

“Nash-in-Nash” Bargaining: A Microfoundation for Applied Work*

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Abstract

A “Nash equilibrium in Nash bargains” has become the workhorse bargaining model in applied analyses of bilateral oligopoly. This paper proposes a non-cooperative foundation for “Nash-in-Nash” bargaining that extends the Rubinstein (1982) model to multiple upstream and downstream firms. Assuming there exists gains from trade for each firm pair, we prove that there exists an equilibrium with immediate agreement and negotiated prices that correspond to Rubinstein prices if and only if the marginal contribution of a set of agreements is weakly greater than the sum of the agreements’ marginal contributions. We provide stronger conditions under which equilibrium prices are unique.

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1 Introduction

Bilateral bargaining between pairs of agents is pervasive in many economic environments. Manufacturers bargain with retailers over wholesale prices, and firms negotiate with unions over wages paid to workers. As an example, in 2013, private insurers in the United States paid hospitals \$348 billion and physicians and clinics \$267 billion for their services.¹ Private prices for medical services are determined neither by perfect competition, nor by take-it-or-leave-it offers (as is assumed in Bertrand competition). Instead, they are predominantly determined by bilateral negotiations between medical providers and insurers. Furthermore, these negotiations are typically interdependent: e.g., an insurer’s value from having one hospital in its network depends on which other hospitals are already in its network.

Given the centrality of bilateral oligopoly market structures and the prevalence of policy questions in such environments, a substantial theoretical literature has sought to understand equilibrium outcomes of bilateral bargaining models. Papers in this literature have derived conditions for when network environments can yield efficient outcomes (Kranton and Minehart, 2001; Corominas-Bosch, 2004), evaluated how information affects pricing (Polanski, 2007), examined how network centrality affects pricing (Manea, 2011), and considered the impact of investment specificity on efficiency (Elliott, 2015).²

Concurrently, an applied literature—both empirical and theoretical—has used bilateral bargaining models to evaluate a range of questions including: the welfare impact of bundling (Crawford and Yurukoglu, 2012), horizontal mergers (Chipty and Snyder, 1999), and vertical integration (Crawford, Lee, Whinston, and Yurukoglu, 2015) in cable television; the effects of price discrimination for medical devices (Grennan, 2013); and the price impact of hospital mergers (Gowrisankaran, Nevo, and Town, 2015) and health insurance competition (Ho and Lee, 2015).³ Increasingly, this applied literature is influencing antitrust and regulatory policy.⁴

¹See Exhibit 1 on p. 4 of “National Health Expenditure Accounts: Methodology Paper, 2013” at <https://www.cms.gov/Research-Statistics-Data-and-Systems/Statistics-Trends-and-Reports/NationalHealthExpendData/Downloads/DSM-13.pdf> accessed on August 2, 2015.

²A related literature considers the more general case of multilateral or coalitional bargaining with more than two players. See, for instance, Chatterjee, Dutta, Ray, and Sengupta (1993); Merlo and Wilson (1995); Krishna and Serrano (1996); Chae and Yang (1994) (c.f. Osborne and Rubinstein (1994); Muthoo (1999)). We restrict attention to bilateral surplus division, as side payments among firms on the same side of the market (or between firms without a contractual relationship) would generally violate antitrust laws.

³There is also a literature on bargaining over wages between many workers and a single firm, with profits only accruing to the firm side (Stole and Zweibel, 1996; Westermarck, 2003).

⁴The Federal Communication Commission used a bargaining model similar to that analyzed in this paper in its analysis of the Comcast-NBC merger (Rogerson, 2013) and in recent hospital merger cases (Farrell, Balan, Brand, and Wendling, 2011). Also, in a recent ruling in a restraint of trade case in sports broadcasting, Judge Shira Scheindlin’s opinion heavily referenced the Crawford and Yurukoglu (2012) bargaining framework as an appropriate way to consider competition in this sector (c.f. *Thomas Laumann v National Hockey League*

The theoretical and applied literatures have proceeded in different directions. In order to derive meaningful predictions, the theoretical literature has primarily focused on evaluating the equilibrium properties of network games, and has often assumed a simple underlying structure to payoffs: e.g., some papers assume that the value of an agreement between two firms is independent of agreements formed by other parties; other papers allow for externalities but assume specific functional forms. These simplifications have allowed the theoretical literature to develop analytically tractable models for complex network environments. In contrast, the applied literature has emphasized the presence of general forms of interdependencies and externalities across firms and agreements, as they are often fundamental to applied questions in bilateral oligopoly environments. For instance, hospital mergers may raise prices in a bargaining context because the loss to an insurance company from removing multiple hospitals is worse than the sum of the losses from removing individual hospitals (Capps, Dranove, and Satterthwaite, 2003); however, a bargaining model without interdependencies would typically rule out a price increase following a merger. Moreover, the pattern of interdependencies in many settings depends on the rich heterogeneity frequently observed in firm and consumer characteristics, and often does not lend itself to being summarized by simple functional forms.

To tractably and feasibly analyze the division of surplus in settings with interdependent payoffs, the applied literature has leveraged the relatively simple solution concept proposed by Horn and Wolinsky (1988) (used originally to study horizontal merger incentives for a downstream duopoly with an upstream monopolist supplier). This bargaining solution is a set of transfer prices between upstream and downstream firms where the price negotiated between any pair of firms is the Nash bargaining solution for that pair *given that all other pairs reach agreement*. Because this solution can be cast as nesting separate bilateral Nash bargaining problems within a Nash equilibrium to a game played among all pairs of firms, we refer to it as the “Nash-in-Nash” bargaining solution. The Nash-in-Nash solution has the benefit of providing easily computable payments in complicated environments with externalities and interdependent payoffs. It also implies that negotiated prices are based on marginal valuations of relationships, which fits well with classical price theory. Yet, the Nash-in-Nash solution has been criticized by some as an ad hoc solution that nests a cooperative game theory concept of Nash bargaining within a non-cooperative Nash equilibrium. Though non-cooperative microfoundations for Nash-in-Nash have been previously developed, all to our knowledge have required that firms not use all the information that may be at their disposal at any point in time; i.e., they often assume that firms involved in multiple bargains use

(J. Scheindlin, S.D.N.Y. 2015 12-cv-1817 Doc. 431)).

“separate representatives” who cannot communicate with one another for each bargain.⁵

The purpose of this paper is to provide support for the Nash-in-Nash solution as a viable surplus division rule in the applied analysis of bilateral oligopoly by specifying a non-cooperative microfoundation that does not require firms to behave independently (or “schizophrenically”) across bargains. We develop a simple extensive form bargaining game that extends the classic Rubinstein (1982) model to the bilateral oligopoly case. Focusing on cases where there are gains from trade between every pair of firms, we prove two main results. The first result provides a necessary and sufficient complementarity condition—given that all other agreements form, the benefit of having any set of agreements exceeds the sum of the gains from having each individual agreement—for there to exist an equilibrium of our game with immediate agreement between all firms, and negotiated prices equivalent to those in Rubinstein (1982) for each pair of firms (given that all other firms reach agreement); furthermore, these prices converge to the Nash-in-Nash solution as the time between offers goes to zero. The second result provides stronger conditions on the value of adding additional agreements that are sufficient for these equilibrium prices to be unique. Importantly, our uniqueness result does not rely on assuming stationary strategies (e.g., as with refinements such as Markov Perfect equilibrium); instead, we leverage the simultaneity of actions within a period among all upstream or downstream firms, and a restriction on firms’ beliefs following off-equilibrium deviations.

We believe that our work has three general takeaways. First, our finding that it is possible to extend the Rubinstein (1982) non-cooperative foundation for the Nash bargaining solution to a setting with multiple upstream and downstream firms shows that the Nash-in-Nash solution may be a reasonable solution concept with which to examine bilateral oligopoly, thus providing a microfoundation for applied models that use Nash-in-Nash. Second, our equilibrium existence result highlights *when* Nash-in-Nash is an appropriate solution concept. Specifically, we believe that it is appropriate for economic problems with declining returns from additional agreements, but perhaps not for other environments (e.g., with strong complementarities across agreements).⁶ In other cases, one might consider models based on average payoffs, such as Myerson-Shapley values.⁷ Finally, our uniqueness result, proven

⁵For instance, Crawford and Yurukoglu (2012) sketch a non-cooperative extensive form game generating this solution, writing: “Each distributor and each conglomerate sends separate representatives to each meeting. Once negotiations start, representatives of the same firm do not coordinate with each other. We view this absence of informational asymmetries as a weakness of the bargaining model,” (p. 659). Appendix A provides a formal analysis of this type of model. See also Chipty and Snyder (1999) (footnote 10) and Inderst and Montez (2014) (Section 4.2).

⁶While we have not proved that other extensive form models cannot allow for a credible Nash-in-Nash representation with complementarities, we view our representation as the most natural representation of Nash-in-Nash, because it generalizes the Rubinstein (1982) model.

⁷De Fontenay and Gans (2013) provide an extensive-form representation for Myerson-Shapley values.

without restricting attention to stationary strategies, suggests that Nash-in-Nash may be a relatively robust solution within certain settings. We thus believe that our results may serve as a launching point for further development of the theoretical bargaining literature with more complex interdependencies.

Overview. We now briefly discuss additional details of our model, results and proofs. Our extensive form game is a natural extension of Rubinstein (1982) with multiple “upstream” and “downstream” players—from now on “firms.” In odd periods, each downstream firm makes simultaneous private offers to each upstream firm with which it has not yet formed an agreement; each upstream firm then accepts or rejects any subset of its offers. In even periods, roles are reversed, with upstream firms making private offers and downstream firms accepting or rejecting them. If an offer is accepted, a payment is made between the two firms and an agreement forms between the two firms.⁸ At the end of each period, the set (or “network”) of agreements that have been formed is observed by all firms, and upstream and downstream firms earn flow profits. These profits, assumed to be a primitive of the analysis, are a function of the entire set of agreements formed up to that point, and allow for flexible interdependencies across agreements. Crucially, our model admits the possibility that a firm can jointly deviate across multiple negotiations and hence optimally respond to information acquired from one of its negotiations in others in which it is engaged.

Our game has imperfect information since offers are private within a period, and any given firm does not see offers that involve solely other firms. We place restrictions on firm beliefs following the observation of an off-equilibrium offer by employing Perfect Bayesian Equilibrium with passive beliefs as our solution concept. Passive beliefs implies that a firm i , upon receiving an off-equilibrium offer from firm j , assumes that j and all other firms have still made equilibrium offers to their other contracting partners. This solution concept has been widely used and employed in the vertical contracting literature to analyze similar types of problems (Hart and Tirole (1990), McAfee and Schwartz (1994); c.f. Whinston (2006)).

For our analysis, we focus on settings where there are gains from trade from all bilateral agreements between upstream and downstream firms given that all other agreements form. The additional sufficient and necessary condition for the first of our two main results—that there exists a passive-belief equilibrium to our game with negotiated prices that converge to the Nash-in-Nash prices as the time between offers shrinks—is that the marginal contribution to each firm from any set of agreements is no less than the sum of the marginal contributions

⁸We restrict our analysis to the case where the prices are lump-sum payments: e.g., if downstream firms engage in price competition for consumers, we assume that the negotiated prices with upstream firms represent fixed fees so that only the presence of agreements, but not the negotiated prices, affect the value of other agreements.

from each individual agreement within that set. We call this assumption *weak conditional declining marginal contribution* (CDMC). If weak CDMC is violated, then there will be a gain to a firm from dropping multiple agreements. The second of our two main results—that equilibrium prices are unique—relies on stronger assumptions on the declining marginal contributions of agreements and nature of externalities across agreements.⁹

While our proof of existence is relatively straightforward, we view our proof of uniqueness as our primary technical contribution. We leverage induction on the set of agreements formed at any point in time by relying on (i) the fact that what we refer to as “Rubinstein prices” between firms makes a firm indifferent between accepting an offer and rejecting it (with a counterproposal in the next period), and (ii) the feature that multiple agreements form simultaneously.

Specifically, our uniqueness proof proceeds by induction on the set of agreements which have not yet been formed at some point in time (which we call “open” agreements). The key step is proving that, under the inductive hypothesis that any subgame with fewer open agreements results in immediate agreement at Rubinstein prices, any given subgame also has immediate agreement at Rubinstein prices. Relying on induction thus allows us to reduce the problem in cases where other offers are accepted even if a firm rejects a given offer by recursively applying known payoffs upon rejection. The base case, of one open agreement, follows directly from Rubinstein (1982) as our model in this case is identical.

Proving the result for subgames involving multiple firms with open agreements requires ruling out equilibria with agreements that are not at Rubinstein prices or that do not occur simultaneously. Using passive beliefs and a CDMC assumption, we show that if a pair does not reach an agreement when other agreements form, one of the firms has the incentive to “pull up” that agreement to the current period since, by induction, the agreement will otherwise form in the following period at Rubinstein prices. The CDMC assumption ensures that such a deviation is surplus-increasing as it implies that the value of an agreement when all other agreements have not yet been formed is weakly higher than the value of that agreement when all other agreements have been formed (where the latter governs the determination of Rubinstein prices). Having shown that all agreements form simultaneously, we then show that every equilibrium must make each receiving firm marginal between accepting all offers and rejecting exactly one—which is exactly the condition from Rubinstein, implying that agreements form at Rubinstein prices (and converge to Nash-in-Nash prices). We then use assumptions that ensure that the full set of agreements is profitable to prove the immediacy of all agreements.

⁹Appendix E provides counterexamples where weak CDMC—though not the additional assumptions—hold and where the equilibrium prices are not unique.

Proving our result when the set of open agreements only involves one firm on one side of the market and multiple firms on the other—e.g., a single downstream firm and many upstream firms—is the most involved. The reason for this is that in a period when the single firm is receiving offers, it may reject *all* offers that it receives; as the set of open agreements remains the same, our induction argument thus does not apply in the following subgame. Instead, by using a type of “exploding offers” argument as in Rubinstein (1982)—i.e., where any rejection must be met with a future credible gain in order to occur in equilibrium—we are able to find lower bounds on the prices that can be agreed upon in equilibrium. We then leverage these bounds to show that agreement must be simultaneous and at Rubinstein prices.

The remainder of our paper is organized as follows. Section 2 describes our extensive form bargaining model and our equilibrium concept. Section 3 provides our assumptions and results for existence of equilibrium. Section 4 provides our assumptions and results for uniqueness of equilibrium. Section 5 concludes.

2 Model

Consider the negotiations between N upstream firms, U_1, U_2, \dots, U_N , and M downstream firms, D_1, D_2, \dots, D_M . Let \mathcal{G} represent the set of agreements between all firms that can be formed.¹⁰ We only permit agreements to be formed between upstream and downstream firms and not between firms on the same “side” of the market.¹¹ This implies that the set of agreements formed can be represented by a bipartite network between upstream and downstream firms. Denote an agreement between U_i and D_j as ij ; the set of potential agreements that U_i can form as \mathcal{G}_i^U ; and the set of agreements that D_j can form as \mathcal{G}_j^D . For any subset of agreements $\mathcal{A} \subseteq \mathcal{G}$, let $\mathcal{A}_j^D \equiv \mathcal{A} \cap \mathcal{G}_j^D$ denote the set of agreements in \mathcal{A} that involve firm D_j , and let $\mathcal{A}_{-j}^D \equiv \mathcal{A} \setminus \mathcal{A}_j^D$ denote the set of all agreements in \mathcal{A} that do not involve D_j . Define \mathcal{A}_i^U and \mathcal{A}_{-i}^U analogously.

We take as primitives profit functions $\{\pi_i^U(\mathcal{A})\}_{\forall i=1, \dots, N, \mathcal{A} \subseteq \mathcal{G}}$ and $\{\pi_j^D(\mathcal{A})\}_{\forall j=1, \dots, M, \mathcal{A} \subseteq \mathcal{G}}$, which represent the surpluses realized by upstream and downstream firms for a set or “network” of agreements that have been formed at any point in time. Importantly, profits from

¹⁰A model that determines the set the agreements that will be formed is outside the scope of this paper. In the vein of Lee and Fong (2013), there may be a prior network formation game that leads to a set of agreements \mathcal{G} being feasible. This paper only concerns itself with the transfers given \mathcal{G} .

¹¹In many market settings, contractual agreements between two firms on the same side of the market can be interpreted as collusion and hence constitute *per se* antitrust violations. Alternatively, agreements between two firms on the same side of the market can be viewed as a horizontal merger, in which case our analysis would treat those merged firms as one entity. We do not explicitly model the determination of such mergers in this paper.

an agreement may depend on the set of other agreements formed; this allows for interdependencies and externalities in profits across agreements. We assume that each upstream firm U_i and downstream firm D_j negotiate over a price p_{ij} , which represents the lump-sum payment made from D_j to U_i for forming an agreement. E.g., in the health care example, an agreement would represent a hospital joining an insurer’s network and serving its patients. Because we are assuming prices are lump-sum, surplus to other parties depends on the set of agreements formed but not on the negotiated prices.¹²

We model a dynamic game with infinitely many discrete periods. Periods are indexed $t = 1, 2, 3, \dots$, and the time between periods is Λ . Total payoffs (profits and prices) for each firm are discounted. The discount factors between periods for an upstream and a downstream firm are represented by $\delta_{i,U} \equiv \exp(-r_{i,U}\Lambda)$ and $\delta_{i,D} \equiv \exp(-r_{i,D}\Lambda)$ respectively.¹³

The game begins in period $t_0 \geq 1$ with no agreements formed: i.e., all agreements are “open.” In odd periods, each downstream firm D_j simultaneously makes private offers $\{p_{ij}\}_{ij \in \mathcal{G}_j^D}$ to each U_i with which it does not yet have an agreement; each upstream firm U_i then simultaneously accepts or rejects any offers it receives. In even periods, each U_i simultaneously makes private offers $\{p_{ij}\}_{ij \in \mathcal{G}_i^U}$ to downstream firms with which it does not yet have an agreement; each D_j then simultaneously accepts or rejects any offers that it receives. If D_j accepts an offer from U_i , or U_i accepts an offer from D_j , then an agreement is formed between two firms, and those two firms remain “linked” with one another for the rest of the game. Each U_i receives its payment from D_j , p_{ij} , immediately in the period in which an agreement is formed.

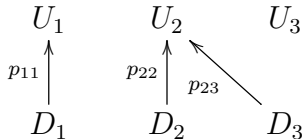
We assume that within a period, a firm only observes the set of contracts that it offers, or that are offered to it. However, at the end of any period, we assume that all firms observe which firms (if any) have come to an agreement. This implies that at the beginning of each period, every firm observes a common *history of play* h^t which contains the sequence of all actions (offers and acceptance/rejections) that have been made by every firm in each preceding period.¹⁴ At the end of each period (after lump sum payments from new agreements

¹²Suppose instead that profits to each firm depends on not only the set of agreements formed by all agents, \mathcal{G} , but also the set of prices agreed upon, $\mathbf{p} \equiv \{p_{ij}\}_{ij \in \mathcal{G}}$: i.e., profits to each D_j are given by $\pi_j(\mathcal{G}, \mathbf{p})$. This would be the case if, for instance, negotiated prices represented wholesale prices or linear fee contracts, and downstream firms engaged in price competition with one another. Dealing with bargaining in a context without transferable utility is difficult. Indeed, to our knowledge, this issue has not been resolved in the context of a two player, Rubinstein (1982) bargaining game, let alone the environment considered in this paper with multiple upstream and downstream firms.

¹³The model can also be recast without discounting but with an exogenous probability of breakdown occurring after the rejection of any offer as in Binmore, Rubinstein, and Wolinsky (1986).

¹⁴Institutionally, the contracted price between U_i and D_j will generally not be observed by $U_k, k \neq i$, either for competitive or antitrust concerns. Relaxing this assumption and allowing for prices or offered contracts to be observed does not change our results, as contracted prices here do not affect the surplus to be divided. All our results will hold as long as the identity of firms reaching an agreement is publicly known

Figure 1: Example: Market with Three Agreements Already Formed



have been made), each upstream firm U_i and downstream firm D_j receives a flow payment equal to $(1 - \delta_{i,U})\pi_i^U(\mathcal{A})$ or $(1 - \delta_{j,D})\pi_j^D(\mathcal{A})$ (respectively), where \mathcal{A} is the set of agreements that have been formed up to that point in time.

Example. Figure 1 provides a graphical representation of a potential market with 3 upstream and 3 downstream firms. Assume that in period 1, the set of agreements $\mathcal{A}^1 \equiv \{11, 22, 23\}$ form. This implies that in period 1, that U_1 receives a payment p_{11} from D_1 , and U_2 receives p_{22} from D_2 and p_{23} from D_3 ; in the same period, each downstream firm D_j receives flow profits $(1 - \delta_{j,D})\pi_j^D(\mathcal{A}^1)$; and each upstream firm U_i receives flow profits $(1 - \delta_{i,U})\pi_i^U(\mathcal{A}^1)$. In period 2, if D_1 forms an agreement with U_2 (and that is the only agreement that is formed), D_1 would then pay U_2 some payment p_{21} , and all firms would earn flow profits as a function of the new realized network of agreements, $\mathcal{A}^2 \equiv \mathcal{A}^1 \cup \{12\}$.¹⁵

Two points about our model are worth noting. First, while the flow profits continue to accrue to all firms forever, the actions in the game stop at the point of the last agreement. Thus, the game can also be formulated to end in the period of last agreement, with a lump-sum payment to each upstream U_i in this period of $(1 - \delta_{i,U})\pi_i^U(\mathcal{G})/(1 - \delta_{i,U}) = \pi_i^U(\mathcal{G})$ (and analogously for downstream firms). Second, if $M = N = 1$, our game is equivalent to the Rubinstein (1982) alternating offers model.

2.1 Equilibrium Concept

Rubinstein (1982) considers subgame perfect equilibria of his model. Because our model has imperfect information (a firm only observes offers within a period that it makes or receives),

at the end of each period.

