

# Interpreting factor models \*

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## Abstract

We argue that empirical tests of reduced-form factor models do not shed light on competing theories of investor beliefs. Since asset returns have substantial commonality, absence of near-arbitrage opportunities implies a stochastic discount factor (SDF) that is a function of a few dominant sources of return variation. Consistent with this view, we show: an SDF based on the first few principal components explains many recently studied anomalies; if this was not true, near-arbitrage opportunities with extremely high Sharpe Ratios would exist; in-sample near-arbitrages vanish out-of-sample. However, a reduced-form factor SDF of this kind is perfectly consistent with an economy in which all cross-sectional variation in expected returns is caused by sentiment. Components of sentiment-investor demand that line up with common factor loadings affect asset prices because it is risky for arbitrageurs to take the opposite position, while components orthogonal to these factor loadings are neutralized. If investor sentiment is time-varying, the SDF can take the form of an ICAPM. For these reasons, tests of reduced-form factor models, horse races between “characteristics” and “covariances,” and firm investment-based models that take as given an arbitrary or a reduced-form factor SDF cannot discriminate between alternative models of investor beliefs.

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# 1 Introduction

Reduced-form factor models are ubiquitous in empirical asset pricing. In these models, the stochastic discount factor (SDF) is represented as a function of a small number of portfolio returns. In equity market research, the three-factor model of Fama and French (1993) is the most prominent representative and recent extensions feature various additional factors. These models are reduced-form because they are not derived from assumptions about investor beliefs, preferences, and technology that prescribe which factors should appear in the SDF. Reduced-form factor models in this sense also include theoretical models that analyze how cross-sectional differences in stocks' covariances with the SDF arise from firms' investment decisions. Berk, Green, and Naik (1999) and many subsequent models of this kind belong into the reduced-form class because they take as given an arbitrary SDF. They make no assumptions about investor beliefs and preferences other than the existence of an SDF.

Which interpretation should one give such an reduced-form factor model if it works well empirically? More precisely, what do we learn from this evidence about the beliefs and preferences of investors? If the law of one price holds, one can, tautologically, always construct a single-factor representation of the SDF in which the single factor is a linear combination of asset payoffs (Hansen and Jagannathan 1991). Thus, the mere fact that a low-dimensional factor model “works” has no economic content beyond the law of one price. However, reduced-form factors like those of Fama and French not only approximate the SDF, but they are also important sources of commonality in asset returns. The evidence that the factors that help explain the cross-section of expected returns are also common risk factors could, potentially, have additional economic content.

Some researchers take the view that a tight link between expected returns and loadings on common risk factors is a prediction of “rational” asset-pricing models that is not shared by “behavioral” approaches to asset pricing. This view motivates empirical tests that look for expected return variation unexplained by multi-factor covariances, as in Daniel and Titman (1997), Brennan, Chordia, and Subrahmanyam (1998). This view is also underlying arguments that a successful test

or calibration of a reduced-form SDF provides a rational explanation of asset pricing anomalies.<sup>1</sup>

In this paper, we argue that the reduced-form factor model evidence does not help in discriminating between alternative hypotheses about investor beliefs. Only minimal assumptions on preferences and beliefs of investors are required for a reduced-form factor model with a small number of factors to describe the cross-section of expected returns. These assumptions are consistent with plausible “behavioral” models of asset prices as much as they are consistent with “rational” ones. Simply put, one cannot hope to learn much about investor beliefs from the empirical evaluation of a reduced-form model that embodies virtually no restrictions on investor beliefs.

For typical sets of assets and portfolios, the covariance matrix of returns is dominated by a small number of factors. In this case, absence of near-arbitrage opportunities—that is, investment opportunities with extremely high Sharpe Ratios (SR)—implies that the SDF can be represented as a function of these few dominant factors.<sup>2</sup> However, absence of near-arbitrage opportunities still leaves a lot of room for belief distortions to affect asset prices. Belief distortions that are correlated with common factor covariances will affect prices, only those that are uncorrelated with common factor covariances will be neutralized by arbitrageurs that are looking to exploit high-SR opportunities.

