \frac{CoV(R_j,R_p)}{(1+\mu_p)^2}\right)\right]$.

($Co\mathbb{A}$), in addition to the mean and variance, is an important ranking criterion for determining portfolio efficiency.

Theorem 5 (Mean- CoV-$Co\mathbb{A}$ MCSD)

Let R_i and R_j be two uncertain prospects, and $Co\mathbb{A}_{i.p} = \left[CoSwV(R_i, R_p) - \frac{CoV(R_i,R_p)}{(1+\mu_p)^2}\right]$. *The inequalities:* $\mu_i \geq \mu_j$, $CoV(R_i, R_p) \leq CoV(R_j, R_p)$, *and* $Co\mathbb{A}(R_i, R_p) \leq Co\mathbb{A}(R_j, R_p)$ *are sufficient conditions for R_i to marginally and conditionally dominate R_j and for all utility functions with* $U' > 0$, $U'' < 0$, *and* $U''' > 0$.

The MCSD in Theorems 4 and 5 shows the essential roles of CoSwV, CoV, and Co$\mathbb{A}$ in portfolio risk diversification. Specifically, the minimization of these co-variations of asset returns maximizes the portfolio efficiency. In the next section, we demonstrate that similar to the portfolio variance, the SwV and $\mathbb{A}$ measures of a portfolio is a weighted sum value of component assets' CoSwV and Co$\mathbb{A}$, respectively. Consequently, the classical approach of portfolio optimization can be applied to the determination of stochastic dominance equilibrium.

IV. SSD and MV$\mathbb{A}$ Optimization & Equilibrium

Based on (9), we show that the *SwV* of a portfolio is a weighted sum of Co*SwV* between returns on an asset and returns in the portfolio:[19]

$$\mathrm{SwV}_p = 2E(R_p - r_p) - 2d\mu_p = \sum_{i=1}^{N} w_i \cdot CoSwV(R_i, R_p) \quad (11)$$

[19] $SwV_p = 2E(R_p - r_p) - 2d\mu_p = 2E\left[\left(\frac{R_p-\mu_p}{R_p-\mu_p}\right)(R_p - r_p)\right] - 2d\mu_p = 2E\left[\left(\frac{\sum_{i=1}^{N} w_i(R_i-\mu_i)}{R_p-\mu_p}\right)(R_p - r_p)\right] - 2d\mu_p = \sum_{i=1}^{N} w_i \times \{2E\left[\left(\frac{R_i-\mu_i}{R_p-\mu_p}\right)(R_p - r_p)\right] - 2d\mu_p\} = \sum_{i=1}^{N} w_i \cdot CoSwV(R_i, R_p)$.

Recall Theorem 2, the SSD is indeed a tradeoff between mean and the *SwV* in which expected risk-averse utility maximizing investors, who has no knowledge about the form of return distributions, prefer an investment alternative with high mean and low SwV. Consequently, the SSD optimization and efficiency can be determined by choosing a securities mix that minimizes the SwV of the portfolio given its expected rate of return.

A. Minimum-SwV Portfolios and MSwV Asset Pricing Model

Proposition 1 (SSD Optimization and Efficiency)

Suppose there are N assets and short selling is allowed. The SSD optimal portfolios can be determined by

$$\underset{w_i}{Min}\, SwV_p,$$
$$\textit{Subjects to } \mu_p = \sum_{i=1}^{N} w_i E(R_i).$$

Based on equations (9) and (11), the SSD optimal portfolios satisfy the following first order condition:

$$E\left[\frac{R_i R_{p*}^{SwV}}{1 + R_{p*}^{SwV}}\right] = 0 \tag{12}$$

where $R_{p*}^{SwV} = \sum_{i=1}^{N} w_{i*}^{SwV} R_i$, *and where* w_{i*}^{SwV} *is the SSD optimal share of wealth on i-th asset that satisfied the condition (12). The SSD efficient portfolios are the minimum-SwV portfolios that offer the highest expected returns for the same level of SwV. The SSD efficient frontier is concave in that the second order condition is equal to* $E\left[\frac{R_i^2}{\left(1+R_{p*}^{SwV}\right)^2}\right]$ *and is non-negative, and the SSD optimal portfolio* R_{p*}^{SwV} *is unique in the MSwV space, if there is a risk-free asset.*

For an illustration, we simulate the SSD efficiency from a sample set of 300 assets randomly and jointly generated from a mixture of three different forms of return distributions: normal, lognormal, and gamma, respectively. Specifically, the sample distributions have a mean ranged from -0.2 to 0.2, and a standard deviation within a range from 0.4 to 0.7. In addition, the correlation coefficient among assets is from -0.3 to 0.7, accordingly. Figure 2 depicts the analytical results. For a comparison, we also identify the MV efficient portfolios in the MSwV space.

[Insert Figure 2 here]

The main implication from our simulation is twofold. Firstly, MSwV is capable of discriminating the stochastic dominated assets where MV can't do, so that the efficient frontier of SSD could lie above that of the MV. Secondly, with the risk-free asset, the portfolio-separation holds in the MSwV space as well, corresponding to an SSD efficient portfolio of risky assets.

Now, suppose a market portfolio is MSwV optimal and satisfies condition (12), where returns on this portfolio is denoted $R_m^{SwV}(=\sum_{i=1}^{N} w_{i*}^{SwV} R_i)$, we then can determine the SSD equilibrium condition for individual assets, shown in the following Proposition, analogous to the CAPM derivation.

Proposition 2 (The SSD Equilibrium)

Assume short-sell is allowed and investors are permitted to borrow and lend at the risk-free rate of return (R_f). Based on equation 3, if investors are risk-averse, then the expected utility is a function of two parameters: μ, and SwV, respectively. If investors are maximizing expected utilities of returns on an MSwV (SSD) efficient market portfolio, R_m^{SwV}, then the risk-premium of risky assets in equilibrium, can be calculated by the following equation:

$$E(R_i - R_f) = \beta_i^{SwV} E(R_m^{SwV} - R_f), \tag{13.1}$$

where

$$\beta_i^{SwV} = \frac{CoSwV(R_i, R_m^{SwV})}{SwV(R_m^{SwV})}. \tag{13.2}$$

The formulation of SwV and CoSwV is shown in equations (1) and (8), accordingly.

The derivation of equations (13.1) and (13.2) is shown in the appendix. The SSD equilibrium is similar to the MV equilibrium in form; however, the benchmark portfolio and the formulation of beta coefficient are different. As we have intensively discussed in Section 2, although the SSD efficiency is robust to risk-aversion with downside asymmetry on returns, some of the SSD inefficient assets with a prospect of upside potential could be efficient for investors have upside skewness preference. Consequently, the single factor equilibrium model in (13) could be biased, if we take the preference behavior of prospective gain into consideration.

B. Minimum-𝔸 Portfolios and MV𝔸 Asset Pricing Model

To identify the efficient set of assets and portfolios that includes those are chosen by investors who like (dislike) asymmetric gains (losses), we employ the MV𝔸 approach. Consider that the return asymmetry of a portfolio is calculated as:

$$\mathbb{A}_p = SwV_p - \left[\left(\frac{\sigma_p}{1+\mu_p}\right)^2\right] = \left[\sum_{k=3}^{\infty}(-1)^k\binom{2}{k}\frac{\mathcal{M}_p^k}{(1+\mu_p)^k}\right]. \tag{14}$$

Since the value of $\mathbb{A}_p$ is positive (negative) if returns on the portfolio are asymmetrically and negatively (positively) distributed, the MV𝔸 efficient portfolios can be determined by minimizing $\mathbb{A}_p$ subject to different levels of μ_p and σ_p^2.

Proposition 3 (MV𝔸 Optimization and Efficiency)

Suppose there are N assets and short selling is allowed. The MV𝔸 optimal portfolios can be determined by

$$\underset{w_i}{Min}\,\mathbb{A}_p,$$

$$\textit{Subject to } \mu_p = \sum_{i=1}^{N} w_i E(R_i), \textit{ and } \sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

Based on equations (9) and (10), the MV𝔸 optimal portfolios satisfy the following first order condition:

$$E\left[\frac{R_i R_{p*}^{\mathrm{MV}\mathbb{A}}}{1+R_{p*}^{\mathrm{MV}\mathbb{A}}}\right]=0 \tag{13}$$

where $R_{p*}^{\mathrm{MV}\mathbb{A}}=\sum_{i=1}^{N} w_{i*}^{MV\mathbb{A}} R_i$*, and where* $w_{i*}^{MV\mathbb{A}}$ *is the* $MV\mathbb{A}$ *optimal share of wealth on i-th asset that satisfied the first order condition (13). The efficient* $MV\mathbb{A}$ *portfolios are the minimum-*$\mathbb{A}$ *portfolios that offer the highest expected returns and the lowest variances for the same level of SwV. The second order condition is non-negative and* $MV\mathbb{A}$ *efficient space is concave.*[20]

For a graphical illustration, Figure 3 shows that the MV$\mathbb{A}$ efficient portfolios are located the spherical surface in a three-dimensional space of mean, variance, and the minimum-$\mathbb{A}$, respectively. The MV$\mathbb{A}$ optimal portfolio $P_{\mathrm{MV}\mathbb{A}}^{*}$ is the point on the MV$\mathbb{A}$ efficient-surface to which the MV$\mathbb{A}$ capital market line (MV$\mathbb{A}$-CML) is the tangent. Intuitively, the MV$\mathbb{A}$ -CML, SSD-CML and the MV-CML all converge to one line, only if *asymmetry-risk* ($\mathbb{A}$) of all assets is zero, or none of the investors care about that.

