Asset Prices and Portfolio Choice with Learning from Experience *

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Abstract

We study asset prices and portfolio choice in a transparent overlapping generations economy in which the young disregard history to learn from own experience. Disregarding history implies less precise estimates of consumption growth which, in equilibrium, leads the young to increase their investment in risky assets after positive returns or act as trend chasers and to lose wealth and consumption shares to the old. Consistent with findings from survey data, the average belief about expected returns in the economy is negatively related to future realized returns.

Keywords: Learning from Experience, Trend Chasing, Survey Based versus Objective Expected Returns

JEL Classification: G10, G11, G12, E2

1 Introduction

Many models or theories in economics and finance, even those that focus on learning, assume away that experience matters, perhaps motivated by the availability of long time series of high quality data or because of modeling choices due to which all agents learn from the same data. However, recent empirical evidence suggests that experience matters for financial decision makings. If we are to take the evidence seriously, we have to assume that people condition their actions, at least partially, on their experience instead of on all available data or more generally on the advice of experts. But what are, broadly speaking, the implications of experience driven financial decisions? More specifically, how do the young share consumption risk with the old? How does experience drive trading? How do asset prices change when more optimistic young or pessimistic young trade?

In response, we study an overlapping generations economy with learning in which cohort specific experience drives beliefs about consumption growth and through that affects equilibrium outcomes. The main findings are that the young act as trend chasers, wealth shifts from young to old, the market price of risk is countercyclical and the risk free rate of return is procyclical, and average beliefs about stock returns are negatively correlated with actual expected returns.

To isolate the influence of experience we assume that all agents are born with the correct prior about the growth rate of consumption.[1] For experience to matter in such an environment, agents have to be unsure about their prior for growth or disregard history. With these two model ingredients, we see that when consumption booms, new generations learn to be optimists. We also see the opposite, when consumption declines period after period then new generations learn to be pessimists. Therefore, young agents buy risky assets after positive returns and sell risky assets after negative returns. Hence, they act optimally as trend chasers, given their beliefs. Older generations learn that they were too optimistic or

[1] Appendix A.1 shows that this assumption is consistent with an economy where there is a large set of agents born every period that draw their prior belief from a normal distribution with a mean given by the true expected consumption growth.

too pessimistic and, therefore, estimate growth with higher precision than the young. The "market view," as measured by the consumption share weighted average belief, summarizes the interplay between optimists and pessimists and young and old.

One implication of our model is that young or inexperienced investors underperform relative to old investors as their estimates about growth vary more than the estimates of experienced investors. Arrondel et al. (2014) provide survey based evidence that is consistent with this feature of our model in that it suggests that investors' measure of information "increases with past experience." Further, Korniotis and Kumar (2011) find that older and experienced investors "follow rules of thumb" that proxies for greater investment knowledge.

To obtain transparent expressions for asset prices and portfolio choice, we assume that agents have logarithmic preferences and that the infinitively lived risky security is in zero net supply. In equilibrium, the market price of risk and the risk-free rate are given by standard formulas from the logarithmic economy with complete information plus a correction term that captures the difference between the "market view" and the objective growth rate. Specifically, in times when the "market view" for expected consumption growth in the economy is higher than the actual growth, we see that the real interest rate increases and the market price of risk decreases relative to the complete information benchmark. In equilibrium, the "market view" considerably drives objective expected stock market returns.

In our model, equilibrium portfolio policies are qualitatively consistent with the empirical evidence in Malmendier and Nagel (2011). According to Malmendier and Nagel (2011), individuals who have experienced high stock market or bond market returns are more likely to take on further financial risks, i.e., are more likely to participate in the stock market or bond market, and allocate a higher proportion of their liquid assets to stocks or bonds. They find that individuals weight recent returns more than distant realizations, but returns many years ago still impact current allocations. Malmendier and Nagel (2011) also provide evidence that point to the importance of experience for beliefs. According to their view, experience effects could be the result of attempts to learn from experiences where all available historical data is used by not entirely trusted.[2] Further, in a follow up paper, Malmendier and Nagel (2014), show that individuals adapt their inflation forecasts to new data but overweight inflation realized during their life-times. They also show that young individuals update expectations more strongly in the direction of recent surprises than the old. Specifically, learning from experience explains substantial disagreement between young and old individuals in periods of high surprise inflation. Such differences in expectations between young and old is the departure point for our work where we study the role of experience on asset prices, consumption, and portfolio choice in a consumption based overlapping generations economy.

Measures of expected returns, such as realized future returns, suggest a negative or countercyclical relation between expected returns and realized returns. Survey based evidence on expected returns, however, are at odds with the data since the mean forecast keeps rising even after a string of positive returns. Cochrane (2011) argues that since "survey reports of people's expectations are certainly unsettling" we might want to disregard them. Of course, one way to explain why some investors expect high returns when low discount rates suggest low returns going forward is irrationality or trend-chasing. Indeed, it appears increasingly difficult to disregard the overwhelming evidence on people's forecasts in Greenwood and Shleifer (2014) that apparently broadens the impression that people's expectations are irrational. Consequently, Barberis et al. (2015) propose a model with extrapolative expectations that reconciles the evidence on expectations with the evidence on volatility and predictability. Our parsimonious model complements Barberis et al. (2015) as it allows tying together several empirical phenomena including people's apparently unsettling expectations. Specifically, it ties together extrapolative expectations with inexperience in a way that is consistent with both the evidence in Greenwood and Shleifer (2014) and in Malmendier and Nagel (2011). In our model all investors learn optimally in a Bayesian sense, given their beliefs,

[2] Using age as measure of managers' investment experience, Greenwood and Nagel (2009) show that around the peak of the technology bubble, mutual funds run by young managers are more heavily invested in technology stocks, relative to their style benchmarks, than old managers. Young managers trend-chase in their technology stock investments. The old managers do not. Consequently, young managers increase their technology holdings during the run-up, and decrease them during the downturn.

and equilibrium expected returns decline after a series of positive returns. Yet, surveying investors in our model produces a mean forecast that shows a positive relation with realized returns unless the mean forecast is consumption share weighted.

Our paper extends the literature about disagreement and asset prices by studying differences in beliefs in an overlapping generations environment. This literature, initiated by Harrison and Kreps (1978) and Detemple and Murthy (1994), among others, directly or indirectly assumes that agents learn from all available data. More recent examples of such models with differences in beliefs that employ belief structures similar to ours include Zapatero (1998), Basak (2000), Basak (2005), and Dumas et al. (2009).

Gârleanu and Panageas (2014) study implications of preference heterogeneity for asset pricing in an overlapping generations economy. Using recursive preferences, they show that without heterogeneity in risk aversion asset price moments are constant instead of counter-cyclical as in the data. Further, for a given amount of heterogeneity in risk aversion, heterogeneity in the elasticity of intertemporal substitution can still impact asset price moments significantly. Kubler and Schmedders (2011) also study asset prices in an overlapping generations economy and find that belief heterogeneity and life-cycle investments can lead to realistic asset price volatility. Motivated by the mounting evidence in Malmendier and Nagel (2011) and Malmendier and Nagel (2014), Collin-Dufresne et al. (2014) also consider an experience driven learning bias in an overlapping generations economy. Their paper and ours are complementary in that they use more general preferences while our model has more general cohort and demographic structure.

2 The Model

2.1 Demographics

We consider a continuous time overlapping generations economy in the tradition of Blanchard (1985) that we extended to a setting with incomplete information. Every period a fraction

ν of the population dies to be replaced by newborn agents of the same mass. Hence, the population $\int_{-\infty}^{t} \nu e^{-\nu(t-s)} ds = 1$ is invariant for all t, and the time t size of the cohort born at time $s < t$ is given by $\nu e^{-\nu(t-s)} ds$.

2.2 Endowments

Agents receive endowment, $y_{s,t}$, continuously from birth at time s until death. Endowments do not depend on the time of birth, i.e., $y_{s,t} = Y_t$ for all $s \leq t$. Endowments evolve as follows:

$$dY_t = Y_t \left(\mu_Y dt + \sigma_Y dz_t \right) \tag{1}$$

where z_t is a shock modeled as a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$. Aggregate endowment at time t is

$$\int_{-\infty}^{t} \nu e^{-\nu(t-s)} y_{s,t} ds = Y_t \int_{-\infty}^{t} \nu e^{-\nu(t-s)} ds = Y_t, \tag{2}$$

thus, the dynamics of aggregate endowment coincide with the dynamics of individual endowments.

2.3 Information, Learning and Disagreement

Agents observe individual —or more precisely cohort specific— endowments, and hence aggregate endowment, but do not know the value of μ_Y.[3] To simplify, we assume that all agents start with the correct prior for μ_Y. However, in Subsection A.1 in the Appendix we show how one can endogenize this assumption by allowing for within cohort heterogeneity that is normally distributed and unbiased on average. Agents do not completely trust their estimates and, thus, prior variances differ from zero. Specifically, an agent born at time s believes the expected endowment growth is normally distributed with mean $\hat{\mu}_{s,s} = \mu_Y$ and

[3] The specification of priors and optimal learning is similar to the models in Detemple and Murthy (1994) and Basak (2000).

variance $\bar{V} > 0$. Starting from birth, agents use Bayes' rule to update beliefs about expected aggregate endowment growth. By standard filtering theory, the dynamics of the expected endowment growth, $\hat{\mu}_{s,t}$, as perceived by an agent born at time s, and its posterior variance are

$$d\hat{\mu}_{s,t} = \frac{V_{s,t}}{\sigma_Y} dz_{s,t}, \quad V_{s,t} = \frac{\sigma_Y^2 \bar{V}}{\sigma_Y^2 + \bar{V}\,(t-s)}, \tag{3}$$

respectively, and where $z_{s,t}$ denotes a Brownian motion under the belief of an agent born at time s with associated probability P^s and information set (or sigma algebra) $\mathcal{F}_{s,t}^Y = \sigma\,(Y(t), s \leq t)$. Perceived shocks relate to z_t through

$$dz_{s,t} = dz_t + \Delta_{s,t} dt, \tag{4}$$

where $\Delta_{s,t} = \frac{\mu_Y - \hat{\mu}_{s,t}}{\sigma_Y}$. The process $\Delta_{s,t}$ summarizes the standardized estimation error of agents born at time s relative to the objective probability measure.