If this reasoning is correct, then it should be possible to obtain a low-dimensional factor representation of the SDF through purely statistical methods. We show that a factor model with a small number of purely statistical factors obtained from an eigenvalue decomposition of the asset return covariance matrix does about as well in explaining the cross-section of expected returns as popular reduced-form factor models. For example, the three Fama and French (1993) factors, MKT, SMB, and HML, are virtually identical to the factors associated with the first three (highest-eigenvalue) principal components (PCs) of the  $5 \times 5$  size and book-to-market (B/M) portfolio return covariance

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<sup>1</sup>On the empirical side examples of papers that take this view include Liu and Zhang (2008) and Ahn, Conrad, and Dittmar (2003). On the theory side, the Berk, Green, and Naik (1999) model is frequently referenced as a rational explanation of asset pricing anomalies. Recent examples include Asness, Moskowitz, and Pedersen (2013) and Vayanos and Woolley (2013).

<sup>2</sup>This result is closely related to the Arbitrage Pricing Theory (APT) of Ross (1976). When discussing the empirical implementation of the APT in a finite-asset economy, Ross (p. 354) suggests bounding the maximum squared Sharpe Ratio of any arbitrage portfolio at twice the squared SR of the market portfolio. However, our interpretation of APT-type models differs from some of the literature. For example, Fama and French (1996) (p. 75) regard the APT as a “rational” pricing model. We disagree with this interpretation.

matrix. In fact, the first three PCs perform slightly better in explaining the average returns on the size-B/M portfolios than the Fama-French factors. We find a similar result for a set of 15 long-short portfolios from Novy-Marx and Velikov (2014) that capture various anomalies discovered in past couple of decades. The factors associated with the first five PCs explain of the cross-section of average returns of these anomaly portfolios better than the Fama-French factors explain the average returns of the  $5 \times 5$  size and book-to-market (B/M) portfolios. Thus, there doesn't seem to be anything special about the construction of the reduced-form factors proposed in the literature. Purely statistical factors do just as well.

We then ask how the maximum SR available from the test asset portfolios change if we keep the cross-sectional variance of expected returns unchanged, but we assume, counterfactually, that expected returns are not aligned with covariances with the first few PCs, but instead with larger number of (lower-eigenvalue) PCs. The answer is that SRs would explode quickly. For typical test asset portfolios, their return covariance structure essentially dictates that the first few PC factors must explain the cross-section of expected returns, otherwise near-arbitrage opportunities would exist.

Within a given sample, an SDF with a small number of factors of course still leaves some variation in expected returns unexplained. As a consequence, the maximum in-sample SR that can be obtained from the test asset returns is substantially higher than the maximum in-sample SR obtainable from the factors. For example, the Fama-French three-factor model is statistically rejected based on the  $5 \times 5$  size and book-to-market (B/M) portfolios (Fama and French 1993). The same is true for our PC-factor model. However, these pricing errors do not appear to be a robust feature of the cross-section of expected returns. In part, the pricing errors simply reflect sampling error, while in part they may reflect short-lived near-arbitrage opportunities or the effects of data mining that do not persist out of sample.

To check this, we perform a pseudo out-of-sample exercise. We split the sample period from 1963 to 2012 into two subperiods. We extract the PCs from the covariance matrix of returns in the first subperiod. Within the first subperiod, the maximum in-sample SR from the first few PC factors accounts for a substantial portion of the maximum SR available from the test asset returns, but the

addition of (lower-eigenvalue) PC factors still raises the maximum SR substantially. Out of sample the picture looks different. If we keep the first subperiod portfolio weights implied by the PCs and their SR-maximizing combination and we apply those weights to returns in the second subperiod, PCs beyond the first few no longer add to the SR. This is true both for the anomaly portfolios and the size and B/M portfolios. In-sample deviations from low-dimensional factor pricing do not appear to be reliable persist out of sample.

