Sequential Auctions with Synergy and Affiliation

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Abstract

This paper studies sequential auctions with synergy in which each bidder's values can be affiliated across auctions, and empirically assesses the revenue effects of bundling. Ignoring affiliation can lead to falsely detecting synergy where none exists. Motivated by data on synergistic pairs of oil and gas lease auctions, where the same winner often wins both tracts, I model a sequence in which a first-price auction is followed by an English auction. At the first auction, bidders know their first value and the distribution of their second value conditional on the first value. At the second auction, bidders learn their second value, which is affiliated with their first value and also affected by potential synergy if they won the first auction. Both synergy and affiliation take general functional forms. I establish nonparametric identification of the joint distribution of values, synergy function, and risk aversion parameter from observed bids in the two auctions. Intuitively, the effect of synergy is isolated by comparing the second-auction behavior of a first-auction winner and first-auction loser who bid the same amount in the first auction. Using the identification results, I develop a nonparametric estimation procedure for the model, assess its finite sample properties using Monte Carlo simulations, and apply it to the oil and gas lease data. I find both synergy and affiliation between adjacent tracts, though affiliation is primarily responsible for the observed allocation patterns. Bidders are risk averse. Counterfactual simulations reveal that bundled auctions would yield higher revenue, with a small loss to allocative efficiency.

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1 Introduction

Consider two synergistic (or complementary) objects being auctioned in sequence by a government agency. Synergy here refers to the value of two objects together being greater than the sum of the individual values, or superadditive. If a single bidder wins both objects, he benefits from this synergy; if different bidders win each object, each winner obtains the stand-alone value of the object he wins. Thus, the presence of synergy across the sequence creates a dynamic problem for bidders as they decide how much to bid. Meanwhile, it also raises policy questions for the government; for instance, an easy-to-implement policy alternative to sequential auctions would be to bundle the two objects and auction the bundle.

The revenue and efficiency effects of policy alternatives are not obvious and need to be assessed empirically. The aforementioned alternative, bundling, awards both objects to a single bidder, ensuring that synergy is realized. But the flip side is that it forces a single bidder to take both tracts, eliminating the possibility of awarding each tract individually to the highest paying bidder. So the cost or benefit of bundling depends on, among other things, how large the synergy is. Another alternative, the Vickrey-Clarke-Groves auction (a type of combinatorial auction), guarantees an efficient allocation, but may underperform in terms of revenue compared to sequential first-price auctions.[1] Finally, the government would want to weigh the revenue gains of a policy against the efficiency losses, or vice versa, which requires some estimate of the size of each.

To address this policy question, this paper proposes a structural analysis of sequential auctions with synergy and empirically assesses the revenue and efficiency effects of bundling. In particular, the model allows each bidder's values to be flexibly affiliated across auctions. This affiliation is motivated by data on auctions of adjacent oil and gas leases, which are neither independent objects nor homogeneous goods.

In the oil and gas lease auctions run by the New Mexico State Land Office, it is sometimes the case that two adjacent halves of a square mile are auctioned on the same day. By convention, one object is sold by a first-price sealed-bid auction, and the other object is sold later (but on the same day) using an English auction. I observe that the same bidder often wins both tracts. It is possible that the two auctions are linked by synergy, for two reasons. First, as noted in Sunnevåg (2000), equipment and crews will already be nearby, reducing the cost of moving them between disparate locations and possibly eliminating duplicates. Second, in recent years much of the drilling in New Mexico has been horizontal; with permission from government authorities, adjacent tracts can be put together to form a "project area" where horizontal wells can be drilled across lease borders. For these reasons, there may be extra value to winning two adjacent tracts beyond the sum of one's values for each tract

[1] The Vickrey-Clarke-Groves auction, while studied much in theory, remains largely unused in practice.

individually.

Meanwhile, since the tracts in a pair are two adjacent halves of a square mile, a bidder's values for the two are likely to be affiliated, even after conditioning on covariates and regardless of whether there is any synergy. For instance, a firm may like certain geological formations because its engineers are especially skilled in drilling that type of geology. If these geological features are geographically clustered, the firm's values for adjacent tracts will be affiliated. This is a concept distinct from synergy, as it concerns the correlation of values for individual tracts and has nothing to say about how they sum. Affiliation of this kind is likely to coexist with synergy in other contexts as well, as synergy often emerges from some sort of adjacency, which is conducive to affiliation.

Synergy and affiliation are observationally similar; in both cases, the winner of the first auction is more likely to win the second. As a result, misspecified models that allow synergy but ignore affiliation (or allow affiliation but ignore synergy) are likely to attribute the observed effects of affiliation to synergy (and vice versa). In light of the policy question we hope to address, this is problematic since synergy and affiliation have different implications for auction revenue and allocative efficiency. As such, relaxing independence of values across auctions is especially meaningful when estimating auction models with synergy.

In light of the data, I model a sequence in which a first-price auction of one tract is followed by an English auction of the adjacent tract, under the private value paradigm.[2] The timeline of the model is as follows. When bidders bid in the first auction, they know their value for the first tract. Meanwhile, they have some uncertainty about what their value will be in the second auction that happens later. This is because there is noise between the two auctions – in the New Mexico data, there are other auctions taking place in between – that can affect bidder values. So bidders do not know their second value exactly at the first auction, but they do know the distribution from which their second value will be drawn. To allow for affiliation, that distribution is conditional on their value for the first tract. I place few restrictions on this conditional distribution, allowing a very flexible relationship between a bidder's values for the first and second tract. Bidders do learn their exact value for the second tract at the beginning of the second auction. This timing, motivated by the data, helps the model retain tractability while being flexible.

The bidder that won the first auction benefits from synergy, so his ultimate value in the second auction is not just the stand-alone value of the second tract, but the synergy-inclusive value. I define a synergy function that gives this synergy-inclusive value as a function of a bidder's stand-alone values for each tract. Since the stand-alone values are idiosyncratic to each bidder, the size of synergy, being a function of the two, is also idiosyncratic and is private information to each bidder. The synergy function takes a general functional form.

[2]Reasons for the private value paradigm are discussed in Section 3.

To characterize equilibrium bidding, I start with the second auction and work backwards. The second auction is an English auction, where it is a dominant strategy for bidders to bid their value for the second tract; so a bidder who lost the first auction would bid his stand-alone value for the second tract, and a bidder who won the first auction would bid his synergy-inclusive value for the second tract. Then in the first auction, bidders bid in light of not only their value for the first tract, but also the expected benefit in the second auction from winning the first auction. Under some assumptions, I show that bids in the first auction are strictly increasing in a bidder's value for the first tract, and that there exists a unique Bayes-Nash equilibrium for bidding in the first auction.

I establish nonparametric identification of the model primitives from observable data. I emphasize that in doing so, I separately identify synergy and affiliation. The primitives are the joint distribution of first-auction and second-auction values and the synergy function, while the observable data include all bids in the first auction, the final price in the second auction, and bidder identities. The identification argument proceeds in multiple steps, beginning with the second auction and working backwards. First, I identify the distribution of a bidder's values in the second auction, conditional on his first-auction bid and whether he won the first auction. Next, the synergy function is identified by comparing the second-auction value distributions of a first-auction winner and first-auction loser *conditional on the same first-auction bid*. This conditioning on the first-auction bid neutralizes affiliation and allows me to isolate the effect of synergy, since the first-auction winner benefits from synergy while the first-auction loser does not. Finally, first-auction values are identified using the first-order condition for bidding in the first auction. To be more precise, the first-order condition can be rewritten as an inverse bid function that expresses a bidder's first-auction value as a function of his first-auction bid, the observed bid distribution, and an additional term representing the added benefit in the second auction from winning the first auction. This additional term is a function of the second-auction value distributions and the synergy function, which were identified in the previous two steps. So I can back out the first-auction values using this inverse bid function.

Closely following the identification steps, I develop a nonparametric multi-step estimation procedure that recovers the structural parameters of the auction model. It begins with a sieve maximum likelihood estimator to estimate bidders' value distributions in the second auction. For the remaining primitives, which are the synergy function and first-auction value distribution, my identification argument is constructive, so the estimation procedure follows the identification argument step-by-step. I assess the finite sample performance of this estimation procedure in a Monte Carlo study.

When I apply the estimation procedure to the New Mexico data, I find both synergy and affiliation between adjacent tracts, though affiliation is primarily responsible for the observed pattern in which the same bidder often wins both tracts. This result highlights the importance of allowing for affiliation across auctions. Also, I allow bidders to be risk averse when I estimate the model, and find that they are risk averse. Counterfactual simulations using the estimated structural parameters reveal that bundled auctions would yield higher auction revenue, accompanied by a small loss to allocative efficiency.

The paper contributes to the literature by analyzing sequential auctions of affiliated objects linked by synergy, and distinguishing synergy and affiliation in the process. While the model and estimation procedure of this paper are tailored to the empirical application at hand, the main insight behind disentangling synergy from affiliation is adaptable to other contexts as long as all bids in the first auction are monotonic in values and observed.

Oil and gas lease auctions are widespread; they are used by many oil and gas producing states, the U.S. federal government, and governments of other countries. In terms of broader relevance, sequential auctions with synergy are not limited to oil and gas leases. The Israeli cable TV licenses described in Gandal (1997), the construction contracts studied by De Silva, Jeitschko, and Kosmopoulou (2005), and the milk contracts sold by Georgia school districts (Marshall et al. (2006))[3] are some other examples where objects with potential synergy have been auctioned sequentially. More generally, examples of synergistic objects sold via auction abound: geographically contiguous PCS licenses or adjacent bands of spectrum (Ausubel et al. (1997), Cramton (1997)), electricity generation in adjacent time periods (Wolfram (1998)), agri-environmental contracts (Saïd and Thoyer (2007)), and long-haul truckloads (Triki et al. (2014)) fall into this category. Also, when competing localities pay recruitment subsidies to firms, there are benefits from agglomeration if multiple firms form an industrial cluster in the same area (Martin (1999)).

The paper is organized as follows. The remainder of Section 1 provides an overview of the related literature. Section 2 describes the data and empirical evidence. Section 3 develops a model of sequential auctions with synergy. Section 4 establishes nonparametric identification of the model. Section 5 develops an estimation procedure and discusses a Monte Carlo study assessing finite sample performance. Section 6 describes estimation details specific to the data at hand, and discusses the estimation results. Section 7 performs counterfactual simulations of interest including those for bundled auctions. Section 8 concludes. The appendix collects all proofs.

Related literature

This paper is preceded by the empirical literature on sequential auctions, which begins with Ashenfelter (1989)'s study of wine auctions, and includes among others Gandal (1997) and De Silva et al. (2005), whose regression analyses find evidence of synergy in Israeli cable

[3] Marshall et al. (2006) model the Georgia milk auctions as simultaneous.

TV license auctions and Oklahoma DOT construction auctions, respectively. Within that literature, this paper is most closely related to the structural econometric work that starts with Jofre-Bonet and Pesendorfer (2003). That literature has mostly focused on sequential auctions of independent objects linked by bidder dynamics, or homogeneous goods with decreasing marginal values.

In the former category, Jofre-Bonet and Pesendorfer (2003) estimate an infinite horizon model of first-price procurement auctions, where capacity constraints generate dynamics across the sequence. The construction contracts being auctioned are otherwise independent, so a firm's cost draws across auctions are also independent conditional on remaining capacity. Balat (2013) builds on the model of Jofre-Bonet and Pesendorfer (2003) to include auction-level unobserved heterogeneity and endogenous participation. He finds that the accelerated release of procurement projects under the American Recovery and Reinvestment Act increased procurement prices by increasing firms' backlogs. As an example with positive synergy, Groeger (2014) estimates a dynamic auction model to measure savings in bid preparation costs that come from having recently prepared bids on contracts of the same type.

In the latter category are papers that study sequential auctions of homogeneous goods like fish and tobacco, where bidders retain the same value in the first and second auction unless they win the first. For the first-auction winner, second-unit value is assumed to be lower than first-unit value, consistent with decreasing marginal values. Donald, Paarsch, and Robert (2006) study sequential English auctions of homogeneous goods, in which a Poisson demand generation process imposes stationarity across auctions in the sequence. Brendstrup and Paarsch (2006) and Brendstrup (2007) study identification and estimation of sequential English auctions using only the last stage of the game, without specifying equilibria for the earlier stages. However, Lamy (2010) finds that identification actually fails in the context of Brendstrup and Paarsch (2006) and Brendstrup (2007). Building on an equilibrium for the whole two-stage game characterized in Lamy (2012), he establishes conditions under which the model is identified, and develops an estimation procedure that uses both stages of the game.

Meanwhile, Donna and Espin-Sanchez (2015) study sequential auctions of identical water units, which fall into a complements regime or a subtitutes regime depending on weather seasonality. Within each regime, a bidder's marginal utility for a second or third unit is a constant multiple of his value for the first unit, and this constant is common to all bidders.

Relative to this literature, this paper addresses sequential auctions with both flexible synergy and flexible affiliation across auctions. This is to accommodate the nature of adjacent oil and gas leases, which do not fit well with existing models: a bidder's values for adjacent tracts are neither independent nor perfectly correlated, and these are not homogeneous goods.

This paper also relates to empirical work on synergy in non-sequential auctions. Ausubel et al. (1997) and others discuss synergy in the simultaneous ascending PCS auctions run by the FCC. Marshall et al. (2006) model and estimate simultaneous first-price auctions with a specific form of synergy in the Georgia school milk market. Gentry, Komarova, and Schiraldi (2015) also study simultaneous first-price auctions with synergy, but take a more general approach, establishing nonparametric identification of the model under certain restrictions. They apply their framework to Michigan highway procurement auctions and find that bidders view small projects as complements but large projects as substitutes. Cantillon and Pesendorfer (2013) study combinatorial first-price auctions of London bus routes, where synergies could exist. They show conditions for nonparametric identification of the combinatorial auction model, and propose a two-stage estimation procedure to recover bidders' costs from bids. Upon applying the procedure, they find evidence of decreasing returns to scale rather than synergy.

The model introduced in this paper is of theoretical interest as well, as it has not been analyzed before. The theory of sequential auctions is more complete for the case of single-unit demand, where bidders demand at most one unit. A number of early papers explore equilibrium price trends when bidders have single-unit demand for identical goods. In particular, Milgrom and Weber (1999) show that the sequence of prices is a martingale under common assumptions, while McAfee and Vincent (1993), Engelbrecht-Wiggans (1994), Jeitschko (1999) and others offer explanations for declining prices. Budish and Zeithammer (2011) study single-unit demand for two non-identical goods. Meanwhile, when it comes to multi-unit demand, equilibrium analysis is challenging and often intractable. As such, most papers restrict their analysis to two auctions and assume either that a bidder's values for the two goods are the same, that all bidders share the same values, that bidders are represented by a single type variable, or that values are independent across auctions and learned one at a time. Examples include Ortega-Reichert (1968), Hausch (1986), and Caillaud and Mezzetti (2004), who study information revelation across a sequence of auctions, as well as Benoit and Krishna (2001) and Pitchik (2009), who study the effect of budget constraints. Exceptions include Katzman (1999) and Lamy (2012). They study sequential second-price auctions of two homogeneous goods with declining marginal values. Values are not independent since they are ordered, and bidders know both values at the start of the sequence. However, Katzman (1999) still restricts to bid functions that depend only on one value, while Lamy (2012) characterizes the set of equilibria more generally. Relative to this literature, the model used in this paper allows for a flexible relationship between values across auctions, but still retains tractability because bidders learn their second value at the second auction.

In the theory of sequential auctions addressing synergy in particular, price trends have been a topic of interest just as in the literature for single-unit demand. Branco (1997), Jeitschko and Wolfstetter (2002), and Menezes and Monteiro (2003) show how prices can decline for identical objects or when values have a two-point support, while Sørensen (2006) show how prices can increase for stochastically equivalent objects. Jofre-Bonet and Pesendorfer (2014) ask whether first-price or second-price auctions achieve lower procurement cost, and find that second-price auctions are better for complements given risk-neutral bidders and independence across auctions. The issue of whether to bundle in the presence of synergy has also been a topic of interest. Grimm (2007) finds that when bidders can subcontract, bundled auctions yield lower procurement costs than sequential auctions. Subramaniam and Venkatesh (2009) analyze a parametric model and suggest that bundling is better than sequential auctions when the number of bidders is low or synergy is strong. Papers that study simultaneous auctions with synergy also shed light on bundling. Levin (1997) finds that when bidders are symmetric and represented by a single type variable, bundling maximizes revenue over other simultaneous mechanisms. Benoit and Krishna (2001), though not principally focused on bundling, provide an example in which bundling decreases auction revenue in the presence of synergy.

2 Data

2.1 Overview

The New Mexico State Trust Lands were granted to New Mexico by Congress under the Ferguson Act of 1898 and the Enabling Act of 1910. In general terms, the state was granted four square miles – sections 2, 16, 32, and 36 – in each 36-section township, as illustrated in Figure 1.[4] As a result, the Trust Lands are not one contiguous piece of land, but a collection of many non-contiguous pieces, often in units of one square mile each. The State Land Office (SLO) administers this land for the beneficiaries of the state land trust, which include schools, universities, hospitals and other public institutions. Its mission statement explicitly references revenue optimization as the core of its goals.[5] In oil and gas producing parts of the land, such as the Permian Basin, the SLO auctions leases for oil and gas development.

While there is some variation, the amount of land most commonly covered by an oil and gas lease is a rectangle of 320 acres, or half a square mile. Therefore, a section, which is a one square mile block, produces two such leases. The SLO prefers this size because it is long enough to allow horizontal drilling.[6] Also, this size is at least as large as the spacing units

[4] http://www.nmstatelands.org/overview-1.aspx

A section is a one-square-mile block of land in the Public Land Survey System.

[5] The SLO's mission statement as stated in its 2015 Annual Report is to "optimize revenues generated from trust lands to support the beneficiaries while ensuring proper land management and restoration to continue the legacy for generations to come."