Proposition 1. *The estimation error of the cohort born at time s is*

$$\Delta_{s,t} = \frac{\bar{V}\,(z_s - z_t)}{\sigma_Y^2 + \bar{V}\,(t-s)}. \tag{5}$$

Moreover, we have that $\Delta_{s,s} = 0$ and $\lim_{t\to\infty} \Delta_{s,t} = 0$.

The estimation error in Proposition 1 is standard. Note, however, that each cohort starts out with the correct belief and in the long run converges back to it.

2.4 Security Markets and Prices

Agents trade in an instantaneously risk-free asset, which is in zero net supply. Its dynamics are given by

$$dB_t = r_t B_t dt, \tag{6}$$

where r_t denotes the equilibrium real short rate.

An infinitely lived risky asset evolves according to

$$\frac{dS_t}{S_t} = \left(\mu_t^S dt + \sigma^S dz_t\right) = \left(\mu_{s,t}^S dt + \sigma^S dz_{s,t}\right) \tag{7}$$

where the last part of the equation represents the dynamics perceived by agents born at time s. In Equation (7), we have $\mu_{s,t}^S = \mu_t^S - \sigma^S \Delta_{s,t}$, in which μ_t^S is determined in equilibrium, while the volatility coefficient, σ^S, is taken as exogenous since it defines the risky security.

Annuity contracts complete the set of available securities as in Yaari (1965). They entitle to an income stream of $\nu W_{s,t}$ per unit of time. In return, the competitive insurance industry receives all financial wealth when the agent dies. Entering such a financial contract is optimal for all agents or cohorts of agents.

It is convenient to summarize the price system in terms of the stochastic discount factor. As agents have different believes, they have individual stochastic discount factors. Still, they agree on Arrow-Debreu prices since $\xi_{s,t} dP_s = \xi_t dP$. The dynamics of the stochastic discount factor as perceived by an agent born at time s follow

$$d\xi_{s,t} = -\xi_{s,t} \left(r_t dt + \theta_{s,t} dz_{s,t}\right), \tag{8}$$

while the dynamics of the stochastic discount factor under the actual probability measure are

$$d\xi_t = -\xi_t \left(r_t dt + \theta_t dz_t\right). \tag{9}$$

Thus, we have that the relation between the market price of risk under the objective probability measure, θ_t, and the market price of risk as perceived by the cohort born at time s, $\theta_{s,t}$, is

$$\theta_{s,t} = \theta_t - \Delta_{s,t}. \tag{10}$$

The relation between the stochastic discount factor under the objective measure and the belief of an agent born at time s is captured by the disagreement process, $\eta_{s,t}$, through the relation $\xi_t = \eta_{s,t}\xi_{s,t}$. Formally, $\eta_{s,t}$ is the Radon Nikodym derivative that allows to move from the probability measure of an agent born at time s to the actual probability measure and vice versa. The dynamics of the disagreement process, $\eta_{s,t}$, follow

$$d\eta_{s,t} = -\Delta_{s,t}\eta_{s,t}dz_t. \tag{11}$$

2.5 Preferences and Individual Optimization

Agents maximize lifetime utility given by

$$E_{s,s}\left[\int_s^\tau e^{-\rho(t-s)}log\left(c_{s,t}\right)dt\right], \tag{12}$$

where τ is the stochastic time of death. In the above, the first time subscript in the expectation operator denotes under which probability measure the expectation is taken. We use the convention that expectation operators with one time subscript are taken under the objective probability measure. By integrating out the stochastic time of death, the expected life-time utility can be written as

$$E_{s,s}\left[\int_s^\infty e^{-(\rho+\nu)(t-s)}log\left(c_{s,t}\right)dt\right]. \tag{13}$$

The dynamics of financial wealth of an agent born at time s follow

$$dW_{s,t} = \left(r_tW_{s,t} + \pi_{s,t}\left(\mu^S_{s,t} - r_t\right) + \nu W_{s,t} + y_{s,t} - c_{s,t}\right)dt + \pi_{s,t}\sigma^S dz_{s,t},\ W_{s,s} = 0, \tag{14}$$

where $\pi_{s,t}$ denotes the dollar amount held in the risky asset.

All agents maximize expected utility from life-time consumption, Equation (12), subject to the wealth dynamics in Equation (14).

2.6 Equilibrium

In this section we derive the equilibrium for the economy.

Definition 1. *Given preferences, endowments and beliefs, equilibrium is a collection of allocations* $(c_{s,t}, \pi_{s,t})$ *and prices* $\left(r_t, \mu_t^S\right)$ *such that the processes* $(c_{s,t}, \pi_{s,t})$ *are optimal when agents maximize Equation (12) subject to the dynamic budget constraint in Equation (14) and markets clear:*

$$\int_{-\infty}^{t} \nu e^{-\nu(t-s)} c_{s,t} ds = Y_t, \tag{15}$$

$$\int_{-\infty}^{t} \nu e^{-\nu(t-s)} \pi_{s,t} ds = 0, \tag{16}$$

$$\int_{-\infty}^{t} \nu e^{-\nu(t-s)} \left(W_{s,t} - \pi_{s,t}\right) ds = 0. \tag{17}$$

As markets are complete, we solve the individual optimization by martingale methods as in Cox and Huang (1989). Consider an agent born at time s. The static optimization problem for this agent can be written as

$$\max_{c_s} \quad E_{s,s}\left[\int_s^{\infty} e^{-(\rho+\nu)(t-s)} log\left(c_{s,t}\right) dt\right]$$

$$s.t.$$

$$E_{s,s}\left[\int_s^{\infty} e^{-\nu(t-s)} \xi_{s,t} c_{s,t} dt\right] = E_{s,s}\left[\int_s^{\infty} e^{-\nu(t-s)} \xi_{s,t} y_{s,t} dt\right].$$

From the first order conditions (FOCs), we have

$$\frac{e^{-(\rho+\nu)(t-s)}}{c_{s,t}} = \kappa_s e^{-\nu(t-s)} \xi_{s,t}, \tag{18}$$

where κ_s denotes the Lagrange multiplier. For $s \leq u \leq t$ the FOCs imply

$$e^{-(\rho+\nu)(t-u)} \left(\frac{c_{s,u}}{c_{s,t}}\right) = e^{-\nu(t-u)} \frac{\xi_{s,t}}{\xi_{s,u}}. \tag{19}$$

The total wealth at time $u \geq s$ of an agent born at time s is the sum of the value of endowment, $H_{s,u} = \frac{1}{\xi_{s,u}} E_{s,u}\left[\int_u^{\infty} e^{-\nu(t-s)} \xi_{s,t} y_{s,t} dt\right]$, and financial wealth, $W_{s,t}$. Let the total wealth at time u be $\hat{W}_{s,u} = H_{s,u} + W_{s,u}$. Using the static budget constraint, we obtain

$$c_{s,u} = (\rho + \nu) \hat{W}_{s,u}. \tag{20}$$

Equation (20) shows that the standard constant wealth to consumption ratio with log utility holds in our overlapping generations setup with incomplete information. Using the market clearing conditions and the individual aggregate endowments, we have

$$Y_t = \int_{-\infty}^{t} \nu e^{-\nu(t-s)} c_{s,t} ds = \int_{-\infty}^{t} \nu e^{-\nu(t-s)} (\rho + \nu) \hat{W}_{s,t} ds = (\rho + \nu) \hat{W}_t, \tag{21}$$

and, consequently, aggregate wealth, $\hat{W}_t$, is given by

$$\hat{W}_t = \frac{Y_t}{\rho + \nu}. \tag{22}$$

Note that total aggregate wealth, $\hat{W}_t$, equals the value of aggregate endowments, H_t. Thus, from the budget condition we have that the consumption of an agent born at time s equates with endowment

$$c_{s,s} = y_{s,s} = Y_s. \tag{23}$$

Using this relation, we obtain the following proposition:

Proposition 2. *Optimal consumption at time t of agents born at time $s \leq t \leq \tau$, where τ denotes the stochastic time of death, is*

$$c_{s,t} = Y_s e^{-\rho(t-s)} \left(\frac{\eta_{s,t}}{\eta_{s,s}} \right) \left(\frac{\xi_s}{\xi_t} \right). \tag{24}$$

The next proposition characterizes the stochastic discount factor.

