A Theory of Liquidity and Risk Management Based on the Inalienability of Risky Human Capital*

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Abstract

We formulate a dynamic financial contracting problem with risky inalienable human capital. We show that the inalienability of the entrepreneur's risky human capital not only gives rise to endogenous liquidity limits but also calls for dynamic liquidity and risk management policies via standard securities that firms routinely pursue in practice, such as retained earnings, possible line of credit draw-downs, and hedging via futures and insurance contracts.

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1 Introduction

Neither an entrepreneur in need of funding, nor anyone else for that matter, can legally agree to enslave herself to a firm in exchange for financing by outside investors. This fundamental observation has led Hart and Moore (1994) to formulate a theory of financial constraints arising from the inalienability of human capital. In a stylized model of a firm with a single fixed project producing deterministic cash flows over a finite time interval, where moreover the entrepreneur's human capital is certain, they show that there is a finite debt capacity for the firm, which is given by the maximum repayment that the entrepreneur can credibly commit to: any higher repayment and the entrepreneur would be better off abandoning the firm. While they can uniquely determine the firm's debt capacity their highly simplified framework does not uniquely tie down the firm's debt maturity structure. They show that there is a continuum of optimal debt contracts involving more or less rapid debt repayment paths. There is only a unique optimal debt contract when the entrepreneur and investors have different discount rates. In addition, while their framework provides a new foundation for a theory of corporate leverage, their model "does not have room for equity per se" [pp 865], as they acknowledge.

We generalize the framework of Hart and Moore (1994) along several important dimensions: first, we introduce risky human capital and cash flows; second, we let the entrepreneur be risk averse (by "entrepreneur," we mean a representative agent for all undiversified agents with inalienable human capital); third, we consider an infinitely-lived firm with ongoing investment; and, fourth we add a limited liability or commitment constraint for investors. In this significantly more realistic yet still tractable framework we derive the optimal investment and consumption policies, and show how this optimal financial contract can be implemented using replicating portfolios of standard liquidity and risk management instruments such as cash, credit line, futures, and insurance contracts. More concretely, the state variable in the optimal financial contracting problem between risk-neutral investors and the risk-averse entrepreneur is the promised wealth to the entrepreneur per unit of capital, w, and the value of the firm to investors per unit of capital is $p(w)$. Moreover, under the optimal contract the firm's investment and financing policies and the entrepreneur's consumption are all expressed as functions of w. As Table 1 below summarizes, we show that this contracting problem can be reformulated as a dual implementation problem with corporate savings per unit of capital, $s = -p(w)$, as the state variable and where the objective function is the entrepreneur's payoff given by $m(s) = w$. The key observation in this transformation is that the firm's optimal financial contracting problem then can be represented as a more operationally grounded liquidity and risk-management policies with endogenously determined liquidity constraints.

Table 1: Equivalence between primal contracting and dual implementation

	Primal Contracting	Dual Implementation
State Variable	w	s
Value Function	$p(w)$	$m(s)$

A first reason for considering this more involved framework is to explore how the Hart and Moore theory of debt based on the inalienability of human capital generalizes and how the introduction of risky human capital modifies the theory. But, more importantly, our framework reveals that Hart and Moore's focus on the notion of a firm's "debt capacity" is reductive. As it turns out, this is only one of the relevant metrics of the firm's optimal financial policy when human capital is risky. What matters more generally for optimal corporate policies is not just the limit on a credit-line commitment the firm has secured with its investors $\underline{s}$, but the size of the firm's financial slack $(s - \underline{s})$ at any moment in time. Accordingly, inalienability of *risky* human capital is not just a foundation for a theory of debt capacity, but also a foundation for a theory of *corporate liquidity and risk management*. More concretely, our analysis can shed new light on corporate policies that at first sight appear to be inconsistent with the theoretical framework of Hart and Moore, such as the large observed retained cash pools at corporations such as Apple, Google and other high-tech firms. One novel explanation our analysis suggests is that these cash pools are necessary to make credible compensation promises to retain highly valued employees with attractive alternative job opportunities. These employees are largely paid in the form of deferred stock compensation. When their stock options vest and are exercised the company may need to engage in a stock repurchase program so as to avoid excessive stock dilution. But such a repurchase requires funding, which could explain why these companies retain so much cash. There are hints of the relevance of corporate liquidity in Hart and Moore's discussion of their theory[1], however they do not emphasize the importance of this variable. Also, as a result of the absence of any risk in their framework they overlook the importance of the firm's hedging policy.

We introduce risk via both productivity and capital depreciation shocks. These shocks give rise to risky inalienable human capital thus generating a sequence of stochastic dynamic limited-commitment constraints for the entrepreneur. That is, whether the entrepreneur is willing to stay with the firm now depends on the history of realized productivity and capital shocks. When there is a positive shock, the entrepreneur's human capital is higher and she must receive a greater promised compensation to be induced to stay. But the entrepreneur is averse to risk and has a preference for smooth consumption. These two opposing forces give rise to a novel dynamic optimal contracting problem between the infinitely-lived risk-averse entrepreneur and the fully diversified (or risk-neutral) investors.

A key step in our analysis is to show that the optimal long-term contracting problem between investors and the entrepreneur can be reduced to a recursive formulation with a single key endogenous state variable w, the entrepreneur's *promised* certainty equivalent wealth W under the optimal contract scaled by the firm's capital stock K. The optimal recursive contract then specifies three state-contingent policy functions: *i*) the entrepreneur's consumption-capital ratio $c(w)$; *ii*) the firm's investment-capital ratio $i(w)$, and; *iii*) the firm's risk exposure $x(w)$ or hedging policy. This contract maximizes investors' payoff while providing insurance to the entrepreneur and retaining her. The optimal contract thus involves a particular form of the well-known tradeoff between risk sharing and incentives in a model of capital accumulation and limited commitment. Here the entrepreneur's inalienability of human capital constraint at each point in time is in effect her incentive constraint. She needs to be incentivized to stay rather than deploy her human capital elsewhere.

If the entrepreneur were able to alienate her human capital, the optimal contract would simply provide her with a constant flow of consumption and shield her from any risk. Under

[1]For example, on pages 864-865 they wrote: "There is some evidence that firms borrow more than they strictly need to cover the cost of their investment projects, in order to provide themselves with a "financial cushion." This fits in with our prediction in Proposition 2 about the nature of the slowest equilibrium repayment path; indeed, it is true of most paths."

this contract the firm's investment policy reduces to the standard ones prescribed by the q-theory models under the Modigliani-Miller (MM) assumption of perfect capital markets. But with inalienable human capital the entrepreneur must be prevented from leaving. To retain the entrepreneur in the states of the world where the entrepreneur may find her outside option to be greater than her promised certainty equivalent wealth W, the optimal contract must promise her sufficiently high w thus exposing her to productivity and capital shocks.

Following the characterization of the optimal dynamic corporate policy $(c(w), i(w), x(w))$ we proceed with the implementation of this policy in terms of familiar standard dynamic financing securities. In particular, we show that the optimal contract can be implemented by delegating control over the firm and transferring equity ownership to the entrepreneur in exchange for a credit line with an endogenously determined stochastic limit $\underline{S}$. The key endogenous state variable for this implementation problem is $s = S/K$, the ratio between financial slack S and the firm's capital stock K. The entrepreneur maximizes her life-time utility by optimally choosing consumption-capital ratio $\mathfrak{c}(s)$, investment-capital ratio $\mathfrak{i}(s)$, and hedge $\phi(s)$ as a function of s. In other words, the optimal long-term contract under risky inalienable human capital can be implemented via a sequence of short-term contracts, which take the form of a continuously revised credit line combined with optimal cash management and dynamic hedging policies. This implementation is simply a particularly realistic illustration of the general result of Fudenberg, Holmstrom and Milgrom (1990) that optimal long-term agency contracts with moral hazard can be implemented via a sequence of short-term contracts. It is also analogous to the implementation results via dynamically replicating portfolios of Arrow-Debreu equilibria of Merton (1973) and Duffie and Huang (1985).

The optimal contract provides the entrepreneur with a (locally) deterministic consumption stream as long as the capital stock does not grow too large. When the capital stock increases as a result of investment or positive shocks to the point where the entrepreneur's inalienability of human capital constraint may be violated the contract provides a higher consumption stream to the entrepreneur. As long as investors can perfectly commit to an optimal stochastic credit-line limit $\underline{S}$ (what we refer to as the one-sided commitment problem), the entrepreneur's consumption and wealth are positively correlated with the capital stock under the optimal contract, and the firm will generally underinvest relative to the first-best MM benchmark of fully alienable human capital.

In the two-sided commitment problem, where a limited liability constraint for investors must also hold, we obtain further striking results. The firm may now *overinvest* and the entrepreneur may *overconsume* (compared with the first-best benchmark). The intuition is as follows. In order to make sure that investors do not have incentives to default on their promised future utility for the entrepreneur, the entrepreneur's scaled promised wealth w cannot be too high otherwise the investors will end up with negative valuations for the firm. As a result, the entrepreneur needs to substantially increase investment and consumption in order to satisfy the investors' limited-liability constraint.

Related literature. Our paper provides foundations for a dynamic theory of liquidity and risk management based on risky inalienable human capital. As such it is obviously related to the early important contributions on corporate risk management by Stulz (1984), Smith and Stulz (1985) and Froot, Scharfstein, and Stein (1993). Unlike our setup, they consider static models with exogenously given financial frictions to show how corporate cash and risk management can create value by relaxing these financial constraints.

Our paper is also evidently related to the corporate security design literature, which seeks to provide foundations for the existence of corporate financial constraints, and for the optimal external financing by corporations through debt or credit lines. This literature can be divided into three separate strands. The first approach provides foundations for external debt financing in a static optimal contracting framework with either asymmetric information and costly monitoring (Townsend, 1979, and Gale and Hellwig, 1985) or moral hazard (Innes, 1990, and Holmstrom and Tirole, 1997).

The second more dynamic optimal contracting formulation derives external debt and credit lines as optimal financial contracts in environments where not all cash flows generated by the firm are observable or verifiable.[2]

The third approach which is closely related to the second provides foundations for debt financing based on the inalienability of human capital (Hart and Moore, 1994, 1998). Harris and Holmstrom (1982) is an early important paper that generates non-decreasing consump-

[2]See Bolton and Scharfstein (1990), DeMarzo and Fishman (2007), Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Sannikov (2006), Piskorski and Tchistyi (2010), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo, Fishman, He and Wang (2012). See Sannikov (2012) and Biais, Mariotti, and Rochet (2013) for recent surveys of this literature.

tion profile in a model where workers are unable to commit to long-term contracts. Berk, Stanton, and Zechner (2010) incorporate capital structure and human capital bankruptcy costs into Harris and Holmstrom (1982). Rampini and Viswanathan (2010, 2013) develop a model of corporate risk management building on similar contracting frictions. A key result in their model is that hedging may not be an optimal policy for firms with limited capital that they can pledge as collateral. For such firms hedging demand, in effect, competes for limited collateral with investment demand. They show that for growth firms the return on investment may be so high that it crowds out hedging demand. Li, Whited, and Wu (2014) structurally estimate optimal contracting problems with limited commitment along the line of Rampini and Viswanathan (2013) providing empirical evidence in support of these class of models.

The latter two approaches are often grouped together because they yield closely related results and the formal frameworks are almost indistinguishable under the assumption of risk-neutral preferences for the entrepreneur and investors. However, as our analysis with risk-averse preferences for the entrepreneur makes clear, the two frameworks are different, with the models based on non-contractible cash flows imposing dynamic incentive constraints that restrict the set of incentive compatible financial contracts, while the models based on inalienable human capital only impose (dynamic) limited-commitment constraints for the entrepreneur. With the exception of Gale and Hellwig (1985) the corporate security design literature makes the simplifying assumption that the contracting parties are risk neutral. By allowing for risk-averse entrepreneurs, we not only generalize the results of this literature on the optimality of debt and credit lines, but we are able to account for the fundamental role of corporate savings and risk management (via futures, options or other commonly used derivatives), and also to provide micro-foundations for executive compensation contracts.

A closely related paper to ours is by Ai and Li (2013) that analyzes a similar contracting framework to study corporate investment and managerial compensation but with very different economic motivation and focus. We show that liquidity and risk management optimally implements the optimal contracting solution and also we characterize the dynamics of optimal corporate liquidity and risk management. Additionally, we incorporate stochastic productivity shocks and focus on the implications due to the inalienability of risky human capital. Also closely related is Lambrecht and Myers (2012), who consider an intertemporal model of a firm run by a risk-averse entrepreneur with habit formation, and derive the firm's optimal dynamic corporate policies. They show that the firm's optimal payout policy resembles the famous Lintner (1956) payout rule of thumb.

Our financial implementation of the optimal financial contract is also related to the portfolio choice literature featuring illiquid productive assets and under-diversified investors in an incomplete-markets setting. Building on Merton's intertemporal portfolio choice framework, Wang, Wang, and Yang (2012) study a risk-averse entrepreneur's optimal consumption-savings decision, portfolio choice, and capital accumulation when facing uninsurable idiosyncratic capital and productivity risks. Unlike Wang, Wang, and Yang (2012), our model features optimal liquidity and risk management policies that arise endogenously from an underlying financial contracting problem.

Our framework also provides a micro-foundation for the dynamic corporate savings models that take external financing costs as exogenously given. Hennessy and Whited (2005, 2007), Riddick and Whited (2009), and Eisfeldt and Muir (2014) study corporate investment and savings with financial constraints. Bolton, Chen, and Wang (2011, 2013) study the optimal investment, asset sales, corporate savings, and risk management policies for a firm that faces external financing costs. It is remarkable that although these models are substantially simpler and more stylized the general results on the importance of corporate liquidity and risk management are broadly similar to those derived in our paper based on more primitive assumptions. Conceptually, our paper shows that to determine the dynamics of optimal corporate investment, a critical variable in addition to the marginal value of capital (marginal q) is the firm's marginal value of liquidity. Indeed, we establish that optimal investment is determined by the ratio of marginal q and the marginal value of liquidity, which reflects the tightness of external financing constraints.[3] Our model thus shares a similar focus on the marginal value of liquidity as Bolton, Chen, and Wang (2011, 2013) and Wang, Wang, and Yang (2012).

Our paper also relates to the macroeconomics literature that studies the implications of dynamic agency problems for firms' investment and financing decisions. Green (1987), Thomas and Worrall (1990), Marcet and Marimon (1992), Kehoe and Levine (1993) and

[3]Pinkowitz and Williamson (2004), Faulkender and Wang (2006), Pinkowitz, Stulz, andWilliamson (2006), Dittmar and Mahrt-Smith (2007), and Bolton, Schaller, and Wang (2014) empirically measure the marginal value of cash.

Kocherlakota (1996) are important early contributions on optimal contracting under limited commitment.[4] Alvarez and Jermann (2000, 2001) study welfare and asset pricing implications of endogenously incomplete markets due to limited contract enforcement constraints. Albuquerque and Hopenhayn (2004), Quadrini (2004) and Clementi and Hopenhayn (2006) characterize firms' financing and investment decisions under limited commitment and/or asymmetric information. Lorenzoni and Walentin (2007) study q theory of investment under limited commitment. Grochulski and Zhang (2011) consider a risk sharing problem under limited commitment. Finally, Miao and Zhang (2013) develop a duality-based solution method for limited commitment problems.

2 The Model

We consider an optimal long-term contracting problem with limited commitment to participate between an infinitely-lived risk-neutral investor (*the principal*) and a financially constrained, infinitely-lived, risk-averse entrepreneur (*the agent*). The investor provides the entrepreneur, who has a proprietary business idea but no funds, with both the initial capital stock K_0 and on-going funding commitments to the venture. We begin by describing the production technology and the entrepreneur's preferences before formulating the dynamic optimal contracting problem between the two agents.

2.1 Capital Accumulation and Production Technology

We adopt the stochastic capital accumulation specification used in Cox, Ingersoll, and Ross (1985) and Jones and Manuelli (2005), among others. The firm's capital stock K accumulates as follows:

$$dK_t = (I_t - \delta K_t)dt + \sigma_K K_t dZ_t, \tag{1}$$

where I is the firm's rate of gross investment, $\delta \geq 0$ is the expected rate of depreciation, Z is a standard Brownian motion, and σ_K is the volatility of the capital depreciation shock. The firm's capital stock can be interpreted as either tangible capital (property, plant and equipment), firm-specific intangible capital (patents, know-how, brand value, and organizational

[4] See Ljunqvist and Sargent (2004) Part V for a textbook treatment on this class of macro models.

capital), or any combination of these.

Production requires combining the entrepreneur's inalienable human capital with the firm's capital stock. The investor's outside option value at time 0 is denoted by F_0^*; this is the value of the capital stock K_0 in an alternative use. When the investor's capital stock and the entrepreneur's human capital are united the firm's revenues are given by A_tK_t, where $\{K_t;\, t \geq 0\}$ is the firm's stochastic capital stock process and $\{A_t;\, t \geq 0\}$ is a stochastic productivity shock process. To keep the analysis simple, we model $\{A_t;\, t \geq 0\}$ as a two-state Markov switching process. Specifically, $A_t \in \left\{A^L, A^H\right\}$ with $0 < A^L < A^H$. Let λ_n be the transition intensity out of state $n = L$ or H to the other state.[5] In other words, given a current value of $A_t = A^L$ the firm's productivity changes to A^H with probability $\lambda_L dt$, and if $A_t = A^H$ the firm's productivity changes to A^L with probability $\lambda_H dt$ in the time interval $(t, t + dt)$. The productivity process $\{A_t;\, t \geq 0\}$ is observable to both the investor and entrepreneur, and is also contractible. To reiterate, the entrepreneur's human capital is critical: Without the entrepreneur's human capital, the capital stock K_0 would not generate any cash flows and the investors would only be able to collect the outside option value F_0^*.

