

Co-movements Between the Real Economy and Financial Markets

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Abstract

The long-term relationship between financial markets and economic conditions is unclear. This paper aims to investigate how movements in the financial markets interact with the broader economy in the long term. Dynamic factor models are implemented to capture unobserved factors—common factor and sector factors. The common factor represents the co-movement between the real economy and the financial markets, and the sector factors indicate co-movements within the economy or within the financial markets. Variance decomposition is performed to show how much of variation in each variable can be explained by the co-movements. The results show that bond indexes are highly co-moved with money/credit related economic indicators, but stock indexes seem only to co-move with one another, and a big portion of variation in the stock market remains unexplained.

Key words: Dynamic factor models, financial markets, co-movements.

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1. Introduction

Financial markets are well known to be volatile in the short term, making forecast almost impossible. But in the longer term, there is substantial linkage between economic growth and the performance of financial markets. This paper is aimed to capture such co-movements between the real economy and financial markets in the past couple of decades. It provides long-term investors useful guidance on how to select their investment portfolios based on related economic variables. Moreover, the investigation of co-movements among financial instruments helps investors with risk hedging by avoiding co-moved instruments in their portfolios.

Correlations between economic conditions and bond and stock returns have attracted enormous attention in the literature. A wide range of literature finds co-movement within the real economy and between economies (Stock & Watson 2002, Kose, Otrok & Whiteman 2003, Jaimowich & Siu 2007, Del Negro & Otrok 2008). Many studies also find co-movement within the financial market (Campbel & Ammer 2003, Pindyck & Rotemberg 1993, Connoly, Stivers & Sun 2005, Baele & Soriano 2009, Bekaert, Hodrick & Zhang 2009).

Especially relevant to our research are studies that focus on finding links between real economy and capital markets (Fama & French 1989, Cochrane 2005). Positive slope of the yield curve is associated with the future increase in the economic activity (Estrella & Hardouvelis 1991). Other studies link different macroeconomic variables to stock returns performance. In particular, expected stock returns have been linked to investment returns (marginal rates of transformation) which are inferred from investment data via a production function (Cochrane 1991), the dividend-earnings ratio (Lamont 1998), investment plans (Lamont 2000), an "output gap" formed from the Federal Reserve capacity index (Cooper and Priestley 2005), and the ratio of consumption to wealth (Lettau and Ludvigson 2001). Dividends and earnings contribute substantial explanatory power at short horizons. For forecasting long-horizon returns, however, only (scaled) stock prices matter (Lamont 1998). Investment plans have substantial forecasting power for excess stock returns, showing that time-varying risk premia affect investment (Lamont 2000). The investment/capital ratio and consumption/wealth ratios are particularly attractive variables. Studies show that the aggregate consumption-wealth ratio can predict long-run stock returns (Lettau & Ludvigson 2001). Baele, Bekaert, & Inghelbrecht (2010) use macroeconomic variables, such as interest rates, inflation, the output gap, and cash flow growth, a "fundamental" risk aversion measure derived from consumption growth data based on Campbell and Cochrane's (1999) model and macroeconomic uncertainty measures derived from survey data on inflation and GDP growth expectations. The paper shows that macroeconomic fundamentals contribute little to explaining stock and bond return correlations

but that other factors, especially liquidity proxies, play a more important role (Baele, Bekaert, & Inghelbrecht 2010).

Many of the above-mentioned papers make strict assumptions. For instance, VAR models assume that financial markets' performance is based on the current and lagged values. In addition, very few studies founded include the time period of financial crisis of 2007 in their sample. In this paper, we relax a lot of these assumptions by implementing a dynamic factor model to capture unobserved co-movements between economic variables and financial benchmark securities over the period of 1987-2014.

We follow the model framework that is widely used in business cycle studies. Kose, M. A., Otrok, C., & Whiteman, C. H. (2003) makes an important contribution to the literature on international business cycles. The authors use Bayesian dynamic latent factor model to estimate common components in macroeconomic aggregates (output, consumption, and investment). This paper introduces a method to study multiple co-movements simultaneously. Del Negro, M., & Otrok, C. (2008) extends the work by introducing time-varying parameters to the dynamic factor models.