Proposition 3. *In equilibrium, the stochastic discount factor is*

$$\xi_t = \frac{X_t}{Y_t}, \tag{25}$$

where X_t solves the integral equation

$$X_t = \int_{-\infty}^{t} \nu e^{-(\rho+\nu)(t-s)} X_s \frac{\eta_{s,t}}{\eta_{s,s}} ds. \tag{26}$$

The fraction of aggregate output at time t consumed by an agent born at time s is $\frac{c_{s,t}}{Y_t}$, and since the measure of agents born at time s equals $\nu e^{-\nu(t-s)}$, we have that the fraction of aggregate output at time t consumed by agents born at time s is

$$f_{s,t} = \nu e^{-\nu(t-s)} \frac{c_{s,t}}{Y_t} = \nu e^{-\nu(t-s)} \frac{Y_s e^{-\rho(t-s)} \left(\frac{\eta_{s,t}}{\eta_{s,s}}\right)\left(\frac{\xi_s}{\xi_t}\right)}{Y_t} = \nu e^{-(\nu+\rho)(t-s)} \left(\frac{X_s}{X_t}\right)\left(\frac{\eta_{s,t}}{\eta_{s,s}}\right). \tag{27}$$

Next, we introduce a decomposition of X_t:

Proposition 4. *Let*

$$d\bar{\eta}_t = -\bar{\Delta}_t \bar{\eta}_t dz_t, \tag{28}$$

where $\bar{\Delta}_t$ is the consumption share weighted average disagreement in the economy given by

$$\bar{\Delta}_t = \int_{-\infty}^{t} f_{s,t} \Delta_{s,t} ds, \tag{29}$$

then

$$X_t = e^{-\rho t} \bar{\eta}_t. \tag{30}$$

The process $\bar{\eta}_t$ captures the change of measure from the "market view" as measured by the consumption share weighted average belief, $\bar{\mu}_t = \int_{-\infty}^{t} f_{s,t} \hat{\mu}_{s,t} ds$, to the actual probability measure. The stochastic discount factor can then be decomposed in the following way:

$$\xi_t = \underbrace{\bar{\eta}_t}_{\text{Effect from disagreement}} \times \underbrace{\frac{e^{-\rho t}}{Y_t}}_{\text{Log utility discount factor}}. \tag{31}$$

The next proposition characterizes the real short rate and the market price of risk.

Proposition 5. *In equilibrium, the real short rate is*

$$r_t = \rho + \mu_Y - \sigma_Y^2 - \sigma_Y \bar{\Delta}_t, \tag{32}$$

and the market price of risk is

$$\theta_t = \sigma_Y + \bar{\Delta}_t. \tag{33}$$

The expression for the risk free rate is intuitive: The real rate is the standard real rate with log utility, $\rho + \mu_Y - \sigma_Y^2$, plus a correction for the disagreement in the economy. Thus, in times when the "market view" of the expected growth in the economy is higher than the actual growth, agents market view translates into demand for borrowing to smooth consumption across time. However, aggregate consumption is fixed and, consequently, the real interest rate increases to clear markets. Using the definition for $\bar{\Delta}_t$ we express the real rate as $\rho + \bar{\mu}_t - \sigma_Y^2$. Hence, the real rate is the same as in the standard log utility case, but now the expected aggregate endowment growth, μ_Y, is replaced by the market view, $\bar{\mu}_t$. The expression for the market price of risk is also intuitive: It is the standard log-utility market price of risk adjusted for disagreement. Again it is useful to rewrite the expression for the market price of risk as $\theta_t = \sigma_Y + \frac{1}{\sigma_Y}(\mu_Y - \bar{\mu}_t)$. From this we see that when the market is relatively optimistic about the expected growth, i.e., $\bar{\mu}_t > \mu_Y$, the market price of risk is low. Indeed, on the objective probability measure, the risky asset is expensive, and thus the market price of risk must be low.

Proposition 6. *The real short rate is pro-cyclical and the market price of risk is counter-cyclical.*

The intuitions for Proposition 6 follow from the dynamics of $\bar{\Delta}_t = \frac{\mu_Y - \bar{\mu}_t}{\sigma_Y}$. When there is a positive shock to aggregate output, i.e., $dz_t > 0$, all agents in the economy revise their expectations upwards and the disagreement process $\bar{\Delta}_t$ decreases. This implies that the risk free rate increases and the market price of risk decreases.

Before proceeding to the optimal portfolio policies, it is convenient to derive the dynamics of the optimal individual consumption.

Proposition 7. *The dynamics of individual consumption are*

$$dc_{s,t} = c_{s,t} \left(\mu_{c_{s,t}} dt + \sigma_{c_{s,t}} dz_t \right), \tag{34}$$

where the drift and the diffusion are

$$\mu_{c_{s,t}} = \mu_Y + \left(\bar{\Delta}_t + \sigma_Y \right) \left(\bar{\Delta}_t - \Delta_{s,t} \right), \quad \sigma_{c_{s,t}} = \sigma_Y + \bar{\Delta}_t - \Delta_{s,t}. \tag{35}$$

Proposition 7 shows that the diffusion of the individual consumption is driven by the difference between the consumption share weighted average disagreement in the economy, $\bar{\Delta}_t$, and the individual estimation error, $\Delta_{s,t}$.

To gain more intuition, we rewrite the diffusion term as a function of the difference between the individual and the market view, $\hat{\mu}_{s,t} - \bar{\mu}_t$. Following an aggregate shock, dz_t, agents adjust their consumption level by an amount equal to $\sigma_Y + \frac{\hat{\mu}_{s,t} - \bar{\mu}_t}{\sigma_Y}$. The direction and the extent of this adjustment are analyzed in Section 3.

Turning to the drift of the individual consumption process, we see again the influence of the difference between the individual and the market view. Further, note that here the difference is scaled by $\bar{\Delta}_t$.

The next proposition characterizes the optimal portfolio policy of an agent born at time s.

Proposition 8. *The optimal dollar amount invested in the risky asset for an agent born at time s is*

$$\pi_{s,t} = \frac{\bar{\Delta}_t - \Delta_{s,t}}{\sigma^S} \hat{W}_{s,t} + \frac{\sigma_Y}{\sigma^S} W_{s,t}. \tag{36}$$

The optimal portfolio has two components. The sign of the first one is determined by the relative disagreement of the market and the agent scaled by the variance of the risky asset.

Thus, if an agent is relatively more optimistic about the aggregate growth he will be long the risky asset, while if he is relatively pessimistic he will be short. The second component is the financial wealth multiplied by the volatility of the aggregate output scaled by the variance of the risky asset.

It is of interest to focus on the optimal portfolio allocation of a new born agent

$$\pi_{s,t} = \frac{\bar{\Delta}_t}{\sigma^S} H_{s,s}. \tag{37}$$

The optimal portfolio for a new born depends exclusively on the relative disagreement in the market since the agent is born with the correct prior about the expected aggregate endowment growth. The dollar amount that he invests in the stock market depends exclusively on his endowment since we assume that financial wealth at birth is zero. In the case of no disagreement between the market view and the actual growth rate, the portfolio allocation is trivially zero. If the market is more optimistic (pessimistic) than the actual growth rate $\bar{\mu}_t > \mu_Y$ ($\bar{\mu}_t < \mu_Y$), then the newborn will short (go long in) the risky asset.

3 Numerical Illustrations

To strengthen the intuition for our results, we simulate an economy populated by a large number of cohorts where one cohort is born every period. One period in the simulation represents one month. After 6000 burn-in periods we obtain an economy with 6000 cohorts. We employ the final values from the burn-in simulation as starting values for simulating the same economy for another 6000 periods forward. We generate data from $500,000$ simulations, each with 6000 periods or 500 years. The data is used to study the relation between portfolio choice and shocks and the relation between perceived and objective expected stock market returns by cohort lifespan. It turns out that the latter relation can explain that survey reports of people's expectations are negatively related to future realized returns. We also study consumption growth by cohort lifespan.

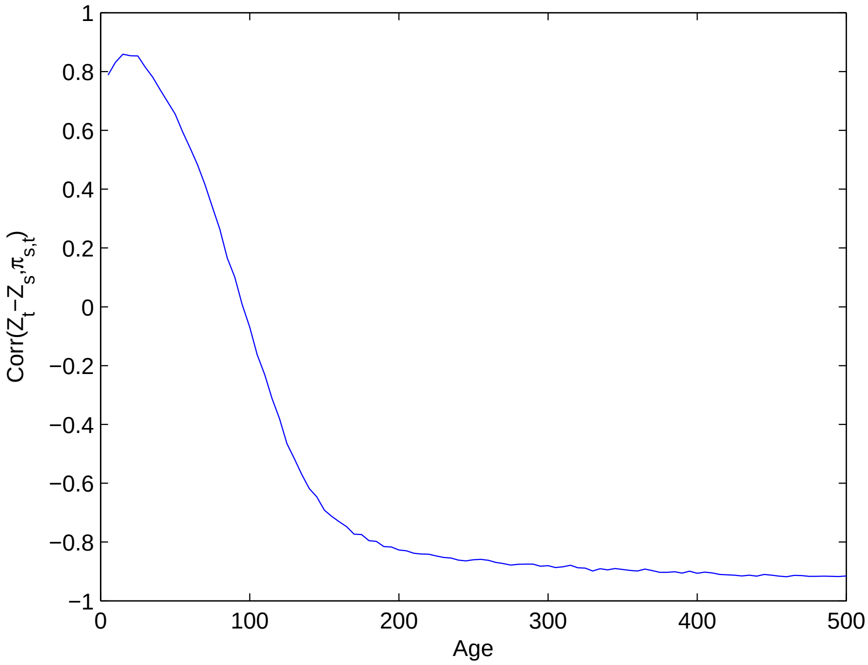
In the numerical illustrations we use the following parameter values: The discount rate is 1%. For the death probability, we use 2%. The drift and volatility of aggregate endowment are set to 2%. Prior variance of the expected aggregate endowment growth, $\bar{V}$, is 0.02^2. The diffusion term for the risky asset equals 15%.