Investment involves both a direct purchase cost and an adjustment cost, so that the firm's cash flows (after capital expenditures) are given by:

$$Y_t = A_tK_t - I_t - G(I_t, K_t), \tag{2}$$

where the price of the investment good is normalized to unity and $G(I, K)$ is the standard adjustment cost function in the q-theory of investment. Importantly, Y_t can be negative, which means that the investor would be financing investment I_t and associated adjustment costs G from other sources than just current realized revenue A_tK_t. We follow the q-theory literature and assume that the firm's adjustment cost $G(I, K)$ is homogeneous of degree one in I and K, so that $G(I, K)$ takes the following homogeneous form:

$$G\,(I, K) = g(i)K, \tag{3}$$

[5]Piskorski and Tchistyi (2010) consider a model of mortgage design in which they use a Markov-switching process to describe interest rates. DeMarzo, Fishman, He, and Wang (2012) use a Markov-switching process to model the persistent productivity shock.

where $i = I/K$ denotes the firm's investment-capital ratio and $g(i)$ is an increasing and convex function. This homogeneity assumption (3) is made purely for tractability reasons. As Hayashi (1982) has first shown, with this homogeneity property Tobin's average and marginal q are equal under perfect capital markets.[6] However, as we will show, under limited commitment to participate an endogenous wedge between Tobin's average and marginal q will emerge in our model.[7]

Hart and Moore (1994) is a special case of our model when we set: *i)* $\sigma_K = 0$ so that there are no shocks to the capital stock; *ii)* $\delta = 0$, so that the capital stock does not depreciate; *iii)* $I_t = 0$, so that there is no endogenous capital accumulation; *iv)* $A_t = A > 0$, so that there are no shocks to earnings; and *v)* $t \in [0, T]$, with $T < \infty$, so that the horizon is finite. In other words, our framework adds to the basic Hart and Moore (1994) setup an endogenous capital accumulation process and shocks to both productivity and capital.

2.2 Preferences

We also generalize the Hart and Moore (1994) setup by introducing risk aversion for the entrepreneur. We assume that the infinitely-lived entrepreneur has a standard concave utility function over positive consumption flows $\{C_t; t \geq 0\}$ given by:

$$V_t = \mathbb{E}_t \left[\int_t^{\infty} \zeta e^{-\zeta(v-t)} U(C_v) dv \right] , \tag{4}$$

where $\zeta > 0$ is the entrepreneur's subjective discount rate, $\mathbb{E}_t[\,\cdot\,]$ is the time-t conditional expectation, and $U(C)$ takes the standard constant-relative-risk-averse (CRRA) utility form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} , \tag{5}$$

[6] Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average q. Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

[7] An endogenous wedge between Tobin's average and marginal q also arises in cash-based optimal financing and investment models such as Bolton, Chen, and Wang (2011) and optimal contracting models such as DeMarzo, Fishman, He, and Wang (2012).

where $\gamma > 0$ is the coefficient of relative risk aversion. We normalize the flow payoff with ζ in (4), so that the utility flow is $\zeta U(C)$, as is standard in dynamic contracting models.[8]

2.3 Entrepreneur's Human Capital and Outside Option

The entrepreneur's human capital is inalienable and she can at any time leave the firm. When the entrepreneur exits she obtains an outside payoff of $\widehat{V}_n(K_t)$ (in utils) in state $n \in \{L, H\}$. In other words, $\widehat{V}_n(K_t)$ is the entrepreneur's *endogenous* outside option: it depends on both accumulated capital K_t and the firm's productivity A^n at the moment of exit. The entrepreneur's inalienability-of-human-capital constraint at each point in time t is therefore given by:

$$V_t \geq \widehat{V}_n(K_t), \quad t \geq 0. \tag{6}$$

Effectively, the entrepreneur's inalienability of human capital generates an outside option value which is the endogenous lower bound on her continuation utility, $\widehat{V}_n(K_t)$, which in turn constrains the set of feasible consumption and investment policies.

This inalienability-of-human-capital constraint can be interpreted in several ways.

1. A first interpretation is that when she quits the entrepreneur can find a new investor to finance a new firm whose initial size is a fraction $\alpha \in (0, 1)$ of the on-going firm's current capital stock. We assume that the production function remains the same at the new firm as in the existing firm. In this narrative, there is no misappropriation involved and the entrepreneur's outside option simply reflects the market value of her accumulated human capital. The key insight here is that the entrepreneur's outside option offers her a larger fraction of a smaller firm upon exit, which the incumbent financier has to take into account in the optimal contract.

2. A second interpretation is that the entrepreneur may abscond with a fraction $\alpha \in (0, 1)$ of the firm's capital stock and start a new firm.

[8] For example, see Sannikov (2008). We can generalize these preferences to allow for a coefficient of relative risk aversion that is different from the inverse of the elasticity of intertemporal substitution, à la Epstein and Zin (1989). Indeed, as Epstein-Zin preferences are homothetic, allowing for such preferences in our model will not increase the dimensionality of the optimization problem.

3. A third common interpretation is that the entrepreneur appropriates the capital stock and continues operating in autarky. She then forgoes intertemporal consumption-smoothing opportunities. She therefore trades off the benefit of appropriation with the cost of losing consumption smoothing.[9]

In our analysis, we will adhere to the first interpretation. We discuss the last interpretation with the entrepreneur living in autarky as the outside option in Appendix D.

2.4 The Contracting Problem

We assume that the output process Y_t is publicly observable and verifiable. In addition, we assume that the entrepreneur cannot privately save, as is standard in the literature on dynamic moral hazard (see Bolton and Dewatripont, 2005 chapter 10). The contracting game begins at time 0 with the investor making a take-it-or-leave-it long-term contract offer to the entrepreneur. The contract specifies an investment process $\{I_t; t \geq 0\}$ and a consumption allocation process $\{C_t; t \geq 0\}$ to the entrepreneur, both of which depend on the entire history of productivity shocks $\{A_t; t \geq 0\}$ and capital stock $\{K_t; t \geq 0\}$.

At the moment of contracting at time 0 the entrepreneur also has a reservation utility V_0^*, so that the optimal contract must also satisfy the constraint:

$$V_0 \geq V_0^* \,. \tag{7}$$

Without loss of generality, we let $V_0^* \geq \widehat{V}_n(K_0)$ for $n \in \{L, H\}$. The investor's problem at time 0 is thus to choose dynamic investment I_t and consumption C_t to maximize the time-0 discounted value of cash flows,

$$F_0 = \max_{I,C} \quad \mathbb{E}_0 \left[\int_0^\infty e^{-rt}(Y_t - C_t)dt \right] , \tag{8}$$

subject to the capital accumulation process (1), the production function (2), the entrepreneur's inalienability-of-human-capital constraint (6) at all t, and the entrepreneur's time-0 participation constraint (7). Additionally, we require that the investor's value at time 0, F_0, is

[9]This interpretation is commonly used in the international macro literature. See Bulow and Rogoff (1989).

(weakly) greater than investors' second-best option F_0^*.

The participation constraint (7) is always binding under the optimal contract. Otherwise, the investor can always increase his payoff by lowering the agent's consumption and still satisfy all other constraints. However, the entrepreneur's inalienability-of-human-capital constraints (6) will often not bind as the investor dynamically trades off the benefits of providing the entrepreneur with risk-sharing/consumption smoothing and the benefits of extracting higher contingent payments from the firm.

By varying the entrepreneur's outside option V_0^*, we can trace out the constrained Pareto optimal frontier, which is the best attainable outcome given the inalienability-of-human-capital constraint. We may interpret each point on the constrained Pareto optimal frontier as an outcome of the bargaining game between the investor and the entrepreneur.

3 The Full-Commitment Benchmark

Before characterizing the optimal contract under inalienability of human capital, we determine the solution under full commitment by both investors and the entrepreneur. In that case our contracting problem generates the first-best outcome. The risk-neutral investor maximizes the present discounted value of the venture's cash flows by asking the risk-averse entrepreneur to choose the first-best investment rule while providing perfect risk-sharing to the risk-averse entrepreneur, achieving the reservation utility V_0^*.

Given the stationarity of the economic environment and the homogeneity of the production technology with respect to K, there is an optimal productivity-dependent investment-capital ratio $i_n = I/K$ in state $n \in \{L, H\}$ that maximizes the present value of the venture. The following proposition summarizes the main results under full commitment.

Proposition 1 *In each state $n \in \{L, H\}$, the firm's value $Q_n^{FB}(K)$ is proportional to K: $Q_n^{FB}(K) = q_n^{FB} K$, where q_n^{FB} is Tobin's q in state n. In state H, q^H solves:*

$$(r+\delta)\, q_H^{FB} = \max_i \quad \left(A^H - i - g(i)\right) + \lambda_H \left(q_L^{FB} - q_H^{FB}\right) \,, \tag{9}$$

and the maximand for (9), denoted by i_H^{FB}, is the first-best investment-capital ratio. The *entrepreneur is perfectly insured with a deterministic consumption stream:*

$$C_t = C_0 \, e^{-(\zeta - r)t/\gamma} \,, \quad t \geq 0 \ . \tag{10}$$

Homogeneity implies that return and present value relations hold for both the whole firm and for each unit of capital K. The first term on the right side of (9), $A^H - i - g(i)$, is the firm's unit cash flow, and the second term, $\lambda_H \left(q_L^{FB} - q_H^{FB}\right)$, is the expected unit capital gain over the interval of time dt. At the optimum, the expected rate of return on capital is given by the sum of the discount rate r and the expected deprecation rate of capital δ, explaining the left side of (9). A similar (and symmetric) valuation equation holds in state L for q_L^{FB}.

Under MM and with homogeneity properties, Tobin's average q in state n, q_n^{FB}, is also the marginal value of capital, often referred to as marginal q. Adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that that $q_n^{FB} \neq 1$ in general. We can express Tobin's q via the first-order condition for investment:

$$q_n^{FB} = 1 + g'(i_n^{FB}), \ \ n \in \{L, H\}, \tag{11}$$

which states that marginal q is equal to the marginal cost of investing, $1 + g'(i)$, at the optimum investment level i_n^{FB}. By jointly solving (9) and (11) and the similar two equations for state L, we obtain the values for q_n^{FB} and i_n^{FB}, where $n \in \{L, H\}$.

Next we turn to the entrepreneur's consumption. With full commitment, the risk-averse entrepreneur is fully insured by the risk-neutral investors and therefore, the entrepreneur's consumption is given by (10), independent of the firm's investment dynamics. To the extent that the investor and entrepreneur have different discount rates, $\zeta \neq r$, the optimal contract will be structured so that the entrepreneur's consumption changes exponentially at a rate $-(\zeta - r)/\gamma$ per unit of time, where $1/\gamma$ should be interpreted as the elasticity of intertemporal substitution. Thus, depending on the sign of $(\zeta - r)$ the entrepreneur's consumption may grow or decline deterministically over time. It is only when the investor and the entrepreneur are equally impatient $(\zeta = r)$ that the entrepreneur's consumption is constant over time under the optimal full-commitment contract.

The only unknown that remains to be solved is the initial consumption C_0. For a given level of the entrepreneur's utility V, we can calculate the corresponding certainty equivalent wealth by inverting the expression $V(W) = U(bW)$ and obtain:

$$W = U^{-1}(V)/b, \tag{12}$$

where $U^{-1}(\,\cdot\,)$ is the inverse of the utility function (5) and b is a constant given by:[10]

$$b = \zeta \left[\frac{1}{\gamma} - \frac{r}{\zeta} \left(\frac{1}{\gamma} - 1 \right) \right]^{\frac{\gamma}{\gamma-1}} . \tag{13}$$

Because the entrepreneur's time-0 participation constraint (6) is binding the initial certainty equivalent wealth W_0^* must satisfy: $W_0^* = U^{-1}(V_0^*)/b$, so that we obtain

$$C_0 = \chi W_0^* = \left(\frac{\zeta}{b} \right)^{\frac{1}{\gamma}} ((1-\gamma)V_0^*)^{\frac{1}{1-\gamma}} , \tag{14}$$

where χ is the marginal propensity to consume (MPC) given by

$$\chi = r + \gamma^{-1} (\zeta - r) . \tag{15}$$

The entrepreneur's utility, denoted by V_t^{FB}, is then given by:

$$V_t^{FB} = U(bW_t) = U(bC_t/\chi) \sim V_0^* \, e^{-(\zeta-r)(1-\gamma)t/\gamma} , \tag{16}$$

where $U(\,\cdot\,)$ is given by (5). For the special case where $\zeta = r$, the entrepreneur's utility is time-invariant, $V_t^{FB} = U(C_0) = V_0$, as consumption is flat at all times.

In summary, with full commitment, the first-best investment-capital ratio i^{FB} depends on the current state $n \in \{L, H\}$ but is independent of capital shocks. Moreover, the investor perfectly insures the entrepreneur's consumption. As we will show next, the entrepreneur's inability to fully commit to the venture significantly alters this solution.

[10] As a special case, when $\gamma = 1$, we have $b = \zeta e^{\frac{r-\zeta}{\zeta}}$.

4 Optimal Dynamic Contracting

The first-best outcome is not achievable when the entrepreneur has inalienable human capital. The intuition is as follows. Under the first-best setting the firm's capital K_t stochastically grows over time. When it reaches the cut-off values $\overline{K}_t^H$ or $\overline{K}_t^L$ for which $\widehat{V}_n(K_t) > V_t^{FB}$, when respectively $K_t > \overline{K}_t^H$ in state H, and $K_t > \overline{K}_t^L$ in state L, and where V_t^{FB} is given by (16), the entrepreneur's limited-commitment constraint will be violated and she will walk away. To prevent such an outcome the investor writes a second-best contract where he commits to a consumption flow $\{C_t : t \geq 0\}$ for the entrepreneur such that $V_t \geq \widehat{V}_n(K_t)$ at all times t in both states H and L. Since $\widehat{V}_n(K_t)$ is a stochastic process, this second-best contract will inevitably expose the entrepreneur to consumption risk. Accordingly, the optimal dynamic contracting problem under limited commitment involves a specific form of the classic agency tradeoff between insurance of the agent's consumption risk and incentive provision for the agent to stay with the firm.

An important difference from the standard dynamic moral hazard problem is that the entrepreneur's and investors' dynamic limited-commitment constraints often will not bind. The reason is that if the contract were to always hold the entrepreneur or the investors down to respective limited-commitment constraints then the entrepreneur's promised consumption would be excessively volatile, reducing total ex-ante surplus.

4.1 Formulating the optimal recursive contracting problem

The second-best dynamic contracting problem includes: *i*) a contingent investment plan $\{I_t; t \geq 0\}$, and *ii*) consumption promises $\{C_t; t \geq 0\}$ to the entrepreneur that maximize the present value of the firm for investors. As is well known (see e.g. DeMarzo and Sannikov, 2006), an important simplification of the contracting problem is to summarize the entire history of the contract in the entrepreneur's promised utility V_t conditional on the history up to time t. Under the optimal contract the dynamics of the agent's promised utility can then be written in the recursive form below. The sum of the agent's utility flow $\zeta U(C_{t-})dt$ and change in promised utility dV_t has the expected value $\zeta V_t dt$, or:

$$\mathbb{E}_{t-}\left[\zeta U(C_{t-})dt + dV_t\right] = \zeta V_{t-}dt\,. \tag{17}$$

To see why (17) must hold, we first construct a stochastic process, $\{\widehat{U}_t, t \geq 0\}$, from the agent's utility V and consumption C processes as follows:

$$\widehat{U}_t = \int_0^t e^{-\zeta v} \zeta U(C_v) dv + e^{-\zeta t} V_t = \mathbb{E}_t \left[\int_0^\infty \zeta e^{-\zeta v} U(C_v) dv \right] . \tag{18}$$

Second, we know that $\{\widehat{U}_t; t \geq 0\}$ is a martingale: $\mathbb{E}_t[\widehat{U}_s] = \widehat{U}_t$ for all s and t such that $s > t$. Third, applying Ito's formula to the process $\widehat{U}$ given in (18), and using the property that a martingale's drift is zero, we then obtain (17). In other words, delivering a marginal unit of consumption to the entrepreneur lowers her promised utility V by reducing its drift ζV_{t-} by the amount $\zeta U(C_{t-})$, and hence we have the following equivalent representation of (17):

$$\mathbb{E}_{t-} \left[dV_t \right] = \zeta \left(V_{t-} - U(C_{t-}) \right) dt. \tag{19}$$

Next, given that there are two shocks–the capital shock (via the Brownian motion Z) and the productivity shock (via the two-state Markov chain)– we may write the stochastic differential equation (SDE) for dV implied by (17) as the sum of: *i*) the expected change (i.e., drift) term $\mathbb{E}_{t-} [dV_t]$; *ii*) a martingale term driven by the Brownian motion Z; and *iii*) a martingale term driven by the productivity shock. Accordingly, if we denote by N_t the cumulative number of productivity changes up to time t, and adopt the convention that the current productivity state at time $t-$ is H, we may write the dynamics of the entrepreneur's promised utility process V as follows:

$$dV_t = \zeta(V_{t-} - U(C_{t-}))dt + x_{t-} V_{t-} dZ_t + \Gamma_H(V_{t-}, A^H)(dN_t - \lambda_H dt) , \tag{20}$$

where $\{x_t; t \geq 0\}$ controls the diffusion volatility of the entrepreneur's promised utility V, and $\Gamma_H(V_{t-}, A^H)$ controls the endogenous adjustment of promised utility V conditional on the change of productivity from A^H to A^L. That is:

1. the first term on the right side of (20) is the expected change of dV_t as implied by (17),

2. the second term is the unexpected change due to capital shock Z, and

3. the last term captures the mean-zero unexpected component of dV_t due to the change of

productivity. Indeed, given that λ_H is the probability per unit of time of a productivity switch from A^H to A^L, the expected value of $(dN_t - \lambda_H dt)$ is zero.

Finally, we can write investors' objective as a value function $F(K, V, A^n)$ with three state variables: *i*) the entrepreneur's promised utility V; *ii*) the venture's capital stock K; and, *iii*) the state of productivity $n \in \{L, H\}$.

The optimal contract then specifies investment I, consumption C, risk exposure x, and insurance adjustment of promised utility Γ_n, to solve the following optimization problem,

$$F(K_t, V_t, A^n) = \max_{C, I, x, \Gamma_n} \quad \mathbb{E}_t \left[\int_t^\infty e^{-r(v-t)} (Y_v - C_v) dv \right], \tag{21}$$

subject to the entrepreneurs' inalienability-of-human-capital constraints (6) for all time t, and the entrepreneur's initial participation constraint (7).

We next characterize the investor's optimization problem in the interior region and then describe the boundary conditions.