[Insert Figure 3 here]

An important implication of Figure 3 is that with the risk-free asset, the optimal portfolio of MV$\mathbb{A}$ is identical to that of M*SwV* (SSD). The optimal portfolio of M*SwV* (SSD) is the tangent point on the efficient curve from the risk-free asset and is the one with largest M*SwV* Sharpe ratio, i.e., mathematically, $\underset{w_i}{Min}\frac{\mu_p-R_f}{SwV_p}=\underset{w_i}{Min}\frac{\mu_p-R_f}{\sigma_p^2+\mathbb{A}_p}$. On the other hand, the optimal portfolio of MV$\mathbb{A}$ in Figure 3 also has the highest Sharpe ratio in the three-dimensional MV$\mathbb{A}$ space, which is determined by the tangent of the angle between the MV$\mathbb{A}$ capital market line and its projection line on $\mathbb{A}_p-\sigma_p^2$ plane. Since $\mathbb{A}_p$ is orthogonal to σ_p^2, the optimal portfolio of MV$\mathbb{A}$ can be determined from this

[20] The derivation of Proposition 3 is similar to that of Proposition 1. We omit the detail description.

minimization process: $\underset{w_i}{Min} \frac{\mu_p - R_f}{\sqrt{(\sigma_p^2)^2 + (\mathbb{A}_p)^2}}$, or equivalently, $\underset{w_i}{Min} \frac{\mu_p - R_f}{\sigma_p^2 + \mathbb{A}_p}$. As a result, the optimal portfolio of MV$\mathbb{A}$ is the SSD optimal portfolio.

Figure 3 highlights that although the tow-fund separation holds between the risk-free fund and the optimal SSD portfolio, the risk-return tradeoff of expected utility maximization is determined by the three parameters: mean, volatility and asymmetry, respectively.[21] Figure 3 also shows the bias of the MV model is from the ignorance of the impact of distributional asymmetry on investment decision making. Thus, the asymmetry in addition mean and volatility is necessary to be jointly considered in the portfolio efficiency, performance analysis as well as capital asset pricing.

For developing the MV$\mathbb{A}$ equilibrium, it is necessary to identify two orthogonal benchmark portfolios that mimic the symmetry and asymmetry of returns on the SSD optimal market portfolio. First, we identify the efficient portfolios with symmetric returns by minimizing SwV$_p$ subject to $\mu_p = \sum_{i=1}^{N} w_i E(R_i)$, and $\mathbb{A}_p = 0$. Then, the optimal symmetry-factor portfolio, with returns denoted $R_m^{\mathbb{S}^\perp}$ $\left(= \sum_{i=1}^{N} w_{i*}^{\mathbb{S}^\perp} R_i\right)$, is the one has the highest Sharp Ratio. Second, returns on the factor portfolio of asymmetry (denoted $R_m^{\mathbb{A}^\perp}$) have to be independent to $R_m^{\mathbb{S}^\perp}$, but the sum of these two returns must be proportionally equal to R_m^{SwV}, the returns on the SSD optimal portfolio. Mathematically, $R_m^{\mathbb{A}^\perp}$ must then satisfy the following two conditions:

$$w_{i*}^{\mathbb{A}^\perp} = (1-\theta) w_{i*}^{SwV} - w_{i*}^{\mathbb{S}^\perp} \quad \text{for all } i\,, \tag{14.1}$$

and

$$CoV\left(R_m^{\mathbb{A}^\perp}, R_m^{\mathbb{S}^\perp}\right) = 0, \tag{14.2}$$

[21] Under the two-fund separation, $EU\left[(1-w)R_f + wR_{p*}^{SwV}\right] = EU\left[(1-w)R_f + wR_{p*}^{\mathrm{MV}\mathbb{A}}\right]$. Then, according to equations (3) and (5), the risk-measure of the optimal investment can be linearly decomposed such that $SwV\left(R_{p*}^{SwV}\right) = \gamma \cdot Var\left(R_{p*}^{\mathrm{MV}\mathbb{A}}\right) + \delta \cdot \mathbb{A}\left(R_{p*}^{\mathrm{MV}\mathbb{A}}\right)$, where $\gamma = \frac{U''}{2\mathbb{U}} > 0$, and $\delta = \frac{\mathbb{U}^{(3)}}{\mathbb{U}} > 0$.

where $0 \leq \theta \leq 1$. Finally, solve for θ from the equations (14.1) and (14.2) simultaneously. We then can calculate returns on the optimal asymmetry-factor such that $R_m^{\mathbb{A}\perp} = \sum_{i=1}^{N} w_{i*}^{\mathbb{A}\perp} R_i$.

From Figure 3 and Theorem 3, although the optimal portfolio of MV𝔸 is equivalent to that of SSD, the efficient set of MV𝔸 is larger than that of SSD. That is, the MV𝔸 equilibrium is more general than the SSD equilibrium in that the MV𝔸 efficiency is valid not only for risk-averse investors but also for those who have upside (downside) asymmetry preference (aversion). We formally derive the formulation of risk-premium for individual assets under the MV𝔸 equilibrium in the following proposition:

Proposition 4 (The MV𝔸 Equilibrium)
Assume short sell is allowed and investors are permitted to borrow and lend at R_f. Based on equation (5), for all utility functions with $U' > 0$, $U'' < 0$, and $U''' > 0$, the expected utility can be transformed as a function of three parameters: μ, σ^2, and $\mathbb{A}$, respectively. Let $R_m^{\mathbb{S}\perp}$ and $R_m^{\mathbb{A}\perp}$ be orthogonal returns on the symmetric and asymmetric factor portfolios determined from the SSD optimal portfolio based on (14.1) and (14.2), respectively. The risk-premium of risky assets, in equilibrium, can be calculated by the following equation:

$$E(R_i - R_f) = \beta_i^{\mathbb{S}} \lambda_{\mathbb{S}} + \beta_i^{\mathbb{A}} \lambda_{\mathbb{A}}, \tag{15.1}$$

where

$$\beta_i^{\mathbb{S}} = \frac{CoV(R_i R_m^{\mathbb{S}\perp})}{Var\left(R_m^{\mathbb{S}\perp}\right)}, \tag{14.2}$$

$$\beta_i^{\mathbb{A}} = \left[\frac{CoSwV\left(R_i, R_m^{\mathbb{A}\perp}\right) - \frac{CoV\left(R_i, R_m^{\mathbb{A}\perp}\right)}{\left[1+E\left(R_m^{\mathbb{A}\perp}\right)\right]^2}}{SwV\left(R_m^{\mathbb{A}\perp}\right) - \frac{Var\left(R_m^{\mathbb{A}\perp}\right)}{\left[1+E\left(R_m^{\mathbb{A}\perp}\right)\right]^2}}\right], \tag{14.3}$$

$\lambda_{\mathbb{S}} = E\left(R_m^{\mathbb{S}\perp} - R_f\right)$, and $\lambda_{\mathbb{A}} = E\left(R_m^{\mathbb{A}\perp} - R_f\right)$.

Serving as an extension of the CAPM, the two-factor linear model (15), derived theoretically from the expected utility maximization, demonstrates that two deterministic components: the symmetry and the asymmetry, are sufficient to explain the market price dynamics. Again, the ignorance of the impact of asymmetry on market return variations causes the failure of the traditional CAPM. Many empirical pricing factors, such as SBM, HML, and others, do successfully fulfill the incompleteness of the CAPM. Nevertheless, all those factors, perhaps, capture just the systematic asymmetry in equilibrium price determination partially.

V. Empirical Analysis

The primary source of sample data for our empirical analysis comes from CRSP equity database that covers all firms incorporated in the US and listed on the NYSE, AMEX, or NASDAQ. We select stocks that have a CRSP share code of 10 or 11 to be consistent with the Fama-French asset pricing factors in Kenneth French's Database. Further, to avoid survivorship bias, stock return information before July 1969 is eliminated. The ending sample period is December 2015. Also, sample returns of fourteen hedge funds listed on the Credit Suisse Hedge Fund Index over a period a from April 1994 to December 2015 are employed as well.