The aim of our paper is to investigate whether or not there exist co-movements between economic conditions and financial market performance between 1987 and 2014. Variance decomposition is performed to capture the percentage of variation in each selected variable explained by co-movements.

The rest of the paper is organized as follows: Section 2 describes the data; Section 3 introduces the model setup and discusses empirical methodologies; Section 4 analyzes the empirical results; Section 5 concludes and provides implications. Appendices and references are listed at the end.

2. Data

The dataset contains monthly data on twelve variables: six variables from financial market and six variables from real economy. Data on stock market returns are obtained through Federal Reserve Economic Data (FRED). Bond market returns come from Dimensional Returns database (Dimensional Returns Database). Data on macroeconomic variables come from Federal Reserve Economic Data (FRED). These variables are observed monthly from October 1987 to May 2014. Variable description is presented in Table 1.

Table 1. Description of variables

Variable	Description
Financial Market: Stocks	
S&P 500	Includes returns from 500 large companies' common stocks (large capitalization)
NASDAQ-100	Includes returns from 100 of the largest non-financial companies listed on the NASDAQ
Russell 2000	Includes returns from small-cap to mid-cap companies
Financial Market: Bonds	
Barclays US Government/Credit Bond Index	Includes broad-based index composed of government and corporate debt issues that are investment grade (rated Baa/BBB or higher). Includes US Credit Index
Barclays US Treasury Bond Index	Is the measure of the public obligations of the U.S. Treasury. Includes: public obligations of the U.S. Treasury, Fixed-rate bullet, puttable, and callable bonds, Soft bullets
Barclays US High-Yield Bond Index	Represents USD-denominated, non-investment grade, fixed-rate, taxable corporate bond market. Securities are classied as high-yield if the middle rating of Moodys, Fitch, and S&P is Ba1/ BB+/BB+ or below
Macroeconomic variables	
M2 Money Supply	Includes M1 money supply plus savings deposits, small-denomination time deposits (those issued in amounts of less than \$100,000), and retail money market mutual fund shares.
Index of Manufacturers' Prices	Percentage of purchasing agents who report paying higher prices in the current month compared with the preceding month. A higher index indicates stronger demand for business inputs relative to their supply.
Consumer Credit Outstanding	Percent change in the amount of consumer debt outstanding during the month from the amount three months earlier. Consumer debt includes auto loans and credit card debt, but not home mortgages or home equity loans. Borrowing is a source of consumer purchasing power
New Housing Permits	New private housing units authorized by building permits. This variable tends to lead construction expenditures.
Initial Claims for Unemployment Insurance	Inverted for analysis. Measures the average

Average Workweek In Manufacturing

number of persons who file first-time claims for unemployment compensation each week in a given month. A decline in general business activity leads to layoffs.

The total of paid labor-hours of manufacturing production workers divided by the number of such workers. Employers tend to reduce the workweek of their labor force before they reduce the size of their workforce.

We obtain the total monthly returns for each variable, and use them in estimation. Descriptive statistics of the variables is presented in Table 2.

Table 2. Descriptive statistics of the data

Variable	Obs	Mean	Std. Dev.	Min	Max
Barclays US Government/Credit Bond Index	321	.0058	.0126	-.0419	.0453
Barclays US Treasury Bond Index	321	.0055	.0132	-.0439	.0531
Barclays US High-Yield Bond Index	321	.0074	.0254	-.1590	.1211
S&P500	321	.0066	.0439	-.2176	.1116
NASDAQ-100	321	.0103	.0787	-.4803	.2498
Russell 2000	321	.0078	.0570	-.3077	.1643
M2 Money Supply	321	.0044	.0036	-.0053	.0225
Index of Manufacturers' Prices	321	.0043	.1032	-.3108	.6111
Consumer Credit Outstanding	321	.0049	.0052	-.0069	.0453
New Housing Permits	321	-8.66e-06	.0521	-.2397	.2293
Initial Claims for Unemployment Insurance	321	.0009	.0418	-.0992	.2122
Average Workweek In Manufacturing	321	.0001	.0052	-.0293	.0403

The descriptive statistics shows that mean total monthly returns are higher in the stock and bond market, compared to the returns in the real economy. Particularly, mean returns to three macroeconomic variables (new housing permits, initial claims for unemployment insurance, and average workweek in manufacturing) are the lowest among all variables. Standard deviations of

most variables are larger than the means. This indicates high dispersion of the returns for all variables, or high volatility.