The interior region. For expositional simplicity, suppose that the current state is H. Then, the following Hamilton-Jacobi-Bellman (HJB) equation holds:

$$\begin{aligned} rF(K, V, A^H) \;=\; \max_{C, I, x, \Gamma_H} \; & \big\{ Y - C + (I - \delta K) F_K + \sigma_K^2 K^2 F_{KK}/2 \\ & + [\zeta(V - U(C)) - \lambda_H \Gamma_H] F_V + \frac{(xV)^2}{2} F_{VV} + \sigma_K x K V F_{VK} \\ & + \lambda_H [F(K, V + \Gamma_H, A^L) - F(K, V, A^H)] \big\}. \end{aligned} \tag{22}$$

The right-hand side of (22) gives the expected change of the investor's value function $F(K, V, A^H)$. The first term is the venture's flow profit $(Y - C)$ for the investor, which can be negative. In this case, the investor is financing operating losses; The second term reflects the expected change of the investor's value $F(K, V, A^H)$ resulting from the expected (net) capital accumulation $(I - \delta K)$; The third term represents the expected change in the investor's value resulting from the volatility of the capital shock; The fourth and fifth terms in turn reflect the change in investor's value from the drift and volatility of the entrepreneur's promised utility V; The sixth term captures how the investor's value is affected by the (perfect) correlation between K and V;[11] Finally, the last term captures the effect of the persistent productivity shock on the value function. Importantly, in addition to the direct effect on the investor's value F, the productivity switch from A^H to A^L also has an indirect effect on the investor's value F due to the endogenous adjustment of the entrepreneur's promised utility from V to $V + \Gamma_H$. As investors earn the rate of return r at all times, the sum of all terms on the right side of (22) must equal $rF(K, V, A^H)$, which is given on the left-hand side of (22).

Differentiating the right-hand side of (22) with respect to C, I, and x we then obtain the following first-order conditions (FOCs):

$$\begin{aligned}
\zeta U'(C^*) &= -\frac{1}{F_V(K, V, A^H)}, && (23)\\
F_K(K, V, A^H) &= 1 + G_I(I^*, K), \text{ and} && (24)\\
x^* &= -\frac{\sigma_K K F_{VK}}{V F_{VV}(K, V, A^H)}. && (25)
\end{aligned}$$

FOC (23) characterizes the entrepreneur's optimal consumption C^*, which must equalize the entrepreneur's marginal utility of consumption $\zeta U'(C^*)$ with $-1/F_V$, which is positive as $F_V < 0$. Multiplying (23) through by $-F_V$, we observe that at the optimum the agent's normalized marginal utility of consumption, $-F_V \zeta U'(C)$, has to equal unity, the risk-neutral investor's marginal cost of providing a unit of consumption.

FOC (24) characterizes optimal investment, which is obtained when the marginal benefit of investing, $F_K(K, V, A^n)$, is equal to the marginal cost of investing, $1 + G_I(I, K)$.

FOC (25) characterizes the optimal exposure of the promised utility V to the shock Z. As we show later, x is closely tied to the firm's optimal risk management policy.

Finally, we turn to the optimal choice of Γ_H, the discrete change in the entrepreneur's promised utility contingent on the change of productivity from H to L. Whenever feasible, the optimal contract equates investors' marginal cost of delivering compensation just before and after the productivity change, so that:

$$F_V(K, V + \Gamma_H^*, A^L) = F_V(K, V, A^H)\,, \tag{26}$$

[11] As there is only one exogenous diffusion shock in the model, V and K are locally perfectly correlated.

which is the FOC with respect to Γ_H implied by (22). Note that the second-order condition (SOC) with respect to Γ_H is given by $F_{VV}(K, V + \Gamma_H^*, A^L) < 0$ which implies that F is concave in V at Γ_H^*. The condition (26) only holds when neither the entrepreneur's limited-commitment constraint nor the investor's limited-liability constraint bind. When either constraint binds, we will have inequalities rather than equalities for the FOC with respect to Γ_H. We return to the corner-solution case later with a more detailed discussion.

Next we turn to the boundary conditions, where either the entrepreneur's limited-commitment constraint or the investors' limited-liability constraint bind.

Boundary conditions. First we define the endogenous lower boundary $\underline{V_n}(K_t)$ where the entrepreneur is indifferent between continuing within the contracting relationship with the investor and walking away with the outside option. Given the entrepreneur's outside option value $\widehat{V}_n(K_t)$, we require:

$$\underline{V_n}(K_t) = \widehat{V}_n(K_t)\,. \tag{27}$$

Second, we can also define an endogenous upper boundary condition for the promised utility to the entrepreneur $\overline{V}_n(K)$ at which the investor's limited-liability condition binds (in the two-sided limited-commitment problem):

$$F(K, \overline{V}_n(K), A^n) = 0\,. \tag{28}$$

Note that to fully solve our problem, we need to specify the entrepreneur's outside option value $\widehat{V}_n(K_t)$. However, without loss of generality, we will assume that

$$\widehat{V}_n(K_t) = \overline{V}_n(\alpha K_t)\,, \tag{29}$$

where $\overline{V}_n(\,\cdot\,)$ is given by (28).

The economic logic behind (29) is that when the entrepreneur starts afresh at time τ with a new venture whose initial capital stock will be of size αK_τ provided by the new investor. Equality (29) holds if she is able to raise on-going funding at competitive terms so that the new investor earns zero profit.

The methodological reason for this assumption is that we can then solve for an endogenous outside option value so that the equilibrium value V and the outside option value $\widehat{V}$ are consistent and simultaneously determined under the same economic environment.

4.2 The Entrepreneur's Promised Certainty Equivalent Wealth W

How do we link the entrepreneur's promised utility V, the key state variable characterizing the optimal contract, to variables that are empirically measurable? As we will show, it is possible to formulate the optimal contract as a problem of liquidity and risk management that can be implemented via standard financial instruments. The corporate liquidity policy can be implemented through a combination of retained earnings and a line of credit commitment by investors. And the risk management policy can be implemented using a combination of futures hedging positions and insurance claims held by investors.

A helpful simplification towards the contracting formulation in terms of corporate liquidity and risk management is to express the entrepreneur's promised utility in units of consumption rather than utils. This involves mapping the promised utility V into the promised (certainty-equivalent) wealth W, defined as the solution to the equation $U(bW) = V$, where b is the constant given by (13). Doing so transforms the investor's value function $F(K, V, A^n)$ in terms of V into the value function $P(K, W, A^n)$ in terms of W via the following identity:

$$P(K, W, A^n) \equiv F(K, U(bW), A^n) = F(K, V, A^n)\,, \quad n \in \{L,\, H\}. \tag{30}$$

As is shown in the appendix, we can reformulate the HJB equation for $F(K, V, A^n)$ (with the corresponding FOCs for C, I, x, Γ_n, and boundary conditions) into an equivalent HJB equation for $P(K, W, A^n)$ with associated FOCs and boundary conditions by using the identity in (30) and applying Ito's formula to $P(K, W, A^n)$.

5 Implementation: Liquidity and Risk Management

Having characterized the optimal contract in terms of the entrepreneur's promised certainty-equivalent wealth W, we show next how to implement the optimal contract via commonly used financial instruments. Similarly to DeMarzo and Fishman (2007), DeMarzo and Sannikov (2006), and Biais, Mariotti, Plantin, and Rochet (2007), where the optimal financial contract can be implemented through a line of credit, in our setup the entrepreneur's inalienability-of-human-capital constraints naturally give rise to a dual optimization problem for the entrepreneur with endogenous liquidity constraints. More precisely, we show that the optimal contracting solution can be implemented as a dynamic entrepreneurial finance problem, where the entrepreneur owns the firm's capital stock and chooses consumption and corporate investment by optimally managing liquidity via short-term borrowing/savings on a bank account and by optimally hedging the firm's risk from operations subject to endogenous liquidity constraints (implied by the contracting problem). Under this implementation the investor holds a claim on the firm through the credit line granted to the firm.[12]

Liquidity management. Consider first the entrepreneur's liquidity management problem. We endow the entrepreneur with a bank account and let S_t denote the account's time-t balance. Naturally $S_t < 0$ corresponds to an overdraft or draw-down on a line of credit (LOC) granted by the bank to the entrepreneur. In the implementation problem the entrepreneur can borrow on this LOC at the risk-free rate r up to a maximal value of $\underline{S_n}(K_t)$, which we refer to as the endogenously determined *liquidity capacity* of the firm in state n. This borrowing limit ensures that the entrepreneur does not walk away from the firm in an attempt to evade her debt obligations. Then, as long as the entrepreneur works at the firm, the firm's credit line is risk free and hence can be financed at the risk-free rate. The liquidity buffer S_t in the risk-free savings/credit account becomes the state variable in the implementation problem, but as the optimal contracting problem highlights, liquidity management alone will only provide partial insurance to the entrepreneur. To replicate the optimal contracting outcome, additional insurance instruments are needed to which we turn next.

Risk management against capital shocks. One instrument the entrepreneur can use to hedge the capital risk Z is a standard futures contract.[13] Since investors are risk neutral the

[12]It is well known that implementation is not unique. To simplify the exposition we focus on one intuitive implementation and later discuss alternative ways of implementing the dynamic optimal contract.

[13]Bolton, Chen, and Wang (2011) analyze the optimal corporate risk management for a financially constrained firm. In that model, they also analyze the dynamic futures trading strategies but their model is not a dynamic contracting framework.

futures price involves no premium given that the futures contract payoffs have zero mean.[14] Moreover, since profits/losses of the futures position are only subject to diffusion shocks that are instantaneously credited/debited from the entrepreneur's bank account, there is no default risk. We normalize the payoff of a unit long position in the futures contract to be $\sigma_K dZ_t$. Given any admissible futures position $\phi_t K_t$ that the entrepreneur takes to hedge the firm's risk exposure to the capital risk Z, we obtain an instantaneous payoff $\phi_t K_t \sigma_K dZ_t$.

Insurance against productivity shocks. Finally, the entrepreneur can take out a contingent claim to hedge the risk with respect to changes in the productivity state. Suppose that the current productivity state is H. If the entrepreneur takes a unit long position in the contingent claim she pays an insurance premium λ_H per unit of time and receives a unit payment from the insurer when the state switches from H to L. Given that insurers are risk neutral, the actuarially fair premium per unit of time for this insurance is then λ_H. Let $\pi_H(S, A^H)K$ denote the entrepreneur's demand for this insurance contract in state H. She then pays a total insurance premium $\pi_H(S, A^H)K\lambda_H$ per unit of time and receives a lump-sum payment $\pi_H(S, A^H)K$ when the state switches from H to L (i.e. when $dN_t = 1$). Therefore, the total stochastic exposure of this contingent claim is $\pi_H(S, A^H)K_t(dN_t - \lambda_H dt)$ where $dN_t \in \{1, 0\}$. Again, here we assume investors are risk neutral and hence there is no risk premium in the insurance contract.[15]

Liquidity dynamics. With these three financial instruments the entrepreneur's savings balance, denoted by S_t, evolves as follows in state n, where $n \in \{L, H\}$:

$$dS_t = (rS_t + Y_t - C_t)dt + \phi_t K_t \sigma_K dZ_t + \pi_n(S, A^n)K_t(dN_t - \lambda_n dt), \tag{31}$$

as long as the limit on the LOC granted by the bank is not violated:

$$S_t \geq \underline{S_n}(K_t). \tag{32}$$

[14] In the standard asset pricing framework, futures have zero value and its payoff has zero mean under the risk-neutral measure. We can generalize our analysis to the setting with risk premium via a standard change of measure. Details are available upon request.

[15] We can extend the model to incorporate a stochastic discount factor (SDF) capturing a risk premium for the stochastic change of the productivity shock.

The first term in (31), $rS_t + Y_t - C_t$, is simply the sum of the firm's interest income rS_t and net operating cash flows, $Y_t - C_t$. In the absence of any risk management and hedging, $rS_t + Y_t - C_t$ is simply the rate at which the entrepreneur saves or draws on the LOC at the risk-free rate r. The second term $\phi_t K_t \sigma_K dZ_t$ in (31) is the exposure from hedging the capital shock Z via the futures position $\phi_t K_t$. The third term, $\pi_n(S, A^n)K_t(dN_t - \lambda_n dt)$, captures the effect of the insurance contract against productivity changes. Note that λ_n is the insurance premium per unit for risk-neutral investors in state n and the unit insurance payment is triggered if and only if $dN_t = 1$.

Dynamic entrepreneurial finance. The implementation problem can now be formulated asfollows: In each period the entrepreneur optimally chooses consumption C_t, investment I_t, futures position $\phi_t K_t$ and insurance demands, $\pi_H K_t$ and $\pi_L K_t$, to maximize her utility function given in (4)-(5), subject to the liquidity accumulation dynamics (31) and the endogenous credit limits (32) implied by the inalienability-of-human-capital constraints.[16]

This dual optimization problem for the entrepreneur is equivalent to the primal problem for the investor in (22) if and only if the borrowing limits, $\underline{S_n}(K)$, are such that:

$$\underline{S_n}(K) = -P(K, \underline{W_n}, A^n)\,, \qquad n \in \{L, H\}, \tag{33}$$

where $P(K, \underline{W_n}, A^n)$ is the investors' value when the entrepreneur's inalienability-of-human-capital constraint binds, that is, when $W = \underline{W_n}$. Accordingly, we characterize the implementation solution for the dual problem by first solving the investors' problem in (22), and then imposing the constraint in (33).

Guided by the observation in the full-commitment case that the value function of the entrepreneur inherits the CRRA form of the entrepreneur's utility function, we conjecture (and later verify) that the entrepreneur's value function $J(K, S, A^n)$ takes the form:

$$J(K, S, A^n) = \frac{(bM(K, S, A^n))^{1-\gamma}}{1-\gamma}\,, \quad n \in \{L, H\}\,, \tag{34}$$

[16] The implementation of the contracting problem can be achieved in a decentralized market setting where the bank, the futures counterparty, and the insurance seller need no coordination among themselves. All three financiers will break even.

where $M(K, S, A^n)$ is the entrepreneur's certainty equivalent wealth and the normalization constant b is given by $U(bM) = J$. In the Appendix, we provide the HJB equation that characterizes $M(K, S, A^n)$ together with the corresponding boundary conditions.

To summarize, the primal optimal contracting problem gives rise to the investor's value function $F(K, V, A^n)$, with the promised utility to the entrepreneur V as the key state variable. By expressing the entrepreneur's promised utility in units of consumption rather than utils, the investor's value function can be rewritten in terms of the entrepreneur's promised certainty-equivalent wealth W: $P(K, W, A^n)$. The dual problem for the entrepreneur gives rise to the entrepreneur's value function $J(K, S, A^n)$, with $S = -P(K, W, A^n)$ as the key state variable. Or, again expressing the entrepreneur's value in units of consumption, the entrepreneur's value function is her certainty equivalent wealth $M(K, S, A^n)$ and the relevant state variable is her savings $S = -P$, as Table 1 in the introduction summarizes. Again, the key attraction of the dual formulation is that it frames the optimal financial contracting problem in terms of a more operational liquidity and risk management problem for the firm.

To simplify the exposition of the key economic mechanism in our model, we next analyze the case with capital (diffusion) risk only, which is the special case with $A^L = A^H = A$.

6 No Productivity Shocks

In this section we consider the special case when the firm's productivity is constant, $A^L = A^H = A$, so that the only shock is the diffusion capital shock Z.

By using our model's homogeneity property, we show that the investors' value function $P(K, W)$ and the entrepreneur's certainty equivalent wealth $M(K, S)$ can be written as:

$$P(K, W) = p(w) \cdot K, \tag{35}$$

where $w = W/K$ is the entrepreneur's certainty-equivalent wealth scaled by the firm's capital stock K, and $p(w)$ is the scaled value function of investors, and

$$M(K, S) = m(s) \cdot K\,, \tag{36}$$

where $s = S/K$ is the entrepreneur's savings S scaled by the firm's capital stock K, and $m(s)$ is the scaled promised (certainty equivalent) wealth.[17] The other variables are also scaled by K, so that $\mathfrak{c}(s)$ is the consumption-capital ratio, $\mathfrak{i}(s)$ the investment-capital ratio, and $\phi(s)$ the hedge ratio. In the interior region we then have:

$$ds_t = \mu^s(s_t)dt + \sigma^s(s_t)dZ_t \,, \tag{37}$$

where the drift and volatility processes $\mu^s(\,\cdot\,)$ and $\sigma^s(\,\cdot\,)$ for s are given by

$$\begin{aligned} \mu^s(s) &= (A - \mathfrak{i}(s) - g(\mathfrak{i}(s)) - \mathfrak{c}(s)) + (r + \delta - \mathfrak{i}(s))s - \sigma_K \sigma^s(s), & (38)\\ \sigma^s(s) &= (\phi(s) - s)\sigma_K \,. & (39) \end{aligned}$$

Note from (39) that the volatility of savings can be controlled by the futures position $\phi(s)$. In particular, by setting $\phi(s) = s$ the entrepreneur can make sure that he faces no risk with respect to the growth of savings $\mu^s(s)$. However, it is generally not optimal to do so.

The following proposition summarizes the solution.

Proposition 2 *In the region where $s > \underline{s}$, the entrepreneur's scaled promised wealth $m(s)$ solves the equation:*

$$\begin{aligned} 0 &= \max_{\mathfrak{i}(s)} \frac{m(s)}{1-\gamma}\left[\gamma\chi(m'(s))^{\frac{\gamma-1}{\gamma}} - \zeta\right] - \delta m(s) + [(r+\delta)s + A]\, m'(s) \\ &\quad + \mathfrak{i}(s)(m(s) - (s+1)m'(s)) - g(\mathfrak{i}(s))m'(s) - \frac{\gamma\sigma_K^2}{2}\frac{m(s)^2 m''(s)}{m(s)m''(s) - \gamma m'(s)^2}\,, \end{aligned} \tag{40}$$

subject to the following boundary conditions:

$$\begin{aligned} \lim_{s\to\infty} m(s) &= q^{FB} + s\,, & (41)\\ m(\underline{s}) &= \alpha m(0)\,, & (42)\\ \lim_{s\to\underline{s}} \sigma^s(s) &= 0 \quad \textit{and} \quad \lim_{s\to\underline{s}} \mu^s(s) \geq 0\,. & (43) \end{aligned}$$

[17] Wang, Wang, and Yang (2012) solve an entrepreneur's optimal consumption-savings, business investment, and portfolio choice problem with endogenous entry and exit decisions. By exploiting homogeneity, they derive the optimal investment policy in a q-theoretic context with incomplete markets. In our model, we optimally implement the solution of the optimal contacting problem.

The ODE given by (40) characterizes the entrepreneur's scaled promised wealth $m(s)$ in the interior region $s > \underline{s}$. As the entrepreneur's savings become infinitely large the entrepreneur's promised wealth must be equal to the first-best value of investment and savings. At that point the entrepreneur's inability to commit no longer affects firm value, as the entrepreneur's self insurance is sufficient to achieve the first-best resource allocation outcome. In this limit the marginal value of liquidity is simply unity as a financially unconstrained entrepreneur does not pay a premium for liquid assets, and the entrepreneur values a unit of capital K at its first-best maximal value q^{FB}.

At the other endogenous boundary, $\underline{s}$, where the entrepreneur runs out of liquidity, the entrepreneur's promised wealth $m(s)$ equals $\alpha m(0)$, the entrepreneur's certainty-equivalent wealth per unit of capital under the outside option.