[Insert Table 1 here]

A. The Impact of Return Asymmetry on Portfolio Performance

The difference between SwV and variance is symmetries in returns as shown in (1). Therefore, to illustrate the impact of asymmetry on portfolio performance, we apply both models of MV and MSwV to hedge fund index data. Table 1 presents the summary statistics of our analysis. Although the SwV of funds seems to be similar to their variance in value, a significant difference appears in the ranking of Sharpe ratios. The number of the ranking inconsistency is nine out of fourteen. This piece of evidence demonstrates the impact of distributional asymmetry of returns on fund performance analysis is significant and should not be ignored. Also, Table 1 shows the considerable distinction between MV-beta and MSwV-beta that highlights the importance of return co-asymmetry (e.g., co-skewness and higher co-moments) in the systematic-risk determination. Specifically, for all hedge fund data, the MV-beta is significantly larger than MSwV-beta, which suggests that the co-asymmetry of hedge funds tends to be positive and provides diversification benefits, according to (7) and (8).

Traditional portfolio theory suggests return volatility (σ) can be reduced by forming portfolios if returns on assets are not perfectly correlated. Is this true as well for the return asymmetry ($\mathbb{A}$)? Further, unlike the volatility, the quantity $\mathbb{A}$ could be negative (positive) corresponding to the prospect of potential upside-gains (downside-loss). Is the risk-diversification concerning return asymmetry different between the aspect of downside and that of upside, and which type of portfolios performs better among positive asymmetry, symmetry, and negative asymmetry? We find the answer to the above questions in Table 2 empirically.

[Insert Table 2 here]

B. Volatility, Asymmetry, Diversification and Stability

Twenty equally weighted portfolios are formed by grouping all stocks in our database according to their $\mathbb{A}$ measures. The average value of individual stocks' $\mathbb{A}$ ranges from -483.28 to 265.50 basis-point (b.p.). To examine the consistency between ex-post and ex-ante measurements, we conduct our analysis under both the in-sample and the out of sample frameworks. Table 2 illustrates that the relationship between $\mathbb{A}$ and σ is truncated and concave; higher (lower) the positive (negative) $\mathbb{A}$, larger the σ. That is consistent with our theory shown in Figure 3. Table 2 also demonstrates that forming a portfolio significantly reduces the magnitude of asymmetry of individual assets. The range of $\mathbb{A}$ of the 20 sorted portfolios decreases to -7.37 (-14.04) b.p. to 6.27 (6.38) b.p. from the in-sample (out-of-the sample) analysis. Interestingly, the portfolio of stocks that have the most negatively asymmetric returns (portfolio #20) has the most significant reduction of $\mathbb{A}$ (from 265 to 3.92). Thus, the implication is that returns on portfolios are more symmetrically distributed than those on individual securities. Since a portfolio's $\mathbb{A}$ is a weighted sum of component assets' $Co\mathbb{A}$ (see Sections 2 and 3), the magnitude deduction of $\mathbb{A}$ from portfolio formation is due to the effects of the co-asymmetry of individual securities. Implicitly, assets have a relatively high prospect of downside losses (upside gains), tends to have more negative (positive) co-asymmetry with the core portfolio.

Table 2 depicts that the ex-post estimate of return asymmetry is consistent with the ex-ante measure. Again, the ranking difference between Sharps ratios of MV and MSwV indicates the importance of asymmetry in portfolio performance analysis. Portfolios with slightly negative asymmetry (positive $\mathbb{A}$) perform better than others. Regardless the in-sample or out-of-sample, the worst performed portfolio (#20) is the one has the most highly negative return-asymmetry. From the MV Sharpe ratios, it shows that portfolio #1 (the one with most positively skewed returns) has the best performance; however, the MSwV and MV$\mathbb{A}$ analyses do not show that. As we have discussed in Section II that highly upside skewed securities (e.g., the lottery-type stocks) are SSD inefficient assets with a substantial volatility. They may be MVA efficient but are probably not dominating alternatives for all risk-averse investors. Also, these securities tend to be positively correlated, and the portfolio #1 should not outperform the overall market. That again highlights the potential bias of the MV model in fund performance analysis that ignores the impact of return-asymmetry.

C. Performance of the Optimal MSwV and MV𝔸 Market Portfolios

One of the main contribution of this article is the development of optimal stochastic dominance portfolios based on simple and basic optimization procedures shown in Propositions 1 and 3, respectively. Since optimal market portfolio shown in Figure 3 should outperform other index portfolios in theory, it is important to examine empirically the performance of MSwV and MV𝔸 Market Portfolios. For developing market indexes, we determine the optimal weights according to the MSwV and MV𝔸 optimization procedures and based on past 60 monthly excess returns on all CRSP stocks. We then apply the optimal weights to the following month to calculate the returns on indexes. Therefore, the MSwV and MV𝔸 optimal indexes are tradeable funds.

[Insert Table 3 here]

For a comparison purpose, Table 3 summarizes the basic statistics and Shape ratios of the MSwV and MV𝔸 optimal market funds vs. Fama-French's factor portfolios as well as major US market indexes including S&P500 (SPX), Dow Jones Industrial Average (DOW) and NASDAQ, accordingly. Noticeably, the quantity 𝔸 for all market and factor indexes is negative in value indicating return distribution of all indexes are positively skewed, and the magnitude of asymmetry is positively correlated with the volatility. Corresponding to Figure 3, the optimal market portfolio of MV𝔸 converges to that of MSwV (SSD) in the MV𝔸 space. Although all other indexes may be located on the MV𝔸 efficient plane, the slope of their capital allocation lines will be lower than that of the optimal portfolio. We find the empirical evidence that supports our theoretical argument from the Sharpe Ratios shown in Table 3. Specifically, the distributional statistics of MSwV and MV𝔸 optimal (market) portfolios are almost identical, and they have the highest Sharpe ratios among all funds.

D. Testing for MSwV (SSD) and MV𝔸 Asset Pricing Models

In this paper, we derive the asset pricing models under the MSwV and MV$\mathbb{A}$ frameworks following the same derivation procedures as that of CAPM. To examine the validity of these models, we follow the Fama-MacBeth (1973) two-stage testing methodology. Based on equations (13.2), (15.2), and (15.3), over the sample period, we first calculate the estimates of factor loadings, $\hat{\beta}_i^{SwV}$, $\hat{\beta}_i^{\mathbb{S}}$, and $\hat{\beta}_i^{\mathbb{A}}$, on the monthly basis for all securities i with respect to the past 60 monthly returns on indexes, R_m^{SwV}, $R_m^{\mathbb{S}}$, and $R_m^{\mathbb{A}}$, accordingly. Secondly, to eliminate the impact of idiosyncratic risk of securities on the analysis and have a focus on the systematic risk premium, at the beginning of each period, we form sixty equal-sized portfolios, sorted by the factor loadings of all individual securities that estimated from the first step. We then regress cross-sectional returns on the sixty portfolios against their factor loadings to estimate the factor premium, $\hat{\lambda}_{SwV}$, $\hat{\lambda}_{\mathbb{S}}$, and $\hat{\lambda}_{\mathbb{A}}$, respectively.

[Insert Table 4 here]

Table 4 presents the results of the Fama-MacBeth test. It appears that all estimates of factor premium are statistical significant at least at the five percent level. Thus, the empirical evidence supports that from a cross-sectional aspect, λ_{SwV} is a common pricing factor for risk-averse and expected utility maximizing investors. If investors also have preference (aversion) for (to) the prospect of potential upside gains (downside losses), then the common factors, $\lambda_{\mathbb{S}}$ and $\lambda_{\mathbb{A}}$, are necessary to be considered in the determination of return generating process for all risky assets. That is, the MV$\mathbb{A}$ asset pricing model can be viewed as an important extension of the traditional approach purely based on risk-aversion assumption in that it takes the main argument of the prospect theory into consideration.