We then provide a concrete demonstration of the limited economic content of absence of near-arbitrage opportunities. We model a multi-asset market in which fully rational risk averse investors (arbitrageurs) trade with investors whose asset demands are driven by distorted beliefs (sentiment investors). We make two plausible assumptions. First, the covariance matrix of asset cash flows features a few dominant factors that drive most of the stocks' covariances. Second, sentiment investors cannot take extreme positions that would require substantial leverage or short-selling. In this model, all cross-sectional variation in expected returns is caused by distorted beliefs and yet a low-dimensional factor model explains this cross-sectional variation. To the extent that sentiment investor demand is orthogonal to covariances with the dominant factors, arbitrageurs elastically accommodate this demand and take the other side with minimal price concessions. Only sentiment investor demand that is aligned with covariances with dominant factors affects prices because it is risky for arbitrageurs to take the other side. As a result, the SDF in this economy can be represented to a good approximation as a function of the first few PC factors that dominate the assets' covariance matrix even though all deviations of expected returns from the CAPM are caused by sentiment. The fact that a low-dimensional factor model holds thus says little about the sources of cross-sectional variation in expected returns. It is consistent with "behavioral" explanations just as much as it is consistent with "rational" explanations.

In this type of world, horse races between "covariances" with reduced-form factors and stock "characteristics" such as B/M that are meant to proxy for mispricing or sentiment investor demand (as, e.g, in Daniel and Titman (1997), Brennan, Chordia, and Subrahmanyam (1998), Davis, Fama, and French (2000), and Daniel, Titman, and Wei (2001)) cannot discriminate between competing hypothesis about investor beliefs. As long as near-arbitrage opportunities are absent, both covari-

ances and characteristics should approximately carry the same information about expected returns. To the extent that one or the other are mis-measured, one of them may appear to dominate, but otherwise it is difficult to imagine an economic reason why they would carry truly distinct information. By looking for mispricing that is orthogonal to reduced-form factor covariances, these tests are setting the bar too high: Even in a world in which belief distortions affect asset prices, expected returns should line up with common factor covariances.

Tests of factor models with ad-hoc macroeconomic factors (as, e.g., in Chen, Roll, and Ross 1986; Cochrane 1996; Li, Vassalou, and Xing 2006; Liu and Zhang 2008) are not more informative either. As shown in Reisman (1992) (see, also, Shanken 1992; Nawalkha 1997; and Lewellen, Nagel, and Shanken 2010), if  $K$  dominant factors drive return variation and that the SDF can be represented as a linear combination of these  $K$  factors, then the SDF can be represented, equivalently, by a linear combination of any  $K$  macroeconomic variables with possibly very weak correlation with the  $K$  factors.

Relatedly, theoretical models that derive relationships between firm characteristics and expected returns, taking as given an arbitrary SDF, do not shed light on the rationality of investor beliefs. Models such as Berk, Green, and Naik (1999), Johnson (2002), Liu, Whited, and Zhang (2009) or Liu and Zhang (2014), apply equally in our sentiment-investor economy as they apply to an economy in which the representative investor has rational expectations. These models show how firm investment decisions are aligned with expected returns in equilibrium, according to *firm's* first-order conditions. But these models do not speak to the question under which types of beliefs—rational or otherwise—*investors* align their marginal utilities with asset returns through their first-order conditions.

The observational equivalence between “behavioral” and “rational” asset pricing with regards to factor pricing also applies, albeit to a lesser degree, to partial equilibrium intertemporal capital asset pricing models (ICAPM) in the tradition of Merton (1973). In the ICAPM, the SDF is derived from the first-order condition of an investor who holds the market portfolio and faces exogenously given time-varying investment opportunities. This leaves open the question how to endogenously generate the time-variation in investment opportunities in a way that is consistent,

in equilibrium, with the ICAPM investor's first-order condition and her choice to hold the market portfolio. We show that one possible avenue is time-varying investor sentiment. If sentiment investor asset demands in excess of market portfolio weights have a single-factor structure and they vary over time with mean of zero, then the arbitrageurs' first-order condition implies an ICAPM that resembles the one in Campbell (1993) and Campbell and Vuolteenaho (2004). In our model, stocks that load highly on the sentiment-demand factor have a high market beta, but only because their discount-rate beta ("good beta") is high. Arbitrageurs do not demand a risk premium for discount-rate beta exposure, because expected return variation only has transitory effects on their wealth. Only the cash-flow beta ("bad beta") is compensated with a risk premium.