¹⁵We express profits in terms of flows, since we believe this is a more accurate depiction of many markets. In contrast, profits are paid as a lump sum in the Rubinstein model. However, our formulation is equivalent to D_1 receiving the incremental profits as a lump sum (e.g., if agreements \mathcal{A} were formed in period 1 and agreements \mathcal{B} were formed in period 2, then D_1 would receive $\pi_1^D(\mathcal{A})$ in period 1 and $\pi_1^D(\mathcal{A} \cup \mathcal{B}) - \pi_1^D(\mathcal{A})$ in period 2. We avoid payments between downstream and upstream firms other than lump sum transfers. Otherwise, since each party has a potentially different discount rate, loans could be made between upstream and downstream agents that lead to unbounded increases in the utilities of both parties.

our solution concept is perfect Bayesian equilibrium. However, perfect Bayesian equilibrium does not place restrictions on beliefs for information sets that are not reached in equilibrium. As an example, it does not restrict beliefs of an upstream firm U_i over offers received by other firms upon receiving an out-of-equilibrium price offer from some D_j . Following the literature on vertical contracting (Hart and Tirole, 1990; McAfee and Schwartz, 1994; Segal, 1999), we assume “passive beliefs”: i.e., each firm U_i assumes that other firms receive equilibrium offers even when it observes off-equilibrium offers from D_j . Henceforth, when we refer to an “equilibrium” of this game, we are referring to a perfect Bayesian equilibrium with passive beliefs.

2.2 Nash-in-Nash and Rubinstein Prices

For exposition, it will be useful to define $\Delta\pi_i^U(\mathcal{A}, \mathcal{B}) \equiv \pi_i^U(\mathcal{A}) - \pi_i^U(\mathcal{A} \setminus \mathcal{B})$, for $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{G}$. This term is the increase in profits to U_i of adding agreements in \mathcal{B} to the set of agreements $\mathcal{A} \setminus \mathcal{B}$. We refer to $\Delta\pi_i^U(\mathcal{A}, \mathcal{B})$ as the “marginal contribution” of agreements \mathcal{B} given that agreements \mathcal{A} have been formed. Correspondingly, let $\Delta\pi_j^D(\mathcal{A}, \mathcal{B}) \equiv \pi_j^D(\mathcal{A}) - \pi_j^D(\mathcal{A} \setminus \mathcal{B})$.

We now define Nash-in-Nash and candidate equilibrium prices for our game. For a given set of agreements \mathcal{G} and set of bargaining weights $\{b_{i,U}\}_{\forall i}$ and $\{b_{j,D}\}_{\forall j}$, Nash-in-Nash prices are a vector of prices $\{p_{ij}^{\text{Nash}}\}_{i \in 1, \dots, N, j \in 1, \dots, M}$ such that:

$$\begin{aligned} p_{ij}^{\text{Nash}} &= \arg \max_p [\pi_j^D(\mathcal{G}) - \pi_j^D(\mathcal{G} \setminus \{ij\}) - p]^{b_{j,D}} \times [\pi_i^U(\mathcal{G}) - \pi_i^U(\mathcal{G} \setminus \{ij\}) + p]^{b_{i,U}} \\ &= \frac{b_{i,U} \Delta\pi_j^D(\mathcal{G}, \{ij\}) - b_{j,D} \Delta\pi_i^U(\mathcal{G}, \{ij\})}{b_{i,U} + b_{j,D}}, \forall i = 1, \dots, N, j = 1, \dots, M. \end{aligned}$$

In words, the Nash-in-Nash payoff p_{ij}^{Nash} maximizes the Nash bargaining product between D_j and U_i given all other agreements in \mathcal{G} form. The terms $b_{i,U}$ and $b_{j,D}$ are the bargaining weights of the Nash bargaining problem, which determine the portion of the surplus accruing to each firm.

We define “Rubinstein prices” as follows:

$$\begin{aligned} p_{ij,U}^R &= \frac{(1 - \delta_{j,D}) \Delta\pi_j^D(\mathcal{G}, \{ij\}) - \delta_{j,D} (1 - \delta_{i,U}) \Delta\pi_i^U(\mathcal{G}, \{ij\})}{1 - \delta_{i,U} \delta_{j,D}} \\ p_{ij,D}^R &= \frac{\delta_{i,U} (1 - \delta_{j,D}) \Delta\pi_j^D(\mathcal{G}, \{ij\}) - (1 - \delta_{i,U}) \Delta\pi_i^U(\mathcal{G}, \{ij\})}{1 - \delta_{i,U} \delta_{j,D}}. \end{aligned}$$

These will be the candidate even and odd offers made in equilibrium by firms; when $M = N = 1$, they correspond to the Rubinstein (1982) offers made in alternating periods. As in Binmore, Rubinstein, and Wolinsky (1986), these candidate prices also converge to the

Nash-in-Nash prices as the time between offers becomes arbitrarily small:

Lemma 2.1 $\lim_{\Lambda \rightarrow 0} p_{ij,U}^R = \lim_{\Lambda \rightarrow 0} p_{ij,D}^R = p_{ij}^{Nash}$, where $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$ and $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$.

(All proofs are contained in the appendices.)

There is one property of the Rubinstein prices that will prove crucial in our proofs: Rubinstein prices make the receiving agent indifferent between accepting its offer or waiting until the next period and having its counteroffer accepted given that all other agreements form. In our case, in an even (upstream-proposing) period, this means that a downstream firm is indifferent between accepting and waiting, or:

$$\underbrace{(1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \{ij\})}_{\text{Loss in profit from waiting}} = \underbrace{p_{ij,U}^R - \delta_{j,D}p_{ij,D}^R}_{\text{Decrease in transfer payment from waiting}}. \quad (1)$$

Correspondingly, for an upstream firm in odd periods,

$$(1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) = \delta_{i,U}p_{ij,U}^R - p_{ij,D}^R. \quad (2)$$

2.3 Gains from Trade

For our analysis, we will assume that the joint surplus created from any U_i and D_j coming to an agreement (given that all other agreements have been formed) is positive:

Assumption 2.2 (A.GFT: Gains From Trade)

$$\Delta\pi_i^U(\mathcal{G}, \{ij\}) + \Delta\pi_j^D(\mathcal{G}, \{ij\}) > 0 \quad \forall i, j.$$

Assumption A.GFT implies that each pair of firms has an incentive to keep an agreement given that all other agreements form. We believe that it is natural in many settings of interest as without A.GFT, firms may prefer to drop any agreements in which there are losses from trade. Indeed, this assumption for two parties is fundamental for the Nash bargaining solution to be defined.

Some manipulations of equations (1) and (2) together with A.GFT imply that prices in even (upstream-proposing) periods are higher than in odd periods, and that the value from agreement to each party is higher than prices paid or received:

$$\begin{aligned} \Delta\pi_j^D(\mathcal{G}, \{ij\}) &> p_{ij,U}^R > p_{ij,D}^R, \\ \Delta\pi_i^U(\mathcal{G}, \{ij\}) &> -p_{ij,D}^R > -p_{ij,U}^R. \end{aligned} \quad (3)$$

(Lemma D.1 in Appendix D provides a proof of this result.)

An Alternative Model with Separate Representatives. In Appendix A, we also formally analyze an alternate extensive-form model in which firms appoint separate representatives for each bilateral bargain; this model is based on previous microfoundations discussed in Section 1 (and in papers such as Crawford and Yurukoglu (2012)). We prove that A.GFT is sufficient for there to exist an equilibrium with prices that converge to Nash-in-Nash prices as the time between offers goes to 0. However, we view this alternative model as unsatisfying, as it requires that any firm does not leverage information and cannot coordinate across different concurrent negotiations.

3 Existence of Equilibrium

Our first main result proves that there exists an equilibrium of the bargaining game with immediate agreement between all firms at Rubinstein prices. The result relies on the following necessary and sufficient condition:

Assumption 3.1 (A.WCDMC: Weak Conditional Decreasing Marginal Contribution)

For all $i = 1, \dots, N$ and all $\mathcal{A} \subseteq \mathcal{G}_i^U$,

$$\Delta\pi_i^U(\mathcal{G}, \mathcal{A}) \geq \sum_{ik \in \mathcal{A}} \Delta\pi_i^U(\mathcal{G}, \{ik\}),$$

and, for all $j = 1, \dots, M$ and all $\mathcal{A} \subseteq \mathcal{G}_j^D$,

$$\Delta\pi_j^D(\mathcal{G}, \mathcal{A}) \geq \sum_{hj \in \mathcal{A}} \Delta\pi_j^D(\mathcal{G}, \{hj\}).$$

A.WCDMC states that the marginal contribution of any set of agreements to a firm is weakly greater than the sum of the marginal contributions of each individual agreement within the set. Although A.WCDMC rules out certain cases of agreements having lower inframarginal values (when certain other agreements are not formed) than marginal values (when all other agreements form), we label it “weak” as it does not rule out all of these cases. While we will impose assumptions that are strictly stronger than A.WCDMC to prove uniqueness, A.WCDMC is the minimum assumption needed for existence with our candidate strategies.

Note that A.WCDMC is satisfied by many of the applications of the Nash-in-Nash solution concept. For instance, in Capps, Dranove, and Satterthwaite (2003), adding another hospital to the choice set increases surplus, but this increase in surplus is decreasing in the

size of the network.¹⁶ Similarly, a declining contribution assumption has also been used in the applied theory literature (e.g., Westermarck (2003) in a setting with one firm bargaining with many workers).

We now state our existence result:

Theorem 3.2 (Existence) *Assume A.GFT. Then there exists an equilibrium of the bargaining game beginning at period t_0 for which:*

- (a) *there is immediate agreement between all agents at every node;*
- (b) *for all i and j , equilibrium prices are $p_{ij}^* = p_{ij,D}^R$ at each odd period node and $p_{ij}^* = p_{ij,U}^R$ at each even period node; and*
- (c) *$p_{ij}^* \rightarrow p_{ij}^{Nash} \forall i,j$ as $\Lambda \rightarrow 0$ regardless of whether t_0 is odd or even, where $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$ and $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$;*

if and only if A.WCDMC holds.

The proof of Theorem 3.2, contained in Appendix C, checks that the equilibrium described is robust to one-shot deviations by any firm, and that if A.WCDMC does not hold, there exists a profitable deviation by a firm. In particular, when A.WCDMC does not hold for a particular firm and subset of agreements, that firm—when deciding to accept or reject that set of offers at Rubinstein prices—would find it profitable to reject those offers.

Discussion. The necessity of A.WCDMC for our equilibrium to exist is best illustrated through counterexamples. Consider first a counterexample where there are strong complementarities across agreements so that A.WCDMC does not hold: assume that there are three upstream “parts suppliers” that each supply a necessary component for production by a downstream “automobile manufacturer.” As the marginal contribution to total surplus of each upstream firm is the total surplus, the Nash-in-Nash prices with equal bargaining weights would give each upstream firm 50% of some “total” surplus. But, this would then leave the downstream firm with a negative payoff since it pays 150% of the total surplus to

¹⁶ Capps, Dranove, and Satterthwaite (2003) show that the profit for an insurer is related to the ex ante surplus received by enrollees from the insurer’s network of hospitals. For a logit model, the total surplus of the insurer’s network \mathcal{H} can be expressed as $\sum_i \log \left(\sum_{j \in \mathcal{H}} u_{ij} \right)$ where u_{ij} is the exponentiated utility (net of an i.i.d. Type I extreme value error) that patient i receives from visiting hospital j and the ‘ i ’ sum is over the patients of the insurer. The marginal contribution of some hospital $k \notin \mathcal{H}$ to the insurer’s network—denoted willingness-to-pay—is thus $WTP = \sum_i \log \left(u_{ik} + \sum_{j \in \mathcal{H}} u_{ij} \right) - \sum_i \log \left(\sum_{j \in \mathcal{H}} u_{ij} \right)$, which can be shown to be decreasing as we add elements to \mathcal{H} . The diminishing returns property also holds more generally, e.g. with random coefficients logit models (Berry, Levinsohn, and Pakes, 1995).

the upstream suppliers, implying that the downstream firm would not wish to reach agreement at these prices with all firms. In this model then, it is implausible that transfers will be based on marginal contributions. Either no agreement will be formed, or surplus division may be based on some other concept, such as average valuation, as in Myerson-Shapley Values.

Now consider a different counterexample with two upstream firms, and a single downstream firm that still earns 45% of some “total” surplus if an agreement is formed with either upstream firm, and earns the full surplus if agreements form with both. As before, there are complementarities across agreements (though weaker), and A.WCDMC does not hold. As opposed to the first counterexample, the downstream firm would rather reach agreement at Nash-in-Nash prices of 27.5% of the full surplus with each upstream firm than not reach agreement at all (as it would obtain positive surplus as opposed to none). At our candidate equilibrium from (2), in any even (upstream-proposing) period, the downstream firm will be exactly indifferent between (a) accepting both offers at prices $p_{ij,U}^R$ with each U_i and (b) accepting one offer at $p_{ij,U}^R$ in this period, and rejecting the other (which is formed in the following period at $p_{ij,D}^R$); however, the downstream firm will prefer instead to reject both offers and wait until the following period and have agreements formed at $p_{ij,D}^R$ with both upstream firms, which “breaks” our candidate equilibrium.¹⁷

In both of these examples, if A.WCDMC held, such deviations would not be profitable. To see this, without loss of generality (WLOG), consider any upstream firm U_i in an odd (downstream-proposing) period and any set of agreements $\mathcal{A} \subseteq \mathcal{G}_i^U$ (that involve U_i). A.WCDMC implies that the gain to U_i from accepting the agreements in \mathcal{A} given that all other agreements are accepted is weakly greater than rejecting these offers, and waiting one period to form these agreements (following the candidate equilibrium strategies):

$$\begin{aligned} & (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \mathcal{A}) + \sum_{ik \in \mathcal{A}} [p_{ik,D}^R - \delta_{i,U}p_{ik,U}^R] \\ & \geq \sum_{ik \in \mathcal{A}} [(1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ik\}) + p_{ik,D}^R - \delta_{i,U}p_{ik,U}^R] = 0. \end{aligned} \tag{4}$$

as the inequality follows from A.WCDMC and the equality follows from (2). A similar inequality can be derived to show that each downstream firm D_j in even (upstream-proposing) periods would not wish to reject any subset of offers that it receives.

¹⁷In this case, although it still may be possible to posit a reasonable extensive-form model where payments between firms are based on marginal contributions, we believe that surplus division rules based only on marginal contributions (as opposed to perhaps average contributions) may be less realistic in environments with complementarities.

4 Uniqueness of Equilibrium Prices

Under stronger assumptions than A.WCDMC, we prove in this section our second main result: *every* perfect Bayesian equilibrium with passive beliefs yields the same equilibrium outcomes described in Theorem 3.2 with immediate agreement between all firms at Rubinstein prices. If there are multiple equilibria of this game, they will only vary in prescribed behavior *off* the equilibrium path, and they will all result in the same outcome on the equilibrium path.¹⁸

We provide two sets of sufficient conditions that we use to prove our uniqueness result. We present both as the first set employs stronger conditions on payoffs (that are easier to understand), while the second set delineates more tightly what we use to prove uniqueness.

Condition Set #1 (A.CDMC, A.LEXT): Our first set of conditions includes strengthening A.WCDMC and limiting externalities so that each agent is unaffected by agreements that do not involve it:

Assumption 4.1 (A.CDMC: Conditional Decreasing Marginal Contribution)

$$\begin{aligned}\Delta\pi_i^U(\mathcal{A}, \{ij\}) &\geq \Delta\pi_i^U(\mathcal{G}, \{ij\}) && \forall ij \in \mathcal{A}, \forall \mathcal{A} \subseteq \mathcal{G}, \\ \Delta\pi_j^D(\mathcal{A}, \{ij\}) &\geq \Delta\pi_j^D(\mathcal{G}, \{ij\}) && \forall ij \in \mathcal{A}, \forall \mathcal{A} \subseteq \mathcal{G}.\end{aligned}$$

Assumption 4.2 (A.LEXT: Limited Externalities)

$$\begin{aligned}\pi_i^U(\mathcal{A} \cup \mathcal{B}) &= \pi_i^U(\mathcal{A} \cup \mathcal{B}') && \forall i; \forall \mathcal{A} \subseteq \mathcal{G}_i^U; \forall \mathcal{B}, \mathcal{B}' \subseteq \mathcal{G}_{-i}^U, \\ \pi_j^D(\mathcal{A} \cup \mathcal{B}) &= \pi_j^D(\mathcal{A} \cup \mathcal{B}') && \forall j; \forall \mathcal{A} \subseteq \mathcal{G}_j^D; \forall \mathcal{B}, \mathcal{B}' \subseteq \mathcal{G}_{-j}^D.\end{aligned}$$

where $\mathcal{G}_{-i}^U \equiv \mathcal{G} \setminus \mathcal{G}_i^U$ and $\mathcal{G}_{-j}^D \equiv \mathcal{G} \setminus \mathcal{G}_j^D$.

A.CDMC states that the marginal contribution of any agreement to a firm is weakly lower when all agreements form than if any subset of agreements form. A.LEXT states that a firm's profits only depend on its own formed agreements.

It is straightforward to prove that A.CDMC implies A.WCDMC (see Lemma D.2). Unlike A.WCDMC, A.CDMC rules out *every* case where adding an agreement has a lower value at an inframarginal point to at the point where all other agreements form. To see the difference, consider an example with a single downstream firm and three upstream firms. Suppose

¹⁸Appendix E provides an example where there are multiple equilibria that vary in off-equilibrium-path actions, but coincide along the equilibrium path.

that the downstream firm earns 25% of maximum surplus with one agreement, 70% with two agreements, and full surplus with all three agreements formed. The example violates A.CDMC: the inframarginal benefit to the downstream firm from a single agreement when no agreements have been formed (25%) is less than the marginal benefit when all agreements have been formed (30%). However, the example does not violate A.WCDMC: the marginal benefit of adding two (75%) or three (100%) agreements exceeds the sum of the individual marginal benefits from these agreements (60% and 90%, respectively) at the full network.

Condition Set #2 (A.SCDMC, A.LNEXT, A.ASR): Our second set of conditions uses a stronger assumption than A.CDMC on the marginal contribution of agreements, limits the extent to which agreements involving other firms can harm a firm's profits, and imposes an equilibrium refinement.

Assumption 4.3 (A.SCDMC: Strong Conditional Decreasing Marginal Contribution)

$$\begin{aligned}\pi_i^U(\mathcal{A} \cup \mathcal{B} \cup \{ij\}) - \pi_i^U(\mathcal{A}' \cup \mathcal{B}) &\geq \Delta\pi_i^U(\mathcal{G}, \{ij\}) && \forall ij \in \mathcal{G}; \mathcal{B} \subseteq \mathcal{G}_{-j}^D; \mathcal{A}, \mathcal{A}' \subseteq \mathcal{G}_j^D \setminus \{ij\}, \\ \pi_j^D(\mathcal{A} \cup \mathcal{B} \cup \{ij\}) - \pi_j^D(\mathcal{A}' \cup \mathcal{B}) &\geq \Delta\pi_j^D(\mathcal{G}, \{ij\}) && \forall ij \in \mathcal{G}; \mathcal{B} \subseteq \mathcal{G}_{-i}^U; \mathcal{A}, \mathcal{A}' \subseteq \mathcal{G}_i^U \setminus \{ij\}.\end{aligned}$$

Assumption 4.4 (A.LNEXT: Limited Negative Externalities) $\forall \mathcal{C} \subseteq \mathcal{G}, \exists ij \in \mathcal{C}$ s.t.:

$$\begin{aligned}\Delta\pi_i^U(\mathcal{G}, \mathcal{C}) &\geq \sum_{ik \in \mathcal{C}_i^U} \Delta\pi_i^U(\mathcal{G}, \{ik\}), \\ \Delta\pi_j^D(\mathcal{G}, \mathcal{C}) &\geq \sum_{hj \in \mathcal{C}_j^D} \Delta\pi_j^D(\mathcal{G}, \{hj\}).\end{aligned}$$

Assumption 4.5 (A.ASR: Acceptance Strategies Refinement) *We restrict attention to equilibria in which: if any firm, given the strategies of all other firms, is weakly willing to accept an offer (holding fixed its other prescribed actions), it accepts that offer.*

The first of these conditions, A.SCDMC, states that, at the full network, the marginal contribution to any U_i from forming an agreement with any D_j is no greater than the contribution to U_i from forming an agreement with D_j at any subnetwork, even if D_j (and only D_j) were to change any of its other agreements. We state a similar condition for the marginal contribution to any downstream firm from an agreement with any upstream firm. It is straightforward to prove that A.SCDMC implies A.CDMC (and hence A.WCDMC; see Lemma D.3).

The second of these conditions, A.LNEXT, states that for any subset of agreements \mathcal{C} , there exists a pair U_i and D_j , $ij \in \mathcal{C}$, such that for each of them, the gains from having all agreements in \mathcal{C} form (including those that do not involve either of them) is at least as great as the sums of marginal contribution of each individual agreement that does involve them. The statement of this assumption is similar to that of A.WCDMC, except that the subsets considered may also include agreements that do not involve the firm in question.

