

Forward Guidance and Credible Monetary Policy*

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Abstract

The effectiveness of forward guidance depends crucially on the credibility of the central bank. Instead of assuming full commitment or full discretion of the central bank, this paper sheds light on the best credible forward guidance policy a central bank can offer by solving for the whole set of sustainable sequential equilibria (SSE) in a standard New Keynesian model with the occasionally binding constraint of a nominal interest rate of zero. The inflation bias resulting from eliminating price distortion under the best SSE is much smaller than the bias under full discretion. In the presence of the zero lower bound (ZLB), while the full commitment solution implies that forward guidance has a longer duration of the nominal rate at zero bound followed by a quick revert-to-normal path, the best SSE features low but non-zero rates, and even a prolonged period of higher inflation after the recession ends. To make forward guidance credible and as stimulative as possible, the adjustment of policy rates, and in particular, the rise of policy rates, should be smooth and gradual. In other words, policy normalization does not mean non-accommodation. Ramsey equilibria are not generally implementable.

Keywords: Forward Guidance, Zero Lower Bound, Credible Monetary Policy, Commitment, Reputation, Discretion

JEL Codes: E32, E52, E61

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1 Introduction

Since the Great Recession, many major central banks, including the Federal Reserve Bank (Fed), the European Central Bank, and the Bank of England, have joined the Bank of Japan in using forward guidance for an economic boost.¹ In contrast to the conventional short-term interest rate policy, which is constrained by the zero lower bound (ZLB), forward guidance is the statement and communication of the central bank's projected future path of short-term interest rates.² An important feature of the recent forward guidance practice is its promise to maintain the interest rate at or near zero even after the economy emerges from recession. For example, the Fed has repeatedly said that it will keep short-term interest rates low for a "considerable" time after the economy emerges from the recession.³ The prolonged stay at the ZLB will then increase expected inflation and stimulate current consumption.⁴

However, the central bank is likely to raise interest rates as the economy strengthens and inflation is high. The failure of the central bank to honor its promises would severely reduce the effectiveness of forward guidance.

The key to having forward guidance work is for households to believe that what the central bank promises today is truly what it will deliver tomorrow, i.e., the central bank must be credible. Specifically, households in the economy must believe that central banks will maintain the nominal rate at zero, as stated in the forward guidance, even after the economy emerges from recession. I show that this belief may turn out to be wrong in a standard New Keynesian model. This model features a benevolent central bank that sets the nominal interest rate, sticky prices due to quadratic price adjustment cost à la Rotemberg (1982), and the occasionally binding constraint of the nominal interest rate at zero due to high discount factor shocks, which are used to proxy financial crisis. Optimal policy of the central bank is then interpreted as forward guidance.

Without the ZLB, the optimal policy of the central bank is always to set the gross nominal rate to the inverse of the discount factor, which is less than 1 at the time of crisis, to eliminate

¹The history of the Bank of Japan using forward guidance dates back to 1999. The Fed also experimented with forward guidance in 2003 and 2004.

²As summarized in Issing (2014), different forms of forward guidance have been adopted by different central banks. Widely speaking, forward guidance also includes inflation targeting.

³For example, the FOMC minutes of Jan 2013 stated: "To support continued progress toward maximum employment and price stability, the Committee expects that a highly accommodative stance of monetary policy will remain appropriate for a considerable time after the asset purchase program ends and the economic recovery strengthens."

⁴A second effect would be a shorter duration of bad times, which will also boost the economy.

the price distortion (i.e., zero inflation). With the ZLB, the central bank's gross short-term policy rate cannot go below 1 to close gaps. However, the central bank under full commitment is able to reduce (negative) inflation and output gaps by promising to keep nominal interest rates at zero even after the economy strengthens, which will lead to positive gaps in the future. The promise at the time of recession is likely time-inconsistent because the central bank may raise the rates to close gaps if this is permissible and the economy strengthens. The full commitment literature ignores the credibility issue by not allowing central banks to pick a policy that differs from the one to which they have committed earlier. Indeed, the first contribution of this paper is to show that the forward guidance policy by assuming full commitment central banks is generally not credible.

Another strand of literature assumes full discretion for the central bank. Under full discretion, the central bank is limited to choosing policy rates solely conditional on predetermined variables and the current value of shocks. Since discount factor shock is the only state variable in the model of this paper, the central bank cannot manipulate the public's future beliefs or its future policy rates. Therefore, when a contractionary (high discount factor) shock occurs, the optimal policy of the central bank is to lower the nominal rate to zero. When the shock disappears, the central bank will set the rate back to its normal level to achieve zero gaps in inflation and output.⁵ Hence, the only credible forward guidance under full discretion is to stay at zero during recession and jump back to the normal level when the economy emerges from recession. In other words, if the Fed has no commitment, then the forward guided zero policy rates it says it will maintain even after the economy strengthens are not credible. After all, agents in the economy never believe any forward guidance other than that saying the rate will revert to its normal level when the recession ends. The announcement of such forward guidance, though credible, does nothing to improve the economy in recession using future policy rates.

Recent episodes of zero interest rates suggest that central banks in industrialized economies have neither full commitment nor full discretion. For example, the Bank of Japan never followed the prescription of the optimal policy under full commitment in 1999/2000. On the other hand, if central banks are fully discretionary, then the forward guidance of the Fed in the summer of 2011 would not have led to a drop of 10-20 basis points for long-term interest rates. With these

⁵Households actually choose to have slightly negative output and inflation gaps to smooth their consumption due to the recurrence of high discount factor shocks.

observations and with the concerns of non-credibility under full commitment and strict limitation of policy choices under full discretion, this paper asks: What does the best credible forward guidance look like?

In contrast to the concepts of full commitment (Ramsey equilibrium) and full discretion (Markov perfect equilibrium, MPE), this paper answers this question by solving for the whole sustainable sequential equilibria (SSE) set of the New Keynesian model discussed above. The equilibrium set is characterized by payoffs to households and central banks. Every equilibrium in the set is supported by credible plans. In equilibrium, the central bank will only follow credible plans and will not deviate from them. I refer to central banks under the SSE concept as banks with reputation.

Given the solved SSE set, I show that Ramsey equilibrium is generally not credible because the corresponding payoff combinations under Ramsey do not always lie in the set of SSE. This conclusion is robust to various properties of shocks (size, frequency, and persistence) and different values for other model parameters, such as Frisch labor elasticity, elasticity of substitution among intermediate goods, and price adjustment cost. It turns out that as long as the time-inconsistency under Ramsey exists, i.e., the ZLB binds under Ramsey equilibrium, the central bank will always deviate. In other words, Ramsey equilibrium is not credible. In contrast, the payoffs of MPE lie within the set of SSE and MPE is indeed credible. In the following, I focus on the best SSE and compare its policy implications with those of Ramsey and MPE.

A central bank with reputation can do much better than one with full discretion. In the absence of fiscal subsidy and the ZLB, the inflation bias under the best SSE is only half of that under MPE. Since lower inflation bias means higher welfare, the central bank with reputation can achieve higher welfare with access to future policy rates, which the central bank with full discretion cannot have. In the presence of full fiscal subsidy and the ZLB, the output gap is on average -4.55% under full discretion but only -1.55% under reputation at the time of recession. This effectiveness of monetary policy in boosting the economy under the SSE is again due to the central bank's access to and credible control and twist of future policy rates.

Under MPE, the policy rate reverts to its normal level immediately after the recession ends. That is, the extra duration of the ZLB after recession is a zero period. Under Ramsey, the policy rate stays at the ZLB for an extended period, however, with a sharp increase and overshooting of its normal level. A natural guess of the policy response for the best SSE is that it will stay at

the ZLB with a time period from zero to the one under Ramsey. Surprisingly, the best forward guidance associated with the best SSE features a prolonged stay with low but non-zero policy rates after the recession ends in most cases. This results in a high and persistent inflation path. In the baseline calibration, it takes 3 to 5 quarters longer for inflation and the interest rate to go back to their normal state under SSE than under Ramsey, which brings the output gap under the best SSE closer to that under Ramsey (-1.55% vs -1.05%) .

The optimal policies also differ in their entry strategies when recession starts. Under MPE and Ramsey, the central bank immediately sets the nominal rate to zero, whereas under the best SSE, the central bank only lowers the rate moderately at the time of the first shock and then continues to lower the rate but keeps it away from zero given the baseline calibration. I summarize the optimal policies by relating nominal interest rates to inflation and output to derive a Taylor-style rule. Given that the only type of shock in this paper is a discount factor shock, both rules from full commitment and the best reputational equilibrium state that the nominal rate should decrease when inflation increases and decrease when output decreases. However, the rule under reputational equilibrium says the central bank should react to inflation and output less aggressively, with elasticities in the nominal rate over inflation and output about one third and half of that under full commitment, respectively.

The less aggressive rule uncovers the key role of credibility: The policy rate cannot be as accommodative as that under Ramsey. A credible central bank sets the rate to a level that will boost the economy as much as possible and in the meanwhile reduce the incentive to deviate. The aggressive rule, in particular, the zero rates under Ramsey, lead to enough incentive for the central bank to deviate. In other words, the central bank under Ramsey has reaped all benefits by having zero interest rate last period while having almost zero cost to deviate today. Therefore, the central bank deviates today. To avoid the deviation, the policy rate should not have been set so low last period.

This gradual stance of normalization is consistent with the Fed's all the time minutes. The two prevailing arguments for gradualism in exit of accommodative monetary policy are worries about the headwinds of the economy and the over-sensitivity of agents to interest rate due to experiences of the last few years. This paper, however, concludes that the gradualism of exit is also the self-evident requirement of making policy credible. That is, the lift of the policy rate does not mean

that the central bank stops being accommodative. In other words, policy normalization does not mean non-accommodation.

While SSE is a definition of equilibrium that is relatively widely used to discuss optimal taxation problems (see Chari and Kehoe (1990) and Phelan and Stacchetti (2001)), it is less frequently discussed in monetary policy, in particular, in policy related to the ZLB. One exception is Nakata (2014), the key question of which is to what degree the Ramsey plan is sustainable and credible. While he proposes one revert-to-discretion plan – if the central bank deviates from Ramsey equilibria, it will switch households’ expectation about future policies, which he assumes, to MPE – to support the Ramsey plan, there is no reason to think that this expectation is the one to which agents in the economy will switch. In contrast, my paper solves for all credible plans. Since households can switch their expectation about future policies to those under the best SSE once the central bank deviates, the Ramsey equilibrium becomes less implementable. In fact, if the revert-to-plan, using Nakata’s definition, is based on the best SSE, the Ramsey equilibrium is not generally implementable.⁶

This paper is also related to Bodenstein et al. (2012), who study how imperfect credibility of the central bank affects the optimal policy at the ZLB. In their paper, they model imperfect credibility by allowing central banks to discard their earlier promises and re-optimize with an exogenous probability. They also assume, as in Nakata (2014), that if the central bank deviates, households will change their expectation and stick to policies under full discretion. In contrast, households in my model have a full understanding of what the central bank does given the state. In equilibrium, the central bank will never deviate.

The method for computing the equilibrium payoff set closely follows Feng (forthcoming), Phelan and Stacchetti (2001), and Chang (1998). The central bank and the public play a dynamic game and the behavior of both is sequentially rational. The reputation mechanism ensures that if the central bank deviates from equilibrium policy at the ZLB and pursues a different policy (i.e., it shortens the duration of zero interest rates when the economy strengthens), although it obtains instant benefits by having lower inflation and smaller output gaps, it will be punished by a lower continuation value. The equilibria payoff set is found by repeatedly applying a monotonic set-valued

⁶Another paper that uses similar strategies to Nakata (2014) is Ireland (1997), which discusses conditions under which the Friedman rule, the optimal policy under commitment, can be supported when the government lacks a commitment technology.

operator, as in Fernández-Villaverde and Tsyvinski (2002) and Yeltekin and Sleet (2000). However, I use a different method to represent the equilibria set on a computer.

The rest of the paper is structured as follows. Section 2 presents the model. Section 3 introduces MPE and Ramsey equilibria while Section 4 presents SSE and the strategies to solve the model. Sections 5 and 6 are devoted to results, without and with the ZLB, respectively. Section 7 concludes.

2 The Model

This section presents a standard New Keynesian model and the definition of competitive equilibrium. The economy is populated by a continuum of households and firms and a monetary authority. The economy at period t is hit by discount factor shocks, β_t , which follows a two-state Markov chain process.⁷ This Markov chain is characterized by the following transition matrix:

$$\mathbf{P} = \begin{bmatrix} p_{LL} & 1 - p_{LL} \\ 1 - p_{HH} & p_{HH} \end{bmatrix}$$

and two states $\{\beta^L, \beta^H\}$, where L and H mean low and high states for β , respectively. At high state β^H , the economy is in recession due to more patience and less consumption of households. At low state β^L , the economy is said in normal state. $1 - p_{LL}$ is the probability of the economy switching from normal state to recession and p_{HH} is the probability of the economy remaining in recession.

Let $s_t = \{\beta_t\}$ be the shock at period t , which have finite realizations and finite support $\mathcal{S} := \{\beta^H, \beta^L\}$. The history of shocks up to time t is thus $s^t = (s_0, s_1, \dots, s_t)$ for given initial shocks s_0 . The probability of each of these histories is given by $\pi(s^t)$. Agents in the economy make decisions after observing shocks s_t and having full information of history s^t and $\pi(s^t)$.

⁷It is also called preference shocks or aggregate demand shocks in the literature, say Nakata (2014), and Burgert and Schmidt (2014) and many others. Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011) view it as standing in for a wide variety of factors that alter households' propensity to save, for example, financial and uncertainty shocks.

2.1 Households

The representative household is to maximize its lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^t \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \frac{l(s^t)^{1+\chi}}{1+\chi} \right\} \quad (1)$$

subject to

$$c(s^t) + \frac{B(s^t)}{P(s^t)} = w(s^t)l(s^t) + R(s^{t-1})\frac{B(s^{t-1})}{P(s^t)} + \tau(s^t) + d(s^t) \quad (2)$$

where $w(s^t)$ is the real wage, $R(s^{t-1})$ the nominal interest rate from period $t-1$ to t , $\tau(s^t)$ a real lump-sum transfer or tax, and $d(s^t)$ real profits of the firms in the economy. The household chooses labor to supply ($l(s^t)$), nominal bonds to buy ($B(s^t)$) and goods to consume ($c(s^t)$) to maximize its lifetime utility. The optimality conditions for the household are:

$$\frac{1}{c(s^t)} = \beta_t R(s^t) E_t \left\{ \frac{1}{c(s^{t+1})} \frac{1}{\Pi(s^{t+1})} \right\} \quad (3)$$

$$w(s^t) = l(s^t)^\chi c(s^t) \quad (4)$$

where $\Pi(s^t) = P(s^t)/P(s^{t-1})$ is the inflation from period $t-1$ to t .

2.2 Firms

The final good producer is to maximize its profits by solving the following problem:

$$\max_{y_{it}} P(s^t)y(s^t) - \int_0^1 P_i(s^t)y_i(s^t)di$$

subject to

$$y(s^t) = \left(\int_0^1 y_i(s^t)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$$

where ϵ is the elasticity of substitution among intermediate goods. Given the prices of final and intermediate goods, the demand for intermediate good i is:

$$y_i(s^t) = \left(\frac{P_i(s^t)}{P(s^t)} \right)^{-\epsilon} y(s^t) \quad (5)$$

The intermediate goods producer has a linear technology to produce with labor as its only input, i.e., $y_i(s^t) = l_i(s^t)$. It maximizes discounted profits by paying a cost to adjust its price, that is,

$$\max_{P_{it}} \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^t \beta(s^i) \right) \pi(s^t) \left\{ \frac{\lambda(s^t)}{\lambda(s^0)} d_i(s^t) \right\}$$

subject to its demand function (5), and

$$d_i(s^t) = \frac{P_i(s^t)y_i(s^t)}{P(s^t)} - (1 - \xi)w(s^t)l_i(s^t) - \frac{\phi}{2} \left\{ \frac{P_i(s^t)}{P_i(s^{t-1})} - 1 \right\}^2 y(s^t)$$

which is the dividend in period t . The coefficient of the quadratic term ϕ captures how costly to adjust prices and $\lambda(s^t)$ is the Lagrangian multiplier for the household at date t . Note that ξ is a subsidy to eliminate steady state distortion due to monopolistic pricing. After solving the problem and using symmetry conditions, the behavior of firm sector can be summarized by the following equation:

$$\left[(\epsilon - 1) - (1 - \xi)\epsilon w(s^t) + \phi (\Pi(s^t) - 1) \Pi(s^t) \right] \frac{l(s^t)}{c(s^t)} = \beta_t E_t [\phi (\Pi(s^{t+1}) - 1) \Pi(s^{t+1})] \frac{l(s^{t+1})}{c(s^{t+1})} \quad (6)$$

Equation (6) states that the marginal cost of adjusting prices (LHS) must equate the marginal benefit (RHS).⁸

2.3 The Central Bank

It is assumed that the central bank is benevolent and chooses the nominal interest rate $\{R_t\}_{t=0}^{\infty}$ to maximize households' lifetime utility (1). However, the central bank cannot reduce the rate below 1. In other words, the net nominal interest rate is bounded below by zero. After all, people can hold cash, which of course pays no interest, rather than lend money out at a negative rate of return (Williams, 2013). At each period after the central bank sets the rate, households then make their decisions about consumption and leisure, and firms set their new prices. In the next section, I will detail the credibility of this policy rate.