To further examine the robustness of the MV$\mathbb{A}$ pricing model, we test the sufficiency of the common factors in explaining the equilibrium returns on assets. If MV$\mathbb{A}$ model shown in (15) is valid, then the residual returns of individual assets, $\varepsilon_{j,t} = R_{j,t} - \left(\hat{\beta}_j^{\mathbb{S}} R_{m,t}^{\mathbb{S}} + \hat{\beta}_j^{\mathbb{A}} R_{m,t}^{\mathbb{A}}\right)$, should be idiosyncratic. Thus, no other common factors would have explanatory power to $\varepsilon_{j,t}$. For an illustration, in this paper, we focus on four important empirical factors of Fama and France (1993, 2015): *SMB*, *HML*, *RMW*, and *CMA*, respectively. Again, by employing Fama-MacBeth approach, we first calculated the factor loadings from the following regression model: $\varepsilon_{j,t} = \alpha_j + \beta_j^{SMB} SMB_t$ + $\beta_j^{HML} HML_t$ +$\beta_j^{RMW} RMW_t$ + $\beta_j^{CMA} CMA_t + e_{j,t}$, over the past 60 months. Then, at the beginning of each period, sixty equal-sized portfolios sorted by the beginning-of-period beta estimates are formed from the entire sample. To calculate estimates of factor-premium, $\hat{\lambda}_{SMB}$, $\hat{\lambda}_{HML}$, and $\hat{\lambda}_{RMW}$, and $\hat{\lambda}_{CMA}$, accordingly, we then, in a cross-sectional framework, regress all portfolio residuals, ε_p, against multiple beta estimates of factor loading, $\hat{\beta}_p^{SMB}$, $\hat{\beta}_p^{HML}$, $\hat{\beta}_p^{RMW}$, and $\hat{\beta}_p^{CMA}$, respectively.

[Insert Table 5 here]

Panel 1 of Table 5 reports factor premium estimates of Fama-French original two-factor approach. We exclusive the market factor in that it is already embedded in the MV𝔸 factors. Panel 2 shows the results from the multiple cross-sectional regression of Fama-French newly proposed four empirical pricing factors. Statistically, we found none of the λ estimates is significant, indicating the Fama-French empirical pricing factors have no impact on the residual returns of assets calculated from the MV𝔸 model. The MV𝔸 asset pricing model is robust in describing the cross-sectional returns on risky equity securities. From the previous discussion, the traditional MV model is a valid approach, if returns on the asset are symmetrically distributed. Therefore, intuitively, the difference between MV𝔸 and MV models is merely the impact of asymmetries in returns, and the asymmetry factor $\lambda_{\mathbb{A}}$ characterizes this difference.

In short, logically, the symmetry is unique; however, the appearance of asymmetry could be infinite; similar to the case that there is an endless number of distributional moments in determining the return asymmetry. Therefore, at least in theory, it is possible to have infinite amounts of empirical factors identified from the sample data that can describe the phenomenon of pricing asymmetry in equilibrium. The contribution of this paper is to provide a simple methodology that converges all possible distributional or pricing asymmetries into a summary statistic or a common factor.

VI. Conclusion

The swap variance (SwV), formulated merely by the twice expected difference of arithmetic and logarithmic returns adjusted by the mean, summarizes the entire probability distribution of returns. Since variance measures the quadratic (symmetric) variation of returns, the difference between SwV and the variance thus characterizes the asymmetries (denoted $\mathbb{A}$) in returns. We prove mathematically that the expected utility can be completely transformed as a function of mean and SwV as well as that of mean, variance, and $\mathbb{A}$, accordingly, without any restriction on the form of utility functions and that of return distributions. Therefore, the MSwV and MVA analyses, consistent with expected utility maximization, serve as an extension of the classical MV model by considering distributional asymmetries. Importantly, the MSwV efficiency is necessarily and sufficiently second-degree stochastic dominance (SSD) efficient for all risk-averse investors. The efficient set of MVA, on the other hand, is much broader than that of SSD for that it also includes investors who prefer (dislike) the prospect of potential upside gains (downside losses). That makes some of the highly risky assets, e.g., the lottery-type securities, to be included in the MVA efficient set but be excluded from the SSD efficient set.

Similar to the portfolio variance, the SwV of a portfolio is also a weighted sum of CoSwV between individual assets and the portfolio, where CoSwV is the covariance pluses a polynomial combination of all higher co-moments. Thus, the CoSwV is different from covariance by the co-asymmetry (denoted CoA), which captures the jointly asymmetric variation between an asset and the core portfolio. The return asymmetry of a portfolio is then a weighted sum of CoA. Consequently, the SSD efficient frontier can be determined from the minimum SwV assets in the mean-SwV space, and the MVA efficient plane can be defined by the minimum A portfolios in the three-dimensional mean-variance-asymmetry space.

By employing conventional methods of expected utility optimization, we develop the equilibrium pricing models under the frameworks of MSwV and MVA, respectively. The MSwV approach is a single factor model and is similar to the CAPM. However, for calculating the beta coefficient, the market factor needs to be replaced by the MSwV optimal portfolio, and the covariance (variance) has to be substituted by the CoSwV (SwV). We show empirically that the MSwV model is superior to the MV approach mainly for returns are asymmetrically distributed.

The MVA asset pricing approach, a two-factor model that serves as an extension of the CAPM, quantifies the deterministically systematic components of equilibrium returns on risky assets between symmetry and asymmetry, respectively. Crucially, the MVA model is unbiased not only to the risk-aversion but the upside gain-preference as well as the downside loss-aversion. Based on the Fama-MacBeth tests, we showed that the MVA is empirically robust. Based on data from the US equity markets, we further find that with the MVA factors of symmetry and asymmetry, the conventional empirical pricing factors lose their explanatory power to the cross-sectional expected returns on assets. Our analysis implies that since only two fundamental factors are sufficient for determining the systematic risk of assets, most empirically defined factors perhaps capture just parts of the phenomenon of pricing asymmetry in equilibrium.

In summary, the simplicity and generality of MSwV and MVA approaches make them as a powerful tool in analyzing investment decision making under risk and uncertainty. Instead of replacing the MV model, the MVA analysis enhances the conventional methods of security selection, asset allocation, portfolio efficiency analysis as well as the asset valuation to a general framework by taking asymmetries in return as well as investors' prospect of gain and loss into consideration.

Appendix

Proof of Theorems 1 and 2

Suppose R_i stochastically dominate R_j such that $EU(R_i) \geq EU(R_j)$. From (3), we have

$$EU(R_i) - EU(R_j) = \quad U(\mu_i) - U(\mu_j) + \theta \cdot (SwV_i - SwV_j) \geq 0 \tag{A-1}$$

where $\theta = \frac{\mathbb{U}(U,\mu_i,R_i^o)SwV_i - \mathbb{U}(U,\mu_j,R_j^o)SwV_j}{SwV_i - SwV_j}$. Based on mean value theorem, θ is a number between $\mathbb{U}(U,\mu_i,R_i^o)$ and $\mathbb{U}(U,\mu_j,R_j^o)$, and thus $\theta \leq 0$. The conditions that $\mu_i \geq \mu_j$, and $SwV_i = SwV_j$ are necessary to ensure the inequality (A-1) for all investors who prefer more to less ($U' \geq 0$) without further restriction on the utility function. For risk-averse investors where $\mathbb{U}(U,\mu,R^o) \leq 0$, in addition to higher mean return, the condition $SwV_i \leq SwV_j$ is necessary for the inequality (A-1).

To prove the sufficiency of FSD, consider risk-neutral investors in which $\frac{\theta}{U'(0)}$ approaches zero, the condition $\mu_i \geq \mu_j$ must hold to have the inequality (A-1). Even for the most risk-averse (risk-loving) investors, where the ratio $\frac{\theta}{U'(0)}$ is extremely positive (negative), the condition $SwV_i = SwV_j$ is sufficient to have the stochastic dominance.

The sufficiency of SSD can be determined by considering the most (least) risk-averse investors. That is, even if the ratio $\frac{\theta}{U'(0)}$ is extremely negative (approaches zero) in value, the condition, $SwV_i \leq SwV_j$ ($\mu_i \geq \mu_j$) must hold for the stochastic dominance.