The rest of the paper is organized as follows. In Section 2 we describe the portfolio returns data that we use in this study. In Section 3 we lay out the implications of absence of near-arbitrage opportunities and we report the empirical results on factor pricing with principal component factors. Section 4 demonstrates the model in which fully rational risk averse arbitrageurs trade with sentiment investors. Section 5 develops a model with time-varying investor sentiment, which results in an ICAPM-type hedging demand.

## 2 Portfolio Returns

To analyze the role of factor models empirically, we use two sets of portfolio returns. First, we use a set of 15 anomaly long-short strategies from Novy-Marx and Velikov (2014). This set of returns captures many of the most prominent features of the cross-section of stock returns discovered over the past few decades. Second, for comparison, we also use the  $5 \times 5$  Size (SZ) and Book-to-Market (BM) sorted portfolios of Fama and French (1993).<sup>3</sup>

Table 1 provides some descriptive statistics for the anomaly long-short portfolios. Mean returns per month range from 0.21% to 1.44%. Annualized squared SRs, shown in the second column, range from 0.02 to 1.08. Since these long-short strategies have low correlation with the market factor, these squared SRs are roughly equal to the incremental squared SR that the strategy would contribute if added to the market portfolio.

The factor structure of returns plays an important role in our subsequent analysis. To prepare the stage, we analyze the commonality in these anomaly strategy returns. We perform an eigenvalue decomposition of the covariance matrix of these 15 long-short portfolio returns and extract the principal components (PCs), ordered from the one with the highest eigenvalue (which explains most of the co-movement of returns) to the one with the lowest. We then run a time-series regression of each long-short strategy return on the first, the first and the second, ... , up to a regression on the PCs one to five. The last five columns in Table 1 report the  $R^2$  from these regressions. While there is a lot of variation in the explanatory power of the first PC, with  $R^2$  ranging from 0.01 for the Asset Growth strategy to 0.74 for the Idiosyncratic Volatility strategy, once the first five PCs are included in the regression, the explanatory power is more uniform, with  $R^2$  ranging from 0.37 for the Investment strategy to 0.74 for the Momentum strategy. Thus, a substantial portion of the time-series variation in returns of these anomaly portfolios can be traced to a few common factors.

For the second set of returns from the size-B/M portfolios, it is well known from Fama and

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<sup>3</sup>We thank Robert Novy-Marx and Ken French for making the portfolio returns available on their websites. From those available on Novy-Marx's website, we use those strategies that are available starting in 1963, are not classified as high turnover strategies, and are not largely redundant. Based on this latter exclusion criterion we eliminate the monthly-rebalanced net issuance (and use only the annually rebalanced one). We also exclude the gross margins and asset turnover strategies which are subsumed, in terms of their ability to generate variation in expected returns, by the gross profitability strategy, as shown in Novy-Marx (2013).

Table 1: Anomalies: Returns and Principal Component Factors

The sample period is August 1963 to December 2012. The anomaly long-short strategy returns are from Novy-Marx and Velikov (2014). Average returns are reported in percent per month. Squared Sharpe Ratios are reported in annualized terms. The principal component factors are extracted from the pooled anomaly long-short strategy returns.