⁸The higher the marginal benefit of adjusting price, the higher cost of inflation, and higher inflation.

2.4 Market Clearing

Finally, the goods market clears:

$$c(s^t) = \left(1 - \frac{\phi}{2}(\Pi(s^t) - 1)^2\right) l(s^t) \quad (7)$$

2.5 Competitive Equilibrium

Definition 2.1 (Competitive Equilibrium) *Suppose the economy starts with $\Upsilon\{s_0, R_0\}$, a competitive equilibrium (CE) for $\Upsilon\{s_0, R_0\}$ is characterized by a state-contingent sequence $(c(s^t), l(s^t), w(s^t), \Pi(s^t))$ such that, for all $t \geq 1$, $s^t \in \mathbf{S}^t$, and $R(s^t) \geq 1$ and equations (3) - (4), (6) and (7) hold.*

Remarks: Given the series of R , the households and firms behave optimally subject to resource constraint.

3 Full Discretion and Full Commitment

From this section on, I present different definitions of equilibrium.

3.1 Ramsey Equilibrium (Ramsey)

Definition 3.1 *A Ramsey equilibrium is where the central bank at time 0 instructs all the policies of future depending on the possible shocks to maximize the lifetime utility of the household (1). Namely,*

$$\max_{\{c(s^t), l(s^t), w(s^t), \Pi(s^t), R(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^t \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \frac{l(s^t)^{1+\chi}}{1+\chi} \right\}$$

subject to (3) - (4), (6) and (7) and $R(s^t) \geq 1$ for $\forall t$.

Denote the maximized lifetime utility of Ramsey planner at period 0 as $V^{RAM}(s^0)$ and discounted utility at period t as $V^{RAM}(s^t)$. By definition, the Ramsey equilibrium delivers the highest lifetime utility at time 0. However, it does not guarantee that for any given period $t > 1$, the discounted utility $V^{RAM}(s^t)$ coincides with the one if the central bank is given the chance to

re-optimize. This time-inconsistency comes from the fact that the central bank has strong incentive to close saving and inflation gaps that are promised earlier to boost the economy due to the zero lower bound constraint. I will detail this in Section 7.

To solve the Ramsey equilibrium of this model, I follow Marcet and Marimon (2011) and Adam and Billi (2006) and first write the Ramsey problem recursively with the introduction of two Lagrangians for the two Euler equations of households and the firms. Time iteration method is then applied on Karush-Kuhn-Tucker conditions to get policy functions, with the transformation of the ZLB constraint following Dong (2012). Linear interpolation is used to approximate values not on the grids pre-assigned.

3.2 Markov Perfect Equilibria (MPE)

Definition 3.2 *A MPE is the case where the central bank at any time t maximizes agent's lifetime utility (1) that period on by choosing consumption ($c(s_t)$), labor ($l(s_t)$), wage ($w(s_t)$), inflation ($\Pi(s_t)$) and interest rate ($R(s_t)$) subject to conditions (3) - (4), (6) and (7), and non-negativity of the nominal interest rate, taking as given the behavior of the future central bank and households' expectations.*

Denote $V^{MPE}(s)$ the discounted lifetime utility with state s . The Bellman equation is:

$$V^{MPE}(s) = \max_{c,l,w,\Pi,R} \log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta \mathbb{E}[V^{MPE}(s')|s]$$

subject to

$$\begin{aligned} \frac{1}{c} &= R\beta \mathbb{E} \left[\frac{1}{c'\Pi'} \right] \\ w &= l^\chi c \\ c &= \left[1 - \frac{\phi}{2}(\Pi - 1)^2 \right] l \\ [(\epsilon - 1) - (1 - \xi)\epsilon w + \phi(\Pi - 1)\Pi] \frac{l}{c} &= \beta \mathbb{E} \left[\phi(\Pi' - 1) \Pi' \frac{l'}{c'} \right] \\ R &\geq 1 \end{aligned}$$

The solution of the MPE is characterized by a sequence of time-invariant value function and policy functions of consumption, labor, wage, inflation and interest rate, i.e., $\{c(s_t), l(s_t), w(s_t), \Pi(s_t), R(s_t), V^{MPE}(s_t)\}$. Since the discretionary central bank re-optimizes every period, MPE is time-consistent. Value function iteration is then used to solve this equilibrium. A similar method can be found in Bodenstein et al. (2012).

4 Sequential Sustainable Equilibria (SSE)

The game is played between households and the central bank.⁹ Denote $\Gamma(s_0)$ the game where the economy starts with s_0 . The public history of the game is $\zeta^t = (\zeta_0, \zeta_1, \dots, \zeta_t)$, where $\zeta_t = (c_t, l_t, w_t, \Pi_t, R_t, s_t)$. Let σ_H be the strategy of households and σ_B that of the central bank. Both are measurable functions. Strategy σ_B maps publicly observed history ζ^{t-1} and the current shock s_t into interest rate for date-event s^t , namely, $R(s^t) = \sigma_B(\zeta^{t-1}, s_t)$. Similarly, strategy σ_H specifies $c(s^t), w(s^t), l(s^t), \Pi(s^t)$ as functions of expanded history $(\zeta^{t-1}, s_t, R(s^t))$; that is, $(c(s^t), w(s^t), l(s^t), \Pi(s^t)) = \sigma_H(\zeta^{t-1}, s_t, R(s^t))$. Further, I use $\Sigma(s_0) = \Sigma_H(s_0) \times \Sigma_B(s_0)$ to denote the set of all symmetric strategy profiles for $\Gamma(s_0)$, where $\Sigma_H(s_0)$ represents the set of strategies for households, and $\Sigma_B(s_0)$ the set of strategies for the central banks. The value of a strategy $\sigma = (\sigma_H, \sigma_B)$ for the central bank is defined as:

$$\Phi_B(s_0, \sigma) = \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^t \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \frac{l(s^t)^{1+\chi}}{1+\chi} \right\} \quad (8)$$

Definition 4.1 A strategy profile σ of the game $\Gamma(s_0)$ is an **SSE** if for any $t \geq 0$ and history ζ^{t-1} :

1. $\Phi_B(s_t, \sigma|_{\zeta^{t-1}}) \geq \Phi_B(s_t, (\sigma_H|_{\zeta^{t-1}}, \gamma))$ for any strategy γ in $\Sigma_B(s_t)$ for the central bank;
2. $\{c(s^j), l(s^j), w(s^j), \Pi(s^j)\}_{j=t}^{\infty}$ is a CE for $\Gamma\{s_t, R_{s^t}\}$, where $R_{s^t} := \{R(s^j)\}_{j=t}^{\infty}$, $R(s^t) \in \sigma_B(\zeta^{t-1}, s_t)$, and $(c(s^t), l(s^t), w(s^t), \Pi(s^t)) \in \sigma_H(\zeta^{t-1}, s_t, R(s^t))$.

In line with Phelan and Stacchetti (2001), $\sigma|_{\zeta^{t-1}}$ denotes the strategy profile in SSE with history ζ^{t-1} , and $(\sigma_H|_{\zeta^{t-1}}, \gamma)$ the strategy profile in which the household plays a SSE strategy under history ζ^{t-1} while the central bank plays an alternative one. The first condition above says that the

⁹The firms are owned by households.

continuation payoff for the central bank's strategy σ_B is better than that from any deviation to a different strategy. The second condition requires that the household always responds to a central bank strategy with decisions that imply a CE since this is the situation that is compatible with feasibility and optimality.

4.1 Recursive Formulation of SSE

Following Feng (forthcoming) and others,¹⁰ define

$$m_1(s^t) = \frac{1}{c(s^t)\Pi(s^t)} \quad (9)$$

$$m_2(s^t) = \phi(\Pi(s^t) - 1)\Pi(s^t)\frac{l(s^t)}{c(s^t)}. \quad (10)$$

These two quantities represent, in period t , the expected derivatives of the household's lifetime discounted utility from period $t + 1$ on with respect to $B_t R/P_t$ and P_{it}/P_t , respectively.¹¹ Using the market clearing condition, equation (10) can be used to calculate Π as a function of m_2 :¹²

$$\Pi = \frac{\phi(1 + m_2) + \sqrt{2\phi m_2^2 + 4\phi m_2 + \phi^2}}{\phi(2 + m_2)}$$

With (9), consumption can be backed out as follows:

$$c = \frac{1}{m_1\Pi}$$

Therefore, m_1 and m_2 are also referred to as expected or promised consumption and inflation. For any s^t , $m_1(s^{t+1})$ and an arbitrary specified interest rate R , households solve the following problem:

$$\max_{c(s^t), l(s^t), B(s^t)} \log(c(s^t)) - \frac{l(s^t)^{1+\chi}}{1+\chi} + \beta_t E_t m_1(s^{t+1}) \frac{B(s^t)}{P(s^t)} R \quad (11)$$

¹⁰Chang (1998) and Phelan and Stacchetti (2001) show that, though in different model setups, equilibria can be characterized in terms of their value to the government and their marginal value of private variables. See Kydland and Prescott etc for the reason and justification of doing so and how to explain it.

¹¹It is important to emphasize that each household is atomistic and an expectation taker.

¹²The other root is $\Pi = \frac{\phi(1+m_2) - \sqrt{2\phi m_2^2 + 4\phi m_2 + \phi^2}}{\phi(2+m_2)}$, the limit of which is $1 - \sqrt{2/\phi}$ as m_2 goes to $+\infty$ and $1 + \sqrt{2/\phi}$ as m_2 goes to $-\infty$. Since this root never visits the range of $[1 - \sqrt{2/\phi}, 1 + \sqrt{2/\phi}]$, and violates the Friedman rule for reasonable values of ϕ , I ignore always this case.

subject to the budget constraint (2). By construction, the recursive problem is equivalent to the sequential problem provided that the transversality condition is satisfied:

$$\lim_{t \rightarrow \infty} \sum_{t=0}^{\infty} E_t \left(\prod_{i=0}^t \beta(s^i) \right) m_1(s^t) \frac{B(s^t)}{P(s^t)} R = 0 \quad (12)$$

This is shown in the following proposition, which is an extension of the results in Feng (forthcoming) and Phelan and Stacchetti (2001).

Proposition 4.2 *Assume $R \in [1, \bar{R}]$ and $l \in [0, \bar{l}]$, where $\bar{R} < +\infty$ and $\bar{l} < +\infty$. Given the functional forms of preference and production function, the recursive and sequential problems are equivalent.*

Proof. See Appendix. ■

The firms also solve a recursive problem appropriately given $m_2(s^{t+1})$.¹³ Now I define a static CE using the two variables introduced.

Definition 4.3 *Let $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ be the static economy in which the current shock is s , the current interest rate set by the central bank is R , and agents have expectations about the future summarized in $\{m_1^+, m_2^+\}$. (c, l, w, Π) is a CE for $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ if and only if the following conditions are satisfied:*

$$\frac{1}{c(s)} = R\beta E\{m_1^+\} \quad (13)$$

$$w(s) = l(s)^\chi c(s) \quad (14)$$

$$c(s) = \left(1 - \frac{\phi}{2}(\Pi(s) - 1)^2\right) l(s) \quad (15)$$

$$\left[(\epsilon - 1) - (1 - \xi)\epsilon \frac{w(s)}{A} + \phi(\Pi(s) - 1)\Pi(s) \right] \frac{l(s)}{c(s)} = \beta E m_2^+ \quad (16)$$

I denote this equilibrium as $(c, l, w, \Pi) \in \mathbf{CE}^S\{s, R, \{m_1^+, m_2^+\}\}$.

The following lemma allows us to think of the original economy as a sequence of static economies with endogenously changing state variables and exogenous stochastic shocks.

¹³See Appendix.

Lemma 4.4 *Given a feasible interest rate policy $R = \{R_t\}_{t=0}^\infty$, suppose that the sequence $\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\}_{t=0}^\infty$ is such that for each t ,*

$$\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\} \in \mathbf{CE}^S\{s_t, R_t, \{m_1(s^{t+1}), m_2(s^{t+1})\}\}$$

where

$$m_1(s^{t+1}) = \frac{1}{c(s^{t+1})\Pi(s^{t+1})} \quad (17)$$

$$m_2(s^{t+1}) = \phi(\Pi(s^{t+1}) - 1)\Pi(s^{t+1})\frac{l(s^{t+1})}{c(s^{t+1})} \quad (18)$$

then $\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\}_{t=0}^\infty$ constitutes a competitive equilibrium for $\Upsilon\{s_0, R_0\}$.

Proof. See Appendix. ■

The lemma says that the promised marginal value of investment in bonds and the promised cost for adjusting prices, will summarize the expectation of households. Let h denote the equilibrium continuation payoff of the central bank Φ_B defined by (8). The equilibria of the economy can be characterized by:

$$V(s) := \{(m_1, m_2, h) \mid \sigma \text{ is a SSE for } \Gamma(s)\}$$

which is a mapping from the values of the states s into set of possible payoffs associated with a strategy profile σ that constitutes a SSE.

4.2 Credible Plans

To recursively characterize $V(s)$, I first introduce two definitions that lead to credible plans.

Definition 4.5 (Consistency) *Let $W : S \rightarrow R^3$ denote the set of all equilibrium payoffs. A vector $\psi = (R, c, l, w, \Pi, \{m_1^+, m_2^+\})$ is consistent wrt W at s if*

$$(c, l, w, \Pi) \in \mathbf{CE}^S(s, R, \{m_1^+, m_2^+\})$$

for $(m_1(s, \psi), m_2(s, \psi), h(s, \psi)) \in W(s)$, and $(m_1^+, m_2^+, h^+) \in W(s^+)$, where the values of m_1, m_2

and h are given by

$$\begin{aligned} m_1(s, \psi) &= \frac{1}{c\Pi} \\ m_2(s, \psi) &= \phi(\Pi - 1)\Pi \frac{l}{c} \\ h(s, \psi) &= \log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta E h^+. \end{aligned}$$

Definition 4.6 (Admissibility) *The vector ψ is admissible wrt W if it is consistent wrt W at s and*

$$h(s, \psi) \geq h(s, \psi')$$

for any other consistent ψ' .

Consistency guarantees that the vector ψ delivers an allocation that is optimal for households and feasible. In addition, it requires that the promised continuation values (m_1^+, m_2^+, h^+) belong to the same equilibrium set as those of (m_1, m_2, h) . Admissibility says that the interest rate set by the central bank is optimal and it has no incentive to deviate. That is, the central bank cannot increase its payoff by setting a different interest rate R' . A credible plan thus is the strategy of households and central banks instructed by an admissible ψ .

With these two definitions, I define an operator \mathbb{B} with its fixed point being the set of equilibrium values V as follows:

For a given set of equilibrium values W ,

$$\mathbb{B}(W)(s) = \{(m_1, m_2, h) \mid \psi \text{ admissible wrt } W \text{ at } s\}$$

The interpretation of the operator and the constraints is as follows. $\mathbb{B}(\cdot)$ is the convex hull of the payoffs (m_1, m_2, h) such that there are associated values of consumption, labor supply, wage, inflation and government policy rates and next period payoffs that belong to the value correspondence W for every possible realization of the shock compatible with the current state and that satisfy certain conditions.