We refer to A.LNEXT as limiting “negative externalities” since it is equivalent to stating that for any subset \mathcal{C} , there exists a U_i such that any harm to U_i from agreements in \mathcal{C} that do not involve U_i forming is not too large (a similar condition holds for some downstream firm D_j). To see why A.LNEXT implies this, note first that $\Delta\pi_i^U(\mathcal{G}, \mathcal{C}) = \Delta\pi_i^U(\mathcal{G}, \mathcal{C}_{-i}^U) + \Delta\pi_i^U(\mathcal{G} \setminus \mathcal{C}_{-i}^U, \mathcal{C}_i^U)$. Restating the condition from A.LNEXT yields:

$$\Delta\pi_i^U(\mathcal{G}, \mathcal{C}_{-i}^U) \geq \left(\sum_{ik \in \mathcal{C}_i^U} \Delta\pi_i^U(\mathcal{G}, \{ik\}) \right) - \Delta\pi_i^U(\mathcal{G} \setminus \mathcal{C}_{-i}^U, \mathcal{C}_i^U),$$

where the RHS of the inequality can be shown to be negative by an application of A.CDMC. Thus, A.LNEXT (with A.CDMC) implies that the change to U_i 's profits from agreements in \mathcal{C} that do not involve U_i ($\Delta\pi_i^U(\mathcal{G}, \mathcal{C}_{-i}^U)$) is bounded below by some negative amount.

Finally, A.ASR rules out equilibria in which strategies prescribe a firm (given the strategies of other firms and its other actions) rejecting an offer that it is indifferent over accepting or rejecting.

We now state our uniqueness result:

Theorem 4.6 (Uniqueness) *Assume A.GFT and either (i) A.CDMC and A.LEXT; or (ii) A.SCDMC, A.LNEXT and A.ASR. Then every equilibrium of the bargaining game beginning at t_0 satisfies the conditions in Theorem 3.2, with immediate agreement at t_0 for all $ij \in \mathcal{G}$ at prices $p_{ij}^* = p_{ij,D}^R$ ($p_{ij,U}^R$) if t_0 is odd (even), where $p_{ij}^* \rightarrow p_{ij}^{Nash}$ as $\Lambda \rightarrow 0$ and $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$ and $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$.*

4.1 Role of Assumptions

We now discuss intuitively how the additional assumptions contained in Condition Set #1 and #2 are used in the proof of Theorem 4.6 (which is outlined in the next subsection and contained fully in Appendix D). Our proof relies on showing that any equilibrium of our game must have all agreements formed simultaneously and immediately at Rubinstein prices.

We leverage A.SCDMC, or A.CDMC and A.LEXT, in proving that all agreements must be formed simultaneously. To illustrate, consider an example with two upstream firms and

two downstream firms where the first period is odd. Assume, by contradiction, that there exists an equilibrium where only agreements between U_1 and D_1 and between U_2 and D_1 form in the first period (i.e., only agreements $\{11, 21\}$ form). We argue that this cannot be an equilibrium as D_2 will find it profitable to make an out-of-equilibrium offer at Rubinstein prices to U_1 in the first period. First, we use A.CDMC (either assumed directly or implied by A.SCDMC) to argue that U_1 will accept such a deviation: by passive beliefs, U_1 believes that 22 will still not form in the first period, and A.CDMC implies that the inframarginal benefit to U_1 from forming an agreement with D_2 when 22 is not formed is at least as high as the marginal benefit when all agreements form (where this latter amount determines the Rubinstein price; note that A.WCDMC is not sufficient to ensure this). Second, we use A.SCDMC or A.LEXT to ensure that D_2 will wish to engage in such a deviation. In the current example, it could be the case that U_1 , upon receiving D_2 's deviant offer in the first period, would no longer form an agreement with D_1 in the first period. A.SCDMC implies that such a response by U_1 would not deter D_2 from making a deviant offer as the inframarginal value to D_2 from forming 12 is at least as great as the marginal value of forming that agreement at the full network even if U_1 were to change its other agreements; A.LEXT, by eliminating externalities across agreements that do not involve D_2 , achieves the same goal.

Next, we use assumptions A.ASR or A.LEXT to rule out equilibria where a proposing firm receives a “worse” price than its Rubinstein price, due to an off-equilibrium threat by the recipient firm to add or drop another offer over which it is indifferent in the event that the proposing firm demands a better agreement.

Finally, we use A.LNEXT (either assumed or implied using A.LEXT) to prove that all agreements form immediately. Suppose we are faced with a candidate equilibrium where all agreements are simultaneous and form at Rubinstein prices, but there is (either finite or infinite) delay. A.LNEXT implies that—given any set of open agreements—there is a pair of firms for which the marginal contribution of all open agreements forming to each firm is weakly higher than the marginal values of the firm's own open agreements forming. Consequently, there is a profitable deviation among this pair of firms that benefits from immediate agreement, yielding a contradiction.

In Appendix E, we provide several examples to further illustrate the role of these assumptions. In particular, we provide a counterexample with immediate and complete agreement where A.SCDMC and A.LEXT do not hold, and a firm receives a price different than the Rubinstein price; however, in this counterexample, all prices still converge to the Nash-in-Nash prices as $\Lambda \rightarrow 0$.¹⁹ We also provide a counterexample where A.SCDMC and A.LNEXT

¹⁹Whether A.GFT and A.WCDMC are sufficient to ensure that all equilibria in which all agreements form

do not hold and there exists an equilibrium without any agreement ever forming.

Additional Remarks. A particular setting where both A.SCDMC and A.LEXT are satisfied is when there are $N \geq 1$ firms on one side of the market each with profits (net of transfers) that are constant (e.g., zero), and only one firm on the other side of the market with profits (net of transfers) satisfying A.CDMC. Indeed this is the setting considered by Westermark (2003).

One example of A.LEXT holding is bargaining between a monopolist cable distributor and many content providers using a model such as in Crawford and Yurukoglu (2012) with lump-sum transfers instead of linear fees: since the content providers typically have zero marginal costs of providing their channels to cable operators, their profits (net of transfers) will typically not depend on the agreements of other channels.

Another example is a special case of negotiations between many hospitals and one managed care organization (MCO), similar to the model used in Capps, Dranove, and Satterthwaite (2003) (which is embedded in the models used by Gowrisankaran, Nevo, and Town (2015) and Ho and Lee (2015)). Suppose that the hospital's cost function has constant marginal costs c , and can be given by $C(q) = F + cq$ (where F is a fixed cost). Moreover, suppose that the MCO reimburses hospitals for the marginal cost of treating each patient, in addition to offering them lump-sum payments for joining their network. In this case, the hospital's profits will not depend on the agreements signed by other hospitals (thus satisfying A.LEXT), and the MCO's profits will generally satisfy A.CDMC (see footnote 16).

4.2 Sketch of Uniqueness Proof

We now sketch our proof of Theorem 4.6, with the full formal proof given in Appendix D. Our proof is based on induction on the set of open agreements at any node of our game. Specifically, our inductive hypothesis is that if there are \mathcal{C} open agreements at any period t and history of play, then any subgame following this node that begins with a subset of open agreements $\mathcal{B} \subset \mathcal{C}$ results in all agreements in \mathcal{B} immediately forming at Rubinstein prices. The heart of our uniqueness result is proving our inductive step, formally provided in Proposition D.7, which states that if the induction hypothesis holds for any subgame with \mathcal{C} open agreements, then all agreements in \mathcal{C} also form immediately at Rubinstein prices.

We separate the possible nodes of our game with N upstream firms and M downstream firms into three distinct cases, and prove that our inductive step holds in each case. Figure 2 provides a graphical representation of the different types of nodes and the basis for our

immediately have prices that converge to the Nash-in-Nash prices is an open question.

argument in each case. It also details where our assumptions are leveraged in our proof.

We now go through these three cases in turn.

4.2.1 Base Case: One Open Agreement (Proposition D.6)

If there is only one open agreement between some U_i and D_j ($ij \in \mathcal{G}$) at period t and A.GFT holds, Rubinstein (1982) proves that the unique equilibrium of this subgame involves immediate agreement at Rubinstein prices ($p_{ij,D}^R$ if t is odd and $p_{ij,U}^R$ otherwise).

4.2.2 Multiple Upstream and Downstream Firms with Open Agreements (Lemmas D.11 – D.13)

Although given our inductive structure the case with one firm on one side of the market and multiple firms on the other side is formally proven first, the logic for the argument when there are multiple upstream firms and multiple downstream firms with open agreements is easier to understand. Thus, for exposition, we present this case first in the main text.

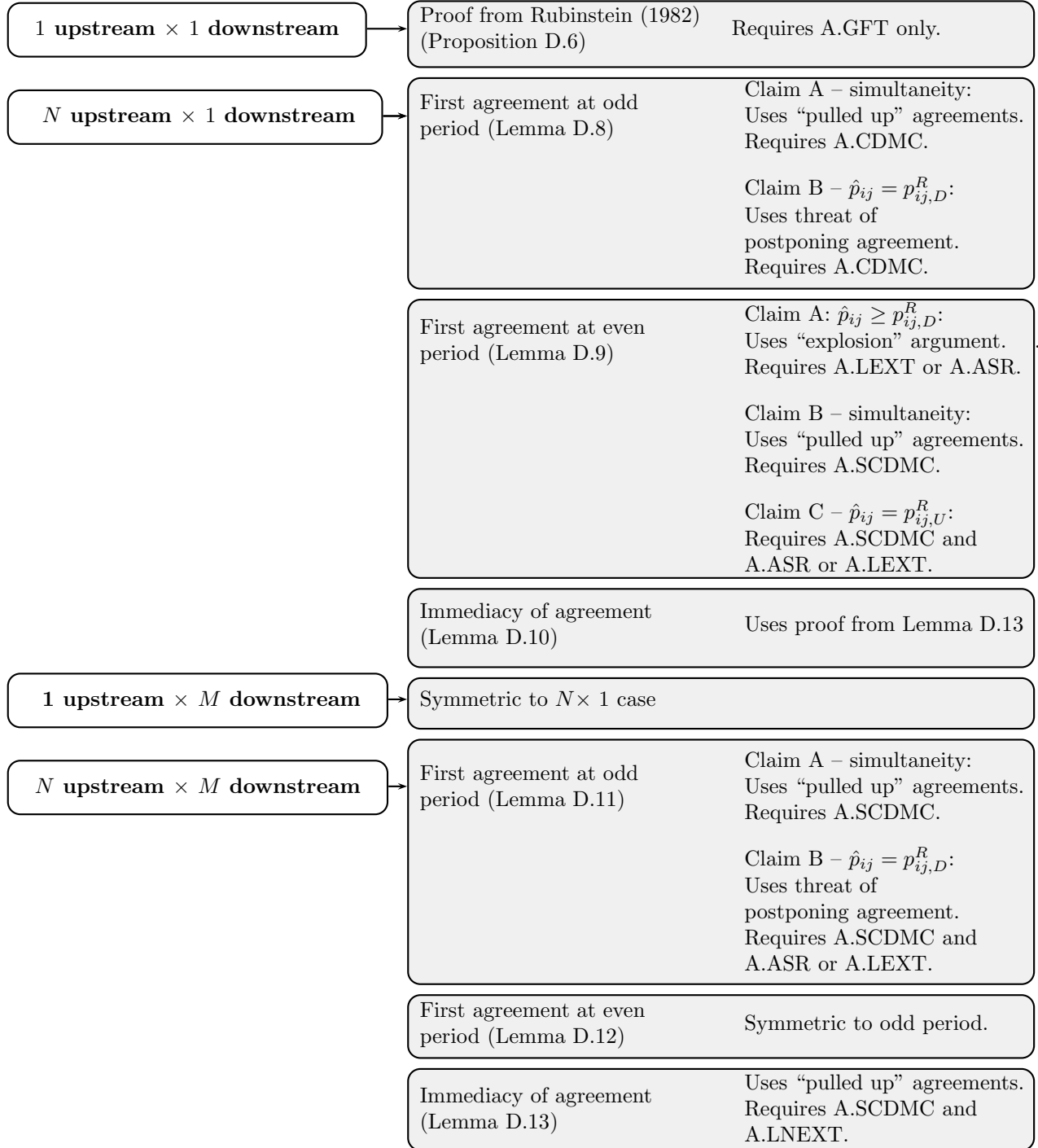
Assume now that there are \mathcal{C} open agreements involving multiple upstream and multiple downstream firms at period t_0 . Assume also that our inductive hypothesis holds, so that any subsequent subgame beginning with fewer open agreements than \mathcal{C} results in all open agreements immediately forming at Rubinstein prices. Given these assumptions, we prove the following three claims: (a) if any agreements form at period t , then all agreements form at period t ; (b) all agreements $ij \in \mathcal{C}$ form at Rubinstein prices; and (c) all agreements form immediately at period t_0 .

(a) All agreements form simultaneously (if any form). We prove this statement by contradiction. Consider a candidate equilibrium in which the first agreement in \mathcal{C} is formed (assuming that one is ever formed) at period t . Assume WLOG that t is an odd (downstream-proposing) period.²⁰ Suppose by contradiction that not all agreements in \mathcal{C} form at period t . By the inductive hypothesis, all agreements ij that are not formed at period t form at period $t + 1$ at prices $p_{ij,U}^R$. Fix some ij and ab such that the ij agreement is formed at period $t + 1$; the ab agreement is formed at period t ; and $i \neq a$ (where j may or may not be the same as b). Such a pair of agreements must exist by the contradictory assumption and the fact that there are multiple upstream firms without an agreement prior to period t .

We now show that D_j has an incentive to pull up its agreement with U_i to period t . First, we show that if D_j makes a deviant offer $\tilde{p}_{ij,D}$ at period t that is higher than $p_{ij,D}^R$

²⁰A symmetric argument, which we omit, would apply if t were an even (upstream-proposing) period.

Figure 2: Diagram of Proof of Theorem 4.6



Note: Downstream firms propose in odd periods and upstream firms propose in even periods. All parts use A.GFT.

but lower than $p_{ij,D}^R$, then U_i will accept the offer. By the nature of Rubinstein prices, if all agreements but ij form, then U_i is indifferent between receiving $p_{ij,D}^R$ at period t and $p_{ij,U}^R$ at period $t + 1$. Although at period t , there are no assurances that all other agreements form, A.CDMC states that the surplus to U_i from accepting D_j 's offer is no worse with some inframarginal set of acceptances than when all other agreements form. Furthermore, by passive beliefs, U_i believes that the ab agreement will still be formed at period t even if it receives off-equilibrium offer $\tilde{p}_{ij,D}$, which then leads it to believe (by the inductive hypothesis) that if it were to reject this offer, it will form the ij agreement at period $t + 1$ for a price $p_{ij,U}^R$. Combining these arguments, it follows that U_i strictly prefers accepting $\tilde{p}_{ij,D}$ at period t to waiting until $t + 1$, and will accept the deviant offer from D_j .

Next, we show that D_j prefers to have $\tilde{p}_{ij,D}$ accepted by U_i at period t instead of having this agreement formed at $p_{ij,U}^R$ at period $t + 1$. Here, by the nature of the Rubinstein prices, since D_j is the proposing firm at period t , it is strictly better off by paying $p_{ij,D}^R$ at period t than by paying $p_{ij,U}^R$ at period $t + 1$ if all other agreements form at period t . But A.CDMC is not sufficient to ensure that D_j is better off when all other agreements are not formed: it is possible that, when U_i receives its offer for $\tilde{p}_{ij,D}$, it changes its other acceptances, in turn harming D_j 's payoffs. To rule out this possibility, we employ the stronger assumption A.SCDMC (either assumed directly or implied by A.CDMC and A.LEXT), which implies that the inframarginal surplus to D_j from the ij agreement is no worse than when all other agreements form, even when U_i is allowed to vary its agreements that do not involve D_j .

Thus, D_j has an incentive to deviate from our candidate equilibrium by offering $\tilde{p}_{ij,D}$ to U_i at period t , which will be accepted. This leads to a contradiction, implying that if there is an equilibrium in which an agreement is formed in any period, all agreements must be formed in that period as well.

(b) All agreements form at Rubinstein prices. Assume again WLOG that the first period in which an agreement in \mathcal{C} is formed in a candidate equilibrium is an odd period. Given 4.2.2(a), all agreements in \mathcal{C} must form at period t in any such equilibrium. Again, we proceed by contradiction.

Suppose first that there is an agreement $ij \in \mathcal{C}$ formed at a price less than $p_{ij,D}^R$ at period t . By passive beliefs, U_i believes that upon rejecting this agreement and accepting all others, ij will be the only open agreement at $t + 1$. By the nature of Rubinstein prices, U_i would prefer to reject this offer as, by the inductive hypothesis, it will receive $p_{ij,U}^R$ in the next period, which makes it strictly better off. This yields a contradiction.

Suppose next that there is an agreement ij formed at a price greater than $p_{ij,D}^R$ at period t . In this case, we show that D_j has a profitable deviation by offering a slightly lower offer

(that is still greater than $p_{ij,D}^R$) to U_i instead. By the nature of the Rubinstein offers, such a deviant offer will be accepted by U_i . However, it is possible that, following this deviation by D_j , U_i rejects an offer from some downstream firm(s) D_k , $k \neq j$, since U_i is indifferent between accepting all offers at prices $\{p_{ik,D}^R\}_{ik \in \mathcal{C}}$ at period t and rejecting any one. To rule out the possibility that such a rejection by U_i harms D_j 's profits, we use either A.LEXT (which implies that D_j 's profits do not depend on U_i 's other agreements) or A.ASR (which implies that U_i will not reject an offer with another firm when indifferent). This deviation is thus profitable for D_j (as it results in D_j paying U_i a strictly lower price), leading to a contradiction.

Thus, if any agreements form in equilibrium, they must be at Rubinstein prices.

(c) All agreements form immediately. By 4.2.2(a) and 4.2.2(b), if any agreements form in a candidate equilibrium, they must all form in the same period at Rubinstein prices. We now show that if agreements do not happen immediately (or at all), the side that is proposing has an incentive to “pull up” agreements to the current period.

Assume again WLOG that the first period t in the subgame with \mathcal{C} open agreements is odd. By contradiction, suppose that a candidate equilibrium has no accepted offers at period t . A.LNEXT (which is assumed directly or implied by A.LEXT) implies that there is a pair of firms D_j and U_i , $ij \in \mathcal{C}$, in which both benefit from having all agreements in \mathcal{C} form at Rubinstein prices. Now consider a strategy by D_j to deviate by offering U_i a price $\tilde{p}_{ij,D}$ that is higher than $p_{ij,D}^R$ but lower than $p_{ij,U}^R$.

We first show that U_i would accept this deviant offer. If U_i rejects all offers including the deviant offer, there are three potential possibilities for the resulting subgame: agreements in \mathcal{C} never form, or all agreements in \mathcal{C} form in either a future even period or odd period. In each of these cases, we show that U_i 's payoffs from each of these outcomes is strictly worse than accepting the deviant offer (and possibly other offers in \mathcal{C}_i^U). The reason for this is first, that A.CDMC implies that receiving the deviant offer at period t is beneficial relative to a Rubinstein offer at period $t+1$ (similar to the argument in 4.2.2(a)); second, that A.LNEXT ensures that delay is costly; and third, that the results of 4.2.2(b) show that firms cannot wait and receive a better price in some future subgame. Finally, if U_i rejects the deviant offer while accepting some other offers, then A.CDMC is sufficient to show that it would be better off from adding the deviant offer to its set of acceptances.

Next, we show that D_j has an incentive to make this deviant offer. If D_j makes the deviant offer, we have shown that U_i will accept it (and perhaps other offers in \mathcal{C}_i^U); all remaining agreements in \mathcal{C} that are not formed at period t will then form at period $t+1$ by the inductive hypothesis. If D_j does not make the deviant offer, then there are three potential

outcomes of the resulting subgame: no agreements in \mathcal{C} are ever formed, or all agreements are formed in either a future even or odd period. Here, A.SCDMC and A.LNEXT ensure that D_j 's payoffs are strictly worse from any of these outcomes than making the deviant offer and having it accepted by U_i at period t ; A.SCDMC (instead of A.CDMC) is necessary here because U_i may change its other acceptances upon receiving D_j 's deviant offer.

Thus, given that there is a profitable deviation by D_j , any equilibrium must have all open agreements formed immediately.

4.2.3 Multiple Open Agreements Involving a Single Upstream or Downstream Firm (Lemmas D.8 – D.10)

Finally, we turn to the case where \mathcal{C} contains multiple open agreements, but on one side of the market, there is only one firm with open agreements. Again, assume that the inductive hypothesis holds, and WLOG, consider the case where there is only a single downstream firm D_j with open agreements. In this case, we show in order that (a) all agreements form simultaneously and at (odd) Rubinstein prices when the first agreement is formed in an odd period; (b) all agreements form simultaneously and at (even) Rubinstein prices when the first agreement is formed in an even period; and (c) all agreements form immediately.

(a) All agreements form simultaneously at Rubinstein prices (if any form in an odd period). We first show that if the first agreement in \mathcal{C} forms in an odd (downstream-proposing) period, then all agreements in \mathcal{C} form in this period at Rubinstein prices. The proof follows the same arguments as in 4.2.2(a) and 4.2.2(b), which is valid as those arguments only conditioned on the presence of multiple upstream firms (as opposed to downstream firms) with open agreements.

(b) All agreements form simultaneously at Rubinstein prices (if any form in an even period). We next show that if the first agreement in \mathcal{C} forms in an even (upstream-proposing) period, then (i) all agreements $ij \in \mathcal{C}$ form at prices $\tilde{p}_{ij} \geq p_{ij,D}^R$; (ii) all agreements in \mathcal{C} form in the same period; and (iii) all agreements form at Rubinstein prices. This case is the most challenging, because the fact that D_j is the only downstream firm with open agreements implies that, in cases where there is a deviation from a candidate equilibrium at period t , we cannot rely on the inductive hypothesis to argue that all agreements will still form at period $t + 1$: i.e., D_j can unilaterally choose to reject all offers at period t (even) so that there will still be the same set \mathcal{C} of open agreements at period $t + 1$. This is why there is an extra step, 4.2.3(b)(i), relative to the multiple upstream and multiple downstream case. We now sketch our proof for each of these arguments.