	Mean Return	Squared SR	PC1	PC factor-model $R^2$			
				PC1-2	PC1-3	PC1-4	PC1-5
Size	0.31	0.05	0.37	0.76	0.76	0.78	0.83
Gross Prof	0.40	0.17	0.06	0.08	0.15	0.64	0.64
Value	0.48	0.14	0.01	0.30	0.73	0.73	0.74
ValProf	0.83	0.54	0.00	0.27	0.43	0.79	0.82
Accruals	0.28	0.09	0.03	0.04	0.07	0.11	0.41
Net Issuance	0.79	0.86	0.21	0.24	0.38	0.46	0.46
Asset Growth	0.36	0.12	0.01	0.18	0.54	0.54	0.63
Investment	0.58	0.41	0.00	0.19	0.33	0.33	0.37
F-score	0.21	0.02	0.30	0.38	0.38	0.52	0.54
ValMomProf	1.44	1.08	0.14	0.76	0.82	0.87	0.87
ValMom	0.94	0.46	0.24	0.81	0.81	0.85	0.85
Idio. Volatility	0.66	0.10	0.74	0.80	0.94	0.94	0.95
Momentum	1.35	0.47	0.47	0.70	0.94	0.96	0.97
Long Run Rev.	0.47	0.11	0.17	0.56	0.73	0.73	0.76
Beta Arbitrage	0.48	0.13	0.09	0.10	0.41	0.50	0.89

French (1993) that three factors, the excess return on the value-weighted market index (MKT), a small minus large stock factor (SMB), and a high minus low BM factor (HML) explain more than 90% of the time-series variation of returns. While Fama and French construct SMB and HML in a rather special way from a smaller set of six size-B/M portfolios, one obtains essentially similar factors from the first three PCs of the  $5 \times 5$  size-B/M portfolio returns.

The first PC is, to a good approximation, a level factor that puts equal weight on all 25 portfolios. Since we focus on cross-sectional differences in expected returns in this paper, not on the equity premium, all of our empirical work below uses the returns on the size-B/M portfolios after removing this level factor (which we label “ex level factor” returns). This makes the portfolio returns similar to the anomaly long-short strategies above, which are also, roughly, market-neutral, i.e., with little exposure to market return movements.

The first two of the remaining PCs after removing the level factor are, essentially, the SMB

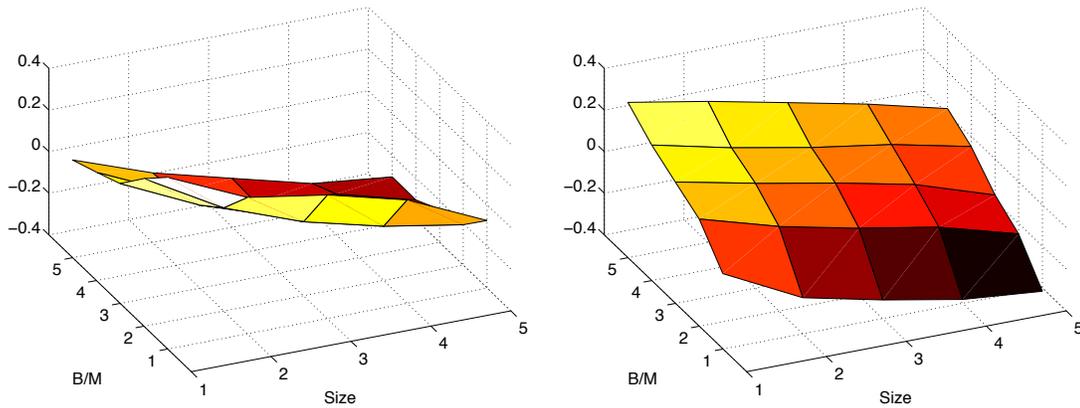


Figure 1: Fama-French 25 SZ/BM portfolio returns ex level factor: First two principal components.

and HML factors. Figure 1 plots the eigenvectors. PC1, shown on the left, has positive weights on small stocks and negative weights on large stocks, i.e., it is similar to SMB. PC2, shown on the right, has positive weights on high B/M stocks and negative weights on low B/M stocks, i.e., it is similar to HML. This shows that the Fama-French factors are not special in any way; they simply succinctly summarize cross-sectional variation in the size-B/M portfolio returns, similar to the first three PCs.<sup>4</sup>

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<sup>4</sup>A related observation appears in Lewellen, Nagel, and Shanken (2010). Lewellen et al. note that three factors formed as linear combinations of the 25 SZ/BM portfolio returns with random weights explain the cross-section of expected returns on these portfolios about as well as the Fama-French factors.

