WHEN IS BAD "BAD ENOUGH"? A FRAMEWORK FOR ANALYZING BENEFITS OF COORDINATION UNDER ENVIRONMENTAL EXTERNALITIES

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Abstract. This paper addresses the problem of determining when coordination is beneficial. I describe a negative externality game containing a "worsening parameter and develop a framework linearizing this parameter for tractable examination. The worsening parameter can be classified according to "own effect" – changing the marginal utility of a players own action, "opponent effect" – altering the marginal externality, or "submodular effect" strengthening the games submodularity. Using this framework, I examine the sufficient conditions for parameter changes to move non-cooperative and cooperative solutions in opposite directions. In a symmetric game, an increase in own effect will increase the distance between utility and action level of the non-cooperative and cooperative solutions. In a non-symmetric game, there are sufficient conditions on the second derivatives which give this pattern as well. I argue that situations behaving in this manner have more benefit to coordination through the increased range in actions.

Keywords: game theory, coordination, linearization, public economics, accelerating externalities, action gaps

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1. Introduction

As with most other resources studied in economics, international cooperation may be considered scarce. There is only so much national effort to expend in the pursuit of negotiation with other countries, whether measured in diplomats' man-hours, dollars spent on transfers, or implementation costs. Thus, a framework is desired for determining which situations are best, most important, or most beneficial to entertain for negotiation. There is an abundance of global externality situations that could benefit from an international treaty, but if there

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is a cost to cooperation – for instance, even the opportunity cost of other things that cannot be negotiated over – then it is vital to know when a situation is more valuable cooperatively.

What does it mean to say an externality is "worse" in one situation than in another? This may be an easy question if social marginal cost is directly estimable. On the other hand, the question may be more difficult to answer if the externality is affected by another parameter more deeply embedded in the model.

This paper presents a framework for the analysis of these questions, as well as sufficient conditions for a situation in which externalities are worse, based on an increasing disparity in actions between coordination and lack thereof. As a preview, this paper finds that an acceleration in the benefit of reduction, quantified by a large positive opponent-directed second derivative, will increase the likelihood of coordination.

Section 2 describes the literature surrounding this problem. In Section 3, I examine how to simplify a severity parameter in a one-stage game, first in a symmetric game and then a non-symmetric game. Section 4 concludes.

2. Literature

The environmental economics literature has long examined international coordination on reducing negative externalities. Many externality problems, including most environmental situations, go beyond the simple explanation of the tragedy of the commons. Analysis often predicates upon modeling particular situations with some detail, such as location and travel hindering resource extraction (Fischer and Mirman, 1992), the development of technology with complementarities driving economic growth (Carlaw and Lipsey, 2002), the market power concentration of resource sale (Mirman and Datta, 1999), managerial risk preferences under catastrophe (Motoh, 2004), or positive spillover effects increasing efficiency after increases in environmental regulation (Galloway and Johnson, 2015). In each of these situations, it is desirable to describe and compare differing environmental details, such as level of hindrance or magnitude of complementarities, and attempt to determine the situation that calls for intervention more urgently.

For example, overfishing is a general environmental externality affecting almost all regions and numerous species. Between 1970 and 2012, the population sizes for marine species have on average declined 49 percent, almost halving over a period of 42 years (World Wildlife Foundation and Zoological Society of London, 2015). This great depletion in ocean stocks is further exacerbated by the growth pattern of aquatic species, which often require a certain minimum population level to guarantee recurrence. Unlike land animals, the actual stock of a marine species can be extremely difficult to assess. Hence, "[u]nless the rate of harvesting can be controlled somehow, the fish population may eventually be reduced (at a profit) to a low level. This in turn may affect the productivity of the resource and greatly reduce future catches" (Clark, 2006). Furthermore, there is evidence that humans are "fishing down the food web," seen in a declining mean trophic level of worldwide catches (Pauly and Palomares, 2005), which is a sign of unsustainable fishing strategies.

Though the problem is widespread, the aggregation hides some of the nuance of which species are most affected. Numbers of species, as well as the number of species, in northern regions have seen some increase, while populations in tropical climes have been declining (Brunel and Boucher, 2007; World Wildlife Foundation and Zoological Society of London, 2015). Apart from region, the biological relationship of fish species can alter the severity of overfishing. Fischer and Mirman (1992) examine the sources of externalities in a model with two national fisheries and find that in a non-cooperative equilibrium, an increase in the reproductive capacity of a country's own fish species leads to a lower catch ratio due to investment value, while the effect of an increase in reproductive capacity of the other country's species depends on the species cross-effects. If the species have a symbiotic or negative interaction (i.e. both prey upon each other), then this causes a lower catch ratio as well, but if the species have a predator-prey relationship, then the increase in reproductive capacity of the other country's fish results in a higher catch ratio. Furthermore, making the species more symbiotic by increasing a positive cross-effect leads to a lower catch ratio; decreasing a negative cross-effect leads to a higher catch ratio; and if there is a predator-prey relationship, it results in a lower catch ratio for the predator, a higher ratio for the prey.

Clearly, understanding the relationship of the two species of fish gives insight into how the negative externality works and how to coordinate to reduce it.

As with this fish species interaction term, in many externality situations there are parameters that change the story which then cause a drastic effect on the externality, such as the threshold effects in an environmental stock externality model (Farzin, 1996) or the tail shape of a distribution in a climate-change model (Weitzman, 2009). Using these varied sources of inspiration, I present a negative externality model in the next section which allows for "worsening" of an externality through a parameter, θ. I also describe and then solve for how changes in this parameter affect the difference between non-coordination and coordination.

3. Model

In developing a whole new framework of analysis, the first setting to examine is the simplest, and it will give basic intuition to guide further study. Here is a one-shot, two player game in which players take an action which exerts an externality. Future work includes full extension to dynamic settings and positive externalities. I leave the exact story, timing, and utility outcomes vague at the moment, since the goal is to describe the most general setting first and investigate individual examples afterward. I formally define the game $\Gamma = \{I, \{A_i\}_{i \in I}, \{w_i\}_{i \in I}, \theta\}$ with the following:

(1) agents $I = \{1, 2\}$;
(2) actions $a_i \in A_i$;
(3) utility functions $w_i(a_i, a_j; \theta) \in W$, which have the properties of
    (a) twice-differentiable continuity, $w_i \in C^2$,
    (b) concavity with respect to own action, $\frac{\partial^2 w_i}{\partial a_i^2} < 0$,
    (c) negative externality, $\frac{\partial w_i}{\partial a_j} < 0$,
    (d) submodularity, $\frac{\partial^2 w_i}{\partial a_i \partial a_j} < 0$, and
    (e) unique Nash equilibrium; and
(4) a "worsening parameter" $\theta \in \Theta$.

Submodularity is for convenience of the analysis, though it can be relaxed in the future. Unique Nash equilibrium allows for ease of examination. Finally, some example of functions and worsening parameters are:

(1) **Fishing Boat 1.** Consider a model of a fishing boat, where a_i is effort that yields a marginal benefit depending on total actions exerted and which has a constant marginal cost:

$$w_i = a_i(1+\theta)v(a_i + a_j) - c \cdot a_i.$$

In this example, θ multiplies the value of the action. In particular, $v(a)$ is decreasing, while a_i and a_j are perfect substitutes, so they enter the valuation function as a sum; increasing either action decreases the marginal benefit of all action. A larger θ means that action is more valuable, and so intuitively, both players would increase actions and thus further diminish $v(\cdot)$.

(2) **Fishing Boat 2.** Another possibility is:

$$w_i = a_i v(a_i, (1+\theta)a_j) - c \cdot a_i.$$

This is example is similar to the previous one, with two small changes: the players' actions are now longer perfect substitutes, and θ now directly magnifies the effect of the opponent's action, decreasing $v(\cdot, \cdot)$. However, unlike the example above, there is no compensating benefit from θ, and it appears that both players will lower their actions, which then gives room for compensation.

(3) **Variance Spread 1.** Now consider a dynamic utility function where θ determines the entrance and effect of shocks:

$$w_i = (1-\beta)a_i[v(a_i + a_j) - c] + \beta V((s - a_i - a_j)((1-\theta)r + \theta h_t)).$$

In this example, there is a dynamic stock which affects value next period, as well as a parameter h_t carried around which affects the variability of next period's input.

As θ gets larger, there is less weight on the static growth rate r and more weight on the series of h_t.

(4) **Variance Spread 2.** Another possibility is:

$$w_i = (1-\beta)a_i[v(a_i + a_j) - c] + \beta E\left[V((s - a_i - a_j)((1-\theta)r + \theta h_{t+1}))|h_t\right].$$

where $h_{t+1} \sim f(h_t)$. This example is similar to the one above, except that h_t is not a known sequence. Agents can no longer perfectly prepare for what will happen, and as θ gets larger, more weight is put toward uncertainty.

(5) **Time Correlation 1.** A final example is:

$$w_i = (1-\beta)u_i(a_i, a_j, s) + \beta E\left[V(a_i, a_j, s, h_{t+1})|h_t, \theta\right].$$

where $h_{t+1} \sim f(h_t, \theta)$. In this final example, which is a further extension of the ones before, θ is not even in the utility function directly, but rather governs the distribution of some shock. If this is a correlation parameter, then this could enhance a dynamic externality.

For now, the problem will remain general. An individual player's Nash optimization problem, which has the unique solution $a_i^N(\theta)$, can be written as follows:

$$\max_{a_i \in A_i} w_i\left(a_i, a_j^N(\theta); \theta\right) \tag{1}$$

Because of the negative externality, the Nash equilibrium is not optimal. A social planner putting equal weight on each player would choose $a^P(\theta) = \left(a_i^P(\theta), a_j^P(\theta)\right)$, which is the unique solution to the following problem:

$$\max_{a_i \in A_i, a_j \in A_j} w_i(a_i, a_j; \theta) + w_j(a_j, a_i; \theta) \tag{2}$$

For both of these problems,[1] there needs to be a baseline of what occurs when the parameter is zero. Only with an understanding of a baseline can a change be measured.

