Noisy Rational Bubbles*

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Abstract

This paper develops a theory of asset price dynamics during bubble-like market episodes. In the model, noise trading breaks the winner's curse and leads to systematic overpricing. Over time, investors gradually learn and asset prices tend to fall toward the fundamentals. Importantly, however, investors also update their expectations regarding the average precision of new information. This mechanism works to drive prices farther away from the intrinsic value. Finally, the model also allows for gradual endogenous investor inflows that greatly amplify predicted price movements. Numerical simulations show that the model can produce various price episodes.

JEL: D82, G12, G14

Keywords: Bubble-like price dynamics, Uncertainty, Investor flows, Noisy rational expectations equilibrium

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1 Introduction

Through history, asset prices have occasionally risen spectacularly and then fallen sharply (and often dramatically). Existing literature on speculative bubbles usually attributes such extreme price behavior to speculation by investors. However, while existing models focus on specific aspects of these episodes, nearly all of them leave the description of the dynamics of asset prices incomplete.[1] Another notable phenomenon accompanying these market episodes that has been largely ignored in the literature is a massive inflow of new investors.[2]

Understanding price dynamics during these episodes is undoubtedly important for further policy analysis. Therefore, this paper seeks to fill these two gaps in the existing literature. It shows how speculation by rational but imperfectly informed investors, together with an endogenous influx of new investors, can lead to a period of rapid run-up in asset prices that is then followed by either a crash or a prolonged downturn.

The model combines two novel features to explain the endogenous asset price dynamics. First, in each period, the asset's fundamental value is fully revealed with some unknown probability, and investors learn about this resolution probability. Investors do not know whether the uncertainty about fundamentals is going to be resolved soon, as it is in the majority of earnings announcement events, or slowly as it was during the IT boom. As time goes by, if the fundamental value is not fully revealed, investors expect the learning process to take longer, and thus begin to speculate more. As a result, this layer of uncertainty gradually pushes asset prices up in the beginning. The second novel feature in this paper is the introduction of endogenous investor inflows that are linked to optimal entry decisions by potential new investors. This endogenous increase in asset demand from new investors is driven by the same uncertainty that drives up asset prices. I show that it can greatly amplify the observed asset price fluctuations.

In equilibrium, both uncertainty and investor entry determine asset prices and the magnitude

[1] Examples include the following, Tirole (1985) generates price bubbles that grow indefinitely in an overlapping generations model; Allen and Gorton (1993) and Allen et al. (1993) focus only on how price bubbles arise in a static framework; Scheinkman and Xiong (2003) studies a stationary economy with an infinite horizon and relies on an exogenous change in parameter values to obtain realistic price dynamics; Abreu and Brunnermeier (2003) assumes that the price grows exogenously over time; Allen et al. (2006) considers a dynamic setting with a finite horizon but predicts declining prices during bubble episodes.

[2] For example, Singleton (2012) documents that positions of index investors increased dramatically during the commodity boom in 2008 with a significant impact on crude-oil futures prices, and Haughwout et al. (2011) finds that during the recent U.S. housing bubble, the share of purchases by investors with multiple first-liens increased dramatically. In addition, Frazzini and Lamont (2008) offers an illustration of how investors increased exposure to tech stocks right before the peak of the IT bubble.

of price movements. Initially, learning about how long the speculation can last dominates, there are gradual investor inflows, and asset prices are pushed up continuously—as long as the fundamental value is not fully revealed. Meanwhile, however, investors are learning about the fundamentals over time. As their beliefs gradually converge, asset prices eventually fall back to their intrinsic value. If the fundamental value is fully revealed in the middle of this learning process, the market crashes as asset prices collapse to their fundamentals.

The model can generate various patterns of price dynamics by adjusting the relative rates of information flow. If learning about the resolution probability dominates for a longer time, the equilibrium price exhibits a slow build-up before collapsing as it did during the IT boom; otherwise, it has a long-lasting downturn after a surge, as was observed during the Japanese real estate bubble of the late 1980s. In addition, investor inflows help us understand trading frenzies. As the influx of new investors bids up asset prices, assets change hands from existing investors to newcomers and trading volume skyrockets.

To demonstrate price formation before discussing price dynamics, I start the analysis with a static benchmark framework in which a continuum of risk-neutral rational investors trade competitively based on their own private information and public information such as price, under short-sale constraints, limits on long positions, and unobservable random asset supply (for instance, caused by noise traders). Within this setting, the equilibrium price reflecting the marginal investor's belief contains an overvaluation component. This is because the equilibrium price is informative about the asset's fundamental value and also about the asset supply, so investors choosing to hold the asset, are on average optimistic about the asset fundamentals, and perceives the asset supply to be very high such that the price is pushed below the fundamentals. As a result, the equilibrium price reflecting the marginal investor's belief is on average higher than the fundamental value. These investors understand that the asset is unconditionally expected to be overpriced, but it is profitable to hold the asset because asset supply is negatively correlated with the price and thus the expected profit is positive. Another way to understand this result is to think of it as a common-value auction in which unobservable random asset supply breaks the winner's curse.

Since this overpricing result is anticipated, the expected equilibrium price in a dynamic setting is declining over time. The reason is that, in a given period, all the future overpricing components are predicted and thus incorporated into the price, and additionally, under the same argument as that in the static case, between the given period and its subsequent period, a new overpricing component arises through the fundamental belief of the marginal investor in the given period. This happens for every period and thus leads to a declining price path.

I further show that when there is resolution risk in each period, if the probability is known, even though the expected future over-valuation is only gradually incorporated into the price, the equilibrium price on average still falls.

I then introduce uncertainty about the resolution probability and endogenous investor flows and show that they can push up the asset price gradually before the price starts to fall. Initially, investors are worried that the uncertainty will get resolved very soon. After one period, if new information is not very informative, investors update their beliefs and their perceived resolution risk becomes smaller. As a result, investors' expected future overvaluation, which was discounted by the resolution risk, now increases, so they are willing to pay higher prices. Moreover, as perceived resolution risk decreases, investors expect speculation to last longer and their total expected speculative profit increases. This leads to investor inflows. Under the influence of these two forces, the equilibrium asset price goes up. In this way, as speculation continues period after period, the asset price is pushed up higher and higher. As time goes by, the change in the investors' beliefs about the resolution probability gets smaller and smaller, so eventually investor inflows stop and the price starts to fall.

Related Literature. This paper addresses the entire rise-and-fall price dynamics observed during bubble-like episodes. Previous literature focuses on bursts of bubbles or market crashes.[3] For example, in Abreu and Brunnermeier (2003), the bubble grows exogenously. Some papers have discussed the price evolution. Avery and Zemsky (1998) shows the possibility of bubble-like price paths generated by herds. Fostel and Geanakoplos (2008) proposes the theory of leverage cycle. Pástor and Veronesi (2009) explains the bubble-like stock price behavior. In their paper, prices always reflect investors' perceived fundamentals, and their study is specific to technological revolutions. In Doblas-Madrid (2012), the price rises because agents invest their exogenously growing endowments into the asset continuously. Burnside et al. (2013) considers social dynamics, that is, better-informed agents are more likely to convert others to their beliefs. In contrast with those papers, here it is two layers of uncertainty and endogenous investor inflows that drive price dynamics.

The result on price formation is related to the discussion on whether bubbles are consistent with rational behavior. Tirole (1985) shows how bubbles can be sustained to grow in overlapping-generation models that have infinite trading opportunities, and Allen and Gorton (1993) produces churning bubbles in which investors always expect resale options in the future. Unlike them, Allen et al. (1993) and Allen et al. (2006) have explained bubbles with

[3] Examples include Grossman (1988), Gennotte and Leland (1990), Romer (1993), Caplin and Leahy (1994), Lee (1998), Zeira (1999), Abreu and Brunnermeier (2003), Barlevy and Veronesi (2003), Hong and Stein (2003) and Veldkamp (2006a).

a finite number of trading opportunities using higher-order uncertainty. In line with their papers on finite bubbles, I provide a different mechanism through short-sale constraints and noise trading. Allen et al. (2006) also uses a dynamic noisy expectations framework, but they emphasize higher-order uncertainty, and they predict that the price declines over time. Another paper relevant to my paper is Albagli et al. (2015). The static setup in my model is close to a specific example found in their paper, and we both emphasize that noise trading under heterogeneous information can lead to systematic misvaluation. The difference is that, in their paper, the price departure comes from asymmetric asset payoff risks, while in this model, the payoff is symmetric, and the price wedge is driven by short-sale constraints and limited asset supply which implies that the marginal investor in on average optimistic.

In light of this overpricing result, the static benchmark framework in this paper shares several features with models of speculative bubbles that rely on heterogeneous beliefs and short-sale constraints. However, while those papers attribute heterogenous beliefs to differences of opinion (Harrison and Kreps (1978)), heterogenous priors (Morris (1996)), or overconfidence (Scheinkman and Xiong (2003)), in this paper, the heterogeneity results from rationally anticipated differential private information, and thus investors do not agree to disagree.

Moreover, this paper makes several other contributions. First, to my knowledge, this is the first theoretical paper to emphasize the importance of investor flows (or capital flows in general) to bubble episodes. Empirically, Singleton (2012) documents that investor flows had a significant impact on crude-oil futures prices. Theoretically, Merton (1987) shows that a larger investor base can reduce the risk premium and increase the asset value, but the paper is not about bubbles or dynamics of investor flows. Second, in my model, the interaction between the two layers of uncertainty leads to gradual investor inflows, which provides an alternative explanation for the phenomenon of slow-moving capital (see Duffie (2010)).

Technically, my model fits within noisy rational expectations models.[4] Based on a tractable dynamic framework, I explore the roles of two novel features in this paper.

Structure. The rest of the paper is organized as follows. I introduce the setup of the model in Section 2. Section 3 starts with the baseline model and then adds those two new features to explain the mechanism of price dynamics. Section 4 discusses the definition of bubbles and model assumptions, and explores further extensions and implications. Section 6 concludes.

[4]The seminal work on classical noisy rational expectations models is Grossman (1976), which was later extended by Hellwig (1980) and Diamond and Verrecchia (1981). He and Wang (1995) provides tractable characterization of dynamics.

2 The Model

The model embeds uncertainty about the resolution probability and endogenous investor flows into the context of a dynamic noisy rational expectations model in which investors have differential private information and face short-sale constraints. The model demonstrates how systematic overvaluation arises when there are a finite number of trading opportunities and also highlights how the new features help to explain the mechanism of price dynamics.

The setup is as follows. Consider an economy in discrete time and with a finite horizon, $t = 0, \ldots, T$. It is populated by a continuum of long-lived risk-neutral investors with a time discount rate ρ. For simplicity, assume $\rho = 0$.

There are two assets. One is riskless with infinitely elastic supply, yielding a return $r = 0$. I take this asset to be the numeraire and normalize its price to one. The other is a risky asset with a stochastic supply. In the last period T, each share of this asset pays Π, which is its fundamental value.[5]

2.1 Information structure.

Assume the fundamental value Π is unobservable and drawn from a normal distribution

$$\Pi \sim N(\Pi_0, 1/\rho_0). \tag{1}$$

In each trading period, first, a public signal is released

$$\begin{aligned} D_t &= \Pi + \epsilon_{D,t} \\ \epsilon_{D,t} &\sim N(0, 1/\rho_{D,t}). \end{aligned} \tag{2}$$

Assume the signal precision is stochastic. In each period, it can grow at a low rate η with probability $1-\lambda$, or a high rate that I put to the extreme, ∞, which implies that the fundamental value is fully revealed immediately and the uncertainty is resolved, with probability λ. Specifically,

$$\rho_{D,t} = \begin{cases} e^{\eta t}\rho_D & \text{with probability } 1-\lambda \\ +\infty & \text{with probability } \lambda. \end{cases} \tag{3}$$

[5] In an earlier version of this paper, I model the fundamental value explicitly as the discounted value of future dividends. The result is robust. Please see Chapter 1 of my PhD dissertation.

The resolution probability λ measures the average learning speed or how long speculation will last. With a lower λ, it is less likely that the fundamental value will be fully revealed in each period, and thus investors expect the learning to be slower. The motivation behind this setup is that investors learn about different events at different speeds. Some events, like earnings announcements, allow for very fast learning, while others, like the IT boom, only allow for very slow learning.

Here, I introduce the second layer of uncertainty, uncertainty about this resolution probability λ. Investors start from a common prior over λ following $Beta(\beta, \gamma)$ with the prior mean $\beta/(\beta+\gamma)$. In each period, after observing $\rho_{D,t}$, they use Bayes' rule to update their beliefs. In period t, if the uncertainty is not resolved, investors' perceived expected resolution probability is

$$\lambda_t = \frac{\gamma}{\beta+\gamma+t+1} \tag{4}$$

which is decreasing over time. The intuition is that, as time goes by, if not much new information arrives, investors will become increasingly confident that the uncertainty will not get resolved very soon.[6]

In the financial market, while some investors trade the risky asset, others do not.[7] I call the latter group *potential investors*. Assume there are unlimited potential investors. In each period, after receiving the public signal, potential investors make entry decisions based on the history of public information $\mathcal{F}_t^e = \{\mathcal{F}_0, P_\tau, D_\tau, \rho_{D,\tau}, D_t, \rho_{D,t} : 0 \leq \tau \leq t-1\}$. If they decide to enter and start to trade the risky asset, they need to pay an entry cost e. We can think of it as a one-time information acquisition cost or the opportunity cost associated with the portfolio adjustment. Right after entry, each new investor receives a private signal

$$\begin{aligned} S^i &= \Pi + \epsilon_S^i \\ \epsilon_S^i &\sim N(0, 1/\rho_S) \end{aligned} \tag{5}$$

where ϵ_S^i is i.i.d across investors.[8] Assume each investor initially endowed with the risky

[6] This intuition is also reflected in the conjugate prior I use. Two shape parameters β and γ represent the number of historical realizations η and ∞ respectively, so the prior mean of the resolution probability is $\gamma/(\beta+\gamma)$. In the next period, if the uncertainty is not resolved, we add 1 to β and the posterior mean of the resolution probability decreases to $\gamma/(\beta+\gamma+1)$.

[7] One justification is that mutual funds have different investment objectives. According to Wiesenberger, Strategic Insight and Lipper Objective codes, mutual funds investing in domestic equity can be classified by sector (e.g., technology or health), by capitalization (e.g., large or small capitalization), or by style (e.g., growth or income).

[8] We can also assume existing investors continue to receive private signals in each period. However, there

asset also receives a private signal following the same distribution. Under this specification, all the existing investors, no matter when they enter, have access to all the public information and equally informative private information. Therefore, their beliefs have the same precision.

Investors then make trading decisions based on their private information and also the public information, which by rational expectations, includes the current equilibrium price.

The timeline for each period is summarized in Figure 1 below.

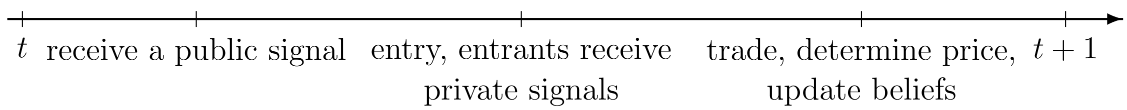

Figure 1: Timeline.