Finally, the third condition (43) ensures that the entrepreneur does not quit as s approaches $\underline{s}$. This condition ensures that the volatility of s evaluated at $\underline{s}$ is zero and that the drift $\mu^s(\underline{s})$ is weakly positive, so as to guarantee that the constraint $s \geq \underline{s}$ is satisfied at all times and that the entrepreneur will not run out of liquidity.

Remark: Note that for these scaled variables the primal and dual optimization problems are linked as follows: $\underline{s} = -p(\underline{w})$. That is, the inalienability-of-human-capital constraint maps to the endogenous liquidity constraint in the dual problem.

6.1 Parameter Choices and Calibration

Our model with no productivity shocks is parsimonious with only eight parameters. Three parameters essential for the contracting tradeoff between risk sharing and limited commitment are the entrepreneur's coefficient of relative risk aversion γ, the volatility of the capital shocks σ_K, and the parameter measuring the degree of human capital inalienability α. The other five parameters (the risk-free rate r, the entrepreneur's discount rate ζ, the depreciation rate δ, the adjustment cost θ, and the productivity parameter A) are basic to any dynamic model with investment. We choose plausible parameter values to highlight the model's mechanism and main insights.

Thus, we take the widely used value for the coefficient of relative risk aversion, $\gamma = 2$; the annual risk-free interest rate $r = 5\%$; and, the entrepreneur's annual subjective discount rate set to equal to the risk-free rate, $\zeta = r = 5\%$. As for investment, we rely on the parameter findings suggested by Eberly, Rebelo, and Vincent (2009): we set the annual productivity A at 20% and the annual volatility of capital shocks at $\sigma_K = 20\%$. While our model is equally tractable for any homogeneous adjustment cost function $g(i)$, we choose the following widely-used quadratic adjustment cost function for illustrational simplicity,

$$g\left(i\right)=\frac{\theta i^{2}}{2}, \tag{44}$$

which gives explicit formulas for Tobin's q and optimal i in the first-best MM benchmark:

$$q^{FB}=1+\theta i^{FB}, \text{ and } i^{FB}=r+\delta-\sqrt{(r+\delta)^{2}-2\frac{A-(r+\delta)}{\theta}}. \tag{45}$$

Fitting the first-best values of q^{FB} and i^{FB} to the sample averages, we set the adjustment cost parameter at $\theta = 2$ and the (expected) annual capital depreciation rate at $\delta = 12.5\%$. These parameters imply $q^{FB} = 1.2$ and an annual investment-capital ratio of $i^{FB} = 0.1$.

Finally, we choose the fraction of capital stock that the entrepreneur may start out with when she quits, α, to be 0.8, in line with some empirical estimates.[18]

The parameter values for our baseline case are summarized in Table 2. Note that all parameter values are annualized when applicable.

6.2 Promised Wealth W and Liquidity S

The primal contracting and dual implementation problems are linked as follows:

$$s=-p(w) \text{ and } w=m(s), \tag{46}$$

where $p(w)$ is the scaled investors' value in the contracting problem, and $m(s)$ is the entrepreneur's scaled certainty equivalent wealth as a function of s in the implementation formulation. Thus, *liquidity* s for the entrepreneur is the payoff the investor is giving up

[18]See Li, Whited, and Wu (2014) for the empirical estimates of α. The averages are 1.2 for Tobin's q and 0.1 for the investment-capital ratio, respectively, for the sample used by Eberly, Rebelo, and Vincent (2009). The imputed value for the adjustment cost parameter θ is 2 broadly in the range of estimates used in the literature. See Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).

Table 2: Summary of Parameters

This table summarizes the parameter values used for numerical illustration.

A. Baseline model with no productivity shocks		
Parameters	Symbol	Value
Risk-free rate	r	5%
The entrepreneur's discount rate	ζ	5%
The entrepreneur's relative risk Aversion	γ	2
Capital depreciation rate	δ	12.5%
Volatility of capital depreciation shock	σ_K	20%
Quadratic adjustment cost parameter	θ	2
Firm's productivity	A	20%
Inalienability of human capital parameter	α	80%

B. General model with productivity shocks			
Parameters	Symbol	State H	State L
Firm's productivity	A_n	20%	18%
State transition intensity	λ_n	10%	0

through the *promised wealth* w to the entrepreneur. More formally, (46) implies that the composition of $-p$ and m, denoted by $-p \circ m$, yields the identity function: $-p(m(s)) = s$.

Scaled promised wealth w and scaled investors' value $p(w)$. Figure 1 plots the investor's scaled value $p(w)$ and the sensitivity of the value to changes in promised wealth $p'(w) = P_W$ in Panels A and B respectively. In the first-best MM world, compensation to the entrepreneur is simply a one-to-one transfer away from investors, as we see from the dotted lines: $p(w) = q^{FB} - w = 1.2 - w$ and $p'(w) = -1$. With inalienability of human capital, investors' value $p(w)$ is decreasing and concave in w. That is, as w increases the entrepreneur is less constrained so that the marginal value $p'(w)$ decreases. Additionally, $p(w)$ approaches $q^{FB} - w$, and $p'(w) \to -1$, as $w \to \infty$. That is, the first-best payoff obtains when the entrepreneur is unconstrained. However, the entrepreneur's inability to fully commit not to walk away *ex post* imposes a lower bound $\underline{w}$ on w. For our parameter values, $w \geq \underline{w} = 0.944$.

Note finally that despite being risk neutral, the investor effectively behaves in a risk-averse manner due to the entrepreneur's inalienability-of-human-capital constraints. This is reflected in the concavity of the investors' scaled value function $p(w)$. This concavity property

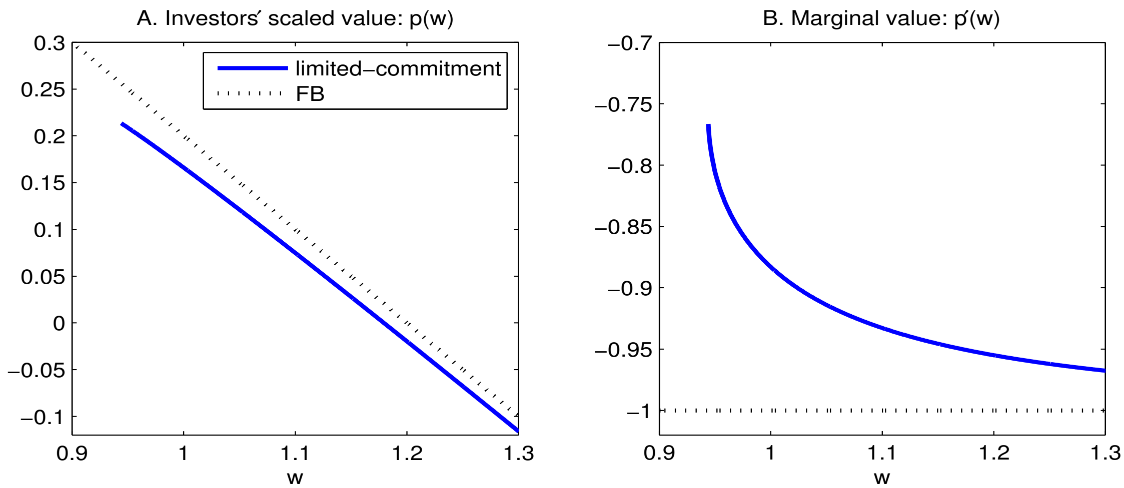

Figure 1: **Investors' scaled value $p(w)$ and the marginal value of w, $p'(w)$.** For the limited-commitment case, $w \geq \underline{w} = 0.944$, and $p(w)$ is decreasing and concave in w. The dotted line depicts the full-commitment MM results: $p(w) = q^{FB} - w$ and $p'(w) = -1$.

is an important difference of the limited commitment problem relative to the neoclassical problem, where volatility has no effect on firm value.

Scaled liquidity s and the entrepreneur's scaled certainty-equivalent wealth $m(s)$. Figure 2 plots the entrepreneur's scaled savings $m(s)$ and the marginal value of liquidity $m'(s)$ in Panels A and B respectively. As one might expect $m(s)$ is increasing and concave in s. The higher the firm's financial slack s the less the entrepreneur is likely to want to walk away and hence the higher the value of $m(s)$ in the long-term bilateral relationship. Moreover, as s increases the entrepreneur is less constrained so that the marginal value of savings $m'(s)$ decreases ($m''(s) < 0$). In the one-sided limited-commitment case the entrepreneur's scaled wealth $m(s)$ approaches $q^{FB} + s$ and $m'(s) \to 1$ as $s \to \infty$.[19] The entrepreneur's LOC limit, or in other words, her risk-free debt capacity $\underline{s} = -p(\underline{w})$ is given by -0.214.

We next discuss the optimal policy rules.

[19]See Wang, Wang, and Yang (2012) for similar conditions in a model with exogenously-specified incomplete-markets model of entrepreneurship.

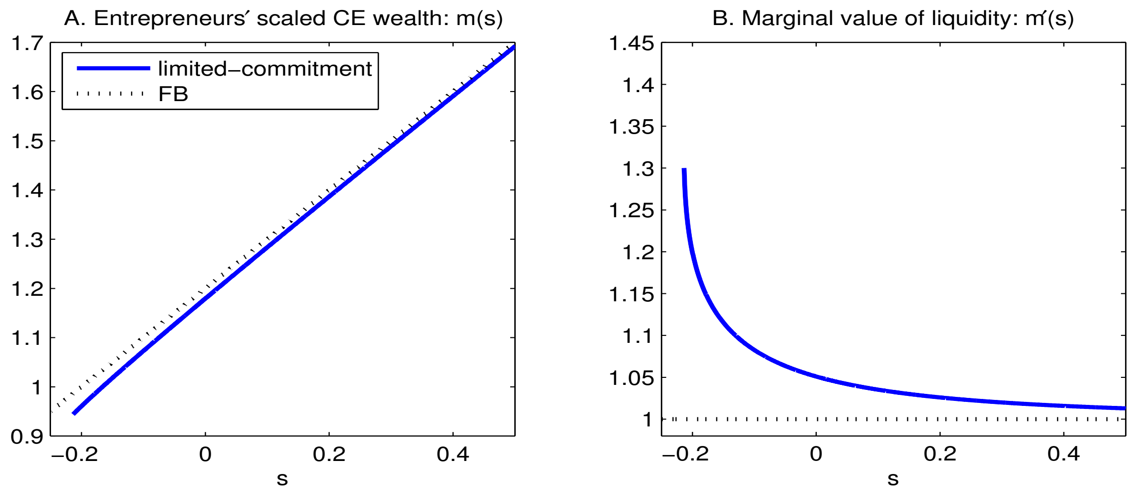

Figure 2: **The entrepreneur's scaled certainty equivalent wealth** $m(s)$ **and marginal (certainty equivalent) value of liquidity,** $m'(s)$**.** For the limited-commitment case, $s \geq \underline{s} = -0.214$, and $m(s)$ is increasing and concave. The dotted line depicts the full-commitment MM results: $m(s) = q^{FB} + s$ and the sensitivity $m'(s) = 1$.

6.3 Investment, Consumption, Liquidity and Risk Management

We first analyze the firm's investment decisions, then the entrepreneur's optimal consumption, and finally corporate liquidity and risk management.

6.3.1 Investment, marginal q, and the marginal value of liquidity $m'(s)$.

We can simplify the FOC for investment to:

$$1 + g'(\mathfrak{i}(s)) = \frac{J_K}{J_S} = \frac{M_K}{M_S} = \frac{m(s) - sm'(s)}{m'(s)}, \tag{47}$$

where the first equality is the investment FOC, the second equality follows from the definition of the value function in (34), and the last equality follows from the homogeneity property of $M(K,S)$ in K. Under perfect capital markets the entrepreneur's certainty equivalent wealth is given by $M(K,S) = m(s) \cdot K = (q^{FB} + s) \cdot K$ and the marginal value of liquidity is $M_S = 1$ at all times. Hence in this case, the FOC (47) specializes to the classical Hayashi condition for optimal investment, where the marginal cost of investing $1 + g'(\mathfrak{i}(s))$ equals marginal q.

Under limited commitment, $M_S > 1$ in general and the FOC (47) then states that the marginal cost of investing (on the left-hand side) equals the ratio between (a) marginal q, measured by M_K, and (b) the marginal value of liquidity measured by M_S. Unlike in the classical q theory of investment, here financing matters and M_S measures the (endogenous) marginal cost of financing generated by limited commitment constraints.

Figure 3 illustrates the effect of inalienability of human capital on marginal q and investment $\mathfrak{i}(s)$. The dotted lines in Panels A and B of Figure 3 give the first-best $q^{FB} = 1.2$ and $i^{FB} = 0.1$, respectively. With limited commitment, $\mathfrak{i}(s)$ is lower than the first-best benchmark $i^{FB} = 0.1$ for all s, and increases from -0.03 to $i^{FB} = 0.1$ as s increases from the left boundary $\underline{s} = -0.214$ towards ∞. This is to be expected: increasing financial slack mitigates the severity of under-investment for a financially constrained firm. Note however

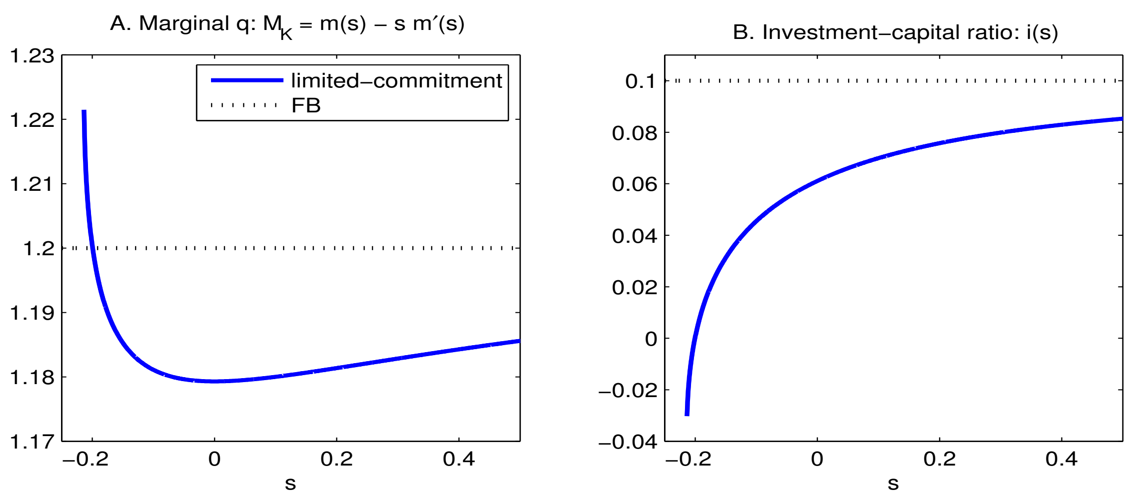

Figure 3: **Marginal q, $M_K = m(s) - sm'(s)$, and the investment-capital ratio $\mathfrak{i}(s)$.** For the limited-commitment case, the firm always under-invests and $\mathfrak{i}(s)$ increases with s. The dotted line depicts the full-commitment MM results where the marginal equals $q^{FB} = 1.2$ and the first-best investment-capital ratio $\mathfrak{i}(s) = i^{FB} = 0.1$.

that, surprisingly, marginal q (that is, M_K) *decreases* with s from 1.22 to 1.18 in the credit region $s < 0$. What is the intuition? When the firm is financing its investment via credit at the margin (when $S < 0$), increasing K moves a negative-valued s closer to the origin thus mitigating financial constraints, which is an additional benefit of accumulating capital.[20]

But why does a high marginal-q firm invest less in the credit region $s < 0$? And how do we reconcile an increasing investment function $\mathfrak{i}(s)$ with a decreasing marginal q function,

[20]Formally, this result follows from $dM_K/ds = -sm''(s) < 0$ when $s < 0$ and from the concavity of $m(s)$.

$M_K = m(s) - sm'(s)$ in the credit region $s < 0$? The reason is simply that in the credit region ($s < 0$) a high marginal-q firm also faces a high financing cost. When $s < 0$ the marginal q and the marginal financing cost $m'(s)$ are perfectly correlated. And investment is determined by the ratio between the marginal q and $m'(s)$ as we have noted. At the left boundary $\underline{s} = -0.214$ marginal q is 1.22 and $m'(s)$ is 1.30 both of which are high. Together they imply that $\mathfrak{i}(-0.214) = -0.03$, which is low compared with the first-best $i^{FB} = 0.10$.

More generally, we consider a measure of investment-cash sensitivity given by $\mathfrak{i}'(s)$. Taking the derivative of investment-capital ratio $i(s)$ in (47) with respect to s, we have

$$\mathfrak{i}'(s) = -\frac{1}{\theta}\frac{m(s)m''(s)}{m'(s)^2} > 0. \tag{48}$$

As $m(s)$ is concave in s regardless of whether $s \geq 0$ or $s < 0$, $\mathfrak{i}(s)$ is increasing in liquidity.[21]

6.3.2 Consumption

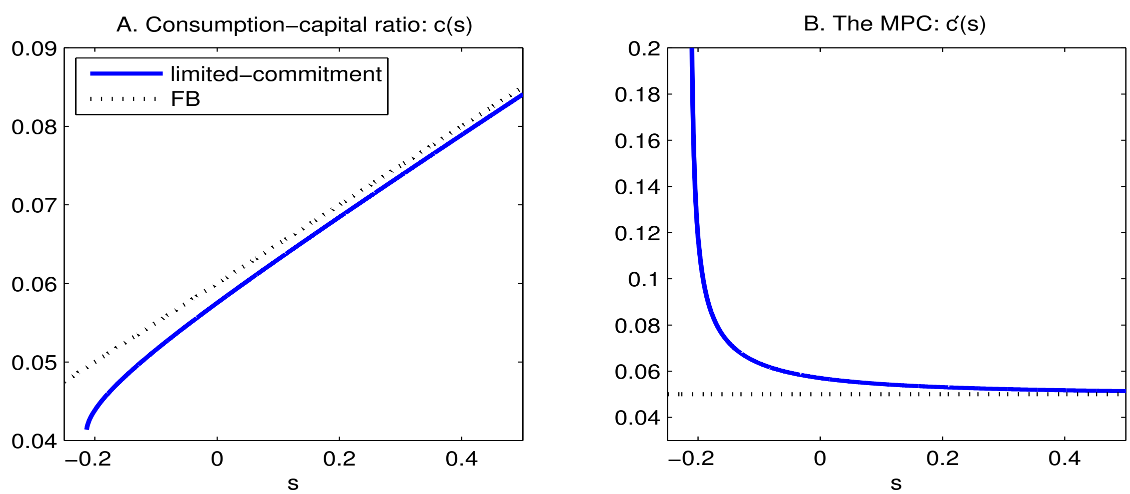

Figure 4: **Consumption-capital ratio $\mathfrak{c}(s)$ and the MPC $\mathfrak{c}'(s)$.** For the limited-commitment case, the entrepreneur always under-consumes compared with the full-commitment case and $\mathfrak{c}(s)$ increases with s. The dotted line depicts the full-commitment consumption-smoothing results: $\mathfrak{c}(s) = \chi(s + q^{FB})$ and the MPC $\mathfrak{c}'(s) = \chi = 5\%$.