[6] It is a misconception that oil and gas are migratory underground; there are plenty of formations where

Figure 1: Sections 2, 16, 32, and 36 of a Township

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

required for oil wells (40 acres) and gas wells (320 acres) by state rules.[7] Meanwhile, larger tracts are rarely offered under a single lease. This is because under current rules, leases do not expire as long as some minimal amount of oil and gas production is sustained, and staff are concerned firms might abuse the system by holding on to large areas of land for long periods of time with minimal or less than full development of the tract.

As mentioned above, a section of land produces two adjacent 320-acre leases. Often, these two leases are auctioned on the same day. Typically they have the same lease terms, which include the royalty rate, rental payments,[8] and length of the lease, and are very similar geologically, as they are adjacent halves of a square mile. I will refer to two such leases as a "pair." The focus of study in this paper are pairs that were auctioned in the Permian Basin area during 2000-2014.

The SLO uses two auction formats, the first-price sealed-bid format and the English auction format. When it comes to pairs, the SLO has a convention of selling one of the leases by first-price sealed-bid, and the other lease by English auction later in the day. The English auction always occurs later. Thus the two leases in a pair are auctioned in a sequence. In this paper, I refer to the earlier auction as the "first auction" and the later auction as the "second auction." The SLO employs a fixed and publicly known reserve price of roughly \$15.625 per acre. To be clear, the two leases of a pair are not the only items being auctioned on a given day, nor are they auctioned back to back; in 2000-2014, the average number of Permian Basin leases auctioned on a single day was 39.

oil and gas do not flow easily. Horizontal drilling accompanied by fracking extract oil and gas from these formations by increasing the surface area of rock exposed and creating fractures so that oil and gas can flow. According to conversations with the New Mexico Oil Conservation Division, roughly 80% of the drilling happening today in the Permian Basin is horizontal.

[7] A spacing unit is a contiguous area of land, in which all parties that own any part of the unit share all the oil and gas produced from any part of the unit. Spacing units are there to eliminate the common pool problem. Rules also specify how far wells must be from unit boundaries; it is forbidden to access oil and gas outside lease boundaries.

[8] The annual rental is nominal, at either \$0.50 or \$1 per acre.

Table 1: Number of pairs 2000-2014, by number of bidders N in the first auction

N	pairs
0	14
1	267
2	247
3	165
4	98
5	50
6	21
7	9
8	1

Table 2: Statistics for paired leases, 2000-2014

Mean winning bid per acre (2009 dollars)	\$239	
For $N \geq 2$:		
Correlation of final price in 1st and 2nd auction	0.91	
Probability that 2nd-auction winner also bid on 1st auction	93%	
Probability that pair is won by same bidder:	observed	even odds
$N = 2$	74%	50%
$N = 3$	62%	33%

In terms of observable data, I observe all bids and bidder identities for the first-price sealed bid auction. For the English auction, I observe the transaction price and the identity of the winner only. Table 1 displays the number of pairs observed by N, which is the number of bidders in the first-price sealed bid auction. Table 2 displays other statistics, including some within-pair statistics that are telling.

The auction prices of paired leases are highly correlated, consistent with the geological similarity of adjacent leases. 93% of bidders winning the second auction ("A2") also participate in the first auction ("A1"), suggesting that by and large, the same set of bidders are bidding on both items. This is consistent with conversations with SLO staff; bidders interested in one half of a section are typically interested in the other half as well. Meanwhile, the probability that both leases in a pair will be won by the same bidder is higher than it would be if all A1 participants had an equal chance of winning A2. This suggests that, at a simple correlation level, the winner of A1 is more likely to win A2 than other bidders.

To check this correlation more formally, I perform a probit analysis where the unit of observation is a bidder-lease in a first auction, and the dependent variable is whether that bidder wins the paired second auction. Only auctions with two or more bidders are used.

Table 3: Probit regression results for probability of winning second auction

	(1) $N \geq 2$	(2) $N \geq 2$	(3) $N \geq 2$	(4) $N = 2$	(5) $N = 3$
Won first auction	1.561***	2.045***	2.041***	1.723***	1.769***
	(0.093)	(0.197)	(0.201)	(0.169)	(0.194)
Number of bidders fixed effects	Y	Y	Y	-	-
Bidder fixed effects	Y	N	N	Y	Y
Bidder-date fixed effects	N	Y	Y	N	N
Lease descriptive covariates	N	N	Y	N	N
Observations	1557	612	612	381	405

Standard errors in parentheses
* p<0.10, ** p<0.05, *** p<0.01

The results are displayed in Table 3. Columns (1)-(3) include number-of-bidders fixed effects, and columns (4) and (5) focus on $N = 2$ and $N = 3$, respectively. Columns (1), (4), and (5) control for bidder fixed effects, and columns (2) and (3) control for bidder-date-of-auction fixed effects.[9] Column (3) also controls for covariates describing the lease, which are listed in Table 7 and defined in section 6.2.

In every column, winning the first auction has a highly significant positive effect on the observed probability of winning the second auction. Using the column (1) specification, the probit coefficient can be interpreted as follows: winning A1 increases the observed probability of winning A2 from 0.17 to 0.72 if $N = 2$, and from 0.13 to 0.66 if $N = 3$, for an average tract and an average bidder. We can conclude that the winner of A1 is more likely to win A2 than other bidders. The cause, however, cannot be diagnosed without further investigation.

2.2 Evidence of synergy and affiliation

Intuitively, synergy gives winners of the first auction ("A1") a boost in winning the second auction ("A2"). However, the mere observation that A1 winners are more likely to win A2 need not indicate synergy. Instead, the phenomenon can be due to affiliation of a bidder's values for the first (v_1) and second item (v_2), which is especially likely in this empirical context as the tracts in question are adjacent halves of a square mile. In order to confirm the presence of synergy, we need to account for the fact that even without synergy, the A1 winner is more likely to have the highest v_2 due to affiliation.

One way to perform such a test is to use a regression discontinuity design. For each

[9] There are 128 bidders in the sample, some of which bid very few times. Bidders or bidder-dates that do not bid enough to compute fixed effects are dropped from the regression.

bidder in the first auction, define

$$z \equiv ln(b) - ln(\text{highest competing } b)$$

where b is his bid in the first auction. Then $z > 0$ indicates an A1 winner, and $z < 0$ indicates an A1 loser. A large $|z|$ indicates a large gap between the first and second highest bids in A1. If bidders' v_1 and v_2 are affiliated, a larger $|z|$ makes it more likely that the same bidder will win both A1 and A2. On the other hand, if $|z|$ is very small, this means the A1 winner just barely won. In the absence of synergy, such a bidder should not be much more likely to win A2 than if he just barely lost. This is the idea I exploit to detect synergy; I look for a discontinuity in the probability of winning A2 at $z = 0$. The test does not necessarily prove or disprove synergy, but can provide suggestive indications. As an earlier example of exploiting the idea of RD in the auctions literature, Kawai and Nakabayashi (2014) examine bidders who narrowly won the first round of a multi-round auction, and find evidence of collusion in their pattern of winning subsequent rounds.

Formally, I seek to measure

$$\beta = y^+ - y^-$$

where $y^+ \equiv lim_{z \to 0^+} E[y_i | z_i = z]$ and $y^- \equiv lim_{z \to 0^-} E[y_i | z_i = z]$. As proposed in Hahn, Todd, and Van der Klaauw (2001), I use local linear regression to estimate y^+ and y^-.

As different bidders may have more or less aggressive bidding strategies in A1, which is a first-price sealed-bid auction (unlike A2, which is English), it is best to compare the same bidder against himself in the two scenarios of $z \to 0^+$ and $z \to 0^-$. The results that follow are for the most frequent bidder, who allows the largest number of data points.[10]

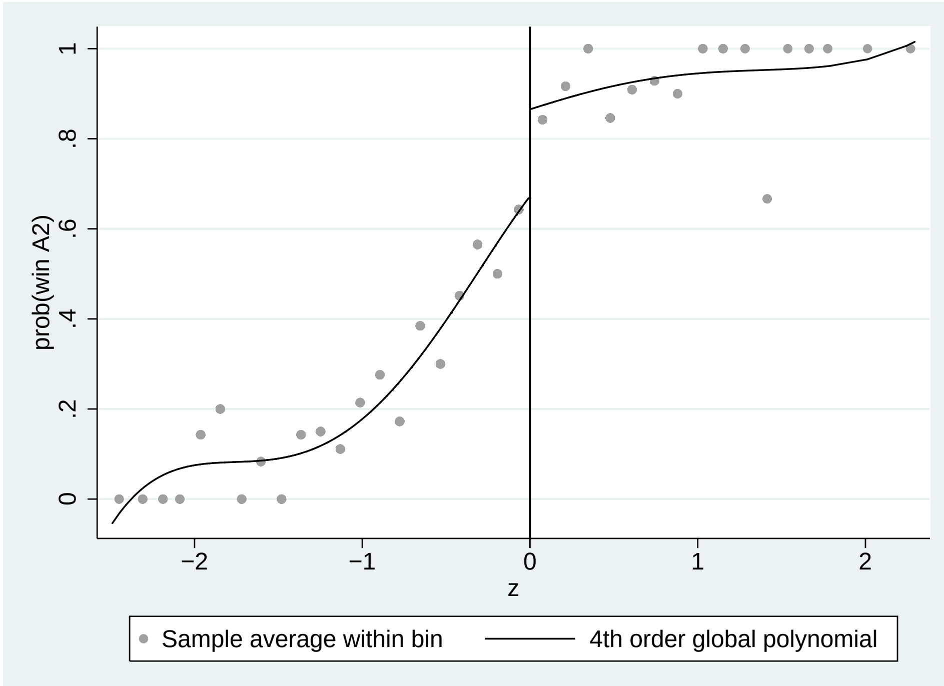
An RD-style plot of the data is displayed in Figure 2.[11] Two features of Figure 1 stand out. First, the probability of winning the second auction is increasing in z. This is consistent with affiliation of values across adjacent tracts, which makes the results of A1 predictive of A2. Second, there seems to be a discontinuity at $z = 0$, consistent with synergy between adjacent tracts.

The local linear regression results are shown in Table 4. The second row of Table 4 corrects for the bias in conventional RD estimates as discussed in Calonico, Cattaneo, and Titiunik (2014b), and the third row increases the standard error to account for the fact that this bias is itself estimated. The columns show different choices of bandwidth selectors: CV represents the cross-validation method proposed by Ludwig and Miller (2007), IK represents

[10] The number of observations drops exponentially going down the ordered list of bidders. See the appendix for more on other bidders.

[11] Figure 2 and Table 4 are obtained using the software packages described in Calonico, Cattaneo, and Titiunik (2014a).

Figure 2: Regression discontinuity plot



Imbens and Kalyanaraman (2012), and CCT represents Calonico et al. (2014b).

Though the null of no synergy cannot be rejected with the robust confidence intervals in the third row, there are nonetheless suggestive indications of synergy, both in the plot of data and in the estimation results. The estimated jump in the probability of winning is roughly 0.2.

3 A model of sequential auctions with synergy and affiliation

Motivated by the empirical setting, I build a model of sequential auctions with synergy. To fix ideas, I introduce the model in the context of risk-neutral, symmetric bidders. Afterwards, I extend the model to asymmetric bidders and risk aversion.

Private values paradigm

I develop the model within the private values paradigm. In common value models of oil and gas leases, the source of interdependence is that each bidder gets a different signal about a value-relevant but unknown characteristic, such as how much oil is underground. However, the Permian Basin in New Mexico is an area where knowledge of the geology is more complete due to a long history of development and production dating back to the 1920s.

Table 4: Sharp RD estimates using local linear regression

Bandwidth selector:	CV	IK	CCT
Conventional	0.215***	0.211**	0.193
	(0.082)	(0.106)	(0.120)
Bias-corrected	0.191**	0.185*	0.176
	(0.082)	(0.106)	(0.120)
Robust	0.191	0.185	0.176
	(0.122)	(0.139)	(0.145)
Observations	545	545	545

Epanechnikov kernel
Standard errors in parentheses
* p<0.10, ** p<0.05, *** p<0.01

Seismic work done by the state is publicly available. Permits for new seismic surveys are no longer requested in the basin, as these are only done in areas that are not well known. Much of the basin has already been drilled in the past. And when land is drilled, electric wireline logs that record geologic formations are submitted to the New Mexico Oil Conservation Division and made public. Conversations with agency staff and bidders suggest that, though the science is never exact and uncertainty remains, the industry has a fairly good idea of oil and gas potential in the basin, and bidders are working with the same, publicly available information when they assess the value of a tract to their firm. As one bidder put it,

> "Bear in mind that New Mexico has been producing oil and gas for over 80 years and there have been thousands of wells drilled. This provides us a lot of historical data. Most tracts that show up on a given monthly sale, are in an area with lots of production history and exploration success, or have the lack thereof."

Meanwhile, valuations of a lease can be idiosyncratic by bidder for firm-specific reasons. Different firms have different niches and areas of interest. They may be interested in different depths or layers of the same tract of land, and engineering teams may design different plans for how to drill it. Firms vary in their leaseholding strategies. In particular, winning a lease does not require the firm to drill; it grants the right, but not the obligation, for five years. As such, one firm may plan on drilling in the first year, while another firm may plan on the last year. The tract may not be drilled at all - this is very common - and different firms may have different probabilities of drilling for each tract. Leasing budgets, operating costs, and infrastructure also vary across firms. In light of firms' use of publicly available information when researching tracts, and the relatively small uncertainty regarding oil and gas potential, the private values paradigm is a decent approximation of this setting; at the least, it does not seem worse than in other auctions typically studied under the private values framework. Nonetheless, it simplifies the complexities of the real environment.

3.1 Setup

I introduce the full model first, and then discuss the merits and demerits of particular features in turn.

A pair of adjacent tracts is leased via auction on the same day. One tract is sold by a first-price sealed-bid auction, and the other is sold by an English auction, which happens later chronologically. Before bidding in the first auction, each bidder draws a value

$$v_1 \sim F^1(\cdot)$$

which is his stand-alone, private value for the first object.

Between the first auction (A1) and second auction (A2), there is noise that affects bidders' values for the second object. Therefore, bidders do not know their value for the second object (v_2) with certainty at the time of the first auction. However, they do know the distribution from which the stand-alone value v_2 will be drawn:

$$v_2 \sim F^2(\cdot|v_1)$$

The distribution F^2 is conditional on v_1, allowing for affiliation between v_1 and v_2. As an example, a special case would be that $E[v_2|v_1] = v_1$, but this model is more general. The exact value of v_2 is learned after the first but before the second auction.

The firm that won the first auction benefits from synergy if he also wins the second auction, so his ultimate value for the second object is not just the stand-alone value v_2 but a synergy-inclusive value

$$s(v_1, v_2).$$

I allow the synergy function s to be a function of both v_1 and v_2 to be as general as possible. The size of synergy is idiosyncratic to each bidder, since it is a function of v_1 and v_2, which are different for each bidder and private information. To simplify notation when expressing the idea that the A1-winner applies synergy to his v_2 when valuing the second tract, I define

$$D(x|v_1) \equiv prob(s(v_1, v_2) \leq x|v_1)$$

and say winners of A1 draw their ultimate value for the second tract from the distribution

$D(\cdot|v_1)$. I assume that the same set of bidders participate in the first and second auction.

Now I discuss the ideas underlying specific parts of this model.

I do not explicitly model the noise between the first and second auction, but in the case of the oil and gas lease auctions, one source of noise is other auctions that take place in between the two sales. The type and number of tracts won and lost in these intervening auctions can lead to adjustments in bidders' values. In other data contexts, noise may come from the passage of time, often months, between the two auctions.[12]

The distribution of v_2 is conditional on v_1, but it is not conditional on any other signal. The underlying assumption is that v_1 is a sufficient statistic for anything known by a bidder at the time of A1 that his F^2 could depend on. This does not require the two objects to be identical; if each object has its own descriptive covariates, then the statement can be made conditional on these covariates. For the oil and gas lease pairs, which are adjacent halves of a square mile, the assumption is a reasonable approximation of reality. More generally, it is reasonable when, as in other contexts involving adjacency, the objects are related and determinants of private value "shocks" are likely to be similar. On the other hand, if the objects are not so related, the model may be too crude of an approximation. The alternative for those cases would be to have a separate signal for the second object at the time of the first auction. However, two-dimensional types introduce significant difficulties to characterizing equilibria, let alone estimating the model. In contexts where it is appropriate, the model introduced here provides a practical way forward.

It is helpful to compare this setup with other sequential or dynamic auction models. The literature on sequential auctions of homogeneous goods, such as Brendstrup and Paarsch (2006) and Lamy (2010), has employed models in which bidders' value for the second item remains v_1 or v_1 times a constant if they do not win the first item. If they do win, the value of the second unit is always less than the value of the first unit. The empirical applications were in auctions of commodities, such as fish and tobacco. One restriction of that model is that only the two highest bidders for the first unit can ever win the second unit. Letting v_2 be a draw from $F_2(\cdot|v_1)$ as in this paper encompasses this special case and imposes no restrictions on who can win the second auction. On the other hand, bidders in Lamy (2010) learn both v_1 and v_2 prior to bidding in the first auction. However, identification and estimation in Lamy (2010) still proceed from establishing that first auction bids depend only on v_1 under certain conditions.

This is also different from the dynamic auction model used in Jofre-Bonet and Pesendorfer (2003), where a bidder's values across auctions are independent conditional on covariates and state variables. Here, the distribution of a bidder's v_2 is directly dependent on his v_1,

[12] In Marshall et al. (2006), school milk procurements take place from May through August of each year, and in Gandal (1997), Israeli cable TV licenses are auctioned over a period spanning 1988-1991.

encompassing independence across auctions as a special case. Allowing for such correlation is critical when attempting to measure synergy. As discussed in section 2.2, ignoring affiliation could lead us to detect synergy where none exists. On the other hand, Jofre-Bonet and Pesendorfer (2003) consider an infinite horizon of auctions, while this paper models just two auctions.

Assumptions

For now, assume all items are homogeneous for expositional ease. Section 5.2 will discuss how to work with heterogeneity across pairs.