Definition. The **baseline utility function** of a game Γ is evaluated at $\theta = 0$, and is formally written as:

$$w_i(a_i, a_j; 0) = u_i(a_i, a_j) \tag{3}$$

The **baseline optima** are as follows:

(1) The non-cooperative Nash equilibrium is denoted as $a^N(0)$, abbreviated as $\mathbf{a^N}$, with components $\left(a_i^N(0), a_j^N(0)\right)$, which may also be abbreviated to $\left(\mathbf{a_i^N}, \mathbf{a_j^N}\right)$. This is the unique solution to the simultaneous maximization problems for all i in I:

$$\max_{a_i \in A_i} u_i\left(a_i, a_j^N\right)$$

(2) The cooperative Nash equilibrium, or Social Planner's solution, is denoted as $a^P(0)$, or shortened to $\mathbf{a^P}$, with components $\left(a_i^P(0), a_j^P(0)\right)$, which may also be abbreviated to $\left(\mathbf{a_i^P}, \mathbf{a_j^P}\right)$. This is the unique solution to the social planner's maximization problem:

$$\max_{a_i \in A_i, a_j \in A_j} u_i(a_i, a_j) + u_j(a_j, a_i)$$

The existence of negative externalities means that coordination, if possible, would be Pareto-improving. One of the main interests is when coordination will happen and its resulting value. It is possible that when value is higher, coordination is more likely. However, what does it mean for value to be higher? To answer this question, I examine the difference between non-coordination and coordination, taking into account gaps in utilities and in actions. The possible objects of interest to pursue are:

[1] Observe that the first order conditions to both of these problems could be summarized as the following set of equations, where the Nash condition is at $t = 0$, while the social planner's condition is at $t = 1$:

$$\frac{\partial w_i}{\partial a_i} + t\frac{\partial w_j}{\partial a_j} = 0, \quad t\frac{\partial w_i}{\partial a_i} + \frac{\partial w_j}{\partial a_j} = 0.$$

(1) **Direct value to coordination:** This seems to be the clear measure of benefit: how much extra total surplus can be created by moving from non-coordination to coordination in situations with varying degrees of externality? If θ characterizes the externality, then of interest is how increases in θ, which make the externality worse, will affect the gap between coordination and non-coordination:

$$\frac{d}{d\theta}\left[w^P(\theta) - \left(w_1^N(\theta) + w_2^N(\theta)\right)\right]$$

where the $w^P(\theta)$ is the total utility evaluated at $a^P(\theta)$, $w_1^N(\theta)$ is the utility to agent one evaluated at $a^N(\theta)$, while $w_2^N(\theta)$ is the utility to agent two evaluated at $a^N(\theta)$.

The difference between the Pareto value and the summed Nash utilities should always be weakly positive, because of the definitions of the two problems. However, the gap could stay the same as θ increases, or could even shrink. Therefore, it is of non-trivial interest to characterize when this gap is strictly increasing with θ.

Unfortunately without careful attention to the structure of the problem, this object could capture changes purely in levels. It appears this "value to coordination" can be arbitrarily manipulated via magnitude of the gap. This leads to a second object of interest.

(2) **Increase in range of coordination:** In the presence of a negative externality, a social planner's actions are generally smaller than the Nash actions. Rather than pursuing changes in utility, one way to think of an externality getting worse would be if the social planner's recommended actions are decreasing as the worsening parameter increases, while agents acting on their own are inclined to do the opposite. An increasing gap between actions taken under coordination and non-coordination can be another sign that an externality is getting worse. Therefore, of interest is how the difference between the Nash equilibrium actions and the Pareto optimal actions changes with respect to the parameter:

$$\frac{d}{d\theta}\left[a_i^N(\theta) - a_i^P(\theta)\right].$$

If the actions are moving further apart from one another, there may be more benefit to coordination. The scope of possible agreements is increased, and there are more reductions that can be made, so this may be another notion of when a treaty is more likely.

The above can also be written as:

$$\frac{\Delta a_i^N(\theta) - \Delta a_i^P(\theta)}{\Delta\theta}$$

Moving from an original utility function, the change in the parameter is simply the value assigned, that is

$$\Delta\theta = \theta - 0 = \theta.$$

The changes in the Nash and Pareto optimal actions can be written as:

$$\Delta a_i^N(\theta) = a_i^N(\theta) - a_i^N(0)$$

$$\Delta a_i^P(\theta) = a_i^P(\theta) - a_i^P(0)$$

Observe that when multiplied by θ, these look like pieces of a first order Taylor expansion around the Nash equilibrium and the social optimum.

For certain problems, even if it is possible to determine how the parameter θ affects actions, the relative changes between coordination and non-coordination may be difficult to characterize. Furthermore, if θ is of a difficult nature to derive, linearization may assist in answering the questions of interest.

In the following subsection, I describe a symmetric game within the basic assumptions described earlier in order to gain some intuition in a simple case. I modify the original utility functions with three different linearized worsening parameters – an "own effect," an "opponent effect," and a "submodular effect" – in order to represent more complicated utility functions. I then derive general conclusions for such parameters in a one-stage symmetric game.

3.1. **Symmetric Game.** There are many ways to model the severity of an externality, depending on the type of influence the action has upon it. For instance, the simplest notion of worsening could consist of "pure hurt," a multiplying factor on the opponent's action which does not affect marginal utility of own action but which lowers utility unambiguously. A more complicated version of worsening could involve a story of correlation in time shocks of a resource stock, and as more information is available, the resource stock is exploited even more and the tragedy of the commons worsens.

As mentioned earlier, this section models the worsening of externalities using three paths: changing how the opponent's action affects utility, changing how the agent's own action affects utility, and changing how the cross-effect of actions affects utility. These three paths offer representation of more complicated stories on their own or through combinations.

With regard to the earlier discussion of utility gaps versus action gaps, there are a few ways to go about adding a linearized term to the utility function. One possibility is to simply add a linear term multiplied by θ. This will change the derivative with respect to that variable in a linear manner. A "pure hurt" term would be represented as subtracting off $\theta \cdot a_j$ from the baseline utility function. This approach will give the correct intuition for the action gaps, but will necessarily affect utility as well. There is some worry that an increase in the utility gaps between non-coordination and coordination is somehow "built in" through this term, so increasing utility gaps should remain circumspect.

Another approach is to model the parameter effects as if they were Taylor expansions around the Nash equilibrium or the Pareto optimal solution, so these changes in externalities can be thought of as affecting the derivatives of a symmetric utility function. These would not affect utility through the additive term unless actions changed. However, while taking a derivative at two different points is a mathematically sound idea, the economic intuition is somewhat murkier. This exercise also captures the correct directional changes in actions, but may cause concern that the utility function under coordination is different than that under non-coordination.

Because of these concerns, I use the simple linearization to study only the action gaps. I ignore the direct value to coordination, because of the limitations mentioned earlier. I do present the alternative Taylor expansion structures in the Appendix, and initial analysis for them appears to be similar.

The *opponent effect* is the most intuitive of the three linearizations. This is where the worsening of the externality rotates the first derivative with respect to the opponent's action. The linearization for the individual problem is:

$$\begin{aligned} w_i^N(\theta_J) &= u_i(a_i, a_j) - \theta_J a_j, \\ w_j^N(\theta_J) &= u_j(a_j, a_i) - \theta_J a_i. \end{aligned} \tag{4}$$

The own effect linearization for the Social Planner's problem is:

$$w^P(\theta_J) = u_i(a_i, a_j) + u_j(a_j, a_i) - \theta_J \left(a_i + a_j\right). \tag{5}$$

As θ_J increases there is more room for an increase in the opponent's action to harm the player. Thus, as θ_J increases, the externality is worsening, particularly compared to level of $\theta_J = 0$.

The *own effect* improves the value of one's own action, incentivizing agents to take larger actions. With a submodular utility function, this enhanced activity decreases the marginal benefit of the opponent, thereby increasing the negative externality. Here, the worsening of the externality is the rotation of the first derivative with respect to own action. The linearization for the individual problem is:

$$\begin{aligned} w_i^N(\theta_I) &= u_i(a_i, a_j) + \theta_I a_i, \\ w_j^N(\theta_I) &= u_j(a_j, a_i) + \theta_I a_j. \end{aligned} \tag{6}$$

The own effect linearization for the Social Planner's problem is:

$$w^P(\theta_I) = u_i(a_i, a_j) + u_j(a_j, a_i) + \theta_I \left(a_i + a_j\right). \tag{7}$$

This equation is very similar to Equation (5); the main difference is that the sign on the worsening parameter is opposite. When linearizing the parameter, the determination of own or opponent effect in the social planner's problem is reduced to the sign on the coefficient. The increase of the parameter θ_I increases the value of acting, which may in fact override the externality at some point, when individual benefit outweighs social cost. Thus it can be expected that changes of this kind eliminate the need for coordination at high levels, though at low parameter values there might still be benefit.

The *submodular effect* changes the cross-partial of both actions, making the utility function more submodular than before and enhancing the negative externality in this manner. The linearization for the individual problem is:

$$\begin{aligned} w_i^N(\theta_{IJ}) &= u_i(a_i, a_j) - \theta_{IJ} a_i a_j, \\ w_j^N(\theta_{IJ}) &= u_j(a_j, a_i) - \theta_{IJ} a_i a_j. \end{aligned} \tag{8}$$

The submodular effect linearization for the Social Planner's problem is:

$$w_i^P(\theta_{IJ}) = u_i(a_i, a_j) + u_j(a_j, a_i) - 2\theta_{IJ} a_i a_j. \tag{9}$$

The effect of each separate modification can be found by comparing the first order conditions of the altered coordination and non-coordination problems. The following theorem gives the direction of the action gap for each effect in a symmetric game.

Theorem 1. *For a symmetric game Γ, an increase in the parameter multiplying the added linearizations has the following effect for each:*

(1) Increasing the opponent effect increases the distance in the actions under non-coordination and coordination, that is, for all i:

$$\frac{d}{d\theta_J}[a_i^N(\theta_J) - a_i^P(\theta_J)] > 0;$$

(2) Increasing the own effect has ambiguous results on the distance in actions under non-coordination and coordination; and

(3) *Increasing the submodular effect also has ambiguous results on the distance in actions under non-coordination and coordination.*

The proof of Theorem 1 is the Appendix, though its intuition is discussed briefly here. As mentioned earlier, the opponent effect is perhaps the most intuitive, and it is easy to see why its effect is unambiguous. Adding the linearized term to the problem of non-coordination does not change the player's own incentives, so the Nash actions are unchanged. However, this term changes the incentives facing a social planner, and coordinated actions decrease. The opponent effect could model a story where there is simply a larger harm from the opponent's action, or a more complicated story where harm from the opponent's action prevails.