Derivation of Proposition 2

Suppose the optimal market portfolio is *MSwV* efficient with returns denoted R_m^{SwV} , we define the Lagrange function of expected utility with respect to its mean and SwV as well as the risk-free rate.

$$L = EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})] - \lambda(1 - \sum_i w_i - f), \quad \text{(A-2)}$$

where λ is the Lagrange multiplier, and f is the weight of risk-free asset. Now, take the partial derivative of the Lagrange function with respect to w_i and f, respectively, we have

$$\frac{\partial L}{\partial w_i} = \frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial \mu(R_m^{SwV})} * \frac{\partial \mu(R_m^{SwV})}{\partial w_i} + \frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial SwV(R_m^{SwV})} * \frac{\partial SwV(R_m^{SwV})}{\partial w_i} - \lambda = 0, \quad \text{(A-3)}$$

and

$$\frac{\partial L}{\partial f} = \frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial \mu(R_m^{SwV})} * \frac{\partial \mu(R_m^{SwV})}{\partial f} + \frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial SwV(R_m^{SwV})} * \frac{\partial SwV(R_m^{SwV})}{\partial f} - \lambda = 0. \quad \text{(A-4)}$$

Subtract (A-4) from (A-3), and note that $\frac{\partial SwV(R_m^{SwV})}{\partial w_i} = CoSwV(R_i, R_m^{SwV})$, $\frac{\partial SwV(R_m^{SwV})}{\partial f} = 0$, and $\left(\frac{\partial \mu(R_m^{SwV})}{\partial w_i} - \frac{\partial \mu(R_m^{SwV})}{\partial f}\right) = E(R_i - R_f)$, we obtain the equilibrium condition for individual assets corresponding to the optimal portfolio as:

$$E(R_i - R_f) = \frac{-CoSwV(R_i, R_m^{SwV})\frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial SwV(R_m^{SwV})}}{\frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial \mu(R_m^{SwV})}}. \quad \text{(A-5)}$$

The condition in (A-5) also holds for the market portfolio that

$$E(R_m^{SwV} - R_f) = \frac{-SwV(R_m^{SwV})\frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial SwV(R_m^{SwV})}}{\frac{\partial EU[\mu(R_m^{SwV}), SwV(R_m^{SwV})]}{\partial \mu(R_m^{SwV})}}. \quad \text{(A-6)}$$

Finally, divide (A-5) by (A-6), we have equations (13.1) and (13.2).

Proof of Proposition 4:

Suppose the optimal market portfolio is MV𝔸 efficient with returns denoted $R_m^{\mathrm{MV}\mathbb{A}}$ (= $R_m^{\mathbb{S}^\perp}$ + $R_m^{\mathbb{A}^\perp}$). We define the Lagrange function of expected utility with respect to its mean, variance and 𝔸, as well as the risk-free rate.

$$L = EU[\mu(R_m^{\mathrm{MV}\mathbb{A}}), Var(R_m^{\mathrm{MV}\mathbb{A}}), \mathbb{A}(R_m^{\mathrm{MV}\mathbb{A}})] - \lambda(1 - \textstyle\sum_i w_i - f), \qquad \text{(A-7)}$$

Since U is the aggregated utility, the optimal portfolio is the market portfolio in equilibrium. Then, the first order condition of (A-7) can be written as:

$$\frac{\partial L}{\partial w_i} = \left[\frac{\partial EU}{\partial \mu(R_m^{\mathrm{MV}\mathbb{A}})} \cdot \frac{\partial \mu(R_m^{\mathrm{MV}\mathbb{A}})}{\partial w_i}\right] + \left[\frac{\partial EU}{\partial Var(R_m^{\mathrm{MV}\mathbb{A}})} \cdot \frac{\partial Var(R_m^{\mathrm{MV}\mathbb{A}})}{\partial w_i}\right] + \left[\frac{\partial EU}{\partial \mathbb{A}(R_m^{\mathrm{MV}\mathbb{A}})} \cdot \frac{\partial \mathbb{A}(R_m^{\mathrm{MV}\mathbb{A}})}{\partial w_i}\right] - \lambda = 0, \qquad \text{(A-8)}$$

and

$$\frac{\partial L}{\partial f} = \frac{\partial EU}{\partial \mu(R_m^{\mathrm{MV}\mathbb{A}})} \cdot \frac{\partial \mu(R_m^{\mathrm{MV}\mathbb{A}})}{\partial f} - \lambda = 0. \qquad \text{(A-9)}$$

Since $Var\left(R_m^{\mathrm{MV}\mathbb{A}}\right) = Var\left(R_m^{\mathbb{S}^\perp}\right)$, and $\mathbb{A}\left(R_m^{\mathrm{MV}\mathbb{A}}\right) = \mathbb{A}\left(R_m^{\mathbb{A}^\perp}\right)$, equation A-8 can be rewritten as:

$$\frac{\partial L}{\partial w_i} = \left[\frac{\partial EU}{\partial \mu(R_m^{\mathrm{MV}\mathbb{A}})} \cdot \frac{\partial \mu(R_m^{\mathrm{MV}\mathbb{A}})}{\partial w_i}\right] + \left[\frac{\partial EU}{\partial Var\left(R_m^{\mathbb{S}^\perp}\right)} \cdot \frac{\partial Var\left(R_m^{\mathbb{S}^\perp}\right)}{\partial w_i}\right] + \left[\frac{\partial EU}{\mathbb{A}\left(R_m^{\mathbb{A}^\perp}\right)} \cdot \frac{\partial \mathbb{A}\left(R_m^{\mathbb{A}^\perp}\right)}{\partial w_i}\right] - \lambda = 0, \qquad \text{(A-10)}$$

Subtract (A-9) from (A-10), and note that $\frac{\partial Var\left(R_m^{\mathbb{S}^\perp}\right)}{\partial w_i} = CoV\left(R_i, R_m^{\mathbb{S}^\perp}\right)$, $\frac{\partial \mathbb{A}\left(R_m^{\mathbb{A}^\perp}\right)}{\partial w_i} = CoSwV\left(R_i, R_m^{\mathbb{A}^\perp}\right) - \frac{CoV\left(R_i, R_m^{A^\perp}\right)}{\left[1+\mu\left(R_m^{\mathbb{A}^\perp}\right)\right]^2}$, and $\left[\frac{\partial \mu\left(R_m^{\mathrm{MV}\mathbb{A}}\right)}{\partial w_i} - \frac{\partial \mu\left(R_m^{\mathrm{MV}\mathbb{A}}\right)}{\partial f}\right] = \left[E(R_i) - R_f\right]$, we obtain the following equation for individual assets corresponding to the optimal portfolio:

$$E(R_i) - R_f = \frac{-\left[\frac{\partial EU}{\partial Var\left(R_m^{\mathbb{S}^\perp}\right)}\right]}{\left[\frac{\partial EU}{\partial \mu\left(R_m^{\mathrm{MV}\mathbb{A}}\right)}\right]} CoV\left(R_i, R_m^{\mathbb{S}^\perp}\right) + \frac{\left[\frac{\partial EU}{\mathbb{A}\left(R_m^{\mathbb{A}^\perp}\right)}\right]}{\left[\frac{\partial EU}{\partial \mu\left(R_m^{\mathrm{MV}\mathbb{A}}\right)}\right]} \left[CoSwV\left(R_i, R_m^{\mathbb{A}^\perp}\right) - \frac{CoV\left(R_i, R_m^{A^\perp}\right)}{\left[1+\mu\left(R_m^{\mathbb{A}^\perp}\right)\right]^2}\right] \qquad \text{(A-11)}$$

$$E\left(R_m^{\mathbb{S}^\perp} - R_f\right) = \frac{-\left[\frac{\partial EU}{\partial Var\left(R_m^{\mathbb{S}^\perp}\right)}\right]}{\left[\frac{\partial EU}{\partial \mu\left(R_m^{\mathrm{MV}\mathbb{A}}\right)}\right]} Var\left(R_m^{\mathbb{S}^\perp}\right) \qquad \text{(A-12)}$$

$$E\left(R_m^{\mathbb{A}^{\perp}} - R_f\right) = \frac{\left[\frac{\partial EU}{\mathbb{A}\left(R_m^{\mathbb{A}\perp}\right)}\right]}{\left[\frac{\partial EU}{\partial \mu\left(R_m^{\mathrm{MV}\mathbb{A}}\right)}\right]}\left[SwV\left(R_m^{\mathbb{A}\perp}\right) - \frac{Var\left(R_m^{\mathbb{A}\perp}\right)}{\left[1+\mu\left(R_m^{\mathbb{A}\perp}\right)\right]^2}\right] \tag{A-13}$$

Finally, substitute (A-12) and (A-13) into (A-11), we have (15.1) and (15.2).