Figure 1 shows the correlation between portfolio allocations and stock market shocks by cohort age profile. The correlation increases during the first years and then declines monotonically over time reaching -1 in ripe old age.This result shows that young individuals optimally update, given their beliefs, more than old agents. When young, each shock to the stock market is perceived as a true shock and, therefore, portfolio re-allocations move in lock step with shocks or price changes.

The reason for the correlation to approach -1 in the long run can be understood from considering the general equilibrium properties of the model. For the market to clear, the old counter-balance the portfolio allocations of the young. Specifically, consider the optimal portfolio allocation for an agent as shown in Equation (8). Using the definitions of $\bar{\Delta}_t$ and $\Delta_{s,t}$ we express the optimal portfolio as $\pi_{s,t} = \frac{\hat{\mu}_{s,t}-\bar{\mu}_t}{\sigma_Y \sigma^S}\hat{W}_{s,t} + \frac{\sigma_Y}{\sigma^S} W_{s,t}$. Thus, the portfolio allocation is driven by the difference between the agents' and the market view, $\hat{\mu}_{s,t} - \bar{\mu}_t$. Following a positive shock to the stock market, $dz_t > 0$, the young expect an aggregate consumption growth that is larger in changes than the market view, since $Var(\hat{\mu}_{s,t}) \geq Var(\bar{\mu}_t)$. Hence, the young increase their allocation in the risky asset. As the old are less sensitive to aggregate shocks their estimates of aggregate consumption growth are lower than the market view, since $Var(\hat{\mu}_{s,t}) \leq Var(\bar{\mu}_t)$. Therefore, they counter-balance the behaviour of the young by reducing their demand for the risky asset.

From Figure 1 above we learn that after a positive aggregate shock young individuals increase their risky asset holdings whereas the old agents decrease it. To assess the consequences of such exposures, we investigate further the equilibrium dynamics of the risky asset. Particularly, we compare the expected return under the objective probability measure, left hand side of Equation (7), with the perceived one, right hand side of Equation (7). Using

Figure 1: *Portfolios and Shocks.* The figure plots the correlation between portfolio allocations and stock market shocks by cohort lifespan. Shocks are model as Brownian increments. The figure is averaged from $500,000$ simulations with 6000 periods or 500 years per simulation.



a no-arbitrage condition, we express the expected return under the objective measure as $\mu_t^S = r_t + \sigma^S \theta_t$. Substituting the equilibrium formulas for the real short rate and the market price of risk into the latter equation, we obtain an intuitive expression for objective expected returns

$$\mu_t^S = A + B\bar{\Delta}_t \quad \text{where} \quad A = \rho + \mu_Y - \sigma_Y^2 + \sigma^S \sigma_Y, \quad B = \sigma^S - \sigma_Y > 0. \tag{38}$$

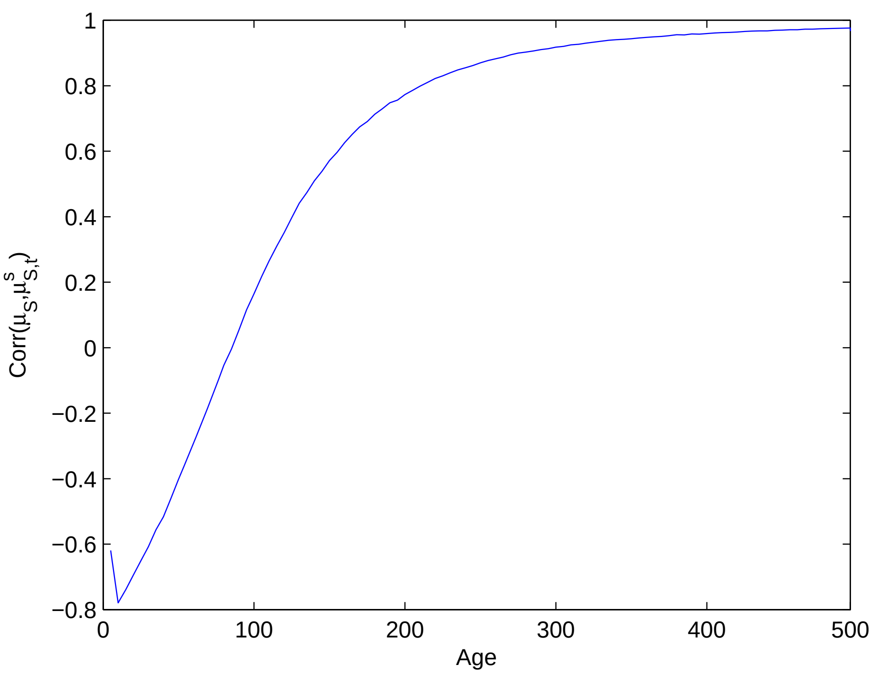
From the above we see that expected returns are high when $\bar{\Delta}_t$ is high. Moreover, expected returns decrease after a positive shock to aggregate output because agents revise their expectation upwards, and hence $d(\mu_Y - \bar{\mu}_t) < 0$ and $\bar{\Delta}_t$ declines. Using the same argument for perceived expected returns, we obtain

$$\mu_{s,t}^S = A + B\bar{\Delta}_t - \sigma^S \Delta_{s,t}. \tag{39}$$

We see that the impact of the last term of $\mu^S_{s,t}$ moves in opposite direction to the one described for the expected return under the objective measure. We also have that $B = \left(\sigma^S - \sigma_Y\right) < \sigma^S$. Taken together, we see that for young agents where $Var(\hat{\mu}_{s,t}) \geq Var(\bar{\mu}_t)$, the perceived expected return will move in opposite direction to the objective returns. For old and experienced agents where $Var(\hat{\mu}_{s,t})$ is small, the second term dominates the third term, and therefore the perceived expcted return is positively correlated with the actual expected return.

Figure 2 plots the correlation between the expected return and the perceived expected return over time, assuming that the volatility of stock market returns is greater than the one of the aggregate consumption, $\sigma^S > \sigma_Y$.

Figure 2: *Perceived versus Objective Returns.* The figure plots the correlation between the expected return under the objective measure and the perceived expected return by cohort lifespan. Shocks are model as Brownian increments. The figure is averaged from 500,000 simulations with 6000 periods or 500 years per simulation.



The correlation is negative and approaches -1 for young agents, from where it increases monotonically in age. When the stock market experiences positive returns, i.e. $dz_t > 0$, then the stock market return under the objective measure is expected to be low because

$\bar{\Delta}_t$ decreases following upward expectation revisions by the agents. Yet, young agents still believes expect the stock market returns are high and this explains the strong negative correlation. The correlation turns positive over time, indicating that old agents perceive expected returns to vary closely with the objective expected return. When young, an agent makes mistakes when estimating the expected stock market returns. In good times, the agent expects stock market returns to be high also in the foreseeable future suggesting a momentum or trend chasing like behaviour for expectation formation. Older generations eventually learn and correctly perceive an expected stock market return close to the objective expected return. Their behaviour apparently resemble that of individuals with correct beliefs. In fact, using the equality $\mu^S_{s,t} = \mu^S_t - \sigma^S \Delta_{s,t}$ and Proposition 1, we observe that $\lim_{t\to\infty} \mu^S_{s,t} = \mu^S_t$.

Our results regarding perceived expected returns provide an additional view at the explanation for the optimal portfolio allocation above. A positive shock to the stock market rises young agents expectations about future stock market return, in turn we see an increase in their demand for the risky asset. Old agents reduce their expectations about expected stock market relative to the young and, coherently, they therefore reduce their portfolio holdings in the risky asset.

Greenwood and Shleifer (2014) show that average beliefs from survey measures are negatively correlated with future realized returns. In the spirit of a mean forecast from a survey, we define the average belief in the economy as

$$\hat{\mu}^S_t = \int_{-\infty}^{t} \nu e^{-\nu(t-s)} \mu^S_{s,t} ds. \tag{40}$$

Table 1 below shows, consistent with Greenwood and Shleifer (2014), that the average belief in the economy is negatively correlated with future realized returns.

The intuition for this result is that the actual expected return is driven by consumption share weighted average belief and not the average belief, $\hat{\mu}^S_t$. Thus, the average belief puts too much weight on the young and inexperienced agents with low wealth. As the young

Table 1: *Average belief and future realized returns.* The table shows coefficient estimate from a regression of future realized returns on the average expected returns: $R_{t+1} = a + b\hat{\mu}_t^S + e_{t+1}$. The regression uses data from 500,000 simulations with 6000 periods or 500 years per simulation.

$\hat{\mu}_{S,t}$	-0.21
	$[-4.971]$
constant	0.04
	[20.755]

agents believe future returns are high after a series of positive shocks to the stock market, the average belief reflects their view and, therefore, predicts future returns with a negative sign.