AS1 (v_1, v_2) are independent across bidders: $(V_{1i}, V_{2i}) \perp (V_{1j}, V_{2j})$

AS2 $F^1(\cdot)$ is differentiable, with continuous density $f^1 = F^{1'}$.

AS3 $F^2(\cdot|v_1)$ and $D(\cdot|v_1)$ are differentiable and have the same support, for every v_1.

AS4 $F^2(\cdot|v_1)$ is stochastically ordered in v_1: $v_1' > v_1$ implies $F^2(\cdot|v_1') \leq F^2(\cdot|v_1)$.

AS5 $|E[v_2|v_1] - E[v_2|v_1']| \leq |v_1 - v_1'|$

AS6 $\frac{\partial s(v_1,v_2)}{\partial v_1} \geq 0$ and $\frac{\partial s(v_1,v_2)}{\partial v_2} \geq 0$.

AS7 The reserve price r is not binding.

AS1 means that while values can be dependent across the first and second auction, values are independent across bidders. AS3, which says all bidders bidding in the second auction draw their values from the same support, is an assumption included for completeness as it provides for full identification of the value distributions. It is not a critical assumption in the sense that, if the supports are different, the value distributions will be identified where the supports overlap, just not at the extremes. I discuss this further in the identification section. AS4 means that a bidder with higher v_1 is more likely to have a higher v_2. This makes sense given that the two tracts in a pair are located in the same square mile; even with intervening noise between the two auctions, v_1 and v_2 are likely to be positively correlated. This assumption also helps establish monotonic bidding in the first auction. AS5 is like a Lipschitz condition that rules out extreme movements or divergence of the expected value of v_2 as a function of v_1. This assumption is easy to verify in data once $F^2(\cdot|\cdot)$ is estimated. AS6 says that $s(v_1, v_2)$, the synergy-included value of the second tract to a firm that won the first tract, is a nondecreasing function of v_1 and v_2. This does not rule out "negative synergy", or $s(v_1, v_2) < v_2$. Rather, it means that if a firm's stand-alone value for a tract increases, the firm's synergy-included value for the tract does not decrease, all else equal.

Notation

It is useful to introduce some notation that simplifies long expressions in the expected profit function. Though the model is different, I follow the style of notation used by Lamy (2012).

The distribution of the highest competing bid in the second auction given that the bidder wins the first auction and the highest competing bid in A1 is t is[13]

$$H_1(u|t) = F^2(u|b \leq t)^{N-2} F^2(u|b = t) \tag{1}$$

To explain, the probability that the highest competing bid in A2 is $\leq u$ is equal to the probability that all bidders other than this bidder have values $\leq u$ for the second item. Since the highest competing bid in A1 is t, the other bidders in A2 consist of one bidder who bid t in A1 and $N-2$ bidders who bid $\leq t$ in A1. The right-hand side of (1) expresses the probability that all of these competing bidders have values $\leq u$. The subscript 1 on H indicates the case where the bidder wins the first auction.

Next, the distribution of the highest competing bid in A2 given that the bidder loses A1 and the highest competing bid in A1 is t is

$$H_2(u|t) = F^2(u|b \leq t)^{N-2} D(u|b = t) \tag{2}$$

The subscript 2 on H indicates the case where the bidder loses the first auction. The right-hand side of (2) is the same as that of (1) except that $D(u|b = t)$ replaces $F^2(u|b = t)$. Having lost A1, the bidder knows he will be competing against the winner of A1, who benefits from synergy. Therefore, H_2 is different from H_1 if synergy exists.

3.2 Bidding in the second auction

Working backwards, I discuss bidding in the second auction before thinking about the first auction. The second auction is an English auction. Under the private value paradigm, it is a dominant strategy for each bidder to bid up to his value for the tract. For the bidder who won the first auction, this is $s(v_1, v_2)$. For all other bidders who bid in the first auction, this is v_2.

[13] H_1 and H_2 are conditional only on the highest competing bid t, and not on any other bids. This is because these expressions will be used in the expected profit function, which is computed by the bidder before the first auction happens. Before the first auction, all he knows is that if he wins with bid b, the highest competing bid must be less than b, and if he loses with bid b, the highest competing bid must be greater than b.

3.3 Bidding in the first auction

Now I consider bidding in the first auction (A1), which is a first-price sealed-bid auction, under the assumption of a symmetric equilibrium. Let $G(\cdot)$ be the distribution of bids in this auction.

Expected profit at the time of the first auction

The expected profit from the two auctions at the time of the first auction, if the bidder bids b is

$$\begin{aligned}\pi(v_1, b) &= \int_{v_2=\underline{v}}^{\bar{v}} \Bigg\{ \int_{t=\underline{b}}^{b} \Big(v_1 - b + \int_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1, v_2) - u) dH_1(u|t) \Big) dG^{N-1}(t) \\ &\quad + \int_{t=b}^{\bar{b}} \int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2(u|t) dG^{N-1}(t) \Bigg\} dF^2(v_2|v_1)\end{aligned}$$

The outer integral over v_2 represents the fact that at the time of the first auction, v_2 is uncertain. The first expression inside the outer integral represents the case where the bidder wins the first auction, and the second expression represents the case where the bidder loses the first auction. Notice that when he wins the first auction, he benefits not only from $v_1 - b$, but also from the fact that the second item is now worth $s(v_1, v_2)$ to him rather than just v_2 due to synergy. $G^{N-1}(\cdot)$ is the distribution of the highest bid out of $N - 1$ bidders.

First-order condition

A bidder will bid the b that maximizes his expected profit $\pi(v_1, b)$. Taking the derivative of $\pi(v_1, b)$ with respect to b and setting it equal to zero gives

$$\begin{aligned}0 &= -G^{N-1}(b) + (N-1)G^{N-2}(b)g(b) \int_{v_2=\underline{v}}^{\bar{v}} \Bigg\{ v_1 - b \\ &\quad + \int_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1, v_2) - u) dH_1(u|b) - \int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2(u|b) \Bigg\} dF^2(v_2|v_1)\end{aligned} \tag{3}$$

Using integration by parts and rearranging, the first-order condition can be simplified to

$$b = v_1 + \int_{v_2=\underline{v}}^{\bar{v}} \Bigg\{ \int_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|b) du - \int_{u=\underline{v}}^{v_2} H_2(u|b) du \Bigg\} dF^2(v_2|v_1) - \frac{G(b)}{(N-1)g(b)} \tag{4}$$

It is instructive to compare this FOC to the FOC of a stand-alone first-price auction.

From Guerre, Perrigne, and Vuong (2000), we know that the FOC for a stand-alone first-price auction is $b = v_1 - \frac{G(b)}{(N-1)g(b)}$. In (4), there is an additional term on the right-hand side that represents the expected benefit due to synergy in the second auction from winning the first auction. In other words, the v_1 in the stand-alone first-price auction is replaced by v_1 plus the expected benefit of synergy. If there is no synergy, i.e. $s(v_1, v_2) = v_2$, then (4) collapses to $b = v_1 - \frac{G(b)}{(N-1)g(b)}$, as in Guerre et al. (2000).

Note that by construction, the bids in the first auction are a function only of v_1; only the distribution of v_2, not the realization, is known at the time of the first auction.

3.4 Equilibrium properties

Strictly increasing bid function

Given the assumptions discussed in 3.1, it can be shown that it is impossible for a strictly lower first-auction bid to be a best response for a strictly higher value.[14] Also, the right-hand side of (4) is strictly increasing in v_1, so a single bid cannot be a best response for two different values. A proof along these lines leads to the following proposition.

Proposition 1. *The bid function $b(v_1)$ in the first auction is strictly increasing in v_1.*

A strictly increasing bid function means that in a given equilibrium, $F^2(v_2|b = b(x)) = F^2(v_2|v_1 = x)$ and $F^2(v_2|b)$ retains the stochastic ordering property of $F^2(v_2|v_1)$. In an abuse of notation, I use the same "F^2" to denote both $F^2(\cdot|b(v_1))$ and $F^2(\cdot|v_1)$. Similarly, if I define a new function $\tilde{s}(\cdot, \cdot)$ such that $\tilde{s}(b(v_1), v_2) = s(v_1, v_2)$, $\tilde{s}(\cdot, v_2)$ retains the weak monotonicity of $s(\cdot, v_2)$. To simplify notation, I use "s" to denote both $\tilde{s}(\cdot, \cdot)$ and $s(\cdot, \cdot)$, since they are easily distinguished by whether the first argument is a bid or a value. Now, replacing $F^2(v_2|v_1)$ with $F^2(v_2|b)$ and $s(v_1, v_2)$ with $s(b, v_2)$ in (4) and rearranging defines an inverse bid function:

$$\xi(b) \equiv b + \frac{G(b)}{(N-1)g(b)} - \int_{v_2=\underline{v}}^{\bar{v}} \left\{ \int_{u=\underline{v}}^{s(b,v_2)} H_1(u|b)du - \int_{u=\underline{v}}^{v_2} H_2(u|b)du \right\} dF^2(v_2|b) = v_1. \tag{5}$$

Uniqueness of equilibrium

Having established monotonic bidding, we can revisit (1) and (2) to see that in a given equilibrium,

$$H_1(u|b(x)) \equiv F^2(u|b(v_1) \leq b(x))^{N-2} F^2(u|b(v_1) = b(x)) = F^2(u|v_1 \leq x)^{N-2} F^2(u|v_1 = x)$$

[14] A best-response bid b is the bid or one of the bids that maximizes $\pi(v_1, b)$, given what the other bidders are doing.

$$H_2(u|b(x)) \equiv F^2(u|b(v_1) \leq b(x))^{N-2} D(u|b(v_1) = b(x)) = F^2(u|v_1 \leq x)^{N-2} D(u|v_1 = x)$$

Then $H_1(u|b)$ and $H_2(u|b)$ in (3) can be replaced with $H_1(u|\xi(b))$ and $H_2(u|\xi(b))$, where $\xi(\cdot)$ is the inverse bid function for that equilibrium, and $G(b)$ can be replaced with $F^1(\xi(b))$. After some algebra, this gives the following differential equation that must be satisfied in equilibrium:

$$P'(v_1) = T(v_1) \frac{d}{dv_1} [F^1(v_1)^{N-1}] \tag{6}$$

where $P(v_1) \equiv b(v_1) F^1(v_1)^{N-1}$ is the bidder's expected payment and

$$T(v_1) \equiv v_1 + \int\limits_{v_2=\underline{v}}^{\bar{v}} \left\{ \int\limits_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|v_1) du - \int\limits_{u=\underline{v}}^{v_2} H_2(u|v_1) du \right\} dF^2(v_2|v_1).$$

Equation (6) is similar to the equilibrium condition for a stand-alone first-price auction as studied in Riley and Samuelson (1981), except that $T(v_1)$ takes the place of what was v. Solving this differential equation leads to Proposition 2.

Proposition 2. *There is a unique symmetric Bayes-Nash equilibrium for the first auction, given by* $b(v_1) = \int_{\underline{v}}^{v_1} T(x) dF^1(x)^{N-1} / F^1(v_1)^{N-1}$ *and* $b(\underline{v}) = T(\underline{v})$. *The bidder's expected payment is* $\int_{\underline{v}}^{v_1} T(x) dF^1(x)^{N-1}$.[15]

Note that $T(v_1) = v_1$ if there is no synergy, and $T(v_1) > v_1$ if synergy is positive. This means the bidder's expected payment $\int_{\underline{v}}^{v_1} T(x) dF^1(x)^{N-1}$ and auction revenue in the first auction are higher when synergy is positive than when synergy is zero; i.e. there is a synergy premium.

As for revenue in the first auction versus the second auction, simulations show that the model does not restrict revenue in A1 to be higher than in A2 or vice versa, even if synergy is strictly positive. The intuition behind this is as follows: on the one hand, anticipating the benefits of synergy in A2 leads to more aggressive bidding in A1; on the other hand, the winner of A1 bidding in light of the synergy he has secured leads to higher prices in A2. The revenue relationship depends on the shape of the value distributions and the size of synergy. This is in line with the theory of Sørensen (2006), who finds that prices need not decrease in sequential second-price auctions of stochastically equivalent complementary objects. This is different from Branco (1997) and Menezes and Monteiro (2003), who consider different models of complementary objects with identical values and find that expected prices decline in the sequence.

[15] I confirm that the second-order condition holds, so the bid that satisfies the first-order condition does achieve the global maximum. See appendix.

The model can be extended to the case where bidders have asymmetric value distributions and synergy functions. The details are discussed in the appendix.

3.5 Risk aversion

The model of sequential auctions with synergy can also be extended to the case where bidders are risk averse. Since the second auction (A2) is an English auction, it remains a dominant strategy for bidders to bid their value in the second auction. However, risk aversion does affect bidding in the first auction (A1), which uses the first-price sealed-bid format.

With risk aversion, the expected profit at the time of the first auction is

$$\begin{aligned}\pi(v_1,b) \quad = \quad & \int_{v_2=\underline{v}}^{\bar{v}} \left\{ \int_{t=\underline{b}}^{b} \int_{u=\underline{v}}^{s(v_1,v_2)} U(v_1-b+s(v_1,v_2)-u)dH_1(u|t)dG^{N-1}(t) \right. \\ & +U(v_1-b)\int_{t=\underline{b}}^{b} \int_{u=s(v_1,v_2)}^{\bar{v}} dH_1(u|t)dG^{N-1}(t) \\ & \left. + \int_{t=b}^{\bar{b}} \int_{u=\underline{v}}^{v_2} U(v_2-u)dH_2(u|t)dG^{N-1}(t) \right\} dF^2(v_2|v_1)\end{aligned} \tag{7}$$

All profits now show up inside the utility function $U(\cdot)$. The first expression inside the outer integral represents the case where a bidder wins both auctions, the second expression is the case of winning only the first auction, and the third expression is the case of winning only the second auction.

Again, taking a derivative of $\pi(v_1,b)$ with respect to b yields the first-order condition for bidding. The mathematical expression is more complex in the risk averse case:

$$\begin{aligned}\frac{G(b)}{(N-1)g(b)} \quad = \quad & \textstyle\int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(v_1,v_2)} U(v_1-b+s(v_1,v_2)-u)dH_1(u|b) \\ & \textstyle + \int_{u=s(v_1,v_2)}^{\bar{v}} U(v_1-b)dH_1(u|b) - \int_{u=\underline{v}}^{v_2} U(v_2-u)dH_2(u|b)\} dF^2(v_2|v_1) / \\ & \textstyle\int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(v_1,v_2)} U'(v_1-b+s(v_1,v_2)-u)dH_1(u|t\le b) \\ & \textstyle + \int_{u=s(v_1,v_2)}^{\bar{v}} U'(v_1-b)dH_1(u|t\le b)\} dF^2(v_2|v_1)\end{aligned} \tag{8}$$

When bidders are risk averse, the uncertain parts of the expected payoff function $\pi(v_1,b)$ - namely, the terms involving the second auction - have a lower certainty equivalent and are discounted relative to the case of risk neutrality. Thus, as bidders grow more risk averse, the second auction will matter less, and bidding in the first auction will become increasingly similar to a regular first-price auction without a second auction.

Regarding the effect of risk aversion on A1 revenue compared to the risk neutral case, no general statement can be made, as there are two opposing forces. As is well known, risk aversion pushes bidders to bid more in a first-price auction because they want to buy insurance against the possibility of losing. On the other hand, risk aversion causes bidders to discount potential payoffs from A2 when they bid in A1, as discussed in the previous paragraph. This decreases the synergy premium in A1 bids.

4 Identification

In this section, I show that the model primitives, meaning the value distributions $F^1(\cdot)$, $F^2(\cdot|\cdot)$, and the synergy function $s(\cdot,\cdot)$, are identified from the observable data, which are the joint distribution of first auction bids and second auction prices, along with bidder identities. The key idea behind this identification result is as follows: suppose we observe two ex-ante symmetric bidders submit identical bids in the first auction, but one of them wins and the other loses. The fact that they bid the same means they had the same v_1 when they started. If winning the first auction has no effect on bidders' values for the second item, the winner should behave no differently from the loser in the second auction. By comparing the behavior of the winner and the loser, we can measure the synergy that comes from having two adjacent tracts.

As in the previous section, I begin with the case of risk-neutral bidders, and then extend to risk aversion. I abstract away from auction-specific heterogeneity, the discussion of which is deferred to section 5.2.

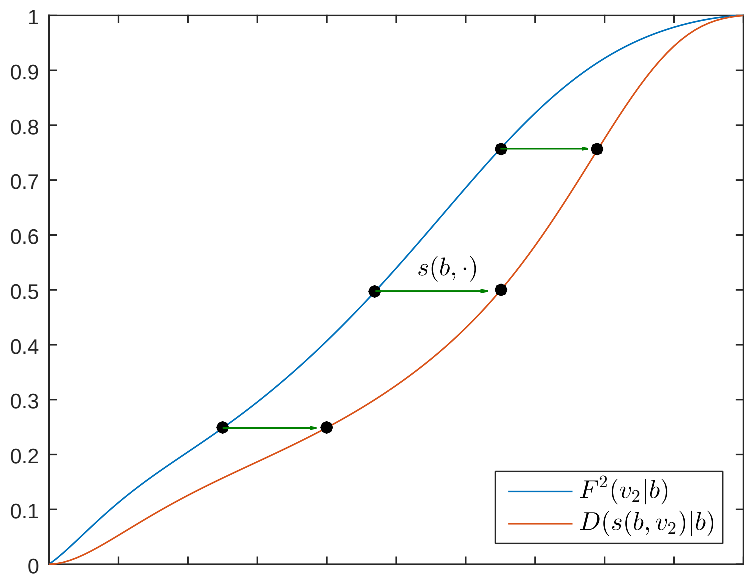
4.1 Identification

To establish identification, I need to show that there is a unique structure $[F^1(\cdot),\ F^2(\cdot|\cdot),\ s(\cdot,\cdot)]$ that rationalizes the data. The identification strategy starts by looking at the second auction, and then proceeds back to the first auction.