3. Model and Methodology

The model implemented in this study is the single-factor and multi-factor version of the dynamic unobserved factor model used in Kose, Otrok & Whiteman (2003).

Let M denote the number of variables, T denote the length of time series. Observable growth rates of these variables are denoted $y_{i,t}$ for variable $i = 1, \dots, 12$ and time period $t=1, \dots, T$.

The dynamic single-factor model decomposes dynamic of observables $y_{i,t}$ into the sum of two unobservable components:

f_t^{world} - common factor, affects all i 's

$\varepsilon_{i,t}$ - idiosyncratic factor, specific to each i

The single-factor model is:

$$y_{i,t} = a_i + b_i^{world} f_t^{world} + \varepsilon_{i,t}$$

Where a_i is a constant, b_i^{world} is exposure or loading of series i to the common factor.

Both components follow autoregressive processes of order 2:

$$\begin{aligned} f_t^{world} &= \phi_{0,1}^{world} f_{t-1}^{world} + \phi_{0,2}^{world} f_{t-2}^{world} + u_{0,t}^{world} \\ \varepsilon_{i,t} &= \phi_{1,1} \varepsilon_{i,t-1} + \phi_{1,2} \varepsilon_{i,t-2} + \sigma_i u_{i,t} \end{aligned}$$

Where σ_i - standard deviation of idiosyncratic component,

$u_{i,t} \sim N(0,1)$ for $i=0$ and $0=1, \dots, n$ – innovations to laws of motions 2 and 3 (Kose, Otrok & Whiteman 2003).

The dynamic multi-factor model decomposes dynamic of observables $y_{i,t}$ into the sum of several unobservable components:

f_t^{world} - common factor, affects all i 's.

f_t^{region} - group-specific factor, affects only a group of i 's. Two factors belong here: one affects only the real economy, while the other one affects only financial market.

$\varepsilon_{i,t}$ - idiosyncratic factor, specific to each i .

The three-factor model is:

$$y_{i,t} = a_i + b_i^{world} f_t^{world} + b_i^{region} f_t^{region} + \varepsilon_{i,t}$$

Where a_i is a constant, b_i^{world} is exposure or loading of series i to the common factor, b_i^{region} is exposure or loading of series i to the group-specific factor.

All components follow autoregressive processes of order 2:

$$\begin{aligned} f_t^{world} &= \phi_{0,1}^{world} f_{t-1}^{world} + \phi_{0,2}^{world} f_{t-2}^{world} + u_{0,t}^{world} \\ f_t^{region} &= \phi_{0,1}^{region} f_{t-1}^{region} + \phi_{0,2}^{region} f_{t-2}^{region} + u_{0,t}^{region} \\ \varepsilon_{i,t} &= \phi_{i,1} \varepsilon_{i,t-1} + \phi_{i,2} \varepsilon_{i,t-2} + \sigma_i u_{i,t} \end{aligned}$$

Where σ_i - standard deviation of idiosyncratic component,

$u_{i,t} \sim N(0,1)$ for $i=0$ and $0=1, \dots, n$ – innovations to laws of motions 2 and 3.

In order to find how significant the common factors are in explaining the variation of the observable variables, we use variance decomposition (Kose, Otrok & Whiteman 2003). We decompose the variance of each observable variable ($y_{i,t}$) into the fraction that is due to common factors ($f_t^{world}, f_t^{region}$), and the idiosyncratic component ($\varepsilon_{i,t}$).

$$Var(y_{it}) = (b_i^{world})^2 var(f_t^{world}) + (b_i^{region})^2 var(f_t^{region}) + var(\varepsilon_{i,t})$$