The own effect is ambiguous, at least in attempting to describe it for the whole range. At small increases, it can result in a positive gap, due to the submodularity in the problem. However, the enhanced benefit from an increase in one's action at some point outweighs the increased negative externality caused by the other player doing the same. A social planner would also increase actions, but more slowly because of the negative externality and submodularity. An example of this would be an improved technology that increases the marginal benefit of own action.

The submodular linearization is also ambiguous unless curvature is examined, but for another reason. While the own effect caused increases in actions under both coordination and non-coordination, the submodular effect causes decreases in both. The direction of change in the distance of action gaps depends on the comparative speeds of reduction.

For this symmetric analysis, the three effects were examined separately, in order to distinctly characterize each. In translating an externality situation into this linearized parameter, the three effects may need to be combined to correctly capture the circumstances. This idea requires further examination.

3.2. **Non-symmetric Game.** The symmetric game places assumptions on the direction that the responses to θ can take. For instance, cases where the same change in θ affects the players differently are not permitted. Thus, in examining the many ways an externality could be worse, non-symmetric games are important as well. Moving to a non-symmetric game opens up more possible outcomes with regard to direction of the players' reactions, and the directions derived in Theorem 1 may no longer hold.

Because there are fewer restrictions, this section will ignore the submodular effect in order to keep the analysis tractable. Using both the own effect and the opponent effect allows for the agents to affect each other asymmetrically. The parameter θ_{xy} represents a deepening of player x's effect on player y. With this adjusted linearization, the agent's utility functions are now:

$$\begin{aligned} w_i^N(\theta) &= u_i(a_i, a_j) + \theta_{ii} a_i - \theta_{ji} a_j, \\ w_j^N(\theta) &= u_j(a_j, a_i) - \theta_{ij} a_i + \theta_{jj} a_j. \end{aligned} \tag{10}$$

The non-symmetric linearization for the social planner is:

$$w^P(\theta) = u_i(a_i, a_j) + u_j(a_j, a_i) + (\theta_{ii} - \theta_{ij}) a_i + (\theta_{jj} - \theta_{ji}) a_j. \tag{11}$$

As briefly alluded to earlier, the expansions of interest have a linear combination of coefficients in front of them. However, while the symmetric game assured that both coefficients collapsed into only one, here there are two distinct coefficient. Therefore, two composite coefficients can be defined as allows:

$$\begin{aligned} \gamma_i &= \theta_{ii} - \theta_{ij} \\ \gamma_j &= \theta_{jj} - \theta_{ji} \end{aligned} \tag{12}$$

With these coefficients, the social planner's problem can be rewritten as:

$$\max_{a_i, a_j} \; u_i(a_i, a_j) + u_j(a_j, a_i) + \gamma_i a_i + \gamma_j a_j \tag{13}$$

Because the two effects linearly combine, whether there is an own effect or an opponent effect for each agent is given by the signs of γ_i and γ_j. There are five main regions of interest: both γ_i and γ_j are positive, both are negative, they are of opposite signs, one is zero while the other is positive, and one is zero while the other is negative.

If both coefficients are negative, then there is an opponent effect only. From the previous section, it is safe to say that coordination will reduce actions, while the non-coordination actions are constant or increasing. If they are of opposite signs, or one is negative while the other is zero, then it is likely that the agent causing an opponent effect will have his action reduced, while the other agent's action may be increased. Also from the previous section came the result that the own effect is ambiguous and depending on curvature. This is of great interest and will now be somewhat resolved for the non-symmetric case. The following analysis will characterize sufficient conditions for diverging actions under own effect, or in areas where $\theta_{ii} > \theta_{ij}$ and $\theta_{jj} > \theta_{ji}$.

The Nash equilibrium, a^N, uniquely solves the following simultaneous best response problems:

$$\max_{a_i} u_i(a_i, a_j) + \theta_{ii}a_i - \theta_{ji}a_j$$

$$\max_{a_j} u_j(a_j, a_i) - \theta_{ij}a_i + \theta_{jj}a_j$$

The Nash first order conditions are:

$$\frac{\partial u_i(a_i, a_j)}{\partial a_i} + \theta_{ii} \equiv 0,$$
$$\frac{\partial u_j(a_j, a_i)}{\partial a_j} + \theta_{jj} \equiv 0.$$

In the symmetric case, the direction of movement of actions could be determined because of the extra assumptions symmetricity imposed. Now, however, the actions could be moving in separate directions, as the parameters θ_{ii} and θ_{jj} can also move around separately. One similarity to the symmetric case is that the opponent effect coefficients wash out, no longer appearing in the first order conditions. Any effect from the opponent will come from their own adjustment of action. The directions of changes can be analyzed with a second order expansion. This process can be found in the Appendix. The comparative statics that result can be summarized as follows:

$$U^N = \begin{bmatrix} \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} \end{bmatrix}$$

$$D_{\theta_{ii}} a^N = \begin{bmatrix} \frac{\partial a_i^N}{\partial \theta_{ii}} \\ \frac{\partial a_j^N}{\partial \theta_{ii}} \end{bmatrix}, \quad D_{\theta_{jj}} a^N = \begin{bmatrix} \frac{\partial a_i^N}{\partial \theta_{jj}} \\ \frac{\partial a_j^N}{\partial \theta_{jj}} \end{bmatrix}, \quad Da^N = \begin{bmatrix} D_{\theta_{ii}} a^N & D_{\theta_{jj}} a^N \end{bmatrix}$$

$$U \cdot Da^N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

All of the entries in U are negative due to concavity and submodularity. This means that the entries in $D_{\theta_{ii}} a^N$ need to be opposite signs, as do the entries in $D_{\theta_{jj}} a^N$, in order to obtain the negative identity matrix when multiplied with U^N.

Since θ_{ii} is not in player j's first order conditions, the effect of θ_{ii} on j's action can be described as follows:

$$\frac{\partial a_j^N}{\partial \theta_{ii}} = \frac{\partial a_j^N}{\partial a_i} \cdot \frac{\partial a_i^N}{\partial \theta_{ii}}$$

Because of the submodularity, $\frac{\partial a_j^N}{\partial a_i} < 0$, and because of the own effect, $\frac{\partial a_i^N}{\partial \theta_{ii}} > 0$. Hence, $\frac{\partial a_j^N}{\partial \theta_{ii}} < 0$, so the two are of opposite signs and the story can hold under proper curvature assumptions. This idea is similar to Huang and Smith's discussion of congestion versus agglomeration and determining the direction of externalities in shrimp fishing Huang and Smith (2014).

For the cooperative problem a social planner chooses unique $a^P(\theta)$ to solve:

$$\max_a \; u_i(a_i, a_j) + u_j(a_j, a_i) + \gamma_i a_i + \gamma_j a_j$$

The first order conditions are:

$$\frac{\partial u_i(a_i, a_j)}{\partial a_i} + \frac{\partial u_j(a_j, a_i)}{\partial a_i} + \gamma_i \equiv 0,$$
$$\frac{\partial u_i(a_i, a_j)}{\partial a_j} + \frac{\partial u_j(a_j, a_i)}{\partial a_j} + \gamma_j \equiv 0.$$

Once again, the second order expansions are in the Appendix. The summary of the comparative statics is:

$$U^P = \begin{bmatrix} \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} + \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} + \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} + \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} + \frac{\partial^2 u_i(a_i,a_j)}{\partial a_j^2} \end{bmatrix}$$

$$U^P = U^N + V, \quad V \equiv \begin{bmatrix} \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_j^2} \end{bmatrix}$$

$$D_{\gamma_i} a^P = \begin{bmatrix} \frac{\partial a_i^P}{\partial \gamma_i} \\ \frac{\partial a_j^P}{\partial \gamma_i} \end{bmatrix}, \quad D_{\gamma_j} a^P = \begin{bmatrix} \frac{\partial a_i^P}{\partial \gamma_j} \\ \frac{\partial a_j^P}{\partial \gamma_j} \end{bmatrix}, \quad Da^P = \begin{bmatrix} D_{\gamma_i} a^P & D_{\gamma_j} a^P \end{bmatrix}$$

$$U^P \cdot Da^P = (U^N + V) \cdot Da^P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Because of submodularity and concavity, all the entries in U^N are negative, and the off-diagonal entries in V are negative as well. As of yet, however, this paper has placed no assumptions on the diagonal entries in V. The diagonals are second derivative with respect to opponent's action, an aspect which is not commonly modeled.

In many common utility functions, the opponent-directed second derivative is zero. Once the opponent's initial effect is known, externality or not, rarely is the "speed" of that effect explicitly described as being central to the problem. If the second derivative is zero, $\frac{\partial^2 u_i}{\partial a_j^2} = 0$, this means that the opponent's effect is constant, and that regardless of the opponent's action, their marginal externality will be the same. For a negative externality, if the second derivative is negative, $\frac{\partial^2 u_i}{\partial a_j^2} < 0$, then the opponent's effect on utility is "accelerating" – as the opponent's action is increasing, the marginal externality is becoming more negative. If the second derivative is positive, $\frac{\partial^2 u_i}{\partial a_j^2} > 0$, this means a negative externality is "decelerating." As the opponent's action is increasing, the marginal negative effect on the player is becoming less negative.[2] Borrowing the term from physics, the second derivative of distance with respect

[2]For a positive externality, a negative second derivative is decelerating the effect of the externality, while a positive second derivative is accelerating.

to time is acceleration; this concept gives an idea to the incentives of reduction of a negative externality or promotion of a positive externality. When the benefits to coordination are not only increasing but accelerating, negotiation is warranted.

In order to determine the worsening parameter effect on the gap between actions, the D matrices need to be understood. It can be shown that $Da^N = -U^{-1}$ and $Da^P = -(U+V)^{-1}$. The main questions are how these behave and where $Da^N - Da^P$ has a definite positive sign.