Proposition 3. *$F^1(\cdot)$, $F^2(\cdot|\cdot)$, and $s(\cdot,\cdot)$ are identified from all the bids in the first auction, the transaction price in the second auction, and bidder identities. They can be recovered by following these steps: (i) The value distributions involved in the second auction, $F^2(\cdot|b)$ and $D(\cdot|b)$ are identified from the observables. (ii) Once $F^2(\cdot|b)$ and $D(\cdot|b)$ are known, the synergy function $s(b,\cdot)$ is nonparametrically identified. (iii) Using $F^2(\cdot|b)$ and $s(b,\cdot)$, $F^1(\cdot)$ is identified nonparametrically from bids in the first auction. (iv) $F^2(\cdot|b)$ and $s(b,\cdot)$ can be converted to $F^2(\cdot|v_1)$ and $s(v_1,\cdot)$.*

The identification argument for step (i), presented in the appendix, is based on Athey and Haile (2002). In their Theorem 2, Athey and Haile (2002) show that the value distributions of

Figure 3: Nonparametric identification of $s(b, \cdot)$



asymmetric IPV bidders are identified from transaction prices and winner identities. When it comes to the second auction in our model, the first auction induces asymmetry between bidders that were ex-ante symmetric. Specifically, the winner $w1$ of the first auction draws his value from $D(\cdot|b_{w1})$, and each loser i from the first auction draws from $F^2(\cdot|b_i)$. For a fixed set of first-auction bids $\{b_i\}$, we can apply Theorem 2 of Athey and Haile (2002), so each of these distributions is identified from transaction prices and winner identities in the second auction.

Step (ii) says that having identified $F^2(\cdot|b)$ and $D(\cdot|b)$, the synergy function $s(\cdot, \cdot)$ is also identified. As mentioned at the beginning of the identification section, the intuition is to compare how a first-auction winner and first-auction loser behave differently in the second auction when they are otherwise identical, even to the point of having the same v_1. We can do just this by comparing $F^2(\cdot|b)$ and $D(\cdot|b)$; by conditioning on $b(v_1)$, we compare two bidders who only differ in that one of them won the first auction while the other did not. Therefore, the difference between $F^2(\cdot|b)$ and $D(\cdot|b)$ can be attributed to synergy. More precisely, recall that $F^2(\cdot|b)$ is the distribution of $v_2|b$ and $D(\cdot|b)$ is the distribution of $s(b, v_2)|b$. Since $s(b, v_2)$ is monotonically increasing in v_2, $s(b, \cdot)$ must map the α-quantile of $F^2(\cdot|b)$ to the α-quantile of $D(\cdot|b)$. Since $F^2(\cdot|b)$ and $D(\cdot|b)$ are identified, this mapping provides for nonparametric identification of $s(\cdot, \cdot)$. Figure 3 illustrates the idea graphically.

In step (iii), having identified $F^2(\cdot|b)$ and $s(b, \cdot)$, $F^1(\cdot)$ can be identified using the inverse bid function (5) for the first auction. The inverse bid function can be computed at this stage because its components - $F^2(\cdot|b)$ and $s(b,\cdot)$ as well as the observed bid distribution $G(b)$ - are all known now. Since bids $b(v_1)$ are monotonic in v_1, any quantile of v_1 can be recovered by computing the inverse bid for that quantile of b, and this gives $F^1(\cdot)$ nonparametrically.

Finally step (iv) ties the remaining loose ends. Denote the α-quantile of v_1 and b by $v_1(\alpha)$ and $b(\alpha)$, respectively. Then with some abuse of notation[16], $F^2(v_2|v_1(\alpha)) = F^2(v_2|b(v_1(\alpha))) = F^2(v_2|b(\alpha))$, and $s(v_1(\alpha),v_2) = s(b(v_1(\alpha)),v_2) = s(b(\alpha),v_2)$. Now all the primitives of the model, $F^1(\cdot)$, $F^2(\cdot|\cdot)$, and $s(\cdot,\cdot)$, are identified.

Identification of the model with asymmetric bidders is discussed in the appendix.

Further discussion

It is interesting to consider whether identification would be possible for models with added randomness. For instance, suppose a stochastic component $\epsilon \sim F_\epsilon(\cdot)$ is added to $s(v_1 v_2)$ so that the synergy-inclusive value of the second tract is $s(v_1,v_2)+\epsilon$. For a dataset generated by such a model, there will generally exist other models that rationalize the same data, one of which is the model with $\epsilon = 0$ that can be constructed by following the identification steps above. In other words, $F_\epsilon(\cdot)$ and the function $s(\cdot,\cdot)$ are not separately identified for such a model. However, even if the true model involves an error term like ϵ, the synergy function computed under the $\epsilon = 0$ model - say we call this $\check{s}(\cdot,\cdot)$ - is likely to have meaning. For instance, if $F_\epsilon(\cdot)$ is a symmetric distribution with mean zero, $\check{s}(v_1,v_2)$ will closely approximate $s(v_1,v_2)$ for most values of v_2; the largest discrepancies will occur near the boundaries of the support, i.e. for values of v_2 near $\underline{v}$ or $\bar{v}$.

Revisiting step (i), one assumption that goes into Athey and Haile (2002)'s result is that all value distributions have the same support. I list this assumption in AS3 to provide for full identification. What are the consequences if this assumption does not hold? Suppose $D(\cdot|\cdot)$ has a larger support than $F^2(\cdot|\cdot)$, with a greater supremum $\bar{v}_D > \bar{v}_{F^2}$. Since English auction prices only reveal the second highest value, we will never observe prices above $\bar{v}_{F^2}$, meaning we gather no information on the shape of $D(\cdot|\cdot)$ in the interval $(\bar{v}_{F^2},\bar{v}_D)$. $D(\cdot|\cdot)$ will be identified on $[\underline{v},\bar{v}_{F^2}]$, where the two supports overlap, but not on $(\bar{v}_{F^2},\bar{v}_D)$.[17] As such, estimation results at the boundaries of the support will be less reliable. The impact of this partial identification on answering questions of interest will depend on how far $D(\bar{v}_{F^2}|\cdot)$ is from 1. In many applications, the upper end of the distribution is very sparse and $D(\bar{v}_{F^2}|\cdot) \sim 1$ even if $\bar{v}_{F^2} \neq \bar{v}_D$.[18]

[16] To be notationally correct, I should write something like $F^2(v_2|v_1(\alpha)) = \tilde{F}^2(v_2|b(v_1(\alpha))) = \tilde{F}^2(v_2|b(\alpha))$. However, I abstract from notational correctness to avoid introducing more notation that is not central to the paper.

[17] Meilijson (1981), on which Theorem 2(a) of Athey and Haile (2002) is based, remarks on non-identical supports.

[18] To get an idea, in the data studied in this paper, the 95-percentile of observed bids is less than one-tenth of the highest observed bid, and the 99-percentile of observed bids is only one-third of the highest observed

4.2 Identification with risk aversion

Is the model identified when bidders are risk averse? Steps (i) and (ii) in Proposition 3 apply even with risk averse bidders, since bidding strategies in the second auction, which is English, are unaffected by risk aversion. This means $F^2(\cdot|b)$ and $s(b,\cdot)$ are identified regardless of risk attitudes. $U(\cdot)$ and $F^1(v_1)$ remain to be identified.

Rewriting the first-order condition for risk averse bidders in (8), replacing $s(v_1, v_2)$ with $s(b, v_2)$ and $F^2(v_2|v_1)$ with $F^2(v_2|b)$, we get

$$\begin{aligned} \frac{G(b)}{(N-1)g(b)} = \; & \int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(b,v_2)} U(v_1 - b + s(b,v_2) - u) dH_1(u|b) \\ & + \int_{u=s(b,v_2)}^{\bar{\bar{v}}} U(v_1 - b) dH_1(u|b) - \int_{u=\underline{v}}^{v_2} U(v_2 - u) dH_2(u|b) \} dF^2(v_2|b) / \\ & \int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(b,v_2)} U'(v_1 - b + s(b,v_2) - u) dH_1(u|t \le b) \\ & + \int_{u=s(b,v_2)}^{\bar{\bar{v}}} U'(v_1 - b) dH_1(u|t \le b) \} dF^2(v_2|b) \end{aligned} \tag{9}$$

Every term on the right-hand side is observed or identified except for v_1 and $U(\cdot)$. And since $U'(\cdot) > 0$ and $U''(\cdot) \le 0$ under risk aversion, the right-hand side is strictly increasing in v_1. This means that if we know $U(\cdot)$, we can use this FOC to uniqely back out the v_1 associated with any bid b. So the missing step is to identify $U(\cdot)$, or the risk aversion level.

One way to identify risk aversion is to exploit variation in the number of bidders. Appealing to ideas in Guerre et al. (2009), it is possible to identify $U(\cdot)$ if either the number of bidders varies exogenously, or if there is an instrument that affects the number of bidders but not the underlying private value distribution. Anothery way is to exploit the auction sequence. I discuss each case in turn.

Using variation in number of bidders with exogenous participation

Suppose the number of bidders N varies exogenously in the data, such that $F^1(v_1; N') = F^1(v_1; N'')$, where $N' \neq N''$.[19] Let $\xi(b, U; N)$ represent the value of v_1 backed out from (9) as a function of b, $U(\cdot)$, and N. Then, the true $U(\cdot)$ must satisfy

$$\xi(b(\alpha|N'), U; N') = \xi(b(\alpha|N''), U; N'') \tag{10}$$

for all quantiles $\alpha \in [0, 1]$. These so-called compatibility conditions provide a basis for identifying $U(\cdot)$.

From (9), it appears that $\xi(\cdot)$ is a complicated function for which we do not have an explicit expression. As a result, it is difficult to provide a nonparametric identification

bid.

[19] If there are covariates describing the auctioned object, then exogeneity here refers to exogenous variation of N conditional on observed covariates.

strategy for $U(\cdot)$ the way Guerre, Perrigne, and Vuong (2009) did. I present a parametric alternative instead.

Suppose $U(\cdot)$ can be represented in parametric form, like the constant relative risk aversion utility $U(x) = x^{1-\rho}$. Then the compatibility conditions become a function of the single parameter ρ:

$$\xi(b(\alpha|N'), \rho; N') = \xi(b(\alpha|N''), \rho; N'') \tag{11}$$

While I cannot show analytically that there is a unique ρ satisfying the compatibility conditions, it is possible to check this numerically as long as ρ is bounded. If numerical computations show that there is a unique best ρ in a bounded range known to contain the true parameter, then ρ is identified.

Using variation in number of bidders with endogenous participation

In real data settings, the number of bidders is often endogenously determined. For instance, there may be auction-specific unobserved heterogeneity u, of which higher realizations lead to more participation. Following Guerre et al. (2009), it is still possible to identify $U(\cdot)$ parametrically in this setting if the following conditions hold (assume items are observably homogeneous for ease of exposition):

1. There is an instrument x such that $N = N(x, u)$ and $F^1(v_1|x, u) = F^1(v_1|u)$
2. N is a sufficient statistic for u given x, e.g. $u = N - E[N|x]$

The second condition allows for recovery of u. After conditioning on u, all remaining variation in N comes from the instrument x, allowing us to return to the logic of the exogenous variation case. The compatibility conditions are the same as before except that they are now conditional on u:

$$\xi(b(\alpha|N', u), \rho; N', u) = \xi(b(\alpha|N'', u), \rho; N'', u) \tag{12}$$

for all quantiles $\alpha \in [0, 1]$. The risk aversion parameter is identified numerically if there is a unique value of ρ that best satisfies the compatibility condition.

Using the auction sequence

Alternatively, if the stand-alone value of auction items, conditional on having the same characteristics, should have similar (though not necessarily identical) distributions regardless of placement in the auction sequence, this can be used to identify the risk aversion parameter ρ. The unconditional distribution of stand-alone values in A2, given by $F^2(v_2) = \int F^2(v_2|b)dG(b)$, is identified independently of ρ. Meanwhile, $\xi(b, \rho)$, the inverse bid function for A1 defined by (9), will generally yield lower values of v_1 as ρ increases, since higher risk aversion leads to higher bidding strategies in first price auctions. The risk aversion parameter is identified if there is a unique value of ρ that satisfies the appropriate criterion of similarity between the distribution $F^2(\cdot)$ and the distribution of $\xi(b, \rho)$ for $b \sim G(b)$.

5 Estimation

5.1 A multi-step estimation procedure

I develop a multi-step estimation procedure that closely follows the identification steps. Following the identification strategy in section 4.1, the first step of estimation is to estimate $D(\cdot|b)$ and $F^2(\cdot|b)$, which are the distributions of second auction values for the first-auction winner and first-auction loser, respectively, given the first-auction bids. For this task I use a sieve maximum likelihood estimator with Bernstein polynomial bases, similar to the one used in Kong (2015). Properties of sieve estimators, including sieve maximum likelihood, are discussed in Chen (2007). Komarova (2013) is an example of using Bernstein polynomials for sieve estimation of distribution functions in an ascending auction framework.

In the second auction, we observe for each item the transaction price p, the identity of the winner, and the identity and first-auction bids of all bidders in the related first auction. Taking the case of $N = 2$ (two bidders in the first auction) as an expositional example, the likelihood of the second-auction price and winner given the first-auction data can be expressed as follows for each item.

If the first-auction winner wins the second auction:

$$L = (1 - D(p|b_{w1}))f^2(p|b_{l1})$$

If the first-auction loser wins the second auction:

$$L = (1 - F^2(p|b_{l1}))d(p|b_{w1})$$

where d, f^2 are the derivatives with respect to the first argument of D, F^2 respectively; and b_{w1}, b_{l1} are the first-auction bids of the first-auction winner and loser, respectively. The log-likelihood of the observed second-auction data is then

$$\mathscr{L} = \sum_i log(L_i)$$

Now, to use sieve estimation, $D(\cdot|\cdot)$ and $F^2(\cdot|\cdot)$ can be approximated with Bernstein polynomials. Specifically, $D(v|b)$ and $F^2(v|b)$ can be approximated by bivariate Bernstein polynomials of the form

$$B(v,b) \equiv \sum_{i=0}^{m}\sum_{j=0}^{n}\gamma_{i,j}\begin{pmatrix} m \\ i \end{pmatrix} v^i(1-v)^{m-i}\begin{pmatrix} n \\ j \end{pmatrix} b^j(1-b)^{n-j} \tag{13}$$

where m and n are the polynomial degrees for v and b, respectively. This approximation does place a restriction that D and F^2 be continuous in b. Finally, D and F^2 are estimated by finding the polynomial parameters γ that maximize $\mathscr{L}$.

A benefit of using Bernstein polynomials is that they are easy to restrict to satisfy required properties. Since D and F^2 are cdf's, I restrict $B(v,b)$ to be weakly increasing in v by applying the restriction $\gamma_{i,j} \leq \gamma_{i',j}$ if $i < i'$. I also impose $\gamma_{0,0} = 0$ (i.e. $F^2(\underline{v}) = 0$) and $\gamma_{m,n} = 1$ (i.e. $F^2(\bar{v}) = 1$).

The second step of the estimation procedure is to estimate $s(b, v_2)$. As the proof of identification for $s(b, v_2)$ is constructive, we can use it directly as an estimator as follows. In the identification section, we said that for a fixed b, $s(b,\cdot)$ maps the α-quantile of $F^2(\cdot|b)$ to the α-quantile of $D(\cdot|b)$, because $s(b, v_2)$ is monotonic in v_2. Therefore, given $\hat{F}(\cdot|\cdot)$ and $\hat{D}(\cdot|\cdot)$ from the first step of the estimation procedure, we obtain $\hat{s}(b,\cdot)$ nonparametrically as the function that maps $\hat{F}^{2,-1}(\alpha|b) \rightarrow \hat{D}^{-1}(\alpha|b)$ for every quantile α on a grid over [0,1]. Since we can repeat this procedure for any b we choose, we have an estimator for $\hat{s}(\cdot,\cdot)$.

The third step of the estimation procedure is to estimate $F^1(\cdot)$, the distribution of v_1, using the inverse bid function $\xi(b)$ derived in (5):

$$\hat{v}_1 = \hat{\xi}(b) \equiv b + \frac{\hat{G}(b)}{(N-1)\hat{g}(b)} - \int_{v_2=\underline{v}}^{\bar{v}} \left\{ \int_{u=\underline{v}}^{\hat{s}(b,v_2)} \hat{H}_1(u|b)du - \int_{u=\underline{v}}^{v_2} \hat{H}_2(u|b)du \right\} d\hat{F}^2(v_2|b)$$

The cdf and pdf of first-auction bids $\hat{G}(\cdot)$ and $\hat{g}(\cdot)$ can be estimated from observed bids nonparametrically. $\hat{s}(\cdot,\cdot)$ and $\hat{F}^2(\cdot|\cdot)$ are known from estimation steps 1 and 2. $\hat{H}_1$ and $\hat{H}_2$, defined in (1) and (2), are functions of $\hat{F}^2(\cdot|\cdot)$ and $\hat{D}(\cdot|\cdot)$. Therefore, we are able to compute $\hat{\xi}(b)$. Since bids are monotonic in v_1, $F^{1,-1}(\alpha) \equiv v_1(\alpha) = \xi(b(\alpha))$ for any quantile α. Upon computing $\hat{\xi}(b(\alpha))$ for a grid of α over [0,1], we obtain $\hat{F}^1(\cdot)$ as the function that maps $\hat{\xi}(b(\alpha)) \rightarrow \alpha$.

If bidders are risk averse, $\hat{F}^1(\cdot)$ must be estimated conditional on a risk aversion parameter ρ, since the FOC for bidding depends on ρ. This ties into the fourth and final step of the estimation procedure.

