Risky Bubbles, Public Debt and Monetary Policies

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Abstract

We analyze the effects of monetary policies in a neoclassical growth model with nominal rigidities and stochastic rational bubbles, where output is determined by capital accumulation. We show that the expectations of the bubble's bursting risk and of the monetary policy stance after the bubble collapses affect the steady state and dynamics of bubbles. A conventional 'leaning-against-the-wind' monetary policy of a surprise rise in the nominal interest rate decreases the bubble asset price and has typical tightening effects on output and inflation. An unconventional monetary policy of purchasing government bonds has different effects on bubbles, output and inflation, depending on whether bubbles are small or large. The model provides new insights on interactions between monetary policy, safe government debt and risky bubbles.

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1 Introduction

The spectacular booms and busts of asset prices that preceded financial crises in many advanced economies, notably in Japan around 1990 and in the U.S. in the 2000s, have led to important and unresolved questions for policymakers. A heated debate has emerged around the role of monetary policies, such as the 'leaning-against-the-wind' policy of raising the nominal interest rates to address asset price booms in stabilizing the economy in bubbly episodes.[1] Moreover, many central banks have engaged in unconventional monetary policies such as the quantitative easing policies of purchasing government bonds.[2] To the best of our knowledge, understanding the theoretical underpinnings of the roles of such unconventional monetary policies in bubbly episodes like those experienced by Japan and the U.S. remains an open question. Overall, there is a demand for macroeconomic theories to analyze the interactions of monetary policies and asset bubbles.

To this end, we develop a simple growth model with risky asset bubbles, safe government bonds and nominal rigidities, where we study the effects of conventional and unconventional monetary policies in bubbly episodes. The model builds on the flexible-price model with rational asset bubbles (Hirano et al. (2015) and Hirano and Yanagawa (2016)), which is a neoclassical growth model with heterogeneous entrepreneurs, financial frictions and a risky asset bubble. The heterogeneity of entrepreneurs gives rise to natural lenders and borrowers. As is standard in the rational bubbles literature, we model a bubble asset as an asset with no fundamental value, but is traded at a positive price. This is because financial frictions à la Kiyotaki and Moore (1997) limit the functioning of the credit market in facilitating borrowing and lending and depress the interest rate in the credit market. The bubble is effectively a speculative investment vehicle for lenders. The bubble is assumed to be risky, as its existence requires a coordination of expectations across agents and across time. As in Weil (1987), we model this fragility by assuming that in each period the price of the bubbly asset can permanently collapse to zero with an exogenous probability.

We then introduce two important ingredients into this environment: nominal rigidities and safe government bonds. There are monopolistic intermediate goods firms that face costly price adjustments à la Rotemberg (1982). There is a positive supply of safe nominal government bonds backed by lump-sum taxes. In each period, entrepreneurs choose a portfolio consisting of holdings of government bonds, holdings of the risky bubbly asset,

[1]See, e.g., Cecchetti et al. (2000), Bernanke and Gertler (2001), Borio and Lowe (2002), Svensson (2011), Galí (2014), Borio (2014) and Allen et al. (2017).

[2]See Reis (2009) for a survey of these unconventional monetary policies.

investment in the capital good and lending/borrowing.

As is standard in the New Keynesian literature, we model a conventional monetary policy as a policy that sets the nominal interest rate according to a Taylor rule. Furthermore, we model an unconventional monetary policy as changing the net supply of nominal government bonds that are held by private agents.[3]

The model provides three insights on monetary policies, government debt and asset bubbles. First, because agents rationally anticipate the bursting probability of bubbles, their expectations of the post-bubble policy stance have permanent effects on the economy – via the effects on the size of the bubble, capital investment and output in the stochastic bubbly steady state. Furthermore, the effects depend on the size of the bubble in the economy.

Starting from the effects of government debt, the government debt has two opposite effects in the bubble-less economy. On the one hand, government bonds are traded between lenders and borrowers and facilitate saving and investment; on the other hand, they crowd out investment by shifting resources away from production. These two effects are cancelled out in the bubble-less economy, and thereby there is no permanent stock effect of changing the amount of government bonds circulating in the market in the bubble-less steady state.

However, the government debt can have permanent impacts in the stochastic bubbly steady state. When the bubble is "large," so that speculative investment in the bubbly asset absorbs lenders funds and completely crowds out lenders' capital investment, an increase in government debt crowds out funds into the bubble asset and thereby shrinks the bubble asset price. This attenuates the expansionary effect of the bubble and causes a decrease in capital and output. Although the government debt is neutral in the bubble-less economy, it has negative impacts on output and capital in the large-bubble economy.

When the bubble is "small," so that speculative investment in the bubbly asset does *not* completely crowd out lenders' capital investment, the government debt is neutral if prices are flexible. However, with nominal rigidities, an increase in the government debt has expansionary effects on the bubble asset price, investment and output. This result, opposite to the case of a large bubble, is induced by an interaction between monetary policy and government debt. In particular, the non-neutrality holds only if the nominal

[3]In practice, quantitative easing usually involves the monetary authority purchasing long-term government bonds. We acknowledge that a limitation of our model is that it abstracts away from the difference between long-term vs. short-term government bonds. Instead, we focus on the role of government bonds that facilitate saving and investment but at the same time crowd out investment by shifting resources away from production.

interest rate is kept below the real interest rate in the stochastic steady state. In this case, as long as the bubble persists, it plays a role of a better asset in facilitating saving and investment than the government bond. Hence, the bubbly asset attracts more funds and the bubble asset price soars.

What keeps the nominal rate in bubbles low is the post-bubble monetary policy stance. A burst of an expansionary bubble which has increased capital and output is associated with a fall in inflation and a rise in the real return of the government bond, where the degree of these changes depends on monetary policy stance on inflation. The real return of investment, however, falls as the marginal cost drops. Because of these effects, the government bond plays a role of a safe asset and an insurance during bubbles, giving rise to the nominal rate lower than the real rate in the stochastic bubbly steady state with a small bubble.

Another factor that keeps the nominal rate low is inflation-targeting policy in the stochastic steady state. The New Keynesian Phillips curve implies that the real economy has to be stimulated above the neutral level – the marginal cost has to be higher than unity – to stabilize current inflation at the target level in face of deflationary pressure in the next period in the case of a bubble burst. To stimulate the economy to stabilize inflation, the nominal rate has to be kept below the risky real rate in the stochastic steady state.

This observation on the real economy implies that the post-bubble monetary policy stance on inflation and the degree of nominal rigidities affects output and investment during bubbles. Specifically, stronger monetary policy stance on inflation mitigates a fall in inflation when the bubble bursts, softens downward pressure on inflation during bubbles, and leads to the marginal cost closer to unity during bubbles. A low degree of nominal rigidities makes the slope of the New Keynesian Phillips curve steep so that less variation in the marginal cost is required to offset the deflationary pressure from the state of a bubble burst tomorrow.

Second, turning to the dynamic effects of monetary policy shocks, a conventional 'leaning-against-the-wind' monetary policy of a surprise rise in the nominal interest rate dampens the bubble asset price. Intuitively, an increase in the nominal interest rate raises the real interest rate and discourages capital investment, resulting in a drop in the marginal cost. The decrease in the marginal cost, in turn, lowers the return on capital and decreases the net worth of lenders and borrowers. With relatively low net worth at hand, lenders have less capacity to purchase a bubble asset, which leads to a decrease in the bubble asset price. Hence, a rise in the nominal rate has standard contractionary effects on output, investment and inflation, and at the same time it reduces the bubble asset price. This result holds irrespective of the size of bubbles. It is worth noting that in this model output is investment-driven as opposed to being labor-driven as in the standard New Keynesian model. Yet, qualitatively, a rise in the nominal rate has similar contractionary effects in this model irrespective of the presence of bubbles.

Third, unconventional monetary policy of a surprise purchase of government bonds has different effects on bubbles, output and investment, depending on the size of bubbles. As we mentioned, a permanent change in the government debt may or may not have *stock effects* depending on the presence and the size of bubbles. However, a temporary change in the government debt does have *flow effects*, which are different depending on the size of bubbles. When the bubble is small (or when there is no bubble), lenders purchase the bubble asset, hold government bonds, lend to borrowers and at the same time invest in the project. With less government bonds available as a result of the central bank's bond purchase, the lenders shift more resources to other use. But lending/borrowing is constrained by financial frictions and a drop in the net worth due to less government bonds reduces the lenders' capacity to purchase the bubble asset. Consequently, resources freed up by the unconventional monetary policy move to investment and stimulate output, while the bubble asset price decreases in accordance with a drop in the net worth. A key assumption underlying this result of circumventing Barro-Wallace irrelevance proposition (Barro (1974), Wallace (1981)) is that the integrated government with the central bank finances funds for purchasing government bonds by taxing workers, who are different from lenders and borrowers.[4]

When the bubble is large so that lenders do not invest in capital anymore, the same unconventional policy of government bond purchases has opposite effects on bubbles, investment and output. With less government bonds available, lenders shift resources to the bubble asset, which leads to a shoot-up of the bubble asset price and a gradual return to the original level. Along the path, the growth rate of the bubble is negative and the real rate should fall in line with the bubble path. But, the central bank, following a standard monetary policy rule, does not adjust the nominal rate to be consistent with the real rate. The real rate implied by the nominal rate does not fall as much as it should to stabilize inflation, and consequently it discourages investment and decreases inflation, the marginal cost and the net worth. Hence, when a bubble is relatively large, government bond purchase

[4]See Christiano and Ikeda (2013) for the effectiveness and transmission mechanisms of unconventional monetary policies in various models with financial frictions.

policy has contractionary effects on investment, output and inflation, although it increases the bubble asset price.

Overall, our model highlights the nontrivial interaction of monetary policies (including policy shocks and policy expectations), risky asset bubbles, safe government debt and aggregate economic activities in a New Keynesian model with capital accumulation and financial frictions.

Related literature. Our paper is mostly related to a new generation of rational bubbles models that features monetary policy, including Galí (2014, 2017), Asriyan, Fornaro, Martin and Ventura (2016), Hanson and Phan (2017), Biswas, Hanson and Phan (2017), Ikeda (2017), Allen, Barlevy and Gale (2017) and Dong, Miao and Wang (2017). To the best of our knowledge, by providing a general framework that allows for the interactions between nominal rigidities, government debt and bubbles, our paper is the first to analyze the effects of both conventional and unconventional monetary policies on bubbles.

Our paper borrows from the insights from a large literature on (real) theories of rational bubbles. The literature dates back to the pioneering overlapping generations framework of Samuelson (1958), Diamond (1965) and Tirole (1985), which was further developed by, for example, Weil (1987), Grossman and Yanagawa (1993), Martin and Ventura (2012), Farhi and Tirole (2012) and Ikeda and Phan (2015, 2016). Models of bubbles with infinite-lived agents, which are relatively closer to the 'workhorse' dynamic stochastic general equilibrium models that are often used for policy analyses, include Kocherlakota (1992, 2009), Kamihigashi (2008), Aoki, Nakajima and Nikolov (2014), Aoki and Nikolov (2015), Miao and Wang (2017) and Guerron-Quintana, Hirano and Jinnai (2017).[5] More specifically, our paper builds upon the real framework of rational bubbles, infinite-lived agents and financial frictions of Hirano, Inaba and Yanagawa (2015) and Hirano and Yanagawa (2016).

Our paper is also related to the literature on unconventional monetary policies, including Reis (2009, 2017), Cúrdia and Woodford (2010, 2016), Gertler and Karadi (2011, 2013), Joyce et al. (2012), Ennis (2014) and Gourio et al. (2017). We contribute to this literature by providing an analysis of the effects of unconventional policies on asset price bubbles.

The plan of paper is as follows. Section 2 describes the environment. Section 3 analyzes the effects of policies on the steady states. Section 4 analyzes the dynamic responses to

[5] For more comprehensive surveys of the literature, see Barlevy (2007); Barlevy et al. (2015), Miao (2014), Brunnermeier (2016) and Martin and Ventura (2017).

policies along the equilibrium paths. Section 5 concludes. Derivations, proofs and numerical methods are delegated to the Appendix.

2 Model

We consider a discrete-time-infinite-horizon economy with two types of good: consumption and capital goods. The economy has four types of agents: entrepreneurs, workers, firms and a government integrated with a central bank. In the following, we describe the behavior of each type of agents in turn.

2.1 Entrepreneurs

There is a continuum of entrepreneurs with measure unity, each indexed by $j \in (0,1)$. For each j, entrepreneur j has the following preferences:

$$E_0\left(\sum_{t=0}^{\infty} \beta^t \log c_t^j\right),$$

where c_t^j is consumption in period t, $\beta \in (0,1)$ is the preference discount factor, and $E_0(\cdot)$ is an expectations operator conditional on information in period 0.

Entrepreneur j has an investment project with productivity a_t^j that produces capital according to:

$$k_t^j = a_t^j i_t^j,$$

where $i_t^j \geq 0$ is investment in terms of the consumption good and k_t^j is newly produced capital. The productivity is either high $a_t^j = a^H$ or low $a_t^j = a^L$, where $a^H > a^L$. In each period entrepreneurs meet the high-productivity investment project (hereinafter the H-project) with probability $h \in (0,1)$ and a low productivity one (hereafter the L-project) with probability $1-h$. The probability, h, is exogenous and independent across entrepreneurs and over time.[6] We call entrepreneurs with the H-project as "the H-type" and entrepreneurs with the L-project as "the L-type".

The heterogeneous productivity gives rise to borrowing and lending among entrepreneurs, but frictions in a financial market limits entrepreneurial capacity of carrying out such trans-

[6]The i.i.d. assumption simplifies the algebra. For example, it allows us to analytically characterize the bubble dynamics. The model with persistent productivity shocks can be solved numerically, as in Hirano and Yanagawa (2016).

actions. Specifically, entrepreneurs can pledge at most a fraction $\theta \in (0, 1]$ of the future return from their investment to creditors as in Kiyotaki and Moore (1997). This limited pledgeability leads to the following borrowing constraint:

$$R_{t+1} d_t^j \leq \theta q_{t+1} a_t^j i_t^j. \tag{1}$$

where d_t^j is the real value of nominal borrowing, R_{t+1} is the state-contingent real interest rate and q_{t+1} is the real price of capital. Constraint (1) holds for each state of the economy.

For analytical simplicity, we impose two assumptions. First, the initial population measure of the H-type and the L-type is h and $1 - h$ in period 0. This assumption implies that the population measure of each type stays constant at h and $1 - h$, respectively, over time. Second, we assume capital depreciates completely after each period.[7]

Entrepreneurs trade three types of financial assets/liabilities: real borrowing, a nominal government bond, and a bubble. The supply of the nominal government bond is non-negative. As is standard in the literature following Samuelson (1958), Diamond (1965) and Tirole (1985), the bubble is an asset that has no fundamental value but is traded at a positive price in a bubbly equilibrium, as it serves as an investment vehicle for agents who have a demand for savings. The supply of the bubble is normalized to unity. Let b_t^j denote a share of a bubbly asset held by entrepreneur j and p_t^b denote the real price of a bubbly asset. Then the entrepreneur's flow budget constraint is written as

$$c_t^j + i_t^j + p_t^b b_t^j = q_t a_{t-1}^j i_{t-1}^j + d_t^j - R_t d_{t-1}^j - \left(g_t^j - \frac{R_{t-1}^n}{\pi_t} g_{t-1}^j \right) + p_t^b b_{t-1}^j, \tag{2}$$

where g_t^j is the real value of the government bond, R_{t-1}^n is the nominal interest rate between $t - 1$ and t, and π_t is a gross inflation rate of the consumption good.[8] The left hand side of (2) consists of expenditure on consumption, investment, and the purchase of bubbly assets. The right hand side is the available funds at date t, which consists of the return

[7]It is straightforward to extend the model with partial capital depreciation, as in Kocherlakota (2009) and Hirano and Yanagawa (2016).