First, some standardization is called for. The derivatives in Da^N are in fact with respect to θ_{ii} and θ_{jj}, while those in Da^P are with respect to γ_i and γ_j, which are the composite coefficients defined earlier. In order to compare the two, it needs to be shown that the hypothetical derivative of a^N with respect to γ_i and γ_j is the same as already taken for θ_{ii} and θ_{jj}.

Lemma 1. *The derivative of a^N with respect to θ_{ii} and θ_{jj} is equal to the derivative of a^N with respect to γ_i and γ_j, i.e.*

$$\begin{bmatrix} \frac{\partial a_i^N}{\partial \theta_{ii}} & \frac{\partial a_i^N}{\partial \theta_{jj}} \\ \frac{\partial a_j^N}{\partial \theta_{ii}} & \frac{\partial a_j^N}{\partial \theta_{jj}} \end{bmatrix} = \begin{bmatrix} \frac{\partial a_i^N}{\partial \gamma_i} & \frac{\partial a_i^N}{\partial \gamma_i} \\ \frac{\partial a_j^N}{\partial \gamma_j} & \frac{\partial a_j^N}{\partial \gamma_j} \end{bmatrix}$$

Proof. Recall the definition of the composite coefficients:

$$\gamma_i = \theta_{ii} - \theta_{ij}$$

$$\gamma_j = \theta_{jj} - \theta_{ji}$$

When taking the total derivative of a_i with respect to γ_i, the following is obtained:

$$\frac{\partial a_i^N}{\partial \gamma_i} = \frac{\partial a_i^N}{\partial \theta_{ii}} - \frac{\partial a_i^N}{\partial \theta_{ij}}$$

It has already been obtained that $\frac{\partial a_i^N}{\partial \theta_{ij}} = 0$, hence we have $\frac{\partial a_i^N}{\partial \gamma_i} = \frac{\partial a_i^N}{\partial \theta_{ii}}$. This can be repeated for γ_j, and then for a_j^N. Thus, the two matrices are equivalent. □

In such a general setting, it is difficult to say where the derivatives are positive or negative. Therefore, instead of looking for necessity, one possible approach is to look for sufficient cases of possible direction. The action gap is certainly increasing if a_i^P decreases while a_i^N grows or remains constant, or if a_i^P remains constant while a_i^N grows. More difficult situations would involve relative speeds of the two and will remain unaddressed in this paper. Thus, this means it is of interest to figure out when Da^N is positive in both entries while Da^P is negative in both entries when both parameters are changed in the same direction, if not by the same magnitude.

First, I examine the Nash actions to find when Da^N is positive. Then, I examine the social planner's actions to find when Da^P is negative. The matrix U^P is a bit more complicated than U^N, so I will use two different approaches.

To find the responses of the Nash actions when the non-symmetric own effects are both increasing, I look for sufficient conditions for the following to be positive in both entries:

$$U \cdot Da^N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Da^N = U^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -U^{-1} \cdot I = -U^{-1}$$

$$= - \begin{bmatrix} \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} \end{bmatrix}^{-1}$$

When Da^N is written as $\begin{bmatrix} D_{\theta_{ii}} a^N & D_{\theta_{jj}} a^N \end{bmatrix}$, if both entries are positive, then Da^N is positive as well. This method of writing Da^N will be a linearization and can be found by the following procedure:

$$\begin{bmatrix} D_{\theta_{ii}} a^N & D_{\theta_{jj}} a^N \end{bmatrix} = - \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} \end{bmatrix}^{-1}$$

If the linearized inverse is negative, then the whole expression will be positive.

Lemma 2. *For Da^N to be positive and for the Nash actions to be increasing in response to an increase in θ, it is sufficient for the own second derivatives to be the same direction in* *comparison to the cross-partials for both agents. That is, the own second derivative can be more negative than the cross partial for both agents:*

$$\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} < \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} \quad \text{and} \quad \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} < \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j}$$

or, the own second derivative can be less negative than the cross partial for both agents:

$$\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} > \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} \quad \text{and} \quad \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} > \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j}$$

The proof of Lemma 2 is in the Appendix. Having found sufficient conditions for Da^N to be increasing, I now examine the movement of Da^P, the actions under coordination. As before, observe that:

$$\begin{aligned} Da^P &= -(U+V)^{-1} \\ &= -\begin{bmatrix} \frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2} & \frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j} & \frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2} \end{bmatrix}^{-1} \end{aligned}$$

Since $(U+V)$ is symmetric, its inverse is also symmetric. Furthermore, a symmetric matrix is diagonalizable, so the eigenvalues of the inverse matrix can be used to figure out the sign of its determinant. Since the matrix is diagonalizable, there is some Q such that:

$$(U+V)^{-1} = Q^T \Lambda Q$$

The sign of this quadratic form is determined by Λ, the matrix of eigenvalues. If the eigenvalues of $(U+V)$ are positive, then the eigenvalues of its inverse will be as well. When multiplied by the outside negative, Da^P will be negative, providing the decreasing effect desired.

Lemma 3. *For U^P to have only positive eigenvalues, it is sufficient that:*

$$\frac{\partial^2 u_i}{\partial a_j^2} > 0 \tag{14}$$

$$\frac{\partial^2 u_j}{\partial a_j^2} > 0 \tag{15}$$

$$\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right) > \left(\frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2 \tag{16}$$

Lemma 3 is proven in the Appendix. Combining it with Lemma 2, the following theorem is obtained.

Theorem 2. *For a symmetric or non-symmetric game* Γ*, it is sufficient for a utility function to satisfy Lemmas 2 and 3 in order for an increase in the parameters multiplying the added linearizations to increase the distance in actions under non-coordination and coordination.*

Proof. If Γ satisfies Lemma 2, then Da^N is positive, so a^N is increasing in θ_{ii} and θ_{jj} and unresponsive to θ_{ij} and θ_{ji}. If Γ satisfies 3, then Da^P is negative, so a^P is decreasing in γ_i and γ_j, while γ_i is increasing in θ_{ii} and decreasing in θ_{ij} and γ_j is increasing in θ_{jj} and decreasing in θ_{ji}. $\square$

Theorems 1 and 2 establish two interesting cases of sufficiency of increasing action gap: the opponent effect only in a symmetric game, and the own effect in a non-symmetric game under certain curvature assumptions.

4. Conclusion

With the number of possibilities for coordination to reduce an externality problem, there must be some method to determine which situations merit that coordination. This paper examines a novel framework for pursuing the answer to how an externality changes given a certain parameter and when the benefit to coordination increases. Since many externality stories can be difficult to analyze, I propose a method of linearization based on three possible effects and analyzed two notable cases of increasing action gaps between coordination and non-coordination. The first effect was a sole opponent effect in a symmetric game, where the optimal action under coordination unambiguously diverges from the non-coordination action. The second was an own effect in a potentially non-symmetric game, where sufficient conditions for divergence include accelerating benefits to reduction of the action that causes the negative externality.

The main extension to pursue is that of centered parameterizations. Unlike the simple linear parameterization, a centered term can give better insight into the utility gap as well. However, there must be careful understanding of how the separate centerings affect economic intuition. In early analysis, the centered Taylor expansion form suggests that the submodular effect is null. In comparison to the ambiguous decreases under the linear parameterization, this departure suggests that the framework should be checked for robustness to parameterization.

The next step is to begin applying this framework to various externality situations. In particular, my main extension of interest is to apply this analysis to a fishery model. For the case of tuna and shrimp, the possible parameter affecting the externality is persistence in growth shocks. The application would then be to develop such an in-depth fishery model and determine whether such a persistence parameter is an own effect, an opponent effect, a submodular effect, or some combination. Any difficulties in applying the framework may suggest an even more complicated problem is at hand than originally thought, but will give some initial validity to the framework. Finally, the framework can very much be incorporated into data analysis of the numerous environmental externalities under study. Further work can assess the efficiency of existing treaties or ascertain which new areas are ripe for coordination.

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Appendix A. Alternative Parametrization with Centered Taylor Expansions

In this paper, simple linear parameters are used. However, an alternative similar to Taylor expansions was suggested. Here are the respective set ups for the three different effects.

(1) Opponent effect:

   (a) Non-coordination:

$$w_i^N(\theta_J) = u_i(a_i, a_j) - \theta_J(a_j - a_j^N(0))$$

$$w_j^N(\theta_J) = u_j(a_j, a_i) - \theta_J(a_i - a_i^N(0))$$

(b) Coordination:

$$w^P(\theta_J) = u_i(a_i, a_j) + u_j(a_j, a_i) - \theta_J \left[(a_i - a_i^P(0)) + (a_j - a_j^P(0))\right]$$

(2) Own effect:

(a) Non-coordination:

$$w_i^N(\theta_I) = u_i(a_i, a_j) + \theta_I(a_i - a_i^N(0))$$

$$w_j^N(\theta_I) = u_j(a_j, a_i) + \theta_I(a_j - a_j^N(0))$$

(b) Coordination:

$$w^P(\theta_I) = u_i(a_i, a_j) + u_j(a_j, a_i) + \theta_I \left[(a_i - a_i^P(0)) + (a_j - a_j^P(0))\right]$$

(3) Submodular effect:

(a) Non-coordination:

$$w_i^N(\theta_{IJ}) = u_i(a_i, a_j) - \theta_{IJ} \left(a_i - a_i^N(0)\right) \left(a_j - a_j^N(0)\right)$$

$$w_j^N(\theta_{IJ}) = u_j(a_j, a_i) - \theta_{IJ} \left(a_i - a_i^N(0)\right) \left(a_j - a_j^N(0)\right)$$

(b) Coordination:

$$w_i^P(\theta_{IJ}) = u_i(a_i, a_j) + u_j(a_j, a_i) - 2\theta_{IJ} \left(a_i - a_i^P(0)\right) \left(a_j - a_j^P(0)\right)$$

Appendix B. Expanded Results from Section 3.1

Proof of Theorem 1

Restatement of Theorem 1 from Section 3.1. *For a symmetric game Γ, an increase in the parameter multiplying the added linearizations has the following effect for each:*

(1) Increasing the opponent effect increases the distance in the actions under non-coordination and coordination, that is, for all i:

$$\frac{d}{d\theta_J}[a_i^N(\theta_J) - a_i^P(\theta_J)] > 0;$$

(2) Increasing the own effect has ambiguous results on the distance in actions under non-coordination and coordination; and

(3) Increasing the submodular effect also has ambiguous results on the distance in actions under non-coordination and coordination.