The last step of the estimation procedure is to estimate the risk aversion parameter ρ. As discussed in section 4.2, ρ can be identified using the compatibility condition (11), $\xi(b(\alpha|N'),\rho;N') = \xi(b(\alpha|N''),\rho;N'')$. For a given value of ρ, we can evaluate the condition by using the estimated inverse bid function $\hat{\xi}(\cdot)$ to compute the left-hand and right-hand sides of the equation. Specifically, $\hat{\xi}(b(\alpha|N'),\rho;N')$ takes $b(\alpha|N')$, the α-quantile of b conditional on $N=N'$, as an argument and returns $\hat{v}(\alpha|N')$, the α-quantile of $\hat{v}_1$ conditional on $N=N'$. Likewise, $\hat{\xi}(b(\alpha|N''),\rho;N'')$ takes $b(\alpha|N'')$ as an argument and returns $\hat{v}(\alpha|N'')$. Given exogenous variation in the number of bidders, $\hat{v}(\alpha|N')=\hat{v}(\alpha|N'')$ should be true when computed using the true value of ρ. We can evaluate this condition for any value of ρ in an interval containing the true value, and find the ρ that satisfies it best, i.e.

$$\hat{\rho}=\arg\min_{\rho}\sum_{\alpha}[\hat{\xi}(b(\alpha|N'),\rho;N')-\hat{\xi}(b(\alpha|N''),\rho;N'')]^2$$

If the number of bidders varies endogenously, we can use the modified compatibility condition as discussed in section 4.2.

Alternatively, we can develop an estimator for ρ based on identification using the auction sequence, as explained at the end of section 4.2. The criterion for similarity between $F^1(\cdot)$ and $F^2(\cdot)$ could be that the distance between the two distributions be minimized in the sense that

$$\hat{\rho}=\arg\min_{\rho}\sum_{\alpha}[\hat{\xi}(b(\alpha),\rho)-\hat{F}^{2,-1}(\alpha)]^2. \tag{14}$$

5.2 Auction heterogeneity

In the model and identification sections, auction-specific heterogeneity was suppressed for expositional ease. In the real data, there are characteristics z that differ across pairs, which must be accounted for in estimation.

Supposing we had a very large sample, the ideal way to deal with heterogeneity would be to estimate separate value distributions for every value of z. However, this approach is usually infeasible given the size of real datasets. As a result, a common approach in the empirical auction literature, as explained in Haile, Hong, and Shum (2003), has been to homogenize bids across auctions by "demeaning" them, i.e. transforming bids to residuals $\epsilon=b-z'\beta$ and working with the residuals in estimation. This allows one to "pool" all the data. The underlying assumptions are that $v=z'\beta+\mu$ (additive separability), and that the distribution of μ is invariant to z (homoskedasticity). Depending on the context, however, these assumptions may be quite strong.

In this paper, I need to homogenize bids in order to perform the first step of estimation, where I recover $F^2(\cdot|b)$ and $D(\cdot|b)$ using a sieve maximum likelihood estimator. As I do so, I seek to make the minimal assumptions that still allow me to pool heterogeneous objects for the estimation task at hand. Instead of transforming bids and prices to demeaned residuals, I transform bids and prices to quantiles conditional on $z'\beta$; that is, $b\rightarrow\tilde{b}\equiv G(b|z'\beta)$ and $p\rightarrow\tilde{p}\equiv J(p|z'\beta)$, where $G(\cdot)$ is the distribution of first auction bids and $J(\cdot)$ is the distribution of second auction prices. Both G and J are observed in the data. I then use these quantiles to perform the first step of estimation. Afterwards, the output from this step is transformed back to real values before proceeding with the other steps of estimation. Note that demeaning is a special case of taking quantiles; under assumptions of additive separability and homoskedasticity, the residuals $\epsilon = b - z'\beta$ map to quantiles of the bid distribution.

Although I would like to make as few assumptions as possible, this method is not without assumptions. The following assumptions underly the homogenizing procedure.

Assumptions

AS8 Single index assumption: $F^1(\cdot|z) = F^1(\cdot|z'\beta)$, $F^2(\cdot|\cdot, z) = F^2(\cdot|\cdot, z'\beta)$

Now define $\alpha_1 \equiv F^1(v_1|z'\beta)$, $\alpha_2 \equiv F^2(v_2|z'\beta)$, and $\alpha_s \equiv F^2(s(v_1, v_2)|z'\beta)$. Also define $\tilde{\alpha}_2 \equiv J(v_2|z'\beta)$ and $\tilde{\alpha}_s \equiv J(s(v_1, v_2)|z'\beta)$, where $J(\cdot)$ is the distribution of second-auction prices.

AS9 Quantile relationships are invariant to z:

1. $C(\alpha_1, \alpha_2|z) = C(\alpha_1, \alpha_2)$
2. $C(\alpha_1, \alpha_s|z) = C(\alpha_1, \alpha_s)$

In assumption AS9, the $C(\cdot, \cdot)$'s are copulas defined on $[0, 1]^2 \to [0, 1]$. AS9 says that the quantile of a bidder's v_1 implies a distribution for what the quantile of his v_2 will be, and that this quantile-to-quantile relationship, or copula, is invariant to $z'\beta$. A8.2 is the strongest part of the assumption, as it implicitly restricts the synergy function $s(\cdot, \cdot)$ to preserve a quantile relationship across different $z'\beta$. For instance, suppose that when $z'\beta = High$, synergy boosts a bidder's value for the second object from the 0.5-quantile to the 0.6-quantile for $z'\beta = High$. Then AS9.2 implies that synergy must also boost a bidder at the 0.5-quantile to the 0.6-quantile when $z'\beta = Low$.

Proposition 4. *Under assumptions AS8 and AS9, $C(\alpha_1, \tilde{\alpha}_2)$ and $C(\alpha_1, \tilde{\alpha}_s)$ are invariant to z.*

Given Proposition 4, it directly follows that the objects to be estimated - $F^2(\tilde{\alpha}_2|\alpha_1)$ and $D(\tilde{\alpha}_s|\alpha_1)$ - are invariant to $z'\beta$, since $F^2(\tilde{\alpha}_2|\alpha_1)$ is just a marginal of $C(\alpha_1, \tilde{\alpha}_2)$, for instance. Therefore, observations with different $z'\beta$ can be pooled in the first step of estimation once the bids and prices have been transformed to quantiles in this way.[20]

[20]If, in addition, $F^2(\cdot|\cdot, z'\beta)$ does not vary with the number of bidders N, observations with different N can be pooled during the first step of estimation. In that case, b should be transformed to $\tilde{b} \equiv G(b|z'\beta, N)$,

To provide a comparison, if we were to take the demeaning approach, we would need all of the assumptions made here and two more in addition: that v_1 and v_2 are additively separable functions of $z'\beta$ and a residual μ, and that these residuals have the same distribution regardless of $z'\beta$. The quantile approach, on the other hand, does not assume additive separability and allows marginal distributions to vary with $z'\beta$.

I restate that only the first step of estimation requires homogenized bids. After the first step, $\hat{F}^2(\tilde{\alpha}_2|\alpha_1)$ and $\hat{D}(\tilde{\alpha}_s|\alpha_1)$ are translated back to their real-valued versions $\hat{F}^2(v_2|v_1)$ and $\hat{D}(s(v_1,v_2)|v_1)$ before proceeding with the other steps of estimation.

5.3 Monte Carlo study

To evaluate the ability of the estimation procedure to recover the synergy function, I simulate datasets of varying size and apply the procedure. The model underlying the simulated data is specified as follows:

- $N = 2$
- $v_1 \sim U[0,1]$
- $v_2 \sim$ Triangular$(0, 1, v_1)$
- $s(v_1, v_2) = \min(v_2 + 0.1, 1)$

Triangular(0,1,v_1) is a triangular distribution with lower limit 0, upper limit 1, and peak at v_1. Synergy takes a simple form in which a constant 0.1 is added to v_2 up to the constraint that $v_2 \leq 1$.

Sieve orders (i.e. polynomial degrees) selected by the AIC (Akaike information criterion) and BIC (Bayesian information criterion) are displayed in Table 5. Although originally derived for parametric models in the asymptotic case, AIC and BIC are sometimes used to select sieve orders for sieve estimation, where established rules of thumb do not exist. Estimation results for 100 Monte Carlo runs are displayed in Figure 4 for both AIC and BIC. The figures depict estimated synergy functions against the known, true synergy function.

When the sample size is small, the estimated synergy function is biased upwards. This is because a small sample size leads to selection of small sieve orders, which may not control for v_1 tightly enough in estimates of $F^2(v_2|v_1)$ and $D(v_2|v_1)$. Since synergy is identified by

conditioning on N as well as $z'\beta$. This is necessary because, in a first-price auction, the bidding strategy changes when N changes. On the other hand, p should be transformed to $J(p|z'\beta, N = n)$, fixing n, regardless of the number of bidders. This is alright because, in the second auction which is English, the bidding strategy does not change with N. In fact, this is necessary for pooling; unlike $G(\cdot)$, $J(\cdot)$ is the distribution of the second highest out of N values, so the meaning of the statistic represented by J changes with N. If we transform each p to the quantile of $J(\cdot|z'\beta, N)$ for its own N, it would be like all the $\tilde{p}$'s are in different units, and it would make no sense to pool across N.

Figure 4: Monte Carlo experiments using estimator with AIC (left) and BIC (right)

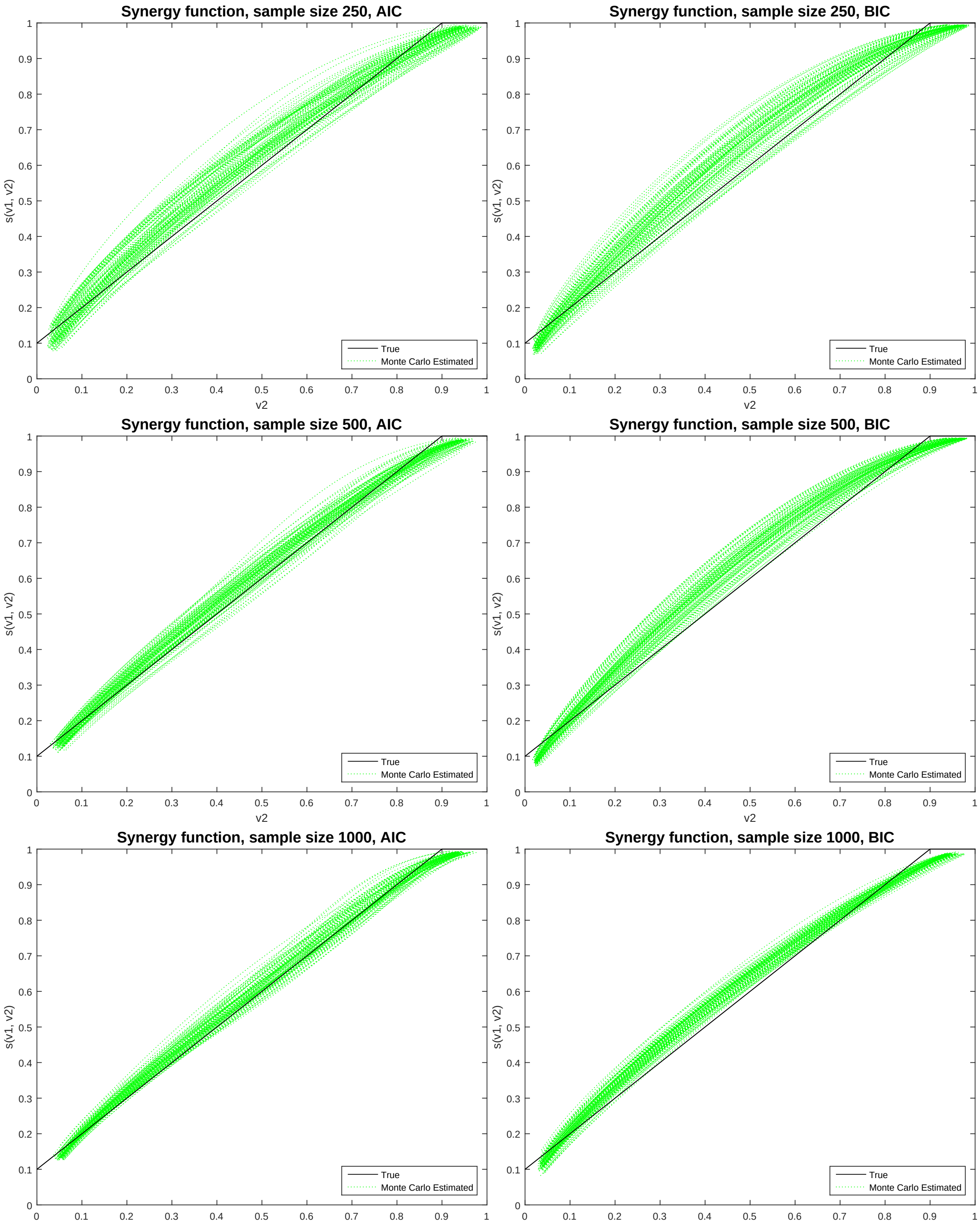


Table 5: Simulation sieve orders selected by AIC and BIC

	AIC		BIC	
sample size	m	n	m	n
250	4	2	3	2
500	5	2	3	2
1000	6	2	4	2

*m and n are defined in equation (13).

Table 6: Simulation sieve orders selected by IC3 and IC4

	IC3		IC4	
sample size	m	n	m	n
250	5	2	5	2
500	7	2	8	3
1000	7	3	10	3

*m and n are defined in equation (13).

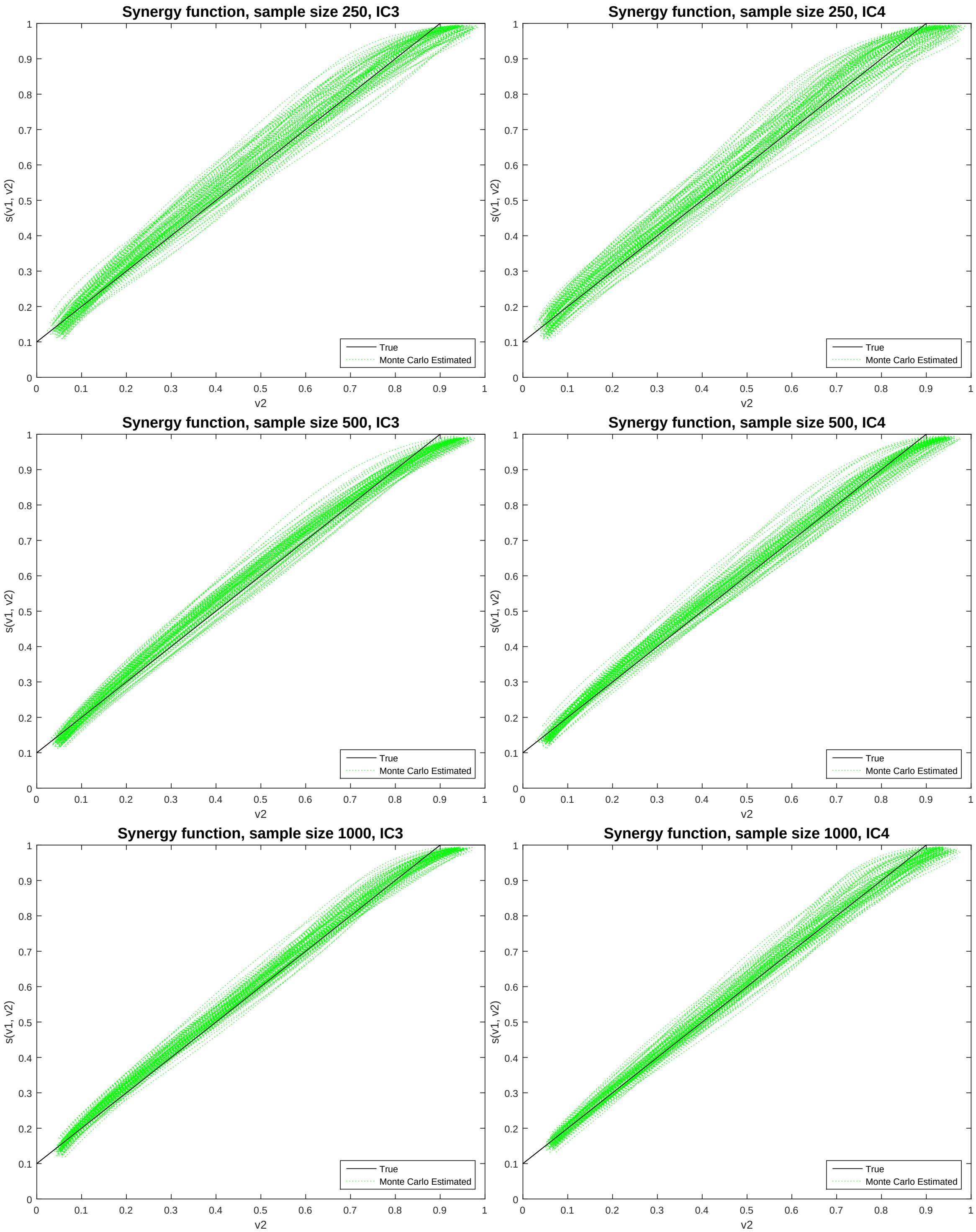
comparing $F^2(v_2|v_1)$ and $D(v_2|v_1)$ for the same v_1, small sieve orders may allow the effects of affiliation to seep into the estimates of synergy, resulting in upward bias. However, as sample sizes and therefore sieve orders increase, the bias goes to zero, and the synergy function is estimated quite well.

AIC, which selects larger sieve orders than BIC, seems to result in smaller bias than BIC. It should be noted that the parameters being estimated are not free and independent. As the polynomials are approximating cdf's, all parameters are bounded by [0,1]. Furthermore, since we restrict F^2 and D to be nondecreasing in v_2, as all cdf's should be, $\gamma_{i,j}$ is bounded by $[\gamma_{i-1,j}$,1]. In light of this, BIC and even AIC may be penalizing the number of parameters more than is optimal.

In order to reduce the bias in smaller samples, I experiment with criteria that penalize the number of parameters less than do AIC and BIC. AIC seeks to minimize $2k-2ln(L)$, and BIC seeks to minimize $ln(n)k-ln(L)$, penalizing k by a multiple of 2 and $ln(n)$, respectively, where k is the number of estimated parameters, $ln(L)$ is the log likelihood of the data, and n is the sample size. I experiment with "IC3" and "IC4", which I define as IC3 $= k-2ln(L)$ and IC4 $= 0.5k-2ln(L)$. I repeat the simulations in Figure 4 with the newly defined criteria. Sieve orders selected by IC3 and IC4 are displayed in Table 6, and Monte Carlo results are displayed in Figure 5.