For any stochastic process $\{Z_t\}$, define $\underline{Z}_t \equiv \{Z_0, \ldots, Z_t\}$ to be the history of Z_t up to and including t. Using this notation, I denote the total public information in period t as

$$\mathcal{F}_t^c = \{\mathcal{F}_0, \underline{P}_t, \underline{D}_t, \underline{\rho}_{D,t}\}$$

and investors' expectation and belief precision based on $\mathcal{F}_t^c$ as

$$E_t^c[\,\cdot\,] = E[\,\cdot\,|\mathcal{F}_t^c],\ \ \rho_t^c = 1/var(\Pi|\mathcal{F}_t^c).$$

Similarly, denote the set of rational investors trading the risky asset in period t as I_t. The total information available to investor $i \in I_t$ is

$$\mathcal{F}_t^i = \{\mathcal{F}_0, \underline{P}_t, \underline{D}_t, \underline{\rho}_{D,t}, S^i\}$$

and the expectation and belief precision based on $\mathcal{F}_t^i$ is

$$E_t^i[\,\cdot\,] = E[\,\cdot\,|\mathcal{F}_t^i],\ \ \rho_t = 1/var(\Pi|\mathcal{F}_t^i).$$

By symmetry, all the existing investors have the same belief precision ρ_t.

is one mechanical issue with constant flows of private information. Under the common prior assumption, the weight on the private information increases first, which leads to the increase in investors' belief dispersion. As a result, prices go up initially when there is only uncertainty about the fundamental value.

Assume the structure of the economy is common knowledge. Thus, investors are only asymmetrically informed about the fundamental value of the risky asset.

2.2 Investors' decisions and asset demand.

Investors behave competitively. Given the initial wealth and asset prices, potential investors make investment (x^i) and entry (t_e^i) decisions to maximize their expected wealth in the last period T (W_T^i), i.e.,

$$\max_{x^i, t_e^i} E_0[W_T^i | \mathcal{F}_0]. \tag{6}$$

Before entry, potential investors earn only the riskless interest rate which is 0, that is, $W_{t+1}^i = W_t^i$, for $0 \leq t < t_e^i$.[9] When they decide to start trading the risky asset, they pay an entry cost. After entry, the return on investors' wealth consists of the zero riskless interest rate and the excess returns from trading the risky asset. This is characterized as

$$W_{t+1}^i = W_t^i + x_t^i (P_{t+1} - P_t) - 1_{\{t = t_e^i\}} e \quad \text{for } t \geq t_e^i.$$

Some investors are initially endowed with the risky asset. They have the same problem except that they do not need to make entry decisions.[10] Specifically,

$$\max_{x^i} E_0[W_T^i | \mathcal{F}_0]. \tag{7}$$

s.t.

$$W_{t+1}^i = W_t^i + x_t^i (P_{t+1} - P_t)$$

and the initial endowment $W_0^i = W_{-1}^i + x_{-1}^i P_0$ is given.

Assume the position investors take on the risky asset x_t^i is restricted to $[0, 1]$. The lower bound is a short-sale constraint. I make this assumption for tractability. It can be relaxed to costly short sales like stock borrowing costs.[11] Since short-sale costs create an asymmetry between buying and selling, it plays an important role in producing the systematic overvaluation result which will be shown later. The upper position bound can be justified, for instance,

[9] Decisions are all conditional on the history which for simplicity I do not write out.

[10] The equilibrium price does not depend on the initial allocation of the risky asset in this model.

[11] Ofek and Richardson (2003) also documents costly derivatives trading to mimic short sales during the dotcom mania.

by assuming wealth constraints.

Risk-neutral investors therefore hold the risky asset as long as its expected excess return is positive. Their trading strategy is characterized as for $t_e^i \leq t \leq T-1$,

$$x_t^i = \begin{cases} 1 & \text{if } E_t^i[P_{t+1}] \geq P_t \\ 0 & \text{o.t.w.} \end{cases} \tag{8}$$

2.3 Asset supply

The supply of this risky asset $Q_t(\epsilon_{q,t})$ is stochastic. For tractability, assume it has the following functional form,

$$Q_t(\epsilon_{q,t}) = n_t(1 - \Phi(q_t - \epsilon_{q,t})) \tag{9}$$

where $q_t = \Phi^{-1}(1 - \frac{1}{n_t})$ is an equivalent measure of n_t, $\epsilon_{q,t} \sim N(0, 1/\rho_q)$, and $\Phi(\cdot)$ is the CDF of a standard normal distribution.[12] The basic feature assumed in this specification is that, given $\epsilon_{q,t}$, $Q_t(\epsilon_{q,t})/n_t$ is decreasing in n_t, or in other words, relative asset supply with respect to the size of existing investor drops as more and more investors flow in to trade this asset, which is intuitive. It is common in the literature to choose specific functional forms to obtain tractability. If we let $n_t = 2$, the functional form I have used becomes $Q_t(\epsilon_{q,t}) \propto 1 - \Phi(-\epsilon_{q,t})$. This is the same as in Hellwig et al. (2006). With investor flows (n_t changes over time), I introduce q_t to offset the scale effect caused by investor flows. To see this, let $\epsilon_{q,t} = 0$, the supply is then 1, which is independent of the measure of investors n_t. In addition, throughout this paper, I assume $n_t > 2$, so the marginal investor holding the asset is on average optimistic about the asset's fundamentals.

2.4 Equilibrium

State variables in this economy are $(\Pi, \underline{\epsilon}_{D,t}, \underline{\epsilon}_{q,t}, \underline{\rho}_{D,t}, (\epsilon_S^i)_{i \in I_t})$. Since the noise in investors' private signals will be smoothed out in the aggregation, the equilibrium price only depends on $\mathcal{F}_t^P = \{\Pi, \underline{\epsilon}_{D,t}, \underline{\epsilon}_{q,t}, \underline{\rho}_{D,t}\}$.[13]

[12]A standard explanation for the stochastic supply is noise traders who trade because of liquidity or hedging demands. Although several finance papers have rationalized the behavior of noise traders, since this greatly complicates the model (see the discussion by Dow and Gorton (2006)), I choose a parsimonious specification in this paper.

[13]Technical problems exist for the validity of the law of large numbers with a continuum of private signals (see Judd (1985)). Several papers have discussed possible remedies (examples include Feldman and Gilles

We define a rational expectations equilibrium as follows:

Definition A rational expectations equilibrium consists of asset prices $\{P_t\}$, the measure of rational investors trading the risky asset $\{n_t\}$; potential investors' trading strategy $\{x_t^i\}$ adapted to $\{\mathcal{F}_t^i\}$, and entry decision t_e^i, which is an optimal stopping time of $\{\mathcal{F}_t^e\}$; originally existing investors' trading strategy $\{x_t^i\}$ adapted to $\{\mathcal{F}_t^i\}$ for $i \in I_{-1}$; and investors' beliefs about the fundamental value and the resolution probability, $\{E_t^c[\Pi], \rho_t^c, \lambda_t, (E_t^i[\Pi], \rho_t)_{i\in I}\}$, s.t.

- Given $\{P_t\}$ and $\{n_t\}$, potential investors' choice $(x_t^i, t_e^i)_{i\in I}$ is the solution to their utility maximization problem (6), and originally existing investors' trading decision $(x_t^i)_{i\in I_{-1}}$ solves their utility maximization problem (7).[14]
- Given $\{P_t\}$ and $\{n_t\}$, investors' beliefs $\{E_t^c[\Pi], \rho_t^c, \lambda_t, (E_t^i[\Pi], \rho_t)_{i\in I}\}$ are updated according to Bayesian rules.
- The asset market clears:[15]

$$\int_{i\in I_t} x_t^i di = Q_t(\epsilon_{q,t}). \tag{10}$$

3 Price Dynamics

In this paper, I study the partially revealing rational expectations equilibrium.[16] To help understand the mechanism of price evolution, I start with a baseline model without the new features and highlight that, under the assumptions of limited stochastic asset supply, short-sale constraints and heterogeneous information, the asset is overpriced and the equilibrium price is unconditionally expected to decline over time. Based on this result, I further show that uncertainty about resolution probability and the induced continuous investor inflows can push up the asset price gradually before it starts falling back toward the asset's fundamentals.

(1985), Bewley (1986), Uhlig (1996) and Al-Najjar (2004)). Here, following Feldman and Gilles (1985), we can relax the independence assumption of private signals across investors to get no aggregate uncertainty.

[14] I is the set of all the potential investors.

[15] Throughout this paper, I always mean almost surely.

[16] There is also a fully revealing equilibrium in which the equilibrium price equals the fundamental value and investors are indifferent between buying and selling. However, it is hard to justify the incorporation of the information into the price when investors' demand schedules contain no information.

3.1 Benchmark with known resolution probability

Consider simplifying the economy by assuming first that there is no uncertainty about the resolution probability λ, and second that there are no investor flows, i.e., $n_t = n > 2$ for $0 \leq t \leq T-1$ and thus $q_t = q > 0$. I first show that the asset is on average overvalued in a static model. After that, I extend the time horizon to two periods and further to multiple periods, and show that the backward accumulation of anticipated overvaluation over time implies a declining price path.

Static model. Let $T = 1$. Investors trade at $t = 0$ and the fundamental value is revealed in the following period. To simplify without losing the intuition, I further assume that there are no public signals D other than the price P. The setup is thus summarized as follows: a continuum of risk-neutral investors with fixed measure n choose their own demand schedule $x^i(P) \in [0,1]$, based on their private information and rational expectations about the information contained in the price, $\mathcal{F}^i = \{\mathcal{F}_0, P, S^i\}$, then the equilibrium price is set such that the aggregate asset demand equals the supply $Q(\epsilon_q)$.

An intuitive explanation of price formation in this static model is as follows. At $t = 0$, knowing that the fundamentals will be revealed in the next period, investors are willing to pay a price no higher than their perceived fundamentals. Since the heterogeneity in their beliefs comes from dispersed private information, intuitively, investors that receive higher private signals perceive higher fundamentals and thus they are willing to pay higher prices. This suggests that there exists a threshold $\hat{S}$ such that the risky asset is held by investors with their private signals higher than $\hat{S}$. In equilibrium, this threshold is determined by the market clearing condition. On the one hand, the private signal across investors follows a normal distribution, so the aggregate demand of the risky asset is $n(1 - \Phi(\sqrt{\rho_S}(\hat{S} - \Pi)))$, where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a standard normal distribution; on the other hand, the asset supply is $n(1 - \Phi(q - \epsilon_q))$. By market clearing condition, the demand equals the supply. Therefore,

$$\hat{S} = \Pi + 1/\sqrt{\rho_S} q - 1/\sqrt{\rho_S} \epsilon_q. \tag{11}$$

In equilibrium, the marginal investor's private signal is exactly at this threshold level. Given ϵ_q, q, which is determined by the measure of existing investors, measures how optimistic the marginal investor is. If $n = 2$, $q = 0$, the marginal investor is on average neutral. If $n \to \infty$, $q \to \infty$, the marginal investor is extremely optimistic.

The equilibrium price reflecting the marginal investor's belief about the asset fundamentals, which by Bayesian rule is a weighted average of the belief based on public information and his private signal, is

$$P = (1 - p_\Pi)E^c[\Pi] + p_\Pi(\Pi + 1/\sqrt{\rho_S}q - 1/\sqrt{\rho_S}\epsilon_q)$$

where $p_\Pi = \rho_S/(\rho_0 + \rho_S\rho_q + \rho_S)$. Since p_Π is the loading of the price on the private signal, $p_\Pi/\sqrt{\rho_S}$ is the standard deviation of price forecast across investors $\sigma(P)$. We then have the next proposition for the unconditional price wedge.

Proposition 3.1. *In a static economy with $\rho_D = 0$, $\lambda = 0$ and $n > 2$, the risky asset is unconditionally expected to be overvalued,*

$$E[P] - E[\Pi] = \sigma(P)q > 0 \tag{12}$$

where $\sigma(P)$ is the standard deviation of price forecast across investors.

Proof. See Appendix A. □

This proposition shows that backward induction argument fails with a finite number of trading opportunities. The reason is as follows. The equilibrium price is determined by both fundamental value and asset supply. Investors with high private signals ascribe the seemingly low price more to ample asset supply. This can be seen from each investor's interpretation of the current price,

$$P = (1 - p_\Pi)E^c[\Pi] + p_\Pi(E^i[\Pi] - 1/\sqrt{\rho_S}E^i[\epsilon_q]) + \sigma(P)q. \tag{13}$$

The higher $E^i[\Pi]$ is, the higher $E^i[\epsilon_q]$ becomes. Thus, investors choosing to hold the asset, are on average optimistic about the asset fundamentals, and perceives the asset supply to be very high such that the asset is not overvalued. As a result, the equilibrium price reflecting the marginal investor's belief is on average higher than the fundamental value. Even though the price is unconditionally expected to decline, investors are making money from this trading process because the existence of a high price when asset holders will lose money corresponds to a low asset supply and vice versa. In other words, investors profit from random asset supply or noise traders. It is in this sense that noise trading partially breaks the winner's curse in a common-value auction.

The proposition also shows that the size of overvaluation depends on two factors, the price forecast dispersion among investors $\sigma(P)$ and the measure of investors q. Larger forecast dispersion resulting from higher fundamental uncertainty ($1/\rho_0$, $1/\rho_S$ and $1/\rho_q$) and more investors trading this risky asset both imply that the marginal investor is more optimistic and thus the asset is more overpriced.

This static benchmark setup is similar to the special risk-neutral normal model presented in Albagli et al. (2015). We both emphasize that noise trading under heterogeneous information leads to systematic misvaluation. However, in their paper, $q = 0$, and they focus on how asymmetric asset payoff leads to a wedge between the price P and the perceived fundamental value based on public information $E[\Pi|\mathcal{F}_0, \hat{S}]$. In my model, the payoff is symmetric, the wedge by their definition is driven by positive q which means limited asset supply such that the marginal investor is on average optimistic, and I focus on a different wedge, that is, the price deviates from the asset's fundamentals systematically.

Dynamic model. It is straightforward to extend the above static model to multiple periods using backward induction. In a dynamic setting, when making trading decisions, investors are comparing their expected next-period price with the current price, so the anticipated overvaluation accumulates backward period by period and thus the expected equilibrium price is decreasing over time.

Let us study a two-period model to explain this result. The setup is the same as that of the static model, except that it has two trading periods. From the static model, we know that the equilibrium price in the last trading period reflects the fundamental belief of the marginal investor,

$$P_1 = (1 - p_{\Pi,1})E_1^c[\Pi] + p_{\Pi,1}(\Pi + 1/\sqrt{\rho_S}q - 1/\sqrt{\rho_S}\epsilon_{q,1}) \tag{14}$$

where $p_{\Pi,1}$ is the same as p_Π in the static case. The fundamental belief based on the public information $\mathcal{F}_1^c = \{\mathcal{F}_0, P_0, P_1\}$, $E_1^c[\Pi]$, can be further decomposed into the belief based on $\mathcal{F}_0^c = \{\mathcal{F}_0, P_0\}$, $E_0^c[\Pi]$, and the information contained in P_1, $\Pi - 1/\sqrt{\rho_S}\epsilon_{q,1}$,

$$E_1^c[\Pi] = (1 - \gamma_1)E_0^c[\Pi] + \gamma_1(\Pi - 1/\sqrt{\rho_S}\epsilon_{q,1})) \tag{15}$$

where $\gamma_1 = (\rho_S\rho_q)/(\rho_0 + 2\rho_S\rho_q)$.