[8]In nominal terms, the budget constraint (2) is written as

$$P_t c_t^j + P_t i_t^j + P_t^b b_t^j = Q_t a_{t-1}^j i_{t-1}^j + D_t^j - R_t \pi_t D_{t-1}^j - \left(G_t^j - R_{t-1}^n G_{t-1}^j \right) + P_t^b b_{t-1}^j,$$

where P_t is the nominal price of the consumption good, $P_t^b \equiv P_t p_t^b$ is the nominal value of the bubble, $Q_t \equiv P_t q_t$ is the nominal return of investment, $D_t^j \equiv P_t d_t^j$ is the nominal amount of borrowing and $G_t^j \equiv P_t g_t^j$ is the nominal value of the government bond. Dividing the both sides of the constraint by P_t yields the flow budget constraint in real terms (2).

from investment in the previous period, the amount of new borrowing minus the debt repayment, and the value of bubbly assets purchased in the previous period. No short sale constraint is imposed for trading the bubbly asset:

$$b_t^j \geq 0. \tag{3}$$

The log utility implies that each entrepreneur consumes a fraction $1-\beta$ of the net worth n_t^j in each period, that is, $c_t^j = (1-\beta)n_t^j$, where

$$n_t^j \equiv q_t a_{t-1}^j i_{t-1}^j - R_t d_{t-1}^j + \frac{R_{t-1}^n}{\pi_t} g_{t-1}^j + p_t^b b_{t-1}^j. \tag{4}$$

Then the budget constraint (2) is written as:

$$i_t^j + p_t^b b_t^j + g_t^j = \beta n_t^j + d_t^j. \tag{5}$$

We restrict our attention to the relevant and most interesting case in which the financial friction parameter θ is sufficiently small that the borrowing constraint (1) binds for the H-type for all periods. This is the case if and only if the following condition holds:

$$E_t \beta \frac{c_t^L}{c_{t+1|L}} q_{t+1} a^L \leq 1 < E_t \beta \frac{c_t^H}{c_{t+1|H}} q_{t+1} a^H, \tag{6}$$

where $c_t^H \equiv \int_{j \in \text{H-type}} c_t^j dj$ and $c_t^L \equiv \int_{j \in \text{L-type}} c_t^j dj$ are the aggregate consumption for the H-type and L-type, respectively, and $c_{t+1|j}$ is the aggregate consumption in period $t+1$ by entrepreneurs who are the j-type in period t for $j \in \{L, H\}$. The discounted return on investment by the L-type – the left-hand-side object of the inequalities (6) – has to be at most unity, otherwise there would be no lending. The discounted return on investment by the H-type – the right-hand-side object of the inequalities (6) – is strictly greater than unity, otherwise the credit constraint is not binding.

Condition (6) implies that the H-types receive a higher return from capital investment than than lending and speculating in the bubbly asset. Consequently, their government bond holding and bubble holding are zero: $g_t^H = b_t^H = 0$. Substituting this into (5) yields $i_t^j = \beta n_t^j + d_t^j$ for the H-types. Further substituting the binding borrowing constraint (1) for d_t^j yields

$$i_t^j = \left[\frac{1}{1 - \theta(q_{t+1}/R_{t+1})a^H} \right] \beta n_t^j. \tag{7}$$

We will confirm later that the ratio q_{t+1}/R_{t+1} is independent of state-$t+1$ variables.

The L-types allocate their savings, βn_t^j, into four components: capital investment, holding of the bubbly asset, lending, and holding of the government bond. The utility maximization of the L-types yields the following first-order conditions:

$$
\begin{aligned}
1 =& E_t \beta \frac{c_t^L}{c_{t+1|L}} R_{t+1}, && (8)\\
1 =& E_t \beta \frac{c_t^L}{c_{t+1|L}} \frac{R_t^n}{\pi_{t+1}}, && (9)\\
1 =& E_t \beta \frac{c_t^L}{c_{t+1|L}} \frac{p_{t+1}^b}{p_t^b}. && (10)
\end{aligned}
$$

Equations (8)-(10) correspond to the Euler equations with respect to lending, the government bond and the bubbly asset, respectively.

2.2 Workers

There is a continuum of workers with measure unity, each indexed by $w \in (0,1)$. As same as the typical entrepreneur, a typical worker has preferences, given by:

$$
E_0 \left(\sum_{t=0}^{\infty} \beta^t \log c_t^w \right).
$$

Workers are endowed with one unit of labor in each period. They supply labor inelastically in labor markets and earn the real wage, w_t. They do not have investment opportunities and cannot borrow against their future labor incomes:

$$
d_t^w \leq 0, \tag{11}
$$

The flow budget constraint is given by

$$
c_t^w + p_t^b b_t^w = w_t + d_t^w - R_t d_{t-1}^w - \left(g_t^w - \frac{R_{t-1}^n}{\pi_t} g_{t-1}^w \right) + p_t^b b_{t-1}^w + f_t - z_t, \tag{12}
$$

where f_t is the profits brought by intermediate goods firms and z_t is the lump-sum tax. As in the entrepreneurial problem no short-sale constraint is imposed:

$$
b_t^w \geq 0, \tag{13}
$$

We restrict our attention to the case in which workers do not save in equilibrium so that the worker consumption is given by $c_t^w = w_t + f_t - z_t$. Workers do not save because the interest rate is too low for them to save due to the financial friction. To see this point, for the purpose of exposition, consider a steady state. For workers to lend, the real interest rate has to be $1/\beta$. Equation (8) implies that the real interest rate is given by $R = (1/\beta)(c_{\cdot|L}/c^L)$, where c^L is the L-type's consumption and $c_{\cdot|L}$ is the entrepreneur's consumption condition on the entrepreneur being the L-type in the previous period. Note that $c_{\cdot|L}/c^L < 1$ because the L-types earn lower returns than the H-types (see inequalities (6)); entrepreneurs switch between the two types in an idiosyncratic manner; entrepreneurs consume a fraction, $1 - \beta$, of the net worth. Specifically, a fraction, h, of the L-types today was the H-type in the previous period, who has earned the higher return than those who were the L-types. The entrepreneurs who were the L-type yesterday have earned less than those who were the H-type and thereby consume less today. Therefore, $c_{\cdot|L}/c^L < 1$ and as a result $R < 1/\beta$ so that the workers do not lend. We focus on a case in which this holds for dynamics out of the steady state as well.

2.3 Firms

There are two types of firms: a consumption good firm and intermediate goods firms. All firms are owned by workers.

Consumption good firm: A representative consumption good firm combines a continuum of intermediate goods, $\{y_t^i\}_{i \in (0,1)}$, to produce output y_t, according to the aggregation technology, $y_t = \left[\int_0^1 (y_t^i)^{\frac{1}{\lambda}} di\right]^{\lambda}$ with $\lambda > 1$. Perfect competition leads to a demand curve:

$$y_t^i = \left(\frac{P_t^i}{P_t}\right)^{-\frac{\lambda}{\lambda-1}} y_t, \tag{14}$$

where P_t^i is the price of intermediate good i and P_t is the price of the consumption good.

Intermediate goods firms: There is a continuum of intermediate goods firms, each indexed by $i \in (0,1)$. Intermediate goods firm i produces the i-th intermediate good y_t^i by combining capital k_t^i and labor l_t^i according to the Cobb-Douglas production function:

$$y_t^i = (k_t^i)^{\sigma}(l_t^i)^{1-\sigma}, \quad 0 < \sigma < 1.$$

The firm minimizes the cost of production, $w_t l_t^i + q_t k_t^i$, subject to the production technology. The solution leads to factor prices, given by:

$$q_t = s_t \sigma k_{t-1}^{\sigma-1} l_t^{1-\sigma}, \tag{15}$$

$$w_t = s_t (1-\sigma) k_{t-1}^{\sigma} l_t^{-\sigma}, \tag{16}$$

where s_t denotes the real marginal cost, $k_{t-1} \equiv \int_{i \in (0,1)} k_t^i di$ denotes the aggregate capital, which is predetermined in period $t-1$, and $l_t \equiv \int_{i \in (0,1)} l_t^i di$ denotes the aggregate employment. The aggregate output is given by

$$y_t = k_{t-1}^{\sigma} l_t^{1-\sigma}. \tag{17}$$

Intermediate goods firm i sets price P_t^i to maximize the net present value of profits net of a quadratic cost of price adjustments à la Rotemberg (1982):

$$E_0 \sum_{t=0}^{\infty} M_{0,t}^w \left[(1+\tau) \frac{P_t^i}{P_t} y_t^i - s_t y_t^i - \frac{\gamma}{2} \left(\frac{P_t^i}{P_{t-1}^i} - 1 \right)^2 y_t \right], \qquad \gamma > 0,$$

subject to the demand curve (14) where $M_{0,t}^w \equiv \beta^t \frac{c_0^w}{c_t^w}$ is the worker's stochastic discount factor between periods 0 and t, and τ is the sales subsidy. The profits are discounted by the workers' stochastic discount factor because the firm is owned by the workers. The subsidy is set at $\tau \equiv \lambda - 1$, so that distortion arising from monopolistic competition is zero in a steady state with zero inflation. The term in square brackets is profits in period t, which are given by the revenue minus the cost of production minus the cost of adjusting prices. The first-order condition for this problem is

$$\gamma \left(\frac{P_t^i}{P_{t-1}^i} - 1 \right) \frac{y_t}{P_{t-1}^i} = \frac{\lambda}{\lambda-1} \frac{s_t y_t}{P_t} - \frac{1+\tau}{\lambda-1} \frac{y_t}{P_t} + E_t M_{t,t+1}^w \gamma \left(\frac{P_{t+1}^i}{P_t^i} - 1 \right) \frac{P_{t+1}^i}{(P_t^i)^2} y_{t+1}.$$

In a symmetric equilibrium this condition is reduced to:

$$(\pi_t - 1)\pi_t = \frac{1}{\gamma} \frac{\lambda}{\lambda - 1} (s_t - 1) + E_t M_{t,t+1}^w \frac{y_{t+1}}{y_t} (\pi_{t+1} - 1) \pi_{t+1}. \tag{18}$$

As in the standard New Keynesian model, current inflation π_t depends positively on the marginal cost s_t and the expected inflation in the next period.[9]

[9]If we were to log-linearize the model around the steady state with $\pi = 1$ and $s = 1$, then we would

2.4 Government

The government consists of a central bank and a fiscal authority. A central bank sets the nominal interest rate, R_t^n, according to the following Taylor rule:

$$\log(R_t^n) = \log\left(\bar{R}^n\right) + \psi_\pi \log(\pi_t) + \upsilon_t, \ \psi_\pi > 1, \tag{20}$$

where $\bar{R}^n$ is constant and υ_t is a persistent shock following an AR(1) process of $\upsilon_t = \rho_{mp}\upsilon_t + \epsilon_{mp,t}$. Here, $\epsilon_{mp,t}$ is an i.i.d. monetary policy shock. This rule implies that the target rate of net inflation rate is zero in steady state if $\bar{R}^n$ is set at the safe real interest rate in steady state.

The fiscal authority issues the government bond of g_t and imposes the lump-sum tax of z_t to finance the subsidy to the intermediate firms, $\tau y_t = (\lambda - 1)y_t$, and the interest payment to the government bond, $(R_{t-1}^n/\pi_t)g_{t-1}$. The government flow budget constraint then is given by:

$$\tau y_t + \frac{R_{t-1}^n}{\pi_t} g_{t-1} = g_t + z_t. \tag{21}$$

Define

$$\phi_t^g \equiv g_t / \left(\beta n_t\right)$$

as the relative size of the government debt. We assume ϕ_t^g follows an AR(1) process

$$\log\left(\phi_t^g/\phi^g\right) = \rho_g \log\left(\phi_{t-1}^g/\phi^g\right) + \epsilon_{g,t},$$

where $0 \leq \rho_g < 1$ and $\epsilon_{g,t}$ is an i.i.d. shock. The government bond issuance is assumed to satisfy the no-Ponzi condition, $\lim_{s\to\infty} E_t M_{t,t+s}^L g_{t+s} = 0$, where $M_{t,t+s}^L$ is the stochastic discount factor of the L-type entrepreneurs, who are the only agents holding the government bond as will be analyzed below. Given a sequence of such g_t, the lump-sum tax z_t is adjusted to satisfy the budget constraint (21).

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have a standard New Keynesian Phillips curve:

$$\hat{\pi}_t = \kappa \hat{s}_t + \beta E_t \hat{\pi}_{t+1}, \tag{19}$$

where $\kappa \equiv \frac{1}{\gamma}\frac{\lambda}{\lambda-1}$ and $\hat{\pi}_t = \log(\pi_t)$, $\hat{s}_t = \log(s_t)$ are the deviations of the variable from its steady state value, respectively. However, log-linearization is not suitable for studying the model with the probability of bubble burst. We will solve the model globally in analyzing the model.

2.5 Bubbles

Following Weil (1987), we consider a stochastic bubble that persists with probability v and bursts with probability $1 - v$. If it bursts, the price of the bubbly asset becomes zero permanently. It is convenient to define

$$\phi_t^b \equiv \frac{p_t^b}{\beta n_t}$$

as the relative size of the bubble.

As mentioned in condition (6), the first-order condition of the L-types satisfies:

$$E_t \beta \frac{c_t^L}{c_{t+1|L}} q_{t+1} a^L \leq 1.$$

There are two cases, depending on whether the inequality is strict or not.

If the inequality holds with equality, then the L-types are indifferent between lending and investing in capital. We say that the bubble is *small* in t if this is the case. The label is intuitive, as we show in the appendix that if the bubble is small, the relative size of the bubble is smaller than or equal to $\phi - \phi_t^g$, i.e.,

$$\phi_t^b \leq \phi - \phi_t^g,$$

where $\phi \equiv 1 - a^L h/(a^L - \theta a^H)$.

In contrast, if the inequality is strict, then the return from lending is so high that the L-types strictly prefer to lend rather than to invest in capital. We say that the bubble is *large* in t if this is the case. Indeed, as shown in the appendix, the relative size of the bubble satisfies:

$$\phi_t^b > \phi - \phi_t^g.$$

The relative size of the bubble becomes greater by absorbing the L-types funds more than what it would be a case for the small bubble.

2.6 Equilibrium

We close the description of the model by stating market clearing conditions. The clearing condition for the consumption good market is:

$$c_t^H + c_t^L + c_t^w + i_t^H + i_t^L = \left[1 - \frac{\gamma}{2}(\pi_t - 1)^2\right] y_t, \tag{22}$$

where $i_t^H \equiv \int_{j \in \text{H-type}} i_t^j dj$ and $i_t^L \equiv \int_{j \in \text{L-type}} i_t^j dj$ are aggregate investment for the H-type and L-type, respectively. As we mentioned, we restrict our attention to a case in which workers do not save: $d_t^w = g_t^w = b_t^w = 0$. Then, the market clearing conditions for the credit market and the government bond market are written as

$$d_t^H + d_t^L = 0, \tag{23}$$

$$g_t^H + g_t^L = g_t, \tag{24}$$

where $d_t^H \equiv \int_{j \in \text{H-type}} d_t^j dj$ and $d_t^L \equiv \int_{j \in \text{L-type}} d_t^j dj$ are the aggregate borrowing for the H-type and L-type, respectively, and $g_t^H \equiv \int_{j \in \text{H-type}} g_t^j dj$ and $g_t^L \equiv \int_{j \in \text{L-type}} g_t^j dj$ are the aggregate holdings of the government bond for the H-type and L-type, respectively. The market clearing conditions for capital, labor and the bubbly asset are

$$k_t = a^H i_t^H + a^L i_t^L, \tag{25}$$

$$l_t = 1, \tag{26}$$

$$b_t \equiv \int_{j \in (0,1)} b_t^j dj = 1. \tag{27}$$

A *competitive equilibrium* for this economy is defined as follows. Given the exogenous probability of bubble burst $1 - v$ and given the monetary policy rule (20) and a sequence of the government bond, the equilibrium consists of a set of prices $\{R_t^n,\ R_t,\ w_t,\ q_t,\ p_t^b,\ \pi_t\}_{t=0}^{\infty}$ and quantities $\{c_t^H,\ c_t^L,\ c_t^w,\ d_t^H,\ d_t^L,\ g_t^H,\ g_t^L,\ i_t^H,\ i_t^L,\ b_t,\ l_t,\ k_t,\ y_t\}_{t=0}^{\infty}$, such that (i) the market clearing conditions (22)-(27) are satisfied in each period; (ii) each entrepreneur solves the problem of maximizing the expected discounted utility subject to the constraints (1)–(3); (iii) each worker solves the problem of maximizing the expected discounted utility subject to the constraints (11)–(13); (iv) the optimality conditions in the production sector, (15)–(18), hold.