For example, the fraction of volatility due to common world factor would be:

$$\frac{(b_i^{world})^2 var(f_t^{world})}{var(y_{it})}$$

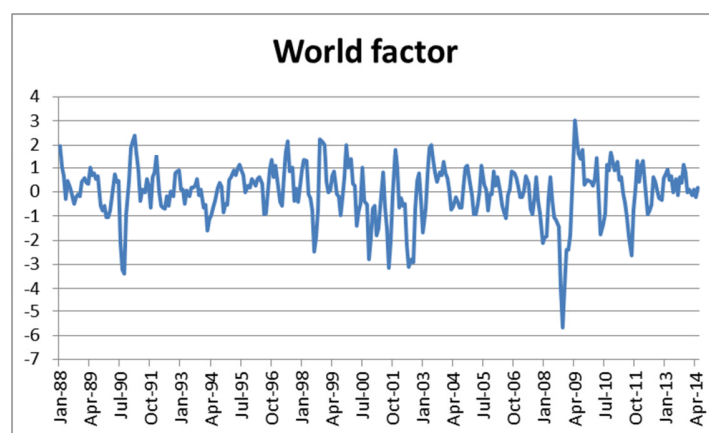
Because the factors are unobservable, special methods must be employed to estimate the model. Our empirical model uses Markov Chain Monte Carlo algorithm and Gibbs sampling. Because the full set of conditional distributions is known, it is possible to generate random samples from the joint posterior distribution for the unknown parameters and the unobserved factor using a Markov-Chain Monte Carlo (MCMC) algorithm. Following Kose, Otrok, and

Whiteman (2003), we use “Gibbs sampling” procedure, which takes this complex problem and decomposes it into a set of tractable ones. We take initial values of the parameters and factors as given, and first sample from the posterior distribution of the parameters conditional on the factors; next we sample from the distribution of the world factor conditional on the parameters and the regional factors; then we sample each regional factor conditional on the world factor. We run 14000 iterations (first 4000 iterations are discarded) to ensure the convergence of results.

4. Empirical Results

We first run the one-factor model, i.e. there is only one common factor that explains co-movement between economic conditions and financial market performance. No regional co-movements are taken into account. Chart 1 shows the three-month moving average of the common factor.

Chart 1: Co-movement captured in the one-factor model



The world factor showed in Chart 1 reveals several economic and financial downturns, for instance, the dot-com bubbles during the late 1990s and early 2000s, and the financial crisis of 2008. But Chart 1 doesn't provide information on significance of the so-called world factor in terms of explaining volatilities of financial markets and the real economy. Variance decomposition is performed to answer this question. See results at Table 3.

Table 3: Shares of variation in each variable explained by the world factor

	World factor share
Barclays US Government/Credit Bond Index	0.11%
Barclays US Treasury Bond Index	0.40%
Barclays U.S. Corporate High Yield Index	18.13%
SP 500	50.99%
NASDAQ 100	38.16%
Russell 2000	52.76%
M2 money supply	0.07%
Index of manufacturers' prices	0.20%
Consumer credit outstanding	0.06%
New housing permits	0.81%
Initial claims for unemployment insurance	1.54%
Average work week in manufacturing	0.03%

Note: 100%-world factor share =idiosyncratic component share

The results show that the stock market variables are dominant in the world factor. Fractions of variation in the stock market that explained by the world factor are substantially larger than any other variable. The corporate high yield index seems to be more correlated with stocks than other bond indexes. The world factor can barely account for variation of any economic variables selected. In other words, evidence of co-movement between the real economy and financial markets is lacking.

One of the reasons that can explain the lack of co-movements is the strong correlation between stock indexes takes over the world factor, leaving little room for the bond indexes and economic variables to participate. Therefore, we adjust the model by introducing three regional factors. Each regional factor captures co-movement within the region. For example, bond market regional factor accounts for the co-movement among bond indexes. There are four factors in the adjusted model: the bond market regional factor, the stock market regional factor, the real economy regional factor, and the world factor. The regional factors are designed to separate regional co-movement effects from the target co-movement. Results are shown at Table 4.