Compared to AIC and BIC, IC3 and IC4 seem to reduce bias at an acceptable cost to variance, and IC4 in turn seems favorable to IC3. I use the IC4 criterion to select sieve orders in my estimation.

Figure 5: Monte Carlo experiments using estimator with IC3 and IC4



6 Estimation using the paired leases in New Mexico

6.1 Sample and model used

Since both winners and losers are needed in order to identify synergy, the number of bidders N must be at least 2 for this estimation procedure. Also, since bid functions in A1 depend on N, a number of steps in the estimation procedure are conducted separately for each value of N. Therefore, there must be a sufficient number of auctions observed in the sample for each value of N considered in the estimation. Looking at Table 1, the sample size is largest at $N = 2$ and $N = 3$, and the number of observations becomes rather small for auctions with $N \geq 4$. Therefore, I use $N = 2$ and $N = 3$ in my estimation, which gives a sample of roughly 400 pairs.

As the model assumes that the set of bidders participating in the first and second auction are the same, observations in which the second-auction winner did not bid in the first auction are dropped from the sieve maximum likelihood estimator. As shown in Table 2, 7% of all pairs with $N \geq 2$ fall into this category. Allowing for an extra bidder in the second auction is not difficult in terms of modeling, but in practice the sample size of the data and scarcity of wins by the extra bidder discourage adding another primitive out of concern for poor estimates.

In light of the relatively small sample size, I estimate the model with symmetric bidders to reduce the burden on the estimator. Meanwhile, as Kong (2015) found risk aversion to be important in the New Mexico oil and gas lease auctions, I do allow bidders to be risk averse. Thus the primitives of the model are the v_1-distribution $F^1(\cdot)$, the conditional v_2-distribution $F^2(\cdot|\cdot)$, the synergy function $s(\cdot,\cdot)$, and the utility function $U(\cdot)$. I choose the constant relative risk aversion (CRRA) specification for utility, so $U(\cdot)$ reduces to a risk aversion parameter ρ.

As explained in section 4.2, the risk aversion parameter can be identified by using variation in the number of bidders or by using the auction sequence. I present estimated values obtained from the latter method using the estimator described in (14), as I find that for this dataset and model the former method yields unstable results.

6.2 Covariates z

As discussed in section 5.2, lease characteristics z will be used to form a single index $z'\beta$ that controls for heterogeneity across pairs. This section explains the z's used to form $z'\beta$. It should be emphasized that the main purpose here is not to study the β coefficient on each variable z. Rather, it is to form an index $z'\beta$ that absorbs as much of the heterogeneity between auction items as possible, such that conditional on the index, remaining variation in bidders' values is due to bidder-level idiosyncracies. The z's are chosen with this purpose in mind.

Observable characteristics of auctioned leases fall into three categories: lease terms, time of auction (industry, economic, local conditions of that time), and location of the tract (encompassing geological features). The royalty rate is indicated by the lease prefix, VA (subregular), V0 (regular), or VB (premium) in this sample; better tracts are assigned higher rates. As the VA prefix was discontinued in 2005, prefixes pre-2005 will be distinguished from prefixes post-2005. The contracted duration of a lease absent production does not vary in this sample, at 5 years.

To represent the effects of time, I include year fixed effects as well as month fixed effects to reflect any seasonality. These are supplemented by oil prices (West Texas Intermediate) and gas prices (natural gas 1 month futures). In addition, average price per acre in the previous month's auctions and average price per acre in the federal Bureau of Land Management's[21] lease sales in the same month are included to reflect local and industry conditions around the time.

The location of the tract implies geological information. As a first-level control, the volume of oil produced on the tract between 1970 and the auction date and the volume of oil produced after the auction date through 2014 are included as indicators of a tract's potential for production. These are truncated measures, because a large fraction of leases are not drilled and production volume can be positive only if the lease-holder decides to drill. Furthermore, these auctions are too recent for us to observe the full post-auction production volume, as oil and gas wells can keep producing for years, even decades. More generally, these are imperfect measures of all the ways the particular geology and location of the tract may affect tract value. To better control for tract heterogeneity, I also construct a geographic "heatmap" of value as follows, using the data available to me. I take deflated sealed bid data from the SLO auctions and fit a smooth surface of these bids on geographic (north-south and east-west) coordinates using local quadratic regression. This procedure is performed once for each auction, excluding own-auction bids from the smoothing procedure, and the heatmap value for each tract is the value predicted by the fitted surface excluding own-auction bids. Smoothing over bids from multiple bidders from different times on multiple tracts in the local geographic area should average out bidder and auction idiosyncracies that are orthogonal to location while retaining the component that is endogenous or persistent to geographic location. This heatmap value serves as a further control for location-determined heterogeneity.

To form the index $z'\beta$, I regress log submitted sealed bids on these covariates. Table 7

[21] The BLM is a bureau that manages federal public lands, and is distinct from the State Land Office that manages state trust lands.

shows the results. The coefficients on heatmap value and production volume are positive as expected. The heatmap value in particular has good explanatory power; it is in units of log dollars, so a 1% increase in heatmap value is associated with a 0.7% increase in bids. The lease prefix dummies are categorical variables, so the lowest-level prefix is omitted: VA is omitted for the years before 2005, and V0 is omitted for 2005 and after when the VA prefix is discontinued. Therefore, the categories included in the regression are premium relative to the omitted category, and the coefficients on them are positive. The coefficients on oil and gas prices and prices in recent lease sales are also positive as expected.

6.3 Empirical results

In this section I discuss the empirical results from applying the estimation procedure to the data. Figure 6 displays the estimated distribution $\hat{F}^2(v_2|v_1)$ conditional on the 25th, 50th, and 75th percentiles of v_1. They are stochastically ordered in v_1, indicating that v_1,v_2 are affiliated. $\hat{F}^2(\cdot|v_1)$ is fairly dispersed. There are a number of possible explanations for dispersion in v_2 even after conditioning on v_1. As mentioned before, the outcomes of auctions that take place between A1 and A2 are one explanation. Bidders' own auction outcomes may affect v_2 through budget constraints or not wanting to "go home with nothing." Outcomes for competitors may also have an influence on v_2 for competitive reasons. Some sort of learning may be taking place as well. On the other hand, it may be that $\hat{F}^2$ is more dispersed than the true F^2 due to approximations made in the estimated model. For instance, I estimate a model of symmetric bidders, but real bidders are not truly symmetric. Also, if the true F^2 is conditional on a second signal besides v_1, not accounting for that signal would result in a more dispersed $\hat{F}^2$.

Figure 7 visualizes the affiliation of v_1 and v_2 by plotting the conditional density $\hat{f}^2(v_2|v_1)$. Lighter colors indicate higher densities, and the tail ends have been trimmed to accentuate the shift of the central mode. We can see that as v_1 increases, so does the modal value of $v_2|v_1$.

Figure 8 plots the estimated synergy function. The estimator measures positive synergy, as $s(v_1, v_2) > v_2$. The added benefit of synergy, i.e. $s(v_1, v_2) - v_2$, appears fairly constant in v_1 and v_2. For the median value of v_1 at median $z'\beta$, $E_{v_2}[s(v_1, v_2) - v_2]$ is estimated to be on the order of $13,000.

The CRRA parameter ρ is estimated using the estimator defined by (14). Figure 9 displays the object the estimator seeks to minimize as a function of ρ, $\frac{1}{n_\alpha}\sum_{i=1}^{n_\alpha}[\hat{\xi}(b(\alpha_i), \rho) - \hat{F}^{2,-1}(\alpha_i)]^2$, where α_i are deciles. ρ appears to be identified, as there is a unique value that minimizes the object, $\hat{\rho} = 0.47$. As a comparison, Kong (2015) estimates CRRA parameters of 0.49 and 0.23 for two subgroups of bidders, Holt and Laury (2002) measure CRRA parameters centered around the 0.3-0.5 range in laboratory experiments, and Lu and

Table 7: Regression of ln(sealed bid) on observable characteristics

	lnbid
heatmap value	0.710***
	(0.057)
ln(oil prod) 1970-auction date	0.022
	(0.014)
ln(oil prod) auction date-2014	0.012
	(0.007)
lease prefix V0 pre-2005	0.091
	(0.065)
lease prefix VB pre-2005	0.102
	(0.280)
lease prefix VB post-2005	0.330***
	(0.076)
ln(nat gas 1 mo futures)	0.317**
	(0.132)
ln(WTI oil price)	0.134
	(0.221)
ln(prior month price/acre)	0.123**
	(0.061)
ln(BLM price/acre)	0.267***
	(0.055)
Constant	0.157
	(0.966)
Year fixed effects	Y
Month fixed effects (seasonality)	Y
Observations	2095
R^2	0.261
Adjusted R^2	0.248

Heteroskedasticity robust standard errors in parentheses
* p<0.10, ** p<0.05, *** p<0.01

Figure 6: $\hat{F}^2(v_2|v_1)$ at median $z'\beta$

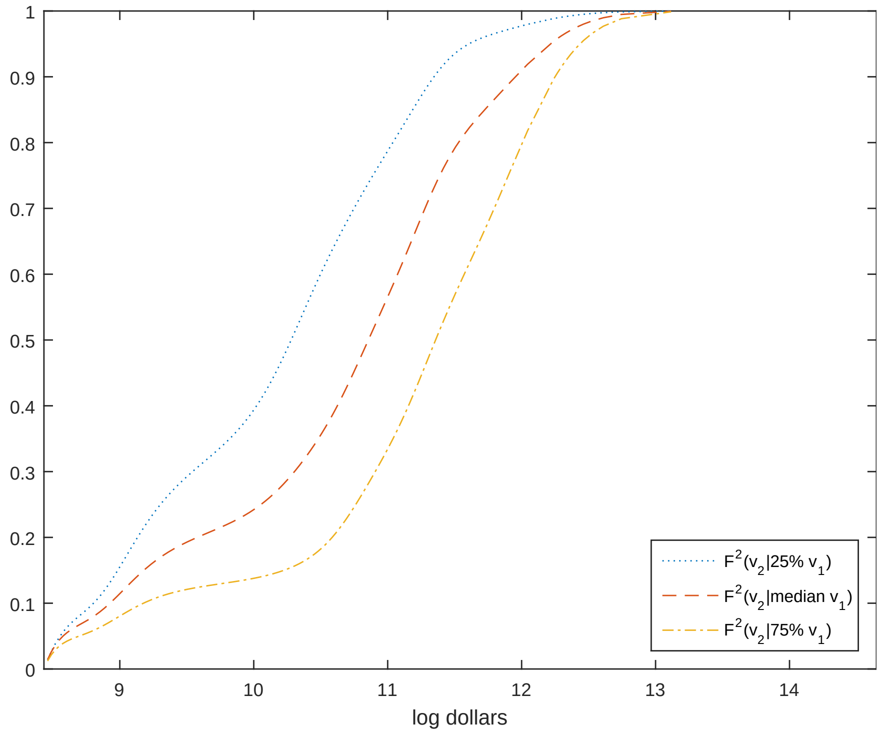


Figure 7: $\hat{f}^2(v_2|v_1)$ at median $z'\beta$

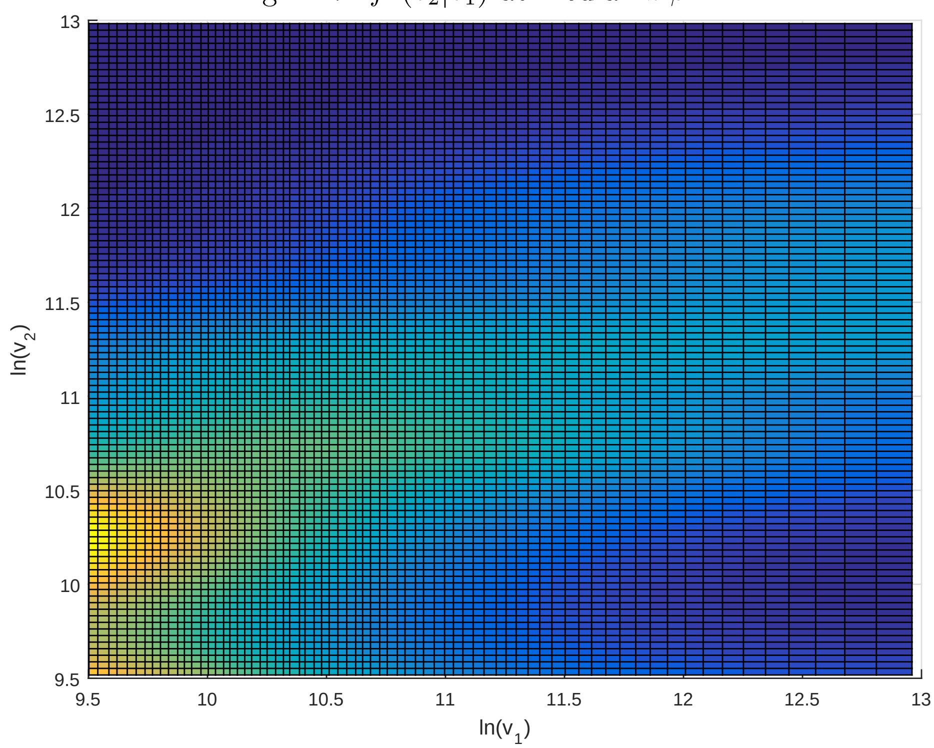
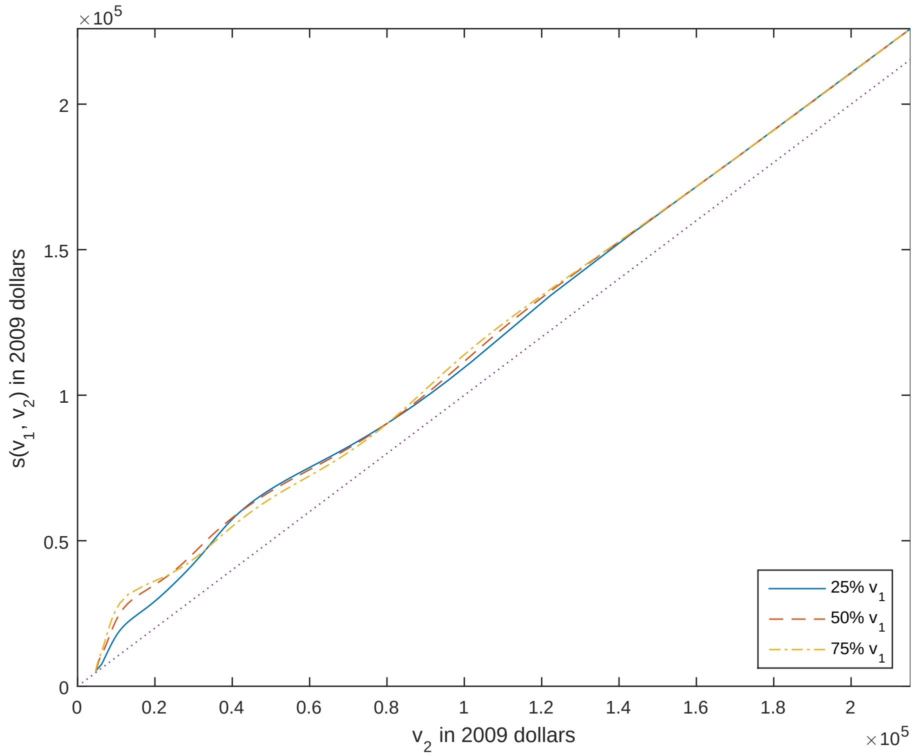


Figure 8: $\hat{s}(v_1, v_2)$ at median $z'\beta$



Perrigne (2008) measure roughly 0.59 for the USFS timber auctions. Figure 10 displays the distribution of v_1 that is estimated given $\hat{\rho} = 0.47$.

7 Counterfactuals

Given the structural estimates obtained, I perform counterfactual simulations to understand the driving forces behind what is observed, and predict the outcome of alternative policies. Table 8 displays the results of counterfactual simulations performed with these objectives in mind.

The "observed" row shows what is observed in the data for pairs with $N = 2$ at median $z'\beta$.[22] Under the "simulated" heading, row (1) displays the expected revenue simulated using the full model; it is the simulated analog of the "observed" row. Subsequent rows show what revenue would be if selected elements of the full model were shut down. Columns (d)-(f) show counterfactual revenue when the pair is auctioned as a bundle, using the first-price sealed-bid format, the English auction format, and an even use of the two formats, respectively. Keeping in mind that first-price auctions yield higher revenue than English auctions when bidders are risk averse, column (f) is useful because it provides a fairer comparison with the

[22] Revenue "at" median $z'\beta$ is computed via kernel regression of revenue on $z'\beta$.