A *bubbly equilibrium* is a competitive equilibrium where the price of the bubble (conditional on not bursting) p_t^b is positive for all t. A *bubble-less equilibrium* is one where $p_t^b = 0$ for all t.

A *steady state* is a competitive equilibrium in which there are no policy shocks and prices and quantities are time-invariant. A *stochastic bubbly steady state* is a steady state where $p_b > 0$, and a *bubble-less steady state* is where $p_b = 0$.

2.7 System of Equations

The equilibrium can be characterized by seven equations for the same number of variables, $\{k_t,\ n_t,\ \phi_t^b,\ s_t,\ \pi_t,\ R_t^n, R_t\}$. The equation that determines the nominal interest rate, R_t^n, is the monetary policy rule (20). The equation that governs the inflation rate, π_t, is the Phillips curve (18). From the consumption good market clearing condition (22) the aggregate workers consumption is given by

$$c_t^w = k_{t-1}^\sigma \left[1 - \frac{\gamma}{2}(\pi_t - 1)^2\right] - \left(1 - \beta\phi_t^b - \beta\phi_t^g\right) n_t,$$

We will state the other five equations for k_t, R_t, n_t, ϕ_t^b and s_t below. The detail of the derivation of the equations is dedicated to Appendix A. The evolution of capital is summarized as:

$$k_t = \begin{cases} \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - \phi_t^b - \phi_t^g\right] a^L \beta n_t & \text{if } \phi_t^b + \phi_t^g \leq \phi. \text{ (small bubble)} \\ a^H \left(1 - \phi_t^b - \phi_t^g\right) \beta n_t & \text{if } \phi_t^b + \phi_t^g > \phi, \text{ (large bubble)} \end{cases} \tag{28}$$

where $\phi \equiv 1 - a^L h/(a^L - \theta a^H)$. The first case of $\phi_t^b + \phi_t^g \leq \phi$ corresponds to the case of a small bubble, while the second case of $\phi_t^b + \phi_t^g > \phi$ corresponds to the case a large bubble. Equation (28) implies that, given the amount of the net worth, capital stock is decreasing in the relative size of bubbles and government debt. This is because resources are allocated away from productive investment to the purchase of bubbles and government debt. This effect is the standard crowding-out effect of bubbles and government debt.

The law of motion for the net worth is derived as

$$n_t = \frac{s_t \sigma k_{t-1}^\sigma + \beta R_{t-1}^n \phi_{t-1}^g n_{t-1}/\pi_t}{1 - \phi_t^b \beta}. \tag{29}$$

Equation (29) implies that, other things being equal, the net worth is increasing in bubbles and government debt. In particular, by selling bubble assets and government bonds to the

L-types, the H-types can invest more in their productive projects. This is the expansionary effect of bubbles and government debt.

The real interest rate is given by

$$R_{t+1} = \begin{cases} s_{t+1} a_L \sigma k_t^{\sigma-1} & \text{if } \phi_t^b + \phi_t^g \leq \phi, \\ s_{t+1} a^H \theta \left(1 - \frac{h}{1-\phi_t^b - \phi_t^g}\right)^{-1} \sigma k_t^{\sigma-1} & \text{if } \phi_t^b + \phi_t^g > \phi. \end{cases} \tag{30}$$

When the L-types invest as well as do the H-types, the real interest rate is equal to the marginal product of capital, as in the first case in equation (30). When the L-types do not invest, the real interest rate is determined by the credit market clearing condition (23), leading to the second case in equation (30). Equations (15) and (30) imply that q_{t+1}/R_{t+1} is independent of time $t+1$ variables as we have conjectured.

The evolution of the relative size of the bubble is given by

$$\phi_t^b = \begin{cases} v\beta(1-h) E_t \left[\frac{\phi_{t+1}^b}{1 - \sigma s_{t+1} k_t^{\sigma-1} \frac{(1-\theta) a^L a^H h \beta}{a^L - \theta a^H} \frac{n_t}{n_{t+1}}} \right] & \text{if } \phi_t^b + \phi_t^g \leq \phi, \\ v\beta(1-h) E_t \left[\frac{\phi_{t+1}^b}{1 - \sigma s_{t+1} k_t^{\sigma-1} (1-\theta) a^H \left(1 - \phi_t^b - \phi_t^g\right) \beta \frac{n_t}{n_{t+1}}} \right] & \text{if } \phi_t^b + \phi_t^g > \phi. \end{cases} \tag{31}$$

This equation is derived from the Euler equation regarding bubbles (9). As long as this condition holds, the L-types are indifferent between holding the bubbly asset and lending to the H-types.

Finally, the Fisher equation – the condition under which lenders (the L-types) are indifferent between lending and purchasing nominal government debt – is given by equation (9), which is written as

$$1 = \begin{cases} E_t \left[\frac{\beta(1-h)}{\frac{n_{t+1}}{n_t} - \sigma s_{t+1} k_t^{\sigma-1} \frac{(1-\theta) a^L a^H h \beta}{a^L - \theta a^H}} \frac{R_t^n}{\pi_{t+1}} \right] & \text{if } \phi_t^b + \phi_t^g \leq \phi, \\ E_t \left[\frac{\beta(1-h)}{\frac{n_{t+1}}{n_t} - \sigma s_{t+1} k_t^{\sigma-1} (1-\theta) a^H \left(1 - \phi_t^b - \phi_t^g\right) \beta} \frac{R_t^n}{\pi_{t+1}} \right] & \text{if } \phi_t^b + \phi_t^g > \phi. \end{cases} \tag{32}$$

It is worth noting that the expectation operator in equation (32) involves bubble and no-bubble states, but in equation (31) the expectation operator pertains only to stochastic shocks $\epsilon_{mp,t+1}$ and $\epsilon_{g,t+1}$.

3 Steady State Analysis

3.1 Bubble-less Steady State

Let us start with the bubble-less economy, where $p_t^b = 0$ and $\phi_t^b = 0$. Throughout, we focus on parameters such that:

$$h \leq \left(1 - \frac{\theta a^H}{a^L}\right)(1 - \phi_t^g), \forall t. \tag{33}$$

This condition guarantees that both types of entrepreneurs invest in capital in the bubble-less economy. The condition requires that the fraction of L-type entrepreneurs h and/or the pledgeability parameter θ should be sufficiently small, so that the aggregate borrowing by the H-types is not sufficient to completely crowd out capital investment by the L-types. Hence, the marginal capital investors in the economy are the L-types. Note that an increase in the relative size of public debt ϕ_t^g relaxes the condition, as it crowds out investment by the L-types.

In the bubble-less steady state, the interest rate and the capital stock are independent of government debt, given by

$$\begin{aligned} R &= \frac{1}{\beta}\left[1 + \frac{(a^H - a^L)h}{a^L - \theta a^H}\right]^{-1} \equiv R_{nb}, \\ k &= \left[\left(1 + \frac{(a^H - a^L)h}{a^L - \theta a^H}\right) a^L \beta\sigma\right]^{\frac{1}{1-\sigma}} \equiv k_{nb}. \end{aligned}$$

3.2 Stochastic Bubbly Steady State with Flexible Prices

It is useful to analyze the effects of monetary policies in the flexible-price special case – where the price adjustment parameter γ is zero. The simplicity of the flexible-price case allows for analytical characterizations and will be a useful benchmark in analyzing the model with nominal rigidities. The system of equations for this economy consists of (28), (29) and (31), where the nominal interest rate over inflation R_t^n/π_{t+1} is replaced by the real interest rate R_{t+1} given by equation (30) and the marginal cost is always unity, $s_t = 1$. The stochastic bubbly steady state is characterized by capital stock k, real interest rate R

and bubble over net worth ratio ϕ^b that solve a system of three equations

$$R = \begin{cases} a_L \sigma k^{\sigma-1} & \text{if } \phi^b + \phi^g \leq \phi \text{ (small bubble)} \\ \theta a^H \left(1 - \frac{h}{1-\phi^b-\phi^g}\right)^{-1} \sigma k^{\sigma-1} & \text{if } \phi^b + \phi^g > \phi \text{ (large bubble)} \end{cases}, \tag{34}$$

$$k^{1-\sigma} = \begin{cases} \left[1 - \phi^b - \phi^g + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h\right] \frac{a^L \beta\sigma}{1-\beta\phi^b-\beta R\phi^g} & \text{if } \phi^b + \phi^g \leq \phi \\ \left(1 - \phi^b - \phi^g\right) \frac{a^H \beta\sigma}{1-\beta\phi^b-\beta R\phi^g} & \text{if } \phi^b + \phi^g > \phi \end{cases}, \tag{35}$$

$$1 - (1-h)v\beta = \begin{cases} \sigma k^{\sigma-1} \frac{(1-\theta)a^L a^H h\beta}{a^L - \theta a^H} & \text{if } \phi^b + \phi^g \leq \phi \\ \sigma k^{\sigma-1}(1-\theta)a^H \left(1 - \phi^b - \phi^g\right)\beta & \text{if } \phi^b + \phi^g > \phi. \end{cases}. \tag{36}$$

where ϕ is a constant:

$$\phi \equiv 1 - \frac{a^L h}{a^L - \theta a^H}.$$

The case $\phi^b + \phi^g \leq \phi$ corresponds to a small bubble, and the opposite case corresponds to a large bubble.

Based on these equations, the conditions under which a stochastic bubbly steady state exist are summarized in the result below. Detailed derivations are proofs are delegated to Appendix B.

Lemma 1 *With flexible prices ($\gamma = 0$), a stochastic bubbly steady state exists if and only if the following two conditions hold:*

$$\theta < v\beta(1-h),$$
$$v > \frac{1}{\beta(1-h)} - \frac{R_{nb}(1-\theta)a^H h}{(1-h)\left(a^L - \theta a^H\right)} \equiv \underline{v}.$$

Intuitively, a stochastic bubbly equilibrium exists when the pledgeability parameter θ is small so that the borrowing constraint is tight enough and when bubbles are not too risky. If the borrowing constraint were less tight, there would be less room for bubbles to mitigate the borrowing constraint and thereby there would be no bubbles. If bubbles assets were too risky, such assets would require a higher return to compensate the high probability of bubble burst and would have to grow rapidly to be sustainable. These conditions are standard in the rational bubbles literature. This is not surprising, as in the absence of nominal rigidities, the model can be effectively mapped to a real model of rational bubbles, such as that in Hirano et al. (2015).

However, a key difference of our model compared to standard rational bubbles models is the presence of a non-zero net supply of government debt. The presence of government debt has impacts on the relative size of bubbles and the regions for small bubbles and large bubbles. We summarize it in the following proposition. Recall that v denotes the probability that the bubble persists.

Proposition 2 *Consider a stochastic bubbly steady state with flexible prices ($\gamma = 0$).*

(i) The bubble is small if and only if $v \in (\underline{v}, \bar{v}(\phi^g)]$, where $\bar{v}(\phi^g)$ is decreasing in ϕ^g. The relative size of the small bubble ϕ^b is increasing in v, but for given $v \in (\underline{v}, \bar{v}(\phi^g)]$ it is independent of the relative size of government debt ϕ^g. Capital is also independent of ϕ^g.

(ii) The bubble is large if and only if $v \in (\bar{v}(\phi^g), 1]$. The relative size of the large bubble ϕ^b is increasing in v, but for given $v \in (\underline{v}, \bar{v}(\phi^g)]$ it is decreasing in the relative size of government debt ϕ^g. In particular, $-1 < d\phi^b/d\phi^g < 0$. Capital is also decreasing in ϕ^g.

Proposition 2 states that an increase in government debt shrinks a region of the small bubbles and expands a region of the large bubbles. In the small bubble region, an increase in the government debt ratio crowds out the L-types' investment. If the bubble is relatively safe, then the relative size of the bubble is rather big because its low probability of burst attracts the L-types demand for the bubble asset. Therefore, for the small bubble with its size big enough, the L-types' capacity to invest is limited and thereby an increase in the government bond ratio completely crowds out the L-types investment. As a result, the small bubble is switched to the large bubble.

Proposition 2 also states that for given v the government debt does not affect the relative size of the small bubble, but it has a negative impact on the relative size of the large bubble. For the small bubbles, the government debt crowds out investment by the L-types, and this crowding-out effect is cancelled by its crowd-in effect, i.e. its positive effect on the net worth and thus on investment, as in the bubble-less economy. The government debt has no impact on the real economy and so does on the relative size of bubbles. However, the government bond does crowd out a bubble if it is the large bubble. Because the L-types do not invest any more in the large bubble economy, an increase in the holdings of government bonds by the L-types has to be compensated by a decrease in the purchase of bubble assets. The less demand for bubble assets causes the relative size of the large bubble to shrink.

Figure 1: Stochastic steady states in the flexible-price model

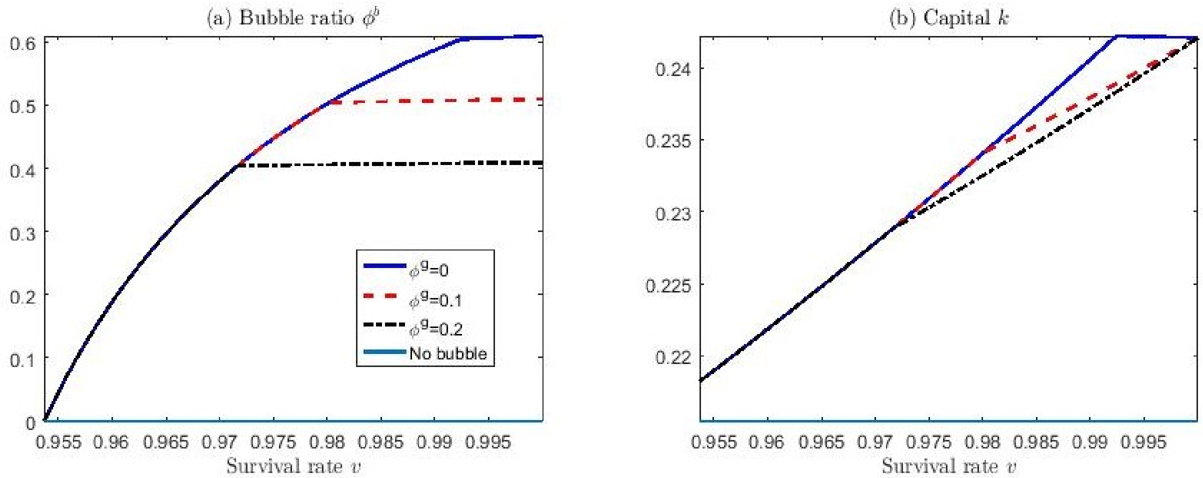


Figure 1 plots the bubble ratio ϕ^b and capital stock k as a function of the bubble survival rate v in the flexible-price model, where we use parameter values set in Section 4.1. As stated in Proposition 2, Figure 1(a) shows that the region for small bubbles (i.e., the parameter region in which the bubbly steady state features a small bubble) shrinks as the government debt ratio ϕ^g increases. For each case of ϕ^g in the figure, there is a kink and the left area from the kink corresponds to the region of the small bubbles and the right area from the kink corresponds to the region of the large bubbles. The figure also shows, as stated in Proposition 2, that given the existence of the small bubble the bubble ratio is independent of the government debt ratio, but for the large bubble it is decreasing in the government bond ratio.