[21] See Bolton, Chen, and Wang (2011) for related discussions on how cash and credit influence the behaviors of investment, marginal q, and marginal value of liquidity.

The entrepreneur's optimal consumption rule $\mathfrak{c}(s)$ is given by:

$$\mathfrak{c}(s) = \chi m'(s)^{-1/\gamma} m(s)\,, \tag{49}$$

where χ is given in (15). Figure 4 plots the optimal consumption-capital ratio $\mathfrak{c}(s)$, and the MPC $\mathfrak{c}'(s)$ in Panels A and B respectively. The dotted lines in Panels A and B of Figure 4 give the first-best $\mathfrak{c}(s) = (s + q^{FB})$ and MPC $\mathfrak{c}'(s) = 5\%$, respectively. The solid blue plot gives the entrepreneur's consumption, which is lower than the first-best benchmark. Additionally, the higher the financial slack s the higher is $\mathfrak{c}(s)$ as seen in the figure. Moreover, we have $m(s) \to q^{FB} + s$ and the marginal value of liquidity $m'(s) \to 1$ as $s \to \infty$, so that $\mathfrak{c}(s) \to \chi\left(q^{FB} + s\right)$, the permanent-income consumption benchmark. Panel B shows that the MPC $\mathfrak{c}'(s)$ decreases significantly with s and approaches the permanent-income benchmark $\chi = 5\%$ as $s \to \infty$. Thus, financially constrained entrepreneurs deep in debt (with s close to $\underline{s}$) have MPCs that are substantially higher than the permanent-income benchmark.

Next we turn to the firm's optimal hedging policy.

6.3.3 Hedging via Futures

Before delving into the analysis, we first review the entrepreneur's total wealth holdings in our implementation, which consist of three parts: (1) a 100% equity stake in the underlying business; (2) a mark-to-market futures position; and (3) a liquidity asset holding in the amount of s (negative when the firm is borrowing.)

The entrepreneur's optimal futures position $\phi(s)$ is given by

$$\phi(s) = \frac{s m''(s) m(s) + \gamma m'(s)(m(s) - s m'(s))}{m(s) m''(s) - \gamma m'(s)^2}\,. \tag{50}$$

Figure 5 plots the futures position $\phi(s)$. First, under full commitment, the risk-averse entrepreneur is fully insured against the idiosyncratic business risk by taking a perfectly offsetting short futures position $\phi(s) = -q^{FB} = -1.2$. See the dotted line in Figure 5.

With limited commitment the entrepreneur cannot fully hedge her equity exposure. How does $\phi(s)$ depend on s in this case? The solid line gives the futures position $\phi(s)$: As the firm becomes less constrained (s increases) the entrepreneur increases the futures hedging position

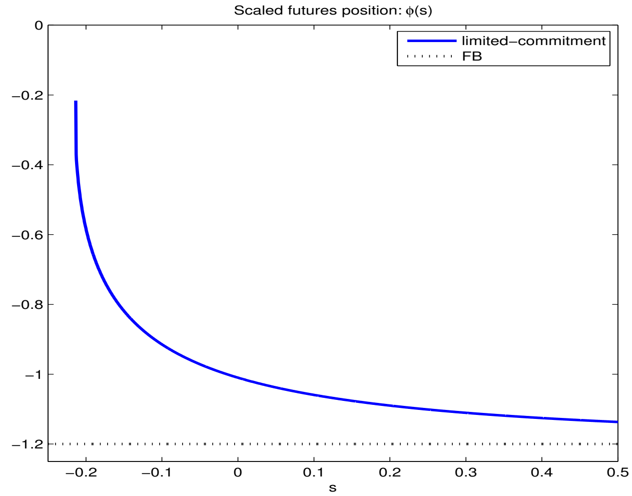

Figure 5: **Futures hedging position** $\phi(s)$**.** For the limited-commitment case, the entrepreneur partially hedges her firm's equity exposure by shorting futures, $\phi(s) < 0$. Note that hedging and liquidity are complements in that $|\phi(s)|$ increases with s. The dotted line depicts the entrepreneur's full-commitment hedging results with $\phi(s) = -q^{FB} = -1.2$.

$|\phi(s)|$. Thus, a less constrained firm has a larger hedging position (after controlling for firm size), and in the limit as $s \to \infty$ the entrepreneur can fully diversity the idiosyncratic business risk by taking a short futures position: $\phi(s) = -q^{FB} = -1.2$, attaining the full-commitment perfect insurance benchmark. Note that here liquidity and hedging are complements.

6.4 Shorting Stocks: An Alternative Implementation

An alternative implementation could be through the holdings of a risky liquid asset that is *perfectly correlated* with the shock Z. Let dR_t denote the incremental return for this risky asset. Given that investors are risk neutral we may write down dR_t as follows,

$$dR_t = rdt + \sigma_K dZ_t \,. \tag{51}$$

Without loss of generality, we choose the volatility of this new risky asset to be σ_K.

Let Ω_t denote the entrepreneur's holdings of this liquid risky asset, and her remaining liquid wealth be $S_t - \Omega_t$, which is invested in a risk-free savings account earning r. Since the entrepreneur can costlessly and continuously rebalance her portfolio, we may write the evolution of the entrepreneur's total liquid wealth S_t as follows:

$$\begin{aligned} dS_t &= (r(S_t - \Omega_t) + Y_t - C_t)dt + \Omega_t(rdt + \sigma_K dZ_t) \\ &= (rS_t + Y_t - C_t)dt + \Omega_t \sigma_K dZ_t \,. \end{aligned} \tag{52}$$

Comparing (52) with (31) it is straightforward to conclude that the entrepreneur's position $\omega(s) = \Omega/K$ under this new implementation is the same as the futures position $\phi(s)$ in the previous implementation, $\omega(s) = \phi(s)$. Unlike in the futures implementation, however, the entrepreneur collects the short-sale proceeds, $-\Omega$, and invests the total liquidity assets $S - \Omega$ in the savings account earning interest at the rate of r.

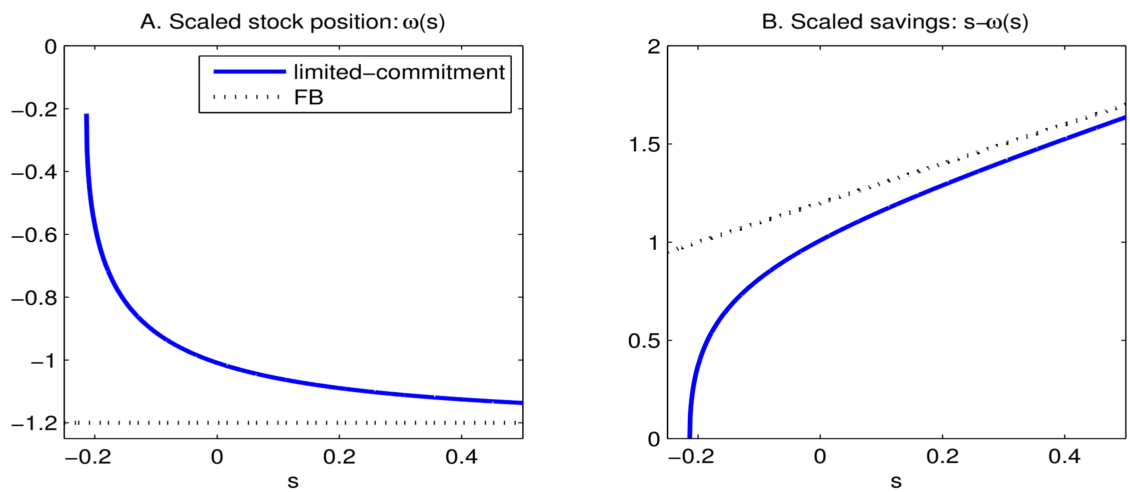

Figure 6: **Optimal hedge** $\omega(s)$ **and savings** $s - \omega(s)$**.** For the limited-commitment case, the entrepreneur partially hedges her firm's equity exposure by shorting the perfectly correlated stock, $\omega(s) < 0$. Note that $\omega(s)$ equals the futures hedging position $\phi(s)$. The dotted line depicts the entrepreneur's full-commitment hedging results with $\omega(s) = -q^{FB} = -1.2$.

Figure 6 plots the hedging position via the risky liquid asset, $\omega(s)$ in Panel A and the entrepreneur's total risk-free asset holdings, $s - \omega(s)$, which earn the risk-free rate r in Panel B. By construction, the risky asset position is the same as the futures hedging position given that the risky asset and the futures contract have the same risk exposures $\sigma_K dZ$. As for the futures position, the entrepreneur takes a short position in the risky asset whose return is given by (51) in order to partially hedge the risk exposure to the underlying illiquid business project. While savings under the futures hedging are simply given by s, the savings in this new implementation equal $s - \omega(s) \neq s$, given that the short position $\omega(s)$ (per unit of capital) generates sales proceeds in the amount of $-\omega(s) > 0$. Panel B of Figure 6 illustrates that in both cases, the entrepreneur stochastically saves, $s - \omega(s) > 0$. In sum, this implementation reveals that borrowing can also take the somewhat unconventional form in corporate finance of shorting a liquid risky asset.

7 Two-sided Limited Commitment

Under the one-sided commitment solution investors must be able to commit to incurring losses. As Figure 1 illustrates $p(w)$ takes negative values when w exceeds 1.18. To be able to retain the entrepreneur, investors then promise such a high wealth w to the entrepreneur that they end up committing to making losses in these states of the world. But, what if they cannot commit to such loss-making wealth promises to the entrepreneur? What if investors are protected by limited liability and cannot commit to a long-term contract that yields a negative net present value at some point in the future? We explore this issue in this section and derive the optimal contract when neither the entrepreneur nor investors are able to fully commit. Specifically, we introduce the additional set of constraints for investors that guarantee at any time t that investors receive a non-negative payoff value under the contract:

$$F_t \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-r(v-t)} (Y_v - C_v) dv \right] \geq 0 \,. \tag{53}$$

As it turns out, solving the two-sided limited commitment problem does not involve major additional complexities. The main change relative to the one-sided problem is that the upper boundary is now $s = 0$. Indeed, any promise of strictly positive savings $s > 0$ is not credible as this involves a negative continuation payoff for investors. Accordingly, we replace condition (41) with the following condition:

$$\lim_{s \to 0} \sigma^s(s) = 0 \quad \text{and} \quad \lim_{s \to 0} \mu^s(s) \leq 0 \,. \tag{54}$$

As before, the volatility $\sigma^s(\,\cdot\,)$ must be zero at $s = 0$ and the drift needs to be weakly negative to pull s to the interior so as to ensure that s will not violate the constraint $s \leq 0$.

7.1 Investment and Risk Management

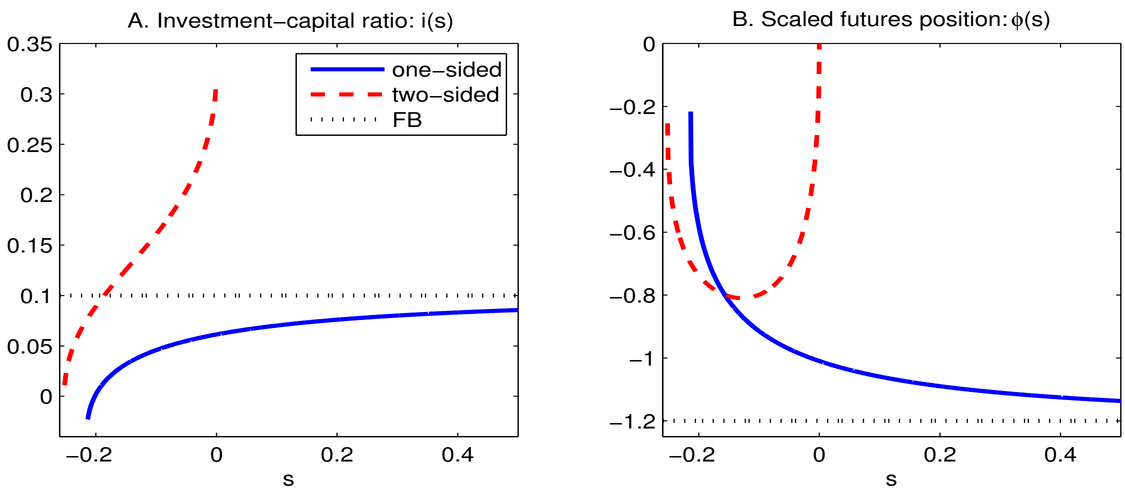

Figure 7: **Optimal investment-capital ratio** $\mathrm{i}(s)$ **and futures hedging** $\phi(s)$**.** For the two-sided limited-commitment case, over-investment (compared with the first-best level) is optimal $-0.2 < s \leq 0$ and hedging is non-monotonic in s.

Figure 7 reports the two-sided limited-commitment solution for investment and futures hedging in Panels A and B, respectively. Comparing the two-sided and one-sided limited commitment solutions for investment in Panel A, we observe that the limited-liability constraint for investors prevents the entrepreneur from owning positive liquid wealth, so that there is only a credit region in the two-sided case: $s \leq 0$. This is necessary for the investors to have positively-valued stake in the firm. Remarkably, in this case the firm may either under-invest or over-invest compared with the first-best benchmark. The firm under-invests when $s < -0.2$ but over-invests when $-0.2 < s \leq 0$. Whether the firm under-invests or over-invests depends on the net effects of the entrepreneur's limited-commitment and the investors' limited-liability constraints. For sufficiently low values of s (when the entrepreneur is deep in debt) the entrepreneur's constraint matters more and hence the firm under-invests. For values of s sufficiently close to zero, the investors' limited-liability constraint has a stronger influence on investment as the investors' value is close to zero. To ensure that s will drift back into the credit region the entrepreneur needs to "save" in the form of the illiquid productive asset (by increasing K) by borrowing more. By over-investing, the firm optimally chooses to keep s between $\underline{s}$ and 0. In summary, given that the entrepreneur cares about the total compensation $W = w \cdot K$ and given that investors are constrained by their ability to promise the entrepreneur w beyond an upper bound, investors reward the entrepreneur along the extensive margin, firm size K, which induces over-investment but allows the entrepreneur to build more human capital.

Panel B of Figure 7 plots the futures hedging position $\phi(s)$. It illustrates that $\phi(s)$ is non-monotonic in s for the two-sided case: Although the entrepreneur can afford to build larger hedging positions when s is larger, investors find these large hedging positions incompatible with their limited-liability constraints. To prevent investors from reneging on their promises volatility must be turned off at $s = 0$, which is achieved by setting $\phi(s) = s$ at $s = 0$, as implied by the volatility boundary condition (39) for $\sigma(s)$. This nonlinear hedging result illustrates the complexity of firms' liquidity and risk management policies and point to the subtle interaction between a firm's risk management and its financial slack.

7.2 Generalizing Investors' Outside Option (Hart and Moore, 1994)

The limited-lability constraints given in (53) implicitly assume that when investors pull the plug on the firm the unit liquidation value of capital ℓ is equal to zero. How is the two-sided limited-commitment solution affected when the liquidation value of capital ℓ is strictly positive, as in Hart and Moore (1994)? We explore this question in this subsection. We only need to modify the investors' limited-liability constraints (53) as follows:

$$F_t \geq \ell K_t \,, \tag{55}$$

where $\ell > 0$ and F_t is investors' value at time t. With these new limited-commitment constraints for investors, the boundary condition (42) must then be replaced with the new boundary condition $m(\underline{s}) = \alpha m(-\ell)$. Figure 8 plots the entrepreneur's scaled value $m(s)$ and optimal investment $\mathfrak{i}(s)$ for three different values of ℓ: $0, 0.2$, and 0.4.

While a higher liquidation value might at first appear to facilitate contracting it has the opposite effect in the two-sided limited-commitment problem. The reason is that investors, going forward, have a higher temptation to pull the plug when the liquidation value of capital is higher. Panel A shows that $m(s)$ decreases with ℓ. Additionally, the domain $[\underline{s}, -\ell]$ shifts

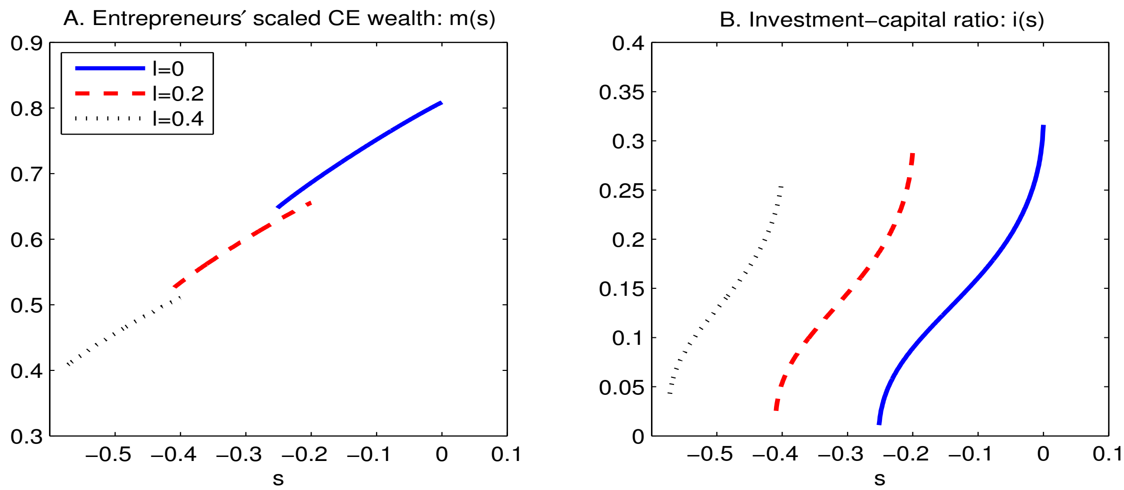

Figure 8: **The effect of investors' outside option value, ℓ.** Panels A and B plot the entrepreneur's scaled certainty equivalent wealth $m(s)$ and the investment-capital ratio $\mathfrak{i}(s)$, respectively. The dotted, solid and dashed lines correspond to the three different levels of liquidation: $\ell = 0, \ell = 0.2$, and $\ell = 0.4$.

to the left as ℓ increases. For $\ell = 0.2$ and $\ell = 0.4$, we have $-0.410 \leq s \leq -0.2$, and $-0.571 \leq s \leq -0.4$, respectively. Panel B demonstrates another important insight from this comparative statics: A higher capital liquidation value ℓ reduces over-investment. Given that the contracting surplus is lower for a higher value of ℓ, investors are less keen to retain the entrepreneur by letting her build her human capital through overinvestment. All in all, these two predictions of our model go against the received wisdom from the static models on financial constraints based on the limited pledgeability of cash flows (see Tirole, 2006).