Following Phelan and Stacchetti (2001) and Abreu et al. (1986, 1990) (APS, henceforth), the operator \mathbb{B} has properties as stated in the following proposition:

Proposition 4.7 *The operator \mathbb{B} has the following properties:*

- 1. *If $W \subseteq \mathbb{B}(W)$, then $\mathbb{B}(W) \subseteq V$;*
- 2. *V is compact and the largest set of equilibrium values W such that $W = \mathbb{B}(W)$;*
- 3. *$\mathbb{B}(\cdot)$ is monotone and preserves compactness;*
- 4. *If we define $W_{n+1} = \mathbb{B}(W_n)$ for all $n \geq 0$, and the equilibrium value correspondence $V \subset W_0$, then $\lim_{n \rightarrow \infty} W = V$.*

Proof. See Appendix. ■

With this proposition, \mathbb{B} is calculated numerically as follows:

$\mathbb{B}(W)(s) = \{(m_1, m_2, h) | \exists R, (c, l, w, \Pi), \text{ and } (m_1^+, m_2^+, h^+) \in W(s) \text{ for all } s^+ \succ s\}$ such that

$$m_1 = \frac{1}{c\Pi} \quad (19)$$

$$m_2 = \phi(\Pi - 1)\Pi \frac{l}{c} \quad (20)$$

$$h = \log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta E h^+ \quad (21)$$

$$(m_1, m_2, h) \in W(s) \quad (22)$$

$$h \geq [u(c', l') + \beta E h^{+'}] (m_1^{+'}, m_2^{+'}, h^{+'}), \forall (m_1^{+'}, m_2^{+'}, h^{+'}) \in W(s^+) \quad (23)$$

$$1/c = R\beta E \{m_1^+\} \quad (24)$$

$$w = l^\chi c \quad (25)$$

$$c = (1 - \frac{\phi}{2}(\Pi - 1)^2)l \quad (26)$$

$$[(\epsilon - 1) - (1 - \xi)\epsilon w + \phi(\Pi - 1)\Pi] \frac{l}{c} = \beta E m_2^+ \quad (27)$$

$$R \geq 1 \quad (28)$$

where $s^+ \succ s$ denotes all possible shocks that follow s . Constraints (19) to (22) are called "regeneration constraints", while (23) is an "incentive constraint". Constraints (24) to (28) are necessary to ensure that continuation of a sustainable plan after any deviation is consistent with a CE. Following

Feng (forthcoming) and Chang (1998), I replace (23) with the following condition,

$$h \geq \tilde{h}(s) \tag{29}$$

where $\tilde{h}(s)$ is the best possible payoff for the central bank when it announces unexpected interest rate R' . In particular, $\tilde{h}(s)$ is defined as

$$\tilde{h}(s) = \max_R \left\{ \min_{\substack{c, l, w, \Pi, \\ (m_1^+, m_2^+, h^+) \in W(s^+)}} [\log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta E h^+] \right\}$$

such that

$$(c, l, w, \Pi) \in CE^S \{s, R, \Pi, \{m_1^+, m_2^+\}, \forall s^+ \succ s\}$$

The idea of replacing (23) with (29) is that: (1) if households punish the central bank for deviations from the claimed policy R , they will punish the latter as worse as available; (2) if the central bank knows the response of households, it will pick the best as long as it decides to deviate. It can be shown that condition (29) is equivalent to (23) in the sense of leading to the same fixed point V by applying the operator \mathbb{B} .

Following Feng (forthcoming), the whole equilibrium set can be characterized by the upper and lower boundaries of $W(s)$, which are:

$$\begin{aligned} \bar{h}(s, m_1, m_2) &= \max_h \{h | (m_1, m_2, h) \in W(s)\} \\ \underline{h}(s, m_1, m_2) &= \min_h \{h | (m_1, m_2, h) \in W(s)\} \end{aligned}$$

I then define the outer approximation of W as follows:

$$\hat{W}(s) = \{(m_1, m_2, h) | h \in [\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]\}$$

Proposition 4.8 For all $(m_1, m_2, h) \in V(s)$,

$$\begin{aligned}\bar{h}(s, m_1, m_2) &= \max_R \{u(c, l) + \beta E \bar{h}(s', m'_1, m'_2)\} \\ \underline{h}(s, m_1, m_2) &= \max_R \{u(c, l) + \beta E \underline{h}(s', m'_1, m'_2)\} \\ \tilde{h}(s) &= \min_{m_1, m_2} \underline{h}(s, m_1, m_2)\end{aligned}$$

subject to the constraint $(c, l, w, \Pi) \in \mathbf{CE}^S\{s, R, m'_1, m'_2\}$ for all $s' \succ s$.

Proof. See Appendix. ■

The task now is to find a new operator \mathbb{F} based on \hat{W} :

Definition 4.9 (The operator \mathbb{F}) For any convex-valued correspondence \hat{W} ,

$$\mathbb{F}(\hat{W})(s) = \{(m_1, m_2, h) | h \in [\underline{h}^1, \bar{h}^1]\}$$

where

$$\begin{aligned}\bar{h}^1 &= \max_{m'_1, m'_2} \{u(c, l) + \beta E \bar{h}^0\} \\ \underline{h}^1 &= \max_{m'_1, m'_2} \{ \max_{m'_1, m'_2} u(c, l) + \beta E \underline{h}^0, \tilde{h}^0 \} \\ \tilde{h}^0 &= \max_{m'_1, m'_2} \{ \min_{m'_1, m'_2} u(c, l) + \beta E \underline{h}^0 \}\end{aligned}$$

such that the vector $(R, c, w, l, \Pi, (m'_1, m'_2, h'))$ is admissible with respect to \hat{W} at s . Define $\underline{h}(s, m_1, m_2) = -\infty$ and $\hat{h}(s, m_1, m_2) = +\infty$ if no such vector exists.

The following theorem shows that this operator has good convergence properties and repeated application of this operator generates a sequence of sets that converge to the equilibrium value correspondence V . The details of the algorithm are postponed in Appendix.

Theorem 4.10 Let \hat{W}_0 be a convex-valued correspondence such that $\hat{W}_0 \supset V$. Let $\hat{W}_n = \mathbb{F}(\hat{W}_{n-1})$. Then $\lim_{n \rightarrow \infty} \hat{W}_n = V$.

Proof. See Appendix. ■

4.3 Recovering Strategies

This subsection shows how to find the strategy that supports the best SSE in deterministic case. The procedure here can be generalized to find strategies supporting any point belonging to the equilibrium value correspondence.

- Step 1: At $t = 0$, find the highest possible value of $h_0 = \sup\{h | (m_{1,0}, m_{2,0}, h_0) \in W^*(s)\}$ and its corresponding $(m_{1,0}, m_{2,0})$. Then search for the central bank's interest rate policy that supports $(m_{1,0}, m_{2,0}, h_0)$, that is, pick R_0 such that

$$u(c_0, l_0) + \beta h_1 = h_0$$

where $h_1 = \bar{h}(m_{1,1}, m_{2,1})$, $m_{1,1} = \frac{u_{\epsilon,0}}{R\beta}$, $m_{2,1} = [(\epsilon - 1) - (1 - \xi)\epsilon w_0 + \phi(\Pi_0 - 1)\Pi_0] l_0 / c_0 / \beta$, and $(m_{1,1}, m_{2,1}, h_1) \in W^*(s)$. Given $(R_0, m_{1,0}, m_{2,0})$, c_0 , l_0 , w_0 , Π_0 can be calculated via definitions of m_1 (9) and m_2 (10), optimality condition of leisure (4), and market clearing condition (7) as:

$$\Pi_0 = \frac{\phi(1 + m_{2,0}) + \sqrt{2\phi m_{2,0}^2 + 4\phi m_{2,0} + \phi^2}}{\phi(2 + m_{2,0})} \quad (30)$$

$$c_0 = \frac{1}{m_{1,0}\Pi_0} \quad (31)$$

$$l_0 = c_0 \left(1 - \frac{\phi}{2}(\Pi_0 - 1)^2\right)^{-1} \quad (32)$$

$$w_0 = l_0^\chi c_0 \quad (33)$$

Therefore, the above problem is well-defined in terms of $(R_0, m_{1,0}, m_{2,0}, h_0)$.

- Step 2: $t = 1$, $m_{1,1}$, $m_{2,1}$, h_1 are given by the solution in step 1. Now search for the central bank's policy R such that

$$u(c_1, l_1) + \beta h_2 = h_1$$

as in step 1.

- Step 3: Repeat step 2 for $t = 2, \dots, T$, for T sufficiently large.

5 Optimal Monetary Policies Without the ZLB

In this section, I will first calibrate the model and then present the solution in the deterministic case and finally the stochastic case without the ZLB. For each case, the solutions are compared with those from Ramsey and MPE. By doing so, it serves as a first step to understand the role of credibility/reputation in conducting monetary policies.

5.1 Calibration

The parameterization here follows mostly the literature. The Frisch elasticity is set to be 1. The elasticity of substitution among intermediate goods is set to be 6. The price adjustment cost ϕ is set to 60 to have a slope of 1/6 for the Phillips curve. There are four parameters to calibrate for the Markov process of the discount factor β . The value of β^L is set to 0.994 to have an average of 2.5% real annual rate during normal times and β^H to 1.011% to have a natural rate of -4.5% for recession times. Setting $p_{HH} = 2/3$ is to have an average of 3 quarters duration of the economy at high shock states as estimated by Kulish et al. (2014). In addition, Williams (2013) reports that expectations from financial markets consistently showed the federal funds rate lifting off from zero within just a few quarters from 2009 to mid-2011. $1 - p_{LL}$ is the frequency of the economy entering the high shock state when the current state is low and set to be 0.0068, the chance of 1 quarter out of 12 years. These two values are within the ranges used by Nakata (2014).¹⁴

Table 1 is here!

5.2 Deterministic Case

Figure 1 shows the whole equilibrium set of the SSE for the benchmark case. The x - and y -axes are the two auxiliary quantities I introduced above, m_1 and m_2 . The z -axis is the payoff to the central bank, h . First, any point within the hill-shape area is an equilibrium. Second, given (m_1, m_2) , there are infinitely many payoffs to the central bank h ranging from $\underline{h}(m_1, m_2)$ to $\bar{h}(m_1, m_2)$. This paper focuses on mostly the best SSE, $\bar{h}(m_1, m_2)$. Third, there are in general infinitely many combinations of (m_1, m_2) that lead to a certain level of payoff \bar{h} . This implies that the central bank can have multiple (and possibly infinitely many) choices of (m_1, m_2) and hence nominal interest

¹⁴The unconditional probability of recession states is $\frac{1 - p_{LL}}{2 - p_{HH} - p_{LL}}$.

rates R to induce a certain equilibria path. Fourth, the shape of the state space also implies that there is a trade-off between m_1 and m_2 . To support a certain same level of payoff, the decrease of m_1 should be accompanied by an increase of m_2 . Since higher m_2 implies higher inflation, and lower m_1 implies lower marginal return of saving, this trade-off means that to achieve the same level of payoff, households have a trade-off between higher inflation and lower return on saving.¹⁵ Finally, given the same level of promised investment return (m_1) or promised inflation (Π), there are in general two levels of the other generating same level of the best SSE. Take the example of $m_1 = 1$. The best SSE \bar{h} supported by deflation (low m_2) can also be supported by inflation (high m_2).

The globally best SSE is the one corresponds to highest payoff h , the top of the hill in Figure 1. The first observation is that the globally best SSE features $(m_1, m_2) = (1, 0)$, of which $m_2 = 0$ means zero inflation. This is also shown in Figure 2, which is the cross-section of the equilibrium set as a function of m_2 when fixing $m_1 = 1$. The maximum h is achieved when $m_2 = 0$. The second observation is that only the globally best SSE is a steady state that can be supported, whereas all other equilibria are not. These two then imply a steady state of prices and allocation that coincides with MPE and Ramsey equilibria, that is, $\Pi = 1$, $c = l = 1$, $w = 1$, and $R = 1/\beta$.

Figure 2 is here!

To better and more straightforwardly show the idea, I will use promised consumption and inflation instead of (m_1, m_2) from now on. Figure 3 shows the supported state space of (c, Π) , which is the bottom plane of the whole set in Figure 1 with appropriate transformation. The area inside the solid line is the space with the ZLB and the one inside the dashed line is the space without the ZLB. It is straightforward to see that the supported state space shrinks due to the ZLB. Given m_1 , the lower bound of m_2 that can be supported shifts up. Figure 4 is cross-section of the equilibria set as a correspondence to m_2 fixing $m_1 = 1$. Therefore, with the ZLB, the public won't believe a lower inflation rate policy that can be achieved in the absence of the ZLB. However, the best SSE and hence the efficient allocation can always be rational and believed by the two agents.

Figure 3 is here!

¹⁵The return on saving of course also depends on inflation. However, higher inflation also leads to higher adjustment cost and thus more resource lost due to price adjustment.

Figure 4 is here!

Many central banks with implicit or explicit inflation targeting have a positive targeted level of inflation in practice. In the US, for example, this target is 2% annually. One justification of this positive target is the inflation bias due to monopolistic pricing distortion under full discretion solution (see Gertler et al. (1999) for example).¹⁶ This bias is at the cost of lower consumption, which leads to lower lifetime utility. A central bank with reputation can do better. In the benchmark, the parameter that governs subsidy ξ is set to completely eliminate steady state distortion due to monopolistic pricing. Figure 5 compares the equilibria set when ξ is reduced to 88% of the benchmark level. Not surprisingly, the absence of full fiscal subsidy leads the central bank to pursue in general higher inflation to eliminate the distortion and households know this, and thus equilibria associated with lower inflation in the presence of full subsidy are no long supported. However, the inflation bias associated with the globally best SSE is much smaller. The globally best SSE now features $m_1 = 1.0057$ and $m_2 = 0.0821$. The positive m_2 corresponds to an inflation of 0.52% annual inflation at steady state. In contrast, the implied annual rate from MPE is 0.94% . In other words, the central bank with reputation can have a much smaller target if it uses inflation to eliminate price distortion due to lack of fiscal subsidy. Table 2 shows the implied inflation target as a function of fiscal subsidy.

Table 2 is here!

Figure 5 is here!

As discussed in the literature, the discount factor is the key to determine the equilibria set and the implementability of Ramsey equilibrium. Given everything else equal, higher discount factor leads to higher expected utility. The cost to lose reputation will thus be higher, which will prevent the central bank from deviating. Therefore, higher discount factor will result in a bigger supported equilibrium set. However, higher discount factor also means a more severe recession, which lowers the expected lifetime utility. This makes the current instant utility more important and hence encourages the central bank to deviate. This will lead to a smaller equilibrium set. It turns out the former effect dominates given the model framework and parameterization. Figure 6 shows the cross-sections of equilibrium sets for low and high β .¹⁷ And indeed, the set associated with high β is bigger, though insignificantly. Note also that the globally best SSE with efficient allocation is

¹⁶In the case of Ramsey equilibrium, inflation bias is always zero.

¹⁷The level of utility is adjusted to make the globally best payoffs the same.

always supported.

Figure 6 is here!

5.3 Stochastic Case

The equilibria set for the stochastic case is a combination of two sets for the low and high β as in the subsection above. The globally best SSE features efficient allocation for any period t and state s . That is, $c_t = l_t = \Pi_t = w_t = 1$ and the associated policy rate and promised values are $R_t = 1/\beta_t$ and $(m_{1t}, m_{2t}) = (1, 0)$ respectively for any t . However, if the central bank picks a slightly different equilibrium at the beginning, the dynamics could be very different.¹⁸

To show this, I start with an equilibrium with promised payoffs $(m_1, m_2) = (0.9996, 1.46 \times 10^{-5})$ at low shock state. The economy is hit by a high discount factor shock at period 0. After period 0, this shock is always in low state. The dynamics after the shock is shown in Figure 7.

Figure 7 is here!

There are several things to pay attention to when compared to the MPE and Ramsey equilibria. First, it takes about 20 years for the consumption and inflation to return their normal levels while in the cases of MPE and Ramsey, those two never change – the nominal rate is always set to $1/\beta_t$ and $c_t = l_t = \Pi_t = w_t = 1$. Second, there is a spike of inflation at the time of shock along with a much milder decrease of nominal interest rates. In the case of Ramsey and MPE, the nominal rate at the time of the contractionary shock is $1/\beta^H = 0.989$. Under the equilibrium I picked above, the rate is 1.004. The reason for this is that expected inflation path changes a lot as the central bank wants and on-impact rates do not have to adjust too much. This gives us the intuition of how optimal policy will work when the ZLB binds, which will be detailed in the following section.

6 Forward Guidance and Credible Monetary Policy

The clear message from discussions above is that when the nominal rate is flexible enough and not constrained by the ZLB, the inflation gap should always be reduced to its minimum zero. The key feature of the optimal policy is zero inflation with nominal (and real) interest rates equal to the inverse of discount factor. The minimum inflation is to reduce the intra-temporal distortion

¹⁸As stated in Rogoff (1987), there is no compelling argument as to why the economy will coordinate on a "good" equilibrium and not a "bad" one out of the continuum of reputational equilibria.

between consumption and leisure, while nominal interest rate is whatever needed to equalize the desirable real rate. In the following, let us call the inverse of discount factor the shadow policy rate, and the discrepancy between real rate and shadow policy rate the shadow policy rate gap, which is proportional to the consumption gap. Without the ZLB, this gap and inflation gap will always be zero. In the presence of the ZLB, there is a trade-off between these two and it is exactly where time-inconsistency comes from.