Figure 1(b) shows, as stated in Proposition 2, that the government debt does not crowd out capital in the small bubble economy, but it does crowd out and decreases capital in the large bubble economy. Although the level of capital is greater than that in the bubble-less economy so that a bubble is expansionary for all v, as the government debt increases from $\phi^g = 0$ to $\phi^g = 0.2$, the capital curve shifts down. To understand this crowding-out effect, it is useful to remind that in the region of the large bubbles capital is decreasing in ϕ^b in this type of model, as analyzed by Hirano et al. (2015). With the government debt, capital is decreasing in $\phi^b + \phi^g$, that is, too much bubbles and government debt crowd out capital. An important observation is that an increase in the government debt does not crowd out the bubble perfectly, i.e. $-1 < d\phi^b/d\phi^g < 0$, so that $1 + d\phi^b/d\phi^g > 0$, causing capital to decrease. In this sense, the bubble and the government debt are not perfect substitutes. The bubble is a risky asset while the government debt is a safe asset. An increase in the safe asset does not crowd out the demand for the risky asset completely because the return on the risky asset is higher than that of the safe asset to compensate the probability of bubble burst in the stochastic bubble steady state where the bubble persists.

Finally, Figure 1(b) points out an interesting observation that with the government debt in place capital is no more decreasing in in the survival rate v in the region of the large bubble.[10] Indeed, it continues to be increasing with the government debt, hence a trade-off between safer bubbles and production efficiency disappears when there is enough government debt. This observation brings an important implication to Hirano et al. (2015), who focus on such a trade-off in the same economy but with no government debt and study the role of government bailout which essentially affects the survival rate v. Figure 1(b) suggests that with the sufficiently large government debt, perfect bailout, i.e. $v = 1$ achieves the highest capital and output.

3.3 Stochastic Bubbly Steady State with Nominal Rigidity

We now turn to the analysis of a stochastic bubbly steady state when there is nominal rigidity ($\gamma > 0$). In the bubbly steady state, agents rationally anticipate that the bubble can burst in the next period. Therefore, in the presence of nominal rigidity, the monetary policy rule (20) is non-neutral even in the stochastic steady state: the anticipation of the post-bubble policy can affect allocations and prices today. Specifically, the real interest rate depends on the nominal interest rate and inflation, the latter of which is determined by the marginal cost and expected inflation, which is, in turn, affected by monetary policy stance. As in a typical steady state in the standard New Keynesian model, we restrict our attention to a case in which the central bank stabilizes inflation completely at the target rate in the stochastic steady state.

First, we consider the impact of monetary policy stance in a bubble burst on inflation and the marginal cost during a bubble period. Suppose that the central bank follows the standard monetary policy rule (20) and a bubble is expansionary so that capital is high in the period of a bubble burst. Then, the bubble burst, which reduces the entrepreneurs' capacity to invest, causes the marginal cost to fall for production to be consistent with the high level of capital which is predetermined from the previous period, and hence the bubble burst is associated with deflation. Although the probability of a bubble burst is

[10] Although it is hard to see, in the case of $\phi^g = 0$ in Figure 1(b), capital is decreasing in v in the region of the large bubble.

small, deflation triggered by the bubble burst affects inflation and the marginal cost during the bubble period through the expectation channel in the Phillips curve (18). In particular, given that deflation is realized in the case of bubble burst, it is impossible to stabilize both inflation π_t and the real economy (the marginal cost s_t) during the bubble period: if $\pi_t = 1$, $s_t > 1$; if $s_t = 1$, $\pi_t < 1$.

Next, we consider the impact of government debt on bubbles, which we summarize in the following proposition.

Proposition 3 *Consider a stochastic bubbly steady state in which a bubble burst is associated with a fall in the marginal cost and inflation. Suppose that in the stochastic steady state the central bank stabilizes inflation completely at zero percent. Then:*

(i) An increase in the government debt increases the relative size of the small bubble.

(ii) Given the nominal interest rate R^n, an increase in the government debt decreases the relative size of the large bubble.

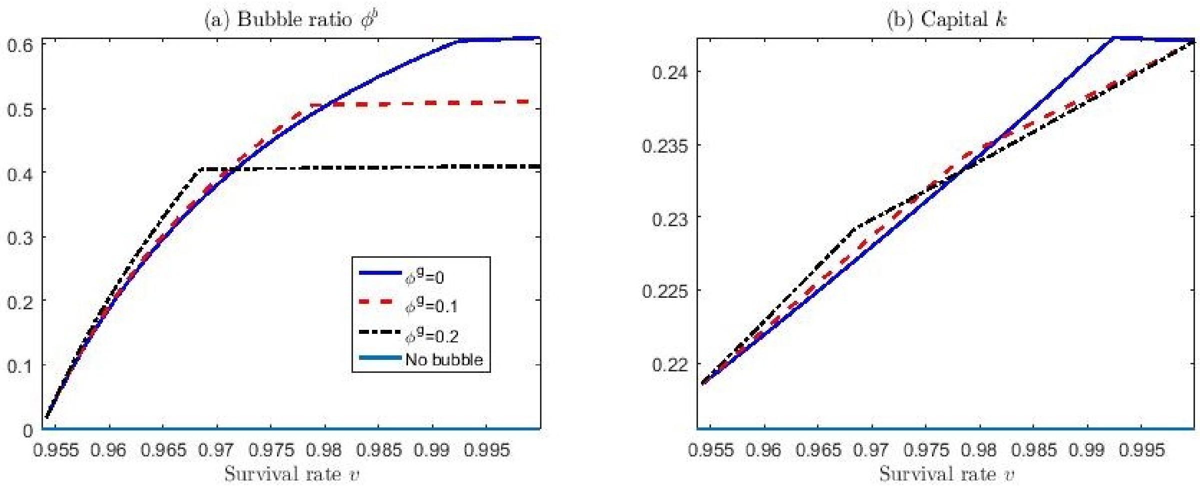
For the large bubble the result is similar to that of the flexible-price economy, but for the small bubble the bubble ratio is increasing in the government debt as opposed to being independent of the government debt in the flexible-price economy. Figure 2(a) plots the relative size of bubbles as a function of the survival rate v for the model with nominal rigidity in a similar manner to Figure 1(a) for the flexible-price model. As the government debt ratio increases from $\phi^g = 0$ to $\phi^g = 0.2$, the relative size of the small bubble increases. Figure 3(a) clearly shows that the bubble ratio is increasing in the government debt ratio for the small-bubble economy. It also shows that nominal rigidity is essential for obtaining the positive relationship. Without it, the bubble ratio is independent of the government debt ratio.

What drives the difference between the model with nominal rigidity and the flexible-price model in terms of the effect of the government debt on bubbles in the small-bubble economy? As shown in the Appendix, the relative size of the small bubble in the stochastic steady state with nominal rigidities is given by:

$$\phi^b = \frac{1 + \beta\left(R - R^n\right)\phi^g - \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right)h\right]\beta R}{\beta\left(1 - R\right)}, \tag{37}$$

where $R = \sigma a^L s k^{\sigma - 1} = \frac{\left[1 - v\beta(1-h)\right]\left(a^L - \theta a^H\right)}{(1-\theta)a^H h\beta}$ is the real interest rate. Equation (37) shows that the bubble ratio is increasing in ϕ^b if the nominal rate is kept below the real interest

Figure 2: Stochastic steady states in the model with nominal rigidity



Note: Parameter values except those for v and ϕ^g are fixed at those set in Section 4.1.

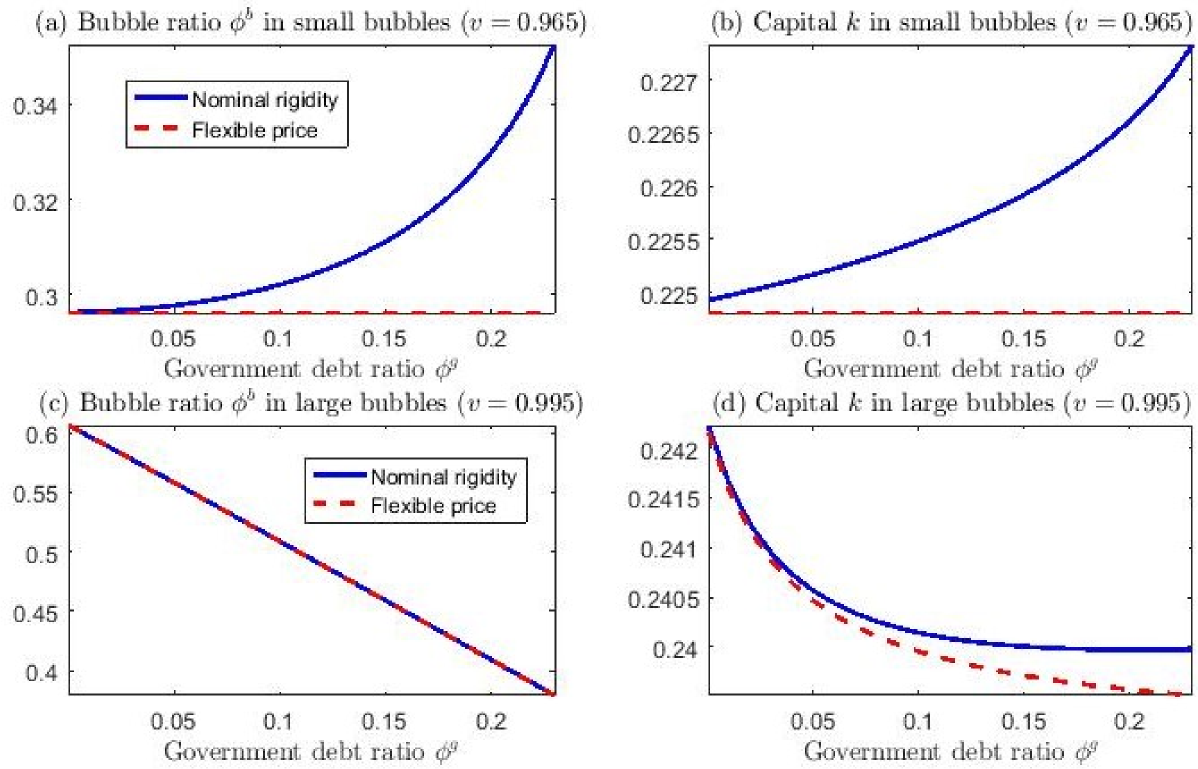
rate: $R^n < R$. Intuitively, when the nominal rate is lower than the real rate, the government bond, which pays the nominal rate, is less useful for preserving the net worth than the bubble asset in the stochastic bubble steady state. A bubble fills in this gap and consequently its size increases.

Then, why is the nominal rate kept below the real rate in the stochastic small bubble steady state? The answer is related to the fact that the government bond is safer than the bubble asset. Unlike the flexible-price model, the real rate in the next period depends on the marginal cost s_{t+1} in the model with nominal rigidity. Given that a bubble is expansionary, its collapse induces s_{t+1} to drop and so does the real rate. In addition, a bubble-burst-led deflation implies that the real return of the nominal rate, R_t^n/π_{t+1}, in the period of a bubble burst is higher, playing as a role of an insurance. Hence, arbitrage between the risky bubble and the safe government bond leads to the lower nominal rate than the real rate in the stochastic steady state.

This observation – inflation in the bubble burst affects the nominal rate and the bubble ratio in the bubble economy – brings an interesting implication for the effects of monetary policy stance on bubbles. In particular, monetary policy with strong inflation stabilization stance, i.e. a high value of the inflation coefficient ψ_π in the monetary policy rule (20), stabilizes inflation in the bubble burst. By doing so it works to raise the nominal rate and decreases the bubble size in the stochastic bubble steady state. Figure 4(a) shows that the bubble ratio in the small-bubble economy is decreasing in the monetary policy stance ψ_π.

Now we consider the effects of government debt and monetary policy stance on capi-

Figure 3: The effects of government debt on bubbles and capital



Note: Parameter values except v, ϕ^g and γ are fixed at hose set in Section 4.1. 'Baseline' γ corresponds to that set in Section 4.1.

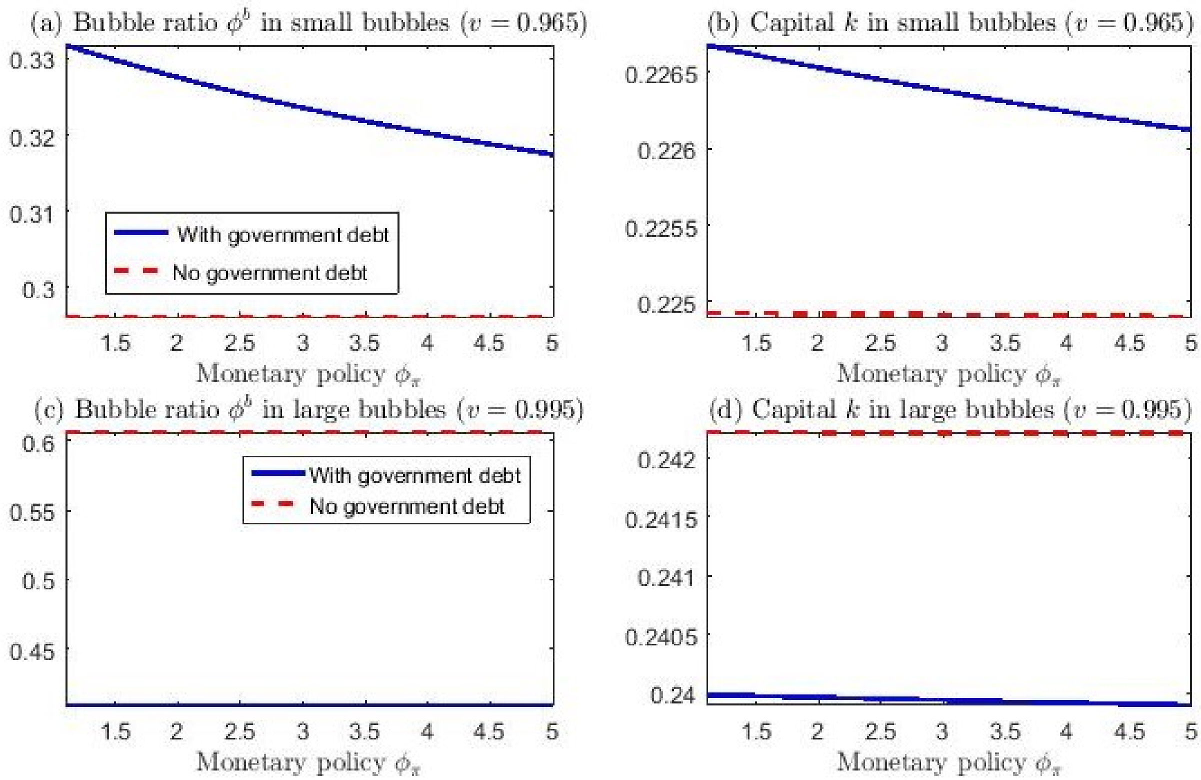
tal/output. For the small-bubble economy, as shown in the Appendix, the level of capital in the stochastic bubbly steady state with nominal rigidities is given by

$$k = \left\{ \frac{1}{\sigma a^L s} \frac{[1 - v\beta(1-h)] \left(a^L - \theta a^H\right)}{(1-\theta) a^H h \beta} \right\}^{\frac{1}{\sigma - 1}}.$$

Hence, capital is increasing in the marginal cost s. Capital is increasing in the government debt ratio (Figure 3(b)) because a higher level of government debt is associated with a higher bubble ratio (Figure 3(a)), causing severer deflation in the bubble burst, which requires a higher marginal cost to achieve zero inflation in the stochastic bubbly steady state. In the flexible-price model, the government debt ratio has no effect on capital because the marginal cost is always unity. The same mechanism also explains why stronger monetary policy stance on inflation reduces capital in the small-bubble stochastic steady state if there is sufficient nominal rigidity (Figure 4(b)).

Turning to the large-bubble economy, an increase in the government debt reduces capital

Figure 4: The effects of monetary policy on bubbles and capital



Note: Parameter values except v, ϕ^g and γ are fixed at hose set in Section 4.1. 'Baseline' γ corresponds to that set in Section 4.1.

in the stochastic steady state (Figure 3(d)) as in the flexible-price economy. However, with nominal rigidity, the effect is mitigated because the positive marginal cost raises capital as in the small-bubble economy.

4 Impulse Responses to Policy Shocks

In this section, we study the effects of two types of monetary policy shocks – interest rate shocks and government-bond-purchase shocks – on bubbles and the economy around the (stochastic) steady state for three types of the economy: the bubble-less economy, the small-bubble economy and the large-bubble economy. We solve the model globally using a function iteration method.[11] The detail of the numerical method is dedicated to the appendix.