It is clear from these results on P_1 and $E_1^c[\Pi]$ that in period 0, an individual investor's problem of price forecast, $E_0^i[P_1]$, reduces to belief formation on the asset fundamentals, $E_0^i[\Pi]$. Therefore, similar to the static case, an additional overvaluation term arises between periods 0 and 1. The equilibrium price in period 0, which equals the price forecast of the marginal investor in that period, who for the same reason as shown before has a private signal $\hat{S}_0 = \Pi + 1/\sqrt{\rho_S}q - 1/\sqrt{\rho_S}\epsilon_{q,0}$, is thus given by

$$P_0 = (1 - p_{\Pi,0})E_0^c[\Pi] + p_{\Pi,0}(\Pi + 1/\sqrt{\rho_S}q - 1/\sqrt{\rho_S}\epsilon_{q,0}) + p_{\Pi,1}/\sqrt{\rho_S}q \tag{16}$$

where $p_{\Pi,0} = (1 - p_{\Pi,1})\gamma_1 + p_{\Pi,1}$.

Comparing prices P_0 and P_1, we have several findings. First, a new overvaluation component arises in each period. This is because, the price in the first period reflects the forecast for the next-period price by the marginal investor of the first period, and this price forecast reflects this marginal investor's fundamental belief which is on average optimistic. It is clear from the price functions that, $p_{\Pi,0}/\sqrt{\rho_S}q$ and $p_{\Pi,1}/\sqrt{\rho_S}q$ are the new overvaluation components arising in periods 0 and 1 respectively. Second, anticipated overvaluation components accumulate backward period by period. In the first period, investors anticipate that the asset will be overpriced in the next period. Since they have this resale opportunity, investors are willing to pay the portion of the anticipated overvaluation contained in the next-period price. Therefore, $p_{\Pi,1}/\sqrt{\rho_S}q$ is also incorporated into the first-period price P_0. Since $p_{\Pi,0}$ is the loading of the price on the private signal in period 0, $p_{\Pi,0}/\sqrt{\rho_S}$ is the standard deviation of price forecast across investors $\sigma_0(P_1)$. Thus, two overvaluation components can be rewritten as $\sigma_0(P_1)q$ and $\sigma_1(\Pi)q$. The above two findings indicate that the price unconditionally declines over time. The next proposition describes this result.

Proposition 3.2. *In the two-period economy with $\rho_{D,t} = 0$, $\lambda_t = \lambda = 0$ and $n_t = n > 2$ for $0 \leq t \leq 1$, the equilibrium price is unconditionally expected to decline over time,*

$$E[\triangle P_0] = E[P_1] - E[P_0] = -\sigma_0(P_1)q < 0 \tag{17}$$

where $\sigma_0(P_1)$ is the standard deviation of price forecast across investors in period 0.

Proof. See Appendix A. □

Another finding by comparing P_0 and P_1 is, the size of the new overvaluation component arising in the first period is larger than that in the second period, since $p_{\Pi,0} > p_{\Pi,1}$. The intuition is as follows. As time goes by, price history becomes longer, and thus public information becomes more informative. Therefore, investors put less weight on their private signals when forecasting the price of the subsequent period. As a result, the overvaluation component coming from the bias in the marginal investor's private signal becomes smaller.

By backward induction, the equilibrium price in each period has the same form and overvaluation components accumulate through the marginal investor's forecast on the fundamental belief of the marginal investor in the subsequent period. Denote $p_{\Pi,t}$ as the loading of the price on the private signal in period t. Thus $p_{\Pi,t}/\sqrt{\rho_S}$ is the standard deviation of price forecast across investors $\sigma_t(P_{t+1})$. the overpricing component can be rewritten as $\sigma_t(P_{t+1})q$. Therefore, the equilibrium price has the following form:

$$P_t = (1 - p_{\Pi,t})E_t^c[\Pi] + p_{\Pi,t}(\Pi - 1/\sqrt{\rho_S}\epsilon_{q,t}) + \sum_{m=t}^{T-1} \sigma_m(P_{t+1})q. \tag{18}$$

The next proposition describes this build-up process of overvaluation components.

Proposition 3.3. *In the T-period economy with $\rho_{D,t} = 0$, $\lambda_t = \lambda = 0$ and $n_t = n > 2$ for $0 \leq t \leq T - 1$, the equilibrium price is unconditionally expected to decline over time,*

$$E[\triangle P_t] = E[P_{t+1}] - E[P_t] = -\sigma_t(P_{t+1})q < 0 \tag{19}$$

where $\sigma_t(P_{t+1})$ is the standard deviation of price forecast across investors in period t.

Proof. See Appendix A. □

Based on this result, it is straightforward to consider public signals with known resolution probability in the above multi-period model. Investors now take into account the possibility that, the asset's fundamental value will be fully revealed and consequently the price will fall back to the fundamental value in any period, so their expected next-period price becomes $\lambda E_t^i[\Pi] + (1 - \lambda)E_t^i[P_{t+1}]$. Considering this, it is easy to show that if the fundamental value is not fully revealed in period t, the equilibrium price is

$$P_t = (1 - p_{\Pi,t})E_t^c[\Pi] + p_{\Pi,t}(\Pi - 1/\sqrt{\rho_S}\epsilon_{q,t}) + \sum_{m=t}^{T-1}(1 - \lambda)^{T-1-m}\sigma_m(P_{t+1})q. \tag{20}$$

For the same reason, overvaluation arises and builds up backward period by period in the equilibrium. However, resolution probability has changed the result such that future overvaluation terms are only gradually incorporated into the price over time. This is because the equilibrium price reflects investors' expected future overvaluation, which has been discounted by the resolution risk. The proposition below shows that in spite of the resolution risk, the equilibrium price on average still falls over time.

Proposition 3.4. *In the T-period economy with $\rho_{D,t} > 0$, $\lambda_t = \lambda$ and $n_t = n > 2$ for $0 \leq t \leq T-1$, under the boundary condition (A5) stated in Appendix A.5, for $0 \leq t \leq T-3$, average price movement if the fundamental value is not fully revealed in period $t+1$ is*

$$\triangle \bar{P}_t = \bar{P}_{t+1} - \bar{P}_t = \underbrace{\lambda \sum_{m=t+1}^{T-1} (1-\lambda)^{m-t-1} \sigma_m(P_{m+1}) q}_{\triangle n_{1,t} > 0} \underbrace{- \sigma_t(P_{t+1}) q}_{\triangle n_{2,t} < 0} < 0. \tag{21}$$

Proof. See Appendix A. □

The proposition demonstrates that the average observed price movement is driven by two forces. One is a negative term $\triangle n_{2,t}$. It is the overvaluation arising in the current period. The other is a positive term $\triangle n_{1,t}$. It captures the change in the anticipated future overvaluation. This term depends on two factors, investors' forecast dispersion $\sigma_t(P_{t+1})$, which decreases over time as learning proceeds, and the total change in the future resolution risk $\lambda \sum_{m=t+1}^{T-1} (1-\lambda)^{m-t-1}$ which is always less than 1. Two factors together imply that the size of the change in the expected future overvaluation terms $\triangle n_{1,t}$ is smaller than that of the current-period overvaluation $\triangle n_{2,t}$, and thus the price declines period after period.

3.2 Resolution probability learning effect and investor flows

In this subsection, we study the role of uncertainty about resolution probability and endogenous investor flows. The problem becomes much more complicated, since there is higher-order uncertainty not only about the asset fundamentals but also about future investor flows. But fortunately equilibrium conditions turn out to be very simple. Next, I first characterize the equilibrium and then discuss the implications of these new elements on the asset price dynamics.

Equilibrium characterization. For tractability, I study the equilibrium in which prices are linear in state variables and the equivalent measure of investors q_t, given by

$$P_t = \mathcal{L}(\Pi_0, \underline{\epsilon}_{D,t}, \underline{\epsilon}_{q,t}) + p_{\Pi,t}\Pi + \sum_{m=t}^{T-1} p_{q,tm} q_m - p_{\epsilon,t}\epsilon_{q,t} \tag{22}$$

where all the coefficients are constant. The next proposition characterizes the equilibrium.

Proposition 3.5. *There exists one and only one linear partially revealing rational expectations equilibrium. In this equilibrium, if uncertainty is not resolved until period t,*

(1) the price has the following form:

$$P_t = (1 - p_{\Pi,t})E_t^c[\Pi] + p_{\Pi,t}(\Pi - 1/\sqrt{\rho_S}\epsilon_{q,t}) + \sum_{m=t}^{T-1}(\prod_{j=t}^{m-1}(1-\lambda_j))\sigma_m(P_{m+1})q_m \tag{23}$$

where $p_{\Pi,t}$ and $\{\sigma_t(P_{t+1})\}_{m=t}^{T-1}$ are constants.

(2) under certain regularity conditions (A1)-(A3) stated in Appendix A.2, for $0 \leq t \leq T-3$, we obtain an influx of new investors which continues for some periods and then stops.

Proof. See Appendix A. □

If we take the equilibrium investor flows as given, the mechanism of price formation is similar to the benchmark case, with the difference that investors use their perceived resolution probability to forecast prices. The equilibrium price reflects the marginal investor's anticipation of future marginal investors' price forecasts. Since marginal investors are on average optimistic about the asset fundamentals, these overvaluation components in their forecasts build up backward over time and are incorporated into the current-period price. Therefore, the equilibrium price consists of two parts: a weighted average of the fundamental belief based on public information and the marginal investor's own private signal in the current period, and expected future overvaluation components,

$$P_t = (1 - p_{\Pi,t})E_t^c[\Pi] + p_{\Pi,t}(\Pi + 1/\sqrt{\rho_S}q_t - 1/\sqrt{\rho_S}\epsilon_{q,t}) + \sum_{m=t+1}^{T-1}(\prod_{j=t}^{m-1}(1-\lambda_j))\sigma_m(P_{m+1})q_m. \tag{24}$$

If the overvaluation term from the current marginal investor's private signal $p_{\Pi,t}/\sqrt{\rho_S}q_t$, or equivalently $\sigma_t(P_{t+1})q_t$, is grouped into the expected future overvaluation terms, the price function is the same as that displayed in the above proposition. As will be shown later, the non-fundamental rise-and-fall price dynamics are driven by these overpricing components.

Now we discuss potential investors' entry decisions and characterize the equilibrium investor flows. The extreme complexity of forecasting investor flows by heterogenous investors is overcome by the following result: it is common knowledge that investors have the same perfect foresight about future investor flows and investor flows in each period are deterministic. This is because, given the equilibrium price function, the expected excess return from holding is

$$E_t^i[P_{t+1}] - P_t = p_{\Pi,t}\epsilon_S^i - \sigma_t(P_{t+1})(q_t - \epsilon_{q,t}), \tag{25}$$

which only depends on the noise in the private signal and the asset supply. Since potential investors only have access to public information before entry, their ex-ante expected excess returns in each period are: first, identical across potential investors; second, unrelated to the fundamental value, thus private information does not help existing investors forecast investor flows, and all the investors, including existing and potential investors, have the same perfect forecast; third, independent of the history of public information, so the equilibrium investor flows in each period are deterministic.

Investors hold the asset if and only if their expected excess return is positive. Thus potential investors' expected excess return from trading in period t, denoted as v_t, is

$$v_t = E[\max\{E_t^i[P_{t+1}] - P_t, 0\}|\mathcal{F}_t^e]. \tag{26}$$

As shown in Appendix A.3, we have a closed-form solution

$$v_t = \sigma_t(P_{t+1})(\sqrt{1+\frac{1}{\rho_q}}\phi(-\frac{q_t}{\sqrt{1+\frac{1}{\rho_q}}}) - q_t\Phi(-\frac{q_t}{\sqrt{1+\frac{1}{\rho_q}}})) \tag{27}$$

with $\sigma_t(P_{t+1}) = p_{\Pi,t}/\sqrt{\rho_S}$, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the PDF and CDF of the standard normal distribution, respectively. v_t is increasing in investors' forecast dispersion $\sigma_t(P_{t+1})$. This is because investors profit from price fluctuations. With a larger forecast dispersion, the asset price is more volatile in response to the asset supply change and thus the expected speculative profit is higher. Besides, this flow payoff is decreasing in the current number of investors q_t. The reason is that, with more investors, the asset price is bid up higher, and this squeezes out speculative profits in the current period.

The equilibrium investor flows are determined by two layers of uncertainty assumed in the model. This can be seen from the indifference condition, that is, if there are continuous investor inflows, potential investors must be indifferent between entering in the current period and in the next period, given by

$$v_t = \lambda_t e. \tag{28}$$

The left-hand side v_t, the flow payoff from speculation, is the benefit from entering in the current period. It equals the right-hand side $\lambda_t e$, the entry cost that can be saved if the uncertainty gets resolved in the next period, which is the benefit from waiting. Mathematically, (27) and (28) show that, if investors' expected resolution probability for the next period λ_t is decreasing faster than investors' belief dispersion $\sigma_t(P_{t+1})$, q_t will increase, which means that there will be investor inflows. Intuitively, the equilibrium investor flows depend on the relative speed at which investors learn about the two layers of uncertainty—the fundamental value and the resolution probability. On the one hand, if the fundamental value is not fully revealed, investors will become increasingly confident that the uncertainty will not get resolved very soon, speculation will last longer, and total expected speculative profits will be higher. This creates an incentive for entry. On the other hand, as learning proceeds, investors' beliefs gradually converge. This reduces speculative profits, so investors have less incentive to enter. These two forces work in opposite ways and the equilibrium investor flows depend on which force is stronger.

Regularity conditions here are similar to a single-crossing condition. They guarantee that learning about the resolution probability is faster initially and slower later, so there are inflows of investors for some time in the equilibrium. One example satisfying these conditions is, when private signals are very precise compared with the initial public signal but over time public signals become more and more informative ($\eta > 0$). In this case, investors' beliefs converge very slowly at the beginning and thus investor inflows sustain for a very long time; later their beliefs converge much faster as the flow of public information speeds up, so investor inflows eventually stop.

For the equilibrium uniqueness, the basic idea is as follows. First, exogenous information flows uniquely determine the coefficients in the equilibrium price function. Second, given these price coefficients, since the two layers of uncertainty have opposite effects on entry decisions, they uniquely determine the equilibrium investor flows. Lastly, the equilibrium investor flows and price coefficients together determine the equilibrium price function.

Price dynamics. Given the equilibrium price function, price dynamics are described by the proposition below:

Proposition 3.6. *In the full model, if uncertainty is not resolved at $t+1$, average price movement is*

$$\triangle \bar{P}_t = \bar{P}_{t+1} - \bar{P}_t = \underbrace{\lambda_t \sum_{m=t+1}^{T-1} (\prod_{j=t+1}^{m-1} (1-\lambda_j))\sigma_m(P_{m+1})q_m}_{\triangle n_{1,t}>0} \underbrace{- \sigma_t(P_{t+1})q_t}_{\triangle n_{2,t}<0} . \tag{29}$$

Under regularity conditions (A1)-(A3) stated in Appendix A.3 and the boundary condition

(A4) stated in Appendix A.3, there exists $\tilde{t}$ *s.t.* $0 \leq \tilde{t} < T-3$, $\triangle \bar{P}_t \geq 0$ *for* $0 \leq t < \tilde{t}$ *and* $\triangle \bar{P}_t < 0$ *for* $\tilde{t} \leq t \leq T-3$.

Proof. See Appendix A. □

The average observed price movement is driven by positive $\triangle n_{1,t}$ and negative $\triangle n_{2,t}$. This is similar to the benchmark case and the explanation is summarized as follows. $\triangle n_{2,t}$ is the overvaluation introduced into the price between period t and $t+1$. It is caused by the noisy aggregation of private information, under the assumption of limited asset supply and short-sale constraints. Specifically, since the equilibrium price is informative about the asset's fundamental value and also about the asset supply, investors holding the asset, who are on average optimistic about the asset fundamentals, perceives the asset supply to be very high, which suppresses the price to be below the mean path. As a result, the equilibrium price reflecting the marginal investor's belief contains an overvaluation component that is captured by $\triangle n_{2,t}$. Since this overvaluation term is anticipated, the equilibrium price in each period also incorporates investors' expected future overvaluation which has been adjusted by the resolution risk. As time goes by, if the fundamental value is not fully revealed, the total resolution risk decreases and thus the expected future overvaluation increases. This change is characterized by $\triangle n_{1,t}$.