Figure 9: Mean squared difference between $\hat{F}_1(\cdot)$ and $\hat{F}_2(\cdot)$ as a function of ρ

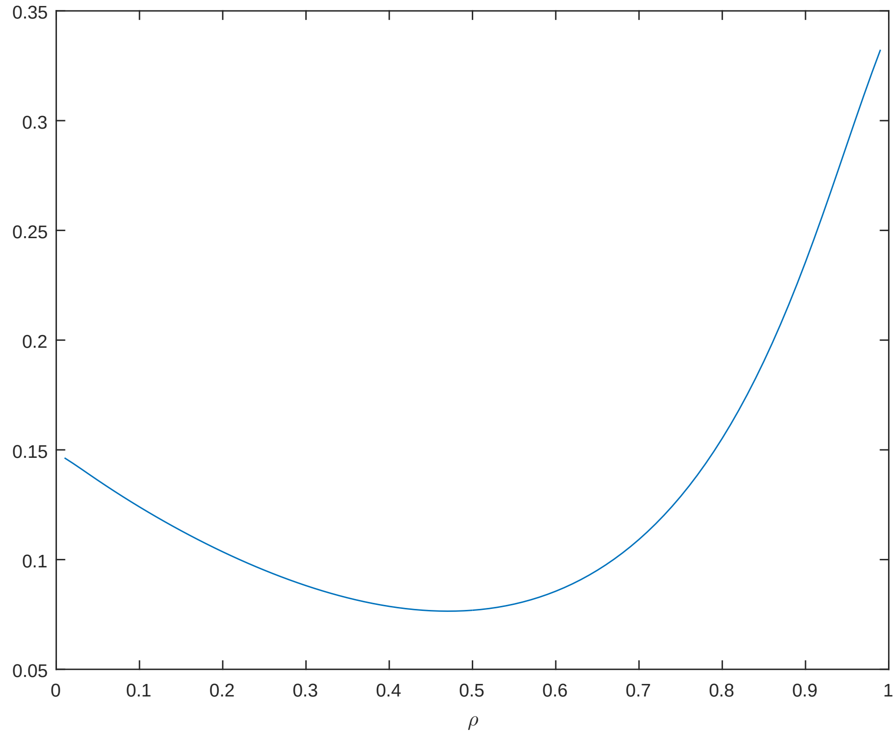


Figure 10: $\hat{F}^1(v_1)$ at median $z'\beta$

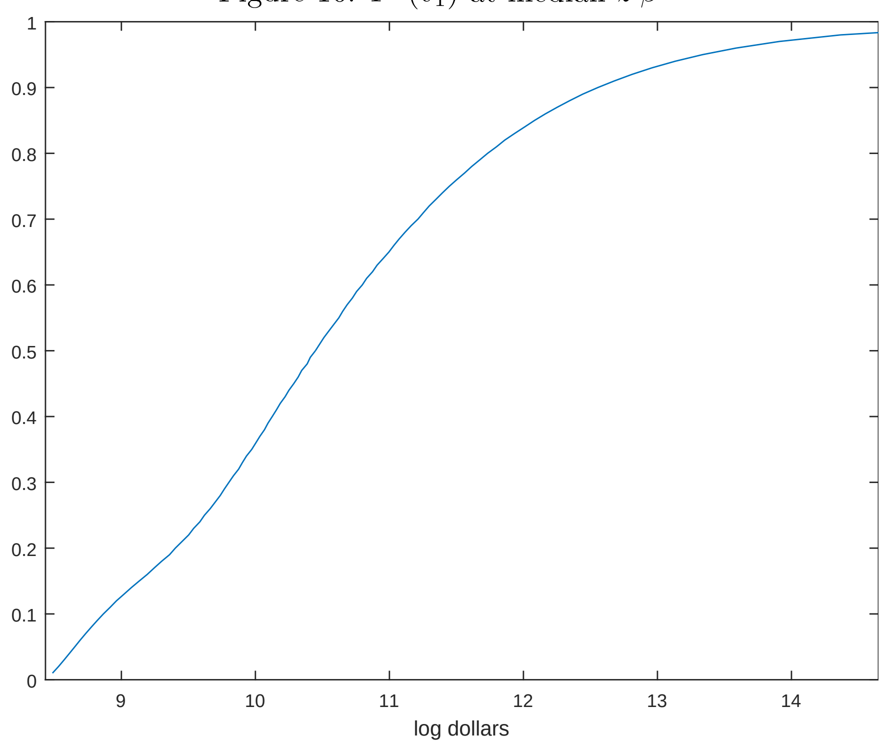


Table 8: Counterfactual revenues for a pair: median $z'\beta$, $N = 2$

		Sequential Revenue			Bundled Revenue			
	% same winner	(a) A1	(b) A2	(c) Total	(d) FPSB	(e) Eng	(f) $\frac{\text{(d)+(e)}}{2}$	(g) $\frac{\text{(f)-(c)}}{\text{(c)}}$
Observed	78%*	52,682	40,155	92,837	-	-	-	-
Simulated								
(1) S + RA + A	75%	53,669	39,082	92,751	120,344	89,571	104,958	13%
(2) RA + A	69%	45,469	36,507	81,976	108,875	79,689	94,282	15%
(3) A	69%	34,261	36,507	70,768	79,000	79,689	79,345	12%
(4) S + RA	55%	50,865	39,645	90,510	129,618	100,571	115,094	27%
(5) S + A	75%	44,409	39,082	83,491	88,831	89,571	89,201	7%

*Excluding cases where A2-winner did not bid in A1, as these cases were not used in estimation.

"S" = synergy

"A" = affiliation

"RA" = risk aversion

sequential auctions, which use both formats. In addition, the State Land Office may have institutional reasons for using both formats, and column (f) respects that constraint. In each of these bundled auction simulations, I assume that v_1,v_2 are both known at the time of bidding, in order to provide a fair comparison with not bundling. The total value of the bundle is $v_1 + s(v_1, v_2)$.

Comparing the "observed" row to row (1) gives a sense of model fit. The model does a good job of fitting the probability that the same bidder wins both tracts and expected revenue in the first and second auction.

We observed in Table 2 that the A1-winner is more likely than other bidders to win A2, but until now we were unable to assess whether this was due to synergy or affiliation. Comparing rows (1) and (2) reveals that if synergy were eliminated, the proportion of cases in which the same bidder wins both tracts would drop from 75% to 69%. On the other hand, row (4) shows that if v_1,v_2 were not affiliated, that percentage would drop to 55%. We can conclude that both synergy and affiliation are responsible for the same-winner %, but affiliation is the primary explanation. This highlights the importance of allowing for and distinguishing affiliation from synergy.

Another phenomenon observed in the pairs data is that revenue is higher in the first auction. Comparing rows (1)-(3) helps us understand the forces behind that observation. First, looking at row (1) relative to row (2), synergy seems to increase revenue in both auctions, but increases A1 revenue more, playing a part in the A1-A2 revenue difference. But second, comparing rows (2) and (3) reveals that a large part of the revenue gap is explained by risk aversion, which increases bidding strategies in A1 (first-price) but not in A2 (English). This is consistent with Kong (2015), which finds that risk aversion is primarily responsible for the dominance of first-price auctions over English auctions in the New Mexico setting overall.

An obvious policy alternative in the presence of synergy would be to auction the pair as a bundle, as this guarantees that the winning bidder will realize synergy. A downside of bundling is that it forces a single bidder to take both tracts, even when the highest-value bidder for each tract is different. A general theoretical comparison of sequential versus bundled auctions that applies to this model does not exist. Ultimately, whether to bundle these tracts is an empirical question that depends on the primitives - including the size of synergy, the shape of the value distributions, the degree of affiliation, and degree of risk aversion - and their interaction. The size of synergy matters, because the computations of Subramaniam and Venkatesh (2009) suggest that the larger the synergy, the more likely that bundled auctions will increase revenue. Also, if synergy is large, a social planner may want to ensure that it is always realized (by using bundled auctions), while if it is small, he may prefer to award each tract separately to the highest-value bidder. Meanwhile, bundling tends to reduce heterogeneity in values (see Schmalensee (1984)) and more generally change the shape of the auction-relevant value distribution. How this will matter depends on the shape of the non-bundled value distributions and degree of affiliation between the two tracts. Risk aversion matters, because it affects bidding strategies in first-price auctions, and also because it changes the way bidders internalize the uncertainty surrounding the second auction and synergy when they bid in the first auction.

Comparing column (f) to column (c) indicates that bundling would increase auction revenue over sequential sales, assuming the State Land Office maintains its policy of using both the first-price sealed-bid and English auction formats evenly. Column (g) computes the percentage increase in auction revenue that would come from this bundling. Judging from rows (1)-(5) of column (g), the benefit of bundling over not bundling seems to be greatest when both synergy and risk aversion are present. I conjecture that this is because bidders can depend on realizing synergy if they win the bundled auction - unlike in sequential auctions, where a bidder may lose the second auction even after winning the first - and this certainty is relatively more valuable when bidders are risk averse than when they are risk neutral. Comparing rows (1) and (4), affiliation reduces the revenue benefits of bundling. The more affiliated v_1 and v_2 are, the more likely that the same bidder will win both tracts anyway, so bundling matters less.

Having used Table 8 to understand the forces at work, I focus exclusively on the question of whether to bundle in Table 9. Row (1) of Table 9 restates row (1) of Table 8. Meanwhile, since these auctions are run by a public institution, revenue considerations must be balanced against allocative efficiency, or the desire to award tracts to the firms that value them most. Row (2) addresses allocative efficiency by computing the total value derived from a pair of

Table 9: Sequential versus bundled auctions, revenue and allocation

		Sequential	Bundled			
		(c)	(d) FPSB	(e) English	(f) $\frac{(d)+(e)}{2}$	(g) $\frac{(f)-(c)}{(c)}$
$N = 2$						
(1)	Revenue per pair	92,751	120,344	89,571	104,958	13%
(2)	Value of tracts to winner(s)	349,070	343,609			-2%
$N = 3$						
(3)	Revenue per pair	155,736	187,302	146,417	166,860	7%
(4)	Value of tracts to winner(s)	423,869	410,250			-3%

At median $z'\beta$

tracts by the winner(s). If a single bidder wins both - which is always the case for bundled auctions - this total value is inclusive of synergy. Row (2) shows that bundling leads to a small loss in this total value, of roughly 2%. One reason the loss is small is that, even in the sequential auctions currently being used, the same bidder often wins both tracts, leading to the same allocative outcome as bundled auctions. In the remaining cases where the allocative outcomes are different, cases that favor bundling and cases that favor sequential auctions seem to balance out. The gains to bundling come from synergy, and the losses come from not giving each tract to its respective highest-valuer. Now, I repeat this counterfactual exercise for three-bidder auctions, considering revenue and allocative efficiency in rows (3) and (4). As was the case for two-bidder auctions, bundling leads to higher revenue with a small loss to allocative efficiency, though the revenue gains of bundling are smaller than in the two-bidder case.

As we saw in Table 1, a large majority of these auctions receive three bids or less. Relating to the bundling literature, the finding in Table 9 that bundling would be better in two-bidder and three-bidder auctions is consistent with the computations of Subramaniam and Venkatesh (2009), which suggest that the smaller the number of bidders, the more likely are bundled auctions to dominate sequential auctions in terms of revenue. This can be reversed for larger N, where it may be optimal to exploit competition twice by selling each tract separately. This helps explain why the revenue gains from bundling in row (3) are smaller than in row (1). The result is also generally consistent with papers that study bundling in contexts without synergy, such as Palfrey (1983) and Chakraborty (1999). Both of these papers find that the smaller the number of bidders, the more likely is bundling to increase revenue in Vickrey auctions.

One caveat in interpreting these results is that this comparison of sequential versus bundled auctions holds the number of bidders constant across the two policies. Leases of the bundled size are too rare in the data to deduce whether and how the act of bundling would change the number of bidders. Here I provide computations for the baseline case of no change.

8 Conclusion

This paper performs a structural analysis of two auctions that take place sequentially, are linked by synergy, and in which each bidder's values can be affiliated across auctions. It explains that ignoring affiliation can lead to falsely detecting synergy where none exists, and distinguishes synergy from affiliation in identifying and estimating the auction model. The model uses general functional forms for synergy and the joint distribution of v_1,v_2 while allowing for risk averse bidders. The paper establishes nonparametric identification of this model and develops a multi-step estimation procedure that recovers all model primitives. Applying the estimation method to oil and gas lease data, I find both synergy and affiliation between adjacent tracts. Affiliation is very important in explaining why the same bidder often wins both tracts. The model predicts, and counterfactual decomposition confirms, that synergy increases revenue in both auctions relative to the case of no synergy.

Meanwhile, bidders are risk averse, and this boosts first-auction revenue substantially, as the first auction is a first-price auction. Interestingly, it seems that the first-price sealed-bid auction - English auction sequence used in New Mexico strikes a balance between revenue and allocative efficiency. The first-price auction in the first stage takes advantage of higher bids generated by risk aversion, while in the second stage, where synergy induces endogenous asymmetry, the English auction maintains allocative efficiency. Counterfactual simulations reveal that bundled auctions would yield higher revenue, given the combination of synergy and risk aversion and the typically low number of bidders.

Groundwork for extending the procedure to asymmetric bidders is provided, but there are a number of issues to be considered when doing so. As the number of primitives multiplies, the sample size needs to grow. Stronger assumptions will be needed when homogenizing bid data. New channels of bias can arise when estimating asymmetric synergy functions with real data, and they should be assessed carefully.

The paper opens the door to analyzing synergy and affiliation in other types of sequential auctions. The main insight for distinguishing synergy from affiliation is adaptable to other auction formats, such as two second-price auctions, as long as first-auction bids are monotonic in values and observed. The equilibrium bidding strategy would have to be worked out separately for each type of sequence. Another very interesting possibility is that of extending the model to a longer sequence of affiliated items. Affiliation of a bidder's values across a longer sequence creates a challenge for analysis, but since bidders do not know future values ahead of time, the model retains hope of tractability, perhaps with the help of some wellplaced assumptions. These questions remain open for future research.

Appendix

Other bidders

In section 2.2, regression discontinuity results using local linear regression were shown for the most frequent bidder. Going down the list of bidders ordered by frequency of bids, the number of observations drops exponentially. For each of the remaining bidders, there were not enough observations near $z = 0$ to perform a meaningful regression discontinuity analysis. However, we can still examine some simple statistics for clues. For the 3 other bidders that had at least 5 observations on each side of $z = 0$ with $|z| < 0.2$, the following table shows the probability of winning the second auction when $z \in [-0.2, 0]$ (i.e. lost first auction with less than 20% bid difference) versus when $z \in [0, 0.2]$ (i.e. won first auction with less than 20% bid difference).

Table 10: Probability of winning the second auction, given $|z| < 0.2$

Bidder name:	CP	DG	DS
Lost first auction	22%	38%	0%
Won first auction	30%	57%	20%

Deriving the first-order condition in section 3.3

A bidder will bid the b that maximizes the expected profit $\pi(v_1, b)$. Taking the derivative of $\pi(v_1, b)$ with respect to b and setting it equal to zero gives

$$\begin{aligned} &\textstyle\int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{t=\underline{b}}^{b} (-1) dG^{N-1}(t) \\ &+ [v_1 - b + \textstyle\int_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1,v_2) - u) dH_1(u|b)](N-1)G^{N-2}(b)g(b) \\ &- \textstyle\int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2(u|b)(N-1)G^{N-2}(b)g(b)\} dF^2(v_2|v_1) \quad = \quad 0 \end{aligned}$$

Rearranging this gives

$$\begin{aligned} \tfrac{G(b)}{(N-1)g(b)} \quad = \quad &\textstyle\int_{v_2=\underline{v}}^{\bar{v}} \{ v_1 - b + \int_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1,v_2) - u) dH_1(u|b) \\ &- \textstyle\int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2(u|b)\} dF^2(v_2|v_1) \end{aligned}$$

Some algebra using integration by parts shows that

$$\int\limits_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1,v_2) - u) dH_1(u|b) = \int\limits_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|b) du$$

$$\int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2(u|b) = \int_{u=\underline{v}}^{v_2} H_2(u|b) du$$

So the first-order condition can be simplified to

$$b = v_1 + \int_{v_2=\underline{v}}^{\bar{v}} \{ \int_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|b)du - \int_{u=\underline{v}}^{v_2} H_2(u|b)du\} dF^2(v_2|v_1) - \frac{G(b)}{(N-1)g(b)}$$

Proposition 1

Proof. First, I show that if $v_1' > v_1$, $b \in BR(v_1)$ (best response set), and $b' \in BR(v_1')$, then it must be that $b' \geq b$. Suppose not; suppose $b' < b$. By definition of best response, $b \in BR(v_1)$ means $\pi(v_1, b) - \pi(v_1, b') \geq 0$ and $b' \in BR(v_1')$ means $0 \geq \pi(v_1', b) - \pi(v_1', b')$. Defining $\kappa(v_1) \equiv \pi(v_1, b) - \pi(v_1, b')$, this means $\kappa(v_1) \geq \kappa(v_1')$. Writing out $\kappa(v_1)$ gives the following expression:

$$\begin{aligned} \textstyle\int_{v_2=\underline{v}}^{\bar{v}} \{ \quad & v_1[G^{N-1}(b) - G^{N-1}(b')] - bG^{N-1}(b) + b'G^{N-1}(b') \\ & + \textstyle\int_{t=b'}^{b} \int_{u=\underline{v}}^{s(v_1,v_2)} (s(v_1, v_2) - u) dH_1(u|t) dG^{N-1}(t) \\ & - \textstyle\int_{t=b'}^{b} \int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2(u|t) dG^{N-1}(t) \, \} dF^2(v_2|v_1) \end{aligned}$$

Then, after some algebra and integration by parts, $\kappa(v_1) - \kappa(v_1')$ is

$$\begin{aligned} & (v_1 - v_1')[G^{N-1}(b) - G^{N-1}(b')] \\ & + \textstyle\int_{t=b'}^{b} [\lambda(v_1, t) - \lambda(v_1', t)] \, dG^{N-1}(t) \\ & - \textstyle\int_{t=b'}^{b} [\mu(v_1, t) - \mu(v_1', t)] \, dG^{N-1}(t) \end{aligned}$$

where $\lambda(v_1, t) \equiv \int \int_{u=\underline{v}}^{s(v_1,v_2)} H_1(u|t)du \, dF^2(v_2|v_1)$ and $\mu(v_1, t) \equiv \int \int_{u=\underline{v}}^{v_2} H_2(u|t)du \, dF^2(v_2|v_1)$. First, since $v_1' > v_1$, $b' < b$, and $G^{N-1}(\cdot)$ is a cdf, the first row of the expression above is negative. Second, since H_1 and H_2 are non-negative, $F^2(v_2|v_1)$ is stochastically ordered in v_1, and $\frac{\partial s(v_1,v_2)}{\partial v_1} \geq 0$, both $\lambda(v_1, t)$ and $\mu(v_1, t)$ are weakly increasing in v_1. Hence the second row of the expression is negative and the third row is positive. Now I focus on this positive third row. $H_2(u|t) < 1$ for all $u < \bar{v}$ because it is a cdf. As a result, $-[\mu(v_1, t) - \mu(v_1', t)]$ is strictly bounded above by $\int \int_{u=\underline{v}}^{v_2} 1du \, dF^2(v_2|v_1') - \int \int_{u=\underline{v}}^{v_2} 1du \, dF^2(v_2|v_1) = \int v_2 dF^2(v_2|v_1') - \int v_2 dF^2(v_2|v_1) = E[v_2|v_1'] - E[v_2|v_1]$. Hence,

$$\begin{aligned} \kappa(v_1) - \kappa(v_1') \quad & < \quad (v_1 - v_1')[G^{N-1}(b) - G^{N-1}(b')] + \textstyle\int_{t=b'}^{b} \{E[v_2|v_1'] - E[v_2|v_1]\} dG^{N-1}(t) \\ & = \quad [G^{N-1}(b) - G^{N-1}(b')][(v_1 - v_1') + E[v_2|v_1'] - E[v_2|v_1]] \\ & \leq \quad 0 \end{aligned}$$

The last line comes from $E[v_2|v_1'] - E[v_2|v_1] \leq v_1' - v_1$, which is given by AS4 and AS5. Then according to the inequality above, $\kappa(v_1) - \kappa(v_1') < 0$. However, this contradicts $\kappa(v_1) \geq \kappa(v_1')$, which must be satisfied by definition of best response. Hence by contradiction, it must be that $b' \geq b$. Note that AS5 is stronger than necessary to arrive at this result; there is slack in the inequality above.