[11] For the computational method, see e.g. Richter et al. (2014).

4.1 Parameters

As is standard in a New Keynesian model with nominal rigidities, the model as described in Section 2 can only be solved numerically. The time periods in the model are quarters. The discount factor is set to be $\beta = 0.96^{1/4}$ and the capital share is set to be $\sigma = 0.36$.

For parameters pertaining to the real economy of the model, we follow the benchmark case of Hirano et al. (2015). The productivity parameters are set to be $a^H = 1.15$ and $a^L = 1$, with the i.i.d. probability of the high productivity state set to be $h = 0.35$. The collateral ratio is set to be $\theta = 0.1$. In the numerical analysis in this section, we set the persisting probability of the bubble as $v = 0.96$ for a small bubble and $v = 0.99$ for a large bubble.

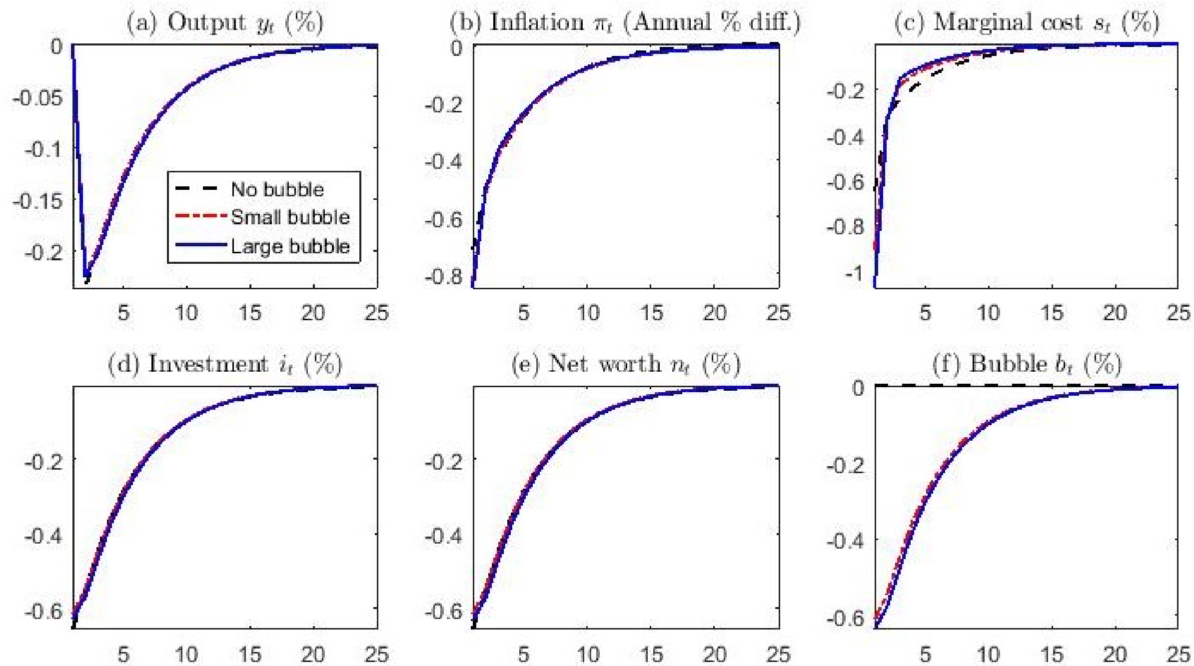
For parameters pertaining to nominal rigidity and monetary policy, we use standard parameter values used in the literature on monetary economics. We set the inflation coefficient on the monetary policy rule as $\psi_\pi = 1.5$ and the gross markup as $\lambda = 1.1$. We set the price adjustment cost parameter γ such that the slope of the Phillips curve in the linearized model (19) is equal to that in the Calvo (1983) model in which the average frequency of price changes is once in a year.[12] We set the ration of government debt to the net worth so that the debt-GDP ratio is about 10 percent. In this economy, the government bonds circulated in the economy would correspond to those for facilitating intermediation and investment in the actual economy. Admittedly it is difficult to infer the holdings of government bonds with such motives. Yet, it would be useful to refer to the ratio of the government bonds held by domestic non-financial sectors and private depository institutions relative to GDP for the US, which is 13.6 percent on average during the period of 2014–16. Although we do not intend to use this model for deriving quantitative implications, we believe that the ratio of government debt holdings for intermediation and investment motives to GDP is not far from 10 percent. Finally, the AR(1) coefficient for nominal interest rate shocks and government bond purchase shocks are both set as $\rho_{mp} = \rho_g = 0.8$.

4.2 Conventional monetary policy shocks

This model, even without bubbles, differs substantially from the standard New Keynesian model. In this model, labor is inelastic and output depends only on capital. In contrast,

[12] Let $1 - \vartheta$ denote the probability of price changes in the Calvo model. Then, the slope of the linearized Phillips curve is given by $(1 - \vartheta)(1 - \beta\vartheta)/\vartheta$. A one-year of the average duration of price changes implies $\vartheta = 0.75$. We set γ such that the slope of the linearized model, $(1/\gamma)\lambda/(\lambda - 1)$, is equal to $(1 - \vartheta)(1 - \beta\vartheta)/\vartheta$.

Figure 5: Impulse responses to a contractionary interest rate shock



capital is constant and output depends only on labor in the standard New Keynesian model. Our focus of this analysis is a mechanism through which conventional monetary policy shocks – interest rate shocks – can affect the economy and bubbles and also possible differences of such a mechanism among the three types of the economy: the bubble-less economy, the small-bubble economy and the large-bubble economy.

Figure 5 shows impulse responses to a 0.25 percent (1 annual percent) rise in the nominal interest rate around the (stochastic) steady state for the three types of the economy. Three observations are worth noting. First, the responses are almost the same for all the three types of the economy. Irrespective of the presence of bubbles or the types of bubbles, the responses are nearly equivalent.

Second, the conventional monetary tightening has contractionary effects on the economy as in the standard New Keynesian model. In response to the exogenous rise in the nominal rate, output, inflation and the marginal cost all decrease (Figure 5(a)-(c)). In spite of the qualitative similarities with the standard New Keynesian model, a mechanism behind the responses is different between the two models. In this model, it is investment and capital that drive output , while in the standard New Keynesian model, it is labor that determines output.

How does a rise in the interest rate dampen investment in this model as shown in Figure

5(d)? Because the entrepreneurs consume a fraction $1-\beta$ of the net worth and the L-types hold bubbles and government bonds, the investment, $i_t \equiv i_t^H + i_t^L$, is given by:

$$i_t = (1 - \phi_t^b - \phi_t^g)\beta n_t,$$

where the government debt ratio ϕ_t^g is constant in the impulse responses in Figure 5. Given that the bubble ratio ϕ_t^b is constant, which is indeed the case as we will explain later, the investment is driven by the net worth (Figure 5(e)), which, in turn, is mainly determined by the marginal cost (Figure 5(c)) as implied by equation (29).

Then, why does the marginal cost drop in response to the contractionary monetary policy shock? An increase in the nominal rate raises the real interest rate, which dampens investment activity by the L-types, until the return on capital investment is equated to the real interest rate. The resulting decrease in the demand for investment leads to a drop in the marginal cost, which causes the net worth to decreases, and slowing down investment further.

Third, the absolute size of a bubble shrinks in response to the monetary policy tightening for both the small-bubble and the large-bubble economies. Because the rise in the nominal rate decreases the net worth, the L-types have less resources to purchase the bubble asset. The demand for the bubble asset falls in line with the net worth, and consequently the absolute size of the bubble decreases with its relative size to the net worth kept constant.

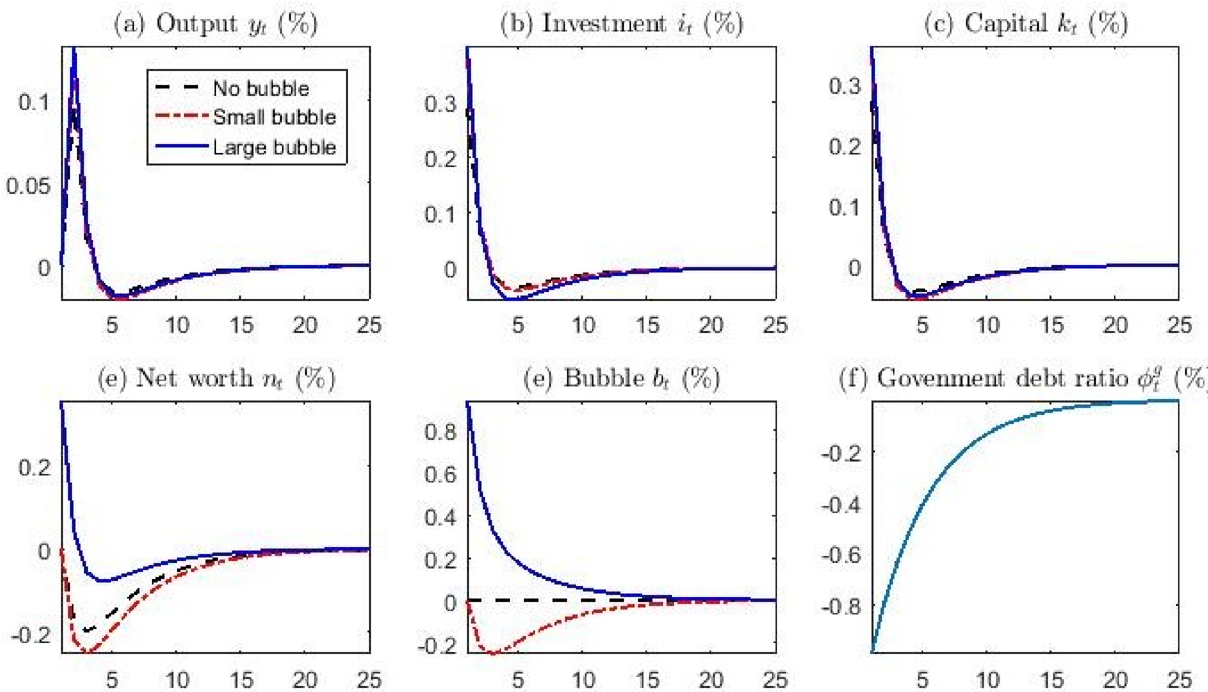
4.3 Unconventional monetary policy shocks

Now we study the effects of unconventional monetary policy shocks – government bond purchases by the central bank – on the economy and bubbles. Because these effects can be analyzed in the flexible-price model as well, we start from the flexible-price model and move onto the model with nominal rigidities.

The flexible-price model. Figure 6 plots impulse responses to a 1 percent decrease in the government debt ratio – a government bond purchase by a 1 percent of the net worth – for the three types of the economy. Two observations are worth noting. First, the government bond purchase has expansionary effects on output and investment for all types of the economy.

To understand the mechanism, it is useful to consider the bubble-less economy because of its analytical solution. Combining equations (28), (29) and (30) in the case of $\phi_t^b + \phi_t^g \leq \phi$

Figure 6: Impulse responses to the government bond purchase shock in the flexible-price model



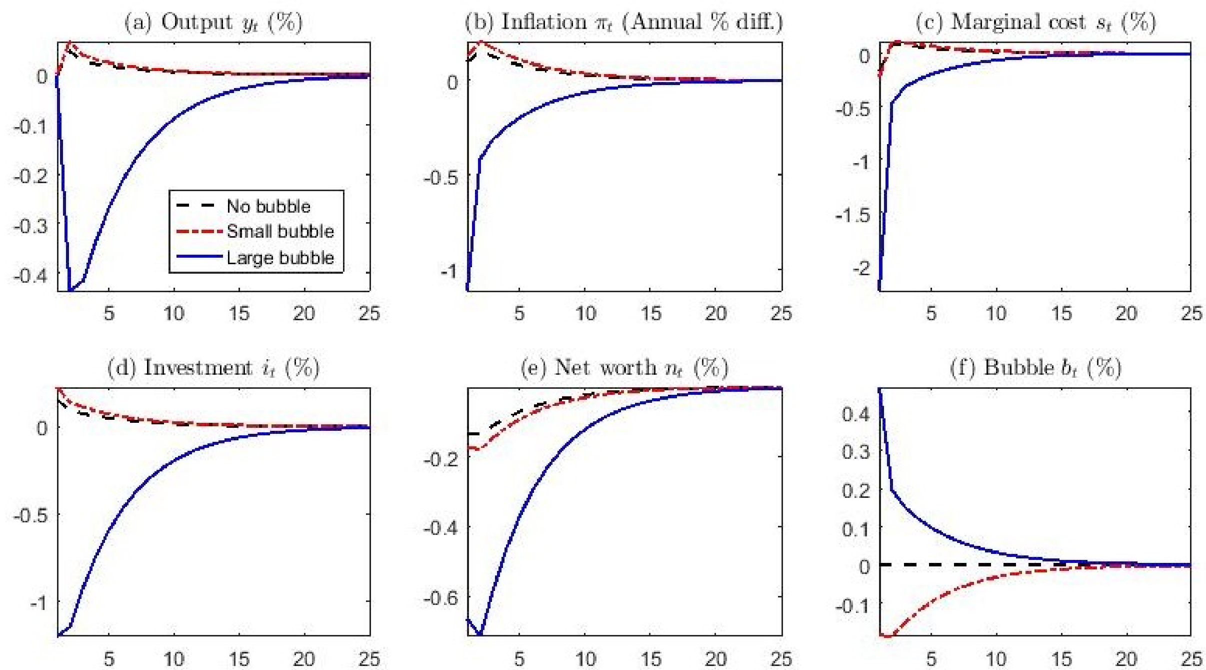
with $\phi_t^b = 0$ and $s_t = \pi_t = 1$ yields

$$k_t = a^L \beta \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - \phi_t^g\right] \left[1 + \frac{\phi_{t-1}^g}{1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - \phi_{t-1}^g}\right] \sigma k_{t-1}^{\sigma}. \tag{38}$$

This equation implies that a government bond purchase by the central bank decreases the government bond ratio ϕ_t^g and stimulates capital in period t and thereby output in period $t+1$. Because the lump sum tax is imposed on workers, the government bond purchase effectively transfers resources from workers to entrepreneurs, breaking the irrelevance result of Wallace (1981). However, from the next period, the opposite effect starts kicking in: a reduction of government bonds decreases the entrepreneurs' net worth, which is summarized by the term in the second bracket in equation (38). Initially, the first effect dominates, but the second effect kicks in and dominates around 5 quarters after the initial shock, and this is why output and investment decrease slightly below the steady state levels around the period.

Although the responses of output, investment and capital are similar (Figure 6(a)-(c)), the mechanism of the expansionary effect differs substantially for the large-bubble economy.

Figure 7: Impulse responses to a government bond purchase shock in the model with nominal rigidities



This is evident in the responses of the net worth and the bubble (Figure 6(d)(e)). In the large-bubble economy, the absolute size of the bubble and also the relative size of the bubble increase in response to the government bond purchase in contrast with the other economies. An increase in the relative size of the bubble boosts the net worth as implied by equation (29), which, in turn, stimulates investment. The bubble ratio increases because the L-types have more resources to purchase the bubble asset as less amount of government bonds are available in the market.

The model with nominal rigidities. Having analyzed how the flexible-price economy responds to government bond purchase shocks, we now move onto the model with nominal rigidities. Figure 7 plots impulse responses to the same government bond purchase shock (Figure 6(f)) for the model with nominal rigidities for the three types of the economy.

Compared with the flexible-price model (Figure 6), the model with nominal rigidities (Figure 7) shows two distinguished features regarding the effects of the government bond purchase. First, the government bond purchase has contractionary effects for the large-bubble economy as opposed to expansionary effects in the flexible price model, while it has expansionary effects for the no-bubble economy and the small-bubble economy as in the flexible-price model.