8 Persistent Productivity Shocks: Insurance

In this section we consider the general case where the firm may be subject to persistent observable productivity shocks. Due to their persistence, it is natural to assume that these productivity shocks are observable and can be contracted on.[22] A key new contractual dimension emerges in this more general setting: the optimality of contingent capital financial

[22] See DeMarzo, Fishman, He, and Wang (2012) for a model of optimal investment in a q-theoretic context with persistent shocks and agency frictions along the line of DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006).

contracts. We derive below the optimal one-sided limited-commitment contract under persistent productivity shocks and the consequences for investment, consumption, liquidity and risk management. As we will show, persistent productivity shocks will naturally give rise to a demand for insurance against the persistent productivity shock, a form of contingent capital financing arrangement. Equivalently, we show that default on a debt claim when productivity decreases from a high level H to a lower level L can be an optimal outcome. We spell out the solution of the optimal contracting problem in the Appendix and illustrate an intuitive financial implementation with commonly used securities below.

8.1 Implementation: Liquidity and Risk management

Again appealing to the homogeneity property of our model, we write the entrepreneur's certainty equivalent wealth function in state $n \in \{L, H\}$, $M(K, S, A^n)$, as follows:

$$M(K, S, A^n) = m_n(s) \cdot K \,. \tag{56}$$

Given the consumption-capital ratio $\mathfrak{c}_n(s)$, the investment-capital ratio $\mathfrak{i}_n(s)$, the hedge ratio $\phi_n(s)$, and the endogenous contingent transfer of liquidity $\pi_n(s)$ when productivity switches out of state n, we can write the dynamic evolution of liquidity s as follows:

$$ds_t = \mu_n^s(s_t)dt + \sigma_n^s(s_t)dZ_t + \pi_n(s)dN_t \,, \tag{57}$$

where the drift and volatility processes $\mu_n^s(\,\cdot\,)$ and $\sigma_n^s(\,\cdot\,)$ for s are given by:

$$\mu_n^s(s) = (A^n - \pi_n(s)\lambda_n - \mathfrak{i}_n(s) - g(\mathfrak{i}_n(s)) - \mathfrak{c}_n(s)) + (r + \delta - \mathfrak{i}_n(s))s - \sigma_K \sigma_n^s(s), \tag{58}$$

$$\sigma_n^s(s) = (\phi_n(s) - s)\sigma_K \,. \tag{59}$$

The one-sided limited-commitment case. The following proposition summarizes the solution for the case where only the entrepreneur cannot commit her human capital.

Proposition 3 *In the region $s > \underline{s_H}$, the scaled value $m_H(s)$ satisfies the following ODE:*

$$
\begin{aligned}
0 &= \max_{\mathfrak{i}_H,\pi_H} \frac{m_H(s)}{1-\gamma}\left[\gamma\chi m'_H(s)^{\frac{\gamma-1}{\gamma}}-\zeta\right]-\delta m_H(s)+\left[(r+\delta)s+A^H-\lambda_H\pi_H\right]m'_H(s) \\
&\quad +\mathfrak{i}_H(m_H(s)-(s+1)m'_H(s))-g(\mathfrak{i}_H)m'_H(s)-\frac{\gamma\sigma_K^2}{2}\frac{m_H(s)^2 m''_H(s)}{m_H(s)m''_H(s)-\gamma m'_H(s)^2} \\
&\quad +\frac{\lambda_H m_H(s)}{1-\gamma}\left(\left(\frac{m_L(s+\pi_H)}{m_H(s)}\right)^{1-\gamma}-1\right), \qquad (60)
\end{aligned}
$$

subject to the following boundary conditions:

$$
\begin{aligned}
\lim_{s\to\infty} m_H(s) &= q_H^{FB}+s\,, \qquad (61)\\
m_H(\underline{s_H}) &= \alpha m_H(0)\,, \qquad (62)\\
\lim_{s\to\underline{s_H}} \sigma_H^s(s) &= 0 \quad \textit{and} \quad \lim_{s\to\underline{s_H}} \mu_H^s(s)\geq 0\,. \qquad (63)
\end{aligned}
$$

An analogous ODE equation for $m_L(s)$ and set of boundary conditions must also hold in state L. We thus jointly solve $m_H(s)$ and $m_L(s)$ given their state dependence.

The two-sided limited-commitment case. Again, we simply modify the upper boundary condition in Proposition 3. The upper boundary is now given by $s = 0$ rather than the natural limiting boundary $s \to \infty$ in the one-sided case. We thus replace condition (61) with the following conditions at the new upper boundary $s = 0$:

$$
\lim_{s\to 0}\sigma_H^s(s) = 0 \quad \text{and} \quad \lim_{s\to 0}\mu_H^s(s)\leq 0\,. \qquad (64)
$$

8.2 Insurance: Hedging against productivity shocks

For illustration, we consider the simplest setting where the productivity jump from H to L is permanent and irreversible, so that $\lambda_L = 0$. We set $\lambda_H = 0.1$ and choose the productivity levels to be $A^L = 0.18$ and $A^H = 0.2$. All the other parameter values remain unchangedl.

Figure 9 plots the entrepreneur's insurance demand $\pi_H(s)$ in state H against the productivity change from state H to L. As we see for all levels of s, the entrepreneur pays a positive but time-varying insurance premium $\lambda_H\pi_H(s)$ per unit of time in state H to investors in or-

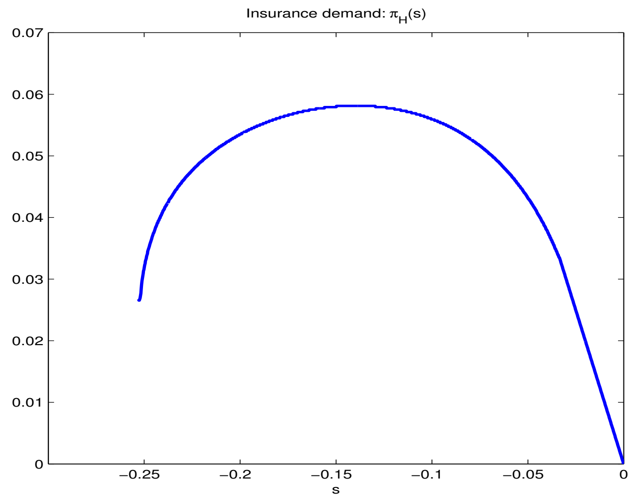

Figure 9: **Insurance demand (against the productivity change from H to L) $\pi_H(s)$.** Parameter values: $A^L = 0.18$, $A^H = 0.2$, $\lambda_L = 0$, and $\lambda_H = 0.1$.

der to receive a lump-sum insurance payment in the amount of $\pi_H > 0$ from investors at the moment when the productivity state switches from H to L. By doing so, the entrepreneur equates the marginal utility before and after the productivity changes whenever feasible. Interestingly, the insurance demand $\pi_H(s)$ is non-monotonic in s as it first increases and then decreases with s. The intuition is as follows. For a severely constrained entrepreneur whose s is close to the left boundary $\underline{s}$, the entrepreneur has limited funds to purchase insurance. Therefore, insurance $\pi_H(s)$ increases as s moves towards the origin turning less negative.

As s approaches the origin sufficiently closely, the entrepreneur's demand for insurance $\pi_H(s)$ decreases for the following reasons. First, the entrepreneurial firm has more liquidity to self-insure and hence demand for additional liquidity decreases. Second, the entrepreneur's decreasing marginal utility also suggests that the entrepreneur's demand for insurance decreases with liquidity, *ceteris paribus*. Finally, the investors' limited-liability constraint requires that $\pi_H(s) \leq -s$, which, in turn, truncates the insurance demand. For these reasons, the insurance demand $\pi_H(s)$ is non-monotonic in liquidity s as shown in Figure 9.

9 Simulation

To deepen understanding of the dynamics of consumption and investment, and value of capital under the optimal contract it is helpful to focus on simulated sample paths of productivity and capital shocks. Figures 10 display the simulation results. In order to value capital stock, we first define the total value of capital and Tobin's average q.

Total value of capital and Tobin's average q. As capital generates payoffs for both investors and the entrepreneur, we calculate the total value of capital via $P_n(K,W)+W$. Alternatively, we may calculate the total value of capital, the firm's enterprise value, in our implementation problem as $M_n(K,S)-S$, the difference between the entrepreneur's certainty equivalent wealth $M_n(K,S)$ and corporate savings S.

Tobin's q, defined as the ratio between value of capital and capital stock, is given by

$$q_n = \frac{P_n(K,W)+W}{K} = p_n(w)+w, \tag{65}$$

$$= \frac{M_n(K,S)-S}{K} = m_n(s)-s, \;\; n \in \{L,H\}. \tag{66}$$

The second line follows from $s_n = -p_n(w)$ and $w_n = m_n(s)$. Note that the two definitions of average q (contracting-based and implementation-based) give the same value of capital and average q. For the first-best benchmark, we uncover the definition of q as in Hayashi (1982).

Panel A plots simulated sample paths for shocks A and Z. As the right vertical axis exhibits, the productivity shock A_t drops from $A_H = 0.2$ to $A_L = 0.18$ in year $t = 9$. The left vertical axis in Panel A plots the realized capital shock Z. All other panels are derived from this simulation of Z and A shocks. Panel B reports the sample path for consumption C_t for the one-sided and two-sided limited-commitment in addition to the full-commitment case. As expected, under full commitment by both parties, consumption C shall be constant at all times to achieve perfect risk sharing. For the one-sided case, consumption C is non-decreasing, remains flat for a period of time, and adjusts upward only when the inalienability-of-human-capital constraint binds. For the two-sided case, we see that consumption can either increase or decrease depending on whose limited-commitment constraints are more costly in a forward-looking sense.

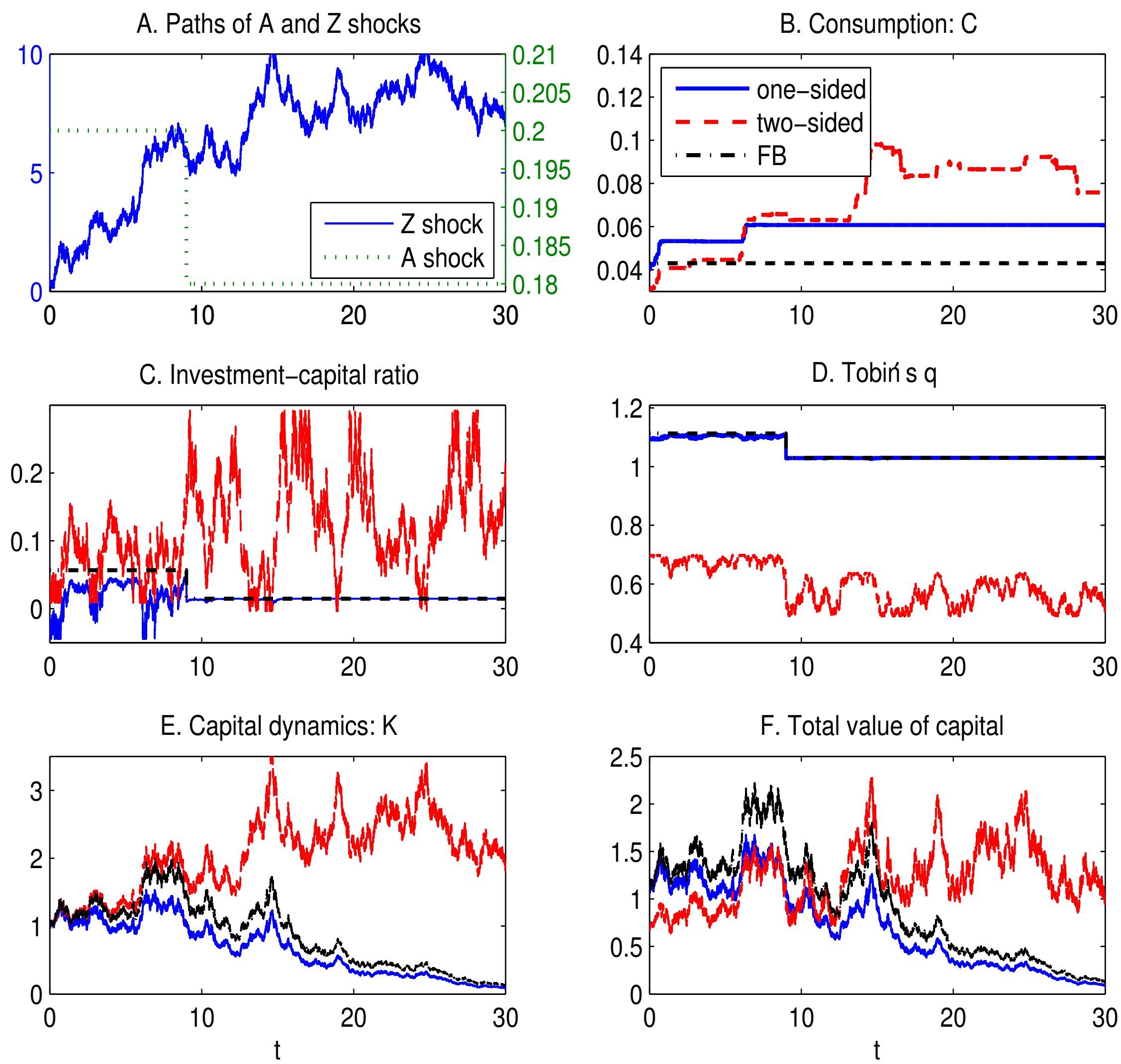

Figure 10: **Simulated sample paths, policy functions, and valuation of capital for both the one-sided and two-sided cases.**

Panel C plots the investment-capital ratio i. Particularly striking is the extent of over-investment in the two-sided case. Note also that in the one-sided case the firm eventually builds sufficient slack that it most reaches the first-best investment level $i^{FB} = 1.5\%$ under state L by year 17. Intuitively, in the one-sided case, the entrepreneur eventually owns more than the whole firm driving the investors into the region with negative present value.

As one expects, Tobin's q in the one-sided case is somewhat lower than under the first-best as underinvestment distorts value of capital. Note also Tobin's q is substantially lower for the two-sided case than for the one-sided case as investment is much more inefficient (featuring both over- and under-investment in the former case), as shown in Panel D.

Panel E plots the dynamics of the capital stock K in the three cases. Note, in particular, that capital K turns out to be higher in the two-sided case due to the over-investment incentives built into the contract. Finally, Panel F reveals that the firm's capitalization $q_n \cdot K$ for the two-sided case may be either higher or lower than for the full-commitment case. However, capital K is always higher under full commitment than under the one-sided commitment. Judging whether corporate investment is efficient or not based on total valuation of capital can be misleading.

10 Deterministic Case *à* la Hart and Moore

The Hart and Moore (1994) model can be viewed as a special case of our model with $\sigma_K = 0$ and $A^L = A^H = A$. When scaled consumption $\mathfrak{c}_t$ and scaled investment $\mathfrak{i}_t$ are set, the entrepreneur's (scaled) liquidity s then grows deterministically as follows:

$$\mu^s(s_t) \equiv \frac{ds_t}{dt} = (r + \delta - \mathfrak{i}_t)s_t + A - \mathfrak{i}_t - g(\mathfrak{i}_t) - \mathfrak{c}_t \,. \tag{67}$$

Let $\mu^s_{FB}(s_t)$ denote the first-best drift $\mu^s(s_t)$ such that $\mathfrak{i}_t = i^{FB}$ given in (45) and $\mathfrak{c}_t = c^{FB} = \chi(s_t + q^{FB})$. It is straightforward to show that we then have:

$$\mu^s_{FB}(s_t) = \left(\delta - i^{FB} - \gamma^{-1}(\zeta - r)\right) m^{FB}(s_t) \,. \tag{68}$$

As the entrepreneur's first-best scaled wealth is nonnegative, $(m^{FB}(s_t) = (s_t + q^{FB}) > 0)$ it immediately follows that the drift $\mu^s_{FB}(s_t) \geq 0$ if and only if the following condition holds:

$$\textbf{Condition A: } i^{FB} \leq \delta + \frac{r - \zeta}{\gamma}, \tag{69}$$

where i^{FB} is given by (45).

There are then two mutually exclusive cases depending on the sign of $\mu^s_{FB}(s_t)$ and equivalently whether Condition A is satisfied or not. In both cases the entrepreneur's scaled certainty equivalent wealth $m(s)$ satisfies the following ODE in the interior region:

$$\begin{aligned} 0 \quad = \quad & \frac{m(s)}{1-\gamma}\left[\gamma\chi(m'(s))^{\frac{\gamma-1}{\gamma}} - \zeta\right] - \delta m(s) + \left[(r+\delta)s + A\right]m'(s) \\ & + \mathfrak{i}(s)(m(s) - (s+1)m'(s)) - g(\mathfrak{i}(s))m'(s)\ . \end{aligned} \tag{70}$$

The differences in the two cases are only reflected in the boundary conditions. In the case where Condition A holds we then obtain the following solution.

Proposition 4 *When Condition A given in (69) is satisfied we have $\mu^s_{FB}(s_t) \geq 0$ and:*

1. *in the one-sided case the entrepreneur chooses the first-best investment and consumption policies despite being financially constrained and her wealth is $m(s) = s + q^{FB}$;*

2. *in the two-sided case, investors' limited-liability constraint implies*

$$\mu^s(0) = 0\,. \tag{71}$$

Figure 11 plots the marginal value of financial slack $m'(s)$ and investment-capital ratio $\mathfrak{i}(s)$ under Condition A. If the first-best allocation is feasible under the optimal contract at all times then it must then be the optimal solution. Hence, the question: under what conditions is the first-best allocation feasible? It is in the one-sided case but only when Condition A given in (69) is satisfied. If $\mu^s_{FB}(s_t) > 0$ then s_t increases with time t under the first-best investment and consumption policies. Therefore, the entrepreneur's limited-commitment constraint never binds, so that the first-best outcome is achieved.