6.1 The Time-inconsistency of Commitment

To illustrate the issue of time-inconsistency, I assume a particular realization of shocks. Suppose that the high shock of β hits the economy at period t and is followed by low shocks thereafter. The shadow policy rate at period t is $1/\beta^H$, which is less than 1. The real interest rate $R_t/E\Pi_{t+1}$ should decrease to close the gap. In the case of MPE, the central bank today cannot affect inflation tomorrow, and therefore the gap will not be closed and consequently real rate is higher than the desirable rate. Households thus save more and consume less. This leads to a weak aggregate demand and deflation, opening the other inflation gap.

Under Ramsey, the central bank has another tool to close the shadow policy rate gap when R is bounded below by 1 – to increase expected inflation by promising to have low nominal rates no matter what happens in period $t + 1$. By doing so, aggregate demand at the current period will increase. This will in turn reduce inflation gap (that is, less deflation). The resulting consequence is increased consumption and less severe deflation.

The time-inconsistency comes at period $t + 1$.¹⁹ With a low shock, the shadow policy rate is $1/\beta^L$. If the central bank follows its promise and sets $R = 1$, then the central bank will have a shadow rate gap of $1/E\Pi_{t+2} - 1/\beta^L$, which is negative. Suppose the central bank can only promise one period, then $E\Pi_{t+2} = 1$ and the gap will be $1 - 1/\beta^L$. This negative gap leads to too high aggregate demand and positive inflation gap. These two gaps are costly to the central bank and they will definitely renege and close them by setting $R = 1/\beta^L$ and $\Pi = 1$ if high discount factor shock never hits the economy again. This is not consistent with what they promised in period t .

The key to the time-inconsistency is the temptation to costlessly close these two gaps when

¹⁹The most likely period for the central bank to fall back on its words is the last period of duration at the ZLB that Ramsey suggests. Here I just assume the central bank deviates the following period when high shock disappears for illustrating purpose.

shock is low. While Ramsey simply exclude the temptation to deviate, the definition of SSE fully takes into account this. The public in the economy basically think about two aspects of what the central bank say: the duration of low nominal rate (and thus implied expected inflation) and how low (high) the nominal interest rate (expected inflation) will be. The (infinitely possible) paths of policies will be a combination of these two.

6.2 Ramsey is Not Implementable

Before examining the optimal policy under the best SSE, it is useful to first show whether the Ramsey equilibria is implementable or not. The strategy is to calculate the corresponding promised values to the private sector and continuation payoff to the central bank as in SSE for the Ramsey case. Namely, for any given realized state, calculate m_1 , m_2 , and h and check if the triplet (m_1, m_2, h) is within the equilibria set of SSE. For the discussion now, assume that shocks materialize like the following: a high β shocks the economy at period 1, followed by low shocks thereafter. I first check the implementability of Ramsey by looking into only the promised values (m_1, m_2) . The associated values and the sustainable sets under SSE are drawn in Figure 8. The area circled by solid line is the sustainable equilibria set for low shock state and the one by dashed line is the set for high shock state. Small circles are the promised value combinations from Ramsey for low shock states. Before the high shock hits the economy, the promised payoffs are denoted by the filled circle. At the time of the high shock, promised payoffs go to the point denoted as star. After the shock, the promised bundle goes back to its original place clockwise as shown in the figure by arrows. The circles lie in the sustainable set for low state. However, at the state of high shock, the promised values, denoted by star, lie beyond the sustainable set for high state, the dashed line area. That is, the promised consumption and inflation bundle under Ramsey is not available. In contrast, for the same realization of shocks, the promised values always lie in the right state space, as shown in Figure 9. Therefore, the Ramsey is not implementable.

Figure 8 is here!

Figure 9 is here!

A second way to check the implementability is to see if the maximized lifetime utility from Ramsey is always higher than or equal to the continuation payoff from a credible plan following Nakata (2014). To be more specific, if the lowest utility from sticking to the Ramsey is at least

as high as the continuation payoff from deviating to a credible plan, then the Ramsey is credible and implementable. Figure 10 draws the lifetime utility of Ramsey for a group of given realizations of shocks and the continuation payoff from SSE if the central bank deviates. For example, the leftmost blue dashed line represents the value sequence of Ramsey equilibrium when the economy is hit by a high shock at period 1 and thereafter stays with low shocks; and the rightmost blue dashed line the case where the high shock remains in the economy for 10 quarters and disappears at the 11th quarter.

Three things should be noted. First, the deviation, if it happens, it will always happen in the first period the high shock disappears because this is the period when the consumption and inflation gaps are largest (see Figures 11 and 12). Second, the longer duration of high shocks, the larger the two gaps after the high shock disappears and hence the more likely the central bank to deviate. This is represented by the blue solid line. Therefore, if the central bank can get higher continued payoff by deviating when high shock hits the economy the first time, it will surely deviate if high shock continues to happen. Finally, however, there is a lower limit for the utility the central bank can get by following Ramsey, represented by the red dash-dotted line. And if the available and credible plan, which the central bank may switch to, cannot obtain higher utility than this limit, then the Ramsey is implementable. Given the baseline calibration, the Ramsey equilibrium is not implementable because the highest payoff under SSE, the red solid line, is higher than the lower limit. To be precise, as long as in the future, the high shock hits the economy successively more than 4 quarters, the central bank will deviate. Again, the Ramsey is not implementable.

The starkly different conclusion about implementability of Ramsey equilibria in contrast to Nakata (2014) largely depends on the credible plan we impose the central bank could deviate to. In his paper, he assumes that if the central bank deviates, households' expectations about the central bank's future policies will change and the policies will be those under MPE. However, MPE is not the best credible plan and indeed it is just an inferior one. In Figure 13, the payoff of this plan is drawn in black line. If the central bank deviates to the best SSE when the contractionary shock disappears, it will gain much higher payoffs. In other words, the existence of other better credible plan than MPE makes the central bank more likely to deviate and less credible. It turns out that as long as the time-inconsistency exists, that is, the ZLB effectively binds under Ramsey equilibrium, the central bank will always deviate (to the best SSE) and thus the Ramsey equilibrium is generally

not implementable. This echoes the argument of Phelan and Stacchetti (2001).²⁰

6.3 Credible Policies at the ZLB

In this subsection, I answer the question raised in the beginning: What does the best credible forward guidance look like? Or in other words, how does the policy rate look like if a central bank has highest reputation?

Since there exist infinitely many policy paths that support the same equilibrium, I use the strategy as described in the Appendix to find the policy path I am interested in. In particular, I find the path that features lowest interest rates when high shock starts to hit the economy.

Figure 14 shows the range of policy rates for a specific realization of shocks, where I assume that at time 1, the economy is hit by a high shock and this shock persists for 6 periods. The length of high shocks is to match the NBER definition of the recent financial recession (December 2007 - June 2009). The dashed line is the upper bound of the nominal rates and the dash-dotted one the lower bound. A first observation is that, under the best SSE, the zero lower bound never binds in recession times. The only chance it will binds is during the period when the high disappears.²² Second, the upper and lower bounds are not smooth. This is because these bounds are conditional on (m_1, m_2) , the promised values, which are moving around. Given the discussion in the last Section, there is no reason this pair of values change in a way to have smooth policy rates.

Figure 14 is here!

Among the many policy paths, I pick one that features lowest nominal rates when recession starts, which is represented by the blue solid line in Figure 15.²³ To compare, I also draw the policy rates under Ramsey and MPE in Figures 16 and 17, respectively.

Figure 15 is here!

Figure 16 is here!

Figure 17 is here!

²⁰This conclusion is robust to a wide range of parameter values. Keeping the baseline parameterization, I experiment with different values for one parameter each time. The experiments with different shock size (β^H), different length of high shock duration (p^{HH}), different frequency ($1 - p^{LL}$) and different price adjustment cost (ϕ) do not show the support of implementability of Ramsey.

²¹From discussions of last section, it is clear that the Ramsey without the ZLB effectively binding coincides with the BSSE, which means the Ramsey is implementable if either $\beta^H \leq 1$ or $1 - p^{LL} = 0$. But this is the case where time-inconsistency disappears.

²²It is true for any length of high shock periods.

²³It is, however, not the path with lowest nominal rates for any time during the recession.

First, when the high shock starts to hit the economy, the central bank with reputation will not react as aggressively as with those with full commitment or full discretion. Instead, it will set the interest rate only about half of that in normal times (1.2% compared to 2.5%). In general, the ZLB will not bind. The only binding case (the ZLB only binds for one period) under SSE is the one that delivers highest nominal rates during recession and accordingly has lowest nominal rates in normal times. Under Ramsey, though the recession ends at period 6, the nominal rate stays low another 4 quarters (3 quarters at zero and 1 quarter close to zero) even after the economy emerges from recession. The nominal rate then jumps to a higher-than-normal level to contain the inflation. Under the best SSE, the nominal rate stays low but non-zero for at least 8 quarters after the recession ends.

Figure 18 is here!

Figure 19 is here!

The intuition behind is simple. If the central bank sets the nominal interest rate too low, then it is not credible. The credible policy path must be relatively smooth in all states to reduce the incentive for the central bank to deviate. At the same time, the reputational central bank will have a longer time span than that under Ramsey to have lower rate to optimize.

Accordingly, the economy has higher and persistent inflation as shown in Figure 18. The inflation on average is higher than that under Ramsey (by 0.12% annually) for the recession time and several periods after the recession. In addition, it takes 3 quarters longer for the inflation to first revert to its normal level under SSE than under Ramsey. However, at normal times, the inflation level is lower than that under Ramsey. Figure 18 also shows that inflation during recession can be higher if it is also higher in normal times, with on average higher interest rates.

Finally, compared to a central bank with full discretion, the bank with reputation can do much better in boosting consumptions during recession, with an average gap of only -1.55% compared to -4.55% under MPE. After all, the consumption gap is still 0.5% bigger than that under Ramsey.

6.3.1 The Simple and Practical Interest Rate Rules

To make the optimal policy rules practical to use, I simulate the economy under the best SSE, Ramsey and MPE 1×10^6 times to get rules similar to Taylor. That is, if the central bank sets the interest rate based on inflation and output gaps, how should the Taylor rule look like? Under the

best SSE with the specific policy rate path picked above, this is,

$$r_t = 0.57 - 1.9336\pi_t + 0.1331y_t + error$$

Under Ramsey, this is

$$r_t = 0.61 - 6.4452\pi_t + 0.2589y_t + error$$

First, as shown in the subsection above, the average inflation under the best SSE is lower than that under Ramsey (0.57% vs 0.61%). However, the former could be higher than the latter if a different policy rate path is selected. Second, the interest rate should respond to inflation negatively under both Ramsey and the best SSE. This is because positive inflation is a sign of recession. Whenever inflation is high, the interest rate should decrease to boost the economy. This is different from what is usually believed as Taylor principle – the nominal rate should increase more than one-to-one to inflation to contain the latter. Finally, the interest rate is reacting more to inflation and output gaps under Ramsey than under the best SSE. Under Ramsey, the elasticity of nominal rates over inflation is about three times as big as that under the best SSE. That is, the central bank decreases roughly 3 times bigger than that under the best SSE for given increases of inflation. Similarly, the reaction of the nominal rate to output gap under Ramsey is about twice as big as that under the best SSE. In other words, if the central bank has full commitment, it can and should act more aggressively to adjust its policy rate in response to inflation and output gaps. By doing so, it will close output and inflation gaps as much as possible in as less as possible time.

The reason for the difference of the rules is the same as I explained above. The aggressive policy is subject to the non-credibility problem, though it is better in the sense of leading to higher welfare. The more aggressive the policy rate responses to inflation and output, the more likely it is non-credible. This should be a general caution for discussions on rule-based (i.e., Taylor rule, price targeting rule) forward guidance, and more generally on determinacy of New Keynesian models with different rules.

Under MPE, since the output gap and inflation gap is one-to-one mapping, it will get multicollinearity if the regression runs like above. Instead, I get the rule only based on inflation or output

as follows:

$$r_t = 0.60 + 0.8841\pi_t + error$$

$$r_t = 0.57 + 0.1355y_t + error$$

In contrast with the rules under Ramsey and the best SSE, the Taylor like rule is very simple: if inflation is negative (output is negative at the same time), decrease the nominal rate appropriately.

It should be emphasized that the rules here only apply to the case where discount factor is the only source of shocks. Since this represents the aggregate demand shock, the rules, in particular, the negative reaction of nominal rates to inflation is to boost the demand by lowering nominal rates.

7 Conclusion

This paper explores the optimal and credible policy, which is interpreted as forward guidance, via a standard New Keynesian model by first characterizing the whole set of SSE when the nominal interest rate occasionally binds at zero. In contrast to Ramsey equilibria and MPE, the best SSE features strong reliance on the expected inflation for a very long time. Forward guidance based on the best SSE states that the interest rate should stay low but non-zero for a prolonged time even when the economy recovers. It takes more than three quarters extra for the economy to return to its normal policy regime. This policy can close the output gap to -1.55%, as compared to 1.05% under full commitment, which is the best a constrained central bank can achieve.

As a theoretical result, I show quantitatively that the Ramsey equilibrium is not generally implementable. The stark difference from the general conclusions in the literature stems mainly from the options to which the government can choose to deviate. While most assume that the MPE is the alternative plan, the best credible plan to which the government can deviate achieves much higher payoffs than the MPE, which makes the Ramsey equilibrium more difficult to implement.

Finally, if the central bank intends to use a simple and practical rule to communicate, the Taylor-style rule says the nominal rate should react less aggressively to inflation and output changes. This rule can be used to estimate the effects of other policies that were used together with forward guidance.

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A Appendix

A.1 The Model with Implementable Policy Rates

In Section 2, I presented a standard New Keynesian model that features private bonds. And at equilibrium, the net position of bonds is always zero. This assumption is criticized for its non-implementability of policy rates, i.e., the central bank in this model has no way to realize the policy target it wants. Here I present a model similar to that in Section 2, but with government bonds and hence central banks’ implementation of policy rates through open market operation.

The representative household is to maximize same lifetime utility as in (1) subject to, however, a different budget constraint,

$$c(s^t) + \frac{B^g(s^t)}{P(s^t)} = w(s^t)l(s^t) + R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)} + \tau(s^t) + d(s^t) + \Delta m(s^t) \quad (34)$$

where $R(s^{t-1})$ is now interpreted as the nominal interest rate paid to government bonds $B^g(s^t)$. I now interpret $\frac{B^g(s^t)}{P(s^t)} - R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)}$ as the change of households’ monetary demand and $\Delta m(s^t)$ the real change of monetary supply. At each period, the demand and supply of money must be equalized, that is

$$\Delta m(s^t) = \frac{B^g(s^t)}{P(s^t)} - R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)} \quad (35)$$

Therefore, any policy target R_t can be implemented by changing $\Delta m(s^t)$ given the households’ bond holdings ($B^g(s^{t-1})$ and $B^g(s^t)$) and price ($P(s^t)$). This change of budget constraint does not change any optimality conditions or feasibility conditions. Note that though I introduce an extra state variable $B^g(s^t)$, it is irrelevant for the analysis of dynamics of the economy.

A.2 Firm's Dynamic Problem

This subsection shows that given the promised values about adjusting prices, the firm solves the same recursive problem as without it. The dynamic problem of an individual firm can be written as follows:

$$V(P_{it-1}) = \lambda_t d_{it} + \beta_t E_t V(P_{it})$$

subject to constraint (5) and the definition of dividend d_{it} . λ_t is the Lagrangian multiplier of households' budget constraint. Solving the problem gives the following Euler equation:

$$\left[(\epsilon - 1) \left(\frac{P_{it}}{P_t} \right)^{-\epsilon} - (1 - \xi) \epsilon w_t + \phi \left(\frac{P_{it}}{P_{it-1}} - 1 \right) \frac{P_t}{P_{it-1}} \right] y_t \lambda_t = \beta_t E_t \left[\left(\frac{P_{it+1}}{P_{it}} - 1 \right) \frac{P_{it+1} P_t}{P_{it}^2} \right] \phi y_{t+1} \lambda_{t+1}$$

Imposing symmetry gives the equation (6) in the text:

$$[(\epsilon - 1) - (1 - \xi) \epsilon w_t + \phi (\Pi_t - 1) \Pi_t] y_t \lambda_t = \beta_t E_t [(\Pi_{t+1} - 1) \Pi_{t+1}] \phi y_{t+1} \lambda_{t+1}$$

Now let $m_2^+ = [(\Pi_{t+1} - 1) \Pi_{t+1}] \phi y_{t+1} \lambda_{t+1}$ be the marginal value (payoff) of adjusting price relative to aggregate price. Then the firm's problems becomes:

$$V(P_{it-1}) = \lambda_t d_{it} + \beta_t E_t m_2^+ P_{it} / P_t$$

Solving and imposing symmetry gives exactly the same solution as the original recursive problem.