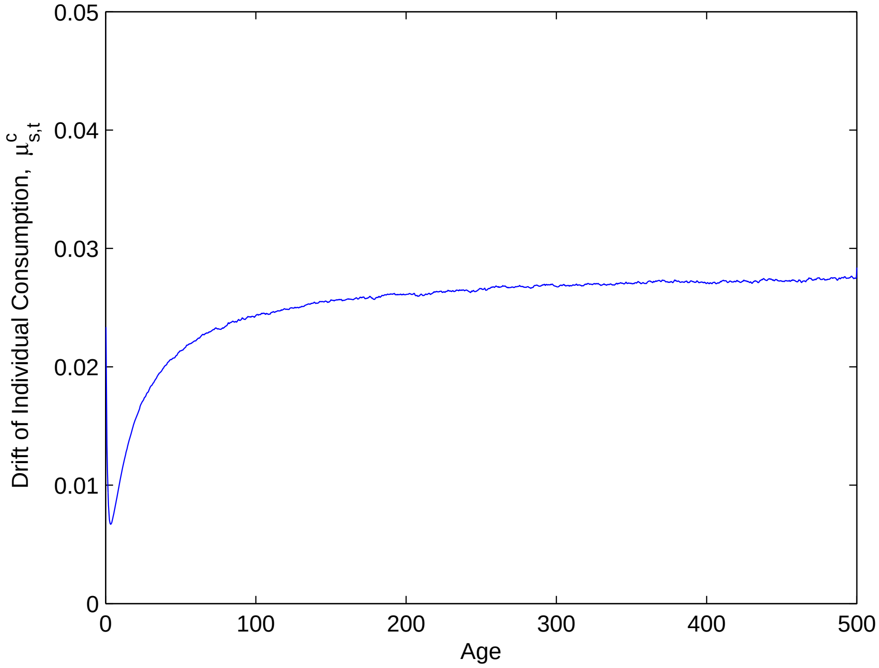
Above we investigate the way young individuals make mistakes in estimating expected returns by comparing the perceived expected returns with the objective expected return. Figure 3 shows how such estimation mistakes affect consumption growth dynamics. We learn that cohort specific consumption growth falls dramatically for young agents but increases as agents learn and age. We note that the drop in consumption for the young is due to the assumption that all cohorts start out with the correct prior. When young, any shock induces a large revision in estimates for growth as well as large revisions to portfolios. As the young trade with experienced agents their consumptions growth is low. Figure 3 shows that the economic cost of learning from experience are potentially large as losses accumulate during the early phases of life, and only after an extended time of learning will an agent's consumption growth catch up.

4 Empirical Implications

In this section, we present an empirical analysis of the predictions of our model. We use data from the Michigan Survey of Consumers and from Kenneth French's website. The Michigan Survey of Consumers (MSC) collects consumers experiences and beliefs regarding individual and aggregate economic conditions.[4] We choose question 24 from the MSC questionnaire as

[4] See http://www.sca.isr.umich.edu/ for a detailed description of the Michigan Survey of Consumers.

Figure 3: *Cohort Specific Consumption Growth.* The figure plots the dynamics of the drift term of consumption under the objective measure by cohort lifespan. Shocks are model as Brownian increments. The figure is averaged from $500,000$ simulations with 6000 periods or 500 years per simulation.



a proxy for agents expectations of future stock market returns. It asks the following:

"The next question is about investing in the stock market. Please think about the type of mutual fund known as a diversified stock fund. This type of mutual fund holds stock in many different companies engaged in a wide variety of business activities. Suppose that tomorrow one were to invest one thousand dollar in such a mutual fund. Please think about how much money this investment would be worth one year from now. What do you think is the percent chance that this one thousand dollar investment will increase in value in the year ahead, so that it is worth more than one thousand dollars one year from now?"

Answers to the question are grouped into intervals and MSC reports the frequency per interval. These frequencies are also available for three age groups or cohorts: $18-34$, $35-54$ and older than 55. We use these data to build two measures for beliefs about future stock returns.

Firstly, we construct an index taking the difference between individuals that replied to the question above with a percentage strictly larger than 50 percent, e.g. optimists, and those who replied with a percentage strictly less than 50 percent, i.e. pessimists. For each month in our panel, we compute the index as $optimists - pessimists + 100$. We do so for each cohort. When the index is below 100 then pessimism prevails, otherwise optimism prevails. In what follows, we refer to this measure as index of individual beliefs (IB).

Secondly, we construct an index measure using a weighted average beliefs (WAB) based on the answers to the questionnaire. Specifically, we compute the product of the number of individuals whose answer falls into a particular interval and the midpoint of the interval; summing over all intervals produces our second measure for individuals expectations.

4.1 Experience Matters

The first empirical test we perform investigates the role of past stock market returns on individual expectations. In doing so, we are guided by the empirical literature as several papers show the impact of experience on individual decisions; Kaustia and Knüpfer (2008), Malmendier and Nagel (2011) and Malmendier and Nagel (2014) are some examples from this literature. Further, our empirical analysis is motivated by the strand of the behavioural finance literature that seeks to understand the role of extrapolative expectations for the determination of security prices; see Greenwood and Shleifer (2014) and references therein.

Using the three cohorts of the MSC, $18-34$, $35-54$ and 55+, we compute the correlations between the past 12-month returns on the U.S. stock market with our measures for cohort expectations. Table 2 presents these pairwise correlations using IB as a measure of beliefs. WAB produces similar results.

From the first column of the table we see that the beliefs of each cohort is strongly positively correlated with past stock market returns. Moreover, beliefs of the $18 - 34$ cohort show the largest correlation with the stock market, followed by the $35 - 54$ and the 55+ cohorts, respectively. This result supports the proposition of our model that young agents are more sensitive to shocks than old individuals. Specifically, it is consistent with the young updating their expectations more than the old following a stock market shock.

Table 2: *On the Role of Past Stock Market Returns for Beliefs Formation.* The table presents pairwise correlations between the past 12-month cumulative stock market return, R, with the IB measure of beliefs for each cohort. Data are from the Michigan Survey of Consumers and from Kenneth French's website. Sample: July 2002 to October 2012.

	R	Cohort $18-34$	Cohort $35-54$	Cohort 55+	Pooled
R	1				
Cohort $18-34$	0.644	1			
Cohort $35-54$	0.629	0.738	1		
Cohort 55+	0.611	0.662	0.804	1	
Pooled	0.694	0.905	0.932	0.8789	1

Figure 4 plots the dynamics of the lagged 12-month cumulative stock market returns and of the IB based measure for cohort beliefs. We note that past realizations and beliefs for the different cohorts track each other closely. WAB produces similar time-series results. As shown in Table 2, the beliefs of the young cohort covaries with past stock market returns more than beliefs of other cohorts do. Qualitatively, our model produces exactly this pattern.

Figure 4: *Beliefs and Realized Return Dynamics.* The figure plots the past 12-month cumulative US stock market returns and the IB measure of beliefs for the $18-34$, $35-54$ and 55+ cohorts. Data are from the Michigan Survey of Consumers and from Kenneth French's website. Sample: July 2002 to July 2012.

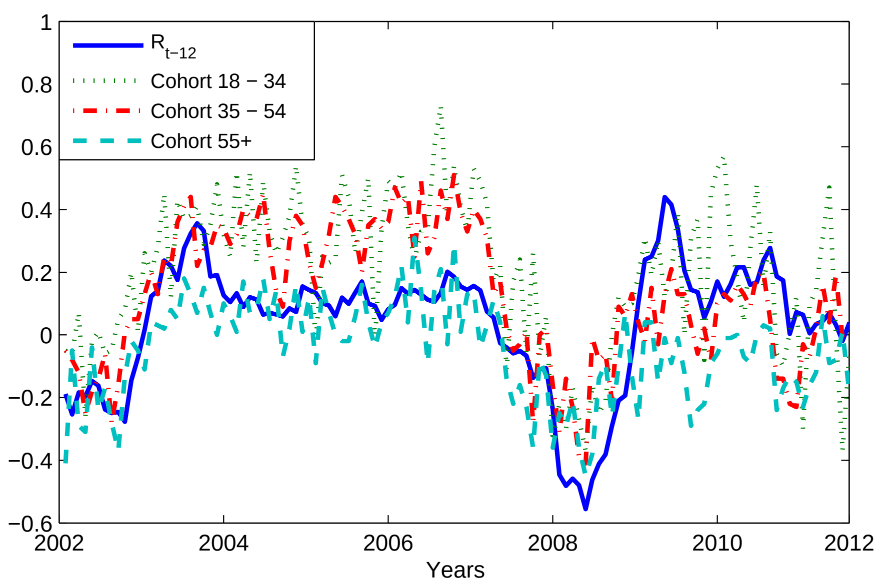


Table 3 presents coefficient estimates of univariate regressions for each cohort:

$$Exp_t = \alpha + \beta R_{t-12} + \epsilon_t, \tag{41}$$

where Exp_t represents beliefs for each cohort at time t and R_{t-12} denotes the cumulative past 12-month returns on the stock market.

Table 3: *Testing for Experience.* The table shows the coefficient estimates of univariate regressions for cohort beliefs proxied by IB on the cumulative past 12-month returns on the stock market. For each cohort we perform a separate regression. Newey-West t-statistics are given in parenthesis. Data are from the Michigan Survey of Consumers and from Kenneth French's website. Sample: July 2001 to April 2012.

	Cohort $18-34$	Cohort $35-54$	Cohort 55+	Pooled
R_{t-12}	78.22	67.58	48.00	193.80
	[7.485]	[5.016]	[5.222]	[6.044]
Constant	16.38	9.90	-6.80	19.48
	[5.745]	[2.681]	[-3.134]	[2.311]
N	117	117	117	117
R^2	0.41	0.39	0.37	0.48

Two results stand out: Firstly, coefficients for each cohort are strongly positive and statistically significant at the 1%-level. Secondly, coefficients decrease monotonically from the youngest cohort ($18-34$) to the oldest (55+). The difference between the slope coefficient of the young and the old is statistically significant at the 5% level. Using WAB instead of IB produces similar results. These results support our model predictions: agents take into account past returns when forming expectations and young agents expectations are more sensitive towards news than the expectations of the old. Hence, according to the data experience matters for forming expectations.[5]

4.2 Predictive Power of Expectations for Returns

In this subsection we test whether cohort beliefs, constructed from MSC data, forecast future returns. We perform univariate regressions of future stock returns on beliefs for each cohort. Recall that under homogeneous and correct beliefs future returns and expected returns should be perfectly positively correlated; hence, the coefficient estimate from a univariate regression in such a world equals to one.