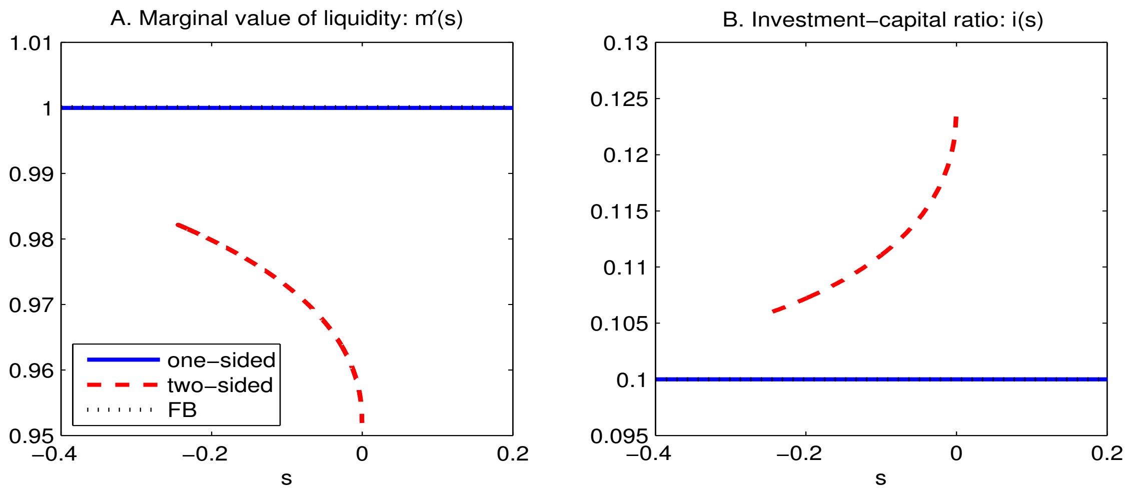

Figure 11: **The marginal value of liquidity $m'(s)$ and investment-capital ratio $\mathfrak{i}(s)$: The deterministic case with $\sigma_K = 0$ and where Condition A is satisfied.** Solutions for the one-sided case coincide with those for the first-best case in the region $s \geq \underline{s} = -q^{FB} = -1.2$. For the two-sided case, the firm over-invests in the entire admissible region $\underline{s} = -0.244 \leq s \leq 0$ compared to the first-best benchmark.

Figure 11 shows that solutions for the one-sided case coincide achieve the first-best level.[23] However, the one-sided limited commitment truncates the support of s by requiring $s \geq \underline{s}$ where $\underline{s} = -q^{FB} = -1.2$. Otherwise, the entrepreneur's net worth $(q^{FB} + s)K$ is negative.

In the two-sided case the optimal contract requires that $s_t \leq 0$ in order to satisfy that investors' limited liability. How to make the drift of s, $\mu^s(\cdot)$, negative? The entrepreneur does this by increasing investment and consumption, as liquidity now has a negative effect on $m(s)$. Over-supply of corporate liquidity causes the entrepreneur to over-consume and over-invest, which we see from Panel B. Finally, the marginal value of liquidity $m'(s)$ decreases with and the degree of over-investment increases as s approaches towards zero.

We next consider the other case when Condition A given in (69) is violated.

Proposition 5 *When Condition A given in (69) is violated, we have* $\mu^s_{FB}(s_t) < 0$ *and*

[23]Technically, the solution for our deterministic case is an initial value problem rather than a boundary value problem. We solve the problem starting from $s = 0$ for the two-sided case. We have two unknowns $m(0)$ and $m'(0)$ which can be solved from two equations, the ODE (70) and $\mu^s(0) = A - i(0) - g(i(0)) - c(0) = 0$. Note that $\mathfrak{i}(s)$ and $\mathfrak{c}(s)$ are also functions of $m(s)$ and $m'(s)$. Once we have the value of $m(0)$ and $m'(0)$, we can then use the ODE (70) to solve $m(s)$ for the entire range of s as an initial value problem.

1. *in the one-sided case, the following conditions hold at the lower boundary $\underline{s}$:*

$$m(\underline{s}) = \alpha m(0)\,, \tag{72}$$

$$\mu^s(\underline{s}) = 0\,. \tag{73}$$

2. *in the two-sided case, conditions (72) and (73) hold also but only in the range $\underline{s} \leq s \leq 0$.*

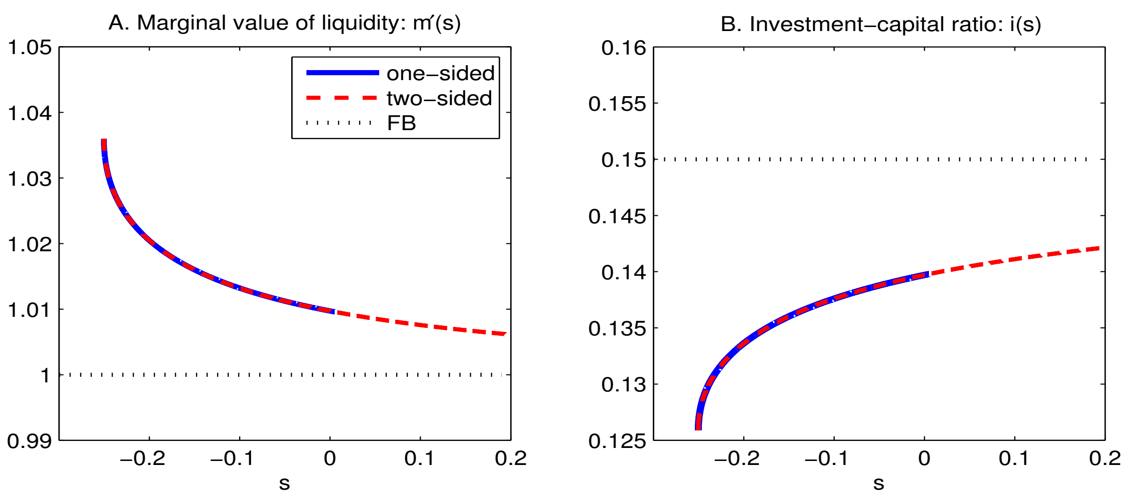

Figure 12: **The marginal value of liquidity $m'(s)$ and investment-capital ratio $\mathfrak{i}(s)$: The deterministic case with $\sigma_K = 0$ and where Condition A is violated.** Productivity $A = 0.205$ implies $i^{FB} = 0.15$ and $q^{FB} = 1.3$ for the first-best case. For the one-sided case, the firm optimally under-invests approaching the first-best level, $i(s) = i^{FB}$, as $s \to \infty$. For the two-sided case, the solution is the same as the one for the one-sided case but the support of s is truncated at the origin, i.e., $-0.250 = \underline{s} \leq s \leq 0$.

Figure 12 plots the marginal value of liquidity $m'(s)$ and investment-capital ratio $\mathfrak{i}(s)$ when Condition A is violated. For this case productivity $A = 0.205$ which is higher than $A = 0.2$ for Figure 11. The corresponding first-best investment-capital ratio i^{FB} increases from 0.1 to 0.15, and hence Condition A is violated $(\mu^s_{FB}(s_t) < 0)$. Therefore, targeting the first-best investment and consumption allocation drains valuable liquidity s. Accordingly, both consumption and investment are below the first-best benchmark. The drift $\mu^s(s)$ only approaches zero as the firm reaches the endogenous left limit $\underline{s}$ where $\mu^s(\underline{s}) = 0$. In fact, $\underline{s}$ is an absorbing boundary and the firm will permanently stay at $\underline{s}$ upon reaching that point.

Moreover, the lower is s the more valuable is liquidity, as reflected in the higher marginal value of liquidity $m'(s)$, and the lower is investment (see Panels A and B).

In the two-sided case, perhaps surprisingly, the solution is identical to that for the one-sided case. The intuition is as follows. In the parameter region where that Condition A is violated, the drift for the one-sided case is already weakly negative for all values of s. Therefore, investors' limited-liability constraints never bind and thus have no additional effect on the optimal contract other than constraint the support of s to the left of the origin.

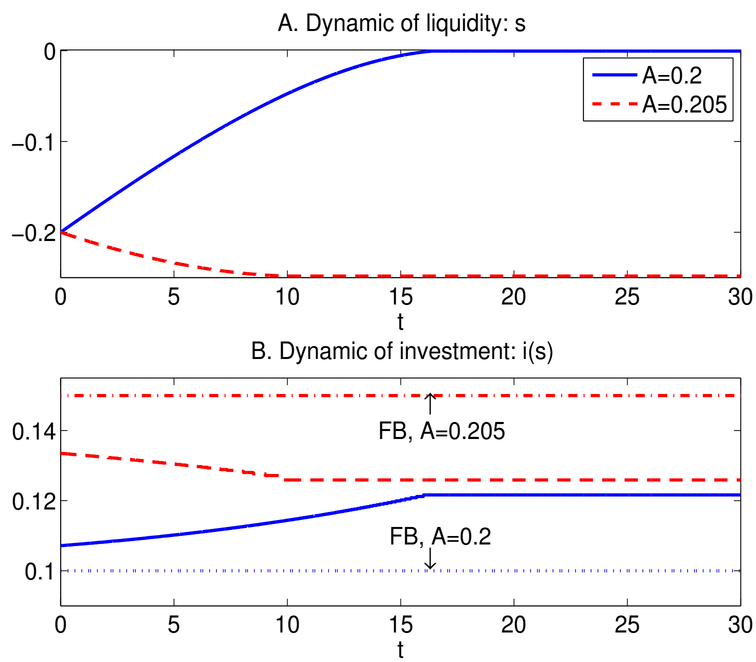

Figure 13: **Deterministic dynamics of liquidity s and investment i: The two-sided limited-commitment cases.** When $A = 0.2$, Condition A is satisfied. The firm increases its investment i_s, accumulates slack and pays down its credit borrowing over time until the investors' limited-liability constraint binds in year 16, reaching the steady state with $s_t = 0$ and $i_t = 0.12$ for $t \geq 16$. With $A = 0.205$, Condition A is violated. The firm decreases its investment i_s and increases its use of credit line over time until the entrepreneur's limited-commitment constraint binds in year 9.8 with $s_t = -0.250$ and $i_t = 0.126$ for $t \geq 9.8$.

Dynamic implications. Next we describe the deterministic dynamics of our model. Figure 13 plots the dynamics of liquidity s_t and investment i_t with $s_0 = -0.2$ for the two-sided case. When $A = 0.20$, Condition A (69) is satisfied. The firm always over-invests and i_t

increases from $i_0 = 0.107$ to $i_{16} = 0.12$, then stays flat at that level for all $t \geq 16$. Despite over-investment, the firm's scaled liquidity s_t increases from $s_0 = -0.2$ to $s_{16} = 0$ and thereafter stays permanently at the origin.

When $A = 0.205$, Condition A given is violated. The firm always under-invests and i_t decreases from $i_0 = 0.134$ to $i_{9.8} = 0.126$, then stays flat at that level for all $t \geq 9.8$. Investment is financed by the firm depleting its liquidity over time from $s_0 = -0.2$ to $s_{9.8} = -0.250$, at which point it stays flat permanently at $\underline{s} = -0.250$. In sum, the firm's investment decisions are always distorted over time and reach the most distorted levels as the limited-commitment constraints permanently bind in the long run.

While our deterministic model shares key features with Hart and Moore (1994), it differs from the Hart and Moore setup in several significant ways. First, unlike in Hart and Moore (1994) the firm's debt capacity constraint only occasionally binds in our formulation. Second, our model generates a unique repayment path for investors and an evolution equation for liquidity s, while Hart and Moore have a continuum of repayment paths. These differences are due to the facts that: i) the entrepreneur is risk averse and therefore values consumption smoothing, and ii) the firm faces convex capital adjustment costs and therefore values investment smoothing. Third, our deterministic model allows for dynamic capital accumulation while Hart and Moore (1994) only have a one-shot investment decision at time 0. Fourth, Hart and Moore (1994) only generates the under-investment result and does not deliver over-investment results.

11 Conclusion

Our generalization of Hart and Moore (1994) to introduce risky human capital, risk aversion, and ongoing consumption reveals the optimality of corporate liquidity and risk management for financially constrained firms. Most of the existing corporate security design literature has confined itself to showing that debt financing and credit line commitments are optimal financial contracts. By adding risky human capital and risk aversion for the entrepreneur, two natural assumptions, we show that corporate liquidity and hedging policies are also an integral part of an optimal financial contract. When productivity shocks are persistent, we find that insurance contracts and/or equilibrium default by the entrepreneur on her debt obligations is part of an optimal contract. We have thus shown that the inalienability-of-human-capital constraint naturally gives rise to a role for corporate liquidity and risk management, dimensions that are typically absent from existing macroeconomic theories of investment under financial constraints following Kiyotaki and Moore (1997).

Although our framework is quite rich, we have imposed a number of strong assumptions, which are worth relaxing in future work. For example, one interesting direction is to allow for equilibrium separation between the entrepreneur and the investors. This could arise, when after an adverse productivity shock the entrepreneur no longer offers the best use of the capital stock. Investors may then want to redeploy their capital to other more efficient uses. By allowing for equilibrium separation our model could be applied to study questions such as the expected and optimal life-span of entrepreneurial firms, the optimal turnover of managers, or the optimal investment in firm-specific or general human capital.

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Appendices

A The Full-Commitment Benchmark

Under full commitment, consumption and investment decisions can be separated. First, investors optimally choose investment I to maximize the firm's value defined by

$$Q(A_t, K_t) = \max_I \quad \mathbb{E}_t \left[\int_t^{\infty} e^{-r(v-t)} Y_v dv \right] . \tag{A.1}$$

Let Q_n denote the firm's value function in state n. Using dynamic programming, we have

$$rQ_H(K) = \max_I \; A^H K - I - G(I,K) + (I - \delta K) Q_H' + \frac{\sigma_K^2 K^2}{2} Q_H'' + \lambda_H (Q_L - Q_H) . \tag{A.2}$$

The FOC with respect to investment I gives $1 + G_I(I,K) = Q_H'$. Using the homogeneity property, we conjecture that the value function $Q_H^{FB}(K)$ is given by

$$Q_H^{FB}(K) = q_H^{FB} \cdot K . \tag{A.3}$$

By substituting (A.3) into (A.2) and (**??**), we obtain (9) and (11).

Now we turn to the consumption rule. First, the entrepreneur's value function $V(W)$ solves:

$$\zeta V(W) = \max_C \; \zeta U(C) + (rW - C) V'(W) . \tag{A.4}$$

where $rW - C$ is the rate of savings. The FOC with respect to consumption C is given by

$$V'(W) = \zeta U'(C) . \tag{A.5}$$

The entrepreneur's value function $V(W)$ takes the following standard homothetic form:

$$V(W) = \frac{(bW)^{1-\gamma}}{1-\gamma} = U(bW) , \tag{A.6}$$

where b is a constant to be determined. By substituting (A.6) and (A.5) into (A.4), we obtain a linear consumption rule, $C = \chi W$, where χ is the MPC out of wealth and given by (15) and the constant b is given by (13). Therefore, wealth accumulation follows

$$dW_t = (rW_t - \chi W_t)dt = -\frac{(\zeta - r)}{\gamma}dt\,, \tag{A.7}$$

which implies the following exponential processes for wealth and consumption:

$$W_t = W_0 e^{-(\zeta - r)t/\gamma} \quad \text{and} \quad C_t = \chi W_t = \chi W_0 e^{-(\zeta-r)t/\gamma} = C_0 e^{-(\zeta-r)t/\gamma}\,.$$

By using (A.6), we obtain (14) and (15).

B Contracting and Implementation

The key step in the reduction of the original contracting problem to a one-dimensional problem is to recognize that the investors' value is homogeneous of degree one in K and W, so that $P(K, W, A^n) = p_n(w)K$, where $w = W/K$. This allows us to focus on the investors' scaled value $p_n(w)$.

B.1 Dynamics of the entrepreneur's scaled promised wealth w

Using Ito's lemma, we have the following dynamics for W:

$$\begin{aligned} dW_t &= \frac{\partial W}{\partial V}dV_t + \frac{1}{2}\frac{\partial^2 W}{\partial V^2} < dV_t, dV_t >\,, & \text{(B.1)} \\ &= \frac{dV_t}{V_W} - \frac{V_{WW}}{2V_W^3} < dV_t, dV_t >\,, & \text{(B.2)} \end{aligned}$$

where $< dV_t, dV_t >$ denotes the quadratic variation of V, (B.2) uses $\partial W/\partial V = 1/V_W$, and

$$\frac{\partial^2 W}{\partial V^2} = \frac{\partial V_W^{-1}}{\partial V} = \frac{\partial V_W^{-1}}{\partial W}\frac{\partial W}{\partial V} = -\frac{V_{WW}}{V_W^2}\frac{1}{V_W} = -\frac{V_{WW}}{V_W^3}\,. \tag{B.3}$$

Substituting the dynamics of V given by (20) into (B.2) yields

$$dW_t = \frac{1}{V_W}\left[\zeta(V_{t-} - U(C_{t-}))dt + x_{t-}V_{t-}dZ_t + \Gamma_n(V_{t-}, A^n)(dN_t - \lambda_n dt)\right] - \frac{V_{WW}(x_{t-}V_{t-})^2}{2V_W^3}dt\,. \tag{B.4}$$

Using the dynamics for W and K, we obtain:

$$dw_t = d\left(\frac{W_t}{K_t}\right) = \mu_n^w(w)dt + \sigma_n^w(w)dZ_t + \psi_n dN_t\,, \tag{B.5}$$

where the drift and volatility processes $\mu^w(\,\cdot\,)$ and $\sigma^w(\,\cdot\,)$ for w are given by

$$\begin{aligned}\mu_n^w(w) \quad &= \quad \frac{\zeta}{1-\gamma}\left(w + c_n(w)\left(\frac{c_n(w)}{\zeta p_n'(w)}\right)^{-\gamma} b^{\gamma-1}\right) - w(i_n(w)-\delta) + \frac{\gamma w}{2}\left(\frac{x}{1-\gamma}\right)^2 + \sigma_K\sigma_n^w(w) \\ &\quad -\frac{\lambda_n w}{1-\gamma}\left[\left(1+\frac{\psi_n}{w}\right)^{1-\gamma} - 1\right],\end{aligned} \tag{B.6}$$

and

$$\sigma_n^w(w) = w\left(\sigma_K - \frac{x}{1-\gamma}\right). \tag{B.7}$$

We next summarize the results on investors' scaled value $p_n(w)$.

The one-sided limited-commitment case.