(i) *All agreements form at prices greater than or equal to Rubinstein prices.* As in earlier cases, we show that it is not credible for D_j to reject any offer from some U_i at a price lower than $p_{ij,D}^R$. However, the reason here is different: we cannot apply induction since D_j can reject all offers, keeping the set of open agreements the same. Instead, we make an argument that a credible rejection at period t implies that D_j would be ready to reject an even lower offer in a future period, and use this to generate a contradiction.

Suppose by contradiction that in a candidate equilibrium, some agreement $ij \in \mathcal{C}$ forms at a price $\hat{p}_{ij} < p_{ij,D}^R$ in an even period t . Consider now a deviation by U_i where it offers \tilde{p}_{ij} instead at t , where \tilde{p}_{ij} is higher than \hat{p}_{ij} but lower than $p_{ij,D}^R$. We first ask the question of whether U_i would prefer to engage in this deviation if this deviant offer were accepted by D_j . Clearly, if the offer were accepted and nothing else changed, U_i would prefer it as it obtains a higher price. However, it is possible that, in the case where D_j receives the \tilde{p}_{ij} offer, D_j changes the other offer(s) that it chooses to accept, and this change subsequently lowers U_i 's payoffs. To avoid this possibility, we employ either A.LEXT or A.ASR. A.LEXT limits externalities and hence implies that U_i 's profits do not depend on D_j 's agreements with other upstream firms. The argument with A.ASR is a little more subtle. If D_j were to accept the offer from U_i at either price \hat{p}_{ij} or \tilde{p}_{ij} , then we have one fewer open agreement, which means that we can apply the inductive hypothesis from period $t + 1$ on, which means that the future prices are the same across these two cases. A.ASR then forces D_j to accept the same set of other offers in both cases, given that it accepts the offer from U_i .

Since U_i prefers to offer \tilde{p}_{ij} instead of \hat{p}_{ij} if this offer were to be accepted, it must be the case that D_j does not accept it, or we would have a profitable deviation from this candidate equilibrium. For D_j to credibly reject such an offer in equilibrium, D_j must anticipate receiving a higher payoff in some subgame following the rejection. Because of discounting, this implies that D_j anticipates paying even lower prices relative to $p_{ij,D}^R$ to some set of firms in some future subgame. By the odd period result in step 4.2.3(a), this cannot occur in odd periods. Hence, this higher payoff must occur in a future even period. But then, the same argument noted in the previous paragraph applies: i.e., there must be a future even period in which D_j obtains agreement at even lower prices. Continuing to apply this argument implies that eventually there must be some future even period in which D_j forms agreements with upstream firms at prices that some U_k , $ik \in \mathcal{C}$, would not find profitable to ever accept. Thus, we have a contradiction.²¹

(ii) *All agreements form at the same period.* The idea of the proof here is quite similar to the proof for 4.2.2(a), but we use the lower bound for prices from 4.2.3(b)(i) instead of

²¹This argument can be seen as a generalization of the subgame perfection argument in Rubinstein (1982) to a setting with multiple agents.

induction in the case of rejection. Specifically, by contradiction, consider the case where the first agreement $ij \in \mathcal{C}$ forms at period t but not all agreements form at period t , with the remaining agreements $kj \in \mathcal{C}$ being formed at period $t + 1$ for prices $p_{kj,D}^R$ (using the inductive hypothesis). Consider one such U_k . It can offer a price $p_{kj,U}^R$ at period t to D_j . Unlike in 4.2.2(a), we cannot apply induction to understand what would occur if D_j rejects this offer as D_j may reject all offers in this case. However, if D_j rejects U_k 's offer, we know that it cannot pay prices any lower than $p_{kj,D}^R$ in any future subgame (from the results in steps 4.2.3(a) and 4.2.3(b)(i)). The fact that 4.2.3(b)(i) only proves that prices are weakly higher to $p_{kj,D}^R$, rather than exactly equal, suffices: A.CDMC again implies that accepting this offer is profitable for D_j . Finally, as in 4.2.2(a), A.SCDMC implies that this acceptance increases U_k 's surplus, and we have a profitable deviation; this yields a contradiction.

(iii) *All agreements form at prices $p_{ij,U}^R$.* The proof follows the same steps as 4.2.2(b). If an agreement ij forms at a price higher than $p_{ij,U}^R$, then D_j will reject the agreement and offer $p_{ij,D}^R$ at period $t + 1$. If an agreement ij forms at lower than the candidate equilibrium price, then A.ASR or A.LEXT implies that U_i can raise price slightly, that this offer will be accepted by D_j , and that U_i will benefit from this acceptance.

(c) All agreements form immediately. Finally, we show that all agreements form immediately. The proof follows the same steps as in 4.2.2(c), as nothing in that proof leveraged the fact that there were multiple downstream firms with open agreements. Here, A.SCDMC implies A.LNEXT and so we do not need to separately assume A.LNEXT.

We have thus proven that there is immediate agreement at Rubinstein prices in our base case (4.2.1), and that our inductive step holds when there are multiple open agreements on both sides (4.2.2) or just one side (4.2.3) of the market. Consequently, by induction, Theorem 4.6 follows.

5 Conclusion

The existence of an equilibrium generating the Nash-in-Nash outcome (Theorem 3.2) relies on two assumptions (gains from trade and weak conditional declining marginal contributions) that we believe are natural for many bilateral bargaining environments. The assumption on the declining marginal contributions from agreements rules out certain complementarities of profits across firms. As discussed in Section 3, if there are strong complementarities across contracting partners, bilateral negotiations over marginal contributions can generate prices that exceed the total contribution of a set of agreements, and induce a firm to unilaterally

drop several of its agreements. In such settings, other surplus division protocols, such as multilateral bargaining or cooperative solution concepts such as the Shapley value, may be more appropriate.²²

To prove the uniqueness of our equilibrium outcome, we leverage stronger assumptions that guarantee that equilibrium prices coincide with “Rubinstein” prices between each bilateral pair. As discussed in Section 4 and in Appendix E, although we have found counterexamples where equilibria outcomes are not unique when our stronger assumptions are violated, all counterexamples that we have found with full and immediate agreement still result in prices that converge to Nash-in-Nash prices as $\Lambda \rightarrow 0$. Whether this convergence property is true more generally under weaker assumptions is an open research question.

For tractability, we have considered only lump sum transfers between agents. In many settings, however, negotiations may occur over linear prices that may affect total surplus: e.g., negotiated prices between upstream and downstream firms may represent wholesale linear fee contracts, and downstream firms engage in price competition with one another for consumers given these contracts.²³ In such settings, profits may depend on not only the set of agreements formed by all agents but also upon the negotiated prices. Understanding the properties of bargaining in these types of environments is another open research question.

In closing, we believe that our results provide justification for the use of the Nash-in-Nash solution as a credible bargaining framework for use in applied work. Instead of requiring that firms cannot coordinate across multiple negotiations, our extensive form allows for firms to engage in deviations across multiple negotiations. We further believe that the mechanisms that we highlight in our extensive form reasonably capture aspects of bargaining protocols that occur in real-world industry settings.

References

BERRY, S., J. LEVINSOHN, AND A. PAKES (1995): “Automobile Prices in Market Equilibrium,” 63(4), 841–890.

BINMORE, K., A. RUBINSTEIN, AND A. WOLINSKY (1986): “The Nash Bargaining Solution in Economic Modelling,” *The RAND Journal of Economics*, 17(2), 176–188.

²²This point relates to a controversy in the discussion of mergers in markets with negotiated prices, such as cable TV, over whether larger firms negotiate better prices. As in Chitty and Snyder (1999), one way to frame this issue is to consider whether the surplus function is convex or concave in the number of agreements. If the surplus function is convex in the number of agreements (so that the marginal contribution of an agreement is increasing as more agreements are signed), we have shown that Nash-in-Nash prices do not arise as an equilibrium outcome, suggesting that a different surplus division mechanism should be considered.

²³See also Iozzi and Valletti (2014) who explore the importance of specifying whether or not disagreement is observable to rivals before a price competition subgame between downstream firms.

- CAPPS, C., D. DRANOVE, AND M. SATTERTHWAITE (2003): “Competition and market power in option demand markets,” *RAND Journal of Economics*, 34(4), 737–763.
- CHAE, S., AND J. YANG (1994): “An N Person Pure Bargaining Game,” *Journal of Economic Theory*, 62(1), 86–102.
- CHATTERJEE, K., B. DUTTA, D. RAY, AND K. SENGUPTA (1993): “A noncooperative theory of coalitional bargaining,” *The Review of Economic Studies*, 60(2), 463–477.
- CHIPTY, T., AND C. M. SNYDER (1999): “The role of firm size in bilateral bargaining: A study of the cable television industry,” *Review of Economics and Statistics*, 81(2), 326–340.
- COROMINAS-BOSCH, M. (2004): “Bargaining in a network of buyers and sellers,” *Journal of Economic Theory*, 115, 35–77.
- CRAWFORD, G., AND A. YURUKOGLU (2012): “The welfare effects of bundling in multi-channel television markets,” *American Economic Review*, 102(2), 643–85.
- CRAWFORD, G. S., R. S. LEE, M. D. WHINSTON, AND A. YURUKOGLU (2015): “The Welfare Effects of Vertical Integration in Multichannel Television Markets,” Unpublished.
- DE FONTENAY, C., AND J. GANS (2013): “Bilateral bargaining with externalities,” *Journal of Industrial Economics*.
- ELLIOTT, M. (2015): “Inefficiencies in networked markets,” *American Economics Journal: Microeconomics*, forthcoming.
- FARRELL, J., D. J. BALAN, K. BRAND, AND B. W. WENDLING (2011): “Economics at the FTC: Hospital Mergers, Authorized Generic Drugs, and Consumer Credit Markets,” *Review of Industrial Organization*, 39, 271–296.
- GOWRISANKARAN, G., A. NEVO, AND R. TOWN (2015): “Mergers When Prices Are Negotiated: Evidence from the Hospital Industry,” *American Economic Review*, 105(1), 172–203.
- GRENNAN, M. (2013): “Price discrimination and bargaining: Empirical evidence from medical devices,” *American Economic Review*, 103(33), 145–177.
- HART, O., AND J. TIROLE (1990): “Vertical Integration and Market Foreclosure,” *Brookings Papers on Economic Activity, Microeconomics*, 1990, 205–286.
- HO, K., AND R. S. LEE (2015): “Insurer Competition in Health Care Markets,” Unpublished.

- HORN, H., AND A. WOLINSKY (1988): “Bilateral monopolies and incentives for merger,” *The RAND Journal of Economics*, 19(3), 408–419.
- INDERST, R., AND J. MONTEZ (2014): “Buyer Power and Dependency in a Model of Negotiations,” Working paper, London Business School.
- IOZZI, A., AND T. VALLETTI (2014): “Vertical Bargaining and Countervailing Power,” *American Economic Journal: Microeconomics*, 6(3), 106–135.
- KRANTON, R. E., AND D. F. MINEHART (2001): “A Theory of Buyer-Seller Networks,” *American Economic Review*, 91(3), 485–508.
- KRISHNA, V., AND R. SERRANO (1996): “Multilateral bargaining,” *The Review of Economic Studies*, 63(1), 61–80.
- LEE, R. S., AND K. FONG (2013): “Markov-Perfect Network Formation An Applied Framework for Bilateral Oligopoly and Bargaining in Buyer-Seller Networks,” Discussion paper, Harvard University.
- MANEA, M. (2011): “Bargaining in Stationary Networks,” *American Economic Review*, 101, 2042–2080.
- MCAFEE, R., AND M. SCHWARTZ (1994): “Opportunism in multilateral vertical contracting: Nondiscrimination, exclusivity, and uniformity,” *American Economic Review*, 84(1), 210–230.
- MERLO, A., AND C. WILSON (1995): “A stochastic model of sequential bargaining with complete information,” *Econometrica*, 63(2), 371–399.
- MUTHOO, A. (1999): *Bargaining Theory with Applications*. Cambridge, Cambridge.
- OSBORNE, M., AND A. RUBINSTEIN (1994): *A course in game theory*. MIT press.
- POLANSKI, A. (2007): “Bilateral bargaining in networks,” *Journal of Economic Theory*, 134(1), 557–565.
- ROGERSON, W. P. (2013): “Vertical Mergers in the Video Programming and Distribution Industry: The Case of Comcast NBCU (2011),” in *The Antitrust Revolution, 6th edition*, ed. by J. E. Kowka, and L. J. White. Oxford University Press, Oxford.
- RUBINSTEIN, A. (1982): “Perfect equilibrium in a bargaining model,” *Econometrica*, 50(1), 97–109.

SEGAL, I. (1999): “Contracting with Externalities,” *Quarterly Journal of Economics*, 64(2), 337–388.

STOLE, L. A., AND J. ZWEIBEL (1996): “Intra-Firm Bargaining under Non-Binding Contracts,” 63(3), 375–410.

WESTERMARK, A. (2003): “Bargaining, Binding Contracts, and Competitive Wages,” *Games and Economic Behavior*, 43(2), 296–311.

WHINSTON, M. (2006): *Lectures on antitrust economics*. MIT Press.

A An Alternative Non-Cooperative Foundation for the Nash-in-Nash Bargaining Solution

In this section, we present an alternative extensive form which involves separate bilateral negotiations between representatives for each firm, and show that this representation also admits the Nash-in-Nash bargaining solution as an equilibrium outcome. For this equilibrium, only A.GFT is required.

Consider the setting introduced in Section 2, where N upstream firms negotiate with M downstream firms. For every pair of firms U_i and D_j , $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$, U_i and D_j send individual representatives who engage in the alternating-offers bargaining protocol of Rubinstein (1982). Although each representative for each firm seeks to maximize her firm’s total expected profits across all bargains, she does not know the outcome of any other bilateral bargain until her own bargain has concluded. One interpretation is that each pair of representatives from different firms are sequestered in separate bargaining rooms, and no one outside the room knows the status of the bargain until it is finished.

In this environment, there exists an equilibrium among *representatives for each firm* which yields the Nash-in-Nash bargaining outcomes:

Theorem A.1 *Assume A.GFT and every firm send representatives to all potential negotiating partners. Then there exists an equilibrium with:*

- (a) *immediate agreement between all representatives for each firm;*
- (b) *equilibrium prices $\hat{p}_{ij} = p_{ij,D}^R \forall i, j$ if the game begins in an odd period, and $\hat{p}_{ij} = p_{ij,U}^R \forall i, j$ if the game begins in an even period; and*
- (c) *$\hat{p}_{ij} \rightarrow p_{ij}^R \forall i, j$ as $\Lambda \rightarrow 0$ regardless of whether the game starts in an odd or even period, where $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$ and $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$.*

To prove the theorem, assume each pair of representatives $U_{i,j}$ and $D_{j,i}$ who negotiate between U_i and D_j employ the following candidate set of strategies: $U_{i,j}$ offers $p_{ij,U}^R$ in even periods and only accepts offers equal to or above $p_{ij,D}^R$ in odd periods; $D_{j,i}$ offers $p_{ij,D}^R$ in odd periods, and accepts offers equal to or above $p_{ij,U}^R$ in even periods. Any off-equilibrium action made by other representatives (including other representatives of their own firms or actions observed by other representative of their own firms) cannot affect or influence the outcomes of other negotiations; as such, given that agreement is expected to occur in all other negotiations, the unique equilibrium for $U_{i,j}$ and $D_{j,i}$ is the set of candidate strategies described (Rubinstein, 1982). As no agent has a profitable deviation, the set of strategies comprise an equilibrium. Part (c) of the theorem follows directly from Lemma 2.1. This proves the theorem.

B Proof of Lemma 2.1

Proof of Lemma 2.1 Using l’Hospital’s rule:

$$\lim_{\Lambda \rightarrow 0} \frac{\delta_{i,U}(1 - \delta_{j,D})}{1 - \delta_{i,U}\delta_{j,D}} = \lim_{\Lambda \rightarrow 0} \frac{e^{-r_{i,U}\Lambda}(1 - e^{-r_{j,D}\Lambda})}{1 - e^{-(r_{i,U} + r_{j,D})\Lambda}} = \frac{r_{j,D}}{r_{i,U} + r_{j,D}},$$

and

$$\lim_{\Lambda \rightarrow 0} \frac{1 - \delta_{j,D}}{1 - \delta_{i,U} \delta_{j,D}} = \lim_{\Lambda \rightarrow 0} \frac{1 - e^{-r_{j,D} \Lambda}}{1 - e^{-(r_{i,U} + r_{j,D}) \Lambda}} = \frac{r_{j,D}}{r_{i,U} + r_{j,D}}.$$

Similarly, it can be shown that:

$$\lim_{\Lambda \rightarrow 0} \frac{\delta_{j,D}(1 - \delta_{i,U})}{1 - \delta_{i,U} \delta_{j,D}} = \lim_{\Lambda \rightarrow 0} \frac{(1 - \delta_{i,U})}{1 - \delta_{i,U} \delta_{j,D}} = \frac{r_{i,U}}{r_{i,U} + r_{j,D}}.$$

which proves the lemma.

C Proof of Theorem 3.2 (Existence)

Claim A: Assume A.GFT and A.WCDMC hold. Then there exists an equilibrium of the model with strategy profiles that follow the statement of the theorem. We first detail the strategies under our candidate equilibrium:

- In every odd period node, each D_j makes offers $p_{ij,D}^R$ to all firms U_i with which it has not already reached agreement. If all price offers that it receives are equal to $p_{ij,D}^R$, U_i accepts all offers. If U_i receives exactly one non-equilibrium offer from some D_j , it accepts all other offers and rejects D_j 's offer if and only if the offer is lower than $p_{ij,D}^R$. Finally, if U_i receives multiple non-equilibrium offers, it plays an arbitrary best response in its acceptance decision, respecting passive beliefs (i.e., assuming that all other offers not involving U_i form at Rubinstein prices).
- In every even period node, each U_i makes offers $p_{ij,U}^R$ to all firms D_j with which it has not already reached agreement. If all price offers that it receives are equal to $p_{ij,U}^R$, D_i accepts all offers. If D_j receives exactly one non-equilibrium offer from some U_i , it accepts all other offers and rejects U_i 's offer if and only if the offer is higher than $p_{ij,U}^R$. Finally, if D_j receives multiple non-equilibrium offers, it plays an arbitrary best response in its acceptance decision, respecting passive beliefs.

The prescribed strategy profile dictates that every firm makes proposals that are the Rubinstein offers: i.e., downstream firms offer $p_{ij,D}^R$ in odd periods, and upstream firms offer $p_{ij,U}^R$ in even periods. On the equilibrium path, all offers are accepted. For off-equilibrium offers, acceptance will depend on the exact set of offers received.

We now prove that no unilateral deviation is profitable.

We start with the decision for an upstream firm U_i of which offers to accept in an odd period t . The information set for U_i at period t contains two elements: (1) the set of agreements \mathcal{C} which are still open at the start of period t , and (2) the price \tilde{p}_{ij} for every open agreement in $ij \in \mathcal{C}_i^U$ that is observed by U_i . Since we examine only one-shot deviations, U_i expects that any non-accepted offers ik will follow the prescribed equilibrium strategies from $t + 1$ onwards and hence will form in period $t + 1$ at prices $p_{ik,D}^R$. We start by defining the gain in one-shot surplus from rejecting a subset $\mathcal{K} \subset \mathcal{C}_i^U$ of agreements relative to accepting all of its open agreements \mathcal{C}_i^U as:

$$F(\mathcal{K}) \equiv - \left[(1 - \delta_{i,U}) \Delta \pi_i^U(\mathcal{G}, \mathcal{K}) + \sum_{ij \in \mathcal{K}} [\tilde{p}_{ij} - \delta_{i,U} p_{ij,U}^R] \right], \quad (5)$$

where we omit the fact that F is implicitly a function of the firm U_i and U_i 's information set at period t . By passive beliefs, U_i assumes that all offers that do not include it (those in \mathcal{C}_{-i}^U) will be accepted at period t . Hence, it chooses the set of agreements to reject that satisfies $\hat{\mathcal{K}} = \arg \max_{\mathcal{K} \subset \mathcal{C}_i^U} F(\mathcal{K})$, as this set of rejections maximizes its profits net of the payoff difference.

Consider first the case where U_i receives candidate equilibrium offers $p_{ij,U}^R$. Substituting these prices

into (5) and then employing (4) (which uses A.WCDMC), we obtain:

$$F(\mathcal{K}) = - \left[(1 - \delta_{i,U}) \Delta \pi_i^U(\mathcal{G}, \mathcal{K}) + \sum_{ij \in \mathcal{K}} [p_{ij,D}^R - \delta_{i,U} p_{ij,U}^R] \right] \leq 0.$$

Since $F(\emptyset) = 0$, $F(\mathcal{K})$ is maximized for $\mathcal{K} = \emptyset$. This implies that at equilibrium prices, U_i maximizes surplus by rejecting no offer, or equivalently, accepting all offers. Thus, in this case, U_i cannot gain by deviating from its candidate equilibrium strategy.

Consider next the case where U_i receives exactly one non-equilibrium offer, $\tilde{p}_{ij} \neq p_{ij,R}^D$. In this case, from (4),

$$F(\mathcal{K}) = - \left[(1 - \delta_{i,U}) \Delta \pi_i^U(\mathcal{G}, \mathcal{K}) + \sum_{ij \in \mathcal{K}} [p_{ij,D}^R - \delta_{i,U} p_{ij,U}^R] \right] + p_{ij,D}^R - \tilde{p}_{ij} \leq p_{ij,D}^R - \tilde{p}_{ij}.$$

Note further that, from (2), $F(ij) = p_{ij,D}^R - \tilde{p}_{ij}$. Hence, with one deviant offer, if $\tilde{p}_{ij} < p_{ij,D}^R$, U_i does not have a profitable deviation from the prescribed strategy of rejecting only the deviant offer, while if $\tilde{p}_{ij} > p_{ij,D}^R$, U_i does not have a profitable deviation from the prescribed strategy of accepting all offers.