How can government bond purchases have negative effects on the large-bubble economy? As in the flexible-price model, the L-types have more resources to purchase the bubble asset as a result of an increase in government bond purchases. The bubble expands initially, but as the government bond purchase is gradually tapered, the bubble also returns to the original level. Along this path, the growth of the bubble p^b_{t+1}/p^b_t falls and as a result the real rate also drops to satisfy the Euler equation (9). In the model with nominal rigidity, unless the nominal rate is adjusted to be consistent with the real rate, the fall in the real rate puts downward pressure on inflation and the marginal cost. Because the central bank follows the monetary policy rule (20), it cannot offset all the downward pressure. Consequently, the real rate implied by the nominal rate rises, which discourages investment and decreases the marginal cost and the net worth. Hence, the model with nominal rigidity brings a totally opposite implication on the effectiveness of government bond purchases, compared with the flexible-price model.

The second notable feature of the model with nominal rigidity is that the responses of output and investment become more persistent for the bubble-less economy and the small-bubble economy (Figure 7(a)(d)) relative to their counterpart in the flexible-price model (Figure 6(a)(b)). Although the government bond purchase decreases the net worth as in the flexible-price model, the effect is mitigated by an increase in the marginal cost (Figure 7(c)). The expansionary impact of the government bond purchase puts upward pressure on the marginal cost, which, in turn, contributes to increasing the net worth, making the actual decrease in the net worth gradual and moderate. In this way, the model with nominal rigidity generates the persistent responses of output and investment.

5 Conclusion

Our paper provides a simple New Keynesian model with investment to analyze the interactions of nominal rigidity, safe government bonds and risky asset bubbles. The model is useful for studying the effects of both conventional and unconventional monetary policies before, during and after bubbly episodes. Given the past, recent and ongoing booms and busts in housing and stock prices in economies around the world, we hope that our model can be useful for future research on optimal policies with respect to risky asset price bubbles. A limitation of the current model is the absence of government bonds at different maturities and the aspect of unconventional monetary policies that involves purchasing government bonds at long maturities. Introducing multiple maturities to the current framework would increase the model's complexity and is left for future research.

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Appendix

A Derivation of Equations

Evolution of capital and the real interest rate By using the budget constraint (5) and reminding that the H-types do not hold the government debt and the bubble asset, the aggregate borrowing for the H-types and that for L-types are given, respectively, as:

$$d_t^H = i_t^H - \beta n_t^H,$$
$$d_t^L = i_t^L + p_t^b + g_t - \beta n_t^L.$$

where $n_t^H \equiv \int_{j \in \text{H-type}} n_t^j dj$ and $n_t^L \equiv \int_{j \in \text{L-type}} n_t^j dj$ are the aggregate net worth for the H-types and L-types, respectively. Substituting these borrowings into the credit market clearing condition (23) yields

$$i_t^H + i_t^L + p_t^b + g_t - \beta n_t = 0, \tag{A1}$$

where $n_t \equiv n_t^H + n_t^L$ is the aggregate net worth. For analytical tractability, we assume $n_0^H = hn_0$. This assumption, combined with the idiosyncratic nature of this model, implies $n_t^H = hn_t$ and $n_t^L = (1-h)n_t$ for all t.

As implied by the inequalities (6), the model has two cases: (i) $1 = E_t\beta(c_t^L/c_{t+1|L})a^L q_{t+1}$ and (ii) $1 > E_t\beta(c_t^L/c_{t+1|L})a^L q_{t+1}$. We analyze each case in turn.

Case (i) $1 = E_t\beta(c_t^L/c_{t+1|L})a^L q_{t+1}$*:* This condition implies that the L-types invest in an investment project. Combining this condition with equation (8) leads to

$$0 = E_t\beta\frac{c_t^L}{c_{t+1|L}}\left(R_{t+1} - a^L q_{t+1}\right). \tag{A2}$$

The binding borrowing constraint (1) in each state of the economy implies that q_{t+1}/R_{t+1} has to be independent of period-$t+1$ variables. Hence, q_{t+1} and R_{t+1}, which satisfy both such a condition and equation (A2), are such that $R_{t+1} = a^L q_{t+1}$. Substituting this equation into equation (7) yields the aggregate investment by the H-types as

$$i_t^H = \left(\frac{a^L}{a^L - \theta a^H}\right)\beta hn_t,$$

where $hn_t = n_t^H$. Substituting this i_t^H into condition (A1) yields

$$i_t^L = \beta n_t - \left(\frac{a^L}{a^L - \theta a^H}\right)\beta hn_t - p_t^b - g_t.$$

Substituting these i_t^H and i_t^L into the evolution of capital (25), we obtain:

$$\begin{aligned} k_t &= \left\{ a^H \left(\frac{a^L}{a^L - \theta a^H} \right) h + a^L \left[1 - \left(\frac{a^L}{a^L - \theta a^H} \right) h - \phi_t^b \right] \right\} \beta n_t - a^L g_t, \\ &= \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H} \right) h - \phi_t^b - \phi_t^g \right] a^L \beta n_t. \end{aligned} \tag{A3}$$

Case (ii) $1 > E_t \beta (c_t^L / c_{t+1|L}) a^L q_{t+1}$*:* In this case, the return on investing in a project is lower than lending, and thereby $i_t^L = 0$. The credit market clearing condition (A1) is then given by:

$$\left(\frac{1}{1 - \theta (q_{t+1}/R_{t+1}) a^H} \right) \beta h n_t + \phi_t^b \beta n_t + \phi_t^g \beta n_t - \beta n_t = 0.$$

This equation determines the discounted value of the return on capital as

$$\frac{q_{t+1}}{R_{t+1}} = \frac{1}{\theta a^H} \left[1 - \frac{h}{1 - \phi_t^b - \phi_t^g} \right]. \tag{A4}$$

Substituting this into equation (7), we obtain:

$$i_t^H = \left(1 - \phi_t^b - \phi_t^g \right) \beta n_t.$$

Again, substituting this investment function and $i_t^L = 0$ into the evolution of capital (25) yields

$$k_t = a^H \left(1 - \phi_t^b - \phi_t^g \right) \beta n_t.$$

Substituting equation (A4) into condition $1 > E_t \beta (c_t^L / c_{t+1|L}) q_{t+1} a^L$, we obtain

$$\phi_t^b + \phi_t^g > 1 - \frac{a^L h}{a^L - \theta a^H} \equiv \phi.$$

Hence, when $\phi_t^b + \phi_t^g > \phi$, only the H-types invest; when $\phi_t^b + \phi_t^g \leq \phi$, the L-types invest as well.

Evolution of net worth Aggregating the individual net worth (4) and using $k_{t-1} = a^H i_{t-1}^H + a^L i_{t-1}^L$, we obtain $n_t = q_t k_{t-1} + (R_{t-1}^n / \pi_t) b_{t-1} + p_t^b = q_t k_{t-1} + (1/\pi_t) R_{t-1}^n \phi_{t-1}^g \beta n_{t-1} + \phi_t^b \beta n_t$. The law of motion for the aggregate net worth is thus given by

$$n_t = \frac{s_t \sigma k_{t-1}^{\sigma} + \beta R_{t-1}^n \phi_{t-1}^g n_{t-1} / \pi_t}{1 - \phi_t^b \beta},$$

where q_t was substituted out by using (15).

Evolution of bubbles The relative size of the bubble ϕ_t^b evolves over time according to

$$\phi_{t+1}^b = \begin{cases} \frac{p_{t+1}^b/p_t^b}{n_{t+1}/n_t}\phi_t^b & \text{if the bubble persists.} \\ 0 & \text{if the bubble bursts.} \end{cases}$$

The Euler equation regarding the bubble (9) is written as:

$$1 = E_t\beta\frac{c_t^L}{c_{t+1|L}}\frac{p_{t+1}^b}{p_t^b} = E_t\beta\frac{(1-\beta)(1-h)n_t}{(1-\beta)n_{t+1|L}}\frac{\phi_{t+1}^b}{\phi_t^b}\frac{\beta n_{t+1}}{\beta n_t} = v\beta(1-h)E_t\left(\frac{\phi_{t+1}^b}{\phi_t^b}\frac{n_{t+1}}{n_{t+1|L}}\right), \tag{A5}$$

where the $t+1$ variables in the final term are those in which the bubble persists and the expectation operator E_t is taken with respect to policy shocks. Now we consider the two cases in deriving the evolution of the bubble size.

Case (i) $\phi_t^b + \phi_t^g \leq \phi$*:* In this case, the net worth in period $t+1$ given that the entrepreneurs are the L-type in period t is given by

$$\begin{aligned} n_{t+1|L} =& n_{t+1} - n_{t+1|H}, \\ =& n_{t+1} - \left(q_{t+1}a^H i_t^H - R_{t+1}d_t^H\right), \\ =& n_{t+1} - \left(q_{t+1} - \theta q_{t+1}\right) a^H i_t^H, \\ =& n_{t+1} - q_{t+1}\frac{(1-\theta)a^L a^H h\beta n_t}{a^L - \theta a^H}. \end{aligned}$$

Then, condition (A5) is written as:

$$1 = v\beta(1-h)E_t\left[\frac{\phi_{t+1}^b/\phi_t^b}{1-\sigma s_{t+1}k_t^{\sigma-1}\frac{(1-\theta)a^L a^H h\beta}{a^L-\theta a^H}\frac{n_t}{n_{t+1}}}\right].$$

where q_{t+1} was substituted out by using (15).

Case (ii) $\phi_t^b + \phi_t^g > \phi$*:* In this case, the net worth in period $t+1$, held by the entrepreneurs who were the L-type in period t, is given by

$$\begin{aligned} n_{t+1|L} =& n_{t+1} - \left(q_{t+1} - \theta q_{t+1}\right) a^H i_t^H, \\ =& n_{t+1} - q_{t+1}(1-\theta)a^H\left(1-\phi_t^b-\phi_t^g\right)\beta n_t. \end{aligned}$$

Then, condition (A5) is written as

$$1 = v\beta(1-h)E_t\left[\frac{\phi_{t+1}^b/\phi_t^b}{1-\sigma s_{t+1}k_t^{\sigma-1}(1-\theta)a^H\left(1-\phi_t^b-\phi_t^g\right)\beta\frac{n_t}{n_{t+1}}}\right]$$

where q_{t+1} was substituted out by using (15).

Fisher equation The Fisher equation is given by equation (9):

$$1 = E_t \beta \frac{c_t^L}{c_{t+1|L}} \frac{R_t^n}{\pi_{t+1}},$$

where $c_t^L = (1-\beta)(1-h)n_t$ and $c_{t+1|L} = (1-\beta)n_{t+1|L}$. The net worth in period $t+1$ given that the entrepreneurs are the L-type in period t is given by

$$n_{t+1|L} = \begin{cases} n_{t+1} - q_{t+1} \frac{(1-\theta)a^L a^H h \beta n_t}{a^L - \theta a^H} & \text{if } \phi_t^b + \phi_t^g \leq \phi \\ n_{t+1} - q_{t+1}(1-\theta)a^H \left(1 - \phi_t^b - \phi_t^g\right) \beta n_t & \text{if } \phi_t^b + \phi_t^g > \phi \end{cases}$$

Hence, the Fisher equation is written as

$$1 = \begin{cases} E_t \frac{\beta(1-h)}{\frac{n_{t+1}}{n_t} - \sigma s_{t+1} k_t^{\sigma-1} \frac{(1-\theta)a^L a^H h \beta}{a^L - \theta a^H}} \frac{R_t^n}{\pi_{t+1}} & \text{if } \phi_t^b + \phi_t^g \leq \phi \\ E_t \frac{\beta(1-h)}{\frac{n_{t+1}}{n_t} - \sigma s_{t+1} k_t^{\sigma-1} (1-\theta)a^H \left(1 - \phi_t^b - \phi_t^g\right)\beta} \frac{R_t^n}{\pi_{t+1}} & \text{if } \phi_t^b + \phi_t^g > \phi \end{cases}$$

where q_{t+1} was substituted out by using (15).

B The Model with Flexible Prices

When there is no nominal rigidity, the system of equations for this economy consists of (28), (29) and (31), where the nominal interest rate over inflation R_t^n/π_{t+1} is replaced by the real interest rate R_{t+1} given by equation (30) and the marginal cost is $s_t = 1$.

In a stochastic bubbly steady state, these four equations are reduced to the following three equations with three unknowns, k, ϕ^b and R, given by

$$k^{1-\sigma} = \begin{cases} \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - \phi^b - \phi^g\right] \frac{a^L \beta \sigma}{1 - \beta \phi^b - \beta R \phi^g} & \text{if } \phi^b + \phi^g \leq \phi \\ \left(1 - \phi^b - \phi^g\right) \frac{a^H \beta \sigma}{1 - \beta \phi^b - \beta R \phi^g} & \text{if } \phi^b + \phi^g > \phi \end{cases}, \tag{B1}$$

$$1 - v\beta(1-h) = \begin{cases} \sigma k^{\sigma-1} \frac{(1-\theta)a^L a^H h \beta}{a^L - \theta a^H} & \text{if } \phi^b + \phi^g \leq \phi \\ \sigma k^{\sigma-1} (1-\theta) a^H \left(1 - \phi^b - \phi^g\right) \beta & \text{if } \phi^b + \phi^g > \phi. \end{cases}, \tag{B2}$$

$$R = \begin{cases} a_L \sigma k^{\sigma-1} & \text{if } \phi^b + \phi^g \leq \phi \\ \theta a^H \left(1 - \frac{h}{1 - \phi^b - \phi^g}\right)^{-1} \sigma k^{\sigma-1} & \text{if } \phi^b + \phi^g > \phi \end{cases}. \tag{B3}$$

Small bubble. Consider the case of $\phi^b + \phi^g \leq \phi$. From equation (B2), capital stock is given by:

$$k = \left[\frac{\sigma}{1 - v\beta(1-h)} \frac{(1-\theta)a^L a^H h\beta}{a^L - \theta a^H}\right]^{\frac{1}{1-\sigma}}.$$

The interest rate is given by equation (B3) as:

$$R = \frac{[1 - v\beta(1-h)](a^L - \theta a^H)}{(1-\theta)a^H h\beta}.$$

The relative size of the bubble is given by equation (B1) as:

$$\phi^b = \frac{1}{\beta(1-R)} \left\{1 - \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h\right] \beta R\right\}.$$

In this economy, capital stock, the interest rate and the relative size of the bubble are all independent of the government debt.

The relative size of the bubble is positive if and only if

$$R < \frac{1}{\beta}\left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h\right]^{-1} \equiv R_{nb}$$

This condition is rewritten as

$$v > \frac{1}{\beta(1-h)} - \frac{R_{nb}(1-\theta)a^H h}{(1-h)\left(a^L - \theta a^H\right)} \equiv \underline{v}.$$

In addition, the condition for the small bubble, i.e. $\phi^b + \phi^g < \phi$, is written as

$$R > \frac{1 - (\phi - \phi^g)\beta}{\left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h\right]\beta - (\phi - \phi^g)\beta},$$

or

$$v < \frac{1}{\beta(1-h)} - \frac{\left[1 - (\phi - \phi^g)\beta\right](1-\theta)a^H h}{\left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - (\phi - \phi^g)\right](a^L - \theta a^H)\beta(1-h)} \equiv \overline{v}.$$

Hence, the existence condition of small bubbles is that the probability of bubble survival lies in the interval $v \in (\underline{v}, \overline{v})$. While the lower bound $\underline{v}$ is independent of ϕ^g, the upper bound is a function of ϕ^g. The effect of ϕ^g on $\overline{v}$ is:

$$\begin{aligned}
\frac{d\overline{v}}{d\phi^g} &\propto -\beta(1-\theta)a^H h\left[1+\left(\frac{a^H-a^L}{a^L-\theta a^H}\right)h-(\phi-\phi^g)\right](a^L-\theta a^H)\beta(1-h) \\
&\quad +\left[1-(\phi-\phi^g)\beta\right](1-\theta)a^H h(a^L-\theta a^H)\beta(1-h) \\
&\propto 1-\beta\left[1+\left(\frac{a^H-a^L}{a^L-\theta a^H}\right)h\right] \\
&= 1-\frac{1}{R_{nb}}<0.
\end{aligned}$$

Therefore, as the government bond ratio ϕ^g increases, a region for the existence of small bubbles shrinks.