Proof. For each type of effect, this proof examines the first order conditions to determine the directions of change in the action gaps. Each effect has separate analysis.

(1) **Opponent Effect**

The opponent effect is set-up in the paper in Equations (4) and (5). First, I examine the Nash first order conditions, and then I examine the social planner's first order conditions.

(a) Non-coordination: The maximization problem for agent i, given the Nash equilibrium action of agent j, is:

$$\max_{a_i \in A_i} \; u_i(a_i, a_j^N(\theta_J)) - \theta_J a_j^N(\theta_J)$$

$a^N(0)$ solves the following:

$$\frac{\partial w_i}{\partial a_i} = \frac{\partial u_i(\cdot, a_j^N(0))}{\partial a_i} \equiv 0$$
$$\frac{\partial w_j}{\partial a_j} = \frac{\partial u_j(\cdot, a_i^N(0))}{\partial a_j} \equiv 0$$

$a^N(\theta_J)$ solves the following:

$$\frac{\partial w_i}{\partial a_i} = \frac{\partial u_i(\cdot, a_j^N(\theta_J))}{\partial a_i} \equiv 0$$
$$\frac{\partial w_j}{\partial a_j} = \frac{\partial u_j(\cdot, a_i^N(\theta_J))}{\partial a_j} \equiv 0$$

Observe that $a^N(\theta_J) = a^N(\theta'_J) = a^N(0)$ for all θ_J and θ'_J in θ_J. Hence,

$$\frac{\partial a_i^N(\cdot)}{\partial \theta_J} = 0.$$

(b) Coordination: The maximization problem for the social planner is:

$$\max_{(a_i, a_j) \in A_i \times A_j} \; u_i(a_i, a_j) + u_j(a_j, a_i) - \theta_J (a_i + a_j)$$

For $\theta_J = 0$, $a^P(0)$ solves the following:

$$\frac{\partial(w_i + w_j)}{\partial a_i} = \frac{\partial u_i(\cdot, a_j^P(\theta_J))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta_J), \cdot)}{\partial a_i} \equiv 0$$
$$\frac{\partial(w_i + w_j)}{\partial a_j} = \frac{\partial u_j(\cdot, a_i^P(\theta_J))}{\partial a_j} + \frac{\partial u_i(a_i^P(\theta_J), \cdot)}{\partial a_j} \equiv 0$$

For $\theta_J > 0$, $a^P(\theta_J)$ solves the following:

$$\frac{\partial(w_i + w_j)}{\partial a_i} = \frac{\partial u_i(\cdot, a_j^P(\theta_J))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta_J), \cdot)}{\partial a_i} - \theta_J \equiv 0$$
$$\frac{\partial(w_i + w_j)}{\partial a_j} = \frac{\partial u_j(\cdot, a_i^P(\theta_J))}{\partial a_j} + \frac{\partial u_i(a_i^P(\theta_J), \cdot)}{\partial a_j} - \theta_J \equiv 0$$

Since the game is symmetric, if the Social Planner changes any agent's action, he will change the other's action in the same manner (i.e same direction and likely magnitude). The next lemma looks at the comparative statics of the whole vector.

Lemma B.1. $a^P(\theta_J)$ *is decreasing in* θ_J.

Proof. Suppose not. Suppose that for $\theta'_J > \theta_J$, $a^P(\theta'_J) \not< a^P(\theta_J)$.

(i) Case i. $a^P(\theta'_J) > a^P(\theta_J)$

Look at the FOC for $\frac{\partial(w_i+w_j)}{\partial a_i}$ (the FOC for $\frac{\partial(w_i+w_j)}{\partial a_j}$ are symmetric):

$$\frac{\partial u_i(a_i^P(\theta_J), a_j^P(\theta_J))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta_J), a_i^P(\theta_J))}{\partial a_i} = \theta_J$$
$$\frac{\partial u_i(a_i^P(\theta'_J), a_j^P(\theta'_J))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta'_J), a_i^P(\theta'_J))}{\partial a_i} = \theta'_J$$

Subtract the first from the second:

$$\left[\frac{\partial u_i(a_i^P(\theta'_J), a_j^P(\theta'_J))}{\partial a_i} - \frac{\partial u_i(a_i^P(\theta_J), a_j^P(\theta_J))}{\partial a_i}\right]$$
$$+ \left[\frac{\partial u_j(a_j^P(\theta'_J), a_i^P(\theta'_J))}{\partial a_i} - \frac{\partial u_j(a_j^P(\theta_J), a_i^P(\theta_J))}{\partial a_i}\right] = \theta'_J - \theta_J$$

Because $\theta'_J > \theta_J$, RHS is greater than zero. Now look at LHS. Take the first bracketed term:

$$\frac{\partial u_i(a_i^P(\theta'_J), a_j^P(\theta'_J))}{\partial a_i} - \frac{\partial u_i(a_i^P(\theta_J), a_j^P(\theta_J))}{\partial a_i}$$

$$\left(\frac{\partial u_i(a_i^P(\theta'_J), a_j^P(\theta'_J))}{\partial a_i} - \frac{\partial u_i(a_i^P(\theta_J), a_j^P(\theta'_J))}{\partial a_i}\right)$$
$$+\left(\frac{\partial u_i(a_i^P(\theta_J), a_j^P(\theta'_J))}{\partial a_i} - \frac{\partial u_i(a_i^P(\theta_J), a_j^P(\theta_J))}{\partial a_i}\right)$$

Since u is concave in own action, if $a_i^P(\theta'_J) > a_i^P(\theta_J)$, then it must be the case that for any a_j:

$$\frac{\partial u_i(a_i^P(\theta'_J), \cdot)}{\partial a_i} < \frac{\partial u_i(a_i^P(\theta_J), \cdot)}{\partial a_i}$$

Furthermore, since u is submodular in opponent action, if

$$a_j^P(\theta'_J) > a_j^P(\theta_J),$$

then it must be the case that for any a_i:

$$\frac{\partial u_i(\cdot, a_j^P(\theta'_J))}{\partial a_i} < \frac{\partial u_i(\cdot, a_j^P(\theta_J))}{\partial a_i}$$

This means that the LHS is negative, so it cannot equal the positive RHS. This is a contradiction, so this case will not occur.

(ii) Case ii. $a^P(\theta'_J) = a^P(\theta_J)$

Look at the FOC for $\frac{\partial(w_i+w_j)}{\partial a_i}$ (the FOC for $\frac{\partial(w_i+w_j)}{\partial a_j}$ are symmetric):

$$\frac{\partial u_i(a_i^P(\theta_J), a_j^P(\theta_J))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta_J), a_i^P(\theta_J))}{\partial a_i} = \theta_J$$
$$\frac{\partial u_i(a_i^P(\theta'_J), a_j^P(\theta'_J))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta'_J), a_i^P(\theta'_J))}{\partial a_i} = \theta'_J$$

If $a^P(\theta'_J) = a^P(\theta_J)$, that means that the two LHS are equal as well. This implies that the two RHS should be equal, so $\theta_J = \theta'_J$. This is a contradiction of $\theta_J < \theta'_J$, so this case cannot occur.

Since both cases are contradictions, it must be that for $\theta'_J > \theta_J$, then $a_i^P(\theta'_J) < a_i^P(\theta_J)$, so the social planner's chosen action is decreasing in θ_J. □

By Lemma B.1, it is seen that:

$$\frac{\partial a_i^P(\cdot)}{\partial \theta_J} < 0.$$

Combining this result with that of non-coordination, it has been obtained that for all agents i:

$$\frac{d}{d\theta_J}\left[a_i^N(\theta_J) - a_i^P(\theta_J)\right] > 0.$$

(2) **Own Effect**

The own effect is set-up in the paper in Equations (6) and (7). First, I examine the Nash first order conditions, and then I examine the social planner's first order conditions.

(a) Non-coordination: The maximization problem for agent i, given the Nash equilibrium action of agent j, is:

$$\max_{a_i \in A_i} \; u_i(a_i, a_j^N(\theta_I)) + \theta_I a_i$$

$a^N(\theta_I)$ solves the following:

$$\frac{\partial w_i}{\partial a_i} = \frac{\partial u_i(\cdot, a_j^N(\theta_I))}{\partial a_i} + \theta_I \equiv 0$$
$$\frac{\partial w_j}{\partial a_j} = \frac{\partial u_j(\cdot, a_i^N(\theta_I))}{\partial a_j} + \theta_I \equiv 0$$

The intuition is that this action is increasing, due to the increased own benefit.[3]

Lemma B.2. *$a_i^N(\theta_I)$ is increasing in θ_I.*

Proof. Suppose not. Suppose that for $\theta_I' > \theta_I$, $a_i^N(\theta_I') \not> a_i^N(\theta_I)$.

(i) Case i. $a_i^N(\theta_I') < a_i^N(\theta_I)$

Look at the FOC for $\frac{\partial w_i}{\partial a_i}$:

$$\frac{\partial u_i(a_i^N(\theta_I), a_j^N(\theta_I))}{\partial a_i} = -\theta_I$$
$$\frac{\partial u_i(a_i^N(\theta_I'), a_j^N(\theta_I'))}{\partial a_i} = -\theta_I'$$

Subtract the first from the second:

[3] In the non-symmetric game, there are cross-partials to check to determine the direction of change, but the symmetric game imposes additional assumptions that assist in making this straightforward.

$$\frac{\partial u_i(a_i^N(\theta_I'), a_j^N(\theta_I'))}{\partial a_i} - \frac{\partial u_i(a_i^N(\theta_I), a_j^N(\theta_I))}{\partial a_i} = -\theta_I' + \theta_I$$

The RHS is negative. Look at the LHS and add/subtract some terms:

$$\begin{aligned}
&\frac{\partial u_i(a_i^N(\theta_I'), a_j^N(\theta_I'))}{\partial a_i} - \frac{\partial u_i(a_i^N(\theta_I), a_j^N(\theta_I))}{\partial a_i} \\
&\frac{\partial u_i(a_i^N(\theta_I'), a_j^N(\theta_I'))}{\partial a_i} - \frac{\partial u_i(a_i^N(\theta_I), a_j^N(\theta_I'))}{\partial a_i} \\
&+ \frac{\partial u_i(a_i^N(\theta_I), a_j^N(\theta_I'))}{\partial a_i} - \frac{\partial u_i(a_i^N(\theta_I), a_j^N(\theta_I))}{\partial a_i}
\end{aligned}$$

If $a_i^N(\theta_I') < a_i^N(\theta_I)$, then because of concavity in own action, the first subtraction pair is positive. Since the agents are symmetric, agent j's action must follow the same pattern. If $a_j^N(\theta_I') < a_j^N(\theta_I)$, then because of submodularity, the second subtraction pair is also positive. Thus the LHS is positive, which contradicts the RHS being negative. Thus, this case cannot occur.