The proposition shows that instead of a declining price path, uncertainty about the resolution probability and the induced investor inflows can gradually push up the asset price before it begins to fall. Initially, investors are worried that the uncertainty will get resolved very soon. After one period, if the asset's fundamental value is not fully revealed, investors adjust their beliefs and their perceived resolution probability becomes lower. Thus, their perceived resolution risk decreases more than it does in the benchmark case with known resolution probability. Therefore, it is possible that investors' expected future overvaluation increases by a large amount so that the asset price goes up. In this way, as speculation continues period after period, the asset price is pushed up higher and higher. As time goes by, the change in the investors' belief about the resolution probability becomes smaller and smaller, so investors' expected future overvaluation increases by a smaller and smaller amount. Thus after some point, this positive force $\triangle n_{1,t}$ becomes dominated by the negative force $\triangle n_{2,t}$, and the price falls. Furthermore, investor inflows amplify price movements, as can be seen from the proposition that price movements are proportional to $\{q_t\}_{t=0}^{T-1}$. The intuition is that, with investor inflows, more optimistic investors are trading the risky asset, and they push the price further away from the asset fundamentals.

To demonstrate price dynamics, I conduct some numerical simulations. Table 1 lists the values of parameters, which will be used for all the numerical results unless otherwise specified. Figure 2 illustrates the results for price dynamics. The horizontal line is the fundamental value. With uncertainty about the fundamentals and known resolution probability, the price is declining over time, as shown by the red dashed line. However, when the resolution probability is unknown and investors are learning about this probability, the price path becomes hump-shaped, as displayed by the green dash-dot line. On top of this, with endogenous investor inflows, price movements are dramatic and resemble a realistic bubble episode.

Table 1. Baseline Parameter Values

Assigned parameters		Value
precision of the prior	ρ_0^c	8
precision of private signals	ρ_S	10
precision of asset supply	ρ_q	0.01
precision of public signals: level	ρ_D	1.6×10^{-4}
precision of public signals: growth rate	η	0.3
mean of the prior about the fundamental value	Π_0	1
fundamental value	Π	1
prior parameter I of the resolution probability	β	0.1
prior parameter II of the resolution probability	γ	9.9
entry cost	e	0.72
number of speculative periods	T	50

Comparative statics. Results for comparative statics are shown in Figure 3. Within the parameters tested, I obtained the following intuitive findings:

If the prior precision ρ_0 increases, price movements are less dramatic due to there being less uncertainty and reduced investor inflows. In addition, the asset price grows for a longer time because the convergence rate of investors' beliefs in each period is lower.

If the asset supply is less noisy, i.e., ρ_q is higher, we have less striking price movements and also an earlier price reversal. This is because prices are more informative and thus investors' beliefs converge faster. Similar results hold for a higher precision of public signals η and ρ_D.

Given a higher precision of private signals ρ_S, investors put more weight on their private

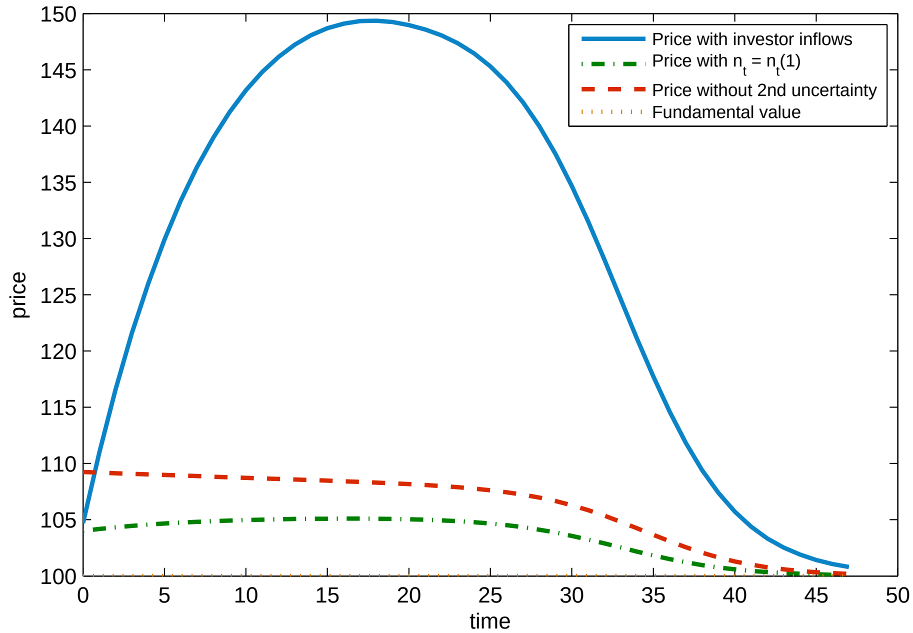

Figure 2: The average observed price path. Normalized by the fundamental value.

information and thus their initial forecast dispersion is larger. This and the induced larger influx of new investors push up prices to a larger extent. Meanwhile, investors' beliefs converge faster, so the price reversal comes earlier.

If the prior mean of the resolution probability is lower, investors are more aggressive in the beginning, and this brings more investors. As a result, initial prices are higher and prices continue to go up to a larger extent. This is shown by the case $(\beta, \gamma) = (0.1, 9.9)$, compared with $(\beta, \gamma) = (5, 5)$. Here we also observe that the price reversal comes later. This is because in each period the change in investors' beliefs regarding the resolution probability is smaller. For cases $(\beta, \gamma) = (0.1, 9.9)$ and $(\beta, \gamma) = (0.2, 19.8)$, the latter implies that investors are more cautious and they change their beliefs by a smaller amount in each period. Thus, there are less incoming investors, and price movements are less pronounced.

As we raise the entry cost e, we reduce investor inflows in each period. This leads to less striking price movements and lower initial prices. Here for $e = 0.82$, there are no investor inflows.

These results on the comparative statics have several cross-sectional implications on asset prices. First, the results on the initial uncertainty indicate that growth stocks undergo more dramatic price movements than income stocks on average. Second, according to the results on the precision of public signals, it is more likely for us to observe striking price movements

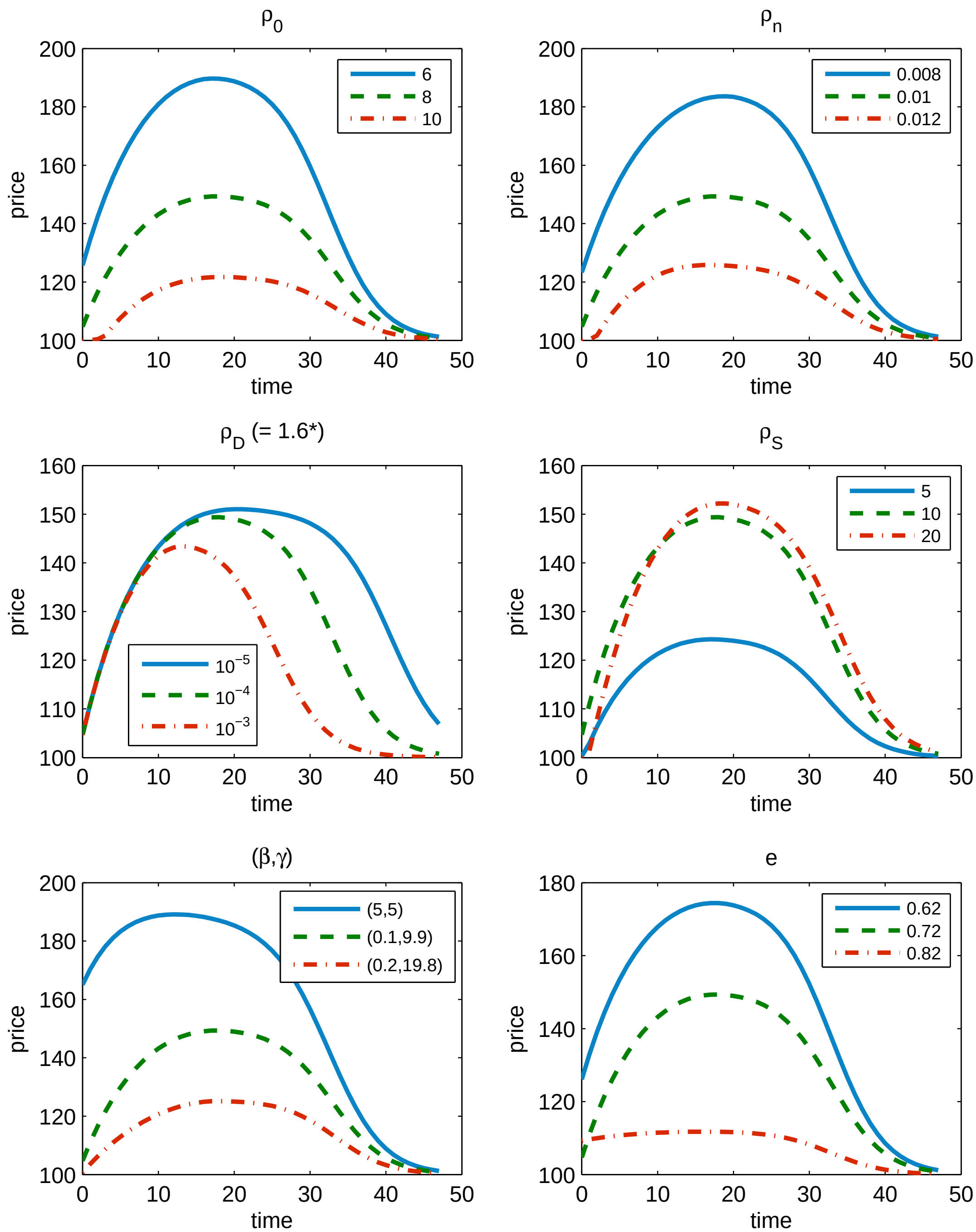

Figure 3: Comparative statics.

in stocks with scarce analyst coverage. Third, although not shown in Figure 3, it is easily seen from the linearity property of equilibrium prices that, investor inflows have a larger impact on prices of small firms and thus their price movements tend to be more pronounced. Moreover, if we think of the entry cost as the downpayment for home purchases, the result shows that intuitively, raising downpayment requirements helps curb speculation.

4 Discussions and Extensions

4.1 Price Formation: Bubbles?

In each period, the equilibrium price is on average above the fundamental value and this is common knowledge among investors. Can we call the price a bubble?

Previous literature on speculative bubbles has defined it in two different ways. In the first, all the investors are speculators and the price is above their perceived fundamental value. (Examples include Harrison and Kreps (1978), Tirole (1982), Allen and Gorton (1993), Allen et al. (1993), Morris (1996), and Scheinkman and Xiong (2003).) The other definition requires only that some of the investors holding the asset are speculators. The rest of the investors do not perceive bubbles for various reasons, such as that they are uninformed about bubbles as in Abreu and Brunnermeier (2003) or they are extremely optimistic as in Allen et al. (2006).

This model has the same property as that appears in Allen et al. (2006): since investors' beliefs are unbounded, extremely optimistic investors always believe that the asset is undervalued. Thus, my definition of bubbles is in line with the second way used in the existing literature. Specifically, I define a speculative bubble to be the price path when the marginal investor is on average willing to pay more than his perceived fundamental value $E_t^i[\Pi]$.

By this definition, there are no bubbles in the last trading period. This result is also shown in Allen et al. (1993). The intuition is that investors know that the asset price in the next period is going to be the fundamental value and thus they are not willing to pay a price higher than their perceived fundamentals. However, bubbles arise if we have one more trading period. In period $T-2$, investors have an opportunity to resell the asset to optimistic investors in period $T-1$. Thus the price acceptable to them is their perceived optimistic marginal investor's belief. This creates speculation and thus bubbles.

To check this definition of bubbles, I plot the fraction of speculators among asset holders

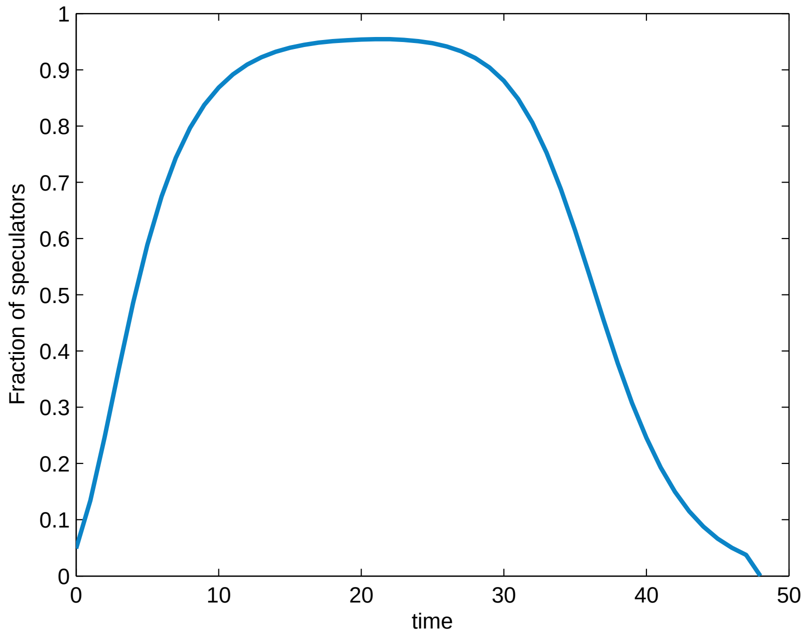

Figure 4: The fraction of speculators. This figure shows the fraction of speculators among asset holders.

in Figure 4. We obtain several findings. First, the plot shows that most investors are speculators in each period. In addition, as discussed earlier, none of the investors holding the risky asset in the last trading period $T-1$ are speculators, thus there are no bubbles in that period.

Another way to study this question is from the view of econometricians. They have access to all the public information. From their point of view, the price can be decomposed into two components: expected fundamental value based on public information, and an error term containing a price bubble, denoted as b_t,[17]

$$P_t = E_t^c[\Pi] + b_t \quad \text{where } b_t = -p_{\Pi,t}/\sqrt{\rho_S} E_t^c[\epsilon_{q,t}] + \sum_{m=t}^{T-1} \left(\prod_{j=t}^{m-1} (1-\lambda_j)\right) \sigma_m(P_{m+1}) q_m. \tag{30}$$

Based on the available public information, the fundamental value component $E_t^c[\Pi]$ is a martingale, that is, the revision in the fundamental expectation is unforecastable. In the error term, the first part $-p_{\Pi,t+1}/\sqrt{\rho_S} E_{t+1}^c[\epsilon_{q,t+1}]$ is caused by random asset supply. Since $\{\epsilon_{q,t}\}$ is i.i.d., this part is unpredictable. The second part $\sum_{m=t}^{T-1}(\prod_{j=t}^{m-1}(1-\lambda_j))\sigma_m(P_{m+1})q_m$, by contrast, is the partially accumulated overvaluation and its change is predicted. Therefore,

[17] I used the result that, since $P_t \subseteq \mathcal{F}_t^c$, $\Pi - p_{\Pi,t}/\sqrt{\rho_S}\epsilon_{q,t} = E_t^c[\Pi] - p_{\Pi,t}/\sqrt{\rho_S} E_t^c[\epsilon_{q,t}]$.

this second part is a price bubble to econometricians. As shown in Proposition 3.5, the non-fundamental rise and fall in the equilibrium price is driven by this component.