Finally, in cases where U_i receives multiple out-of-equilibrium offers, the strategy profile specified above states that U_i picks an arbitrary $\hat{\mathcal{K}}$ that maximizes $F(\mathcal{K})$. Note that such a node of the game is not reachable on the equilibrium path by a unilateral deviation by any single downstream firm, and prescribed play here does not affect the incentives for any firm to unilaterally deviate.

Thus, no unilateral deviation by any upstream firm in an odd period is profitable.

Next, we consider the decision for a downstream firm D_j of what offers to propose in an odd period t when there are \mathcal{C} open agreements. Consider the possibility that D_j deviates from the candidate equilibrium strategies and offers prices different from $\tilde{p}_{ij} \neq p_{ij,D}^R$ for all ij in some $\mathcal{K} \subseteq \mathcal{C}_j^D$. By passive beliefs, each firm U_i receiving \tilde{p}_{ij} perceives that it is the only one to have received an out-of-equilibrium offer. Given the candidate equilibrium strategies, if $\tilde{p}_{ij} > p_{ij,D}^R$, then it will be accepted, while if $\tilde{p}_{ij} < p_{ij,D}^R$, then it will be rejected, with no impact on the acceptance of offers from other downstream firms. Clearly, D_j will never choose to offer $\tilde{p}_{ij} > p_{ij,D}^R$, since it can always offer $p_{ij,D}^R$ instead, without affecting the set of acceptances. The only possible profitable deviation left is for D_j to offer $\tilde{p}_{ij} < p_{ij,D}^R$ for all $ij \in \mathcal{K}$. Again, given the candidate equilibrium strategies, these offers will be rejected at period t , and then accepted at $t+1$ at prices $p_{ij,U}^R$.

Thus, D_j can effectively choose which agreements \mathcal{K} to postpone from forming in period t at prices $p_{ij,D}^R$ to forming in period $t+1$ at prices $p_{ij,U}^R$. The decrease in surplus doing so for some set of offers $\mathcal{K} \subseteq \mathcal{C}_j^D$ is:

$$\begin{aligned} & (1 - \delta_{j,D}) \Delta \pi_j^D(\mathcal{G}, \mathcal{K}) - \sum_{ij \in \mathcal{K}} [p_{ij,D}^R - \delta_{j,D} p_{ij,U}^R] \\ & > (1 - \delta_{j,D}) \Delta \pi_j^D(\mathcal{G}, \mathcal{K}) - \sum_{ij \in \mathcal{K}} [p_{ij,U}^R - \delta_{j,D} p_{ij,D}^R] \\ & \geq \sum_{ij \in \mathcal{K}} [(1 - \delta_{j,D}) \Delta \pi_j^D(\mathcal{G}, \{ij\}) - p_{ij,U}^R + \delta_{j,D} p_{ij,D}^R] = 0, \end{aligned} \tag{6}$$

where the first inequality follows from Lemma D.1 (which uses A.GFT to show that $p_{ij,U}^R > p_{ij,D}^R$), the second inequality follows from the analog of (4) for the downstream side (which uses A.WCDMC), and the equality follows from (1) (which uses A.GFT). Thus, D_j is strictly worse off from any deviation from our candidate equilibrium strategies in an odd period.

We omit the claim for even periods, as it is symmetric. Since there are no profitable one-shot deviations for any agent in both odd and even periods, the candidate set of strategies is indeed an equilibrium.

Claim B: If A.WCDMC does not hold, then there is no equilibrium of the model with prices $p_{ij}^* = p_{ij,D}^R$ at all odd period nodes and $p_{ij}^* = p_{ij,U}^R$ at all even period nodes.

Assume that A.WCDMC does not hold. In this case, there exists a firm (WLOG, an upstream firm U_i) and a set of agreements $\mathcal{K} \subseteq \mathcal{G}_i^U$ such that $\Delta\pi_{ij}^U(\mathcal{G}, \mathcal{K}) < \sum_{ij \in \mathcal{K}} \Delta\pi_i^U(\mathcal{G}, \{ij\})$. Consider again the gain in one-shot surplus from U_i rejecting all $ij \in \mathcal{K}$, denoted $F(\mathcal{K})$, evaluated at period $t_0 = 1$. From the candidate equilibrium, we know that U_i has obtained offers $p_{ij,D}^R$ for all $ij \in \mathcal{G}_i^U$. $F(\mathcal{K})$ is given by:

$$\begin{aligned} F(\mathcal{K}) &\equiv - \left[(1 - \delta_{i,U}) \Delta\pi_i^U(\mathcal{G}, \mathcal{K}) + \sum_{ij \in \mathcal{K}} [p_{ij,D}^R - \delta_{i,U} p_{ij,U}^R] \right] \\ &> - \left[\sum_{ij \in \mathcal{K}} (1 - \delta_{i,U}) \Delta\pi_i^U(\mathcal{G}, \{ij\}) + \sum_{ij \in \mathcal{K}} [p_{ij,D}^R - \delta_{i,U} p_{ij,U}^R] \right] = 0, \end{aligned}$$

where the inequality follows from the assumption that A.WCDMC does not hold and the equality from (2). Hence, it is a profitable deviation for U_i to reject the offers in \mathcal{K} at period 1 when A.WCDMC does not hold.

Finally, part (c) of the theorem follows directly from Lemma 2.1. □

D Proofs of Theorem 4.6 (Uniqueness)

D.1 Supporting Lemmas

Lemma D.1

$$\begin{aligned} \Delta\pi_j^D(\mathcal{G}, \{ij\}) &> p_{ij,U}^R > p_{ij,D}^R, \\ \Delta\pi_i^U(\mathcal{G}, \{ij\}) &> -p_{ij,D}^R > -p_{ij,U}^R. \end{aligned}$$

Proof Since $(\Delta\pi_j^D(\mathcal{G}, \{ij\}) - p_{ij,D}^R) = \frac{(1 - \delta_{i,U})}{(1 - \delta_{i,U}\delta_{j,D})} (\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \Delta\pi_i^U(\mathcal{G}, \{ij\}))$, $(\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \Delta\pi_i^U(\mathcal{G}, \{ij\})) > 0$ by A.GFT, and $\frac{(1 - \delta_{i,U})}{(1 - \delta_{i,U}\delta_{j,D})} > 0$, it follows that $\Delta\pi_j^D(\mathcal{G}, \{ij\}) > p_{ij,D}^R$.

Also, since $(\Delta\pi_i^U(\mathcal{G}, \{ij\}) + p_{ij,U}^R) = \frac{(1 - \delta_{j,D})}{(1 - \delta_{i,U}\delta_{j,D})} (\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \Delta\pi_i^U(\mathcal{G}, \{ij\}))$, $(\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \Delta\pi_i^U(\mathcal{G}, \{ij\})) > 0$ by A.GFT, and $\frac{(1 - \delta_{j,D})}{(1 - \delta_{i,U}\delta_{j,D})} > 0$, it follows that $p_{ij,U}^R > \Delta\pi_i^U(\mathcal{G}, \{ij\})$.

As well, if we combine the terms of the previous two equations, we obtain:

$$\begin{aligned} p_{ij,U}^R - p_{ij,D}^R &= (\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \Delta\pi_i^U(\mathcal{G}, \{ij\})) \left[\frac{(1 - \delta_{i,U})}{(1 - \delta_{i,U}\delta_{j,D})} + \frac{(1 - \delta_{j,D})}{(1 - \delta_{i,U}\delta_{j,D})} - 1 \right] \\ &= \frac{1}{1 - \delta_{i,U}\delta_{j,D}} (\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \Delta\pi_i^U(\mathcal{G}, \{ij\})) (1 - \delta_{i,U} - \delta_{j,D} + \delta_{j,D}\delta_{i,U}). \end{aligned}$$

The fact that $0 < \delta_{i,U} < 1$ and $0 < \delta_{j,D} < 1$ implies that $\frac{1}{1 - \delta_{i,U}\delta_{j,D}} > 0$ and $1 - \delta_{i,U} - \delta_{j,D} + \delta_{j,D}\delta_{i,U} > 0$. Since $\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \Delta\pi_i^U(\mathcal{G}, \{ij\}) > 0$ by A.GFT, it follows that $p_{ij,U}^R > p_{ij,D}^R$.

Lemma D.2 *A.CDMC implies A.WCDMC.*

Proof For a downstream firm, A.WCDMC states that $\Delta\pi_j^D(\mathcal{G}, \mathcal{A}) \geq \sum_{kj \in \mathcal{A}} \Delta\pi_j^D(\mathcal{G}, \{kj\})$ for all $\mathcal{A} \subseteq \mathcal{G}_j^D$. First, index agreements in \mathcal{A} from $k = 1, \dots, |\mathcal{A}|$, and let a_k represent the k th agreement in \mathcal{A} . This allows us to create a sequence of sets of agreements, starting at $\mathcal{B} \equiv \mathcal{G} \setminus \mathcal{A}$, in which we add in each agreement one at a time, given by $\mathcal{D}_0 \equiv \mathcal{B}$, and $\mathcal{D}_k = \mathcal{D}_{k-1} \cup \{a_k\}$ for $k = 1, \dots, |\mathcal{A}|$. We can then decompose:

$$\Delta\pi_j^D(\mathcal{G}, \mathcal{A}) = \Delta\pi_j^D(\mathcal{A} \cup \mathcal{B}, \mathcal{A}) = \sum_{k=1}^{|\mathcal{A}|} \Delta\pi_j^D(\mathcal{D}_k, \{a_k\}).$$

By A.CDMC, $\Delta\pi_j^D(\mathcal{D}_k, \{a_k\}) \geq \Delta\pi_j^D(\mathcal{G}, \{a_k\})$ for all a_k since $\mathcal{D}_k \subseteq \mathcal{G}$. This implies that $\Delta\pi_j^D(\mathcal{G}, \mathcal{A}) = \Delta\pi_j^D(\mathcal{A} \cup \mathcal{B}, \mathcal{A}) \geq \sum_{ij \in \mathcal{A}} \Delta\pi_j^D(\mathcal{G}, \{ij\})$. A symmetric argument can be used for upstream firms.

Lemma D.3 *A.SCDMC implies A.CDMC.*

Proof For a downstream firm, A.SCDMC states that:

$$\pi_j^D(\mathcal{A} \cup \mathcal{B} \cup \{ij\}) - \pi_j^D(\mathcal{A}' \cup \mathcal{B}) \geq \Delta\pi_j^D(\mathcal{G}, \{ij\}) \quad \forall ij \in \mathcal{G}; \mathcal{B} \subseteq \mathcal{G}_{-i}^U; \mathcal{A}, \mathcal{A}' \subseteq \mathcal{G}_i^U \setminus \{ij\}.$$

Consider the case where $\mathcal{A} = \mathcal{A}'$. Then A.SCDMC states that:

$$\begin{aligned} \pi_j^D(\mathcal{A} \cup \mathcal{B} \cup \{ij\}) - \pi_j^D(\mathcal{A} \cup \mathcal{B}) &\geq \Delta\pi_j^D(\mathcal{G}, \{ij\}) \\ \Rightarrow \Delta\pi_j^D(\mathcal{A} \cup \mathcal{B}, \{ij\}) &\geq \Delta\pi_j^D(\mathcal{G}, \{ij\}), \end{aligned}$$

which is the statement of A.CDMC. A symmetric argument can be used for upstream firms.

Lemma D.4 *A.CDMC and A.LEXT imply A.SCDMC and A.LNEXT.*

Proof A.CDMC states that $\Delta\pi_j^D(\mathcal{A} \cup \mathcal{B}, \{ij\}) \geq \Delta\pi_j^D(\mathcal{G}, \{ij\})$ for all $\mathcal{A} \subseteq \mathcal{G}_j^D, \mathcal{B} \subseteq \mathcal{G}_{-j}^D$. A.LEXT, by assuming that profits π_j^D are invariant to the set of agreements that firms other than D_j form, implies that $\pi_j^D(\mathcal{A} \cup \mathcal{B}) = \pi_j^D(\mathcal{A} \cup \mathcal{B}')$ for all $\mathcal{A} \subseteq \mathcal{G}_j^D$ and $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{G}_{-j}^D$. A.SCDMC directly follows. A symmetric argument can be used for upstream firms.

A.LEXT implies that $\Delta\pi_j^D(\mathcal{G}, \mathcal{C}) = \Delta\pi_j^D(\mathcal{G}, \mathcal{C}_j^D)$ and $\Delta\pi_i^U(\mathcal{G}, \mathcal{C}) = \Delta\pi_i^U(\mathcal{G}, \mathcal{C}_i^U)$ for any $\mathcal{C} \subseteq \mathcal{G}$ and any D_j, U_i . The statement of A.LNEXT then follows directly from A.CDMC (which, by Lemma D.2, implies A.WCDMC).

Lemma D.5 *Assume A.GFT and A.LNEXT. Then $\forall \mathcal{C} \subseteq \mathcal{G}, \exists ij \in \mathcal{C}$ such that:*

$$\begin{aligned} \Delta\pi_j^D(\mathcal{G}, \mathcal{C}) &> \sum_{hj \in \mathcal{C}_j^D} p_{hj,U}^R, \quad \text{and} \\ \Delta\pi_i^U(\mathcal{G}, \mathcal{C}) &> - \sum_{ik \in \mathcal{C}_i^U} p_{ik,D}^R. \end{aligned}$$

Proof By A.LNEXT, $\forall \mathcal{C} \subseteq \mathcal{G}, \exists ij \in \mathcal{C}$ such that:

$$\begin{aligned} \Delta\pi_j^D(\mathcal{G}, \mathcal{C}) &\geq \sum_{hj \in \mathcal{C}_j^D} \Delta\pi_j^D(\mathcal{G}, \{hj\}), \quad \text{and} \\ \Delta\pi_i^U(\mathcal{G}, \mathcal{C}) &\geq \sum_{ik \in \mathcal{C}_i^U} \Delta\pi_i^k(\mathcal{G}, \{ik\}). \end{aligned}$$

By A.GFT, $\Delta\pi_j^D(\mathcal{G}, \{hj\}) > p_{hj,U}^R$ and $\Delta\pi_i^U(\mathcal{G}, \{ik\}) > -p_{ik,D}^R$ for all agreements $hj, ik \in \mathcal{G}$ (see (3)). The lemma immediately follows.

D.2 Inductive Structure and Base Case

In the following proofs, we will use both A.CDMC and A.SCDMC (with the understanding that A.SCDMC implies A.CDMC; see Lemma D.3) to emphasize the circumstances in which the stronger assumption (A.SCDMC) is required. We will also use A.SCDMC when only A.CDMC and A.LEXT are assumed to hold (since A.CDMC and A.LEXT imply A.SCDMC; see Lemma D.4) as it serves to emphasize how these assumptions are leveraged.

We first provide the structure of the general proof where there are $N \geq 1$ upstream firms and $M \geq 1$ downstream firms.

For any $\mathcal{C} \subseteq \mathcal{G}$, let $\Gamma_{\mathcal{C}}^t(h^t)$ represent the subgame beginning at period $t \geq t_0$ when there are still \mathcal{C} “open” agreements, or agreements that have not been reached (i.e., all agreements $ij \in \mathcal{G} \setminus \mathcal{C}$ have been formed prior to period t), and history of play h^t . Recall the history at period t contains the sequence of actions, which include offers and acceptances/rejections, that have been made by all firms in all preceding periods. We will prove Theorem 4.6 by induction on \mathcal{C} for any arbitrary t and history of play h^t .

The base case is provided by analyzing $\Gamma_{\mathcal{C}}^t(\cdot)$ when $|\mathcal{C}| = 1$: i.e., there is only one agreement in \mathcal{G} that has not yet been reached at period t .

Proposition D.6 (Base Case) *Let $|\mathcal{C}| = 1$ with only one open agreement: $\mathcal{C} \equiv \{ij\}$. Then the subgame $\Gamma_{\mathcal{C}}^t(h^t)$ for any $t \geq t_0$ and any history of play h^t (consistent with \mathcal{C} being the set of open agreements at t) has a unique equilibrium involving immediate agreement at t with prices $\hat{p}_{ij} = p_{ij,D}^R$ if t is odd, and $\hat{p}_{ij} = p_{ij,U}^R$ if t is even.*

Proof With only one open agreement $ij \in \mathcal{C}$, U_i and D_j engage in a 2-player Rubinstein alternating offers bargaining game over joint surplus $\Delta\pi_i^U(\mathcal{G}, \{ij\}) + \Delta\pi_j^D(\mathcal{G}, \{ij\})$, and the result directly follows from Rubinstein (1982). \square

We now state the inductive hypothesis and inductive step used to prove Theorem 4.6.

Inductive Hypothesis. *Fix $\mathcal{C} \subseteq \mathcal{G}$, t , and h^t . For any $\mathcal{B} \subset \mathcal{C}$ such that $|\mathcal{B}| < |\mathcal{C}|$, any equilibrium of $\Gamma_{\mathcal{B}}^{t'}(h^{t'})$, where $t' > t$ and $h^{t'}$ contains h^t (and is consistent with \mathcal{B} being the set of open agreements at t'), results in immediate agreement between U_i and $D_j \forall ij \in \mathcal{B}$ at prices $\hat{p}_{ij} = p_{ij,D}^R$ if t' is odd, and $\hat{p}_{ij} = p_{ij,U}^R$ if t' is even.*

The inductive hypothesis states that any subgame involving fewer open agreements than $|\mathcal{C}|$ results in immediate agreement at the Rubinstein prices. It implies that if any non-empty set of agreements are reached at any point during the subgame $\Gamma_{\mathcal{C}}^t(h^t)$ at period $t' \geq t$ so that only a strict subset $\mathcal{B} \subset \mathcal{C}$ of open agreements remain, then all remaining agreements $ij \in \mathcal{B}$ are reached in the subsequent period $t' + 1$ at $p_{ij,D}^R$ ($p_{ij,U}^R$) if $t' + 1$ is odd (even).

Proposition D.7 (Inductive Step) *Assume A.GFT and either (i) A.CDMC and A.LEXT; or (ii) A.SCDMC, A.LNEXT and A.ASR. Consider any subgame $\Gamma_{\mathcal{C}}^t(h^t)$ where $\mathcal{C} \subseteq \mathcal{G}$, $t \geq t_0$. Given the inductive hypothesis, every equilibrium of $\Gamma_{\mathcal{C}}^t(h^t)$ results in immediate agreement between U_i and $D_j \forall ij \in \mathcal{C}$ at prices $\hat{p}_{ij} = p_{ij,D}^R$ if t is odd, and $\hat{p}_{ij} = p_{ij,U}^R$ if t is even.*

The inductive step states that if the inductive hypothesis holds for any subgame with \mathcal{C} open agreements, then this subgame also results in immediate agreement for all open agreements $ij \in \mathcal{C}$ at the Rubinstein prices. Note that Propositions D.6 (Base Case) and D.7 (Inductive Step) imply Theorem 4.6 by induction: as we have established that the theorem holds when $|\mathcal{C}| = 1$, the inductive step implies that the theorem will hold for any $\mathcal{C} \subseteq \mathcal{G}$ when $|\mathcal{C}| \geq 1$.

To prove Proposition D.7 (and by consequence, Theorem 4.6), we proceed in three steps: we first focus on subgames $\Gamma_{\mathcal{C}}^t(\cdot)$ where \mathcal{C} contains only agreements involving one downstream firm; we then focus on subgames where \mathcal{C} contains only agreements involving one upstream firm; and finally, we focus on subgames where \mathcal{C} contains more than one upstream and more than one downstream firm. For expositional convenience, we will drop the history of play argument from $\Gamma_{\mathcal{C}}^t$ for the remainder of the text, acknowledging that these subgames will be for any arbitrary history of play consistent with there being \mathcal{C} open agreements at t (though we will still allow for history-dependent strategies to be played).

D.3 One Downstream Firm, Many Upstream Firms

We prove Proposition D.7 in this case using three lemmas. For these lemmas, consider a candidate equilibrium of a subgame beginning at period $\tilde{t} \geq t_0$ with the first agreement $ij \in \mathcal{C}$ reached in period $t \geq \tilde{t}$, and accepted prices denoted $\{\hat{p}_{1j}, \dots, \hat{p}_{mj}\}$. Let $\mathcal{A} \subseteq \mathcal{C}$ denote the set of agreements reached at period t . By the inductive hypothesis, all agreements $ij \in \mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ not reached at period t will be reached in period $t + 1$ at prices $p_{ij,D}^R$ ($p_{ij,U}^R$) if $t + 1$ is odd (even).

Lemma D.8 (Odd, simultaneous.) *In any equilibrium of $\Gamma_{\mathcal{C}}^{\bar{t}}$ with the first agreement occurring in an odd period (i.e., the downstream firms propose), all agreements must occur at the same period with $\hat{p}_{ij} = p_{ij,D}^R \forall ij \in \mathcal{C}$.*

Proof. Suppose the first agreement occurs in some odd period $t \geq \bar{t}$. We prove all agreements occur simultaneously by contradiction.

Assume $\mathcal{B} \neq \emptyset$, implying that not all agreements in \mathcal{C} are reached in period t . By the inductive hypothesis, all U_i such that $ij \in \mathcal{B}$ will reach agreement with D_j at $t+1$ at prices $p_{ij,U}^R$.