(ii) Case ii. $a_i^N(\theta_I') = a_i^N(\theta_I)$

Look at the FOC for $\frac{\partial w_i}{\partial a_i}$:

$$\begin{aligned}
\frac{\partial u_i(a_i^N(\theta_I), a_j^N(\theta_I))}{\partial a_i} &= -\theta_I \\
\frac{\partial u_i(a_i^N(\theta_I'), a_j^N(\theta_I'))}{\partial a_i} &= -\theta_I'
\end{aligned}$$

If $a_i^N(\theta_I') = a_i^N(\theta_I)$ and $a_j^N(\theta_I') = a_j^N(\theta_I)$, then the LHS of both of these are equal. This means the RHS should be equal too. This is a contradiction of the assumption that $\theta_I' > \theta_I$. Therefore, this case cannot occur.

Since both cases cannot occur, it must be the case that $a_i^N(\theta)$ is increasing in θ. This holds symmetrically for $a_J^N(\theta)$. □

By Lemma B.2, it is obtained that:

$$\frac{\partial a_i^N(\cdot)}{\partial \theta_I} > 0.$$

(b) Coordination: The maximization problem for the social planner is:

$$\max_{(a_i,a_j)\in A_i\times A_j} u_i(a_i,a_j)+u_j(a_j,a_i)+\theta_I\,(a_i+a_j)$$

For $\theta_I \geq 0$, $a^P(\theta_I)$ solves the following:

$$\frac{\partial(w_i+w_j)}{\partial a_i}=\frac{\partial u_i(\cdot,a_j^P(\theta_I))}{\partial a_i}+\frac{\partial u_j(a_j^P(\theta_I),\cdot)}{\partial a_i}+\theta_I\equiv 0$$
$$\frac{\partial(w_i+w_j)}{\partial a_j}=\frac{\partial u_j(\cdot,a_i^P(\theta_I))}{\partial a_j}+\frac{\partial u_i(a_i^P(\theta_I),\cdot)}{\partial a_j}+\theta_I\equiv 0$$

The Social Planner may also want to increase actions, because of the increased benefit, but will be wary of the submodularity's effect as well. Recall here, because this is a symmetric game, the action changes will go in the same direction for both agents. The next lemma posits that the SP's actions are also increasing.

Lemma B.3. *$a^P(\theta_I)$ is increasing in θ_I.*

Proof. Suppose not. Suppose that for $\theta_I' > \theta_I$, $a^P(\theta_I') \not> a^P(\theta_I)$.

(i) Case i. $a^P(\theta_I') < a^P(\theta_I)$

Look at the FOC for $\frac{\partial(w_i+w_j)}{\partial a_i}$ (the FOC for $\frac{\partial(w_i+w_j)}{\partial a_j}$ are symmetric):

$$\frac{\partial u_i(a_i^P(\theta_I),a_j^P(\theta_I))}{\partial a_i}+\frac{\partial u_j(a_j^P(\theta_I),a_i^P(\theta_I))}{\partial a_i}=-\theta_I$$
$$\frac{\partial u_i(a_i^P(\theta_I'),a_j^P(\theta_I'))}{\partial a_i}+\frac{\partial u_j(a_j^P(\theta_I'),a_i^P(\theta_I'))}{\partial a_i}=-\theta_I'$$

Subtract the second from the first:

$$\frac{\partial u_i(a_i^P(\theta_I'),a_j^P(\theta_I'))}{\partial a_i}-\frac{\partial u_i(a_i^P(\theta_I),a_j^P(\theta_I))}{\partial a_i}+\frac{\partial u_j(a_j^P(\theta_I'),a_i^P(\theta_I'))}{\partial a_i}$$
$$-\frac{\partial u_j(a_j^P(\theta_I),a_i^P(\theta_I))}{\partial a_i}=-\theta_I'+\theta_I$$

The RHS is negative. Examine the first subtraction pair of the LHS:

$$\frac{\partial u_i(a_i^P(\theta_I'),a_j^P(\theta_I'))}{\partial a_i}-\frac{\partial u_i(a_i^P(\theta_I),a_j^P(\theta_I))}{\partial a_i}$$

$$\left(\frac{\partial u_i(a_i^P(\theta_I'), a_j^P(\theta_I'))}{\partial a_i} - \frac{\partial u_i(a_i^P(\theta_I), a_j^P(\theta_I'))}{\partial a_i}\right)$$
$$+\left(\frac{\partial u_i(a_i^P(\theta_I), a_j^P(\theta_I'))}{\partial a_i} - \frac{\partial u_i(a_i^P(\theta_I), a_j^P(\theta_I))}{\partial a_i}\right)$$

If $a^P(\theta_I') < a^P(\theta_I)$, then by concavity wrt own action, the first subtraction pair is positive, and by submodularity, the second pair is positive. This holds for agent j's first derivatives as well, so the LHS of the previous statement is positive. This contradicts the negative LHS, so this case cannot occur.

(ii) Case ii. $a^P(\theta_I') = a^P(\theta_I)$

Look at the FOC for $\frac{\partial(w_i+w_j)}{\partial a_i}$ (the FOC for $\frac{\partial(w_i+w_j)}{\partial a_j}$ are symmetric):

$$\frac{\partial u_i(a_i^P(\theta_I), a_j^P(\theta_I))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta_I), a_i^P(\theta_I))}{\partial a_i} = -\theta_I$$
$$\frac{\partial u_i(a_i^P(\theta_I'), a_j^P(\theta_I'))}{\partial a_i} + \frac{\partial u_j(a_j^P(\theta_I'), a_i^P(\theta_I'))}{\partial a_i} = -\theta_I'$$

If $a^P(\theta_I') = a^P(\theta_I)$, then the LHS of both functions must be the same. This means the RHS must be the same, i.e. $\theta_I' = \theta_I$, but this is a contradiction. Therefore, this case cannot occur.

Since both of these cases cannot occur, it must be that a_i^P is increasing in θ_I. □

By Lemma B.3, it is obtained that:

$$\frac{\partial a_i^P(\cdot)}{\partial \theta_I} > 0.$$

Both the Nash actions and the efficient actions are increasing in θ_I. At each θ_I, it should be that the efficient actions are smaller than the Nash actions because of the negative externality. Intuition says that the Nash increases are larger, because the agents ignore the externality, but this really depends on the curvature of the utility function. Therefore, though the directions actions take are known, as is the increase in utility for social planner problem, the ambiguity in utility for the Nash problem makes it difficult to say whether the Nash increases are larger or smaller than the efficient increases, rendering the comparison ambiguous.

Thus, the own effect is the confusing type of externality. One the one hand, the direct benefit increases utility, but on the other hand, the agents then exert more of the externality on each other.

(3) **Submodular Effect**

The submodular effect is set-up in the paper in Equations (8) and (9). First, I examine the Nash first order conditions, and then I examine the social planner's first order conditions.

(a) Non-coordination: The maximization problem for agent i, given the Nash equilibrium action of agent j, is:

$$\max_{a_i \in A_i} \; u_i(a_i, a_j^N(\theta_{IJ})) + \theta_{IJ} a_i a_j^N(\theta_{IJ})$$

$a^N(0)$ solves:

$$\frac{\partial w_i}{\partial a_i} = \frac{\partial u_i\left(\cdot, a_j^N(0)\right)}{\partial a_i} \equiv 0$$
$$\frac{\partial w_j}{\partial a_j} = \frac{\partial u_j\left(\cdot, a_i^N(0)\right)}{\partial a_j} \equiv 0$$

$a^N(\theta_{IJ})$ solves:

$$\frac{\partial w_i}{\partial a_i} = \frac{\partial u_i\left(\cdot, a_j^N(\theta_{IJ})\right)}{\partial a_i} - \theta_{IJ} a_j^N(\theta_{IJ}) \equiv 0$$
$$\frac{\partial w_j}{\partial a_j} = \frac{\partial u_j\left(\cdot, a_i^N(\theta_{IJ})\right)}{\partial a_j} - \theta_{IJ} a_i^N(\theta_{IJ}) \equiv 0$$

Going off of the structure above, the next lemma posits that the submodular effect is rendered Null for the Nash equilibrium.

Lemma B.4. *For $\theta'_{IJ} > \theta_{IJ}$, $a_i^N(\theta'_{IJ}) < a_i^N(\theta_{IJ})$.*

Proof. Suppose not. Suppose that for $\theta'_{IJ} > \theta_{IJ}$, $a_i^N(\theta_{IJ}) \geq a_i^N(\theta'_{IJ})$. Because of symmetric utility functions, agents actions go in the same direction.

(i) Case i. $a_i^N(\theta'_{IJ}) > a_i^N(\theta_{IJ}) \, \forall i$. Look at the FOC for $\frac{\partial w_i}{\partial a_i}$:

$$\frac{\partial u_i\left(a_i^N(\theta_{IJ}), a_j^N(\theta_{IJ})\right)}{\partial a_i} = \theta_{IJ} a_j^N(\theta_{IJ})$$

$$\frac{\partial u_i\left(a_i^N(\theta'_{IJ}), a_j^N(\theta'_{IJ})\right)}{\partial a_i} = \theta'_{IJ} a_j^N(\theta'_{IJ})$$

Subtract the second from the first:

$$\frac{\partial u_i\left(a_i^N(\theta'_{IJ}), a_j^N(\theta'_{IJ})\right)}{\partial a_i} - \frac{\partial u_i\left(a_i^N(\theta_{IJ}), a_j^N(\theta_{IJ})\right)}{\partial a_i} = \theta'_{IJ} a_j^N(\theta'_{IJ}) - \theta_{IJ} a_j^N(\theta_{IJ})$$

RHS is positive. If both Nash actions are larger, then by concavity and submodularity, LHS is negative. This case cannot occur.