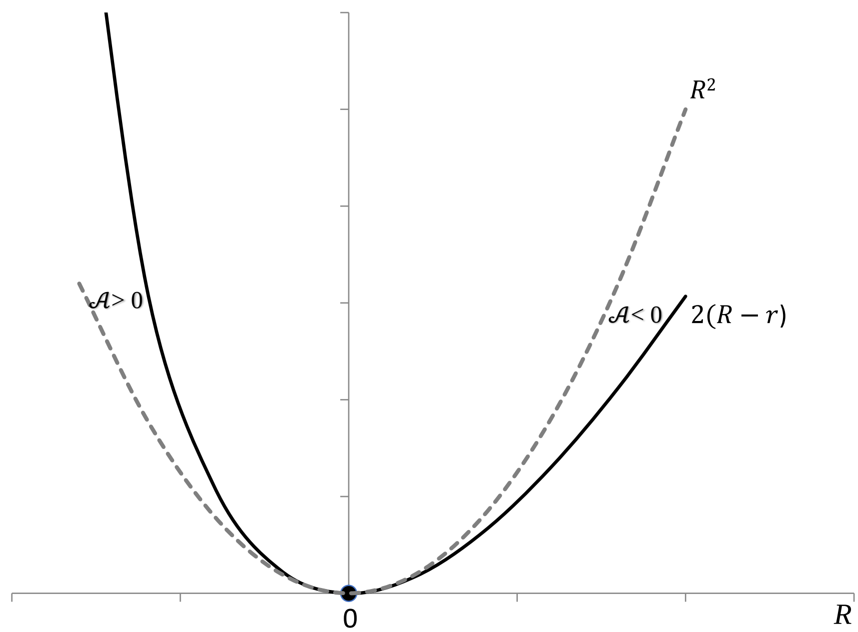

Figure 1: Symmetric R^2 vs. Asymmetric $2(R - r)$

This figure shows that the asymmetries in return, quantified by the difference between $2(R-r)$ and R^2, where $R > -1$, $r = ln(1+R)$, and $\mathcal{A} = [2(R-r) - R^2] = \sum_{k=3}^{\infty}(-1)^k \binom{2}{k} R^k$. R^2 is less (higher) than $2(R-r)$ if R is negative (positive). That implies the variance understates (overstates) the downside (upside) risk if returns are asymmetrically distributed.

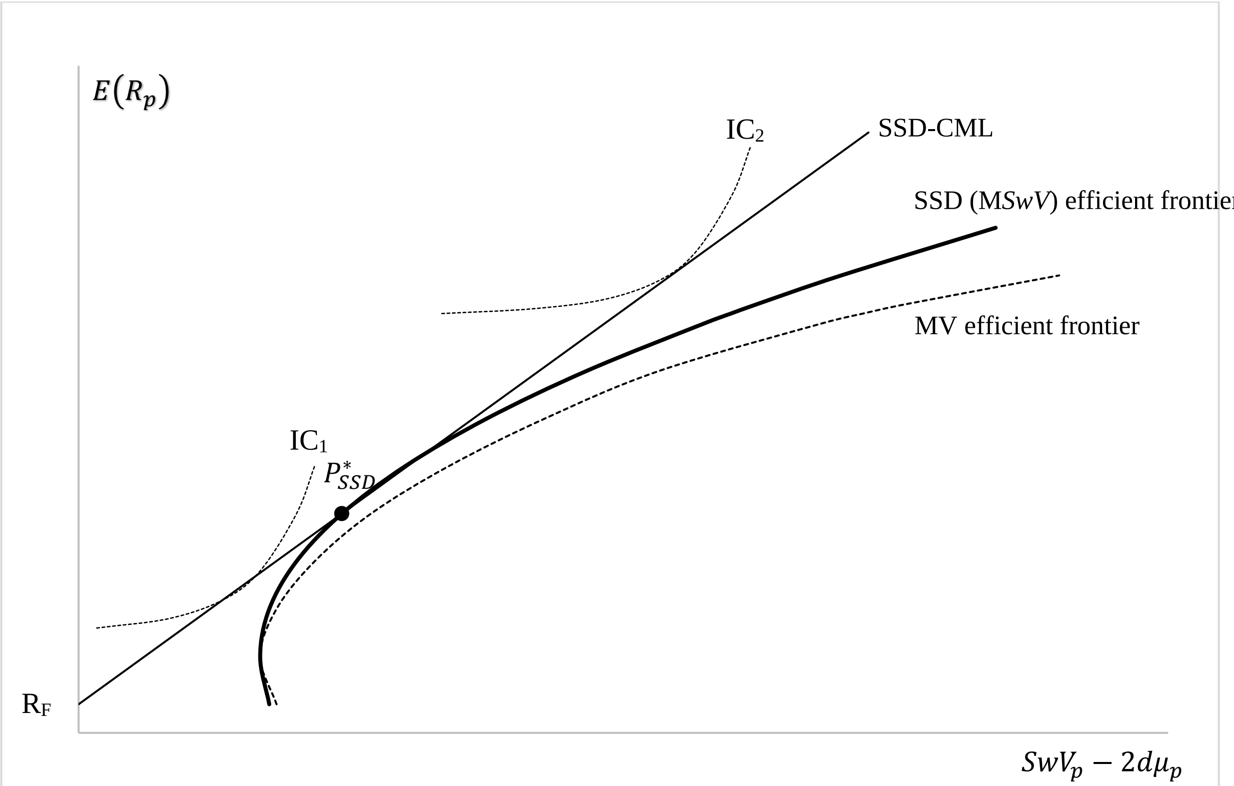

Figure 2. SSD Efficiency and Portfolio Separation This figure depicts a simulated M*SwV* efficient frontier of portfolios (solid line) generated from random returns of three hypothetical distributions: normal, lognormal, and gamma, respectively. The random sample has means ranging from -0.2 to 0.2, standard deviations from 0.4 to 0.7, and correlation coefficients from -0.3 to 0.7, accordingly. For a comparison, we also plot the MV efficient frontier (dash line) in the MSwV space. With a risk-free asset, an optimal SSD portfolio of risky assets, denoted P^*_{SSD}, can then be determined by the point on the SSD efficient curve to which the SSD capital market line (SSD-CML) is the tangent with the highest risk-adjusted mean return. The convex curves, IC_1 and IC_2, are indifference curves. The convexity of indifference curve presents the diminishing marginal rates of substitution between expected return and *SwV*.

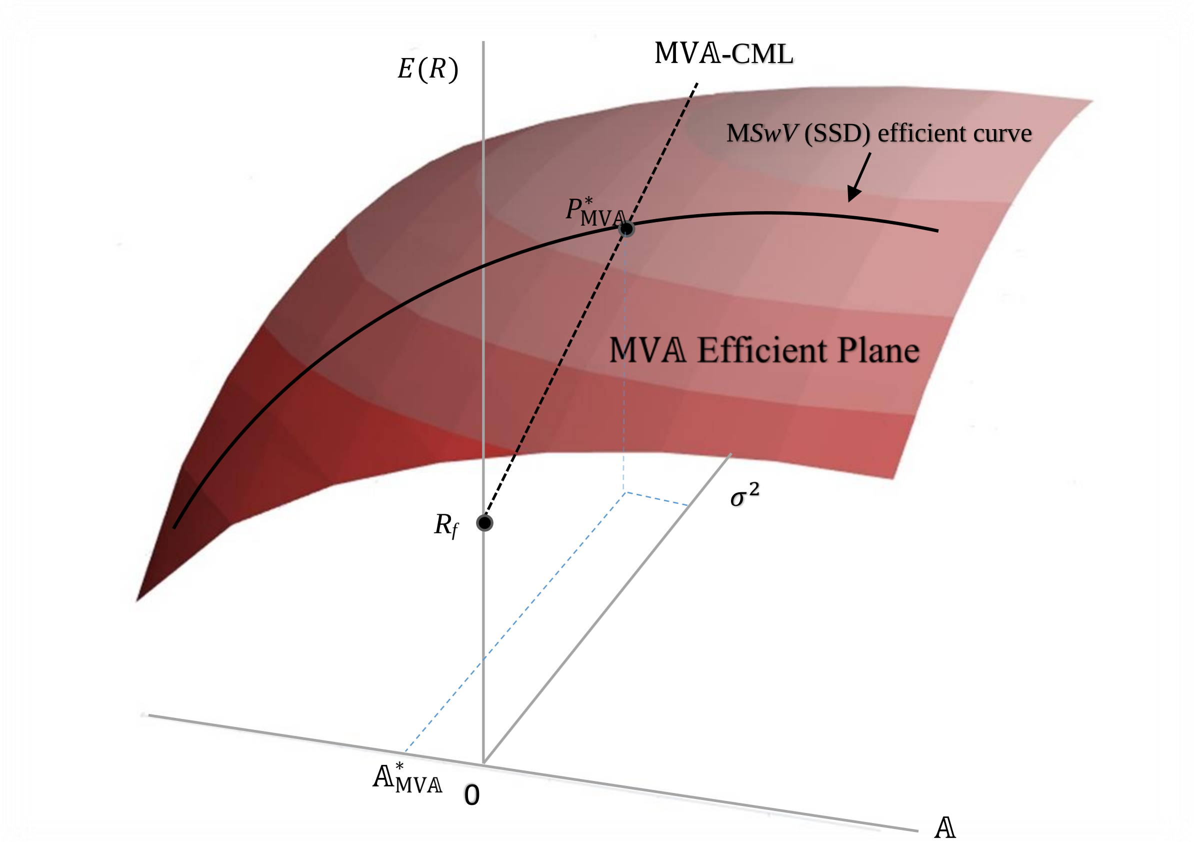

Figure 3. MV𝔸 Efficient Plane and Capital Market Line (CML) This figure depicts the efficient set of portfolios a three-dimensional space concerning mean, variance and 𝔸, where $\mathbb{A}_p = SwV_p - \left(\frac{\sigma_p}{1+\mu_p}\right)^2$. The MV𝔸 efficient assets are the minimum-𝔸 portfolios for every level of the mean return and those of the variance. Both the SSD and MV efficient frontiers lie on the MV𝔸 efficient plane. With a risk-free asset, the MV𝔸 optimal portfolio is unique and is the point on the MV𝔸 efficient plane to which the MV𝔸 capital market line (MV𝔸-CML) is the tangent with the highest risk-adjusted mean return.