A.3 Numerical implementation of the operator \mathbb{F}

Let $\mathbf{S} \times \mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{H}$ denote the space of all equilibrium state vectors and associated payoffs to the central bank (s, m_1, m_2, h) . $\mathbf{W} : \mathbf{S} \rightarrow \mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{H}$ is a correspondence from \mathbf{S} to $\mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{H}$.

With an initial guess $\mathbf{W}^0(s) = \{(m_1(s), m_2(s), h(s))\}$ and a pre-determined tolerance level ϵ , the algorithm goes as follows:

- Step 1: For $\forall s \in \mathbf{S}$, find $\mathbf{\Omega}(s) := \{(m_1, m_2, h) | (m_1, m_2, h) \in \mathbf{W}^0(s), \exists R \in [\underline{R}, \bar{R}] \text{ and } (m'_1, m'_2, h') \in \mathbf{W}^0(s') \text{ such that :}$

$$\begin{aligned}
h &= u(c, l) + \beta \mathbb{E}h' \geq \tilde{h}^0(s) \\
Em'_1 &= \frac{1}{c} \frac{1}{R\beta} \\
Em'_2 &= \frac{1}{\beta} [(\epsilon - 1) - (1 - \xi)\epsilon w + \phi(\Pi - 1)\Pi] \frac{l}{c} \\
m_1 &= \frac{1}{c\Pi} \\
m_2 &= \phi(\Pi - 1)\Pi \frac{l}{c} \\
w &= l^x c \\
c &= \left(1 - \frac{\phi}{2}(\Pi - 1)^2\right)l
\end{aligned}$$

where

$$\begin{aligned}
\bar{h}^0(s, m_1(s), m_2(s)) &= \max_h \{h | (m_1, m_2, h) \in \mathbf{W}^0(s)\} \\
\underline{h}^0(s, m_1(s), m_2(s)) &= \min_h \{h | (m_1, m_2, h) \in \mathbf{W}^0(s)\} \\
\tilde{h}^0(s) &= \min_{(m_1, m_2)} \underline{h}^0(s, m_1, m_2)
\end{aligned}$$

- Step 2: For $\forall s \in \mathcal{S}$, and $\Omega(s)$, denote $\Omega^M(s, h) := \{(m_1, m_2) | (m_1, m_2, h) \in \Omega(s), h = h(s, m_1, m_2)\}$, and define

$$\begin{aligned}
\bar{h}^1(s, m_1, m_2) &= \max_R \max_{\substack{c, l, \Pi, w, \\ (m'_1, m'_2, h') \in \mathbf{W}^0(s')}} u(c, l) + \beta \mathbb{E} \bar{h}^0(s', m'_1, m'_2) \\
\underline{h}^1(s, m_1, m_2) &= \max_R \{ \max_{\substack{c, l, \Pi, w, \\ (m'_1, m'_2, h') \in \mathbf{W}^0(s')}} \min_h \{ u(c, l) + \beta \mathbb{E} \underline{h}^0(s', m'_1, m'_2), \tilde{h}^0(s) \} \}
\end{aligned}$$

for all $(m_1, m_2) \in \Omega^M(s, h)$. Otherwise, set

$$\begin{aligned}
\bar{h}^1(s, m_1, m_2) &= +\infty \\
\underline{h}^1(s, m_1, m_2) &= -\infty
\end{aligned}$$

Further, let

$$\tilde{h}^1(s) = \min_{(m_1, m_2) \in \Omega^M(s, h)} \underline{h}^1(s, m_1, m_2)$$

- Step 3: Define $\mathbf{W}^1(s) = \{(m_1, m_2, h) | (m_1, m_2) \in \Omega^M(s, h), h \in [\min\{\bar{h}^0(s, m_1, m_2), \underline{h}^1(s, m_1, m_2)\}, \max\{\underline{h}^0(s, m_1, m_2), \bar{h}^1(s, m_1, m_2)\}]\}$
- Step 4: Set $\mathbf{W}^* = \mathbf{W}^1$ if $\|\mathbf{W}^1 - \mathbf{W}^0\| < \epsilon$; otherwise, set $\mathbf{W}^0 = \mathbf{W}^1$ and repeat the steps above.

In the deterministic case, I set the number of grids for m_1 and m_2 to be 200, the number of points for interest rate 500 with the range $[1, 1.1]$, implying a 8 basis point change when optimizing. In stochastic case, I reduce the number of grids for m_1 and m_2 to be 50 while keeping the grids for interest rates the same.

A.4 Algorithm to Find All $R(m_1, m_2)$

Give m_2 , Π is fixed. With m_1 , c , l , and w are also determined. Expected payoffs to the firms are also prefixed as $Em'_2 = [(\epsilon - 1) - \epsilon(1 - \xi)w - \phi(\Pi - 1)\Pi] \frac{l}{c\beta}$. Expected payoffs to households, however, depend on R as $Em'_1 = \frac{1}{\beta c} \frac{1}{R}$. Since the best SSE can be supported by many (m_1, m_2) , there may exist different (m'_1, m'_2) that lead to the same level payoff to the central bank. To find all such payoffs and hence interest rate, I find out all $(m'_{11}, m'_{21}, m'_{12}, m'_{22})$ that satisfy the following two equations:

$$\omega \bar{h}(m'_{11}, m'_{21}) + (1 - \omega) \bar{h}(m'_{12}, m'_{22}) = E \bar{h}(m'_1, m'_2)(m_1, m_2) \quad (36)$$

$$\omega m'_{21} + (1 - \omega) m'_{22} = Em'_2(m_1, m_2) \quad (37)$$

where ω is the probability of the economy in state 1 next period given the state today. Note that the right hand sides of the two equations above are functions of (m_1, m_2) and are fixed for the current problem.

The algorithm goes as follows:

- Step 1: Pick (m_1, m_2) and hence $Em'_2(m_1, m_2)$ and $E \bar{h}(m'_1, m'_2)(m_1, m_2)$. Solve for $m'_{21} = (Em'_2(m_1, m_2) - (1 - \omega)m'_{22})/\omega$.

- Step 2: Pick I and J grids for Em'_1 and m'_{11} within their range. Given the i^{th} and j^{th} grids, calculate the corresponding $m'_{11} = (Em'_1 - (1 - \omega)m'_{12})/\omega$. Substituting m'_{11} and m'_{21} back to (36), I define a new function:

$$g = \omega \bar{h}((Em'_1 - (1 - \omega)m'_{12})/\omega, (Em'_2(m_1, m_2) - (1 - \omega)m'_{22})/\omega) + (1 - \omega) \bar{h}(m'_{12}, m'_{22}) - E \bar{h}(m'_1, m'_2)(m_1, m_2)$$

- Step 3: Fixing m'_{11} , g is a function of m'_{22} . Use root finding solver to solve for the roots of $g = 0$. (There are at most 2 roots because h is convex along the dimension of m_2 .)
- Step 4: Keep the roots if any. Go to step 3 if $j \leq J$. Go to step 2 if $j > J$. Go to step 1 if $i > I$.

A.5 Proofs

Proposition 4.2. To simplify the exposition, I abstract away uncertainties. In the text, I have shown that the sequential and recursive problems lead to the same Euler equation. The only thing left is to show that the transversality condition holds.

Let $b_t = B_t/P_t$. To prevent Ponzi schemes, I assume that there are debt limits for b_t , that is, $b_t \in [\underline{b}, \bar{b}]$, where $\underline{b}_t > -\infty$ and $\bar{b}_t < \infty$. Since $R \leq \bar{R} < \infty$, it is sufficient to show that $m_t \leq \bar{m} < \infty$. From the goods market clearing condition, the upper (lower) bound of inflation is $\bar{\Pi} = 1 + \sqrt{2/\phi}$ ($\underline{\Pi} = 1 - \sqrt{2/\phi}$). If inflation is out of the range, then all the goods produced will be used for compensating price adjustment.

Given the preference of the household, I have $\lim_{c \rightarrow 0} u_c(c, l) = -\infty$ and $\lim_{l \rightarrow 0} u_l(c, l) = 0$, which means that the household will be better off by spending a strictly positive amount of time $l > 0$ in working so that he can obtain some income to finance a positive amount of consumption. From the first-order condition $w = cl^\chi$, I have $c = wl^{-\chi}$. So what is left to show is there is a lower limit for the wage w . From the firms' first-order condition (6), it is easy to see that $w = \frac{(1-\epsilon)-(1+\beta)\phi\Pi(\Pi-1)}{\epsilon(1-\xi)}$ is lower bounded. Therefore, there exists upper bound \bar{m} . Finally, $\lim_{t \rightarrow \infty} \beta^t m_t b_t R_t \leq \lim_{t \rightarrow \infty} \beta^t \bar{m} \bar{b} \bar{R} = 0$ ■

Lemma 4.4. The proof follows closely Phelan and Stacchetti (2001). ■

Proposition 4.7. The proof follows closely Phelan and Stacchetti (2001). ■

Proposition 4.8. The proof here follows Feng (forthcoming). By definition, $\bar{h}(s, m_1, m_2)$ is the maximum value of h given (s, m_1, m_2) , which is:

$$\begin{aligned}\bar{h}(s, m_1, m_2) &= \max_R \max_{\{m'_1, m'_2, h'\}} [u(c, l) + \mathbb{E}\{h(s', m'_1, m'_2)\}] \\ &= \max_R \left[u(c, l) + \max_{\{m'_1, m'_2, h'\}} \mathbb{E}\{h(s', m'_1, m'_2)\} \right] \\ &= \max_R [u(c, l) + \mathbb{E}\{\bar{h}(s', m'_1, m'_2)\}]\end{aligned}$$

where the first equality follows the definition of $\bar{h}(s, m_1, m_2)$, the second equality follows the fact that the instant utility only depends on promised values (m_1, m_2) , and the last equality uses the definition of \bar{h} .

A similar argument applies to $\underline{h}(s, m_1, m_2)$. A few comments go as follows. First, $\underline{h}(s, m_1, m_2) = \max_R \min_{\{m'_1, m'_2, h'\}} u(c, l) + \beta \mathbb{E}h'$. Second, it should be noted that the value of $u(c, l) + \beta \mathbb{E}\underline{h}'$ at given R might be smaller than $\tilde{h}(s)$, which says that the incentive constraint is not satisfied when the government has the lowest continuation value. When this happens, the government needs a higher continuation value so that the incentive constraint is satisfied. However, the corresponding payoff for the present government cannot be higher than $h(s)$. This is because only the minimization operates when R is given. There always exists $h' \in [\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]$ to bind the incentive constraint when the worst continuation value violates the incentive constraint. Otherwise, (m_1, m_2) should not belong to the equilibrium value correspondence. Note that $\tilde{h}(s)$ is the payoff of the worst SSE and must lie in the lower boundary of $h(s, m_1, m_2)$. Because it is the worst of all, it must be equal to $\min_{\{m_1, m_2\}} \underline{h}(s, m_1, m_2)$. ■

Theorem 4.10. The proof follows Feng (forthcoming) and first shows that the sequence of \hat{W}_n is decreasing and $\hat{W}_n \ni \hat{W}_{n+1}$. Since W_n is convex-valued, it is sufficient to show that the upper boundary is decreasing and the lower one increasing. The upper boundary is decreasing because of the fact that $\bar{h}^1(s, m_1, m_2)$ is defined as $\max_R u(c, l) + \beta \mathbb{E}\bar{h}^0(s', m'_1, m'_2)$ such that $\psi = (R, c, l, w, \Pi, \{m'_1, m'_2, h'\})$ is admissible wrt \hat{W}^0 at s . The admissibility of the vector ψ implies that $(m_1, m_2, \bar{h}^1(s, m_1, m_2)) \in \hat{W}_0(s)$. Therefore, $\bar{h}^1(s, m_1, m_2) \leq \max\{h | (m_1, m_2, h) \in \hat{W}^0(s)\} = \bar{h}^0(s, m_1, m_2)$. Similarly, I can show that the lower boundary is increasing, i.e., $\underline{h}^1(s, m_1, m_2) \geq \underline{h}^0(s, m_1, m_2)$. The same argument thus holds for $\hat{W}_n(s)$. Since the sequence is decreasing, it

has a limit \hat{W}_∞ . Proposition 4.7 implies that $\mathbb{F}(\mathbf{V}) = \mathbf{V}$. By a simple limit argument, I have $\lim_{n \rightarrow \infty} \hat{W}_\infty = V$. ■

A.6 Tables and Figures

Table 1: Parameterization

Parameters	Explanations	Values	Targets/Sources
χ	Frisch elasticity of labor	1	Hall (2009a,b)
ϵ	elasticity among inter. goods	6	Fernández-Villaverde et al. (2012)
ϕ	price adjustment cost	60	slope of Phillips curve= 1/6
β^H	high shock of preference	1.011	-4.5% natural rate
β^L	low shock of preference	0.994	2.5% natural rate
p_{HH}	persistence of high shock	2/3	aver. duration of 3Qs at high shock
$1 - p_{LL}$	frequency of high shock	0.0068	1 quarter out of 12 years

Table 2: Fiscal Subsidy and Implied Inflation Targets (Annual %)

Fiscal Subsidy	MPE	SSE
100%(Full Subsidy)	0	0
88%(Partial Subsidy)	0.94	0.52
76%(Partial Subsidy)	1.85	1.12
64%(Partial Subsidy)	2.74	1.60