Consider the following deviation by D_j in period t : D_j offers $\tilde{p}_{ij} \equiv p_{ij,D}^R + \varepsilon$ to some U_i , $ij \in \mathcal{B}$. U_i will accept this deviation at period t if it obtains higher profits, or:

$$\begin{aligned} (1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{B}) \cup \{ij\}) + \delta_{i,U}\pi_i^U(\mathcal{G}) + \tilde{p}_{ij} &> (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{B}) + \delta_{i,U}\pi_i^U(\mathcal{G}) + \delta_{i,U}p_{ij,U}^R \\ \Leftrightarrow \tilde{p}_{ij} &> \delta_{i,U}p_{ij,U}^R - (1 - \delta_{i,U})\Delta\pi_i^U((\mathcal{G} \setminus \mathcal{B}) \cup \{ij\}, \{ij\}) \end{aligned}$$

which holds since $\tilde{p}_{ij} \equiv p_{ij,D}^R + \varepsilon = \delta_{i,U}p_{ij,U}^R - (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) + \varepsilon$ (see (2)) and $\Delta\pi_i^U((\mathcal{G} \setminus \mathcal{B}) \cup \{ij\}, \{ij\}) \geq \Delta\pi_i^U(\mathcal{G}, \{ij\})$ by A.CDMC, for sufficiently small ε .

This deviation will be profitable for D_j if D_j 's profit gains from reaching agreement with U_i one period earlier is greater than D_j 's difference in payments:

$$\begin{aligned} (1 - \delta_{j,D})\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{B}) \cup \{ij\}, \{ij\}) &> \tilde{p}_{ij} - \delta_{j,D}p_{ij,U}^R = p_{ij,D}^R - \delta_{j,D}p_{ij,U}^R + \varepsilon \\ \Leftrightarrow (1 - \delta_{j,D})\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{B}) \cup \{ij\}, \{ij\}) &> [(\delta_{i,U} - \delta_{j,D})p_{ij,U}^R - (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\})] + \varepsilon \\ \Leftrightarrow (1 - \delta_{j,D})\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{B}) \cup \{ij\}, \{ij\}) + (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) &> (\delta_{i,U} - \delta_{j,D})p_{ij,U}^R + \varepsilon \end{aligned}$$

(where the second line follows from (2)). Since $\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{B}) \cup \{ij\}, \{ij\}) \geq \Delta\pi_j^D(\mathcal{G}, \{ij\}) > p_{ij,U}^R$ and, $\Delta\pi_i^U(\mathcal{G}, \{ij\}) > -p_{ij,U}^R$ by equation (3), this inequality holds for sufficiently small ε and the deviation is profitable for D_j ; a contradiction. Thus, if the first agreement occurs in odd period t , all agreements must occur at period t .

Now suppose all agreements occur at period t (odd), but $\hat{p}_{ij} \neq p_{ij,D}^R$ for some ij . We will show this leads to a contradiction:

1. If $\hat{p}_{ij} < p_{ij,D}^R$ for some ij , U_i can reject this offer and, as all other upstream firms will agree in equilibrium at period t , obtain a price of $p_{ij,U}^R$ at $t+1$ by the inductive hypothesis. This is a profitable deviation if U_i 's gains in prices exceed its profit gains from coming to agreement one period early:

$$\delta_{i,U}p_{ij,U}^R - \hat{p}_{ij} > (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\})$$

Since the RHS is equal to $\delta_{i,U}p_{ij,U}^R - p_{ij,D}^R$ by (2), this inequality holds leading to a contradiction.

2. If $\hat{p}_{ij} > p_{ij,D}^R$ for some ij , D_j can profitably reduce its offer to $p_{ij,D}^R + \varepsilon$ for $\varepsilon \in (0, \hat{p}_{ij} - p_{ij,D}^R)$; U_i will still accept if:

$$p_{ij,D}^R + \varepsilon - \delta_{i,U}p_{ij,U}^R > -(1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\})$$

Since the RHS is equal to $p_{ij,D}^R - \delta_{i,U}p_{ij,U}^R$ by (2), this inequality holds and leads to a contradiction.

Thus, $\hat{p}_{ij} = p_{ij,D}^R \forall i$ if the first agreement occurs in an odd period. \square

Lemma D.9 (Even, simultaneous.) *In any equilibrium of $\Gamma_{\mathcal{C}}^{\bar{t}}$ with the first agreement occurring in an even (upstream proposing) period, all agreements must occur at the same period with $\hat{p}_{ij} = p_{ij,U}^R \forall ij \in \mathcal{C}$.*

Proof. This Lemma will be proven with 3 claims. First, we prove that $\hat{p}_{ij} \geq p_{ij,D}^R$ for all $ij \in \mathcal{C}$. Second, we prove all agreements $ij \in \mathcal{C}$ occur at period t . Finally, we prove all agreements occur at prices $\hat{p}_{ij} = p_{ij,U}^R$.

Claim A: Equilibrium prices $\hat{p}_{ij} \geq p_{ij,D}^R$ for all $ij \in \mathcal{C}$.

Given a candidate equilibrium, for any subgame Γ^t of $\Gamma_{\mathcal{C}}^{\bar{t}}$ beginning at $t \geq \bar{t}$, let ϕ represent the *total discount* from prices $p_{kj,D}^R$ that D_j can obtain in this equilibrium: i.e., $\phi_{\Gamma} \equiv \sum_{kj \in \mathcal{C}} (p_{kj,D}^R - \hat{p}_{kj})$. Let $\bar{\phi}$ be

the maximum total discount that D_j could achieve under any equilibrium in any subgame of $\Gamma_{\mathcal{C}}^t$; $\bar{\phi}$ is finite, as no upstream firm would offer more than its own total achievable profits in any equilibrium strategy.

Assume that $\bar{\phi} > 0$ and consider the equilibrium and subgame in which this maximum discount is reached.²⁴ We will show that the assumption that $\bar{\phi} > 0$ leads to a contradiction, which implies that in any equilibrium in which the first agreement is reached in an even period, prices cannot be lower than $p_{ij,D}^R$ for any agreement $ij \in \mathcal{C}$.

Without loss of generality, let this subgame be denoted Γ^t ($t \geq \tilde{t}$), and assume that the period in which the first agreement occurs in this subgame is t . By Lemma D.8, t cannot be odd since this would imply that $\phi = 0$ as all agreements would occur at Rubinstein prices. Thus, t is even.

Let $\mathcal{A} \subseteq \mathcal{C}$ denote the set of agreements reached in period t at prices \hat{p}_{ij} . By the inductive hypothesis, all other agreements $kj \in \mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ occur at period $t + 1$ at prices $p_{kj,D}^R$. Thus, by our definition of ϕ , $\sum_{ij \in \mathcal{A}} \hat{p}_{ij} = (\sum_{ij \in \mathcal{A}} p_{ij,D}^R) - \bar{\phi}$. For these prices $\{\hat{p}_{ij}\}_{ij \in \mathcal{A}}$ to have been equilibrium offers, it must be the case that D_j would have rejected any alternative offer $\tilde{p}_{ij} > \hat{p}_{ij}$ from any U_i , $ij \in \mathcal{A}$, at period t . If not, U_i would have a profitable deviation by offering $\tilde{p}_{ij} \equiv \hat{p}_{ij} + \varepsilon$.

- We first show that if D_j accepts \tilde{p}_{ij} as defined above, then U_i would wish to engage in this deviation, leading to a contradiction.

Under A.LEXT, this is straightforward to show: U_i obtains strictly higher payments under this deviation from D_j without changing the timing of its own agreements, and U_i 's profits do not depend on whether or not D_j makes changes to its other agreements.

Under A.ASR, note that it cannot be the case that some other set of agreements $\mathcal{A}' \neq \mathcal{A}$ would be reached at period t if D_j accepted \tilde{p}_{ij} . By the inductive hypothesis, any agreements $\mathcal{B}' \equiv \mathcal{C} \setminus \mathcal{A}'$ not reached at t would occur in period $t + 1$; as a result, if D_j would reach a different set of agreements $\mathcal{A}' \neq \mathcal{A}$ subsequent to accepting the higher deviant offer \tilde{p}_{ij} at t , then it would have obtained strictly higher payoffs by reaching agreements \mathcal{A}' as opposed to \mathcal{A} at period t in the original candidate equilibrium; this is a contradiction.²⁵ Consequently, if \tilde{p}_{ij} were accepted at period t , U_i would obtain the same flow profits as in the candidate equilibrium (since the same set of agreements would be reached at period t and $t + 1$), but it would obtain a strictly higher price as $\tilde{p}_{ij} > \hat{p}_{ij}$.

Thus, if D_j accepted \tilde{p}_{ij} , U_i would prefer to make such a deviant offer.

Thus, D_j needs to credibly reject \tilde{p}_{ij} if such an offer is made.

Consider now $ij \in \mathcal{A}$ such that $p_{ij,D}^R - \hat{p}_{ij} > 0$; such an ij exists since we have assumed $\bar{\phi} > 0$, and means we can construct $\tilde{p}_{ij} \equiv \hat{p}_{ij} + \varepsilon < p_{ij,D}^R$ for some $\varepsilon > 0$. Since D_j must reject \tilde{p}_{ij} at period t if it were offered by U_i , this implies D_j also must either subsequently (i) reach agreements $\mathcal{A}' \subseteq \mathcal{C} \setminus \{ij\}$ at period t , or (ii) reject *all* offers upon rejecting \tilde{p}_{ij} . We show now that either action by D_j leads to a contradiction.

1. Suppose D_j rejects \tilde{p}_{ij} , but reaches agreements $\mathcal{A}' \subseteq \mathcal{C} \setminus \{ij\}$. By the inductive hypothesis, D_j would reach all other agreements $kj \in \mathcal{B}' \equiv \mathcal{C} \setminus \mathcal{A}'$ at $t + 1$ at prices $p_{kj,D}^R$. However, D_j would rather accept \tilde{p}_{ij} and reach agreements $\mathcal{A}' \cup \{ij\}$ instead of rejecting U_i 's deviation if the gains to coming to an agreement earlier exceeded the additional payment required:

$$(1 - \delta_{j,D}) \Delta \pi_j^D((\mathcal{G} \setminus \mathcal{B}') \cup \{ij\}, \{ij\}) > \tilde{p}_{ij} - \delta_{j,D} p_{ij,D}^R$$

By assumption, the RHS is strictly less than $(1 - \delta_{j,D}) p_{ij,D}^R$; since $\Delta \pi_j^D((\mathcal{G} \setminus \mathcal{B}') \cup \{ij\}, \{ij\}) > \Delta \pi_j^D(\mathcal{G}, \{ij\})$ by A.CDMC and since $\Delta \pi_j^D(\mathcal{G}, \{ij\}) > p_{ij,D}^R$ by A.GFT, this inequality holds, leading to a contradiction.

2. Suppose D_j rejects *all* offers at period t upon receiving the deviant offer \tilde{p}_{ij} from U_i .

Let Γ_{RA}^{t+1} denote the subgame following D_j 's rejection of all offers at period t , and let Π^{RA} denote D_j 's payoffs in this subgame (discounted to period t). Note if D_j rejects all offers in \mathcal{C} at period t

²⁴Notice that this proof assumes that the maximum payoff $\bar{\phi}$ is achieved by some equilibrium (i.e., $\bar{\phi}$ is a maximum rather than a supremum). If this is not the case, we can consider any subgame in which the total discount from Rubinstein prices is greater than or equal to $\delta_{j,D} \bar{\phi}$ and substitute this value for $\bar{\phi}$ in the proof.

²⁵A.ASR rules out the possibility that D_j is indifferent between \mathcal{A} and \mathcal{A}' .

upon receiving out-of-equilibrium deviation \tilde{p}_{ij} , D_j must expect to obtain in the subsequent subgame at least as much as it would have obtained had it accepted \tilde{p}_{ij} and all other offers \hat{p}_{kj} , $kj \in \mathcal{A} \setminus \{ij\}$, at t , but rejected all other offers. This lower bound is:

$$\begin{aligned} \underline{\Pi}^D &\equiv (1 - \delta_{j,D})\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - \tilde{p}_{ij} - \sum_{kj \in \mathcal{A} \setminus \{ij\}} \hat{p}_{kj} + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C} \setminus \mathcal{A}} p_{kj,D}^R \right) \\ &= (1 - \delta_{j,D})\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - \sum_{kj \in \mathcal{A}} p_{kj,D}^R + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C} \setminus \mathcal{A}} p_{kj,D}^R \right) + \bar{\phi} - \varepsilon \end{aligned} \quad (7)$$

where $\underline{\Pi}^D$ represents D_j 's expected payoffs if D_j only accepted U_i 's deviant offer at period t , and accepted all other offers $kj \in \mathcal{C} \setminus \{ij\}$ at $t+1$ at prices $p_{kj,D}^R$ (by the inductive hypothesis). For D_j to prefer rejecting all offers at t , it must be the case that $\Pi^{RA} \geq \underline{\Pi}^D$.

In the subgame Γ_{RA}^{t+1} , the first accepted offer in \mathcal{C} can occur at an odd (downstream proposing) period, or an even (upstream proposing) period. We go through each case in turn.

- (a) D_j cannot reject all offers and earn a payoff greater than $\underline{\Pi}^D$ by having the first agreement $kj \in \mathcal{C}$ reached in any subsequent odd period $t + \tau$ ($\tau \geq 1$, odd), as Lemma D.8 implies all agreements in \mathcal{C} would also be realized in the same period at prices $p_{kj,D}^R$; this would yield (discounted to period t) payoffs to D_j of:

$$\begin{aligned} \Pi^{RA} &\equiv \sum_{p=0}^{\tau-1} \delta_{j,D}^p (1 - \delta_{j,D}) \pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R \right) \\ &\leq (1 - \delta_{j,D}) \pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R \right) \end{aligned} \quad (8)$$

where the last inequality is implied from A.GFT (i.e., $\Delta \pi_j^D(\mathcal{G}, \mathcal{C}) - \sum_{ij \in \mathcal{C}} p_{kj,D}^R > 0$). Using (7) and (8) implies $\underline{\Pi}^D - \Pi^{RA} > 0$ if:

$$(1 - \delta_{j,D}) \Delta \pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}, \mathcal{A}) + \bar{\phi} - \varepsilon > \sum_{kj \in \mathcal{A}} (1 - \delta_{j,D}) p_{kj,D}^R$$

Since $\Delta \pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}, \mathcal{A}) \geq \Delta \pi_j^D(\mathcal{G}, \mathcal{A}) \geq \sum_{kj \in \mathcal{A}} p_{kj,D}^R$ by A.GFT and A.CDMC, this last inequality holds, and D_j cannot earn higher profits by rejecting all offers at t and reaching agreement in some subsequent odd period.

- (b) Thus, in order for D_j to credibly reject \tilde{p}_{ij} and all other offers in \mathcal{C} at period t , D_j must expect the first agreement $kj \in \mathcal{C}$ to occur in some subsequent even period $t + \tau$ ($\tau \geq 2$, even) and obtain some payoff $\Pi^{RA} > \underline{\Pi}^D$. Since the set of open agreements at $t + \tau$ is the same as at t , the same logic of rejecting all offers still holds. Thus this strategy must be supported by ever increasing future payoffs and ever decreasing payments, which ultimately leads to a contradiction.

Suppose all agreements $kj \in \mathcal{A}' \subseteq \mathcal{C}$, $\mathcal{A}' \neq \emptyset$ are reached at even period $t + \tau$ at prices p'_{kj} , and (by the inductive hypothesis) the remaining agreements $lj \in \mathcal{B}' \equiv \mathcal{C} \setminus \mathcal{A}'$ are reached in the next period $t + \tau + 1$ at prices $p_{lj,D}^R$. Then D_j 's payoffs (discounted to period t) are

$$\begin{aligned} \Pi^{RA} &\equiv \sum_{p=0}^{\tau-1} \delta_{j,D}^p (1 - \delta_{j,D}) \pi_j^D(\mathcal{G} \setminus \mathcal{C}) \\ &\quad + \delta_{j,D}^\tau \left[(1 - \delta_{j,D}) \pi_j^D(\mathcal{G} \setminus \mathcal{B}') - \sum_{kj \in \mathcal{A}'} p'_{kj} + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{B}'} p_{kj,D}^R \right) \right] \end{aligned} \quad (9)$$

Combining (7) and (9) yields:²⁶

$$\begin{aligned}
(\underline{\Pi}^D - \Pi^{RA}) &= (1 - \delta_{j,D}) \left[\Delta\pi^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}, \mathcal{A}) - \sum_{kj \in \mathcal{A}} p_{kj,D}^R \right] \\
&+ (1 - \delta_{j,D}) \sum_{\rho=1}^{\tau-1} \delta_{j,D}^\rho \left[\Delta\pi_j^D(\mathcal{G}, \mathcal{C}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R \right] \\
&+ (1 - \delta_{j,D}) \delta_{j,D}^\tau \left[\Delta\pi_j^D(\mathcal{G}, \mathcal{B}') - \sum_{kj \in \mathcal{C}} p_{kj,D}^R + \sum_{kj \in \mathcal{A}'} p'_{kj} \right] \\
&+ \delta_{j,D}^{\tau+1} \left[\sum_{kj \in \mathcal{A}'} p'_{kj} - p_{kj,D}^R \right] + \bar{\phi} - \varepsilon
\end{aligned} \tag{10}$$

We show that this expression is positive, leading to a contradiction. We go through this expression line by line. The first term is:

$$T_1 \equiv (1 - \delta_{j,D}) \underbrace{\left[\Delta\pi^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}, \mathcal{A}) - \sum_{kj \in \mathcal{A}} p_{kj,D}^R \right]}_{\Xi_1}$$

Since $\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}, \mathcal{A}) \geq \sum_{kj \in \mathcal{A}} \Delta\pi_j^D(\mathcal{G}, \{kj\}) > \sum_{kj \in \mathcal{A}} p_{kj,D}^R$ by A.GFT and A.CDMC, Ξ_1 is strictly positive, and thus T_1 is as well.

The second term of equation (10) is:

$$T_2 \equiv (1 - \delta_{j,D}) \sum_{\rho=1}^{\tau-1} \delta_{j,D}^\rho \underbrace{\left[\Delta\pi_j^D(\mathcal{G}, \mathcal{C}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R \right]}_{\Xi_2}$$

which, again by A.GFT (see (3)) and A.CDMC, $\Xi_2 > 0$, and thus T_2 is strictly positive.

The third term of equation (10) is:

$$\begin{aligned}
T_3 &\equiv (1 - \delta_{j,D}) \delta_{j,D}^\tau \left[\Delta\pi_j^D(\mathcal{G}, \mathcal{B}') - \sum_{kj \in \mathcal{C}} p_{kj,D}^R + \sum_{kj \in \mathcal{A}'} p'_{kj} \right] \\
&= (1 - \delta_{j,D}) \delta_{j,D}^\tau \underbrace{\left[\Delta\pi_j^D(\mathcal{G}, \mathcal{B}') - \sum_{kj \in \mathcal{B}'} p_{kj,D}^R + \sum_{kj \in \mathcal{A}'} [p'_{kj} - p_{kj,D}^R] \right]}_{\Xi_3}
\end{aligned}$$

²⁶Here, and in other expressions, we will leverage the identity: $\delta = (1 - \delta) \sum_{\rho=1}^{\tau-1} \delta^\rho + \delta^\tau$.

and the fourth term of equation (10) is:

$$T_4 \equiv \delta_{j,D}^{\tau+1} \left[\sum_{kj \in \mathcal{A}'} p'_{kj} - p_{kj,D}^R \right] + \bar{\phi} - \varepsilon$$

Note that $\Xi_3 > 0$ by A.GFT and A.CDMC. Thus, it is straightforward to show that a sufficient condition for $T_3 + T_4 > 0$ is:

$$\varepsilon < \bar{\phi} - \delta_{j,D}^{\tau} \left[\sum_{kj \in \mathcal{A}'} [p_{kj,D}^R - p'_{kj}] \right]$$

Since $\bar{\phi}$ is the maximum discount from Rubinstein prices obtainable in *any* subgame and $\delta_{j,D} < 1$, the RHS will be strictly greater than 0: thus, there is some $\varepsilon > 0$ such that $T_1 + T_2 + T_3 + T_4 > 0$ and $\underline{\Pi}^D \leq \Pi^{RA}$, thus implying that D_j will not wish to reject all offers at period t upon receiving the deviant offer $\tilde{p}_{ij} = \hat{p}_{ij} + \varepsilon$ from U_i .

Consequently, D_j cannot credibly reject $\tilde{p}_{ij} \equiv p_{ij,D}^R + \varepsilon$ if U_i offered it at period t . Since offering \tilde{p}_{ij} at period t is a profitable deviation for U_i , the original assumption that $\bar{\phi} > 0$ and that there exists some $\hat{p}_{ij} < p_{ij,D}^R$ leads to a contradiction. Hence, $\hat{p}_{ij} \geq p_{ij,D}^R$ for all $ij \in \mathcal{C}$.

Claim B: All agreements will occur simultaneously.

Let $\mathcal{A} \subseteq \mathcal{C}$ denote the set of agreements reached at period t in a candidate equilibrium. By the inductive hypothesis, all agreements in $\mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ are reached at period $t + 1$ with prices $p_{ij,D}^R$.

We will show by contradiction that $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \emptyset$: i.e., if one agreement occurs at period t (even), all agreements $ij \in \mathcal{C}$ must occur at period t . Suppose not and $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \neq \emptyset$. By the inductive assumption, all U_i such that $ij \in \mathcal{B}$ will reach agreement at $t + 1$ at prices $p_{ij,D}^R$.

Consider the deviation where some U_i , $ij \in \mathcal{B}$, offers $\tilde{p}_{ij} = p_{ij,D}^R$ at period t .

1. *Such a deviation will be accepted by D_j .*

Suppose not, and D_j rejects \tilde{p}_{ij} .

- (a) Suppose that D_j rejects \tilde{p}_{ij} but accepts some non-empty set of offers $\mathcal{A}' \subseteq \mathcal{C} \setminus \{ij\}$ at period t . However, D_j would find it more profitable to accept \tilde{p}_{ij} while still accepting all agreements $kj \in \mathcal{A}'$ if D_j 's profit gains from reaching agreement with U_i one period earlier is greater than D_j 's difference in payments:

$$\begin{aligned} & (1 - \delta_{j,D}) \Delta \pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}' \cup \{ij\}, \{ij\}) > \tilde{p}_{ij} - \delta_{j,D} p_{ij,D}^R \\ \Leftrightarrow & (1 - \delta_{j,D}) \Delta \pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}' \cup \{ij\}, \{ij\}) > (1 - \delta_{j,D}) p_{ij,D}^R \end{aligned}$$

By A.GFT and A.CDMC, this inequality holds. Thus, D_j cannot reject \tilde{p}_{ij} while still accepting \mathcal{A}' at period t .