Table 1
Summary Statistics

This table summarized the statistics of monthly portfolio returns from 14 hedge fund indexes. The sample period is from January 1995 to December 2015. The numbers after Sharpe ratios are the ranks among hedge funds. The MSwV Sharpe ratio is the expected excess return over the *SwV*. If the ranks are different under the two framework, the ranking numbers are in bold font. The M*SwV* beta is calculated based on (13.2) with respect to the M*SwV* optimal portfolio returns. To test the difference between MV-beta and M*SwV*-beta, we use Monte Carlo approach by data randomization. We randomly select 1,000 sets of 500 random returns from the total 559 observations in the sample and compute the t-test statistics.

Hedge Fund Portfolios	Mean (%)	σ^2 ($\times 10^4$)	*SwV* ($\times 10^4$)	Sharpe Ratio MV	Rank	Sharpe Ratio M*SwV*	Rank	Beta Coefficient MV	Beta Coefficient M*SwV*	Difference
Hedge Fund Index	0.717	4.098	2.116	0.251	5	12.562	6	0.281	0.258	0.023***
Convertible Arbitrage	0.618	3.534	1.981	0.218	9	11.322	7	0.159	0.171	-0.012***
Dedicated Short Bias	-0.408	22.381	4.679	-0.130	14	-2.783	14	-0.860	-0.863	0.004***
Emerging Markets	0.611	14.339	4.016	0.106	11	2.739	11	0.505	0.431	0.075***
Equity Market Neutral	0.454	7.903	3.209	0.088	12	2.358	12	0.184	0.185	-0.001***
Event Driven	0.722	3.185	1.906	0.288	3	15.932	4	0.266	0.250	0.016***
Distressed	0.805	3.279	1.973	0.330	2	17.976	3	0.256	0.268	-0.012***
Multi-Strategy	0.686	3.762	2.035	0.247	6	12.586	5	0.275	0.239	0.036***
Risk Arbitrage	0.480	1.385	1.255	0.231	8	19.672	2	0.142	0.115	0.027***
Fixed Income Arbitrage	0.451	2.372	1.612	0.158	10	9.830	9	0.115	0.138	-0.023***
Global Macro	0.926	6.760	2.714	0.276	4	10.815	8	0.145	0.168	-0.023***
Long/Short Equity	0.832	7.256	2.767	0.232	7	8.716	10	0.448	0.391	0.057***
Managed Futures	0.471	11.524	3.371	0.077	13	2.301	13	-0.054	-0.012	-0.041***
Multi-Strategy	0.678	1.900	1.588	0.341	1	24.633	1	0.137	0.098	0.039***

***, ** and * denote statistics significant at 1 percent, 5 percent and 10 percent levels, respectively.

Table 2
MV, SSD and MV$\mathbb{A}$ Sharpe Ratios

This table reports the rankings of Sharpe ratios of asymmetry portfolios (PFL) that formed according to their in-sample asymmetry measure (*Pre-*$\mathbb{A}$). We conduct a two-stage analysis. On the first stage, 20 portfolios are formed based on sorted annualized $\mathbb{A}$, calculated from the monthly returns on all CRSP stocks with share code 10 and 11. We then report the summary statistics as well as Sharp ratios for the in-sample as well as the out-of-sample (one-month lag) calculations. The sample period covers from 1969 to 2015.

	In-Sample Rankings								*Ave. Stock*	*Out-of-Sample Rankings*							
					Sharp Ratios									Sharp Ratios			
PFL	μ (%)	σ^2 ($\times 10^4$)	SwV ($\times 10^4$)	$\mathbb{A}$ ($\times 10^4$)	MV	Rank	MSwV	Rank	*Pre-*$\mathbb{A}$ ($\times 10^4$)	μ (%)	σ^2 ($\times 10^4$)	SwV ($\times 10^4$)	$\mathbb{A}$ ($\times 10^4$)	MV	Rank	MSwV	Rank
1	4.72	251.54	222.01	-7.37	1.719	8	1.948	8	-483.28	4.08	253.13	219.63	-14.04	1.455	10	1.677	8
2	1.54	104.65	98.80	-2.70	1.093	11	1.158	11	-85.20	1.36	105.68	99.60	-3.26	0.912	13	0.968	11
3	1.02	68.89	67.08	-0.43	0.905	13	0.930	12	-42.61	0.94	69.89	68.06	-0.53	0.778	15	0.799	12
4	0.68	51.70	50.98	-0.02	0.549	15	0.557	15	-25.44	0.69	52.13	51.41	-0.01	0.563	17	0.571	15
5	0.59	39.82	39.69	0.34	0.487	16	0.488	16	-16.46	0.59	40.20	40.07	0.34	0.482	18	0.483	16
6	0.52	30.91	30.91	0.32	0.400	18	0.400	18	-11.09	0.59	31.36	31.47	0.48	0.618	16	0.615	18
7	0.46	24.90	25.00	0.33	0.256	19	0.255	19	-7.62	0.51	25.50	25.70	0.46	0.446	19	0.442	19
8	0.49	20.07	20.34	0.46	0.467	17	0.461	17	-5.26	0.56	20.61	20.88	0.50	0.794	14	0.784	17
9	0.50	16.89	17.14	0.42	0.614	14	0.605	14	-3.49	0.59	17.39	17.72	0.54	1.114	12	1.093	14
10	0.53	14.29	14.67	0.53	0.936	12	0.912	13	-2.05	0.61	14.90	15.37	0.65	1.434	11	1.391	13
11	0.56	13.10	13.62	0.66	1.249	10	1.202	10	-0.71	0.66	13.84	14.44	0.78	1.906	8	1.826	10
12	0.75	14.52	15.44	1.14	2.437	6	2.290	6	0.75	0.81	14.90	15.92	1.26	2.777	6	2.599	6
13	0.99	17.22	18.66	1.78	3.447	4	3.181	4	2.56	1.05	17.39	18.92	1.89	3.759	4	3.455	4
14	1.22	21.25	23.33	2.59	3.876	2	3.531	1	5.04	1.24	21.34	23.52	2.70	3.953	2	3.587	1
15	1.38	25.10	27.88	3.46	3.919	1	3.529	2	8.48	1.42	25.20	27.98	3.48	4.062	1	3.658	2
16	1.62	32.15	35.52	4.39	3.806	3	3.445	3	13.70	1.62	31.92	35.40	4.49	3.833	3	3.457	3
17	1.63	41.47	45.29	5.14	2.975	5	2.724	5	21.99	1.64	41.22	45.02	5.13	3.018	5	2.762	5
18	1.60	54.61	58.52	5.62	2.204	7	2.057	7	36.71	1.65	53.73	57.76	5.76	2.333	7	2.171	7
19	1.47	79.74	83.72	6.27	1.346	9	1.282	9	69.03	1.54	76.56	80.64	6.38	1.494	9	1.418	9
20	0.11	160.53	164.10	3.92	-0.178	20	-0.174	20	265.50	0.25	148.11	151.54	4.16	-0.099	20	-0.097	20

Table 3
A Comparison of Market and Factor Portfolios

This table reports the summary statistics of symmetry and asymmetry of monthly returns on the MSwV (SSD) optimal portfolio, the MV$\mathbb{A}$ optimal portfolio, the MV optimal portfolio, Fama-French (FF) factor portfolios, and the key US market indexes. The Sharp ratios according to MSwV and MV$\mathbb{A}$ models are also reported. To find the SSD and MV$\mathbb{A}$ optimal portfolios, we determine the optimal weights based on past 60 monthly excess returns on all CRSP stocks, and apply the optimal weights to the *following month* to form the portfolios. The sample period covers from 1995 to 2016.