4.2 Various episodes

In the model, depending on the relative speed at which the investors learn the fundamentals versus the resolution probability, price dynamics exhibit different patterns. If learning about the resolution probability dominates for a long period of time, the price builds up slowly before it plummets; otherwise, the price rapidly runs up and then enters a long-lasting downturn. Figure 5 displays these two different cases. In this example, the slow build-up pattern is initially driven by a lower level ρ_D and a higher growth rate η in the precision of public signals. Intuitively, a lower ρ_D indicates slower convergence of investors' beliefs and thus the price goes up for a longer time, and a higher η means that at a later stage investors' beliefs converge much faster so that the price falls more quickly. By contrast, the prolonged downturn pattern occurs with a higher ρ_D and a lower η.[18]

In reality, price processes during various historical bubble episodes have distinct features. As shown in Figure 5, the IT bubble belongs to the slow build-up category, which may have been caused by high uncertainty and slow learning about the value of this revolutionary technology. The Japanese real estate bubble in the late 1980s represents the prolonged downturn pattern, for which one possible reason is the delayed policy response to reduce non-performing loans after the land price turned around in 1991.

In addition, the model can produce exogenous market crashes. In each period, the fundamental value may get fully revealed with a certain probability. If this happens, the price goes back to the fundamental value immediately, as shown in Figure 6. The key to the crash is not the revelation of the fundamental value, but the precision of public information that aligns investors' beliefs. One example is, according to Neal (1990), during the peak of the South Sea Bubble in August of 1720, the South Sea company used the Bubble Act which prohibited any company from engaging in activities outside those authorized in its original charter to fight against its competitors, which had switched their activities from building waterworks to underwriting insurance. However, this strategy also affected the company's own banker which was only authorized to make sword blades. As a result, the credit market became very tight. The growing credit shortage, called a "coordinating event" by Temin and

[18] The key is the relative rate of information flow. Thus, different patterns can also result from the difference in (β, γ) which controls the investors' learning speed with respect to the second layer of uncertainty.

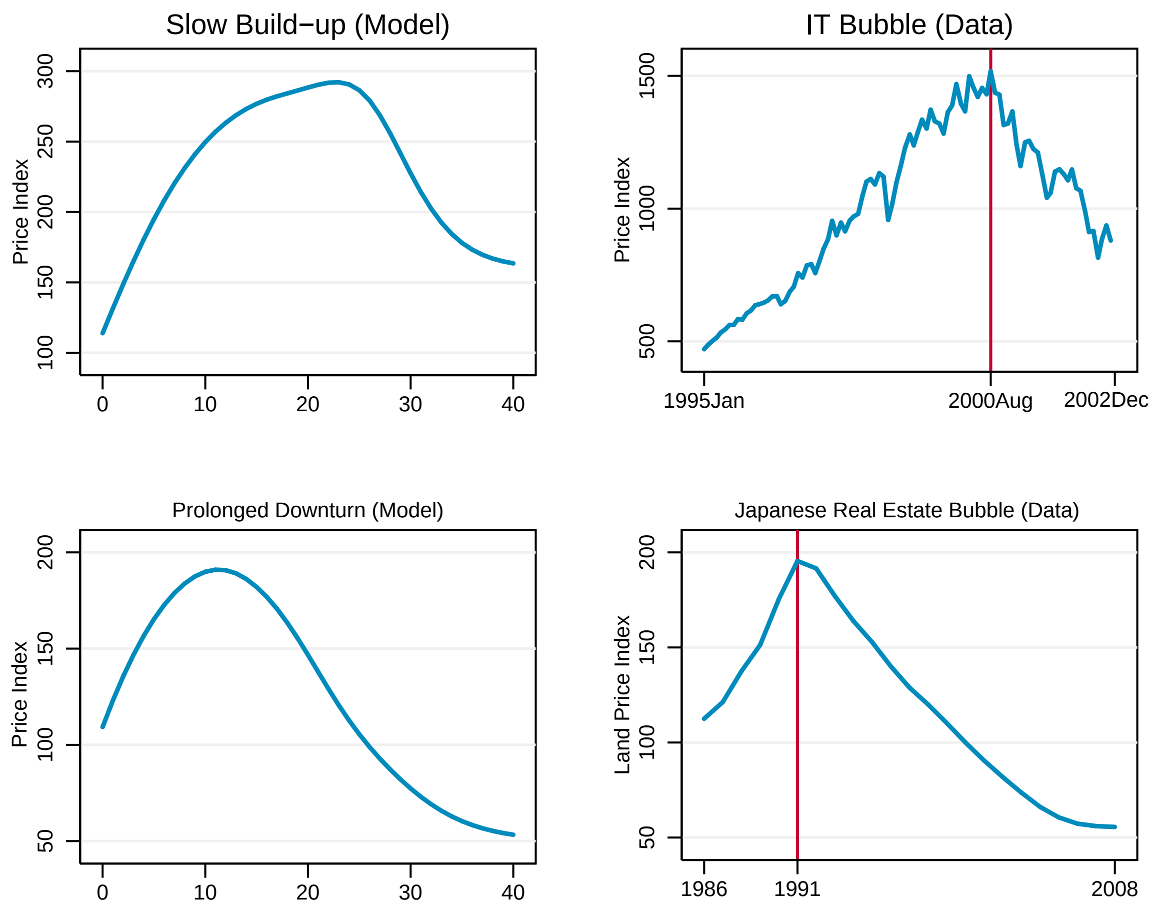

Figure 5: Bubbles of different shapes. The parameters that drive these two different shapes are the level parameter ρ_D and the growth rate η of the precision of public signals. I set $\rho_D = 1.6 \times 10^{-4}/2$ and $\eta = 0.35$ for the slow build-up pattern, and $\rho_D = 1.6 \times 10^{-4} \times 10$ and $\eta = 0.25$ for the prolonged downturn pattern. In addition, I made the following two adjustments to match the IT bubble and the Japanese real estate bubble: first, by comparing prices in the initial and the last period, I set the fundamental value $\Pi = 1.6$ and its prior mean $\Pi_0 = -8.4$ for the IT bubble, and $\Pi = 0.5$ and $\Pi_0 = 2.09$ for the Japanese real estate bubble; second, to match the magnitude of price movements, I choose the initial uncertainty $\rho_0 = 0.5$ and the entry cost $e = 1.3$ accordingly for the IT bubble, and $\rho_0 = 4$ and $e = 0.92$ correspondingly for the Japanese real estate bubble. In the data, the price index for the IT bubble is the S&P 500 Index, and for the Japanese real estate bubble, it is the urban land price of nationwide: commercial from the Japan Statistical Yearbook 2014, Statistics Bureau, Ministry of Internal Affairs and Communications, Japan.

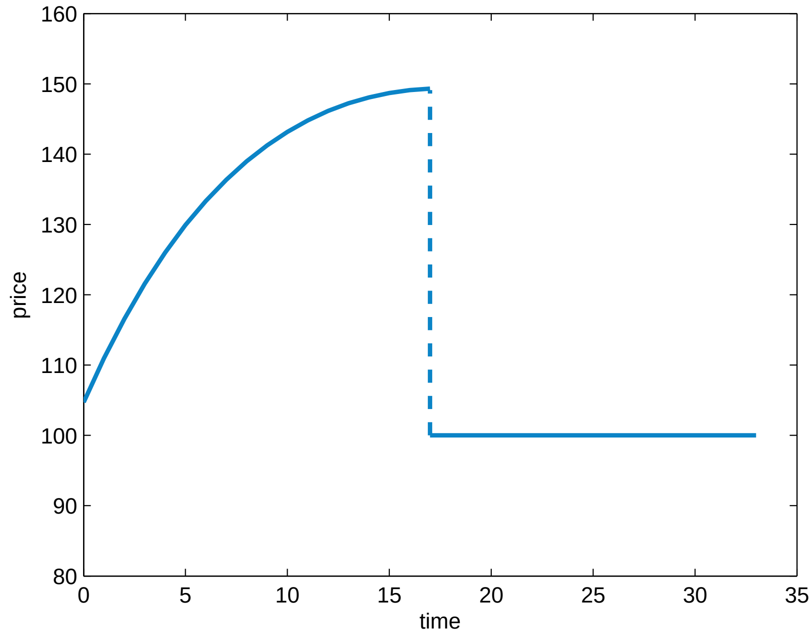

Figure 6: Market crashes.

Voth (2004), signaled the end of speculation, and consequently the stock price of the South Sea company dropped from 820 on August 24, 1720 to 370 on September 24, 1720.

4.3 Model specification

Limits to arbitrage. In general, market anomalies are explained by stating that mispricing cannot be corrected because of limits to arbitrage (see the survey by Gromb and Vayanos (2010)). Among the various types of costs faced by arbitrageurs that have been discussed in the previous literature, short-sale constraints and limited holding positions are two assumptions that have also been made in this model. However, in the current context, investors are able to bring the price back to the fundamental value with these two elements alone. To see this, consider the static framework without noise traders. As argued in Diamond and Verrecchia (1987), the information of pessimistic investors will be incorporated into the equilibrium price, and thus investors can learn from the price and then take action. For the limited holding positions, since the total measure of investors is sufficiently large, the equilibrium price can only be the fundamental value. Therefore in spite of these two assumptions, the key to the overvaluation emphasized in this paper is noise trading.

Moreover, as shown in the static case, the assumption of entry costs is unrelated to the rise of overvaluation in the equilibrium. Investors are compensated by profiting from noise traders.

Investor outflows. In the equilibrium, there are no investor outflows. To explore the impact of exit, I consider a case of exogenous outflows in this subsection. In each period, assume 5% of rational investors randomly choose to exit and stop trading the risky asset. From Figure 7, which plots the measure of rational investors in the risky asset sector, indicated by q_t, and the average observed price path, we can see that the main difference is that the price starts to decline at an earlier date and then falls faster. The dramatic rise-and-fall pattern is preserved for two reasons: first, instead of entry, expected speculation profits lead to net entry; second, investor outflows in the distant future are only partially incorporated into the current-period asset price, that is, the outflows are multiplied by the probability that uncertainty is not resolved by then.

Another concern with investor outflows is that they reflect pessimistic investors' beliefs and thus contain information about the fundamental value. In general, we can treat investor outflows as another noisy signal, like prices.[19] There is no reason to believe that this will change the model results, but technically it will be much more involved.

Information acquisition and aggregation. The model can be justified by an endogenous information acquisition framework. Take the entry cost e as a one-time information acquisition cost and assume an upper bound for the signal precision that can be acquired. It is obvious that in equilibrium investors will choose this upper bound, which is equivalent to exogenous information flows. The rest of the results carry over into this setup.

Admittedly, multi-equilibria might arise in a different endogenous information acquisition setting. This has been discussed at length in the previous literature.[20] In the current framework, consider one example in which the information acquisition cost is increasing in the precision of the signal to be acquired. With endogenous investor flows, we may get two equilibria, one with a high signal precision and a small number of rational investors, and one with a low signal precision and a large influx of new investors.

Moreover, the specification for random asset supply that has been assumed for tractability introduces the independence between the informativeness of price and investor flows. If we

[19]Noises, for instance, may come from risky outside production possibilities, as in Wang (1990).

[20]Examples include the following, Grossman and Stiglitz (1980) obtains the uniqueness result through information substitutability; Barlevy and Veronesi (2000) and Chamley (2008) provide a complementarity example in which more information makes prices less informative by introducing the correlation between fundamentals and the noise; and Veldkamp (2006a) generates strategic complementarities through fixed information production costs.

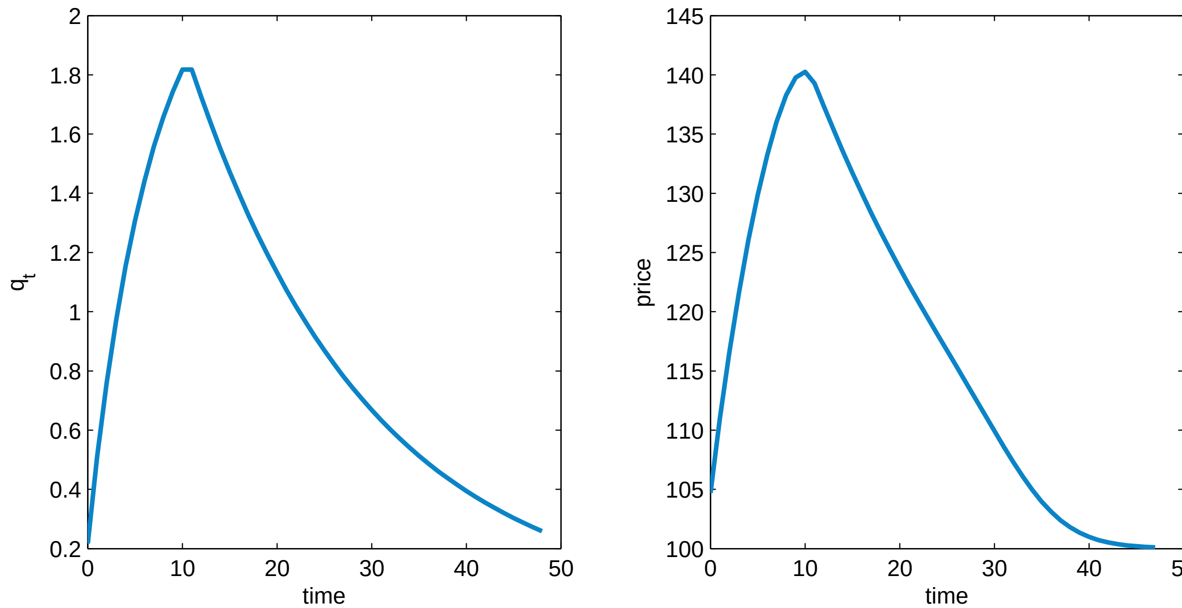

Figure 7: Exogenous investor outflows: flows and the average observed price path. Assume 5% exogenous investor outflows in each period. The left panel displays the measure of rational investors in the risky asset sector indicated by q_t. The right panel shows the average observed price path normalized by the fundamental value.

allow the correlation, the indifference condition should still hold and thus equilibrium investor flows should be determined by the relative speed of learning about the fundamentals and the resolution probability. But the flow size now depends on the direction of correlation. For example, if the random asset supply itself follows a normal distribution, the equilibrium price becomes more informative with investor inflows. This implies that the speculative profits fall more and thus in equilibrium there are fewer new investors.

4.4 Trading volume

As standard noisy rational expectations models show, in contrast with the no-trade theorem by Milgrom and Stokey (1982), stochastic asset supply not only by itself brings about trade but also leads to speculative trade because the information in the price does not "swamp" private information. To compute trading volume, we can keep track of the selling side. In this model, trading volume is given by

$$TV_t = n_{t-1} Pr(E^i_{t-1}[P_t] \geq P_{t-1}, E^i_t[P_{t+1}] < P_t) + (Q(\epsilon_{q,t-1}) - Q(\epsilon_{q,t}))^+. \tag{31}$$

Since trading volume is informative about both the amount of asset supply and fundamentals, it can be treated as a noisy signal on the fundamental value, and there is no reason to believe that including this signal will change the mechanism. Because of this and also because of the computation complexity, following previous literature, I assume investors do not observe trading volume in the model.