- (b) Thus, the only way D_j can reject \tilde{p}_{ij} is for D_j to reject all offers at period t .

If D_j were to reject all offers at period t , the inductive hypothesis does not apply in the subgame beginning at $t + 1$, and the first agreement can occur either in a subsequent odd or even period. We examine both cases.

First, note that D_j can always accept only \tilde{p}_{ij} at period t , and receive all other offers $kj \in \mathcal{C} \setminus \{ij\}$

at period $t + 1$ at prices $p_{ij,D}^R$ (by the inductive hypothesis), yielding profits:

$$\begin{aligned}\tilde{\Pi}_j^D &= (1 - \delta_{ij,D})\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \{ij\}) - \tilde{p}_{ij} + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C} \setminus \{ij\}} p_{kj,D}^R \right) \\ &= (1 - \delta_{j,D}) \left[(\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \{ij\}) - \tilde{p}_{ij}) + \left(\sum_{\rho=1}^{\tau-1} \delta_{j,D}^\rho \left[\pi_j^D(\mathcal{G}) - \tilde{p}_{ij} - \sum_{kj \in \mathcal{C} \setminus \{ij\}} p_{kj,D}^R \right] \right) \right] \\ &\quad + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \tilde{p}_{ij} - \sum_{kj \in \mathcal{C} \setminus \{ij\}} p_{kj,D}^R \right),\end{aligned}$$

for $\tau = 1, 2, 3, \dots$

- i. *First agreement occurs in an odd period.* If the first accepted offer in \mathcal{C} is at an odd period $t + \tau$ ($\tau \geq 1$), then Lemma D.8 implies all agreements in \mathcal{C} would also be realized in the same period at prices $p_{kj,D}^R$, yielding profits (discounted to period t):

$$\begin{aligned}\bar{\Pi}_j^{D,o} &= (1 - \delta_{j,D}^\tau)\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R \right) \tag{11} \\ &= (1 - \delta_{j,D}) \left[\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \left(\sum_{\rho=1}^{\tau-1} \delta_{j,D}^\rho \pi_j^D(\mathcal{G} \setminus \mathcal{C}) \right) \right] + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R \right)\end{aligned}$$

D_j would prefer to accept only \tilde{p}_{ij} at period t instead if $\tilde{\Pi}_j^D - \bar{\Pi}_j^{D,o} > 0$, or:

$$\left[\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \{ij\}, \{ij\}) - \tilde{p}_{ij} \right] + \left[\sum_{\rho=1}^{\tau-1} \delta_{j,D}^\rho \left(\Delta\pi_j^D(\mathcal{G}, \mathcal{C}) - \tilde{p}_{ij} - \sum_{kj \in \mathcal{C} \setminus \{ij\}} p_{kj,D}^R \right) \right] > 0$$

Each term of this expression is positive by A.CDMC and A.GFT since $\Delta\pi((\mathcal{G} \setminus \mathcal{C}) \cup \{ij\}, \{ij\}) \geq \Delta\pi(\mathcal{G}, \{ij\}) > p_{ij,D}^R$, and since $\Delta\pi(\mathcal{G}, \mathcal{C}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R > 0$.

Thus, D_j will not find it profitable to reject all offers and reach agreement in a subsequent odd period.

- ii. *First agreement occurs in an even period.* If the first accepted offer in \mathcal{C} is at an even period $t + \tau$ ($\tau \geq 2$), then the lowest possible payments it can make in equilibrium to all $ij \in \mathcal{C}$ are $\hat{p}_{ij} \geq p_{ij,D}^R$ (by Claim A). Let $\bar{\Pi}_j^{D,e}$ denote the most that D_j can achieve in any subgame in which the first agreement in \mathcal{C} is reached in some future even period $t + \tau$ (discounted to period t).²⁷ However, it is straightforward to show that the expression for $\bar{\Pi}_j^{D,e}$ is identical to that of $\bar{\Pi}_j^{D,o}$ in (11) (for τ even instead of odd); hence, using the same analysis as before, it must be that $\tilde{\Pi}_j^D - \bar{\Pi}_j^{D,e} > 0$. Thus, D_j will not find it profitable to reject all offers and reach agreement in a subsequent even period.

Thus D_j will accept the deviation \tilde{p}_{ij} from U_i .

2. *Such a deviation is profitable for U_i if D_j accepts.*

Assume if D_j accepts the deviation from U_i at period t , D_j also accepts agreements $\mathcal{A}' \subseteq \mathcal{C} \setminus \{ij\}$ at period t as well as ij ; by the inductive hypothesis, D_j then reaches agreements $\mathcal{B}' \equiv \mathcal{C} \setminus [\mathcal{A}' \cup \{ij\}]$ in the following period $t + 1$.

²⁷By the inductive hypothesis, if D_j reaches any agreements at period $t + \tau$, all other agreements would occur in the following period $t + \tau + 1$ at prices $p_{ij,D}^R$. However, D_j would strictly prefer reaching agreement at period $t + \tau$ instead of $t + \tau + 1$ at prices $p_{ij,D}^R$ as $\Delta\pi_j^D(\mathcal{G}, \{ij\}) \geq p_{ij,D}^R$ by A.GFT and A.CDMC.

U_i will find the deviation profitable in this case if and only if:

$$\begin{aligned} \tilde{p}_{ij} + (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{B}') &> \delta_{i,U}p_{ij,D}^R + (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{B}) \\ \Leftrightarrow \tilde{p}_{ij} - \delta_{i,U}p_{ij,D}^R &> (1 - \delta_{i,U})(\pi_i^U(\mathcal{G} \setminus \mathcal{B}) - \pi_i^U(\mathcal{G} \setminus \mathcal{B}')) \\ \Leftrightarrow (1 - \delta_{i,U})p_{ij,D}^R &> (1 - \delta_{i,U})(\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - \pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}' \cup \{ij\})) \end{aligned}$$

By A.SCDMC:

$$\pi_i^U((\mathcal{G} \setminus \mathcal{C}') \cup \mathcal{A}' \cup \{ij\}) - (\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A})) \geq \Delta\pi_i^U(\mathcal{G}, \{ij\})$$

and so the desired inequality will hold since $p_{ij}^R > -\Delta\pi_i^D(\mathcal{G}, \{ij\})$ by A.GFT. Hence, U_i will find it profitable to make the deviation.

Since D_j must accept the deviant offer in any equilibrium, and U_i finds it profitable to make the deviation if it is accepted, then the original candidate equilibrium is not an equilibrium; contradiction. Consequently, if the first agreement occurs in an even period, all agreements must occur simultaneously.

Claim C: If all agreements $ij \in \mathcal{C}$ occur simultaneously in an even period t , then $\hat{p}_{ij} = p_{ij,U}^R \forall ij \in \mathcal{C}$.

1. Assume that $\hat{p}_{ij} > p_{ij,U}^R$ for some ij . Consider the following deviation for D_j : D_j rejects ij and accepts all other offers at t ; D_j will then come to agreement with U_i in $t+1$ for payment $p_{ij,D}^R$ by the inductive hypothesis. This is profitable for D_j if $(1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \{ij\}) < \hat{p}_{ij} - \delta_{j,D}p_{ij,D}^R$. Since the LHS of this inequality is equal to $p_{ij,U}^R - \delta_{j,D}p_{ij,D}^R$ (see (1)), this inequality will hold if $\hat{p}_{ij} > p_{ij,U}^R$. This yields a contradiction.
2. Assume $\hat{p}_{ij} < p_{ij,U}^R$ for some ij . Consider a deviant offer by U_i , $\tilde{p}_{ij} = p_{ij,U}^R - \varepsilon > \hat{p}_{ij}$. We now show that this deviation is profitable to U_i , leading to a contradiction.

If U_i offers \tilde{p}_{ij} instead of \hat{p}_{ij} at period t , we show that D_j accepting all offers (including \tilde{p}_{ij}) at period t is more profitable than:

- (a) D_j rejecting all offers at period t . By Lemma D.8 and Claim B of this lemma, if D_j rejects all offers at t , then all agreements $ij \in \mathcal{C}$ will form in the same future period (if at all). Moreover, if they occur at a future odd period, they will be for the prices $\{p_{kj,D}^R\}_{kj \in \mathcal{C}}$ (Lemma D.8), and if they occur at an even period, they will be for prices of $\{p_{kj,D}^R\}_{kj \in \mathcal{C}}$ or higher (Claim A of this lemma). Thus, the most profitable case for D_j by rejecting all offers at t is to have agreement immediately in the following period $t+1$ and pay no more than $p_{kj,D}^R$ to all $kj \in \mathcal{C}$. In this case, the loss in profits from delay is:

$$(1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \mathcal{C}) \geq (1 - \delta_{j,D}) \sum_{kj \in \mathcal{C}} \Delta\pi_j^D(\mathcal{G}, \{kj\}) = \sum_{kj \in \mathcal{C}} (p_{kj,U}^R - \delta_{j,D}p_{kj,D}^R), \quad (12)$$

from A.CDMC and (1) respectively. But the change in payments is less than $\sum_{kj \in \mathcal{C}} p_{kj,U}^R - \delta_{j,D}p_{kj,D}^R - \varepsilon$, implying that D_j would be better off accepting all offers at period t . This yields a contradiction.

- (b) D_j rejecting offers $\mathcal{B} \subset \mathcal{C}$ at period t where $ij \in \mathcal{B}$. By the inductive hypothesis, all remaining offers $kj \in \mathcal{B}$ occur in period $t+1$ at prices $\{p_{kj,D}^R\}_{kj \in \mathcal{B}}$. D_j would rather accept all offers at period t (including the deviant offer \tilde{p}_{ij}) if:

$$(1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \mathcal{B}) > \tilde{p}_{ij} - \delta_{j,D}p_{ij,D}^R + \sum_{kj \in \mathcal{B} \setminus \{ij\}} (\hat{p}_{kj} - \delta_{j,D}p_{kj,D}^R).$$

Similar to (12), the LHS can be shown to be greater than $\sum_{kj \in \mathcal{B}} p_{kj,U}^R - \delta_{j,D}p_{kj,D}^R$. Since $\tilde{p}_{ij} < p_{ij,U}^R$ and $\hat{p}_{kj} \leq p_{kj,U}^R \forall kj \in \mathcal{B} \setminus \{ij\}$, it follows that this inequality holds.

The only case not ruled out yet is if D_j accepts the deviation from U_i at period t , but rejects some other subset of offers $\mathcal{B}, ij \notin \mathcal{B}$. Under A.LEXT, U_i would be indifferent between this case and having

D_j accept all its offers, since U_i 's payoff is unaffected by these agreements, and so this deviation remains profitable for U_i . Under A.ASR, if D_j chooses this set of acceptances with the deviant ij offer, then it would have strictly preferred rejecting $\mathcal{B} \subset \mathcal{C}$ in the original candidate equilibrium (as again, A.ASR rules out indifference on the part of D_j), which contradicts the fact that the original strategies formed an equilibrium.

Consequently, we have shown that D_j will accept the deviant offer from U_i , and U_i would prefer this deviation. This yields a contradiction.

Thus, $\hat{p}_{ij} = p_{ij,U}^R \forall ij \in \mathcal{C}$ for agreements reached in an even period.

Claims A-C prove the lemma. □

Lemma D.10 (Immediate agreement.) *Any equilibrium of $\Gamma_{\mathcal{C}}^{\tilde{t}}$ results in immediate agreement for all $ij \in \mathcal{C}$ at period \tilde{t} .*

Proof. The proof here is a special case of the proof of immediacy for Lemma D.13 when there are multiple downstream and upstream firms, given below. As that proof does not leverage the presence of multiple downstream firms with open agreements in \mathcal{C} , it is applicable here and not replicated.

D.4 One Upstream Firm, Many Downstream Firms

Consider any subgame $\Gamma_{\mathcal{C}}^{\tilde{t}}$ where $\mathcal{C} \subseteq \mathcal{G}$ contains only open agreements involving one upstream firm U_i , and $|\mathcal{C}| = n$ so that there are $n > 1$ remaining agreements that have not yet been reached at period \tilde{t} .

This case is exactly symmetric to the one downstream, many upstream firm case proved in Section D.3, and thus the proof that Proposition D.7 holds in any subgame with multiple downstream firms and one upstream firm follows immediately.²⁸

D.5 Many Upstream and Many Downstream Firms

Having proven Proposition D.7 holds for all subgames with open agreements involving either only one upstream firm or one downstream firm, we now focus on subgames $\Gamma_{\mathcal{C}}^{\tilde{t}}$ where $\mathcal{C} \subseteq \mathcal{G}$ involves more than one upstream *and* more than one downstream firm. As before, we will prove Proposition D.7 with three lemmas.

Lemma D.11 (Odd, simultaneous.) *If the first agreement occurs at an odd period $t \geq \tilde{t}$, then all agreements $ij \in \mathcal{C}$ must occur at t with $\hat{p}_{ij} = p_{ij,D}^R \forall ij \in \mathcal{C}$.*

Proof. In the candidate equilibrium, let \mathcal{A} indicate the set of agreements reached first in period t (odd), and $\mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ the set of agreements reached at some later date. By the inductive hypothesis, all agreements $ij \in \mathcal{B}$ will occur at $t + 1$ at prices $p_{ij,U}^R$.

We first prove that all agreements $ij \in \mathcal{C}$ occur at the same period (i.e., $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \emptyset$), and then prove all agreements occur at the Rubinstein prices.

Claim A: All agreements occur at the same time. We prove the claim by contradiction.

Suppose all agreements are not simultaneous so that $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \neq \emptyset$. Since there are multiple upstream firms with open agreements at period t (by assumption), we can find agreements $ab \in \mathcal{A}$ and $ij \in \mathcal{B}$ s.t. $U_a \neq U_i$: i.e., we can find an agreement formed at period t and another agreement formed at $t + 1$ involving different upstream firms.²⁹ Consider the following deviation by D_j at period t : D_j offers $\hat{p}_{ij} \equiv p_{ij,D}^R + \varepsilon$ to U_i .

²⁸The only difference is that the sign on prices, since payments are from downstream to upstream firms, is reversed.

²⁹If we cannot find two agreements $ab \in \mathcal{A}$, $ij \in \mathcal{B}$ involving two different upstream parties, it must have been that there was only one upstream firm in \mathcal{C} (which is ruled out by assumption).

1. *Such a deviant offer will be accepted by U_i .*

To show this, assume not, and assume that U_i rejects \tilde{p}_{ij} but accepts some set of agreements $\mathcal{A}' \subseteq \mathcal{C}_i^U \setminus \{ij\}$. Even if \mathcal{A}' is empty, by passive beliefs U_i still anticipates ab will come to agreement at t ; thus, by induction, all remaining agreements will be formed in the next period at prices $p_{ij,U}^R$.

Instead of rejecting \tilde{p}_{ij} , U_i can do better by accepting $\mathcal{A}' \cup \{ij\}$ at period t than accepting only \mathcal{A}' if the change in payoffs exceeds the change in prices, if:

$$(1 - \delta_{i,U})\Delta\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}' \cup \{ij\}, \{ij\}) > \delta_{i,U}p_{ij,U}^R - \tilde{p}_{ij}.$$

Since $\Delta\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}' \cup \{ij\}, \{ij\}) \geq \Delta\pi_i^U(\mathcal{G}, \{ij\})$ by A.CDMC, $(1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) = \delta_{i,U}p_{ij,U}^R - p_{ij,D}^R$ by (2), and $\tilde{p}_{ij} = p_{ij,D}^R + \varepsilon$, this inequality holds. This implies a contradiction, and thus U_i cannot reject the deviant offer.

2. *Such a deviation is profitable for D_j if accepted by U_i .*

Assume that upon receipt of the deviant offer from D_j , U_i accepts some set of offers $\mathcal{A}'_i^U \subseteq \mathcal{C}_i^U$, $ij \in \mathcal{A}'_i^U$. Let $\mathcal{A}' \equiv \mathcal{A}'_i^U \cup \mathcal{A}'_{-i}$ denote the full set of offers accepted at period t . By the inductive hypothesis, all remaining agreements $\mathcal{C} \setminus \mathcal{A}'$ will be formed at period $t+1$ at prices $p_{k,U}^R$. D_j will find the deviation of offering $\tilde{p}_{ij} = p_{ij,D}^R + \varepsilon$ at period t profitable if $(1 - \delta_{j,D})(\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}') - \pi_j^D(\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - \tilde{p}_{ij} + \delta_{j,D}p_{ij,U}^R > 0$.

But,

$$\begin{aligned} & (1 - \delta_{j,D})(\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}') - \pi_j^D(\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - \tilde{p}_{ij} + \delta_{j,D}p_{ij,U}^R \\ & > (1 - \delta_{j,D})(\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}') - \pi_j^D(\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - p_{ij,U}^R + \delta_{j,D}p_{ij,D}^R \\ & \geq (1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \{ij\}) - p_{ij,U}^R + \delta_{j,D}p_{ij,D}^R = 0, \end{aligned}$$

where the first inequality follows from the definition of \tilde{p} and Lemma D.1, the second inequality follows from A.SCDMC, and the equality follows from equation (1). Note that that application of A.SCDMC follows since \mathcal{A}' and \mathcal{A} differ only in agreements involving U_i , and $ij \in \mathcal{A}'$ while $ij \notin \mathcal{A}$.

Since there is a profitable deviation for D_j , there is a contradiction. Thus all agreements happen at the same time.

Claim B: $\hat{p}_{ij} = p_{ij,D}^R$.

1. Assume that $\hat{p}_{ij} < p_{ij,D}^R$ for some ij . Consider the following deviation for U_i : U_i rejects ij and accepts all other offers at t ; U_i will then come to agreement with D_j in $t+1$ for payment $p_{ij,U}^R$ by the inductive hypothesis. This is profitable for U_i if $(1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) > -\hat{p}_{ij} + \delta_{i,U}p_{ij,U}^R$. Since the LHS of this inequality is equal to $-p_{ij,D}^R + \delta_{i,U}p_{ij,U}^R$ (see (2)), this inequality will hold if $\hat{p}_{ij} < p_{ij,D}^R$. This yields a contradiction.
2. Assume $\hat{p}_{ij} > p_{ij,D}^R$ for some ij . Consider a deviant offer by D_j , $\tilde{p}_{ij} = p_{ij,D}^R + \varepsilon < \hat{p}_{ij}$. We now show that this deviation is profitable to D_j , leading to a contradiction.

If D_j offers \tilde{p}_{ij} instead of \hat{p}_{ij} at period t , we show that U_i accepting all offers (including \tilde{p}_{ij}) at period t is more profitable than U_i rejecting offers $\mathcal{B} \subseteq \mathcal{C}_i^U$ at period t where $ij \in \mathcal{B}$. By passive beliefs, U_i believes that all offers in \mathcal{C}_{-i}^U —and hence at least one offer in \mathcal{C} —will conclude at period t . Thus, by the inductive hypothesis, in case of this rejection, all agreements in \mathcal{B} form in period $t+1$ for prices $\{p_{ik,U}^R\}_{ik \in \mathcal{B}}$. U_i would rather accept all offers at period t (including the deviant offer \tilde{p}_{ij}) if:

$$(1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \mathcal{B}) > -\tilde{p}_{ij} + \delta_{i,U}p_{ij,U}^R + \sum_{ik \in \mathcal{B} \setminus \{ij\}} (-\hat{p}_{ik} + \delta_{i,U}p_{ik,U}^R).$$

From (2), the LHS is greater than $\sum_{kj \in \mathcal{B}} p_{kj,U}^R - \delta_{j,D}p_{kj,D}^R$. Since $\tilde{p}_{ij} < p_{ij,U}^R$ and $\hat{p}_{kj} \leq p_{kj,U}^R \forall kj \in \mathcal{B} \setminus \{ij\}$, it follows that this inequality holds.

We have not yet ruled out the possibility that U_i accepts the deviation from D_j at period t , but rejects some other subset of offers $\mathcal{B}, ij \notin \mathcal{B}$. Under A.LEXT, D_j would be indifferent between this case and having U_i accept all its offers, since D_j 's payoff is unaffected by these agreements, and so this deviation remains profitable for D_j . Under A.ASR, if U_i chooses this set of acceptances with the deviant ij offer, then it would have strictly preferred rejecting $\mathcal{B} \subset \mathcal{C}$ in the original candidate equilibrium (as again, A.ASR rules out indifference on the part of U_i), which contradicts the fact that the original strategies formed an equilibrium.

Consequently, we have shown that U_i will accept the deviant offer from D_j , and D_j would prefer this deviation. This yields a contradiction.

Hence, for all $ij \in \mathcal{C}$, $\hat{p}_{ij} = p_{ij,D}^R$. □

Lemma D.12 (Even, simultaneous.) *If the first agreement occurs at an even period $t \geq \tilde{t}$, then all agreements $ij \in \mathcal{C}$ must occur at t with $\hat{p}_{ij} = p_{ij,U}^R \forall ij \in \mathcal{C}$.*

Proof. The proof here is symmetric to the case considered in Lemma D.11.

Lemma D.13 (Immediate agreement.) *Any equilibrium of $\Gamma_{\mathcal{C}}^{\tilde{t}}$ results in immediate agreement for all $ij \in \mathcal{C}$ at period \tilde{t} .*

Proof. We prove this lemma by contradiction. Consider first the case where \tilde{t} is odd. Let agreement $ij \in \mathcal{C}$ satisfy the conditions of A.LNEXT (either directly or through A.LEXT).