[5] These results are also consistent with a broad literature on extrapolative expectations, e.g., Greenwood and Shleifer (2014), Barberis et al. (2015) and references therein.

Table 4 shows the coefficients for the three cohort specific univariate regressions when IB is used as independent variable. We see in Table 4 that all coefficients are negative. Importantly, the coefficient for the young is the only significant one. Recall, that the negative signs appear puzzling since it implies that people's expectations are consistently negatively related to "objective" expectations. In our model, however, the coefficient for the young is predicted to be negative as they act optimally as trend chasers, given their beliefs. WAB produces results that are similar to the results in Table 4.

Table 4: *Predictability of Returns.* The table shows the coefficient estimates of univariate regressions of cumulative 12-month-ahead stock market returns on cohort beliefs proxied by IB. For each cohort we perform a separate regression. Newey-West t-statistics are given in parenthesis. Data are from the Michigan Survey of Consumers and from Kenneth French's website. Sample: July 2002 to April 2012.

	$R(t+12)$	$R(t+12)$	$R(t+12)$	$R(t+12)$
Cohort $18-34$	-0.002			
	$[-2.047]$			
Cohort $35-54$		-0.002		
		$[-1.618]$		
Cohort 55+			-0.0026	
			$[-1.616]$	
Pooled				-0.0001
				$[-1.829]$
N	117	117	117	117
R^2	0.05	0.04	0.04	0.05

Overall, our model provides a way of reconciling several of these pieces of evidence. On the one hand, the model generates a counter-cyclical market price of risk. As explained in Proposition 6, after a positive shock to aggregate output, all agents revise upwards their expectations about the expected consumption growth, thus the market view, $\bar{\mu}_t$, soars. This has the effect to push down the market price of risk. On the other hand, we are qualitatively able to reproduce the empirical evidence emerging from surveys on individual expectations. Firstly, we show that young agent expectations predict future returns negatively. In the model this mechanism is shown in Equations (38) and (39) and in Figure 1. Following a positive aggregate shock the expected return under the true measure decreases, whereas the expected return perceived by young agents increases. This mechanism explains the negative correlation between the expected return and the perceived one as shown in Figure 1 for the first years. Secondly, our empirical evidence shows that expectations of the old have no predictive power for future returns.

5 Conclusions

We make two innocuous assumptions concerning how young agents learn from data: First, to isolate the influence of experience we assume that all agents share the true prior about consumption growth. Second, agents are unsure about their estimate for growth and not all data is used for learning about consumption growth. Therefore, early on in life new data can have a large impact on expectations. Hence, the young are more likely to be overly optimistic or pessimistic than the old and experienced. Put simply, experience matters. Specifically, these two assumptions imply in an overlapping generations economy with optimal learning, given beliefs, among other results that the young act as trend chasers, wealth shifts from young to old, and average beliefs are negatively correlated with the true value for expected returns.

Recent literature on portfolio choice suggests that experience matters for portfolio choice, Malmendier and Nagel (2011). Our experience driven equilibrium portfolio policies are motivated by and consistent with their evidence. The inexperienced optimally use extrapolative expectations and thus their expectations are negatively related to the true expected return. Therefore, our model predicts that the inexperienced bias survey forecasts. In light of this interpretation, perhaps survey reports of people's expectations are, after all, in line with the view that people do try to make optimal decisions or inference.

A Extension: Within-Cohort Heterogeneity

We extend the model to allow for within-cohort heterogeneity. We assume that within each cohort agents are be born with different priors about the mean and the variance of aggregate consumption growth.

In this set up, the agent maximizes

$$E_{a,s,s}\left[\int_s^\infty e^{-(\rho+\nu)(t-s)} log\left(c_{a,s,t}\right)dt\right], \tag{42}$$

subject to the static budget constraint

$$E_{a,s,s}\left[\int_s^\infty e^{-\nu(t-s)}\xi_{a,s,t}c_{a,s,t}dt\right] \leq E_{a,s,s}\left[\int_s^\infty e^{-\nu(t-s)}\xi_{a,s,t}y_{a,s,t}dt\right]. \tag{43}$$

The expectation operator at time s and the stochastic discount factor, ξ, are taken under the probability measure of one specific agent a born at time s. We assume that agents within a cohort are distributed following a generic distribution $g(a)$ defined over the domain $[\underline{a};\bar{a}]$, such that $\int_{\underline{a}}^{\bar{a}} g(a)da = 1$. Therefore, the time-$t$ size of the population is

$$\int_{-\infty}^t \int_{\underline{a}}^{\bar{a}} \nu e^{-\nu(t-s)} g(a)dads = \int_{-\infty}^t \nu e^{-\nu(t-s)} \int_{\underline{a}}^{\bar{a}} g(a)dads = 1. \tag{44}$$

Given this economy, the definition of equilibrium changes from Definition (1) to take into account the dispersion of beliefs within cohorts.

Definition 2. *Given preferences, endowments and beliefs, equilibrium is a collection of allocations* $(c_{a,s,t},\pi_{a,s,t})$ *and prices* $\left(r_t,\mu_t^S\right)$ *such that the processes* $(c_{a,s,t},\pi_{a,s,t})$ *are optimal when agents maximize Equation (42) subject to the static budget constraint in Equation (43) and* *markets clear:*

$$\begin{aligned}
\int_{-\infty}^{t}\int_{\underline{a}}^{\bar{a}} \nu e^{-\nu(t-s)} g(a) c_{a,s,t}\, dads &= Y_t, &(45)\\
\int_{-\infty}^{t}\int_{\underline{a}}^{\bar{a}} \nu e^{-\nu(t-s)} g(a) \pi_{a,s,t}\, dads &= 0, &(46)\\
\int_{-\infty}^{t}\int_{\underline{a}}^{\bar{a}} \nu e^{-\nu(t-s)} g(a) \left(W_{a,s,t} - \pi_{a,s,t}\right) dads &= 0. &(47)
\end{aligned}$$

From the first order conditions, we have

$$\frac{e^{-(\rho+\nu)(t-s)}}{c_{a,s,t}} = \kappa_{a,s} e^{-\nu(t-s)} \xi_{a,s,t}, \tag{48}$$

where $\kappa_{a,s}$ denotes the Lagrange multiplier. Note that for $s \leq u \leq t$ the FOCs imply

$$e^{-(\rho+\nu)(t-u)} \left(\frac{c_{a,s,u}}{c_{a,s,t}}\right) = e^{-\nu(t-u)} \frac{\xi_{a,s,t}}{\xi_{a,s,u}}. \tag{49}$$

Denoting $\hat{W}_{a,s,u}$ the total wealth at time t for an agent a belonging to a cohort s, and using the static budget constraint, we obtain

$$c_{a,s,u} = (\rho+\nu)\hat{W}_{a,s,u}. \tag{50}$$

Equation (50) shows that the result in Equation (20) extends to the case with within-cohort heterogeneity. As in the baseline model, plugging Equation (50) into the market clearing conditions, leads to

$$Y_t = \int_{-\infty}^{t}\int_{\underline{a}}^{\bar{a}} \nu e^{-\nu(t-s)} g(a) \left(\rho+\nu\right) \hat{W}_{a,s,t}\, dads = \left(\rho+\nu\right) \hat{W}_t, \tag{51}$$

and, consequently, aggregate wealth, $\hat{W}_t$, is given by

$$\hat{W}_t = \frac{Y_t}{\rho+\nu}. \tag{52}$$

We keep entertaining the assumption that endowment does not depend on the time of birth imposing that all the agents within each cohort receive the same endowments, i.e., $y_{a,s,t} = y_{s,t}$ for all $s \leq t$. Hence, individual endowments evolve as in Equation (1).

With these assumptions, we have that

$$\hat{W}_s = E_{a,s,s}\left[\int_s^\infty e^{-\nu(t-s)}\xi_{a,s,t}Y_t dt\right] = \hat{W}_{a,s,s}. \tag{53}$$

Hence, after substituting Equation (52) into Equation (50) we obtain that the consumption of a new born agent a equates with his endowment

$$c_{a,s,s} = y_{a,s,s} = Y_s. \tag{54}$$

The following proposition gives the optimal consumption level for an agent a within cohort s under the true probability measure.

Proposition 9. *In equilibrium, optimal consumption for agent a born into cohort s is*

$$c_{a,s,t} = Y_s e^{-\rho(t-s)}\frac{\eta_{a,s,t}}{\eta_{a,s,s}}\frac{\xi_s}{\xi_t}. \tag{55}$$

The stochastic discount factor in this economy is given by the following proposition.