It can be shown that

$$TV_t = n_{t-1}|\Phi(q_t - \epsilon_{q,t}) - \Phi(q_{t-1} - \epsilon_{q,t-1})| + (Q(\epsilon_{q,t-1}) - Q(\epsilon_{q,t}))^+. \tag{32}$$

The first term captures speculative trading and the second term represents noise trading. This paper focuses on the volume of speculative trading which is measured by the equilibrium trading volume with $\epsilon_{q,t} = 0$, $\forall 0 \leq t \leq T-1$.[21]

Proposition 4.1. *In period t, if the uncertainty is not resolved, the equilibrium volume of speculative trading is*

$$\overline{TV}_t = 1 - \frac{n_{t-1}}{n_t}. \tag{33}$$

It is increasing in the speed of investor inflows.

Proof. See Appendix A. □

The intuition is as follows. Investor inflows bring more investors, especially optimistic investors, to the market. These new investors bid up asset prices and induce assets to change hands from existing investors to newcomers. Therefore, the faster investor inflows are, the higher the volume of speculative trading is.

This result helps understand trading frenzies. To explain this phenomenon, researchers have discussed various ways to generate strategic complementarities in speculators' information acquisition behavior. Examples include complementarities that result from short trading horizons as described in Froot et al. (1992), from the riskiness of positions as described in Hirshleifer et al. (1994), from fixed information acquisition costs as described in Veldkamp (2006a,b), from the extra dimension of supply information as described in Ganguli and Yang (2009), from relative wealth concerns as described in García and Strobl (2011), and from the feedback effect from financial markets to the real investment decision as described in Goldstein et al. (2013). By contrast, this paper describes trading frenzies as resulting from large investor inflows. Another explanation not depending on strategic complementarities is

[21] To be precise, it is the median path of the volume of speculative trading.

provided in a recent paper by Biais et al. (2014). They show a different mechanism, that is, higher trading volume comes from round-trip trades caused by preference uncertainty.

5 Conclusion

In this paper, I provide a mechanism of price dynamics to explain bubble-like market episodes. The equilibrium price is informative about the asset's fundamental value and also about the asset supply. Thus in a static framework, investors choosing to hold the asset, are on average optimistic about the asset fundamentals, and perceives the asset supply to be very high such that the price is pushed below the fundamentals. As a result, the equilibrium price reflecting the marginal investor's belief contains an overvaluation component. Since this overpricing component is anticipated, it accumulates backward period by period in a dynamic setting. Therefore, the price tends to decline over time. I show, however, that uncertainty regarding the probability with which the fundamental value is fully revealed in each period and induced investors inflows can push the price up continuously at the beginning. If the fundamental value is not fully revealed period after period, investors perceive lower resolution risk and thus they are willing to pay more for the asset. Meanwhile, investors become increasingly confident that speculation will last longer, their expected speculative profits increase. This leads to gradual investor inflows. Because of these two forces, the price can run up dramatically before it eventually falls. If the fundamental value is fully revealed in the middle of this speculation process, the market crashes. Moreover, by adjusting the relative speed of learning about the two layers of uncertainty, the model equilibrium can produce various bubble-like episodes. The importance of uncertainty and also of investor flows demonstrated in this paper sheds light on the future policy analysis of bubble-like events.

References

D. Abreu and M. K. Brunnermeier. Bubbles and crashes. *Econometrica*, 71(1):173–204, 2003.

N. I. Al-Najjar. Aggregation and the law of large numbers in large economies. *Games and Economic Behavior*, 47(1):1–35, 2004.

E. Albagli, C. Hellwig, and A. Tsyvinski. A theory of asset pricing based on heterogeneous information. NBER Working Paper 17548, 2015.

F. Allen and G. Gorton. Churning bubbles. *Review of Economic Studies*, 60(4):813–836, 1993.

F. Allen, S. Morris, and A. Postlewaite. Finite bubbles with short sale constraints and asymmetric information. *Journal of Economic Theory*, 61(2):206–229, 1993.

F. Allen, S. Morris, and H. S. Shin. Beauty contests and iterated expectations in asset markets. *Review of Financial Studies*, 19(3):719–752, 2006.

C. Avery and P. Zemsky. Multidimensional uncertainty and herd behavior in financial markets. *American Economic Review*, 88(4):724–748, 1998.

G. Barlevy and P. Veronesi. Information acquisition in financial markets. *Review of Economic Studies*, 67(1):79–90, 2000.

G. Barlevy and P. Veronesi. Rational panics and stock market crashes. *Journal of Economic Theory*, 110(2):234–263, 2003.

T. Bewley. Stationary monetary equilibrium with a continuum of independently fluctuating consumers. In W. Hildenbrand and A. Mas-Colell, editors, *Contributions to mathematical economics: in honor of Gerard Debreu.* North-Holland, New York, 1986.

B. Biais, J. Hombert, and P.-O. Weill. Equilibrium pricing and trading volume under preference uncertainty. *The Review of Economic Studies*, 81(4):1401–1437, 2014.

C. Burnside, M. Eichenbaum, and S. Rebelo. Understanding booms and busts in housing markets. Working Paper, 2013.

A. Caplin and J. Leahy. Business as usual, market crashes, and wisdom after the fact. *American Economic Review*, 84(3):548–565, 1994.

C. Chamley. On "acquisition of information in financial markets". *Review of Economic Studies*, 75(4):1081–1084, 2008.

D. W. Diamond and R. E. Verrecchia. Information aggregation in a noisy rational expectations economy. *Journal of Financial Economics*, 9(3):221–235, 1981.

D. W. Diamond and R. E. Verrecchia. Constraints on short-selling and asset price adjustment to private information. *Journal of Financial Economics*, 18(2):277–311, 1987.

A. Doblas-Madrid. A robust model of bubbles with multidimensional uncertainty. *Econometrica*, 80(5):1845–1893, 2012.

J. Dow and G. Gorton. Noise traders. NBER Working Paper 12256, 2006.

D. Duffie. Presidential address: Asset price dynamics with slow-moving capital. *Journal of Finance*, 65(4):1237–1267, 2010.

M. Feldman and C. Gilles. An expository note on individual risk without aggregate uncertainty. *Journal of Economic Theory*, 35(1):26–32, 1985.

A. Fostel and J. Geanakoplos. Leverage cycles and the anxious economy. *American Economic Review*, 98(4):1211–44, 2008.

A. Frazzini and O. A. Lamont. Dumb money: Mutual fund flows and the cross-section of stock returns. *Journal of Financial Economics*, 88(2):299–322, 2008.

K. A. Froot, D. S. Scharfstein, and J. C. Stein. Herd on the street: Informational inefficiencies in a market with short-term speculation. *Journal of Finance*, 47(4):1461–1484, 1992.

J. V. Ganguli and L. Yang. Complementarities, multiplicity, and supply information. *Journal of the European Economic Association*, 7(1):90–115, 2009.

D. García and G. Strobl. Relative wealth concerns and complementarities in information acquisition. *Review of Financial Studies*, 24(1):169–207, 2011.

G. Gennotte and H. Leland. Market liquidity, hedging, and crashes. *American Economic Review*, 80(5):999–1021, 1990.

I. Goldstein, E. Ozdenoren, and K. Yuan. Trading frenzies and their impact on real investment. *Journal of Financial Economics*, 109(2):566–582, 2013.

D. Gromb and D. Vayanos. Limits of arbitrage: the state of the theory. *Annual Review of Financial Economics*, (2):251–275, 2010.

S. Grossman. On the efficiency of competitive stock markets where trades have diverse information. *Journal of Finance*, 31(2):573–585, 1976.

S. Grossman. An analysis of the implications for stock and futures price volatility of program trading and dynamic hedging strategies. *Journal of Business*, 61(3):275–298, 1988.

S. J. Grossman and J. E. Stiglitz. On the impossibility of informationally efficient markets. *American Economic Review*, 70(3):393–408, 1980.

J. M. Harrison and D. M. Kreps. Speculative investor behavior in a stock market with heterogeneous expectations. *Quarterly Journal of Economics*, 92(2):323–336, 1978.

A. Haughwout, D. Lee, J. Tracy, and W. van der Klaauw. Real estate investors, the leverage cycle, and the housing market crisis. Working Paper, Federal Reserve Bank of New York, 2011.

H. He and J. Wang. Differential informational and dynamic behavior of stock trading volume. *Review of Financial Studies*, 8(4):919–972, 1995.

C. Hellwig, A. Mukherji, and A. Tsyvinski. Self-fulfilling currency crises: The role of interest rates. *American Economic Review*, 96(5):1769–1787, 2006.

M. F. Hellwig. On the aggregation of information in competitive markets. *Journal of Economic Theory*, 22(3):477–498, 1980.

D. Hirshleifer, A. Subrahmanyam, and S. Titman. Security analysis and trading patterns when some investors receive information before others. *Journal of Finance*, 49(5):1665–1698, 1994.

H. Hong and J. C. Stein. Differences of opinion, short-sales constraints, and market crashes. *Review of Financial Studies*, 16(2):487–525, 2003.

K. L. Judd. The law of large numbers with a continuum of iid random variables. *Journal of Economic Theory*, 35(1):19–25, 1985.

I. H. Lee. Market crashes and informational avalanches. *Review of Economic Studies*, 65(4): 741–759, 1998.

R. C. Merton. A simple model of capital market equilibrium with incomplete information. *Journal of Finance*, 42(3):483–510, 1987.

P. Milgrom and N. Stokey. Information, trade and common knowledge. *Journal of Economic Theory*, 26:17–27, 1982.

P. R. Milgrom. Good news and bad news: Representation theorems and applications. *The Bell Journal of Economics*, 12(2):380–391, 1981.

S. Morris. Speculative investor behavior and learning. *Quarterly Journal of Economics*, 111 (4):1111–1133, 1996.

L. Neal. *The rise of financial capitalism: International capital markets in the age of reason.* Cambridge: Cambridge University Press, 1990.

E. Ofek and M. Richardson. Dotcom mania: The rise and fall of internet stock prices. *Journal of Finance*, 58(3):1113–1138, 2003.

Ľ. Pástor and P. Veronesi. Technological revolutions and stock prices. *American Economic Review*, 99(4):1451–83, 2009.

D. Romer. Rational asset-price movements without news. *American Economic Review*, 83 (5):1112–1130, 1993.

J. A. Scheinkman and W. Xiong. Overconfidence and speculative bubbles. *Journal of Political Economy*, 111(6):1183–1220, 2003.

K. J. Singleton. Investor flows and the 2008 boom/bust in oil prices. Working Paper, 2012.

P. Temin and H.-J. Voth. Riding the south sea bubble. *American Economic Review*, 94(5): 1654–1668, 2004.

J. Tirole. On the possibility of speculation under rational expectations. *Econometrica*, 50 (5):1163–1181, 1982.

J. Tirole. Asset bubbles and overlapping generations. *Econometrica*, 53(6):1499–1528, 1985.

H. Uhlig. A law of large numbers for large economies. *Economic Theory*, 8(1):41–50, 1996.

L. L. Veldkamp. Media frenzies in markets for financial information. *American Economic Review*, 96(3):577–601, 2006a.

L. L. Veldkamp. Information markets and the comovement of asset prices. *Review of Economic Studies*, 73(3):823–845, 2006b.

J. Wang. *Intertemporal asset prices under asymmetric information.* PhD thesis, University of Pennsylvania, 1990.

J. Zeira. Informational overshooting, booms, and crashes. *Journal of Monetary Economics*, 43(1):237–257, 1999.

Appendix

Appendix A: Proofs

A.1 Proof of Proposition 3.1

Proof. By Milgrom (1981), since the density of the normal distribution ϕ has the property that ϕ'/ϕ is decreasing and unbounded from above and below, investors' posterior belief on the asset's fundamental value, $E^i[\Pi]$, which is based on their information $\mathcal{F}^i = \{\mathcal{F}_0, S^i, P\}$, is increasing in their private signal S^i, and there exists a unique solution $\hat{S}$ such that $E^i[\Pi|\mathcal{F}_0, \hat{S}, P] = P$.

Since the payoff of holding the asset is the asset's fundamental value, risk-neutral investors choose to hold the asset if and only if $S^i > \hat{S}$. Investors' private signal follows a normal distribution, $S^i \sim N(\Pi, 1/\rho_S)$, so the aggregate demand for the asset is $n(1-\Phi(\sqrt{\rho_S}(\hat{S}-\Pi)))$. By market clearing condition, demand equals supply,

$$n(1 - \Phi(\sqrt{\rho_S}(\hat{S} - \Pi))) = n(1 - \Phi(q - \epsilon_q))$$

This determines the threshold $\hat{S}$,

$$\hat{S} = \Pi + 1/\sqrt{\rho_S}q - 1/\sqrt{\rho_S}\epsilon_q$$

As shown in Theorem 1 in Albagli et al. (2015), the equilibrium price reveals the marginal investor's private signal. After adjusting the price for the concern of the winner's curse, the information content in the price is $\xi = \Pi - 1/\sqrt{\rho_S}\epsilon_q$.

Throughout the paper, we focus on the equilibrium in which the price only partially reveals the fundamental value. Therefore, the equilibrium price reflecting the marginal investor's fundamental belief is,

$$P = E^i[\Pi|\mathcal{F}_0, \hat{S}, P] = (1 - p_\Pi)E^c[\Pi] + p_\Pi(\Pi + 1/\sqrt{\rho_S}q - 1/\sqrt{\rho_S}\epsilon_q)$$

where $p_\Pi = \rho_S/(\rho_0 + \rho_S\rho_q + \rho_S)$ and $E^c[\Pi] = E[\Pi|\mathcal{F}_0, P]$.

Since p_Π is the loading of the price on the marginal investor's private signal, $p_\Pi/\sqrt{\rho_S}$ is the standard deviation of price forecast across investors $\sigma(P)$. The unconditional wedge between the price and the fundamental value is thus

$$E[P] - E[\Pi] = \sigma(P)q > 0$$

Another way to prove this proposition is to guess and verify. $\square$

A.2 Proof of Proposition 3.2 and 3.3

Proof. The result can be shown by backward induction. First I study a two-period economy, then extend the result to a multi-period model.

In a two-period economy, given P_1,

$$\begin{aligned} P_1 &= (1 - p_{\Pi,1})E_1^c[\Pi] + p_{\Pi,1}(\Pi + 1/\sqrt{\rho_S}q - 1/\sqrt{\rho_S}\epsilon_{q,1}) \\ &= (1 - p_{\Pi,1})E_1^c[\Pi] + p_{\Pi,1}\Pi + \sigma_1(\Pi)(q - \epsilon_{q,1}) \end{aligned}$$

where $\sigma_1(\Pi) = p_{\Pi,1}/\sqrt{\rho_S}$ is the standard deviation of fundamental belief (or price forecast in the last trading period) across investors, in period 0, an individual investor's price forecast is

$$E_0^i[P_1] = (1 - p_{\Pi,1})E_0^i[E_1^c[\Pi]] + p_{\Pi,1}E_0^i[\Pi] + \sigma_1(\Pi)(q - E_0^i[\epsilon_{q,1}])$$

Since $\epsilon_{q,1} \sim N(0, 1/\rho_q)$, $E_0^i[\epsilon_{q,1}] = 0$. In addition, by Bayes' rule, the fundamental belief based on the public information $\mathcal{F}_1^c = \{\mathcal{F}_0, P_0, P_1\}$, $E_1^c[\Pi]$, can be further decomposed into the belief based on $\mathcal{F}_0^c = \{\mathcal{F}_0, P_0\}$, $E_0^c[\Pi]$, and the information contained in P_1, $\Pi - 1/\sqrt{\rho_S}\epsilon_{q,1}$,

$$E_1^c[\Pi] = (1 - \gamma_1)E_0^c[\Pi] + \gamma_1(\Pi - 1/\sqrt{\rho_S}\epsilon_{q,1})$$

with $\gamma_1 = \rho_S\rho_q/(\rho_0 + 2\rho_S\rho_q)$, and thus,

$$\begin{aligned} E_0^i[E_1^c[\Pi]] &= (1 - \gamma_1)E_0^i[E_0^c[\Pi]] + \gamma_1(E_0^i[\Pi] - 1/\sqrt{\rho_S}E_0^i[\epsilon_{q,1}]) \\ &= (1 - \gamma_1)E_0^c[\Pi] + \gamma_1 E_0^i[\Pi] \end{aligned}$$

where the first term uses the property that, $\mathcal{F}_0^c \subset \mathcal{F}_0^i$, $E_0^i[E_0^c[\Pi]] = E_0^c[\Pi]$.