Consider a candidate equilibrium where no agreements are formed at period \tilde{t} (as, by the previous results, if any agreement is formed in period \tilde{t} , all agreements are formed in that period). We propose a deviant strategy by D_j from this candidate equilibrium and then verify that it will increase D_j 's payoff. Suppose D_j offers \tilde{p}_{ij} satisfying $p_{ij,R}^D < \tilde{p}_{ij} < p_{ij,U}^R$ to some U_i where $ij \in \mathcal{C}$. We first show that U_i will accept this offer and then show that it will increase D_j 's surplus relative to the candidate equilibrium.

Suppose that U_i accepts the offer \tilde{p}_{ij} . Then, by passive beliefs, it believes that this is the only offer to be accepted at period \tilde{t} and, by the inductive hypothesis, that the remaining agreements will form at period $\tilde{t} + 1$. Hence, its payoffs—in period \tilde{t} units—from accepting the offer are:

$$\begin{aligned}
& \underbrace{\tilde{p}_{ij} + (1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \{ij\})}_{\text{Payoff at } \tilde{t}} + \underbrace{\delta_{i,U} \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U \setminus \{ij\}} p_{ik,U}^R \right)}_{\text{Payoff from } \tilde{t}+1 \text{ on}} \\
&= \tilde{p}_{ij} + (1 - \delta_{i,U})\Delta\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \{ij\}, \{ij\}) + \delta_{i,U} \left(\sum_{ik \in \mathcal{C}_i^U \setminus \{ij\}} p_{ik,U}^R + \pi_i^U(\mathcal{G}) \right) + (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{C}) \\
&> p_{ij,D}^R + (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) + \delta_{i,U} \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U \setminus \{ij\}} p_{ik,U}^R \right) + (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{C}) \\
&= \delta_{i,U} \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) + (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{C}),
\end{aligned}$$

where the second line simply adds and subtracts the $(1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{C})$ term, the third line follows from A.CDMC and the definition of \tilde{p}_{ij} , and the final line uses equation (2) and then combines the $p_{ij,U}^R$ terms in the sum.

We next show that this payoff from acceptance is higher than the payoff from rejecting D_j 's deviant offer. In case U_i rejects the deviant offer, there are the following four potential outcomes given equilibrium play in the resulting subgame:

1. U_i rejects all offers in \mathcal{C}_i^U at period \tilde{t} and no offers in \mathcal{C} are ever accepted.

In this case, the payoffs to U_i are:

$$\begin{aligned}
\pi_i^U(\mathcal{G} \setminus \mathcal{C}) &= \delta_{i,U} \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + (1 - \delta_{i,U}) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) \\
&< \delta_{i,U} \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,D}^R \right) + (1 - \delta_{i,U}) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) \\
&< \delta_{i,U} \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) + (1 - \delta_{i,U}) \pi_i^U(\mathcal{G} \setminus \mathcal{C}),
\end{aligned}$$

where the first inequality follows from Lemma D.5 (which uses A.LNEXT) and the second inequality follows from Lemma D.1. Thus, the payoffs to U_i from rejection are less than from accepting D_j 's deviant offer in this case.

2. U_i rejects all offers in \mathcal{C}_i^U at period \tilde{t} and all offers \mathcal{C} are formed in some even period $\tilde{t} + \tau$, for $\tau = 1, 3, 5, \dots$

If U_i accepts no other offers at period \tilde{t} (and by passive beliefs, U_i believes that no offers in are formed at \tilde{t}), the payoffs to U_i are:

$$\begin{aligned}
&(1 - \delta_{i,U}^\tau) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + \delta_{i,U}^\tau \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&= (1 - \delta_{i,U}) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + (\delta_{i,U} - \delta_{i,U}^\tau) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + \delta_{i,U}^\tau \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&< (1 - \delta_{i,U}) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + (\delta_{i,U} - \delta_{i,U}^\tau) \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&\quad + \delta_{i,U}^\tau \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&= (1 - \delta_{i,U}) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + \delta_{i,U} \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right),
\end{aligned}$$

where lines two and four follow by rearranging terms and line three follows again from Lemma D.5. Thus, the payoffs to U_i from rejection are less than from accepting D_j 's deviant offer in this case as well.

3. U_i rejects all offers in \mathcal{C}_i^U at period \tilde{t} and all offers \mathcal{C} are formed in some odd period $\tilde{t} + \tau$, for $\tau = 2, 4, 6, \dots$

In this case, the payoffs to U_i are:

$$\begin{aligned}
&(1 - \delta_{i,U}^\tau) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + \delta_{i,U}^\tau \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,D}^R \right) \\
&< (1 - \delta_{i,U}^\tau) \pi_i^U(\mathcal{G} \setminus \mathcal{C}) + \delta_{i,U}^\tau \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right)
\end{aligned}$$

$$\begin{aligned}
&= (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{C}) + (\delta_{i,U} - \delta_{i,U}^\tau)\pi_i^U(\mathcal{G} \setminus \mathcal{C}) + \delta_{i,U}^\tau \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&< (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{C}) + (\delta_{i,U} - \delta_{i,U}^\tau) \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&\quad + \delta_{i,U}^\tau \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&= (1 - \delta_{i,U})\pi_i^U(\mathcal{G} \setminus \mathcal{C}) + \delta_{i,U} \left(\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{C}_i^U} p_{ik,U}^R \right),
\end{aligned}$$

where the first inequality follows from Lemma D.1 and the remaining logic is identical to case 2. Thus, the payoffs to U_i from rejection are less than from accepting D_j 's deviant offer in this case too.

4. U_i accepts some non-empty set of offers $\mathcal{B} \subseteq \mathcal{C}_i^U \setminus \{ij\}$ at period \tilde{t} .

In this case, by the inductive hypothesis, all remaining agreements $\mathcal{A} \equiv \mathcal{C} \setminus \mathcal{B}$ form in the following (even) period $\tilde{t} + 1$ at Rubinstein prices. Thus, we can express the payoff to U_i from this strategy as $(1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B}) + \sum_{ik \in \mathcal{B}} \hat{p}_{ik} + \delta_{i,U} \left[\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{A}_i^U} p_{ik,U}^R \right]$, where $\hat{p}_{ik} \forall ik \in \mathcal{B}$ are the period \tilde{t} candidate equilibrium prices offered to U_i . But,

$$\begin{aligned}
&(1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B}) + \sum_{ik \in \mathcal{B}} \hat{p}_{ik} + \delta_{i,U} \left[\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{A}_i^U} p_{ik,U}^R \right] \\
&= (1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B}) + \sum_{ik \in \mathcal{B}} \hat{p}_{ik} + \delta_{i,U} \left[\pi_i^U(\mathcal{G}) + p_{ij,U}^R + \sum_{ik \in \mathcal{A}_i^U \setminus \{ij\}} p_{ik,U}^R \right] \\
&= (1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B}) + \sum_{ik \in \mathcal{B}} \hat{p}_{ik} \\
&\quad + p_{ij,R}^D + (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) + \delta_{i,U} \left[\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{A}_i^U \setminus \{ij\}} p_{ik,U}^R \right] \\
&< (1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B}) + \sum_{ik \in \mathcal{B}} \hat{p}_{ik} + \tilde{p}_{ij} + (1 - \delta_{i,U})\Delta\pi_i^U(\mathcal{G}, \{ij\}) + \delta_{i,U} \left[\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{A}_i^U \setminus \{ij\}} p_{ik,U}^R \right] \\
&\leq (1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B}) + \sum_{ik \in \mathcal{B}} \hat{p}_{ik} + \tilde{p}_{ij} + (1 - \delta_{i,U})\Delta\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B} \cup \{ij\}, \{ij\}) \\
&\quad + \delta_{i,U} \left[\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{A}_i^U \setminus \{ij\}} p_{ik,U}^R \right] \\
&= (1 - \delta_{i,U})\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B} \cup \{ij\}) + \sum_{ik \in \mathcal{B}} \hat{p}_{ik} + \tilde{p}_{ij} + \delta_{i,U} \left[\pi_i^U(\mathcal{G}) + \sum_{ik \in \mathcal{A}_i^U \setminus \{ij\}} p_{ik,U}^R \right],
\end{aligned}$$

where lines two and six follow by rearranging terms, line three follows from equation (2), line four follows from the the definition of the deviant offer, and line five follows from A.CDMC.

Since the final line is the value of accepting D_j 's deviant offer and all offers in \mathcal{B} , the payoff to U_i from accepting D_j 's deviant offer and all offers in \mathcal{B} is higher than the payoff from accepting just the offers in \mathcal{B} . This implies that U_i is better off from accepting D_j 's deviant offer even if it also wanted

to accept some other offers.

Since, in each of the cases, U_i is strictly better off from accepting the deviant offer than from rejecting it, U_i will accept this offer. Note that we have not ruled out the possibility that U_i may also choose to accept additional offers in \mathcal{C}_i^U at period \tilde{t} upon accepting deviant offer \tilde{p}_{ij} ; we return to this below.

Having verified that the \tilde{p}_{ij} offer will be accepted by U_i , we now check that the acceptance of this deviant offer will be profitable for D_j . D_j knows that U_i is the only firm that will accept offer(s) at period \tilde{t} and, by the inductive hypothesis, that the remaining agreements will form at period $\tilde{t} + 1$. However, it is possible that upon receiving the deviant offer, U_i will also accept some other offers $\mathcal{B} \subset \mathcal{C}_i^U \setminus \{ij\}$. Hence, D_j 's payoff—in period \tilde{t} units—from making the deviant offer satisfies:

$$\begin{aligned}
& \underbrace{-\tilde{p}_{ij} + (1 - \delta_{j,D})\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{B} \cup \{ij\})}_{\text{Payoff at } \tilde{t}} + \underbrace{\delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D \setminus \{ij\}} p_{kj,U}^R \right)}_{\text{Payoff from } \tilde{t}+1 \text{ on}} \\
& \geq -\tilde{p}_{ij} + (1 - \delta_{j,D})\Delta\pi_j^D((\mathcal{G}, \{ij\})) + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D \setminus \{ij\}} p_{kj,U}^R \right) + (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) \\
& > -p_{ij,U}^R + (1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \{ij\}) + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D \setminus \{ij\}} p_{kj,U}^R \right) + (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) \\
& = -\delta_{j,D}p_{ij,D}^R + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D \setminus \{ij\}} p_{kj,U}^R \right) + (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) \\
& > (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right),
\end{aligned}$$

where the second line applies A.SCDMC, the third line follows from the definition of \tilde{p}_{ij} , the fourth line uses equation (1) on the construction of Rubinstein prices, and the final line uses Lemma D.1 and then combines the $p_{kj,U}^R$ terms in the sum.

Next, we show that the lower bound on payoffs from this deviant offer being accepted (given by the last line of the previous set of equations) is higher than the payoff from the candidate equilibrium. In case D_j does not deviate from the equilibrium with an offer \tilde{p}_{ij} , there are again three possibilities for the equilibrium payoffs at this node:

1. No further offers are accepted.

In this case, the payoffs to D_j from the candidate equilibrium are:

$$\begin{aligned}
\pi_j^D(\mathcal{G} \setminus \mathcal{C}) &= \delta_{j,D}\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) \\
&< (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right),
\end{aligned}$$

where the inequality follows from Lemma D.5. Thus, the payoffs to D_j from the candidate equilibrium are less than from accepting U_i 's deviant offer in this case.

2. All remaining offers are accepted in some even period $\tilde{t} + \tau$, for $\tau = 1, 3, 5, \dots$

In this case, the payoffs to D_j from the candidate equilibrium are:

$$(1 - \delta_{j,D}^\tau)\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right)$$

$$\begin{aligned}
&= (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + (\delta_{j,D} - \delta_{j,D}^\tau)\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right) \\
&< (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + (\delta_{j,D} - \delta_{j,D}^\tau) \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right) \\
&\quad + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_i^U} p_{ik,U}^R \right) \\
&= (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right),
\end{aligned}$$

where lines two and four follow by rearranging terms and line three follows again from Lemma D.5. Thus, the payoffs to D_j from the candidate equilibrium are less than from making the deviant offer in this case also.

3. All remaining offers are accepted in some odd period $\tilde{t} + \tau$, for $\tau = 2, 4, 6, \dots$

In this case, the payoffs to D_j from the candidate equilibrium are:

$$\begin{aligned}
&(1 - \delta_{j,D}^\tau)\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^\tau \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,D}^R \right) \\
&= (1 - \delta_{j,D}^{\tau-1})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{\tau-1} \left((1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}\pi_j^D(\mathcal{G}) - \delta_{j,D} \sum_{kj \in \mathcal{C}_j^D} p_{kj,D}^R \right) \\
&= (1 - \delta_{j,D}^{\tau-1})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{\tau-1} \left(\pi_j^D(\mathcal{G}) - (1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \mathcal{C}) - \delta_{j,D} \sum_{kj \in \mathcal{C}_j^D} p_{kj,D}^R \right) \\
&< (1 - \delta_{j,D}^{\tau-1})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{\tau-1} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} [(1 - \delta_{j,D})\Delta\pi_j^D(\mathcal{G}, \{kj\}) + \delta_{j,D}p_{kj,D}^R] \right) \\
&= (1 - \delta_{j,D}^{\tau-1})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{\tau-1} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right) \\
&= (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + (\delta_{j,D} - \delta_{j,D}^{\tau-1})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{\tau-1} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right) \\
&< (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + (\delta_{j,D} - \delta_{j,D}^{\tau-1}) \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right) \\
&\quad + \delta_{j,D}^{\tau-1} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right) \\
&= (1 - \delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}_j^D} p_{kj,U}^R \right),
\end{aligned}$$

where the second, third, and sixth lines follow by rearranging terms, the fourth line follows from A.WCDMC, the fifth line from equation (1), the seventh line from Lemma D.5, and the final line also by rearranging terms. Thus, the payoffs to D_j from the deviant offer are greater than its equilibrium payoffs in this case too.

So D_j has a profitable deviation, leading to a contradiction in the case where \tilde{t} is odd.

We next consider the case where \tilde{t} is even. An exactly symmetric argument applies here, and hence we do not repeat it.

Thus, any equilibrium involves immediate agreement for all $ij \in \mathcal{C}$. □

E Counterexamples

E.1 Counterexample to Unique Equilibrium

This subsection provides an example of a game with multiple equilibria and immediate and complete agreement even when the assumptions from Theorem 4.6—A.GFT and either (i) A.CDMC and A.LEXT or (ii) A.SCDMC, A.LNEXT and A.ASR—hold. As discussed in the text, the equilibria will differ only in their prescribed off-equilibrium play and hence realized outcomes are the same across equilibria.

Let $M = 1$ and $N = 2$ so that there is one downstream firm D_1 and two upstream firms U_1, U_2 . In this case, if there are multiple and simultaneous deviations by both upstream firms in an even (upstream-proposing) period—which will reach a node off the equilibrium path—then D_1 's best response may be to accept only one and not both of these deviations, and the choice of which offer to accept may be arbitrary.

Numerical Example. Let $\mathcal{G} = \{11, 21\}$. Let $\pi_i^U(\mathcal{K}) = 0, \forall \mathcal{K} \subseteq \mathcal{G}$ and for U_1 and U_2 , so profits accrue only to D_1 . Let $\pi_1^D(\emptyset) = 0, \pi_1^D(\{11\}) = \pi_1^D(\{21\}) = 6$, and $\pi_1^D(\{11, 21\}) = 8$. Note that this example satisfies A.ASR, A.CDMC and A.LEXT.

Suppose that $\delta_{1,U} = \delta_{1,D} = \delta_{2,D} = 0.9$. Note that $p_{1,U}^R = p_{2,U}^R \approx 1.0526$. Now consider the even-period node where U_1 and U_2 have both deviated from their equilibrium strategies and offered $p_1^* = p_2^* = 1$, and D_1 is deciding which offer(s) to accept. It is easy to verify that, at this node, D_1 should accept either offer but not both offers. Thus, one equilibrium involves D_1 accepting U_1 's offer at this node, while another equilibrium involves D_1 accepting U_2 's offer at this node.

The underlying logic is that the difference in D_1 's payoffs between one and two agreements, which is 2, is smaller than the difference in D_1 's payoffs between zero and one agreements, which is 6. The p_U^R payoffs are designed to make D_1 indifferent between accepting both offers and only one—but D_1 strictly prefers one agreement to none at these prices.

E.2 Counterexample to Unique Equilibrium Payoffs With Immediate and Complete Agreement

This subsection provides an example of a game with an equilibrium where a firm can be paid more than the Rubinstein price with immediate agreement when the assumptions of Theorem 3.2 hold but the assumptions of Theorem 4.6 do not.

Let $N = 1$ and $M = 2$ so that there is one upstream firm U_1 and two downstream firms D_1, D_2 . Assume that the first period, t_0 , is odd so that downstream firms make initial proposals, and that $\Delta\pi_1^D(\mathcal{G}, \{12\}) > 0$, so that there is a positive externality to D_1 from D_2 contracting.

Consider the strategy profile prescribed in the proof of Theorem 3.2 in Appendix C, and alter it so that:

- In odd periods, D_1 offers U_1 $\hat{p}_{11,D}^R \equiv p_{11,D}^R + \varepsilon$, where:

$$\varepsilon \in (0, \min\{(1 - \delta_{1,D})\Delta\pi_1^D(\mathcal{G}, \{12\}), (1 - \delta_{1,D})\Delta\pi_1^D(\mathcal{G}, \{11\}) - (p_{11,D}^R - \delta_{1,D}p_{11,U}^R)\})$$

- In odd periods, U_1 accepts any offer $p_{11} \geq p_{11,D}^R$ from D_1 , and rejects otherwise; however, if U_1 accepts p_{11} and $p_{11} \neq \hat{p}_{11}$, then U_1 rejects p_{12} if $p_{12} = p_{12,D}^R$.

This change to the strategies implies that D_1 offers more than its Rubinstein price in an odd period, and that U_1 threatens to reject D_2 's offer of $p_{12,D}^R$ if D_1 makes a deviant offer that is greater than or equal to the Rubinstein price, but different than \hat{p}_{11} . Since U_1 is indifferent over accepting and rejecting $p_{12,D}^R$ from D_2 given it accepts D_1 under the strategy profiles given, U_1 's off-equilibrium threat is credible. The premiums over the Rubinstein price $p_{11,D}^R$ made in the first period can be no higher than either D_1 's gain from U_1 reaching agreement with D_2 immediately, or D_1 's option of offering such a high price in period 1 so that U_1 rejects it, and then reaching agreement with U_1 in the following period at $p_{11,U}^R$.

As long as A.GFT and A.CDMC hold for the remainder of the underlying payoffs, it is straightforward to show that this strategy profile will comprise an equilibrium. Essentially, it is sustained by the positive externality on D_1 that is generated by U_1 coming to agreement with D_2 ; U_1 can leverage this to extract a higher price from D_1 when negotiating in an odd period.

Note that as $\Lambda \rightarrow 0$, the outcome of this equilibrium also converges to the one detailed in the uniqueness proof: i.e., Nash-in-Nash prices for all firms, and immediate agreement.

Numerical Example. Let $\mathcal{G} = \{11, 12\}$. Let $\pi_1^U(\emptyset) = \pi_1^D(\emptyset) = \pi_1^D(\{12\}) = 0$, $\pi_1^U(\{11\}) = \pi_1^U(\{12\}) = 5$, $\pi_1^U(\{11, 12\}) = 8$, $\pi_1^D(\{11\}) = 1$, and $\pi_1^D(\{11, 12\}) = 2$, and let D_2 have symmetric payoffs to D_1 .

This example satisfies both A.ASR and A.CDMC but not A.SCDMC or A.LEXT. Suppose again that $\delta_{1,U} = \delta_{1,D} = \delta_{2,D} = .9$. Then the strategies prescribed above with $\varepsilon \leq 0.1$ comprise an equilibrium.

E.3 Counterexample to Complete and Immediate Agreement

This subsection provides an example of a game where there is an equilibrium without complete and immediate agreement when the assumptions of Theorem 3.2 hold but the assumptions of Theorem 4.6 do not.

Let $N = 1$ and $M = 2$ so that there is one upstream firm U_1 and two downstream firms D_1, D_2 . Consider the case where D_1 is harmed by an offer between U_1 and D_2 , and vice versa. In such a setting, it may be the case that neither downstream firm has an incentive to form an agreement if the other agreement has not yet been formed, and there may exist an equilibrium without any agreement ever forming.

Numerical Example. Let $\mathcal{G} = \{11, 12\}$. Let $\pi_1^U(\mathcal{K}) = 0, \forall \mathcal{K} \subseteq \mathcal{G}$, so profits accrue only to D_1 and D_2 . Let $\pi_1^D(\emptyset) = 0$, $\pi_1^D(\{11\}) = 1$, $\pi_1^D(\{12\}) = -100$, and $\pi_1^D(\{11, 12\}) = -99$, and let D_2 have symmetric payoffs to D_1 .

Note that this example satisfies A.GFT and A.CDMC, but not A.SCDMC, A.LNEXT, or A.LEXT. Suppose again that $\delta_{1,U} = \delta_{1,D} = \delta_{2,D} = .9$. Since the model satisfies A.GFT and A.CDMC there will be an equilibrium with immediate agreement at Rubinstein prices—e.g., if D_2 believes that D_1 will form an agreement with U_1 at Rubinstein prices, D_2 will wish to form an agreement as well. However, there will also be an equilibrium with no agreements ever formed. To see this, note that no downstream firm has an incentive to form an agreement at Rubinstein prices (or at any prices that U_1 would agree to) given the other has not formed an agreement (given discount factors are sufficiently close to 1): even though an agreement may yield positive surplus for one period, any resulting subgame results in the other agreement forming at Rubinstein prices immediately thereby resulting in negative profits for the rest of the game.