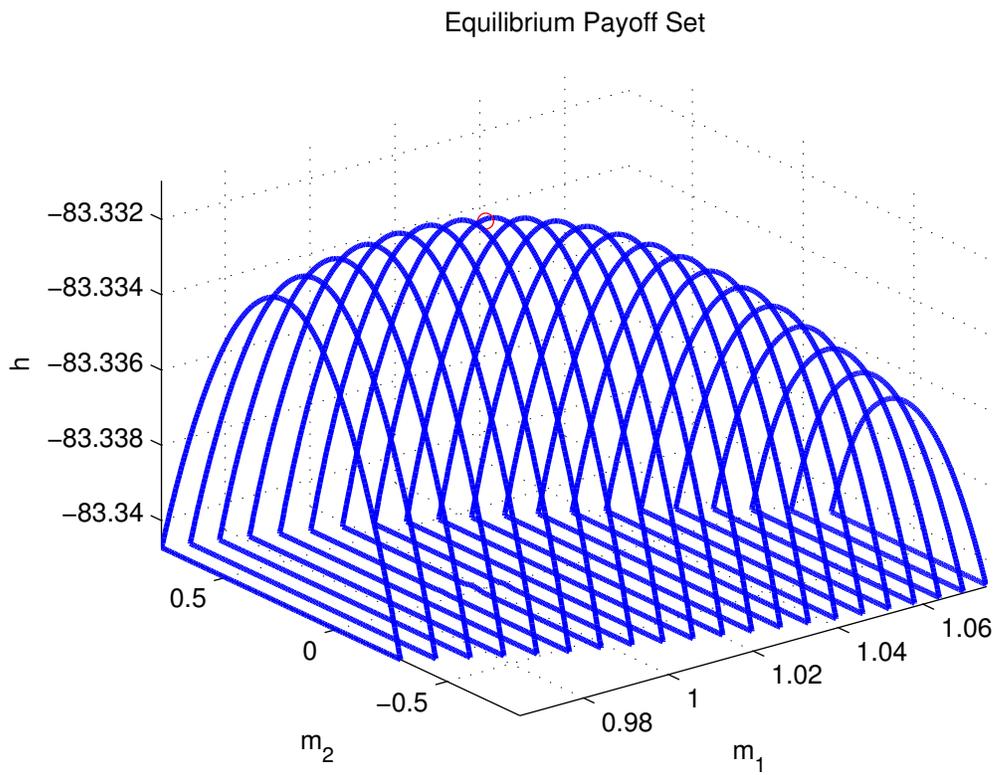


Figure 1: Equilibrium Payoff Set: Deterministic Case

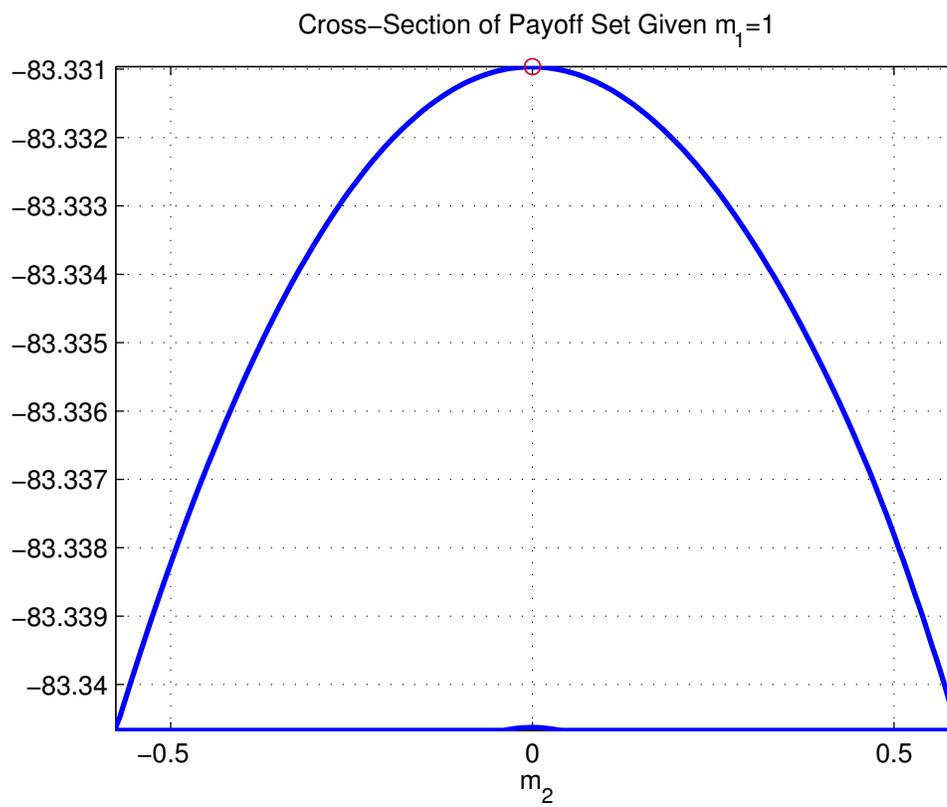


Figure 2: Equilibrium Payoff Set: A Cross Section when $m_1 = 1$

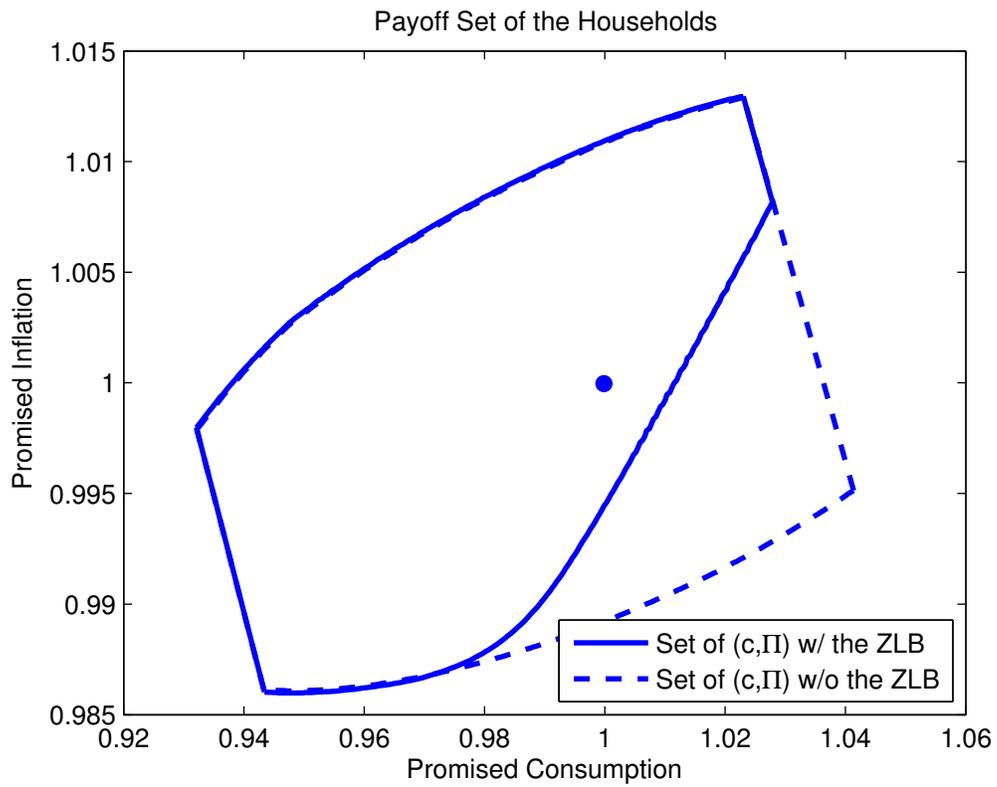


Figure 3: The Sets of Promised Payoffs to the Households (c, Π)

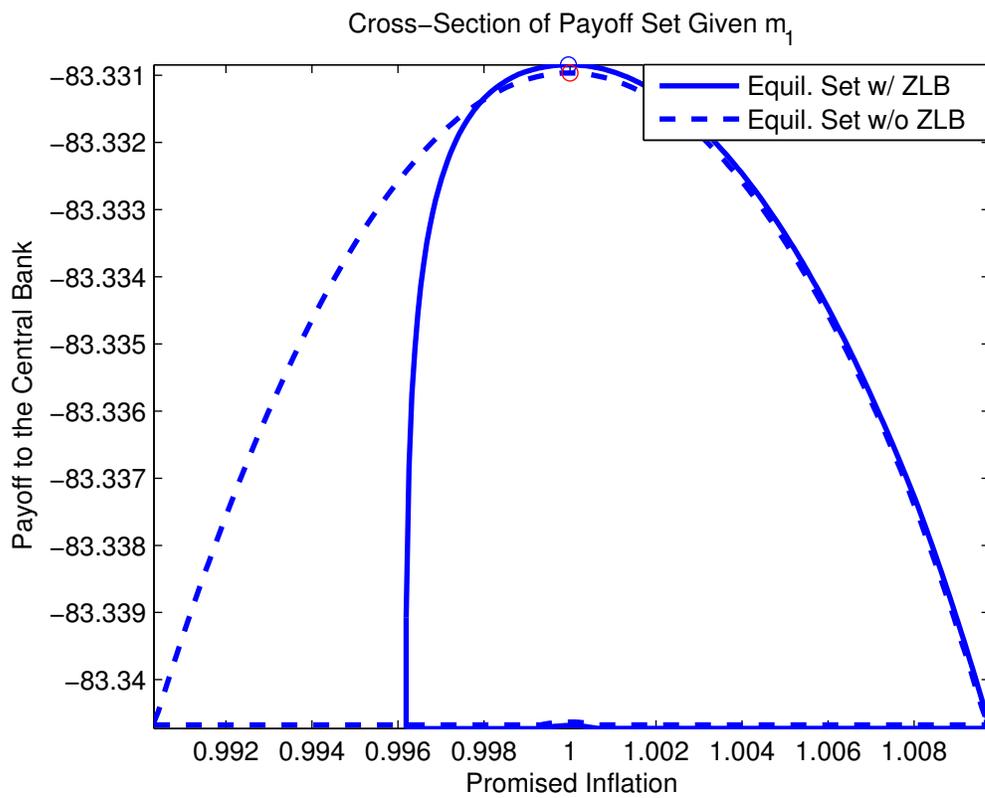


Figure 4: Equilibrium Payoff Set: A Cross Section when $m_1 = 1$

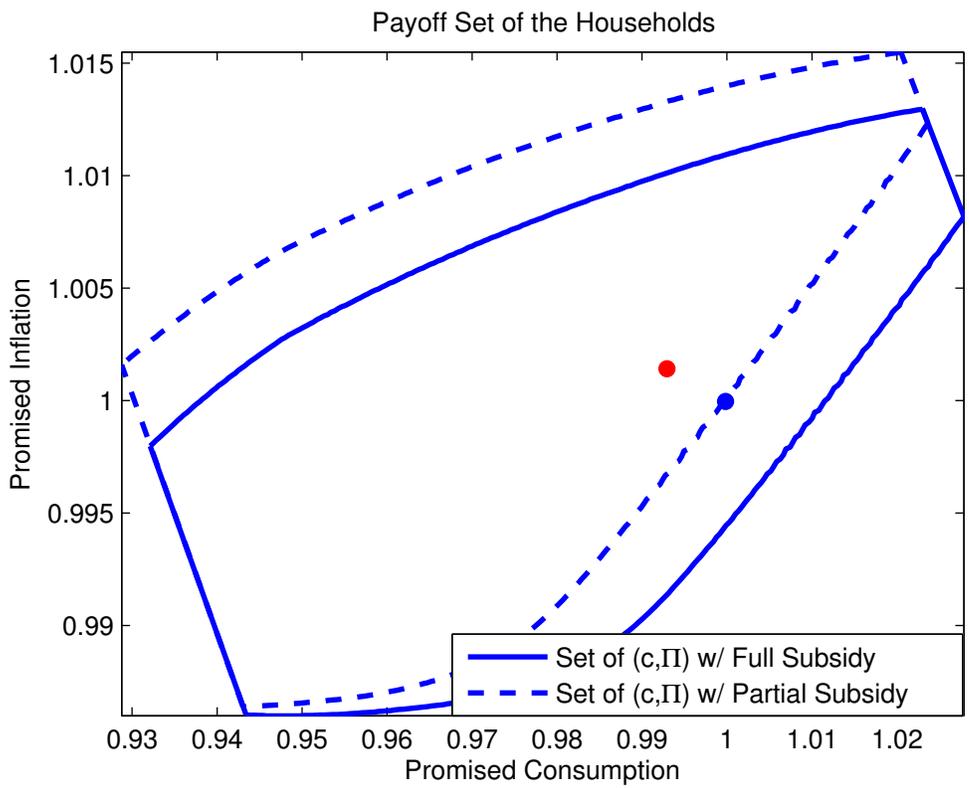


Figure 5: Comparison of Sets of SSE with Different Subsidy (Solid line: full subsidy; dashed line: 88% subsidy; the blue (red) dot is corresponding to the best SSE under full (partial) subsidy)

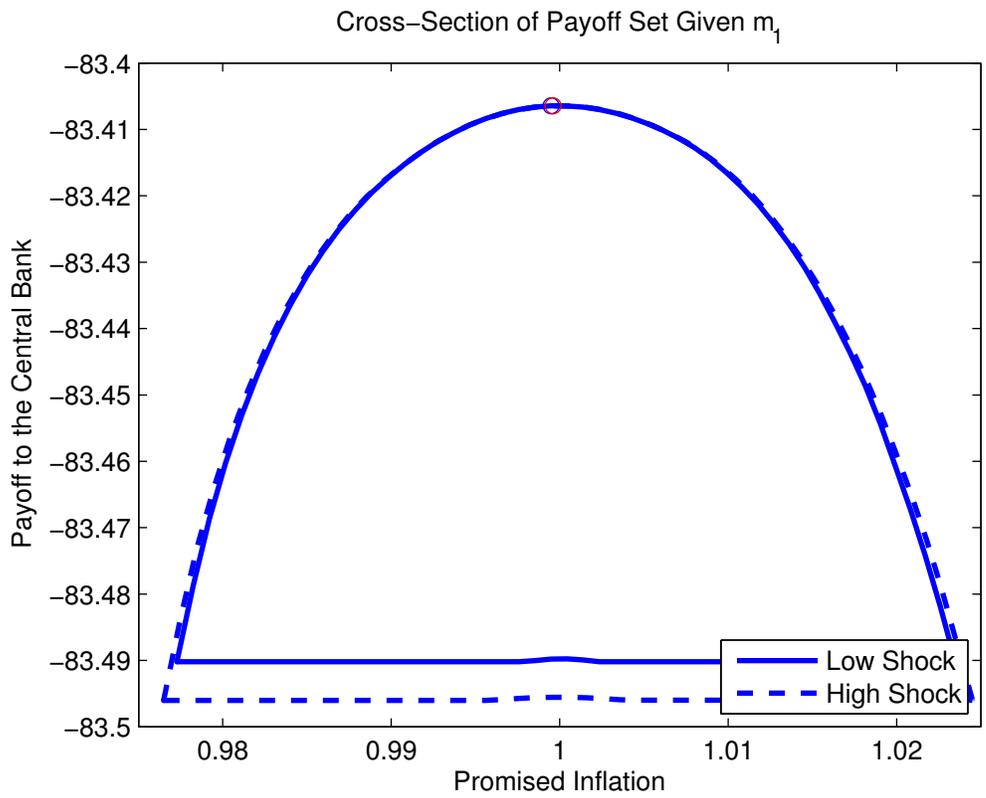


Figure 6: Cross-Sections of Equilibrium Sets when $m_1 = 1$ (The level of payoff for the high shock state is adjusted upward to make the highest payoff in that state equal to the highest payoff in low state.)

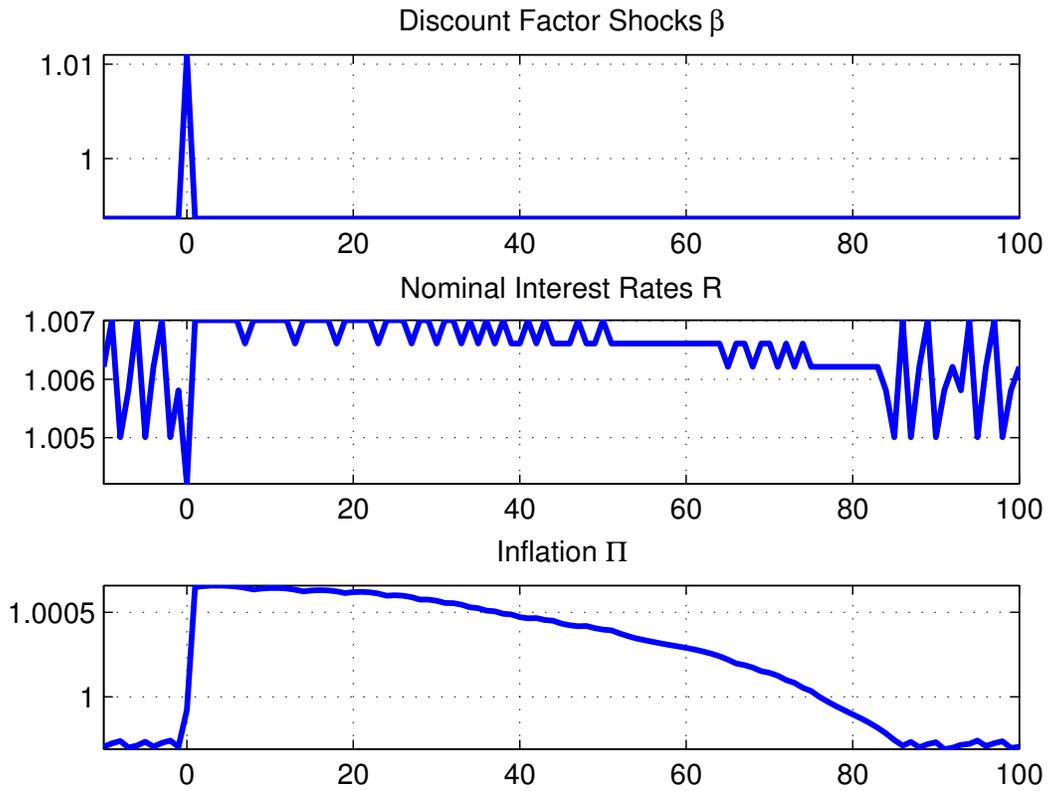


Figure 7: One Simulated Path after a High Shock hits only at time 1 under SSE without the ZLB

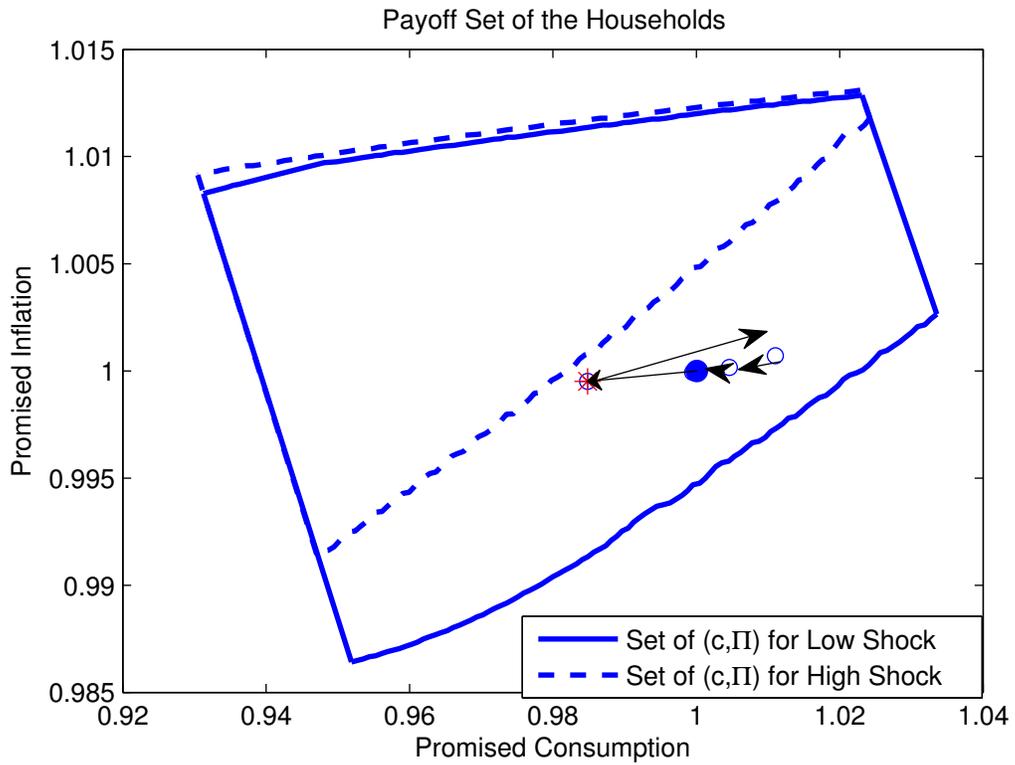


Figure 8: Non-Implementability of Ramsey (The solid line area is the supported payoff set associated with low shock, while the dashed line area the set with high shock. The circles are the calculated corresponding payoffs from the Ramsey equilibrium for low state and the star the corresponding payoff for high state. The star stays beyond the set for high state from the SSE. The dynamics are shown by arrows.)