(ii) Case ii. $a_i^N(\theta'_{IJ}) = a_i^N(\theta_{IJ}) \, \forall i$.

$$\frac{\partial u_i\left(a_i^N(\theta'_{IJ}), a_j^N(\theta'_{IJ})\right)}{\partial a_i} - \frac{\partial u_i\left(a_i^N(\theta_{IJ}), a_j^N(\theta_{IJ})\right)}{\partial a_i} = \theta'_{IJ} a_j^N(\theta'_{IJ}) - \theta_{IJ} a_j^N(\theta_{IJ})$$

If actions are equal, then LHS is equal to zero. The statement can be rewritten as:

$$0 = \left(\theta'_{IJ} - \theta_{IJ}\right) a_j^N(\theta_{IJ})$$

In order for RHS to be zero, need $\theta'_{IJ} = \theta_{IJ}$. This is a contradiction.

Therefore the only possibility is that $a_i^N(\theta_{IJ})$ to be decreasing in θ_{IJ}.

□

By Lemma B.4, it is obtained that:

$$\frac{\partial a_i^N(\cdot)}{\partial \theta_{IJ}} < 0.$$

(b) Coordination: The maximization problem for the social planner is:

$$\max_{(a_i,a_j)\in A_i\times A_j} u_i(a_i,a_j)+u_j(a_j,a_i)+2\theta_{IJ}a_ia_j$$

For $\theta_{IJ}\geq 0$, $a^P(\theta_J)$ solves the following:

$$\frac{\partial(w_i+w_j)}{\partial a_i}=\frac{\partial u_i(\cdot,a_j^P(\theta_{IJ}))}{\partial a_i}+\frac{\partial u_j(a_j^P(\theta_{IJ}),\cdot)}{\partial a_i}-2\theta_{IJ}a_j^P(\theta_{IJ})\equiv 0$$
$$\frac{\partial(w_i+w_j)}{\partial a_j}=\frac{\partial u_j(\cdot,a_i^P(\theta_{IJ}))}{\partial a_j}+\frac{\partial u_i(a_i^P(\theta_{IJ}),\cdot)}{\partial a_j}-2\theta_{IJ}a_i^P(\theta_{IJ})\equiv 0$$

Lemma B.5. *For $\theta'_{IJ}>\theta_{IJ}$, $a_i^P(\theta'_{IJ})<a_i^P(\theta_{IJ})$.*

Proof. Suppose not. Suppose $a^P(\theta'_{IJ})\geq a^P(\theta_{IJ})$.

(i) Case i. $a_i^P(\theta'_{IJ})>a_i^P(\theta_{IJ})\,\forall i$. Look at the FOC for $\frac{\partial w_i}{\partial a_i}$:

$$\frac{\partial u_i\left(a_i^P(\theta_{IJ}),a_j^P(\theta_{IJ})\right)}{\partial a_i}+\frac{\partial u_j\left(a_j^P(\theta_{IJ}),a_i^P(\theta_{IJ})\right)}{\partial a_i}=2\theta_{IJ}a_j^P(\theta_{IJ})$$
$$\frac{\partial u_i\left(a_i^P(\theta'_{IJ}),a_j^P(\theta'_{IJ})\right)}{\partial a_i}+\frac{\partial u_j\left(a_j^P(\theta'_{IJ}),a_i^P(\theta'_{IJ})\right)}{\partial a_i}=2\theta'_{IJ}a_j^P(\theta'_{IJ})$$

Subtract the second from the first:

$$\begin{aligned}&\frac{\partial u_i\left(a_i^P(\theta'_{IJ}),a_j^P(\theta'_{IJ})\right)}{\partial a_i}+\frac{\partial u_j\left(a_j^P(\theta'_{IJ}),a_i^P(\theta'_{IJ})\right)}{\partial a_i}\\&-\frac{\partial u_i\left(a_i^P(\theta_{IJ}),a_j^P(\theta_{IJ})\right)}{\partial a_i}-\frac{\partial u_j\left(a_j^P(\theta_{IJ}),a_i^P(\theta_{IJ})\right)}{\partial a_i}\\&=2\theta'_{IJ}a_j^P(\theta'_{IJ})-2\theta_{IJ}a_j^P(\theta_{IJ})\end{aligned}$$

If $a_i^P(\theta_{IJ})>a_i^P(\theta'_{IJ})$, then RHS is positive.

Because of concavity and submodularity, when both $a_i^P(\theta'_{IJ})>a_i^P(\theta_{IJ})$ and $a_j^P(\theta'_{IJ})>a_j^P(\theta_{IJ})$, we have that:

$$\frac{\partial u_i(a_i^P(\theta'_{IJ}),a_j^P(\theta'_{IJ}))}{\partial a_i}<\frac{\partial u_i(a_j^P(\theta_{IJ}),a_j^P(\theta_{IJ}))}{\partial a_i}$$

and that:

$$\frac{\partial u_j(a_i^P(\theta'_{IJ}), a_j^P(\theta'_{IJ}))}{\partial a_i} < \frac{\partial u_j(a_j^P(\theta_{IJ}), a_j^P(\theta_{IJ}))}{\partial a_i}$$

This means that LHS is negative, which is a contradiction. This case cannot occur.

(ii) Case ii. $a_i^P(\theta'_{IJ}) = a_i^P(\theta_{IJ})$

If the actions are equal for both agents, then LHS is zero, and the subtracted FOC can be written as:

$$0 = 2\left(\theta'_{IJ} - \theta_{IJ}\right) a_j^P(\theta_{IJ})$$

The only way for RHS to equal LHS is for $\theta'_{IJ} = \theta_{IJ}$. This is a contradiction, so this case cannot occur.

□

By Lemma B.5, it is obtained that:

$$\frac{\partial a_i^P(\cdot)}{\partial \theta_{IJ}} < 0.$$

Both the Nash actions and the efficient actions are decreasing in θ_{IJ}. Similar as with the own effect, the efficient actions should be smaller. From the extra two in the social planner's first order conditions, it is suspected that the efficient actions are decreasing more quickly than the non-coordination actions, but this depends on the curvature of the utility function. Thus, the submodular effect is ambiguous as well.[4]

Combined, these three results give Theorem 1.

□

Appendix C. Expanded Results from Section 3.2

Derivation of Second-Order Expansions

(1) Non-coordination first order conditions:

[4]For a centered Taylor expansion version of this problem, the submodular effect would be null, as opposed to ambiguous.

Recall the FOC are:

$$\frac{\partial u_i(a_i, a_j)}{\partial a_i} + \theta_{ii} \equiv 0$$
$$\frac{\partial u_j(a_j, a_i)}{\partial a_j} + \theta_{jj} \equiv 0$$

The derivatives of these with respect to agent i's own effect, θ_{ii}, are:

$$\begin{aligned}
\frac{\partial^2 u_i(a_i, a_j)}{\partial a_i^2} \cdot \frac{\partial a_i^N(\theta)}{\partial \theta_{ii}} + \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i \partial a_j} \cdot \frac{\partial a_j^N(\theta)}{\partial \theta_{ii}} + 1 \quad &= 0 \\
\frac{\partial^2 u_j(a_j, a_i)}{\partial a_j \partial a_i} \cdot \frac{\partial a_i^N(\theta)}{\partial \theta_{ii}} + \frac{\partial^2 u_j(a_j, a_i)}{\partial a_j^2} \cdot \frac{\partial a_j^N(\theta)}{\partial \theta_{ii}} \quad &= 0
\end{aligned}$$

and with respect to agent i's opponent effect, θ_{ij}, are:

$$\begin{aligned}
\frac{\partial^2 u_i(a_i, a_j)}{\partial a_i^2} \cdot \frac{\partial a_i^N(\theta)}{\partial \theta_{ij}} + \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i \partial a_j} \cdot \frac{\partial a_j^N(\theta)}{\partial \theta_{ij}} \quad &= 0 \\
\frac{\partial^2 u_j(a_j, a_i)}{\partial a_j \partial a_i} \cdot \frac{\partial a_i^N(\theta)}{\partial \theta_{ij}} + \frac{\partial^2 u_j(a_j, a_i)}{\partial a_j^2} \cdot \frac{\partial a_j^N(\theta)}{\partial \theta_{ij}} \quad &= 0
\end{aligned}$$

In the second set of expansions, those with respect to θ_{ij}, since the function is concave and submodular, then in both top and bottom two negative numbers multiplied by the derivatives. In order for any set of numbers other than zero to solve this set of equations, this would require the own second derivatives to equal the cross-partials, which is possible, but a small set of functions. Furthermore, since the parameter θ_{ij} does not appear in the first order conditions, this proof will proceed with the case of:

$$\frac{\partial a_i^N(\theta)}{\partial \theta_{ij}} = \frac{\partial a_j^N(\theta)}{\partial \theta_{ij}} = 0.$$

The derivatives of the FOC with respect to agent j's opponent effect on i, θ_{ji}, are:

$$\begin{aligned}
\frac{\partial^2 u_i(a_i, a_j)}{\partial a_i^2} \cdot \frac{\partial a_i^N(\theta)}{\partial \theta_{ji}} + \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i \partial a_j} \cdot \frac{\partial a_j^N(\theta)}{\partial \theta_{ji}} \quad &= 0 \\
\frac{\partial^2 u_j(a_j, a_i)}{\partial a_j \partial a_i} \cdot \frac{\partial a_i^N(\theta)}{\partial \theta_{ji}} + \frac{\partial^2 u_j(a_j, a_i)}{\partial a_j^2} \cdot \frac{\partial a_j^N(\theta)}{\partial \theta_{ji}} \quad &= 0
\end{aligned}$$

and with respect to agent j's own effect , θ_{jj}, are:

$$\begin{aligned}
\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2}\cdot\frac{\partial a_i^N(\theta)}{\partial\theta_{jj}}+\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i\partial a_j}\cdot\frac{\partial a_j^N(\theta)}{\partial\theta_{jj}} &= 0\\
\frac{\partial^2 u_j(a_j,a_i)}{\partial a_j\partial a_i}\cdot\frac{\partial a_i^N(\theta)}{\partial\theta_{jj}}+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_j^2}\cdot\frac{\partial a_j^N(\theta)}{\partial\theta_{jj}}+1 &= 0
\end{aligned}$$