### 3 Factor pricing and absence of near-arbitrage

We start by showing that if we have assets with a few dominating factors that drive much of the covariances of returns (i.e., small number of factors with large eigenvalues), then those factors must explain asset returns. Otherwise near-arbitrage opportunities would arise, which would be implausible, even if one entertains the possibility that prices could be influenced substantially by the subjective beliefs of sentiment investors.

Consider an economy with discrete time  $t = 0, 1, 2, \dots$ . There are  $N$  assets in the economy indexed by  $i = 1, \dots, N$  with a vector of returns in excess of the risk-free rate of  $R$ . Let  $\mu \equiv E[R]$  and denote the covariance matrix of excess returns with  $\Omega$ .

Assume that the Law of One Price (LOP) holds. The LOP is equivalent to the existence of an SDF  $M$  such that  $E[MR] = 0$ . Note that  $E[\cdot]$  represents objective expectations of the econometrician, but there is no presumption here that  $E[\cdot]$  also represents subjective expectations of investors. Thus, the LOP does not embody an assumption about beliefs, and hence about the rationality of investors (apart from ruling out beliefs that violate the LOP).

Now consider the minimum-variance SDF in the span of excess returns, constructed as in Hansen and Jagannathan (1991) as

$$M = 1 - \mu' \Omega^{-1} (R - \mu). \quad (1)$$

Since we work with excess returns here, the SDF can be scaled by an arbitrary constant, and we normalized it to have  $E[M] = 1$ . The variance of the SDF,

$$\text{Var}(M) = \mu' \Omega^{-1} \mu, \quad (2)$$

equals the maximum squared Sharpe Ratio (SR) achievable from the  $N$  assets.

Now define absence of near-arbitrage as the absence of extremely high-SR opportunities (under objective probabilities) as in Cochrane and Saá-Requejo (2000). Ross (1976) also proposed a bound on the squared SR for an empirical implementation of his Arbitrage Pricing Theory in a finite-asset economy. He suggested ruling out squared SR greater than  $2 \times$  the squared SR of the market

portfolio. Such a bound on the maximum squared SR is equivalent, via (2), to an upper bound on the variance of the SDF  $M$  that resides in the span of excess returns.

Our perspective on this issue is different than in some of the extant literature. For example, MacKinlay (1995) suggests that the SR should be (asymptotically) bounded under “risk-based” theories of the cross-section of stock returns, but stay unbounded under alternative hypotheses that include “market irrationality.” A similar logic underlies the characteristics vs. covariances tests in Daniel and Titman (1997) and Brennan, Chordia, and Subrahmanyam (1998). However, ruling out extremely high-SR opportunities implies only weak restrictions on investor beliefs and preferences, with plenty of room for “irrationality” to affect asset prices. Even in a world in which many investors’ beliefs deviate from rational expectations, near-arbitrage opportunities should not exist as long as some investors (“arbitrageurs”) with sufficient risk-bearing capacity have beliefs that are close to objective beliefs. We can then think of the pricing equation  $E[MR] = 0$  as the first-order condition of the arbitrageurs’ optimization problem and hence of the SDF as representing the marginal utility of the arbitrageur.

For example, for an arbitrageur with exponential utility, as we show further below in Section 4, the first-order condition implies  $M = 1 - a[R^A - E(R^A)]$ , where  $R^A$  represents the return on the arbitrageur’s wealth portfolio and  $a$  is the arbitrageur’s risk aversion. As long as the arbitrageur can hold a relatively diversified and not too highly levered portfolio,  $R^A$  will not have extremely high volatility, which keeps the variance of  $M$  bounded from above. Extremely high volatility of  $M$  can occur only if the wealth of arbitrageurs in the economy is small and the sentiment investors they are trading against take huge concentrated bets on certain types of risk. Our model in Section 4 makes these arguments more precise, but for now it suffices to say that an upper bound on the Sharpe Ratio is perfectly consistent with asset prices that are largely sentiment-driven.