Proposition 10. *In equilibrium, the stochastic discount factor is*

$$\xi_t = \frac{X_t}{Y_t}, \tag{56}$$

where X_t solves the integral equation

$$X_t = \int_{-\infty}^t \int_{\underline{a}}^{\bar{a}} \nu e^{-(\rho+\nu)(t-s)} g(a) X_s \frac{\eta_{a,s,t}}{\eta_{a,s,s}} dads. \tag{57}$$

Again, before proceeding with the optimal prices, it is useful to define the following two objects. Firstly, the fraction of aggregate output at time t consumed by an agent of type, a, born at time s is

$$f_{a,s,t} = \nu e^{-\nu(t-s)} g(a) \frac{c_{s,t}}{Y_t} = \nu e^{-\nu(t-s)} g(a) \frac{c_{a,s,s} e^{-\rho(t-s)} \frac{\eta_{a,s,t}}{\eta_{a,s,s}} \frac{\xi_s}{\xi_t}}{Y_t}, \tag{58}$$

and secondly, the consumption share weighted average disagreement in the economy

$$\bar{\Delta}_t^{'} = \int_{-\infty}^{t} \int_{\underline{a}}^{\bar{a}} f_{a,s,t} \Delta_{a,s,t} dads, \tag{59}$$

where

$$\Delta_{a,s,t} = \frac{\mu_Y - \mu_{a,s,t}}{\sigma_Y}, \tag{60}$$

represents the standardized estimation error of one agent born at time s relative to the true probability measure. It is important to notice that, as in the baseline case, we can define the concept of "market view" as the consumption weighted average belief in the economy. In fact,

$$\bar{\mu}_t^{'} = \int_{-\infty}^{t} \int_{\underline{a}}^{\bar{a}} f_{a,s,t} \hat{\mu}_{a,s,t} dads, \tag{61}$$

extends the definition of "market view" given in the baseline case to take into account the within-cohort heterogeneity.

The next proposition characterizes the real short rate and the market price of risk.

Proposition 11. *In equilibrium, the real short rate is*

$$r_t = \rho + \mu_Y - \sigma_Y^2 - \sigma_Y \bar{\Delta}_t^{'}, \tag{62}$$

and the market price of risk is

$$\theta_t = \sigma_Y + \bar{\Delta}_t^{'}. \tag{63}$$

We conclude that the same intuitions as for the baseline case without heterogeneity within a cohort of agents apply here.

A.1 Gaussian Type Distribution

In this subsection we follow Atmaz (2014), and specify the distribution function, $g(a)$, to be Gaussian. Specifically, Atmaz (2014) derives equilibrium with a continuum of agents differing in their beliefs. However, in Atmaz (2014) agents are infinitively lived, so there are no generation specific beliefs. To be more precise, we assume that the distribution of agents, $g(a)$, is given by

$$g(a) = \frac{1}{\sqrt{2\pi\nu_0^2}} e^{-\frac{1}{2}\frac{a^2}{\nu_0^2}}. \tag{64}$$

Note that this is a Normal distribution with mean zero and variance ν^2. The belief of an agent of type a when born is $\hat{\mu}_{a,s,s} = \mu_Y + a$. We assume that agents are homogeneous with respect to the prior variance and that this is given by $\bar{v}$. As the bias parameter a has mean zero, the average agent is born with the correct prior. By standard filtering, one can show the the error process at time t of an agent born at time s with initial belief a is

$$\Delta_{a,s,t} = -\frac{\sigma_Y}{\sigma_Y^2 + \bar{v}\,(t-s)} a + \frac{\bar{v}\,(z_s - z_t)}{\sigma_Y^2 + \bar{v}\,(t-s)}. \tag{65}$$

The dynamics of the disagreement process is

$$d\eta_{a,s,t} = -\eta_{s,a,t}\Delta_{s,a,t} dz_t. \tag{66}$$

We want to aggregate to a cohort specific representative agent. Using Equation (55) we have

$$\begin{aligned} c_{s,t} = \int_{-\infty}^{\infty} c_{a,s,t} da &= \int_{-\infty}^{\infty} g(a) Y_s e^{-\rho(t-s)} \frac{\eta_{a,s,t}}{\eta_{a,s,s}} \frac{\xi_s}{\xi_t} da \\ &= Y_s e^{-\rho(t-s)} \frac{\xi_s}{\xi_t} \int_{-\infty}^{\infty} g(a) \frac{\eta_{a,s,t}}{\eta_{a,s,s}} da. \end{aligned} \tag{67}$$

As we can see from Equation (68) the only term that differs between the agent types born at time s is the disagreement process. Define the aggregate disagreement process for the cohort born at time s as

$$\eta_{s,t} = \int_{-\infty}^{\infty} g(a) \frac{\eta_{a,s,t}}{\eta_{a,s,s}} da. \tag{68}$$

By following the same approach as in Atmaz (2014) one can show that

$$d\eta_{s,t} = -\Delta_{s,t} \eta_{s,t} dz_t, \tag{69}$$

where

$$\Delta_{s,t} = \frac{\left(\bar{v} + \nu^2\right)\left(z_s - z_t\right)}{\sigma_Y^2 + \left(\bar{v} + \nu^2\right)(t - s)}. \tag{70}$$

Note that Equation (70) has the same form as in the base case in Proposition 1 with $\bar{V} = \bar{v} + \nu^2$. Hence, we can interpret the base case model as a model with heterogeneous initial beliefs that are normally distributed with zero bias on average. Note that the dynamics of the cohort specific belief will behave similar to the base case even without learning, i.e., when $\bar{v} = 0$. In this case, $\bar{V} = \nu^2$, and thus it only depends on the within cohort cross-sectional heterogeneity. Consequently, the convergence of the cohort specific belief does not happen because of learning of individual agents, but due to market selection. The agents that start with a relatively more correct initial belief will have a higher consumption growth and will eventually dominate the cohort.

B Proofs of Propositions

B.1 Proof of Proposition 1

Following standard filtering theory, Liptser and Shiryaev (1974a,b), the dynamics of the expected consumption growth as perceived by an agent born at time s are given by Equation (3). Defining the disagreement process as

$$\Delta_{s,t} = \frac{\mu_Y - \hat{\mu}_{s,t}}{\sigma_Y}, \tag{71}$$

and applying Ito's lemma to it we have

$$d\Delta_{s,t} = -\frac{\bar{V}}{\sigma_Y^2 + \bar{V}(t-s)}\Delta_{s,t}dt - \frac{\bar{V}}{\sigma_Y^2 + \bar{V}(t-s)}dz_t. \tag{72}$$

The solution to this stochastic differential equation is found by applying Ito's lemma to

$$\Delta_{s,t} = \frac{\bar{V}(z_s - z_t)}{\sigma_Y^2 + \bar{V}(t-s)} \tag{73}$$

which, then, yields the desired result. By the strong law of large numbers we have that

$$lim_{t\to\infty}\frac{z_t}{t} = 0, \tag{74}$$

and hence $lim_{t\to\infty}\Delta_{s,t} = 0.$

B.2 Proof of Proposition 2

We solve for equilibrium using the martingale method, Cox and Huang (1989). An agent born at time s *solves the following static optimization problem*

$$\max_{c_s} E_{s,s}\left[\int_s^\infty e^{-(\rho+\nu)(t-s)} log(c_{s,t})\, dt\right]$$

$$s.t.$$

$$E_{s,s}\left[\int_s^\infty e^{-\nu(t-s)}\xi_{s,t}c_{s,t}dt\right] = E_{s,s}\left[\int_s^\infty e^{-\nu(t-s)}\xi_{s,t}y_{s,t}dt\right].$$

From the first order conditions (FOCs) we have

$$\frac{e^{-(\rho+\nu)(t-s)}}{c_{s,t}} = \kappa_s e^{-\nu(t-s)}\xi_{s,t}, \tag{75}$$

where κ_s denotes the Lagrange multiplier. Note that for $s \leq t$ the FOCs imply

$$e^{-(\rho+\nu)(t-s)} \left(\frac{c_{s,s}}{c_{s,t}} \right) = e^{-\nu(t-s)} \frac{\xi_{s,s}}{\xi_{s,t}}. \tag{76}$$

Rearranging leads to

$$c_{s,t} = c_{s,s} e^{-\rho(t-s)} \frac{\xi_{s,s}}{\xi_{s,t}}. \tag{77}$$

Using Equation (23) and the Radon-Nikodym derivative to move from the probability measure of an agent born at time s to the actual probability measure, we get the optimal consumption at time t of an agent born at time $s \leq t$

$$c_{s,t} = Y_s e^{-(\rho+\nu)(t-s)} \frac{\eta_{s,t}}{\eta_{s,s}} \frac{\xi_s}{\xi_t} \tag{78}$$

where

$$\eta_{s,t} = \frac{\xi_t}{\xi_{s,t}}. \tag{79}$$

B.3 Proof of Proposition 3

The expression for the equilibrium stochastic discount factor is obtained taking the market clearing condition for the goods market, Equation (15), and plugging the optimal consumption at time t of an agent born at time s, Equation (24), into it. The resulting expression is as follows

$$Y_t = \int_{-\infty}^{t} \nu e^{-(\rho+\nu)(t-s)} Y_s \frac{\xi_s}{\xi_t} \frac{\eta_{s,t}}{\eta_{s,s}} ds. \tag{80}$$

Then, defining

$$X_t = \int_{-\infty}^{t} \nu e^{-(\rho+\nu)(t-s)} Y_s \xi_s \frac{\eta_{s,t}}{\eta_{s,s}} ds, \tag{81}$$

and rearranging Equation (80) leads to

$$\xi_t = \frac{1}{Y_t} \int_{-\infty}^{t} \nu e^{-(\rho+\nu)(t-s)} Y_s \xi_s \frac{\eta_{s,t}}{\eta_{s,s}} ds. \tag{82}$$

Substituting into the integrand above the expression $X_s = Y_s \xi_s$ yields the result.