Table 4. Shares of variation in each variable explained by the world factor and the regional factor

	World factor share	Regional factor share
Barclays US Government/Credit Bond Index	99.05%	0.08%
Barclays US Treasury Bond Index	98.95%	0.09%
Barclays U.S. Corporate High Yield Index	54.53%	0.12%
SP 500	30.30%	37.19%
NASDAQ 100	9.27%	37.14%
Russell 2000	1.22%	51.53%
M2 money supply	78.81%	1.03%
Index of manufacturers' prices	26.40%	10.10%
Consumer credit outstanding	91.25%	0.05%
New housing permits	0.35%	21.05%
Initial claims for unemployment insurance	2.46%	31.14%

Average work week in manufacturing	7.24%	1.11%
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Note: $100\% - \text{world factor share} - \text{regional factor share} = \text{idiosyncratic component share}$

The dominant co-movement among stock indexes in the previous model is transformed to the stock market regional factor. But S&P 500 still shows high correlation with other variables besides its correlation with other two stock indexes. This is to say that S&P 500 is more correlated with the bond market and economic conditions than NASDAQ 100 and RUSSELL 2000.

Another important observation is that after removing the effect of regional co-movement in the stock market, bond indexes appear highly co-moved with economic variables. Almost 99 percent of variation of the US government/credit bond index and US Treasury bond index can be accounted for by the co-movement between financial markets and the real economy. The US corporate high yield index seems less relevant to other bond indexes, but more related to stock indexes.

On the economy side, the M2 money supply and consumer credit make a significant contribution to the co-movement, meaning that these economic indicators share the same trend of movement with financial markets. A regional co-movement within the economy is also found, which is mainly contributed by the initial claims for unemployment insurance, new housing permits, and index of manufacturers' prices.

Nevertheless, the variable that indicates average work week in manufacturing is shown to be little correlated with both financial markets and other economic conditions. In the future work, we will consider substituting this indicator with other variables.

5. Conclusions

Efficient market hypothesis states that it is impossible to predict returns in the financial market. However, understanding the nature of the financial market dynamic is important for policy-making, investment, and financial market research.

This paper investigates co-movements between economic indicators and financial market in the U.S. in the period 1987 - 2014. We use the dynamic factor model to find the co-movement between twelve variables: six representing the real economy, and six representing the financial markets. The estimation procedure is Bayesian and parametric, and employs a Gibbs sampler techniques to draw from the exact finite sample joint posterior distribution of the parameters and factors.

The results of the multi-factor model show high correlation between the financial market and the real economy. Bond indexes appear highly co-moved with economic variables. The stock market shows significant co-movement both with the economic variables, and within the stock market itself. On the economy side, the M2 money supply and consumer credit make a significant contribution to the co-movement, meaning that these economic indicators share the same trend of movement with financial markets.

This study provides a comprehensive evidence of broad long-term co-movement between a set of financial market and macroeconomic variables. It fills the gap in the existing literature and shows the presence of common factor that affects both financial market and real economy.

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Appendix:

The baseline model can be rewritten in state space model pattern:

$$y_{it} = a_i + [\lambda_{i1} \ 0 \ \lambda_{i2} \ 0 \ 1 \ 0] \begin{bmatrix} f_t^w \\ f_{t-1}^w \\ f_t^r \\ f_{t-1}^r \\ u_{it} \\ u_{i,t-1} \end{bmatrix}$$

Subject to:

$$\begin{bmatrix} f_t^w \\ f_{t-1}^w \\ f_t^r \\ f_{t-1}^r \\ u_{it} \\ u_{i,t-1} \end{bmatrix} = \begin{bmatrix} \phi_1^w & \phi_2^w & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_1^r & \phi_2^r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{i1} & \phi_{i2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{t-1}^w \\ f_{t-2}^w \\ f_{t-2}^r \\ f_{t-2}^r \\ u_{i,t-1} \\ u_{i,t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^w \\ 0 \\ \varepsilon_t^r \\ 0 \\ e_{it} \\ 0 \end{bmatrix}$$

Gibbs-sampling for estimating parameters is showed in this section as follows:

For generating Ψ for each country i , we know that

$$y_i = A + \Lambda f_i + u_i$$

$$u_{it} = \psi_{i1} u_{i,t-1} + \psi_{i2} u_{i,t-2} + e_{it}$$

So, in matrix notation, we can get

$$\tilde{u}_{iT} = U_i \tilde{\psi}_i + \tilde{e}_{iT}, \quad \tilde{e}_{iT} \sim N(0, \sigma_i^2 I_T)$$

Prior distribution is assumed to be $\tilde{\psi}_i \sim N(a_i, b_i)$.