Large bubble. Consider the case of $\phi^b+\phi^g>\phi$. Arranging equations (B1)–(B3) yields the following equation for ϕ^b:

$$\left(\phi^b\right)^2-\left[1-h+\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g\right]\phi^b+(1-h)\left[\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-v\phi^g\right]=0.$$

The solution is given by

$$\begin{aligned}
\phi^b &= \frac{1-h-\phi^g+\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\sqrt{\left[1-h+\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g\right]^2-4(1-h)\left(\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-v\phi^g\right)}}{2} \\
&\geq \frac{1-h-\phi^g+\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\sqrt{\left[(1-h)-\left(\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g\right)\right]^2}}{2}, \\
&= \frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g,
\end{aligned}$$

where the inequality holds with equality when $v=1$. The inequality implies $\phi^b+\phi^g\geq\phi$ so that the condition of large bubbles is satisfied. The effect of the government debt on the size of bubble is negative, as it is summarized by

$$\begin{aligned}
\frac{d\phi^b}{d\phi^g} &= -\frac{1}{2}-\frac{1}{2}\frac{-\left(1-h+\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g\right)+2(1-h)v}{\sqrt{\left[1-h+\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g\right]^2-4(1-h)\left(\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-v\phi^g\right)}} \\
&\leq -\frac{1}{2}+\frac{1}{2}\frac{\left(\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g-(1-h)\right)+2(1-h)(1-v)}{(1-h)-\left(\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g\right)} \\
&= -1+\frac{2(1-h)(1-v)}{(1-h)-\left(\frac{v\beta(1-h)-\theta}{\beta(1-\theta)}-\phi^g\right)}>-1.
\end{aligned}$$

Hence, as the government debt ratio increases, the size of bubbles decreases, but the sum of the ratios, $\phi^b + \phi^g$, increases. This result and condition (B2) imply that an increase in the government debt ratio decreases capital.

C Model with Nominal Rigidities: Numerical solution methods

We use a function iteration method to solve the model globally. This appendix describes a solution algorithm for the bubble-less economy and the two types of the bubble economy.

C.1 The bubble-less economy

The system of equations for the bubble-less economy is given by

$$k_t = \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - \phi_t^g\right] a^L \beta n_t, \tag{C1}$$

$$n_t = s_t \sigma k_{t-1}^{\sigma} + \beta R_{t-1}^n \phi_{t-1}^g n_{t-1}/\pi_t, \tag{C2}$$

$$1 = E_t \frac{\beta(1-h)}{\frac{n_{t+1}}{n_t} - \sigma s_{t+1} k_t^{\sigma-1} \frac{(1-\theta)a^L a^H h \beta}{a^L - \theta a^H}} \frac{R_t^n}{\pi_{t+1}}, \tag{C3}$$

$$(\pi_t - 1)\pi_t = \frac{1}{\gamma}\frac{\lambda}{\lambda - 1}(s_t - 1) + E_t M_{t,t+1}^w \frac{y_{t+1}}{y_t}(\pi_{t+1} - 1)\pi_{t+1}, \tag{C4}$$

$$\log(R_t^n) = \log(\bar{R}^n) + \psi_\pi \log(\pi_t) + \upsilon_t, \tag{C5}$$

where $M_{t,t+1}^w = \beta c_t^w / c_{t+1}^w$, $y_t = k_{t-1}^{\sigma}$ and c_t^w is given by

$$c_t^w = k_{t-1}^{\sigma}\left[1 - \frac{\gamma}{2}(\pi_t - 1)^2\right] - (1 - \beta\phi_t^g)\, n_t, \tag{C6}$$

and υ_t and ϕ_t^g follow the AR(1) processes with their shocks given by $\epsilon_{mp,t}$ and $\epsilon_{g,t}$, respectively. Define $x_{t-1} \equiv R_{t-1}^n \phi_{t-1}^g n_{t-1}$. For mitigating computational burdens, we consider the model with one shock only, either $\epsilon_{mp,t}$ or $\epsilon_{g,t}$. We solve for policy functions for the marginal cost and inflation as a function of states. The states consist of k_{t-1}, x_{t-1}, υ_{t-1} and $\epsilon_{mp,t}$ in the case of $\epsilon_{mp,t}$ shock only, while the states consist of k_{t-1}, x_{t-1}, ϕ_{t-1}^g and $\epsilon_{g,t}$ in the case of $\epsilon_{g,t}$ shock only. Here we consider the latter case. We discretize the state space as $k_{t-1} \in \boldsymbol{K} \equiv [K_1, ...K_{n_K}]$, $x_{t-1} \in \boldsymbol{X} \equiv [X_1, ..., X_{n_X}]$, $\phi_{t-1}^g \in \boldsymbol{\Phi} \equiv \left[\Phi_1, ..., \Phi_{n_\phi}\right]$ and $\epsilon_{g,t} \in \boldsymbol{\epsilon} \equiv \left[\epsilon_1, ..., \epsilon_{n_g}\right]$. Let $\boldsymbol{S} \equiv \boldsymbol{K} \times \boldsymbol{X} \times \boldsymbol{\Phi} \times \boldsymbol{\epsilon}$ denote the discretized state space. The model is solved in six steps.

(i) Solve the log-linearized version of the model and use the solution for an initial guess of policy functions for the marginal cost, $s^0 : \boldsymbol{S} \rightarrow \mathcal{R}$, and inflation, $\pi^0 : \boldsymbol{S} \rightarrow \mathcal{R}$. The log-linearized system of equations is given by

$$
\begin{aligned}
0 =& \hat{k}_t - \hat{n}_t + \frac{a^L \beta n \phi_g}{k} \hat{\phi}^g_t, \\
0 =& \hat{n}_t - \frac{s\sigma k^\sigma}{n} \hat{s}_t + \frac{\beta R^n \phi_g}{\pi} \hat{\pi}_t - \frac{s\sigma^2 k^\sigma}{n} \hat{k}_{t-1} - \frac{\beta R^n \phi_g}{\pi} \hat{x}_{t-1} \\
0 =& \hat{R}^n_t - E_t \hat{\pi}_{t+1} - \frac{1}{1 - \sigma s k^{\sigma-1} \frac{(1-\theta) a^L a^H h \beta}{a^L - \theta a^H}} \left(E_t \hat{n}_{t+1} - \hat{n}_t\right) \\
&+ \frac{\sigma s k^{\sigma-1} \frac{(1-\theta) a^L a^H h \beta}{a^L - \theta a^H}}{1 - \sigma s k^{\sigma-1} \frac{(1-\theta) a^L a^H h \beta}{a^L - \theta a^H}} \left(E_t \hat{s}_{t+1} - (1-\sigma) \hat{k}_t\right), \\
0 =& -\pi_t + \frac{1}{\gamma} \frac{\lambda}{\lambda - 1} \hat{s}_t + \beta E_t \hat{\pi}_{t+1}, \\
0 =& -\hat{R}^n_t + \psi_\pi \hat{\pi}_t + \upsilon_t
\end{aligned}
$$

(ii) For each grid point on $\boldsymbol{S}$, take s_t and π_t as given. Compute n_t, R^n_t and ϕ^g_t from equations (C2), (C5) with $\upsilon_t = 0$ and $\log\left(\phi^g_t/\phi^g\right) = \rho_g \log\left(\phi^g_{t-1}/\phi^g\right) + \epsilon_{g,t}$, respectively. Compute k_t and c^w_t from equations (C1) and (C6), respectively. Compute x_t, y_t and y_{t+1} as $x_t = R^n_t \phi^g_t n_t$, $y_t = k^\sigma_{t-1}$ and $y_{t+1} = k^\sigma_t$.

(iii) With k_t, x_t and ϕ^g_t in hand for each grid point on $\boldsymbol{S}$, compute a vector of π_{t+1} and s_{t+1} using the policy functions and linear interpolation for all possible shocks in the next period $\epsilon_{g,t+1} \in \boldsymbol{\epsilon}$. Similarly, compute a vector of n_{t+1} and ϕ^g_{t+1} from equations (C2) and $\log\left(\phi^g_{t+1}/\phi^g\right) = \rho_g \log\left(\phi^g_t/\phi^g\right) + \epsilon_{g,t+1}$. Use k_t, π_{t+1}, ϕ^g_{t+1} and n_{t+1} to compute c^w_{t+1}.

(iv) Compute the time-t expected terms in equations (C3) and (C4) using the Gauss-Hermite quadrature.

(v) For each point on $\boldsymbol{S}$, adjust s_t and π_t so that equations (C3) and (C4) hold. The solution yields new policy functions $s'(\cdot)$ and $\pi'(\cdot)$.

(vi) If $\left|s^t(\cdot) - s'(\cdot)\right| < c$ and $\left|\pi^t(\cdot) - \pi'(\cdot)\right| < c$ for some convergence criterion c, stop. The policy functions $s'(\cdot)$ and $\pi'(\cdot)$ are a solution. Otherwise, update $s^{t+1}(\cdot)$ and $\pi^{t+1}(\cdot)$ as $s^{t+1}(\cdot) = \rho_c s^t(\cdot) + (1-\rho_c) s'(\cdot)$ and $\pi^{t+1}(\cdot) = \rho_c \pi^t(\cdot) + (1-\rho_c) \pi'(\cdot)$ for some $0 \le \rho_c < 1$, and go back to step 2.

C.2 The small bubble economy

Global solution. The system of equations for the small bubble economy in which both the L-types and H-types invest is given by

$$k_t = \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - \phi_t^b - \phi_t^g\right] a^L \beta n_t, \tag{C7}$$

$$n_t = \frac{s_t \sigma k_{t-1}^{\sigma} + \beta R_{t-1}^n \phi_{t-1}^g n_{t-1}/\pi_t}{1 - \phi_t^b \beta}, \tag{C8}$$

$$1 = E_t \left[\frac{\beta(1-h)}{\frac{n_{t+1}}{n_t} - \sigma s_{t+1} k_t^{\sigma-1} \frac{(1-\theta)a^L a^H h\beta}{a^L - \theta a^H}} \frac{R_t^n}{\pi_{t+1}}\right], \tag{C9}$$

$$(\pi_t - 1)\pi_t = \frac{1}{\gamma}\frac{\lambda}{\lambda - 1}(s_t - 1) + E_t M_{t,t+1}^w \frac{y_{t+1}}{y_t}(\pi_{t+1} - 1)\pi_{t+1}, \tag{C10}$$

$$\log(R_t^n) = \log(\bar{R}^n) + \psi_\pi \log(\pi_t) + \upsilon_t, \tag{C11}$$

$$1 = v\beta(1-h)E_t \left[\frac{\phi_{t+1}^b/\phi_t^b}{1 - \sigma s_{t+1} k_t^{\sigma-1} \frac{(1-\theta)a^L a^H h\beta}{a^L - \theta a^H} \frac{n_t}{n_{t+1}}}\right] \tag{C12}$$

where $M_{t,t+1}^w = \beta c_t^w / c_{t+1}^w$, $y_t = k_{t-1}^\sigma$ and c_t^w is given by

$$c_t^w = k_{t-1}^\sigma \left[1 - \frac{\gamma}{2}(\pi_t - 1)^2\right] - \left(1 - \beta\phi_t^b - \beta\phi_t^g\right) n_t, \tag{C13}$$

and υ_t and ϕ_t^g follow the AR(1) processes with their shocks given by $\epsilon_{mp,t}$ and $\epsilon_{g,t}$, respectively. We assume that the constant term $\bar{R}^n$ in the monetary policy rule (C11) is the real interest rate in the stochastic bubble steady state so that net inflation is stabilized at zero in the stochastic bubble steady state.

For expository purposes, we consider a case of shock $\epsilon_{g,t}$ only. Define $x_{t-1} \equiv R_{t-1}^n \phi_{t-1}^g n_{t-1}$. We discretize the state space as $k_{t-1} \in \boldsymbol{K} \equiv [K_1, ...K_{n_K}]$, $x_{t-1} \in \boldsymbol{X} \equiv [X_1, ..., X_{n_X}]$, $\phi_{t-1}^g \in \boldsymbol{\Phi} \equiv \left[\Phi_1, ..., \Phi_{n_\phi}\right]$ and $\epsilon_{g,t} \in \boldsymbol{\epsilon} \equiv \left[\epsilon_1, ..., \epsilon_{n_g}\right]$. Let $\boldsymbol{S} \equiv \boldsymbol{K} \times \boldsymbol{X} \times \boldsymbol{\Phi} \times \boldsymbol{\epsilon}$ denote the discretized state space.

Let π_{t+1}^*, s_{t+1}^*, n_{t+1}^* and c_{t+1}^{w*} denote inflation, the marginal cost, the net worth and worker consumption when the bubble bursts. These variables are given by

$$\begin{aligned}
\pi_{t+1}^* &= \pi^*\left(k_t, x_t, \phi_t^g, \epsilon_{g,t+1}\right), \\
s_{t+1}^* &= s^*\left(k_t, x_t, \phi_t^g, \epsilon_{g,t+1}\right), \\
n_{t+1}^* &= s_{t+1}^* \sigma k_t^\sigma + \beta R_t^n \phi_t^g n_t / \pi_{t+1}^*, \\
c_{t+1}^{w*} &= k_t^\sigma \left[1 - \frac{\gamma}{2}(\pi_{t+1}^* - 1)^2\right] - \left(1 - \beta\phi_{t+1}^g\right) n_{t+1}^*,
\end{aligned}$$

where $\pi^*(\cdot)$ and $s^*(\cdot)$ are the policy functions for inflation and the marginal cost in the bubbleless economy, respectively, derived in Section C.1. With these variables in hand, equations (C9) and (C10) are written as

$$1 = E_t \left[\frac{v\beta(1-h)}{\frac{n_{t+1}}{n_t} - \sigma s_{t+1} k_t^{\sigma-1} \frac{(1-\theta)a^L a^H h\beta}{a^L - \theta a^H}} \frac{R_t^n}{\pi_{t+1}} + \frac{(1-v)\,\beta(1-h)}{\frac{n_{t+1}^*}{n_t} - \sigma s_{t+1}^* k_t^{\sigma-1} \frac{(1-\theta)a^L a^H h\beta}{a^L - \theta a^H}} \frac{R_t^n}{\pi_{t+1}^*} \right], \tag{C14}$$

$$(\pi_t - 1)\pi_t = \frac{1}{\gamma}\frac{\lambda}{\lambda-1}(s_t - 1) \tag{C15}$$

$$+ E_t \left[v\beta \frac{c_t^w}{c_{t+1}^w} \frac{k_t^\sigma}{k_{t-1}^\sigma} (\pi_{t+1} - 1)\pi_{t+1} + (1-v)\,\beta \frac{c_t^w}{c_{t+1}^{w*}} \frac{k_t^\sigma}{k_{t-1}^\sigma} (\pi_{t+1}^* - 1)\pi_{t+1}^* \right].$$

The model is solved in six steps.

(i) Set an initial guess of policy functions for the marginal cost, $s^0 : \boldsymbol{S} \to \mathcal{R}$, inflation, $\pi^0 : \boldsymbol{S} \to \mathcal{R}$, and the bubble, $\phi^{b0} : \boldsymbol{S} \to \mathcal{R}$. We set s^0 and π^0 as a solution to the linearized model of the bubble-less economy, approximated around the stochastic bubble steady state. We set $\phi^{b0}(\cdot) = \phi^b$ for all states of the economy, where ϕ^b is the bubble in the stochastic bubble steady state.

(ii) For each grid point on $\boldsymbol{S}$, take s_t, π_t and ϕ_t^b as given. Compute n_t, R_t^n and ϕ_t^g from equations (C8), (C11) with $v_t = 0$ and $\log(\phi_t^g/\phi^g) = \rho_g \log\left(\phi_{t-1}^g/\phi^g\right) + \epsilon_{g,t}$, respectively. Compute k_t and c_t^w from equations (C7) and (C13), respectively. Compute x_t, y_t and y_{t+1} as $x_t = R_t^n \phi_t^g n_t$, $y_t = k_{t-1}^\sigma$ and $y_{t+1} = k_t^\sigma$.