					Sharp Ratios	
	μ (%)	$SwV \times 10^4$	$\sigma^2 \times 10^4$	$\mathbb{A} \times 10^4$	MSwV	MV$\mathbb{A}$
SSD* MKT	1.106	5.918	10.736	-5.798	19.26	7.55
MV$\mathbb{A}$* MKT	1.107	5.916	10.731	-5.798	19.32	7.57
FF-MKT	0.872	10.196	19.556	-9.779	7.12	3.12
FF-SMB	0.210	5.163	10.424	-5.262	0.20	0.09
FF-HML	0.260	4.959	9.934	-4.991	1.24	0.55
FF-RMW	0.332	4.447	8.749	-4.354	3.69	1.36
FF-CMA	0.283	2.340	4.681	-2.394	3.69	1.59
FF-MOM	0.417	14.449	27.091	-12.591	1.52	0.73
SPX	0.695	9.476	18.373	-9.125	5.51	2.44
DOW	0.714	9.172	17.803	-8.887	5.94	2.62
NASDAQ	0.979	22.890	44.608	-21.809	3.55	1.59

Table 4
Fama-MacBeth Tests for MSwV (SSD) and MV$\mathbb{A}$ Pricing Models

Following Fama-MacBeth (1973), this table reports the estimated factor premium of MSwV model ($\hat{\lambda}_{SwV}$) and that of MV$\mathbb{A}$ model ($\hat{\lambda}_{\mathbb{S}}$, and $\hat{\lambda}_{\mathbb{A}}$), respectively. First, estimated factor loadings, $\hat{\beta}^{SwV}$, $\hat{\beta}^{\mathbb{S}}$, and $\hat{\beta}^{\mathbb{A}}$ are calculated, based on (13.2), (14.2), and (14.3), from returns on individual securities to returns on indexes, R_m^{SwV}, $R_m^{\mathbb{S}}$, and $R_m^{\mathbb{A}}$, respectively, over the past 60 months. Second, at the beginning of each period, sixty equal-sized portfolios sorted by the beginning-of-period estimated betas are formed from the entire sample. We then regress all portfolio returns for the period against the estimated betas to determine the risk-premium for each factor. The sample contains all CRSP-listed ordinary common equities from July 1969 to December 2015. The mean coefficient estimates ($\hat{\lambda}$) across the sample period are reported with their *t*-statistics.

Panel 1: **M*SwV* (SSD) Model**

	$\hat{\boldsymbol{\lambda}}_{SwV}$	$\boldsymbol{R^2}$	**Adj.** $\boldsymbol{R^2}$
$\hat{\boldsymbol{\beta}}_p^{SwV}$-sorted	0.0057 (2.04) **	0.4971	0.4886

Panel 2: **MV$\mathbb{A}$ Model**

	$\hat{\boldsymbol{\lambda}}_{\mathbb{S}}$	$\hat{\boldsymbol{\lambda}}_{\mathbb{A}}$	$\boldsymbol{R^2}$	**Adj.** $\boldsymbol{R^2}$
$\hat{\boldsymbol{\beta}}_p^{\mathbb{S}}$-sorted	0.0057 (2.33) **	-0.0508 (-3.24) ***	0.5779	0.5634
$\hat{\boldsymbol{\beta}}_p^{\mathbb{A}}$-sorted	0.0078 (2.14) **	-0.0072 (-1.98) **	0.6247	0.6118

***, ** and * denote statistics significant at 1 percent, 5 percent and 10 percent levels, respectively.

Table 5
Fama-MacBeth Tests for the Validity of MV𝔸 Pricing Model

If MV𝔸 asset pricing model shown in (14) is valid, then the residual returns of individual assets, $\varepsilon_{j,t} = R_{j,t} - \left(\hat{\beta}_j^{\mathbb{S}} R_{m,t}^{\mathbb{S}} + \hat{\beta}_j^{\mathbb{A}} R_{m,t}^{\mathbb{A}}\right)$, should be idiosyncratic. Thus, no other common factors could explain the residual returns. To examine the validity of the MV𝔸 Model, we first calculated the factor loadings of $\varepsilon_{j,t}$ with respect to multiple common factors (e.g., *SMB* and *HML*) from the following regression model: $\varepsilon_{j,t} = \alpha_j + \beta_j^{SMB} SMB_t + \beta_j^{HML} HML_t + \beta_j^{RMW} RMW_t + \beta_j^{CMA} CMA_t + e_{j,t}$, over the past 60 months. Then, at the beginning of each period, sixty equal-sized portfolios sorted by the beginning-of-period estimated betas are formed from the entire sample. We then regress all portfolio returns for each period against the estimated betas (e.g., $\hat{\beta}_p^{SMB}$ and $\hat{\beta}_p^{HML}$) to determine the risk-premium for each factor. The sample contains all CRSP-listed ordinary common equities from July 1969 to December 2015. The mean coefficient estimates ($\hat{\lambda}$) across the sample period are reported with their t-statistics.

Panel 1: Two-Factor Test

Portfolios	$\hat{\lambda}_{SMB}$	$\hat{\lambda}_{HML}$	R^2	Adj. R^2
$\hat{\beta}_p^{SMB}$-sorted	0.0003 (0.22)	0.0008 (0.27)	0.2607	0.2348
$\hat{\beta}_p^{HML}$-sorted	-0.0034 (-1.10)	0.0002 (0.09)	0.3112	0.2870

Panel 2: Four-Factor Test

Portfolios	$\hat{\lambda}_{SMB}$	$\hat{\lambda}_{HML}$	$\hat{\lambda}_{RMW}$	$\hat{\lambda}_{CMA}$	R^2	Adj. R^2
$\hat{\beta}_p^{SMB}$-sorted	-0.0005 (-0.26)	0.0020 (0.69)	0.0018 (0.89)	-0.0020 (-1.05)	0.3156	0.2659
$\hat{\beta}_p^{HML}$-sorted	0.0005 (0.17)	0.0013 (0.54)	0.0026 (1.17)	-0.0006 (-0.32)	0.3091	0.2589
$\hat{\beta}_p^{RMW}$-sorted	-0.0016 (-0.60)	0.0002 (0.07)	0.0005 (0.24)	-0.0031 (-1.58)	0.3274	0.2786
$\hat{\beta}_p^{CMA}$-sorted	0.0001 (0.04)	0.0008 (0.32)	0.0010 (0.51)	-0.0007 (0.09)	0.3093	0.2591

***, ** and * denote statistics significant at 1 percent, 5 percent and 10 percent levels, respectively.

Reference

Bawa, V. S., & Lindenberg, E. B. (1977). Capital market equilibrium in a mean-lower partial moment framework. *Journal of Financial Economics*, *5*(2), 189-200.

Chung, Y. P., Johnson, H., & Schill, M. J. (2006). Asset pricing when returns are non-normal: Fama-french factors versus higher-order systematic comoments. *The Journal of Business*, *79*(2), 923-940.

Dittmar, R. F. (2002). Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns. *The Journal of Finance*, *57*(1), 369-403.

Fama, E. F., & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of financial economics*, *33*(1), 3-56.

Fama, E. F., & French, K. R. (2015). A five-factor asset pricing model. *Journal of financial economics*, 116(1), 1-22.

Gul, F. (1991). A theory of disappointment aversion. *Econometrica: Journal of the Econometric Society*, 667-686.

Harlow, W. V., & Rao, R. K. (1989). Asset pricing in a generalized mean-lower partial moment framework: Theory and evidence. *Journal of Financial and Quantitative Analysis*, *24*(3), 285-311.

Harvey, C. R., & Siddique, A. (2000). Conditional skewness in asset pricing tests. *The Journal of Finance*, *55*(3), 1263-1295.

Jiang, G. J., & Oomen, R. C. (2008). Testing for jumps when asset prices are observed with noise–a "swap variance" approach. *Journal of Econometrics*, *144*(2), 352-370.

Kumar, A. (2009). Who gambles in the stock market? *The Journal of Finance*, *64*(4), 1889-1933.

Levy, H. (2015). *Stochastic dominance: Investment decision making under uncertainty*. Springer.

Neuberger, A. (1994). The log contract. *The Journal of Portfolio Management*, *20*(2), 74-80.

Post, T. (2003). Empirical tests for stochastic dominance efficiency. *The Journal of Finance*, *58*(5), 1905-1931.

Price, K., Price, B., & Nantell, T. J. (1982). Variance and lower partial moment measures of systematic risk: some analytical and empirical results. *The Journal of Finance*, *37*(3), 843-855.

Shalit, H., & Yitzhaki, S. (1984). Mean-Gini, portfolio theory, and the pricing of risky assets. *The Journal of Finance*, *39*(5), 1449-1468.

Shalit, H., & Yitzhaki, S. (1994). Marginal conditional stochastic dominance. *Management Science*, *40*(5), 670-684.

Vendrame, V., Tucker, J., & Guermat, C. (2016). Some extensions of the CAPM for individual assets. *International Review of Financial Analysis*, *44*, 78-85.

Yitzhaki, S. (1982). Stochastic dominance, mean variance, and Gini's mean difference. *The American Economic Review*, *72*(1), 178-185.