Therefore, price forecast $E_0^i[P_1]$ is

$$\begin{aligned} E_0^i[P_1] &= (1 - p_{\Pi,1})((1 - \gamma_1)E_0^c[\Pi] + \gamma_1 E_0^i[\Pi]) + p_{\Pi,1}E_0^i[\Pi] + \sigma_1(\Pi)q \\ &= (1 - p_{\Pi,1})(1 - \gamma_1)E_0^c[\Pi] + ((1 - p_{\Pi,1})\gamma_1 + p_{\Pi,1})E_0^i[\Pi] + \sigma_1(\Pi)q \end{aligned}$$

Since by Bayes' rule,

$$E_0^i[\Pi] = (1-\alpha_0)E_0^c[\Pi] + \alpha_0 S^i$$

with $\alpha_0 = \rho_S/(\rho_0 + \rho_S\rho_q + \rho_S)$, follows a normal distribution, we have the price forecast across investors follows a distribution $N(\bar{E}_0[P_1], \sigma_0^2(P_1))$ with

$$\begin{aligned}
\bar{E}_0[P_1] &= (1-p_{\Pi,0})E_0^c[\Pi] + p_{\Pi,0}\Pi + \sigma_1(\Pi)q \\
\sigma_0(P_1) &= p_{\Pi,0}/\sqrt{\rho_S}
\end{aligned}$$

where $p_{\Pi,0} = ((1-p_{\Pi,1})\gamma_1 + p_{\Pi,1})\alpha_0$.

Since investors hold one unit of the risky asset iff $E_0^i[P_1] > P_0$, we have the following market-clearing condition:

$$n(1-\Phi(\frac{P_0 - \bar{E}_0[P_1]}{\sigma_0(P_1)})) = Q(\epsilon_{q,0})$$

Plugging in the specification for asset supply $Q(\epsilon_{q,0}) = n(1-\Phi(q-\epsilon_{q,0}))$, we get

$$\begin{aligned}
P_0 &= \bar{E}_0[P_1] + \sigma_0(P_1)(q-\epsilon_{q,0}) \\
&= (1-p_{\Pi,0})E_0^c[\Pi] + p_{\Pi,0}\Pi + \sigma_1(\Pi)q + \sigma_0(P_1)(q-\epsilon_{q,0})
\end{aligned}$$

Therefore, the unconditional price movement between periods 0 and 1 is

$$E[P_1] - E[P_0] = -\sigma_0(P_1)q < 0$$

The proof for the multi-period economy follows the same steps, except that the price coefficients are different,

$$p_{\Pi,t} = ((1-p_{\Pi,t+1})\gamma_{t+1} + p_{\Pi,t+1})\alpha_t$$

where $\gamma_{t+1} = \rho_S\rho_q/(\rho_0 + (t+2)\rho_S\rho_q)$ and $\alpha_t = \rho_S/(\rho_0 + (t+1)\rho_S\rho_q + \rho_S)$. □

A.3 Proof of Proposition 3.5

Proof. The proof consists of three steps. I first derive the functional form of the equilibrium price given the measure of investors, next characterize the dynamics of equilibrium investor flows, and lastly show the existence and uniqueness of the linear partially revealing rational expectations equilibrium.

Step 1: price. We take the measure of rational investors trading the risky asset $\{n_t\}$ (and equivalently $\{q_t\}$) as given. As will be shown in step 2, $\{q_t\}$ is common knowledge.

First, we derive the belief-updating rules. Given the following linear form for the price,

$$P_t = \mathcal{L}(\Pi_0, \underline{\epsilon}_{D,t}, \underline{\epsilon}_{q,t}) + p_{\Pi,t}\Pi + \sum_{m=t}^{T-1} p_{q,tm} q_m - p_{\epsilon,t}\epsilon_{q,t}$$

where all the coefficients are constant, we have the information contained in the price to be

$$\xi_t = \Pi - \mu_t \epsilon_{q,t}$$

where

$$\mu_t = \frac{p_{\epsilon,t}}{p_{\Pi,t}}.$$

Notice that $\epsilon_{q,t}$ is i.i.d., which implies that signals $\{\xi_t\}$ and $\{D_t\}$ are independent and i.i.d.. Thus, instead of the standard Kalman filter, we can directly apply the Bayesian rule and get the posterior belief on Π based on $\mathcal{F}_t^c$ to follow $N(E_t^c[\Pi], 1/\rho_t^c)$ with

$$E_t^c[\Pi] = \frac{\rho_{t-1}^c}{\rho_t^c} E_{t-1}^c[\Pi] + \frac{\rho_q/\mu_t^2}{\rho_t^c}\xi_t + \frac{\rho_{D,t}}{\rho_t^c} D_t$$

and

$$\rho_t^c = \rho_{t-1}^c + \rho_{D,t} + \frac{\rho_q}{\mu_t^2}.$$

For the belief based on the total information available to investor i, $i \in I_t$, since private signals are i.i.d. and also independent of other noise terms, we have the following Bayesian updating rule:

$$E_t^i[\Pi] = (1 - \alpha_t) E_t^c[\Pi] + \alpha_t S^i$$

where

$$\alpha_t = \frac{\rho_S}{\rho_t^c + \rho_S}$$

and

$$\rho_t = \rho_{t-1} + \rho_{D,t} + \frac{\rho_q}{\mu_t^2} + \rho_S.$$

Given the belief-updating rules, now we show that the price function has the linear form as given in the Proposition. The method we use here is backward induction.

At $t = T$, uncertainty is fully resolved and thus $P_T = \Pi$. Given this, at $t = T - 1$, if the fundamental value is not fully revealed, we have

$$E^i_{T-1}[P_T] = E^i_{T-1}[\Pi] = (1 - \alpha_{T-1})E^c_{T-1}[\Pi] + \alpha_{T-1}S^i.$$

Across investors, the price forecast follows a distribution $N(\bar{E}_{T-1}[P_T], \sigma^2_{T-1}(P_T))$ with

$$\begin{aligned}
\bar{E}_{T-1}[P_T] &= (1 - \alpha_{T-1})E^c_{T-1}[\Pi] + \alpha_{T-1}\Pi \\
\sigma_{T-1}(P_T) &= \alpha_{T-1}\sqrt{\frac{1}{\rho_S}}.
\end{aligned}$$

Since investors hold one unit of the risky asset iff $E^i_{T-1}[P_T] \geq P_{T-1}$, we have the following market-clearing condition:

$$n_{T-1}(1 - \Phi(\frac{P_{T-1} - \bar{E}_{T-1}[P_T]}{\sigma_{T-1}(P_T)})) = Q(\epsilon_{q,T-1}).$$

Plugging in the specification for asset supply, we get

$$P_{T-1} = \bar{E}_{T-1}[P_T] + \sigma_{T-1}(P_T)(q_{T-1} - \epsilon_{q,T-1}).$$

Using the result for the belief distribution, we have

$$P_{T-1} = (1 - \alpha_{T-1})E^c_{T-1}[\Pi] + \alpha_{T-1}(\Pi + \sqrt{\frac{1}{\rho_S}}(q_{T-1} - \epsilon_{q,T-1})).$$

Matching the coefficients gives us

$$\begin{aligned}
p_{\Pi,T-1} &= \alpha_{T-1} \\
p_{q,(T-1)(T-1)} &= p_{\epsilon,T-1} = \alpha_{T-1}\sqrt{\frac{1}{\rho_S}}.
\end{aligned}$$

Thus, the equilibrium price at period $T - 1$ has the functional form given in the proposition.

Now assume that the equilibrium price at period $t + 1$ has the linear form given in the proposition, we want to show that it also holds for period t.

If uncertainty is not resolved at t, investors expect the resolution probability in the next period to be λ_t. Thus

$$E_t^i[P_{t+1}] = (1-\lambda_t)E_t^i[(1-p_{\Pi,t+1})E_{t+1}^c[\Pi] + p_{\Pi,t+1}\Pi + \sum_{m=t+1}^{T-1} p_{n,(t+1)m}q_m - p_{\epsilon,t+1}\epsilon_{q,t+1}] + \lambda_t E_t^i[\Pi].$$

By the belief-updating rule for $E_{t+1}^c[\Pi]$,

$$\begin{aligned} E_t^i[E_{t+1}^c[\Pi]] &= \frac{1}{\rho_{t+1}^c}E_t^i[\rho_t^c E_t^c[\Pi] + \frac{\rho_q}{\mu_{t+1}^2}\xi_{t+1} + \rho_{D,t+1}D_{t+1}] \\ &= \frac{\rho_t^c}{\rho_{t+1}^c}E_t^c[\Pi] + \frac{\rho_q/\mu_{t+1}^2 + \rho_{D,t+1}}{\rho_{t+1}^c}E_t^i[\Pi] \end{aligned}$$

we have

$$E_t^i[P_{t+1}] = (1-A_t)E_t^c[\Pi] + A_t E_t^i[\Pi] + (1-\lambda_t)\sum_{m=t+1}^{T-1} p_{q,(t+1)m}q_m$$

where

$$A_t = 1 - (1-\lambda_t)(1-p_{\Pi,t+1})\frac{\rho_t^c}{\rho_{t+1}^c}.$$

Using the belief-updating rule for $E_t^i[\Pi]$, we further have

$$E_t^i[P_{t+1}] = (1-A_t\alpha_t)E_t^c[\Pi] + A_t\alpha_t S^i + (1-\lambda_t)\sum_{m=t+1}^{T-1} p_{q,(t+1)m}q_m. \tag{34}$$

Across investors, it follows $N(\bar{E}_t[P_{t+1}], \sigma_t^2(P_{t+1}))$ with

$$\bar{E}_t[P_{t+1}] = (1-A_t\alpha_t)E_t^c[\Pi] + A_t\alpha_t\Pi + (1-\lambda_t)\sum_{m=t+1}^{T-1} p_{q,(t+1)m}q_m \tag{35}$$

$$\sigma_t(P_{t+1}) = A_t\alpha_t\sqrt{\frac{1}{\rho_S}}.$$

This implies the market-clearing condition

$$n_t(1-\Phi(\frac{P_t - \bar{E}_t[P_{t+1}]}{\sigma_t(P_{t+1})})) = Q(\epsilon_{q,t})$$

from which we have

$$P_t = (1-A_t\alpha_t)E_t^c[\Pi] + A_t\alpha_t(\Pi + \sqrt{\frac{1}{\rho_S}}(q_t - \epsilon_{q,t})) + (1-\lambda_t)\sum_{m=t+1}^{T-1} p_{q,(t+1)m}q_m.$$

This confirms our conjecture about the price function, and by matching the coefficients, we have

$$\begin{aligned}
p_{\Pi,t} &= A_t \alpha_t \\
p_{q,tm} &= (1-\lambda_t) p_{q,(t+1)m} \text{ for } t+1 \leq m \leq T-1 \\
p_{q,tt} &= p_{\epsilon,t} = A_t \alpha_t \sqrt{\frac{1}{\rho_S}}.
\end{aligned}$$

If we reorganize the terms, we get the price function stated in the proposition. This completes the proof.

Step 2: investor flows. The proof has two parts: I first characterize the properties of equilibrium investor flows, and after that I provide regularity conditions to describe the dynamics of investor flows.

We first show that the expected excess return from trading the risky asset at period t if uncertainty is not resolved is the same conditional on the information set of potential investors in any period before and including t, that is, any $\mathcal{F}^e_m$ for $0 \leq m \leq t$.

Notice that by the proof of step 1,

$$P_t = \bar{E}_t[P_{t+1}] + \sigma_t(P_{t+1})(q_t - \epsilon_{q,t})$$

we have

$$E^i_t[P_{t+1}] - P_t = E^i_t[P_{t+1}] - \bar{E}_t[P_{t+1}] - \sigma_t(P_{t+1})(q_t - \epsilon_{q,t}) = p_{\Pi,t}\epsilon^i_S - \sigma_t(P_{t+1})(q_t - \epsilon_{q,t})$$

where the second equals sign uses (34) and (35).

Since ϵ^i_S and $\epsilon_{q,t}$ are independent of any $\mathcal{F}^e_m$ for $0 \leq m \leq t$, we know that

$$(E^i_t[P_{t+1}] - P_t)|\mathcal{F}^e_m \sim N(-\sigma_t(P_{t+1})q_t, \sigma^2_t(P_{t+1})(1 + \frac{1}{\rho_q}))$$

so the expected excess return from trading the risky asset at period t is the same for any $\mathcal{F}^e_m$, $0 \leq m \leq t$, and can be denoted as v_t. It is given by

$$v_t = E[\max\{E^i_t[P_{t+1}] - P_t, 0\}|\mathcal{F}^e_t].$$

Using the distribution we have derived above,

$$v_t = \int_0^\infty x\phi(\frac{x+\sigma_t(P_{t+1})q_t}{\sigma_t(P_{t+1})\sqrt{1+\frac{1}{\rho_q}}})dx$$

which gives

$$v_t = \sigma_t(P_{t+1})(\sqrt{1+\frac{1}{\rho_q}}\phi(-\frac{q_t}{\sqrt{1+\frac{1}{\rho_q}}}) - q_t\Phi(-\frac{q_t}{\sqrt{1+\frac{1}{\rho_q}}}))$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are PDF and CDF of the standard normal distribution, respectively.

As a result, the total expected excess return discounted to period t from entering at t is

$$V_t = \sum_{m=t}^{T-1}(\Pi_{j=t}^{m-1}(1-\lambda_j))v_m.$$

Investors want to maximize their discounted expected wealth at period t. Thus, potential investors will enter only if

$$V_t \geq e.$$

Moreover, they also need to decide when to enter. Since potential investors make the same entry decision, if there are new entrants at both period t and $t+1$, the following indifference condition must hold:

$$V_t - e = E_t[\max\{V_{t+1} - e, 0\}|\mathcal{F}_t^e]$$

which implies

$$v_t = \lambda_t e$$

where $\lambda_t = \gamma/(\beta+\gamma+t+1)$ is investors' expectation regarding the probability that the uncertainty will be fully resolved in the next period, in which case they will not enter.

In addition, in this model, investors can always choose not to hold the risky asset, so it is obvious that there are always positive expected gains for investors who trade the risky asset before uncertainty gets resolved. Thus, investors will not exit, which implies that n_t is nondecreasing during those T periods.

Lastly, we show some properties of v_t. Since

$$\frac{\partial v_t}{\partial q_t} = -\Phi(-\frac{q_t}{\sqrt{1+\frac{1}{\rho_q}}}) < 0$$

v_t is decreasing in q_t (and thus in n_t). This and the indifference condition indicate that if $\lambda_t/\sigma_t(P_{t+1})$ is decreasing over time, we have gradual inflows of investors, as long as $V_t \geq e$.