Proposition 6 *In the region $w > \underline{w_H} = \alpha\overline{w}_H$, investors' scaled value $p_H(w)$ in state H solves:*

$$\begin{aligned} rp_H(w) \quad &= \quad A^H + \frac{\chi\gamma}{1-\gamma}\left(-p_H'(w)\right)^{1/\gamma} w + i_H(w)(p_H(w) - wp_H'(w) - 1) - g(i_H(w)) + \frac{\zeta}{1-\gamma}wp_H'(w) \\ &\quad -\delta(p_H(w) - wp_H'(w)) + \frac{\sigma_K^2 w^2}{2}\frac{\gamma p_H'(w)p_H''(w)}{wp_H''(w) + \gamma p_H'(w)} - \frac{\lambda_H w}{1-\gamma}\left[\left(1+\frac{\psi_H}{w}\right)^{1-\gamma} - 1\right]p_H'(w) \\ &\quad +\lambda_H[p_L(w+\psi_H) - p_H(w)]\,, \end{aligned} \tag{B.8}$$

subject to the following boundary conditions:

$$\lim_{w\to\infty} p_H(w) \quad = \quad q_H^{FB} - w\,, \tag{B.9}$$

$$p_H(\overline{w}_H) \quad = \quad 0\,, \tag{B.10}$$

$$\lim_{w\to\alpha\overline{w}_H} \sigma_H^w(w) \quad = \quad 0 \quad \textit{and} \quad \lim_{w\to\alpha\overline{w}_H} \mu_H^w(w) \geq 0\,. \tag{B.11}$$

We briefly comment on the boundary conditions. First, (B.9) states that as w approaches infinity, the entrepreneur's limited commitment does not bind, and the first-best resource allocation is achieved. The right boundary condition (B.10) provides the condition that new investors (financing the entrepreneur upon the latter quitting) make zero profits. The left boundary condition (B.11) sets the volatility of w_t to zero and the drift to be non-negative at $w = \underline{w_H}$ to ensure that the entrepreneur will stay with current investors.

The two-sided limited-commitment case. When investors face the limited-liability constraint, the contract requires the volatility at the endogenous upper boundary $\overline{w}_H = 0$ to be zero and additionally the drift to be non-positive in order for the investors not to walk away from the contractual agreement:

$$\lim_{w \to \overline{w}_H} \sigma_H^w(w) = 0 \quad \text{and} \quad \lim_{w \to \overline{w}_H} \mu_H^w(w) \leq 0\,. \tag{B.12}$$

The arguments for (B.12) are essentially the same as those we have sketched out for the lower boundary $\underline{w_H} = \alpha \overline{w}_H$.

Derivation for Proposition 6. Applying the Ito's formula to (30) and transforming (22) for $F(K, V, A^H)$ into the following HJB equation for $P(K, W, A^H)$, we obtain

$$\begin{aligned} rP(K,W,A^H) = \max_{C,I,x,\Psi_H} & \Bigg\{ Y - C + \frac{\zeta(U(bW) - U(C)) - \lambda_H[U(b(W+\Psi_H)) - U(bW)]}{bU'(bW)} P_W \\ & + (I - \delta K)P_K + \frac{\sigma_K^2 K^2}{2} P_{KK} + \frac{(xU(bW))^2}{2} \frac{P_{WW}U'(bW) - P_W bU''(bW)}{U'(bW)^3} \\ & + \sigma_K \frac{xKU(bW)}{bU'(bW)} P_{WK} + \lambda_H[P(K, W+\Psi_H, A^L) - P(K,W,A^H)] \Bigg\}\,, \end{aligned} \tag{B.13}$$

where $\Psi_H = \frac{U^{-1}(V+\Gamma_H) - U^{-1}(V)}{b}$. And then using the FOCs for I, C, and x respectively, we obtain

$$1 + G_I(I,K) = P_K\,, \tag{B.14}$$

$$U'(bW) = -\frac{\zeta}{b} P_W U'(C)\,, \tag{B.15}$$

$$x = -\frac{\sigma K P_{WK} bU'(bW)}{U(bW)[P_{WW} - P_W bU''(bW)/U'(bW)]}\,. \tag{B.16}$$

Whenever feasible, the optimal adjustment of W to account for the productivity change satisfies

$$\frac{U'(b(W+\Psi_H))}{U'(bW)} = \frac{P_W(K, W+\Psi_H, A^L)}{P_W(K,W,A^H)}\,. \tag{B.17}$$

When the entrepreneur's limited-commitment constraint binds, $\Psi_H = \underline{W_L} - W$. And when the investor's limited-liability constraint binds, $\Psi_H = \overline{W}_L - W$.

Substituting $P(K, W, A^H) = p_H(w)K$ and the optimal policies into the PDE (B.13), we obtain the ODE (B.8). The constraint $V_t \geq \widehat{V}_n(K_t)$ implies $W_t \geq \underline{w_n} K_t$ where $\underline{w_n}$ is denoted by $\underline{w_n} = \frac{U^{-1}(\widehat{V}_n(K_t))}{bK_t}$. To ensure $V_{t+dt} \geq \widehat{V}_n(K_{t+dt})$ with probability one, the drift of $V_t/\widehat{V}_n(K_t)$ should be weakly positive (negative) if $V_t > 0$ ($V_t < 0$) and the volatility of $V_t/\widehat{V}_n(K_t)$ should be zero at the boundary $V_t = \widehat{V}_n(K_t)$. To summarize, the condition (B.11) ensures that the entrepreneur will not walk away at the left boundary $\underline{w_n}$ and the right boundary condition (B.12) ensures that investors will stay. Finally, condition (28) implies (B.10) and condition (27) implies $\underline{w_n} = \alpha \overline{w}_n$.

B.2 Derivations for Proposition 2 and Proposition 3

Because Proposition 2 is the special case of Proposition 3 by setting $A^H = A^L = A$ and $\lambda_H = \lambda_L = 0$ in Proposition 3, we will thus only sketch out the arguments for Proposition 3.

Applying the Ito's formula to $s = S/K$, we obtain the following dynamics for s in state n:

$$ds_t = d\left(\frac{S_t}{K_t}\right) = \mu^s(s)dt + \sigma^s(s)dZ_t + \pi_n dN_t\,, \tag{B.18}$$

where

$$\mu_n^s(s) = (A^n - \pi_n(s)\lambda_n - \mathfrak{i}_n(s) - g(\mathfrak{i}_n(s)) - \mathfrak{c}_n(s)) + (r + \delta - \mathfrak{i}_n(s))s - \sigma_K \sigma_n^s(s)\,, \tag{B.19}$$

$$\sigma_n^s(s) = (\phi_n(s) - s)\sigma_K\,. \tag{B.20}$$

Now, we apply the HJB equation to the entrepreneur's value function $J(K, S, A^H)$ as follows:

$$\begin{aligned}
\zeta J(K, S, A^H) = \max_{C, I, \phi, \pi_H} \quad & \zeta U(C) + (I - \delta K)J_K + \frac{\sigma_K^2 K^2}{2} J_{KK} \\
& + (rS + AK - I - G(I, K) - C - \lambda_H \pi_H K)J_S + \sigma_K^2 K^2 \phi J_{KS} \\
& + \frac{K^2 \sigma_K^2 \phi^2}{2} J_{SS} + \lambda_H (J(K, S + \pi_H K, A^L) - J(K, S, A^H))\,.
\end{aligned} \tag{B.21}$$

Substituting the conjectured value function (34) for $J(K,S,A^H)$ into (B.22), we have:

$$\begin{aligned}
0 \;=\; \max_{C,I,\phi,\pi_H} \quad & \zeta M \frac{\left(\frac{C}{bM}\right)^{1-\gamma}-1}{1-\gamma} + (I-\delta K)M_K + (rS + AK - I - G(I,K) - C - \lambda_H \pi_H K)M_S \\
& + \frac{\sigma_K^2 K^2}{2}\left(M_{KK} - \frac{\gamma M_K^2}{M}\right) + \sigma_K^2 \phi K^2 \left(M_{KS} - \frac{\gamma M_K M_S}{M}\right) + \frac{K^2\phi^2\sigma_K^2}{2}\left(M_{SS} - \frac{\gamma M_S^2}{M}\right) \\
& + \frac{\lambda_H M(K,S,A^H)}{1-\gamma}\left(\left(\frac{M(K,S+\pi_H K, A^L)}{M(K,S,A^H)}\right)^{1-\gamma} - 1\right). \qquad \text{(B.22)}
\end{aligned}$$

Using the homogeneity property of $M(K,S,A^H) = m_H(s)\cdot K$, we then obtain the ODE (60).

At $\underline{S_n}(K)$ the entrepreneur's inalienability-of-human-capital constraint binds, which implies

$$J(K,\underline{S_n},A^n) = \widehat{V}_n(K) = J(\alpha K, 0, A^n)\,. \qquad \text{(B.23)}$$

By substituting the value function (34) into (B.23), we obtain $M(K,\underline{S_n},A^n) = M(\alpha K,0,A^n)$, which implies (62). To ensure that the entrepreneur does not walk away at the left boundary, we require (63) by using essentially the same argument as the one we have used for $p_n(w)$ in the optimal contacting problem. The boundary conditions (62) and (63) are necessary for both t cases. The upper boundaries differ for the two cases. For the one-sided case, we also require (61) as $s\to\infty$. For the two-sided case, we require (64) for investors not to renege on the contract.

C Equivalence between the Primal and Dual Problems

We provide a sketch of the arguments underlying the equivalence result between the two problems.

First, we postulate the following relations between s and w hold in both $n\in\{L,H\}$:

$$s = -p_n(w), \text{ and } m_n(s) = w\,. \qquad \text{(C.1)}$$

Then, the standard chain rule implies:

$$m_n'(s) = -\frac{1}{p_n'(w)}, \qquad m_n''(s) = -\frac{p_n''(w)}{p_n'(w)^3}\,. \qquad \text{(C.2)}$$

Substituting (C.1) and (C.2) into the ODE (60) for $m_n(s)$, we obtain the ODE (B.8) for $p_n(w)$.

Similarly, substituting (C.1) and (C.2) into the boundary condition (43) for $m_n(s)$, we obtain (B.12), the boundary condition for $p_n(w)$. Substituting (C.1) into the boundary condition (41), we obtain $\lim_{s\to\infty} p_n(w) = q_n^{FB} - w$. Finally, by using (C.1) and (C.2), we may show that the optimal consumption and investment policies for the primal and dual problems are indeed equivalent.

D Autarky as the Entrepreneur's Outside Option

In this appendix, we consider an alternative specification for the entrepreneur's outside option. The entrepreneur may divert the firm's capital stock and manage it by herself. However, by doing so, the entrepreneur will permanently live in autarky (Bulow and Rogoff, 1989). We show that our model still gives rise to a theory of liquidity and risk management with an endogenous credit limit.

Let $\widehat{J}(K_t)$ denote the entrepreneur's value function under autarky defined as follows,

$$\widehat{J}(K_t) = \max_{I} \ \mathbb{E}_t \left[\int_t^\infty \zeta e^{-\zeta(v-t)} U(C_v) dv \right] , \tag{D.1}$$

where autarky implies that the entrepreneur's consumption C equals output Y_t, in that

$$C_t = Y_t = A_t K_t - I_t - G(I_t, K_t) \, . \tag{D.2}$$

The following proposition summarizes the main results.

Proposition 7 *Under autarky, the entrepreneur's value function $\widehat{J}(K)$ is given by*

$$\widehat{J}(K) = \frac{(b\widehat{M}(K))^{1-\gamma}}{1-\gamma} \, , \tag{D.3}$$

where b is given by (13) and $\widehat{M}(K)$ is the entrepreneur's certainty equivalent wealth given by

$$\widehat{M}(K) = \widehat{m} K \, , \tag{D.4}$$

where

$$\widehat{m} = \frac{(\zeta(1+g'(\widehat{i}))(A - \widehat{i} - g(\widehat{i}) \,)^{-\gamma})^{\frac{1}{1-\gamma}}}{b} \, , \tag{D.5}$$

and $\widehat{i}$ is the optimal investment-capital ratio solving the following implicit equation:

$$\zeta = \frac{A - \widehat{i} - g(\widehat{i})}{1 + g'(\widehat{i})} + (\widehat{i} - \delta)(1-\gamma) - \frac{\sigma_K^2 \gamma (1-\gamma)}{2}\,. \tag{D.6}$$

Similar to the analysis in Section 6, we show that the lower boundary $\underline{s}$ satisfies:

$$m(\underline{s}) = \widehat{m}. \tag{D.7}$$

For both the one-sided and two-sided cases, we only need to replace the boundary condition (42) in Proposition 2 with (D.7) and keep all the other conditions unchanged. Note that the lower boundary $\underline{s}$, determined by (D.7), is independent of the upper boundary $\overline{s}$.

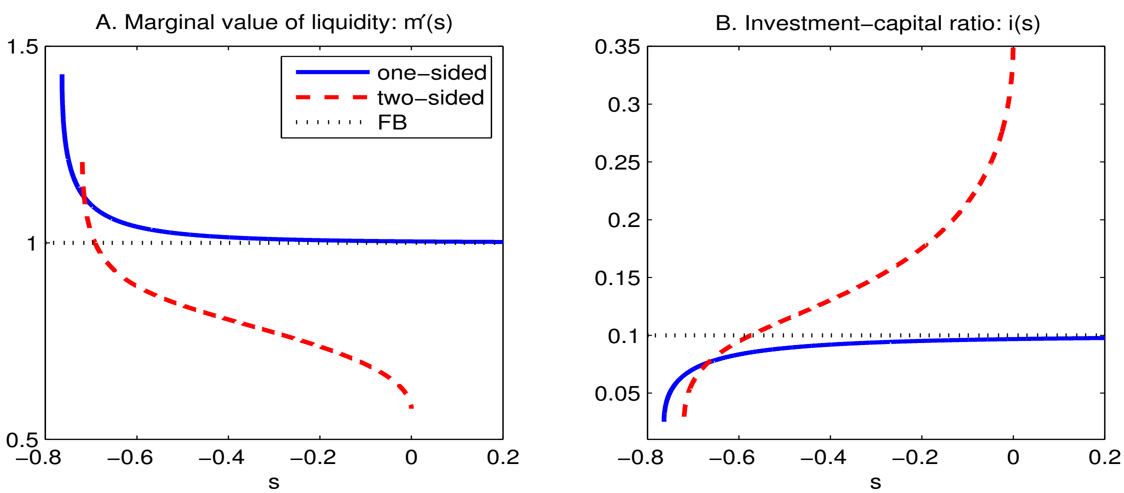

Figure 14: **Autarky as the entrepreneur's outside option: No productivity shocks.** Panels A and B plot the marginal (certainty equivalent) wealth of s, $m'(s)$, and the investment-capital ratio $\mathfrak{i}(s)$, respectively. The solid and dashed lines correspond to the one-sided and two-sided cases, respectively. For both cases, $m(s)$ is increasing and concave. For the one-sided case, $s \geq -0.764$. For the two-sided case, $-0.720 \leq s \leq 0$.

Figure 14 plots the entrepreneur's marginal value of liquidity $m'(s)$ and the optimal investment-capital ratio $\mathfrak{i}(s)$ for both one-sided and two-sided limited-commitment cases. The general patterns reported in the text remain valid. For example, for the one-sided case, the firm always underinvests and the marginal value of liquidity $m'(s)$ is always greater than one. Additionally, the degree of underinvestment weakens and the marginal value of liquidity $m'(s)$ decreases, both of which eventually approach the first-best levels $i^{FB} = 0.10$ and unity, respectively, as $s \to \infty$.

Finally, for the two-sided case, the firm may choose to over-invest (compared with the first-best benchmark) as the marginal value of liquidity $m'(s)$ could be less than unity due to the investors' limited commitment.

E Comparative statics: Risk aversion

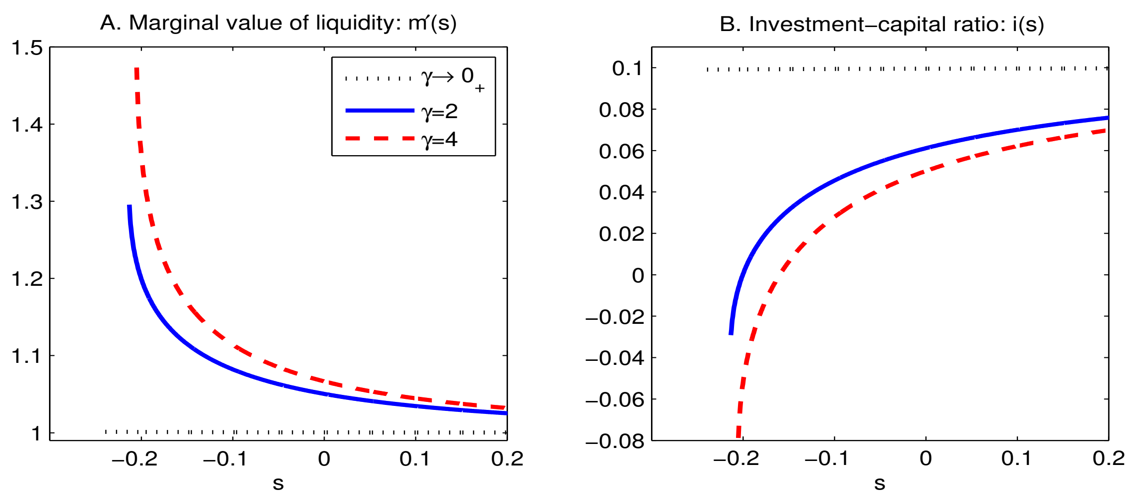

Figure 15: **Effects of the entrepreneur's risk aversion.** Panels A and B plot the marginal (certainty equivalent) wealth of s, $m'(s)$, and the investment-capital ratio $\mathfrak{i}(s)$, respectively. The dotted, solid, and dashed lines correspond to three different levels of risk-aversion, $\gamma \to 0_+$, $\gamma = 2$, and $\gamma = 4$, respectively.

Figure 15 illustrates the effect of risk aversion on the marginal value of liquidity $m'(s)$ and investment-capital ratio $\mathfrak{i}(s)$ in Panel A and B, respectively. First, the more risk-averse the entrepreneur the less the firm invests *ceteris paribus.* And additionally, liquidity is more valuable for more risk-averse entrepreneurs, implying that the marginal value of liquidity $m'(s)$ increases with γ. As $\gamma \to 0_+$, marginal value of liquidity approaches unity and investment approaches the first-best level i^{FB}. Finally, we see that liquidity capacity $|\underline{s}|$ decreases with risk aversion. For example, liquidity limit $\underline{s} = -0.24$ for $\gamma \to 0_+$, $\underline{s} = -0.214$ for $\gamma = 2$, and $\underline{s} = -0.205$ for $\gamma = 4$.