Next, I show that two different values cannot share the same best response b. Consider $\pi_b(v_1, b)$, the derivative of the expected profit function with respect to b. For any $v_1' > v_1$, $\pi_b(v_1', b) - \pi_b(v_1, b)$ is

$$\begin{aligned}
\pi_b(v_1', b) - \pi_b(v_1, b) &= \{v_1' - v_1 + \lambda(v_1', b) - \lambda(v_1, b) - [\mu(v_1', b) - \mu(v_1, b)]\}(N-1)G^{N-2}(b)g(b) \\
&> \{v_1' - v_1 + \lambda(v_1', b) - \lambda(v_1, b) - [E[v_2|v_1'] - E[v_2|v_1]]\}(N-1)G^{N-2}(b)g(b) \\
&> 0
\end{aligned}$$

Again, the second line comes from the fact that $\mu(v_1', b) - \mu(v_1, b)$ is bounded above by $E[v_2|v_1'] - E[v_2|v_1]$, and the third line comes from $E[v_2|v_1'] - E[v_2|v_1] \leq v_1' - v_1$, $\lambda(v_1', b) - \lambda(v_1, b) > 0$, and $(N-1)G^{N-2}(b)g(b) > 0$. So $\pi_b(v_1, b)$ is strictly increasing in v_1. Hence, for any given bid b and bid distribution $G(\cdot)$, there can only be one v_1 that satisfies equation (4); two different values cannot share the same best response b. This rules out $b' = b$. We already established that $b' \geq b$, so it must be that $b' > b$.

Finally, I show that for each v_1, there cannot be more than one $b \in BR(v_1)$. Suppose not; suppose $b'' \in BR(v_1)$ as well, and without loss of generality, $b'' > b$. Given what we already established, the probability of winning does not change whether the bidder bids b'' or b; bidders with values lower than v_1 will bid lower than all elements of $BR(v_1)$ and bidders with higher values will bid higher than all elements of $BR(v_1)$. On the other hand, bidding more still decreases the bidder's payoff upon winning. As a result, $\pi(v_1, b'') < \pi(v_1, b)$. However, this contradicts the premise that $b'' \in BR(v_1)$. Therefore, there cannot be more than one $b \in BR(v_1)$. Hence we can define a bid function $b(v_1)$, which is strictly increasing. $\square$

Checking the second-order condition

In the proof of Proposition 1, I established that the derivative $\pi_b(v_1, b)$ is strictly increasing in v_1. Let $\xi(\cdot)$ be the inverse bid function. By monotonic bidding, if $x < b(v_1)$, then $\xi(x) < v_1$. Then, since $\pi_b(v_1, b)$ is strictly increasing in v_1, $\pi_b(v_1, x) > \pi_b(\xi(x), x) = 0$. Likewise, if $x > b(v_1)$, then $\xi(x) > v_1$ by monotonic bidding, so $\pi_b(v_1, x) < \pi_b(\xi(x), x) = 0$. In summary, $\pi_b(v_1, x) > 0$ if $x < b(v_1)$ and $\pi_b(v_1, x) < 0$ if $x > b(v_1)$, so $b(v_1)$ does achieve the global maximum of $\pi(v_1, \cdot)$.

Proposition 2

Proof. First I derive (6). Replacing $H_1(u|b)$, $H_2(u|b)$, $G(b)$, and $g(b)$ with $H_1(u|v_1)$, $H_2(u|v_1)$, $F^1(v_1)$, and $f^1(v_1)/b'(v_1)$, respectively in (6) gives $0 = -F^1(v_1)^{N-1} + (N-1)F^1(v_1)^{N-2}\frac{f^1(v_1)}{b'(v_1)}(T(v_1) - b)$.

Rearranging this gives $b'(v_1)F^1(v_1)^{N-1} + b(N-1)F^1(v_1)^{N-2}f^1(v_1) = T(v_1)(N-1)F^1(v_1)^{N-2}f^1(v_1)$, which is exactly (6). With the boundary condition $P(\underline{v}) = b(\underline{v})F^1(\underline{v})^{N-1} = 0$, the solution to that differential equation is $P(v_1) = \int_{\underline{v}}^{v_1} T(x)dF^1(x)^{N-1}$. Now, since $P(v_1) \equiv b(v_1)F^1(v_1)^{N-1}$, this means $b(v_1)F^1(v_1)^{N-1} = \int_{\underline{v}}^{v_1} T(x)dF^1(x)^{N-1}$. Hence $b(v_1) = \int_{\underline{v}}^{v_1} T(x)dF^1(x)^{N-1}/F^1(v_1)^{N-1}$. This is a bid function that must be satisfied in any symmetric equilbrium; therefore the equilibrium given by this bid function is the only symmetric Bayes-Nash equilibrium. □

Model with asymmetric bidders

The model of sequential auctions with synergy can be extended to the case where bidders have asymmetric value distributions and synergy functions. Here I extend the model to the case of two asymmetric subgroups. Nothing prevents us from going to larger numbers of subgroups, though mathematical expressions will become increasingly long and complex.

The asymmetric model requires additional notation. First, a subscript m will indicate the subgroup to which value distributions and synergy functions belong, so $v_1 \sim F^1_m(\cdot)$, $v_2 \sim F^2_m(\cdot|v_1)$, $s_m(v_1, v_2)$, and $D_m(x|v_1) \equiv \mathit{prob}(s_m(v_1, v_2) \le x|v_1)$.
Then, the distribution of the highest competing bid in the second auction conditional on the highest competing bid in the first auction being t can be expressed as one of the following, depending on whether the bidder wins the first auction and on the identity of his highest competitor.

$$\begin{aligned}
H_1^{m,m}(u|t) &= F^2_m(u|b_m \le t)^{N_m-2}F^2_{-m}(u|b_{-m} \le t)^{N-m}F^2_m(u|b_m = t)\\
H_1^{m,-m}(u|t) &= F^2_m(u|b_m \le t)^{N_m-1}F^2_{-m}(u|b_{-m} \le t)^{N-m-1}F^2_{-m}(u|b_{-m} = t)\\
H_2^{m,m}(u|t) &= F^2_m(u|b_m \le t)^{N_m-2}F^2_{-m}(u|b_{-m} \le t)^{N-m}D_m(u|b_m = t)\\
H_2^{m,-m}(u|t) &= F^2_m(u|b_m \le t)^{N_m-1}F^2_{-m}(u|b_{-m} \le t)^{N-m-1}D_{-m}(u|b_{-m} = t)
\end{aligned}$$

Subcript 1 applies if the bidder wins the first auction, and subscript 2 applies if the bidder loses the first auction. The first superscript indicates the subgroup of the bidder being considered, and the second superscript indicates the subgroup of the bidder who submits the highest competing first-auction bid t.

Additionally, for a bidder from subgroup m, the probability that the highest competing bid in the first auction is less than or equal to t is $G_m(t)^{N_m-1}G_{-m}(t)^{N-m}$, where G_m is the distribution of first auction bids from subgroup m. Then, for a bidder from subgroup m, the probability that the highest competing bid in the first auction is equal to t is $\frac{\partial G_m(t)^{N_m-1}G_{-m}(t)^{N-m}}{\partial t}$, and can be expressed as $j_m(t) + k_m(t)$, where $j_m(t) \equiv (N_m - 1)G_m(t)^{N_m-2}g_m(t)G_{-m}(t)^{N-m}$ is the probability that the highest competing bid in the first auction is equal to t and from subgroup m, and $k_m(t) \equiv N_{-m} G_{-m}(t)^{N_{-m}-1} g_{-m}(t) G_m(t)^{N_m-1}$ is the probability that the highest competing bid in the first auction is equal to t and from subgroup $-m$.

Using the above notation, the expected profit at the time of the first auction for a bidder from subgroup m is

$$\pi_m(v_1, b) = \int\limits_{v_2=\underline{v}}^{\bar{v}} X_m(v_1, v_2, b) dF_m^2(v_2|v_1)$$

where

$$\begin{aligned} X_m(v_1, v_2, b) \quad \equiv \quad & \textstyle\int_{t=\underline{b}}^{b} [v_1 - b + \int_{u=\underline{v}}^{s_m(v_1,v_2)} (s_m(v_1, v_2) - u) dH_1^{m,m}(u|t)] j_m(t) dt \\ & + \textstyle\int_{t=\underline{b}}^{b} [v_1 - b + \int_{u=\underline{v}}^{s_m(v_1,v_2)} (s_m(v_1, v_2) - u) dH_1^{m,-m}(u|t)] k_m(t) dt \\ & + \textstyle\int_{t=b}^{\bar{b}} \int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2^{m,m}(u|t) j_m(t) dt \\ & + \textstyle\int_{t=b}^{\bar{b}} \int_{u=\underline{v}}^{v_2} (v_2 - u) dH_2^{m,-m}(u|t) k_m(t) dt \end{aligned}$$

In the equation defining X_m, the first two parts account for the probability that the bidder wins the first auction and the last two parts account for the probability that he loses the first auction. There are two parts to each case because with asymmetry, the identity (subgroup) of the highest competing bidder in the first auction matters for the bidder's expected profit in the second auction.

Taking a derivative of the expected profit function $\pi_m(v_1, b)$ with respect to b yields the first-order condition for bidding. After simplifying and rearranging, the FOC for subgroup m can be rewritten as follows

$$\begin{aligned} G_m(b)^{N_m-1} G_{-m}(b) \quad = \quad & (v_1 - b)(j_m(b) + k_m(b)) + \\ & \textstyle\int_{v_2=\underline{v}}^{\bar{v}} \{ j_m(b) [\int_{u=\underline{v}}^{s_m(v_1,v_2)} H_1^{m,m}(u|b) du - \int_{u=\underline{v}}^{v_2} H_2^{m,m}(u|b) du] \\ & + k_m(b) [\int_{u=\underline{v}}^{s_m(v_1,v_2)} H_1^{m,-m}(u|b) du - \int_{u=\underline{v}}^{v_2} H_2^{m,-m}(u|b) du] \} dF_m^2(v_2|v_1) \end{aligned} \tag{15}$$

The FOC for asymmetric bidders is structurally similar to the FOC for symmetric bidders in (4), but breaks down terms to account for differences between subgroups. If the two subgroups are identical, equation (15) reduces to (4).

The logic of Proposition 1 still applies in the asymmetric case, albeit with longer algebra, so bidding in the first auction is monotonic in v_1 within each subgroup, and the single crossing condition as described in Athey (2001) is satisfied. On the other hand, with asymmetry, uniqueness of the equilibrium is not guaranteed and remains to be studied.

Proposition 3

Proof. Step (i): For a fixed set of first auction bids $\{b_i\}$, values in the second auction are drawn from $D(\cdot|b_{w1})$ for the A1-winner $w1$, and from $F^2(\cdot|b_i)$ each loser $i \neq w1$. These draws are independent across bidders. Furthermore, by assumption AS4, all value distributions involved are continuous and have the same support. Hence, we can apply Theorem 2 of Athey and Haile (2002), which establishes identification of asymmetric value distributions from transaction prices and bidder identities. Theorem 3 of Athey and Haile (2002) extends this to auctions with auction-specific covariates.

Step (ii): By assumption AS6, $s(b, v_2)$ is weakly increasing in v_2. So if we define $v_2(\alpha|b) \equiv F^{2,-1}(\alpha|b)$, i.e. the α-quantile of v_2 conditional on b, then $s(b, v_2(\alpha|b))$ must be the α-quantile of s conditional on b, $D^{-1}(\alpha|b)$. That is, for any quantile α,

$$s(b, F^{2,-1}(\alpha|b)) = D^{-1}(\alpha|b)$$

Since b is observed and $F^2(\cdot|b)$ and $D(\cdot|b)$ are identified from step (i), we know the function $s(\cdot, \cdot)$.

Step (iii): Consider (5), the inverse bid function. From steps (i) and (ii), every component of the right-hand side is either observed or identified from data, so $\xi(b)$ can be computed. Since bids are monotonic in v_1, the α-quantile of v_1, $v_1(\alpha)$, corresponds to $\xi(b(\alpha))$. Now, since the distribution of b is observed and $\xi(b)$ can be computed for any b, we can compute $v_1(\alpha)$ for any quantile α. Hence, the distribution of v_1 is identified nonparametrically.

Step (iv) is explained fully in the text. □

Identification of the model with asymmetric bidders

Uniqueness of equilibrium is not guaranteed with asymmetric bidders; assume that the data at hand does come from a single equilibrium being played. Let the subscript m denote the subgroup of a bidder. The main idea for identification is to split the sample into subsamples depending on who won the first auction; for instance, if there are two subgroups of bidders, there would be one subsample of cases where subgroup 1 won the first auction, and another subsample where subgroup 2 won the first auction. Of course, these subsamples are not random; by definition there is selection on the first auction bids b. However, since the value distributions being identified from the subsamples are conditional on b anyway (i.e. $F^2_m(\cdot|b)$ and $D_m(\cdot|b)$), that selection does not introduce problems.

Proposition 5. *Provided that a single equilibrium is being played, the primitives of the asymmetric model, $F^1_m(\cdot)$, $F^2_m(\cdot|\cdot)$, and $s_m(\cdot,\cdot)$, are identified from the observables, which are all the bids in the first auction and the transaction price in the second auction, along with bidder identities.*

Proof. Split the data into two subsamples, one where the first auction winner is from subgroup m, and the other where the first auction winner is from subgroup $-m$. Take the first subsample. In the first subsample, bidders in the second auction are either the first auction winner from subgroup m, a first auction loser from subgroup m, or a first auction loser from subgroup $-m$. Following Proposition 3(i), the value distributions from which each of these bidders draws their second auction values, $D_m(\cdot|b)$, $F^2_m(\cdot|b)$, and $F^2_{-m}(\cdot|b)$, are identified. Similarly, $D_{-m}(\cdot|b)$ is additionally identified from the second subsample.

Then, following Proposition 3(ii), the synergy function $s_m(b,\cdot)$ is identified from $D_m(\cdot|b)$ and $F^2_m(\cdot|b)$, and $s_{-m}(b,\cdot)$ is identified from $D_{-m}(\cdot|b)$ and $F^2_{-m}(\cdot|b)$.

Finally, $F^1_m(v_1)$ and $F^1_{-m}(v_1)$ are identified using each subgroup's FOC for bidding in the first auction, equation (15). If we replace $s_m(v_1,v_2)$ with $s_m(b,v_2)$ and $F^2_m(v_2|v_1)$ with $F^2_m(v_2|b)$, every component of (15) other than v_1 is either observed or identified. Therefore, we can back out any quantile of v_1 by computing the equation using the same quantile of b_m.

Once $F^1_m(v_1)$ and $F^1_{-m}(v_1)$ are identified, we can convert $s_m(b,v_2)$ and $F^2_m(v_2|b)$ back to $s_m(v_1,v_2)$ and $F^2_m(v_2|v_1)$ by replacing $b_m(\alpha)$ with $v_1(\alpha)$. This completes the identification. □

Proposition 4

Proof. Consider the $N=2$ case as an example. $J(\cdot|z'\beta)$ is the distribution of the second highest value out of $\{s(v_1,v_2),\, v_2\}$, which can be rewritten $\{F^{2,-1}(\alpha_s|z'\beta),\, F^{2,-1}(\alpha_2|z'\beta)\}$, where the -1 superscript indicates the inverse function. Now for any v, define $\alpha \equiv F^2(v|z'\beta)$ and $\tilde{\alpha} \equiv J(v|z'\beta)$. Then $\tilde{\alpha} \equiv J(v|z'\beta) = J(F^{2,-1}(\alpha|z'\beta)|z'\beta) = prob(\{F^{2,-1}(\alpha_s|z'\beta), F^{2,-1}(\alpha_2|z'\beta)\}_{(2)} \leq F^{2,-1}(\alpha|z'\beta)|z'\beta) = prob(\{\alpha_s,\alpha_2\}_{(2)} \leq \alpha|z'\beta)$.[23] From AS9, the distributions of α_s, α_2 are invariant to $z'\beta$, so we can simplify $\tilde{\alpha} = prob(\{\alpha_s,\alpha_2\}_{(2)} \leq \alpha|z'\beta)$ to $prob(\{\alpha_s,\alpha_2\}_{(2)} \leq \alpha)$. Hence $\tilde{\alpha}$ is a function only of α, invariant to z. Furthermore, since $C(\alpha_1,\alpha_2)$ is invariant to z according to AS9, and $\tilde{\alpha}$ is a function only of α, $C(\alpha_1,\tilde{\alpha}_2)$ is also invariant to z. The same applies for $C(\alpha_1,\tilde{\alpha}_s)$. □

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[23] The $\{\}_{(2)}$ subscript indicates the second order statistic out of the values in $\{\}$.

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