Now I provide regularity conditions. First, we introduce some notations:

$$B_t = \frac{1}{\lambda_{t-1}}(1-(1-\lambda_{t-1})\frac{\rho^c_{t-1}}{\rho^c_t})\alpha_{t-1}/(\frac{1}{\lambda_t} - \frac{1}{\lambda_{t-1}}(1-\lambda_{t-1})\frac{\rho^c_{t-1}}{\rho^c_t}\alpha_{t-1}),\ D_t = B_t/\lambda_t$$
$$C_t = (B_{t-1} - (1-(1-\lambda_{t-1})\frac{\rho^c_{t-1}}{\rho^c_t})\alpha_{t-1})/((1-\lambda_{t-1})\frac{\rho^c_{t-1}}{\rho^c_t}\alpha_{t-1}),\ O_t = \sigma_t(P_{t+1})/\lambda_t.$$

Assume the following regularity conditions:

(A1) There exists $\tilde{t}$ such that $0 < \tilde{t} \leq T-2$, $D_{t-1} < D_t$ for $0 \leq t < \tilde{t}$, and $D_{t-1} > D_t$ for $\tilde{t} \leq t \leq T-2$.

(A2) $p_{\Pi,T-2} = (1-(1-\lambda_{T-2})(1-\alpha_{T-1})\frac{\rho^c_{T-2}}{\rho^c_{T-1}})\alpha_{T-2} < B_{T-2}$.

(A3) $e \leq V_0(q_t = q_{n,-1}, 0 \geq t \geq T)/(1-\lambda_0)$.

Here O_t over time captures the convergence rate of investors' beliefs about the fundamental value relative to how fast investors become confident that uncertainty will not get resolved very soon. Condition (A1) puts restrictions on the relative rates of information flow so that investors' beliefs about the resolution probability decrease faster than the convergence rate of their beliefs about the fundamental value at the beginning but slower than the convergence rate later. Conditions (A2) and (A3) are boundary conditions that guarantee investor inflows during the first trading period and no flows at period T-2. Now we show that under those regularity conditions, O_t increases first then decreases, and thus the indifference condition (28) implies that investor inflows continue for some periods and then stop.

Using the results for $\sigma_t(P_{t+1})$ and $p_{\Pi,t}$, we have

$$O_{t+1} - O_t = \frac{\sigma_{t+1}(P_{t+2})}{\lambda_{t+1}} - \frac{\sigma_t(P_{t+1})}{\lambda_t} = \frac{p_{\Pi,t+1}\sqrt{\frac{1}{\rho_S}}}{\lambda_{t+1}} - \frac{(1-(1-\lambda_t)(1-p_{\Pi,t+1})\frac{\rho^c_t}{\rho^c_{t+1}})\alpha_t\sqrt{\frac{1}{\rho_S}}}{\lambda_t}$$

and thus

$$sgn(O_{t+1} - O_t) = sgn(p_{\Pi,t+1} - B_{t+1}). \tag{36}$$

Similarly,

$$sgn(O_t - O_{t-1}) = sgn(p_{\Pi,t} - B_t)$$

plugging in

$$p_{\Pi,t} = (1 - (1 - \lambda_t)(1 - p_{\Pi,t+1})\frac{\rho_t^c}{\rho_{t+1}^c})\alpha_t$$

we have

$$sgn(O_t - O_{t-1}) = sgn(p_{\Pi,t+1} - C_{t+1}). \tag{37}$$

Besides,

$$\begin{aligned} B_{t+1} - C_{t+1} &= \frac{1}{(1-\lambda_t)\frac{\rho_t^c}{\rho_{t+1}^c}\alpha_t}(((1-\lambda_t)\frac{\rho_t^c}{\rho_{t+1}^c}\alpha_t B_{t+1} + (1 - (1-\lambda_t)\frac{\rho_t^c}{\rho_{t+1}^c})\alpha_t) - B_t) \\ &= \frac{1}{(1-\lambda_t)\frac{\rho_t^c}{\rho_{t+1}^c}\alpha_t}(B_{t+1}\frac{\lambda_t}{\lambda_{t+1}} - B_t) \end{aligned}$$

which gives

$$sgn(B_{t+1} - C_{t+1}) = sgn(D_{t+1} - D_t). \tag{38}$$

Assumption (A2) and (36) imply $O_{T-2} < O_{T-3}$. By Assumption (A1), $D_{T-3} > D_{T-2}$, and thus $B_{T-2} < C_{T-2}$. This and Assumption (A2) indicate $p_{\Pi,T-2} < C_{T-2}$, and with (37), this gives $O_{T-3} < O_{T-4}$. By (36), we have $p_{\Pi,T-3} < B_{T-3}$. We can continue this process until period $\tilde{t} - 1$, and we have $O_{\tilde{t}-2} > O_{\tilde{t}-1} > \ldots > O_{T-2}$.

For periods before $\tilde{t}$, there are two possibilities. One is $p_{\Pi,t} < C_t$ for all $1 \leq t \leq \tilde{t} - 1$. This and the assumption that D_t is increasing before $\tilde{t}$ imply that O_t is declining over time. Together with Assumption (A3), this implies investor inflows only during the first period.

The other possibility is that there exists a $\tilde{t}'$ s.t. $1 < \tilde{t}' \leq \tilde{t} - 1$ and $p_{\Pi,\tilde{t}'} > C_{\tilde{t}'}$. Thus, we have $O_{\tilde{t}'-2} < O_{\tilde{t}'-1}$ and thus by (36), $p_{\Pi,\tilde{t}'-1} > B_{\tilde{t}'-1}$. Since D_t is increasing before $\tilde{t}$ which implies $B_{\tilde{t}'-1} > C_{\tilde{t}'-1}$, $p_{\Pi,\tilde{t}'-1} > C_{\tilde{t}'-1}$ and thus by (37), $O_{\tilde{t}'-3} < O_{\tilde{t}'-2}$. Continuing this process until period 0, we have $O_0 < \ldots < O_{\tilde{t}'-1}$. Therefore, O_t is increasing from period 0 to $\tilde{t}' - 1$ and decreasing afterwards. This means that there are gradual inflows of investors until period $\tilde{t}' - 1$ or earlier when $V_t < e$. This completes the proof.

Step 3: existence and uniqueness. The proof consists of two parts. We first show the existence and uniqueness of the equilibrium price function, then for investor flows.

Recall step 1, we have

$$\mu_t = \frac{p_{\epsilon,t}}{p_{\Pi,t}} = \sqrt{\frac{1}{\rho_S}}$$

thus,

$$\rho_t^c \;=\; \rho_{t-1}^c + \rho_{D,t} + \frac{\rho_q}{\mu_t^2} = \rho_{t-1}^c + \rho_{D,t} + \rho_S \rho_q.$$

This gives

$$\rho_t^c = \rho_0^c + \sum_{m=0}^{t} \rho_{D,m} + (t+1)\rho_S \rho_q.$$

By step 1, ρ_t^c uniquely determines $\{p_{\Pi,t}, \{p_{q,tm}\}_{m=t}^{T-1}, p_{\epsilon,t}\}$. This suggests the existence and uniqueness of the parameters in the price function.

The existence of the measure of rational investors trading the risky asset $\{n_t\}$ is obvious. Now we show its uniqueness. Suppose not, then there exist two equilibria $N_t^1 \equiv \{n_t^1\}$ and $N_t^2 \equiv \{n_t^2\}$ s.t. $N_t^1 \neq N_t^2$.

There must exist $0 \leq \tilde{t} \leq T-1$, s.t. $n_{\tilde{t}}^1 \neq n_{\tilde{t}}^2$ and $n_t^1 = n_t^2$ for $\tilde{t}+1 \leq t \leq T-1$. Without loss of generality, assume $n_{\tilde{t}}^1 > n_{\tilde{t}}^2$. Since $n_t^1 = n_t^2$ for $\tilde{t}+1 \leq t \leq T-1$ and the flow payoff v_t is strictly decreasing in n_t, we have

$$V_{\tilde{t}}(n_{\tilde{t}}^1, n_{\tilde{t}+1}^1, \ldots, n_{T-1}^1) < V_{\tilde{t}}(n_{\tilde{t}}^2, n_{\tilde{t}+1}^2, \ldots, n_{T-1}^2) \leq e.$$

This and for any $0 \leq t \leq T-1$, $V_t(n_t, \ldots, n_{T-1}) \leq e$ with "$<$" only if $n_t = n_{t-1}$ imply

$$n_{\tilde{t}}^1 = n_{\tilde{t}-1}^1.$$

Thus

$$n_{\tilde{t}-1}^1 = n_{\tilde{t}}^1 > n_{\tilde{t}}^2 \geq n_{\tilde{t}-1}^2$$

Continuing this process, we will get

$$n_0^1 > n_0^2 \geq n_{-1},\; V_0(N_t^1) < V_0(N_t^2) \leq e.$$

From $n_0^1 > n_0^2 \geq n_{-1}$, we know that for $\{n_t^1\}$, there are capital inflows at period 0. This implies $V_0(N_t^1) = e$, contradicting $V_0(N_t^1) < V_0(N_t^2) \leq e$.

Therefore, $\{n_t\}$ must be unique. □

A.4 Proof of Proposition 3.6

Proof. Recall the proof of Proposition 3.5, we have for $0 \leq t \leq T-2$,

$$p_{\Pi,t} = (1-(1-\lambda_t)(1-p_{\Pi,t+1})\frac{\rho_t^c}{\rho_{t+1}^c})\alpha_t < \alpha_t.$$

However, since at period T, the fundamental value is certain to be fully revealed, i.e., $\lambda_T = 1$, we have

$$p_{\Pi,T-1} = \alpha_{T-1}.$$

Thus with finite trading periods, we have discontinuity in the dispersion of investors' price forecast $\sigma_t(P_{t+1})$ and the dispersion may have a sharp increase in the last trading period, which is not appealing. To get rid of this boundary effect, I assume the following condition:

(A4) $\triangle \bar{P}_{T-3} = \lambda_{T-3}(\sigma_{T-2}(P_{T-1})q_{T-2} + (1-\lambda_{T-2})\sigma_{T-1}(P_T)q_{T-1}) - \sigma_{T-3}(P_{T-2})q_{T-3} < 0.$

Notice that if the number of trading periods T is large enough, $\alpha_t \to 0$, $\lambda_t \to 0$, and thus the above condition is easily satisfied.

Under regularity conditions (A1)-(A3), from the proof of Proposition 3.5, we know that $O_t = \sigma_t(P_{t+1})/\lambda_t$ is increasing first then decreasing. Thus for period t,

(1) If $O_{t+1} > O_t$, assume $\triangle \bar{P}_t < 0$, with nondecreasing q_m, we have

$$\triangle \bar{P}_{t+1} = \triangle n_{1,t+1} + \triangle n_{2,t+1} = \frac{\lambda_{t+1}}{1-\lambda_{t+1}}(\frac{\triangle \bar{P}_t}{\lambda_t} - (O_{t+1}q_{t+1} - O_t q_t)) < 0.$$

This implies that if $\triangle \bar{P}_t < 0$, $\triangle \bar{P}_m < 0$ for any $t \leq m \leq T-1$.

(2) If O_m is decreasing during $t \leq m \leq T-1$, q_m will not change at $t \leq m \leq T-1$. Since $\triangle \bar{P}_{T-3} < 0$, by

$$\triangle \bar{P}_t = \lambda_t(\frac{1-\lambda_{t+1}}{\lambda_{t+1}}\triangle \bar{P}_{t+1} + (O_{t+1}q_{t+1} - O_t q_t))$$

we have $\triangle \bar{P}_{T-4} < 0$ and by backward induction, $\triangle \bar{P}_t < 0$.

Therefore, there exists $\tilde{t}$ s.t. $0 \leq \tilde{t} < T-3$, $\triangle \bar{P}_t \geq 0$ for $0 \leq t < \tilde{t}$ and $\triangle \bar{P}_t < 0$ for $\tilde{t} < t \leq T-3$. □

A.5 Proof of Proposition 3.4

Proof. By the same argument used in the proof of Proposition 3.5, to get rid of the boundary effect, instead of (A2), I assume a slightly different condition below:

(A5) $\rho^c_{T-2} < (\rho_S\rho_q + \rho_{D,T-2})(2\rho_S\rho_q + 2\rho_{D,T-1} + 3\rho_S)/(\rho_S + \rho_{D,T-1} - \rho_{D,T-2})$.

This guarantees $p_{\Pi,T-3} > p_{\Pi,T-2}$, and it is easy to verify that $p_{\Pi,t}$ and thus $O_t = \sigma_t(P_{t+1})/\lambda = p_{\Pi,t}\sqrt{1/\rho_S}/\lambda$ is decreasing for $0 \le t \le T-2$. As a result, first, by the indifference condition (28), there are no gradual investor inflows, that is, q_t is constant and I denote it as q; second, using a similar argument to that used in Proposition 3.5, we have $\triangle\bar{P}_{T-3} < 0$ and furthermore $\triangle\bar{P}_t < 0$ for $0 \le t \le T-3$. This completes the proof. □

A.6 Proof of Proposition 5.1

Proof. From the proof of Proposition 3.5, we have

$$E^i_{t-1}[P_t] - P_{t-1} = p_{\Pi,t-1}(S^i - \Pi) - \sigma_{t-1}(P_t)(q_{t-1} - \epsilon_{q,t-1}).$$

Thus, $E^i_{t-1}[P_t] \ge P_{t-1}$ implies

$$\sqrt{\rho_S}(S^i - \Pi) \ge \sqrt{\rho_S}\frac{\sigma_{t-1}(P_t)}{p_{\Pi,t-1}}(q_{t-1} - \epsilon_{q,t-1}) = q_{t-1} - \epsilon_{q,t-1}.$$

Here, $\sqrt{\rho_S}(S^i - \Pi) \sim N(0,1)$.

Similarly, $E^i_t[P_{t+1}] < P_t$ implies

$$\sqrt{\rho_S}(S^i - \Pi) \le q_t - \epsilon_{q,t}.$$

Let $\epsilon_{q,t} = 0$, $\forall 0 \le t \le T$. At time t, the volume of speculative trading is

$$\overline{TV}_t = n_{t-1}Pr(E^i_{t-1}[P_t] \ge P_{t-1}, E^i_t[P_{t+1}] < P_t) = n_{t-1}(\Phi(q_t) - \Phi(q_{t-1})).$$

Using the specification for q_t, we have

$$\overline{TV}_t = 1 - \frac{n_{t-1}}{n_t}.$$

This equals the speed of investor inflows $(n_t - n_{t-1})/n_{t-1}$. □

Appendix B: Computation

The computation procedure for solving the equilibrium is:

(1) Compute $\{\lambda_t\}$ using Proposition 2.1.

(2) By Proposition 3.5, compute $\{\rho_t^c\}$ first, then $\{p_{\Pi,t}, \sigma_t(P_{t+1})\}$.

(3) Since the linear partially revealing equilibrium is unique, we can compute equilibrium investor flows by construction. By Proposition 3.5, we have gradual inflows of rational investors and they stop after a certain number of periods. Thus, first, using the indifference condition (28), solve $\{n_t\}$ and find the period t_1 after which n_t starts to decrease; next, compute the expected total payoff of entry V_t if investor inflows stop at period t, find the first period t_2 when $V_t < e$; $\tilde{t} = \min\{t_1, t_2\}$. This is the period when investor inflows stop. Re-solve $n_{\tilde{t}}$ s.t. $V_{\tilde{t}} = e$. Thus, the equilibrium measure of rational investors trading the risky asset is $\{n_1, \ldots, n_{\tilde{t}}, \ldots, n_{\tilde{t}}\}$.