Posterior distribution can be calculated as

$$\tilde{\psi}_i | \Lambda_i, \sigma_i^2, f_i, y_i \sim N(a_i^*, b_i^*)$$

where

$$a_i^* = (b_i^{-1} + \sigma_i^{-2} U_i' U_i)^{-1} (b_i^{-1} a_i + \sigma_i^{-2} U_i' \tilde{u}_{iT})$$

$$b_i^* = (b_i^{-1} + \sigma_i^{-2} U_i' U_i)^{-1}$$

For generating Φ , we have

$$f_t = \Phi_1 f_{t-1} + \Phi_2 f_{t-2} + \varepsilon_t$$

Prior distribution: $\tilde{\phi}_i \sim N(c_i, d_i)$

Posterior distribution:

$$\tilde{\phi}_i | f_t, y_i \sim N(c_i^*, d_i^*) \quad i=1,2,3$$

where

$$c_i^* = (d_i^{-1} + F_i' F_i)^{-1} (d_i^{-1} c_i + F_i' f_{iT})$$

$$d_i^* = (d_i^{-1} + F_i' F_i)^{-1}$$

For generating σ_i^2 , we know from above

$$\tilde{u}_{iT} = U_i \tilde{\psi}_i + \tilde{\varepsilon}_{iT}, \quad \tilde{\varepsilon}_{iT} \sim N(0, \sigma_i^2 I_T)$$

Prior distribution is $1/\sigma_i^2 \sim \Gamma\left(\frac{v_i}{2}, \frac{w_i}{2}\right)$

Posterior distribution is

$$1/\sigma_i^2 | \tilde{\psi}_i, \Lambda_i, f_i, y_i \sim \Gamma\left(\frac{v_i + (T-2)}{2}, \frac{w_i + (\tilde{u}_{iT} - U_i \tilde{\psi}_i)' (\tilde{u}_{iT} - U_i \tilde{\psi}_i)}{2}\right)$$

For generating Λ , we need to do some adjustment. Substitute $y_t = \Lambda + \Lambda f_t + u_t$ into $u_t = \Psi_1 u_{t-1} + \Psi_2 u_{t-2} + e_t$. Take $i=1$ for example,

$$y_{1t} = \lambda_{11} f_t^w + \lambda_{12} f_t^d + u_{1t}$$

$$u_{1t} = \psi_{11} u_{1,t-1} + \psi_{12} u_{1,t-2} + e_{1t}$$

Then, we can get

$$y_{1t} - \lambda_{11} f_t^w + \lambda_{12} f_t^d = \psi_{11} (y_{1,t-1} - \lambda_{11} f_{t-1}^w + \lambda_{12} f_{t-1}^d) + \psi_{12} (y_{1,t-2} - \lambda_{11} f_{t-2}^w + \lambda_{12} f_{t-2}^d) + e_{1t}$$

$$y_{1t} - \psi_{11} y_{1,t-1} - \psi_{12} y_{1,t-2} = \lambda_{11} (f_t^w - \psi_{11} f_{t-1}^w - \psi_{12} f_{t-2}^w) + \lambda_{12} (f_t^d - \psi_{11} f_{t-1}^d - \psi_{12} f_{t-2}^d) + e_{1t}$$

$$y_{1t}^* = \lambda_{11} f_{1t}^{w*} + \lambda_{12} f_{1t}^{d*} + e_{1t}$$

By using the same method of generating Φ , we can get the sampling for Λ .

For estimating unobserved factors, we rewrote the model into a state space pattern and Kalman Filter is applied to achieve the estimate of factors.

It's important to monitor the convergence of the computation. We did so in a number of ways. First, we restart the computation from a number of different initial values, and the procedure always converges to the same results. Second, we discard the first 4,000 drawings and take the next 10,000 drawings. We try more drawings and the results show the same.