(iii) With k_t, x_t and ϕ_t^g in hand for each grid point on $\boldsymbol{S}$, compute a vector of π_{t+1} and s_{t+1} using the policy functions and linear interpolation for all possible shocks in the next period $\epsilon_{g,t+1} \in \boldsymbol{\epsilon}$. Similarly, compute a vector of n_{t+1} and ϕ_{t+1}^g from equations (C8) and $\log\left(\phi_{t+1}^g/\phi^g\right) = \rho_g \log(\phi_t^g/\phi^g) + \epsilon_{g,t+1}$. Use k_t, π_{t+1}, ϕ_{t+1}^g and n_{t+1} to compute c_{t+1}^w. Similarly, for all possible shocks in the next period $\epsilon_{g,t+1} \in \boldsymbol{\epsilon}$, compute a vector of π_{t+1}^*, s_{t+1}^*, n_{t+1}^* and c_{t+1}^{w*}.

(iv) Compute the time-t expected terms in equations (C12), (C14) and (C15) using the Gauss-Hermite quadrature.

(v) For each point on $\boldsymbol{S}$, adjust s_t, π_t and ϕ_t^b so that equations (C12), (C14) and (C15) hold. The solution yields new policy functions $s'(\cdot)$, $\pi'(\cdot)$ and $\phi^{b\prime}(\cdot)$.

(vi) If $\left|s^t(\cdot) - s'(\cdot)\right| < c$, $\left|\pi^t(\cdot) - \pi^*(\cdot)\right| < c$ and $\left|\phi^{b,t}(\cdot) - \phi^{b\prime}(\cdot)\right| < c$ for some convergence criterion c, stop. The policy functions $s'(\cdot)$, $\pi'(\cdot)$ and $\phi^{b\prime}(\cdot)$ are a solution. Otherwise, update $s^{t+1}(\cdot)$, $\pi^{t+1}(\cdot)$ and $\phi^{b,t+1}$ as $s^{t+1}(\cdot) = \rho_c s^t(\cdot) + (1-\rho_c)\, s'(\cdot)$, $\pi^{t+1}(\cdot) = \rho_c \pi^t(\cdot) +$

$(1-\rho_c)\,\pi'(\cdot)$ and $\phi^{b,t+1}(\cdot) = \rho_c \phi^{b,t}(\cdot) + (1-\rho_c)\,\phi^{b\prime}(\cdot)$ for some $0 \leq \rho_c < 1$, and go back to step 2.

Stochastic steady state. We consider a case in which the constant term $\bar{R}^n$ in the monetary policy rule is set in such a way that net inflation is stabilized at zero, i.e. $\pi = 1$. Then, the above system of equations suggests that the stochastic steady state $\left\{k, n, s, R^n, \phi^b\right\}$ is given by

$$k = \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h - \phi^b - \phi^g\right] a^L \beta n, \tag{C16}$$

$$n = \left(1 - \frac{\beta R^n \phi^g}{1 - \phi^b \beta}\right)^{-1} \frac{s \sigma k^\sigma}{1 - \phi^b \beta}, \tag{C17}$$

$$\frac{1}{R^n} = 1 + \frac{(1-v)\,\beta(1-h)}{\frac{n^*}{n} - \sigma s^* k^{\sigma-1} \frac{(1-\theta) a^L a^H h \beta}{a^L - \theta a^H}} \frac{1}{\pi^*}, \tag{C18}$$

$$0 = \frac{1}{\gamma} \frac{\lambda}{\lambda - 1}(s-1) + \beta\,(1-v)\,\frac{c^w}{c^{w*}}(\pi^* - 1)\pi^*, \tag{C19}$$

$$\sigma a^L s k^{\sigma-1} = \frac{\left[1 - v\beta(1-h)\right]\left(a^L - \theta a^H\right)}{(1-\theta) a^H h \beta}, \tag{C20}$$

where

$$c^w = k^\sigma - \left(1 - \beta\phi^b - \beta\phi^g\right) n.$$

The variables in the bubble-less economy, π^*, s^*, n^* and c^{w*}, are given by

$$\begin{aligned}
\pi^* &= \pi^*\left(k, x, \phi^g, 0\right), \\
s^* &= s^*\left(k, x, \phi^g, 0\right), \\
n^* &= s^* \sigma k^\sigma + \beta R^n \phi^g n / \pi^*, \\
c^{w*} &= k^\sigma \left[1 - \frac{\gamma}{2}(\pi^* - 1)^2\right] - \left(1 - \beta\phi^g\right) n^*,
\end{aligned}$$

where $x \equiv R^n \phi^g n$ and $\pi^*(\cdot)$ and $s^*(\cdot)$ are the policy functions for inflation and the marginal cost in the bubble-less economy, respectively, derived in Section C.1.

The system (C16)-(C20) is solved for $\left\{k, n, s, R^n, \phi^b\right\}$ as follows. First, fix R^n. Combining (C16), (C17) and (C20) leads to

$$\phi^b = \frac{1 + \beta\left(R - R^n\right)\phi^g - \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right) h\right] \beta R}{\beta\left(1 - R\right)},$$

where $R = \sigma a^L s k^{\sigma-1} = \frac{[1-v\beta(1-h)]\left(a^L-\theta a^H\right)}{(1-\theta)a^H h\beta}$ is the real interest rate in the stochastic steady state. Now fix s. From equation (C20), k is given by

$$k = \left\{\frac{1}{\sigma a^L s}\frac{[1-v\beta(1-h)]\left(a^L-\theta a^H\right)}{(1-\theta)a^H h\beta}\right\}^{\frac{1}{\sigma-1}}.$$

Equation (C17) determines n. Calculate x as $x \equiv R^n \phi^g n$. With k, x and ϕ^g in hand, calculate π^*, s^*, n^* and c^{w*}. Adjust s until equation (C19) holds. Finally, adjust R^n until equation (C18) holds.

C.3 The large bubble economy

Global solution. The system of equations for the large bubble economy in which only the H-types invest is given by (C8), (C10), (C11) and

$$k_t = a^H\left(1-\phi_t^b-\phi_t^g\right)\beta n_t, \tag{C21}$$

$$1 = E_t\left[\frac{\beta(1-h)}{\frac{n_{t+1}}{n_t}-\sigma s_{t+1}k_t^{\sigma-1}(1-\theta)a^H\left(1-\phi_t^b-\phi_t^g\right)\beta}\frac{R_t^n}{\pi_{t+1}}\right], \tag{C22}$$

$$1 = v\beta(1-h)E_t\left[\frac{\phi_{t+1}^b/\phi_t^b}{1-\sigma s_{t+1}k_t^{\sigma-1}(1-\theta)a^H\left(1-\phi_t^b-\phi_t^g\right)\beta\frac{n_t}{n_{t+1}}}\right] \tag{C23}$$

where $M_{t,t+1}^w = \beta c_t^w/c_{t+1}^w$, $y_t = k_{t-1}^\sigma$ and c_t^w is given by (C13), and υ_t and ϕ_t^g follow the AR(1) processes with their shocks given by $\epsilon_{mp,t}$ and $\epsilon_{g,t}$, respectively. As in the small bubble economy, we assume that the constant term $\bar{R}^n$ in the monetary policy rule (C11) is the real interest rate in the stochastic bubble steady state so that net inflation is stabilized at zero in the stochastic bubble steady state.

As in the small bubble economy, here we consider a case of shock $\epsilon_{g,t}$ only. Showing the probability of bubble burst explicitly, the Phillips curve (C10) is written as (C15) and equation (C22) is written as

$$\begin{aligned}1 = E_t\Bigg[&\frac{v\beta(1-h)}{\frac{n_{t+1}}{n_t}-\sigma s_{t+1}k_t^{\sigma-1}(1-\theta)a^H\left(1-\phi_t^b-\phi_t^g\right)\beta}\frac{R_t^n}{\pi_{t+1}}\\ &+\frac{(1-v)\beta(1-h)}{\frac{n_{t+1}^*}{n_t}-\sigma s_{t+1}^*k_t^{\sigma-1}(1-\theta)a^H\left(1-\phi_t^b-\phi_t^g\right)\beta}\frac{R_t^n}{\pi_{t+1}^*}\Bigg],\end{aligned} \tag{C24}$$

The model is solved in six steps.

(i) Set an initial guess of policy functions for the marginal cost, $s^0 : \boldsymbol{S} \to \mathcal{R}$, inflation, $\pi^0 : \boldsymbol{S} \to \mathcal{R}$, and the bubble, $\phi^{b0} : \boldsymbol{S} \to \mathcal{R}$. We set s^0 and π^0 as a solution to the linearized model of the bubble-less economy, approximated around the stochastic bubble steady state. We set $\phi^{b0}(\cdot) = \phi^b$ for all states of the economy, where ϕ^b is the bubble in the stochastic bubble steady state.

(ii) For each grid point on $\boldsymbol{S}$, take s_t, π_t and ϕ_t^b as given. Compute n_t, R_t^n and ϕ_t^g from equations (C8), (C11) with $v_t = 0$ and $\log(\phi_t^g/\phi^g) = \rho_g \log\left(\phi_{t-1}^g/\phi^g\right) + \epsilon_{g,t}$, respectively. Compute k_t and c_t^w from equations (C21) and (C13), respectively. Compute x_t, y_t and y_{t+1} as $x_t = R_t^n \phi_t^g n_t$, $y_t = k_{t-1}^\sigma$ and $y_{t+1} = k_t^\sigma$.

(iii) With k_t, x_t and ϕ_t^g in hand for each grid point on $\boldsymbol{S}$, compute a vector of π_{t+1} and s_{t+1} using the policy functions and linear interpolation for all possible shocks in the next period $\epsilon_{g,t+1} \in \boldsymbol{\epsilon}$. Similarly, compute a vector of n_{t+1} and ϕ_{t+1}^g from equations (C8) and $\log\left(\phi_{t+1}^g/\phi^g\right) = \rho_g \log(\phi_t^g/\phi^g) + \epsilon_{g,t+1}$. Use k_t, π_{t+1}, ϕ_{t+1}^g and n_{t+1} to compute c_{t+1}^w. Similarly, for all possible shocks in the next period $\epsilon_{g,t+1} \in \boldsymbol{\epsilon}$, compute a vector of π_{t+1}^*, s_{t+1}^*, n_{t+1}^* and c_{t+1}^{w*}.

(iv) Compute the time-t expected terms in equations (C15), (C23) and (C24) using the Gauss-Hermite quadrature.

(v) For each point on $\boldsymbol{S}$, adjust s_t, π_t and ϕ_t^b so that equations (C15), (C23) and (C24) hold. The solution yields new policy functions $s'(\cdot)$, $\pi'(\cdot)$ and $\phi^{b\prime}(\cdot)$.

(vi) If $\left|s^t(\cdot) - s'(\cdot)\right| < c$, $\left|\pi^t(\cdot) - \pi^*(\cdot)\right| < c$ and $\left|\phi^{b,t}(\cdot) - \phi^{b\prime}(\cdot)\right| < c$ for some convergence criterion c, stop. The policy functions $s'(\cdot)$, $\pi'(\cdot)$ and $\phi^{b\prime}(\cdot)$ are a solution. Otherwise, update $s^{t+1}(\cdot)$, $\pi^{t+1}(\cdot)$ and $\phi^{b,t+1}$ as $s^{t+1}(\cdot) = \rho_c s^t(\cdot) + (1-\rho_c) s'(\cdot)$, $\pi^{t+1}(\cdot) = \rho_c \pi^t(\cdot) + (1-\rho_c)\pi'(\cdot)$ and $\phi^{b,t+1}(\cdot) = \rho_c \phi^{b,t}(\cdot) + (1-\rho_c)\phi^{b\prime}(\cdot)$ for some $0 \le \rho_c < 1$, and go back to step 2.

Stochastic steady state. As in the small bubble economy, we consider a case in which the constant term $\bar{R}^n$ in the monetary policy rule is set in such a way that net inflation is stabilized at zero, i.e. $\pi = 1$. Then, the above system of equations suggests that the stochastic steady state

$\{k, n, s, R^n, \phi^b\}$ is given by (C17), (C19) and

$$k = a^H \left(1 - \phi^b - \phi^g\right) \beta n, \tag{C25}$$

$$\frac{1}{R^n} = 1 + \frac{(1-v)\beta(1-h)}{\frac{n^*}{n} - \sigma s^* k^{\sigma-1}(1-\theta)a^H\left(1-\phi^b-\phi^g\right)\beta}\frac{1}{\pi^*}, \tag{C26}$$

$$\sigma s k^{\sigma-1} = \frac{1 - v\beta(1-h)}{(1-\theta)a^H\left(1-\phi^b-\phi^g\right)\beta} \tag{C27}$$

where

$$c^w = k^\sigma - \left(1 - \beta\phi^b - \beta\phi^g\right) n.$$

The variables in the bubble-less economy, π^*, s^*, n^* and c^{w*}, are given similarly as in the small bubble economy.

The system (C17), (C19) and (C25)-(C27) is solved for $\{k, n, s, R^n, \phi^b\}$ as follows. First, fix R^n. Combining (C17), (C25) and (C27) leads to

$$\phi^b = \frac{v\beta(1-h) - \theta}{\beta(1-\theta)} - R^n\phi^g$$

Now fix s. From equation (C27), k is given by

$$k = \left[\frac{1}{\sigma s}\frac{1 - v\beta(1-h)}{(1-\theta)a^H\left(1-\phi^b-\phi^g\right)\beta}\right]^{\frac{1}{\sigma-1}}.$$

Equation (C17) determines n. Calculate x as $x \equiv R^n\phi^g n$. With k, x and ϕ^g in hand, calculate π^*, s^*, n^* and c^{w*}. Adjust s until equation (C19) holds. Finally, adjust R^n until equation (C26) holds.

C.4 The small bubble economy in the flexible-price model

The system of equations for this economy consists of

$$k_t = \left[1 + \left(\frac{a^H - a^L}{a^L - \theta a^H}\right)h - \phi_t^b - \phi_t^g\right] a^L\beta n_t,$$

$$n_t = \frac{\sigma k_{t-1}^\sigma + \beta R_t\phi_{t-1}^g n_{t-1}}{1 - \phi_t^b\beta},$$

$$1 = v\beta(1-h)E_t\left[\frac{\phi_{t+1}^b/\phi_t^b}{1 - \sigma k_t^{\sigma-1}\frac{(1-\theta)a^La^Hh\beta}{a^L-\theta a^H}\frac{n_t}{n_{t+1}}}\right],$$

where the real interest rate is given by $R_t = a_L\sigma k_{t-1}^{\sigma-1}$. The system is solved similarly to the previous models. Define $x_t \equiv \phi_t^g n_t$. We discretize the state space as $k_{t-1} \in K \equiv [K_1, ...K_{n_K}]$,

$x_{t-1} \in X \equiv [X_1, ..., X_{n_X}]$, $\phi^g_{t-1} \in \Phi \equiv \left[\Phi_1, ..., \Phi_{n_\phi}\right]$ and $\epsilon_{g,t} \in \epsilon \equiv \left[\epsilon_1, ..., \epsilon_{n_g}\right]$, and define $S \equiv K \times X \times \Phi \times \epsilon$ as the discretized state space. In solving the system globally, what is needed is a policy function for ϕ^b_t. Given this policy function, and for each grid point, the first two equations of the above system can be solved for n_t, k_t and R_t, and ϕ^g_t is set to satisfy the third equation of the above system.

C.5 The large bubble economy in the flexible-price model

The system of equations for this economy consists of

$$
\begin{aligned}
k_t &= a^H \left(1 - \phi^b_t - \phi^g_t\right) \beta n_t, , \\
n_t &= \frac{\sigma k^\sigma_{t-1} + \beta R_t \phi^g_{t-1} n_{t-1}}{1 - \phi^b_t \beta}, \\
1 &= v\beta(1-h)E_t \left[\frac{\phi^b_{t+1}/\phi^b_t}{1 - \sigma k^{\sigma-1}_t (1-\theta) a^H \left(1 - \phi^b_t - \phi^g_t\right) \beta \frac{n_t}{n_{t+1}}}\right],
\end{aligned}
$$

where the real interest rate is given by $R_t = \theta a^H \left(1 - \frac{h}{1-\phi^b_t-\phi^g_t}\right)^{-1} \sigma k^{\sigma-1}_{t-1}$. The system is solved similarly to the small bubble economy in the flexible-price model.