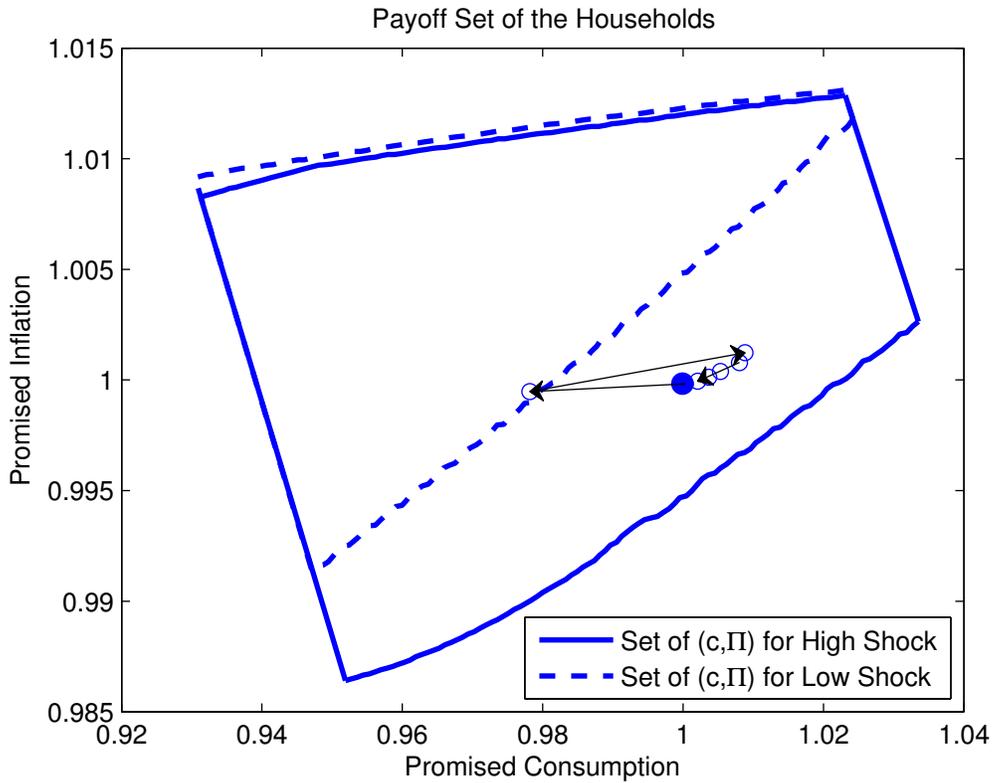


Figure 9: The Path of Promised Consumption and Inflation under SSE (The economy starts with low shock at the point with filled circle. Then it is hit by a high shock for 1 quarter. After the shock, the economy stays at low shock state for ever. The arrows show how the promised bundle moves. And when the shock happens, the promised bundle does lie inside the state space for high shock.)

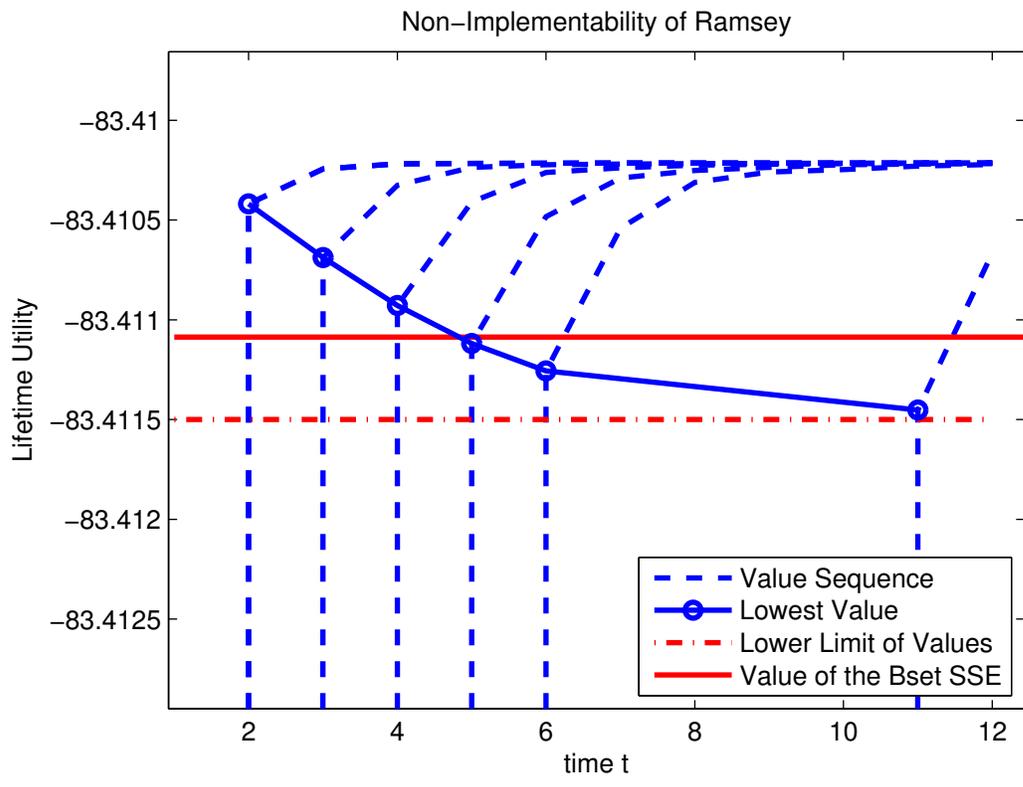


Figure 10: The Non-Implementability of Ramsey: Compare Payoffs to the Central Bank (The dashed lines represent value sequences for realizations of 1, 2, 3, 4, 5 and 10 periods of high shocks, the blue solid line the lowest payoff after each realization of shocks, the red dashed the lower limit of value when sticking to Ramsey, and the red solid line the payoff deviating to the best SSE.)

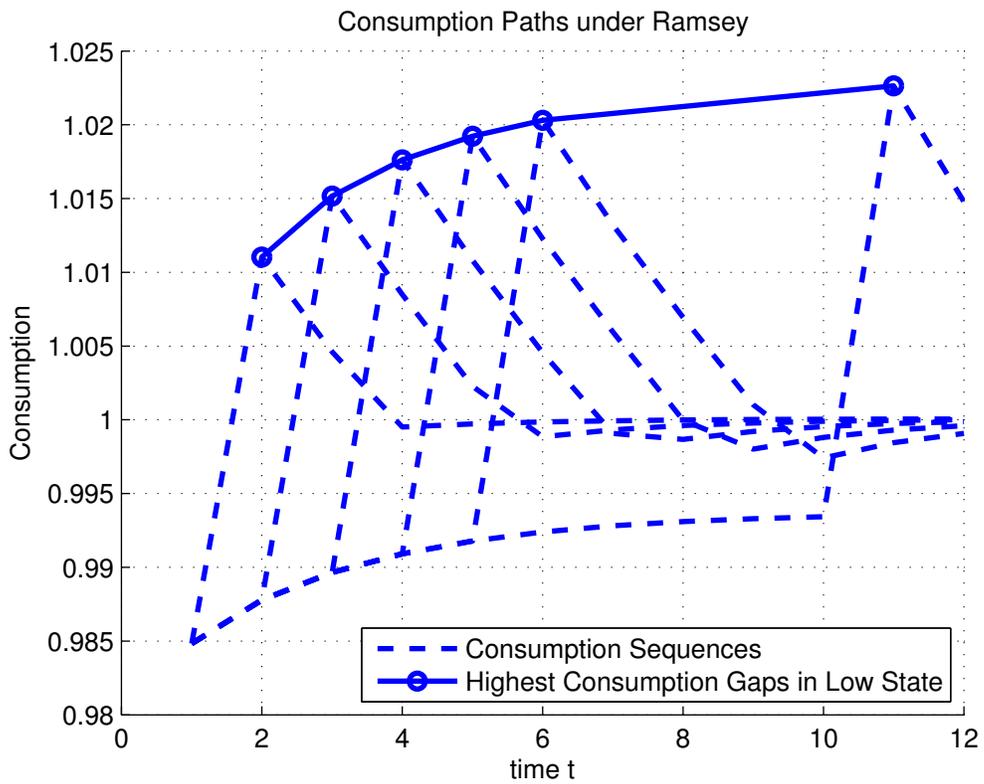


Figure 11: Consumption Paths after Contractionary Shocks Disappear (The dashed lines represent consumption paths after high shock hits the economy 1, 2, 3, 4, 5 and 10 quarters, respectively; the solid line connects the highest consumption gaps associated with each realization of shocks)

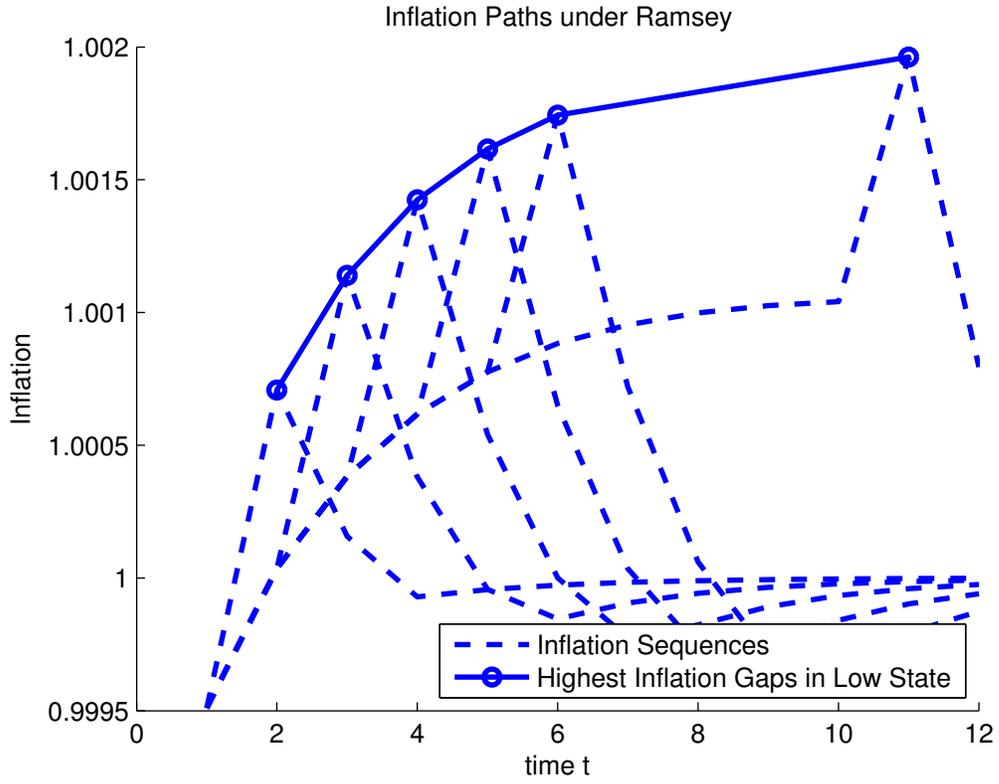


Figure 12: Inflation Paths after Contractionary Shocks Disappear (The dashed lines represent inflation paths after high shock hits the economy 1, 2, 3, 4, 5 and 10 quarters, respectively; the solid line connects the highest inflation gaps associated with each realization of shocks)

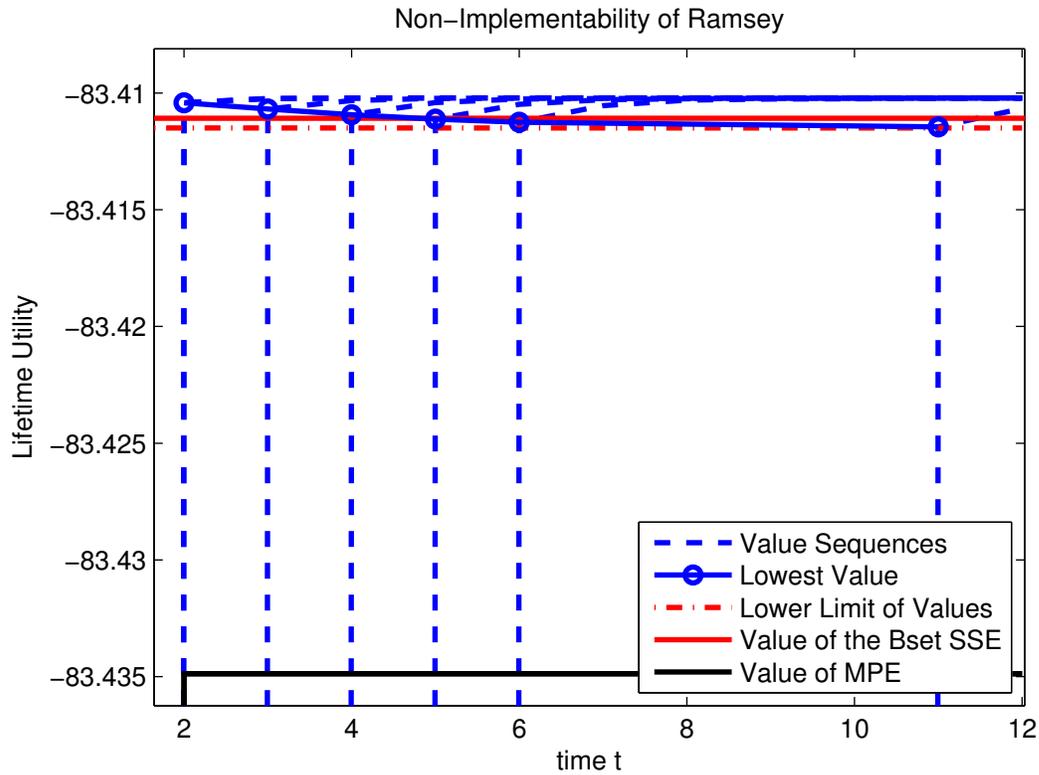


Figure 13: The Non-Implementability of Ramsey: Compare Payoffs to the Central Bank (The dashed lines represent value sequences for realizations of 1, 2, 3, 4, 5 and 10 periods of high shocks, the blue solid line the lowest payoff after each realization of shocks, the red dashed the lower limit of value when sticking to Ramsey, and the red (black) solid line the payoff deviating to the best SSE (MPE).)