B.4 Proof of Proposition 4

To prove Proposition 4, we start by obtaining the dynamics of X_t. Applying Ito's lemma to Equation (26), we obtain

$$\frac{dX_t}{X_t} = -\rho dt + \int_{-\infty}^{t} \nu e^{-(\rho+\nu)(t-s)} \frac{X_s}{X_t} \frac{d\eta_{s,t}}{\eta_{s,s}} ds = -\rho dt - \bar{\Delta}_t dz_t \tag{83}$$

where

$$\bar{\Delta}_t = \int_{-\infty}^{t} f_{s,t} \Delta_{s,t} ds, \tag{84}$$

and

$$f_{s,t} = \nu e^{-\nu(t-s)} \frac{Y_s e^{-\rho(t-s)} \left(\frac{\eta_{s,t}}{\eta_{s,s}}\right) \left(\frac{\xi_s}{\xi_t}\right)}{Y_t} = \nu e^{-(\rho+\nu)(t-s)} \left(\frac{X_s}{X_t}\right) \left(\frac{\eta_{s,t}}{\eta_{s,s}}\right) = \nu e^{-(\rho+\nu)(t-s)} \frac{c_{s,t}}{Y_t}, \tag{85}$$

which represents the share of aggregate output at time t that accrues to agents born at time s. X_t has dynamics satisfying the above stochastic differential equation and has the well-known analytic solution

$$X_t = X_0 e^{-\int_0^t \left(\rho + \frac{\bar{\Delta}_s^2}{2}\right) ds - \int_0^t \bar{\Delta}_s dz_s}. \tag{86}$$

Assuming that $X_0 = 1$, the expression for X_t can be written as

$$X_t = e^{-\rho t} \bar{\eta}_t \tag{87}$$

where

$$\bar{\eta}_t = e^{-\frac{1}{2}\int_0^t \bar{\Delta}_s^2 ds - \int_0^t \bar{\Delta}_s dz_s}. \tag{88}$$

Applying Ito's lemma to (88) gives the dynamics of $\bar{\eta}_t$,

$$d\bar{\eta}_t = -\bar{\eta}_t \bar{\Delta}_t dz_t. \tag{89}$$

Equations (87) and (89) yield the required results.

B.5 Proof of Proposition 5

From the expression of the equilibrium stochastic discount factor, Equation (56) we have

$$\xi_t = \frac{X_t}{Y_t}. \tag{90}$$

Using Equation (87), the stochastic discount factor can be decomposed in the following way

$$\xi_t = \bar{\eta}_t \frac{e^{-\rho t}}{Y_t}. \tag{91}$$

Then, by applying Ito's lemma to Equation (91) we have

$$d\xi_t = d\left(\bar{\eta}_t \frac{e^{-\rho t}}{Y_t}\right). \tag{92}$$

Using Equations (2) and (89), we get

$$d\left(\bar{\eta}_t \frac{e^{-\rho t}}{Y_t}\right) = \left(\bar{\eta}_t \frac{e^{-\rho t}}{Y_t}\right)\left[\left(-\rho - \mu_Y + \sigma_Y^2 + \sigma_Y \bar{\Delta}_t\right) dt - \left(\sigma_Y + \bar{\Delta}_t\right) dz_t\right]. \tag{93}$$

Matching the drift and diffusion terms of the state price density, Equation (9), with Equation (93), we obtain

$$r_t = \rho + \mu_Y - \sigma_Y^2 - \sigma_Y \bar{\Delta}_t \tag{94}$$

and

$$\theta_t = \sigma_Y + \bar{\Delta}_t. \tag{95}$$

These are the equilibrium real short rate and market price of risk, respectively.

B.6 Proof of Proposition 6

To prove Proposition 6 we need to compute the dynamics of $\bar{\Delta}_t$. Applying Ito's lemma to Equation (29) gives

$$d\bar{\Delta}_t = \frac{\partial \bar{\Delta}_t}{\partial f_{s,t}} df_{s,t} + \frac{\partial \bar{\Delta}_t}{\partial \Delta_{s,t}} d\Delta_{s,t} + \frac{\partial \bar{\Delta}_t}{\partial f_{s,t} \partial \Delta_{s,t}} \left(df_{s,t} d\Delta_{s,t}\right). \tag{96}$$

Thus, to evaluate Equation (96) we need the dynamics of $f_{s,t}$ and $\Delta_{s,t}$. From Equation (29), using Ito's lemma gives

$$df_{s,t} = f_{s,t}[\left(-\nu + \bar{\Delta}_t^2 - \bar{\Delta}_t \Delta_{s,t}\right) dt + \left(\bar{\Delta}_t - \Delta_{s,t}\right) dz_t] \tag{97}$$

Next, we compute from Equation (3) and making use of the fact that $\Delta_{s,t} = \frac{\mu_Y - \hat{\mu}_{s,t}}{\sigma_Y}$

$$d\Delta_{s,t} = -\Delta_{s,t} \frac{V_{s,t}}{\sigma_Y^2} dt - \frac{V_{s,t}}{\sigma_Y^2} dz_t. \tag{98}$$

Inserting the dynamics of $f_{s,t}$ and $\Delta_{s,t}$ into Equation (96) yields

$$d\bar{\Delta}_t = \int_{-\infty}^{t} \Delta_{s,t} df_{s,t} ds + \int_{-\infty}^{t} f_{s,t} d\Delta_{s,t} ds + \int_{-\infty}^{t} d\Delta_{s,t} df_{s,t} ds. \tag{99}$$

Focusing on the diffusion term we get

$$d\bar{\Delta}_t = \ldots dt + \int_{-\infty}^{t} \Delta_{s,t} f_{s,t} \left(\bar{\Delta}_t - \Delta_{s,t}\right) ds dz_t - \int_{-\infty}^{t} f_{s,t} \left(\frac{V_{s,t}}{\sigma_Y^2}\right) ds dz_t. \tag{100}$$

The last term of this equation is easy to study. In case of positive (negative) shock to aggregate output, i.e. $dz_t > 0\,(< 0)$, it is always negative (positive). The first term needs additional manipulations. Notice that the first term can be written as

$$\left(\bar{\Delta}_t^2 - \int_{-\infty}^{t} \Delta_{s,t}^2 f_{s,t} ds\right) dz_t. \tag{101}$$

It follows from the Jensen's inequality that

$$\left(\bar{\Delta}_t^2 - \int_{-\infty}^{t} \Delta_{s,t}^2 f_{s,t} ds\right) \leq 0. \tag{102}$$

Thus, the integrand will always be greater or equal to zero and the entire term will be less than or equal to zero. Then, a positive (negative) shock to aggregate output, i.e. $dz_t > 0\,(< 0)$, *will cause the consumption share weighted average disagreement in the economy to decrease (increase),* $d\bar{\Delta}_t < 0\,(> 0)$. *This will drive up (down) the interest rate and down (up) the market price of risk. This proves the proposition.*

B.7 Proof of Proposition 7

Applying Ito's lemma to Equation (24), yields the dynamics for optimal consumption,

$$dc_{s,t} = c_{s,t}\left[\left(\rho + r_t + \theta_t^2 - \Delta_{s,t}\theta_t\right)dt + \left(\theta_t - \Delta_{s,t}\right)dz_t\right]. \tag{103}$$

Substituting the optimal market price of risk, Equation (33), and real short rate, Equation (32), in the PDE above, we get

$$dc_{s,t} = c_{s,t}\left(\mu_{c_{s,t}}dt + \sigma_{c_{s,t}}dz_t\right), \tag{104}$$

where

$$\mu_{c_{s,t}} = \mu_Y + \left(\bar{\Delta}_t + \sigma_Y\right)\left(\bar{\Delta}_t - \Delta_{s,t}\right), \quad \sigma_{c_{s,t}} = \sigma_Y + \bar{\Delta}_t - \Delta_{s,t}, \tag{105}$$

which yields the result.

B.8 Proof of Proposition 8

To derive the optimal portfolio allocation at time t for an agent born at time s we start with

$$\hat{W}_{s,t} = \frac{1}{\xi_t} E_t \left[\int_t^{\infty} e^{-\nu(u-t)} \xi_u c_{s,u} du \right]. \tag{106}$$

Using

$$\hat{W}_{s,t} = \frac{c_{s,t}}{\rho + \nu}, \tag{107}$$

and substituting in optimal consumption, Equation (24), and rearranging terms we have

$$\xi_t \hat{W}_{s,t} = Y_s e^{-(\rho+\nu)(t-s)} \frac{\eta_{s,t}}{\eta_{s,s}} \frac{\xi_s}{\rho + \nu}. \tag{108}$$

Applying Ito's lemma to both sides of the above equation and matching the diffusions leads to

$$\xi_t \left(\pi_{s,t} \sigma^S + H_{s,t} \sigma_Y - \hat{W}_{s,t} \theta_t \right) = -\Delta_{s,t} Y_s e^{-(\rho+\nu)(t-s)} \frac{\eta_{s,t}}{\eta_{s,s}} \frac{\xi_s}{\rho + \nu}. \tag{109}$$

Rearranging and substituting the terms yields

$$\pi_{s,t} \sigma^S = \hat{W}_{s,t} \left(\sigma_Y + \bar{\Delta}_t - \Delta_{s,t} \right) - H_{s,t} \sigma_Y. \tag{110}$$

Finally, solving for the optimal portfolio $\pi_{s,t}$

$$\pi_{s,t} = \frac{\bar{\Delta}_t - \Delta_{s,t}}{\sigma^S} \hat{W}_{s,t} + \frac{\sigma_Y}{\sigma^S} W_{s,t}, \tag{111}$$

yields the result.

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