The parameter θ_{ji} displays a similar pattern as did θ_{ij}, and so the proof will proceed under the following:

$$\frac{\partial a_i^N(\theta)}{\partial\theta_{ji}}=\frac{\partial a_j^N(\theta)}{\partial\theta_{ji}}=0.$$

The results with respect to θ_{ii} and θ_{jj} can be condensed into matrix form:

$$\begin{bmatrix}\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i\partial a_j}\\ \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i\partial a_j} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2}\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial a_i^N}{\partial\theta_{ii}} & \frac{\partial a_i^N}{\partial\theta_{jj}}\\ \frac{\partial a_j^N}{\partial\theta_{ii}} & \frac{\partial a_j^N}{\partial\theta_{jj}}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

(2) Coordination first order conditions:

Recall the FOC are:

$$\frac{\partial u_i(a_i,a_j)}{\partial a_i}+\frac{\partial u_i(a_i,a_j)}{\partial a_j}+\frac{\partial u_j(a_j,a_i)}{\partial a_i}+\frac{\partial u_j(a_j,a_i)}{\partial a_j}+\gamma_i+\gamma_j\equiv 0$$

The derivatives of these with respect to agent i's total effect, γ_i, are:

$$\begin{aligned}
&\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2}\cdot\frac{\partial a_i^P}{\partial\gamma_i}+\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i\partial a_j}\cdot\frac{\partial a_j^P}{\partial\gamma_i}\\
&\qquad+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_i^2}\cdot\frac{\partial a_i^P}{\partial\gamma_i}+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_i\partial a_j}\cdot\frac{\partial a_j^P}{\partial\gamma_i}+1 = 0\\
&\frac{\partial^2 u_i(a_i,a_j)}{\partial a_j\partial a_i}\cdot\frac{\partial a_i^P}{\partial\gamma_i}+\frac{\partial^2 u_i(a_i,a_j)}{\partial a_j^2}\cdot\frac{\partial a_j^P}{\partial\gamma_i}\\
&\qquad+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_j\partial a_i}\cdot\frac{\partial a_i^P}{\partial\gamma_i}+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_j^2}\cdot\frac{\partial a_j^P}{\partial\gamma_i} = 0
\end{aligned}$$

and the derivatives with respect to agent j's total effect, γ_j, are:

$$\begin{aligned}
&\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2}\cdot\frac{\partial a_i^P}{\partial \gamma_j}+\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j}\cdot\frac{\partial a_j^P}{\partial \gamma_j}\\
&\qquad+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_i^2}\cdot\frac{\partial a_i^P}{\partial \gamma_j}+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_i \partial a_j}\cdot\frac{\partial a_j^P}{\partial \gamma_j} &= 0\\
&\frac{\partial^2 u_i(a_i,a_j)}{\partial a_j \partial a_i}\cdot\frac{\partial a_i^P}{\partial \gamma_j}+\frac{\partial^2 u_i(a_i,a_j)}{\partial a_j^2}\cdot\frac{\partial a_j^P}{\partial \gamma_j}\\
&\qquad+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_j \partial a_j}\cdot\frac{\partial a_i^P}{\partial \gamma_j}+\frac{\partial^2 u_j(a_j,a_i)}{\partial a_j^2}\cdot\frac{\partial a_j^P}{\partial \gamma_j}+1 &= 0
\end{aligned}$$

The results with respect to γ_i and γ_j can be condensed into matrix form:

$$\begin{bmatrix} \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2}+\frac{\partial^2 u_j(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j}+\frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j}+\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2}+\frac{\partial^2 u_i(a_i,a_j)}{\partial a_j^2} \end{bmatrix}\cdot\begin{bmatrix} \frac{\partial a_i^P}{\partial \gamma_i} & \frac{\partial a_i^P}{\partial \gamma_j} \\ \frac{\partial a_j^P}{\partial \gamma_i} & \frac{\partial a_j^P}{\partial \gamma_j} \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proof of Lemma 2

Restatement of Lemma 2. *For Da^N to be positive and for the Nash actions to be increasing in response to an increase in θ, it is sufficient for the own second derivatives to be the same direction in comparison to the cross-partials for both agents. That is, the own second derivative can be more negative than the cross partial for both agents:*

$$\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2}<\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j}\quad\text{and}\quad\frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2}<\frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j}$$

or, the own second derivative can be less negative than the cross partial for both agents:

$$\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2}>\frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j}\quad\text{and}\quad\frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2}>\frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j}$$

Proof. Recall the set-up of the linearization:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} \end{bmatrix}$$

$$ad - bc = \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i^2} \cdot \frac{\partial^2 u_j(a_i, a_j)}{\partial a_j^2} - \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i \partial a_j} \cdot \frac{\partial^2 u_j(a_i, a_j)}{\partial a_i \partial a_j}$$

$$\begin{bmatrix} d - c & a - b \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 u_j(a_i,a_j)}{\partial a_j^2} - \frac{\partial^2 u_j(a_i,a_j)}{\partial a_i \partial a_j} & \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i^2} - \frac{\partial^2 u_i(a_i,a_j)}{\partial a_i \partial a_j} \end{bmatrix}$$

There are two cases to consider:

(1) $ad - bc > 0$ while $d - c < 0, a - b < 0$

$$ad - bc = \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i^2} \cdot \frac{\partial^2 u_j(a_i, a_j)}{\partial a_j^2} - \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i \partial a_j} \cdot \frac{\partial^2 u_j(a_i, a_j)}{\partial a_i \partial a_j}$$

$$ad - bc > 0 \Rightarrow$$

$$\frac{\partial^2 u_i(a_i, a_j)}{\partial a_i^2} \cdot \frac{\partial^2 u_j(a_i, a_j)}{\partial a_j^2} > \frac{\partial^2 u_i(a_i, a_j)}{\partial a_i \partial a_j} \cdot \frac{\partial^2 u_j(a_i, a_j)}{\partial a_i \partial a_j}$$

By concavity and submodularity, these are individually negative. So, the above statement could hold under some sort of dominant effect idea, where the own second derivative is more negative ("larger") than the cross partial. Then $d - c$ and $a - b$ would be negative, because both d and a would be smaller (more negative) than c and b.

(2) $ad - bc < 0$ and $d - c > 0, a - b > 0$

On the other hand, with the opposite of dominant effect, or some second order opponent effect, then ad would be smaller than bc, but c would be smaller from d (as well as b from a, which would be positive). This would give the same required sign.

□

Proof of Lemma 3

Restatement of Lemma 3. *For U^P to have only positive eigenvalues, it is sufficient that:*

$$\frac{\partial^2 u_i}{\partial a_j^2} > 0$$

$$\frac{\partial^2 u_j}{\partial a_j^2} > 0$$

$$\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right) > \left(\frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2$$

Proof. Recall the method for calculating eigenvalues:

$$\begin{bmatrix} \frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2} - \lambda & \frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j} \\ \frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j} & \frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2} - \lambda \end{bmatrix}$$

The characteristic function is:

$$\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2} - \lambda\right)\left(\frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2} - \lambda\right) - \left(\frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2 = 0$$

Expanding:

$$\lambda^2 - \lambda\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2} + \frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right) + \left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right) - \left(\frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2 = 0$$

Using quadratic function, we know that the values for λ are as follows:

$$\lambda = \frac{1}{2}\left[\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2} + \frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right) \pm \sqrt{\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2} + \frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right)^2 - 4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right) - \left(\frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]}\right]$$

When are both λ positive? First, check under the square root.

$$\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2} + \frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right)^2 \overset{?}{>} 4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2} + \frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2} + \frac{\partial^2 u_j}{\partial a_j^2}\right) - \left(\frac{\partial^2 u_i}{\partial a_i \partial a_j} + \frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]$$

It is known that the eigenvectors of symmetric matrices are real, so the above equation must hold. Therefore, what is under the square root must be positive. Hence, the following condition that must be also true:

$$\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)^2+\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2+\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2$$
$$>2\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)$$

The two eigenvalues can be denoted as:

$$\lambda_1=\frac{1}{2}\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)+\sqrt{\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2 -4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]}\right]$$

$$\lambda_2=\frac{1}{2}\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\sqrt{\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2 -4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]}\right]$$

From this, it is is clear that if $\lambda_2>0 \Rightarrow \lambda_1>0$ (adding a positive amount vs. subtracting it). Hence, for both to be positive, the minimum is to check when λ_2 is positive.

$$0<\frac{1}{2}\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\sqrt{\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2 -4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]}\right]$$

$$0<\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\sqrt{\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2 -4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]}$$

$$\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)>\sqrt{\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2 -4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]}$$

The square root must be positive, so that means

$$\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)>0$$

as well, and since the utility function is concave, need $\frac{\partial^2 u_i}{\partial a_j^2}>0$ and $\frac{\partial^2 u_j}{\partial a_j^2}>0$. With those added assumptions, square both sides:

$$\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2>\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}+\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)^2$$
$$-4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]$$

This then becomes:

$$0>-4\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]$$
$$0<\left[\left(\frac{\partial^2 u_i}{\partial a_i^2}+\frac{\partial^2 u_j}{\partial a_i^2}\right)\left(\frac{\partial^2 u_i}{\partial a_j^2}+\frac{\partial^2 u_j}{\partial a_j^2}\right)-\left(\frac{\partial^2 u_i}{\partial a_i \partial a_j}+\frac{\partial^2 u_j}{\partial a_i \partial a_j}\right)^2\right]$$

Hence, convex opponent derivative and the above condition are sufficient for positive eigenvalues.

□