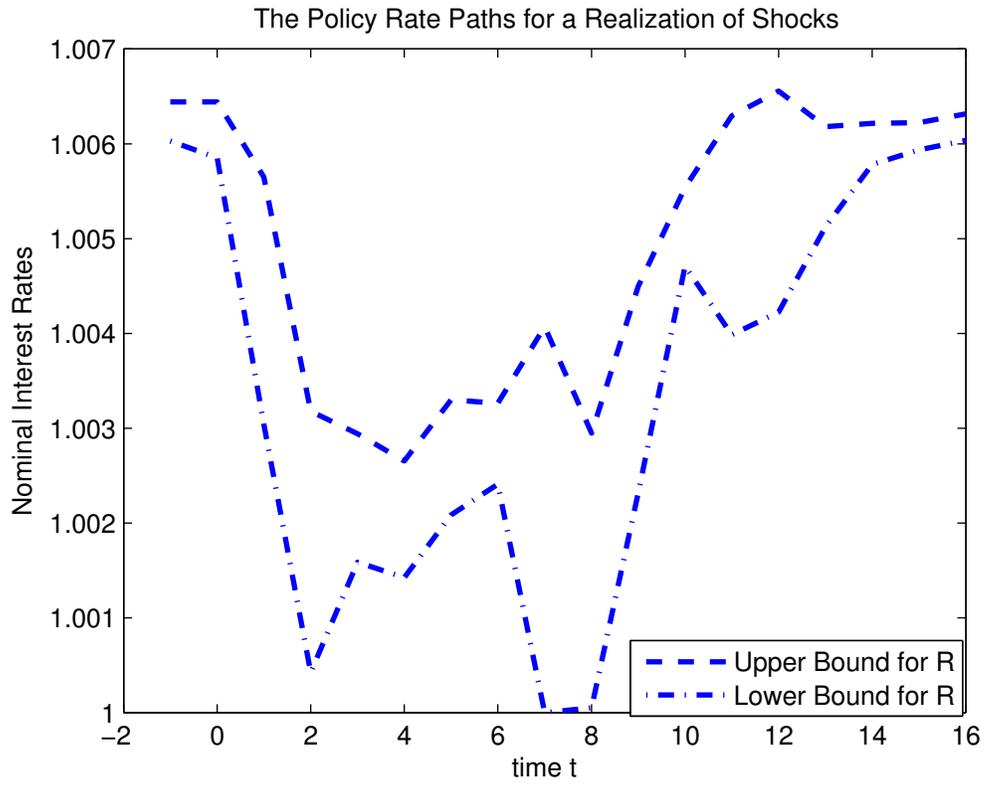


Figure 14: The Upper and Lower Bounds of Policy Rates for the Best SSE (The high shock starts to hit the economy at time 1 and stays for another 5 quarters.)

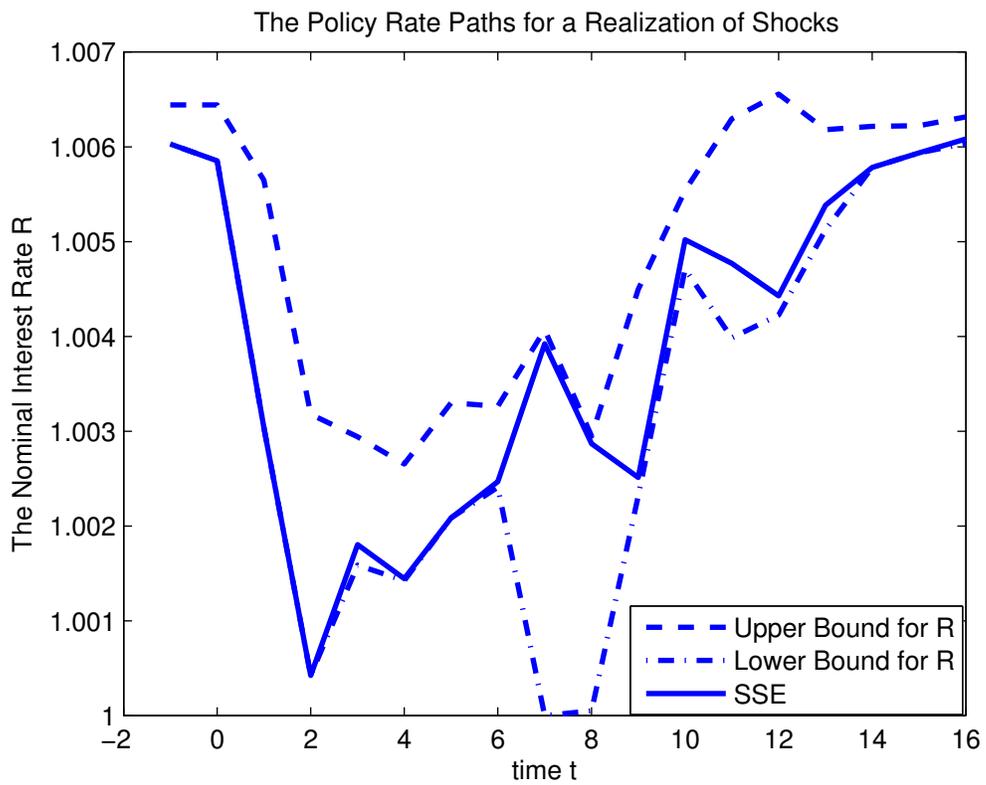


Figure 15: The Upper and Lower Bounds and One Specific Path of Policy Rates for the Best SSE(The high shock starts to hit the economy at time 1 and stays for another 5 quarters.)

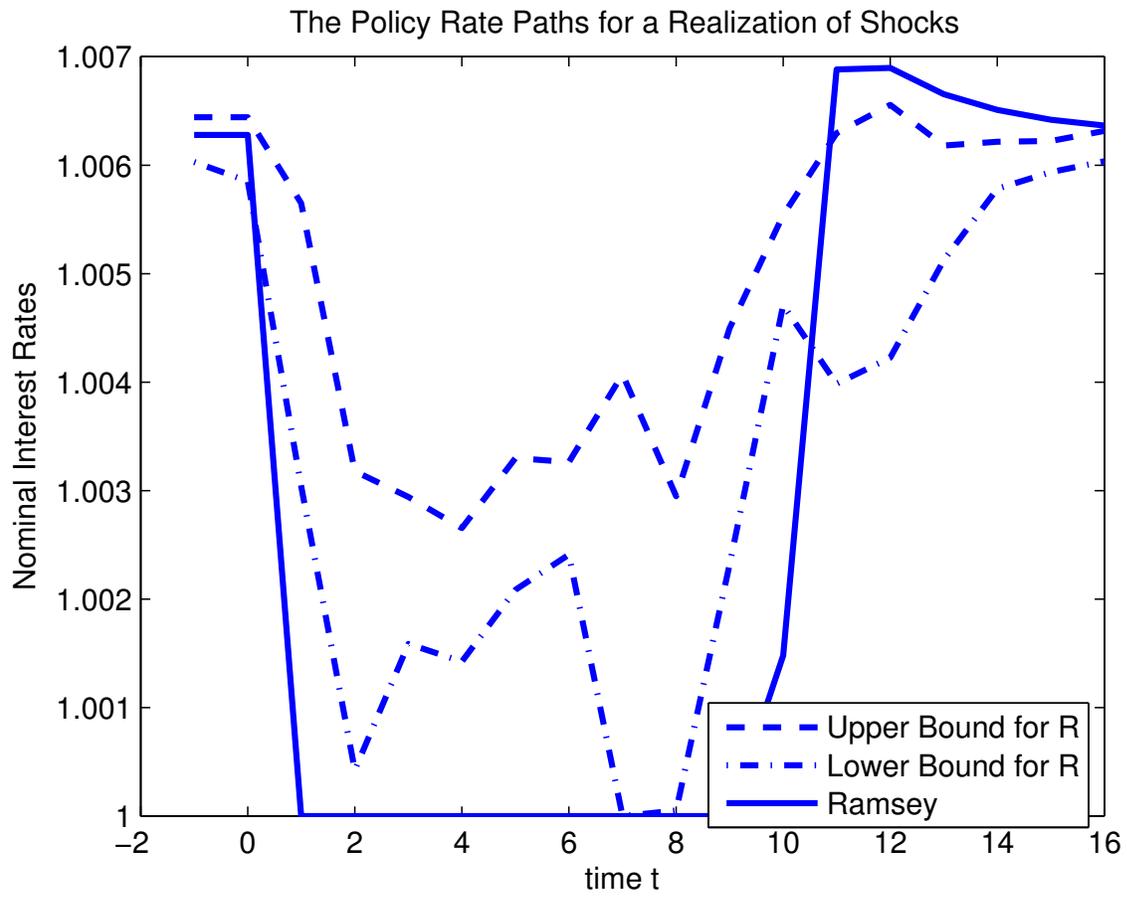


Figure 16: Dynamics of Policy Rates under Ramsey

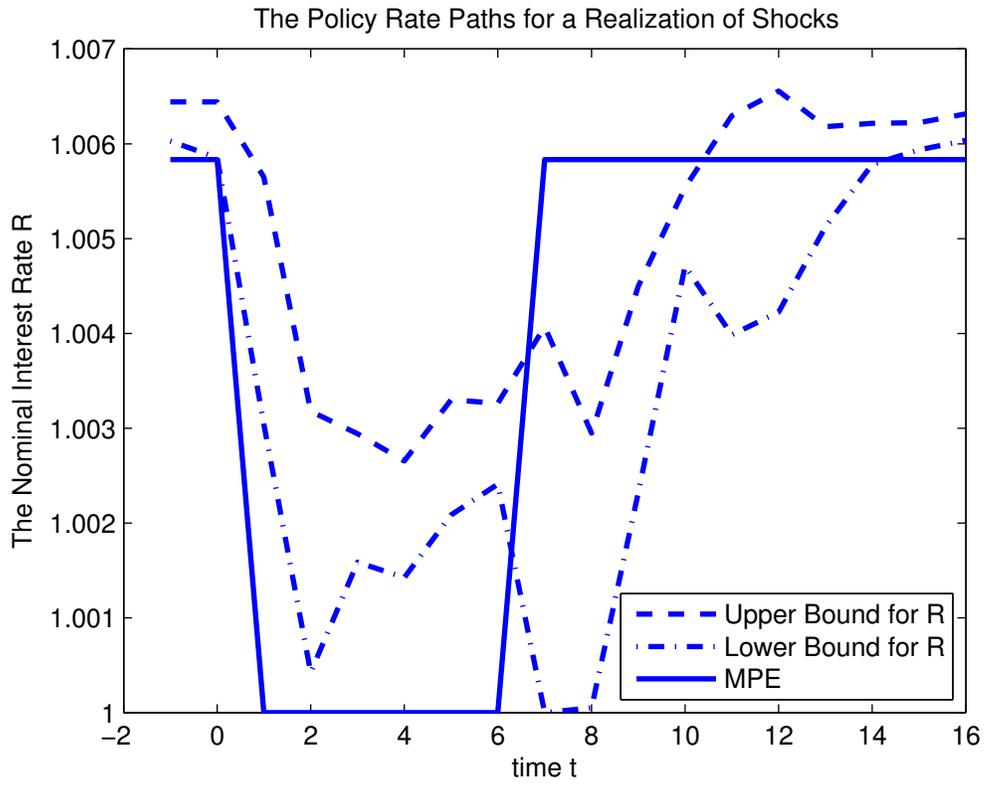


Figure 17: Dynamics of Policy Rates under MPE

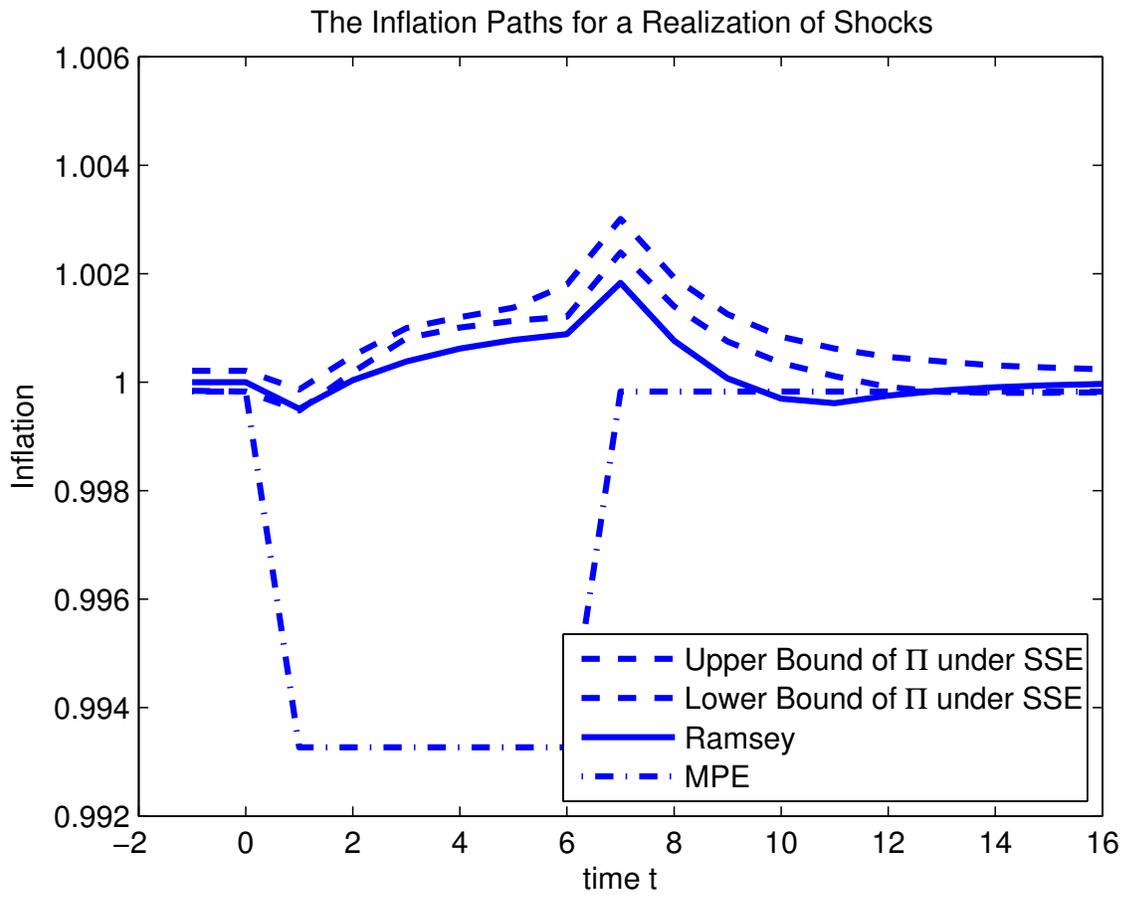


Figure 18: Dynamics of Inflation

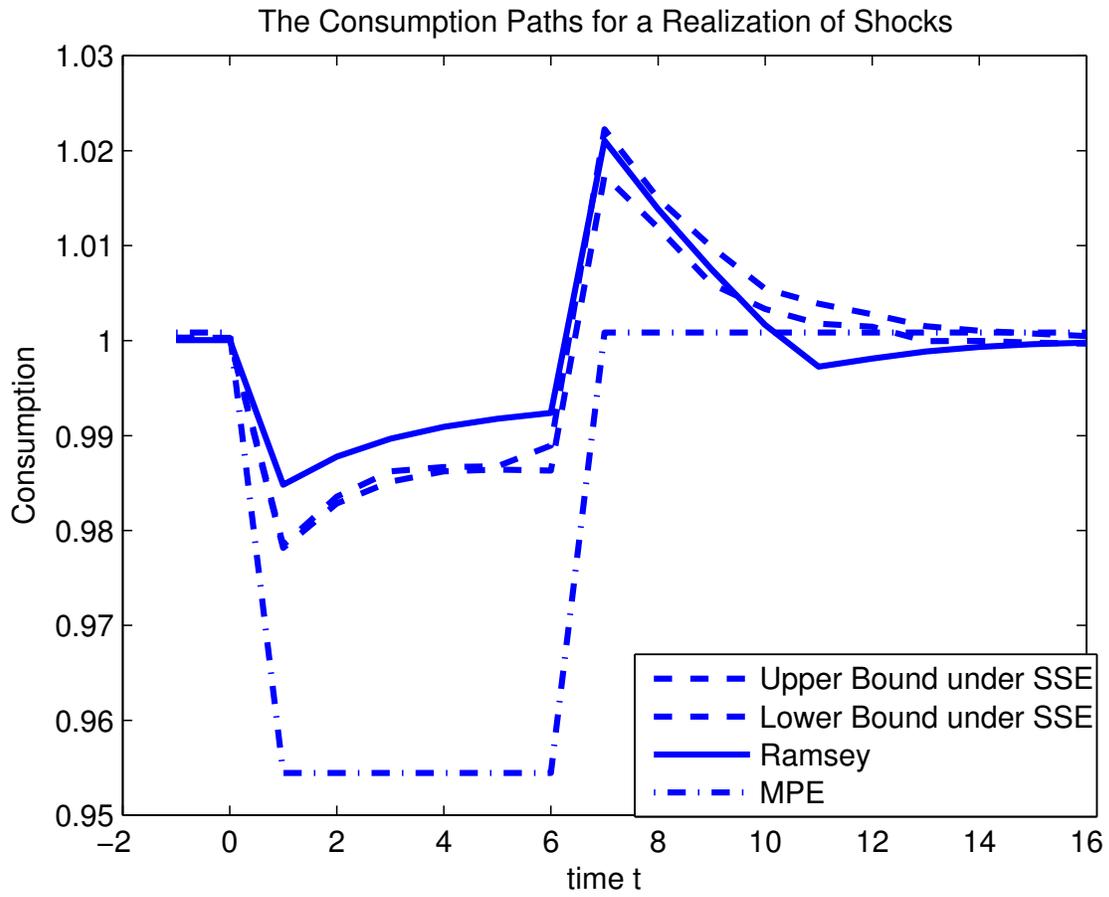


Figure 19: Dynamics of Consumption