We now show that the absence of near-arbitrage opportunities implies that one can represent the SDF as a function of the dominant factors driving return variation. Consider the eigen-decomposition of the excess returns covariance matrix

$$\Omega = Q\Lambda Q' \quad \text{with} \quad Q = (q_1, \dots, q_N)' \quad (3)$$

and  $\lambda_i$  as the diagonal elements of  $\Lambda$ . Assume that the first principal component (PC) is a level factor, i.e.,  $q_1 = \frac{1}{\sqrt{N}}\iota$ , where  $\iota$  is a conformable vector of ones. This implies  $q'_k \iota = 0$  for  $k > 1$ , i.e., the remaining PCs are long-short portfolios. In the Appendix, Section A.1 we show that

$$\begin{aligned} \text{Var}(M) &= (\mu' q_1)^2 \lambda_1^{-1} + \mu' Q_z \Lambda_z^{-1} Q'_z \mu \\ &= \frac{\mu_m^2}{\sigma_m^2} + N \text{Var}(\mu_i) \sum_{k=2}^N \frac{\text{Corr}(\mu_i, q_{ki})^2}{\lambda_k}, \end{aligned} \quad (4)$$

where the  $z$  subscripts stand for matrices with the first PC removed and  $\mu_m = \frac{1}{\sqrt{N}} q'_1 \mu$ ,  $\sigma_m^2 = \frac{\lambda_1}{N}$ . The variance of expected returns and correlations of expected returns with eigenvectors in the second line of (4) are cross-sectional. This expression for SDF variance shows that expected returns must line up with the first few (high-eigenvalue) PCs, otherwise  $\text{Var}(M)$  would be huge. To see this, note that the sum of the squared correlations of  $\mu_i$  and  $q_{ki}$  is always equal to one. But the magnitude of the sum weighted by the inverse  $\lambda_k$  depends on which of the PCs the vector  $\mu$  lines up with. If it lines up with high  $\lambda_k$  PCs then the sum is much lower than if it lines up with low  $\lambda_k$  PCs. For typical test assets, eigenvalues decay rapidly beyond the first few PCs. In this case, a high correlation of  $\mu_i$  with a low-eigenvalue  $q_{ki}$  would lead to an enormous Sharpe Ratio. We now turn to an empirical analysis that demonstrates this point.

### 3.1 Principal components as reduced-form factors: Evidence from anomaly portfolios

Based on the no-near-arbitrage logic developed above, it should not require a judicious construction of factor portfolios to find a reduced-form SDF representation. Brute statistical force should do. We already showed earlier in Figure 1 that the first three principal components of the  $5 \times 5$  size-B/M portfolios are similar to the three Fama-French factors. Here we now investigate the pricing performance of the principal component factor models.

Table 2 shows that first few PCs do a good job of capturing cross-sectional variation in expected returns of the anomaly portfolios. We run time-series regressions of the anomaly excess returns on the principal component factors. The upper panel in Table 2 reports the pricing errors, i.e., the

Table 2: Explaining Anomalies with Principal Component Factors

The sample period is August 1963 to December 2012. The anomaly long-short strategy returns are from Novy-Marx and Velikov (2014). Average returns and factor-model alphas are reported in percent per month. Squared Sharpe Ratios are reported in annualized terms. The principal component factors are extracted from the pooled anomaly long-short strategy returns.

	Mean Return	PC1	PC factor-model alphas			
			PC1-2	PC1-3	PC1-4	PC1-5
Size	0.31	0.81	0.07	0.09	0.00	-0.05
Gross Prof	0.40	0.26	0.38	0.42	0.13	0.13
Value	0.48	0.56	-0.01	-0.15	-0.10	-0.11
ValProf	0.83	0.88	0.39	0.31	0.02	-0.01
Accruals	0.28	0.18	0.15	0.12	0.20	0.28
Net Issuance	0.79	0.57	0.44	0.38	0.29	0.30
Asset Growth	0.36	0.30	-0.07	-0.17	-0.14	-0.09
Investment	0.58	0.55	0.23	0.17	0.19	0.22
F-score	0.21	-0.23	0.09	0.09	-0.13	-0.15
ValMomProf	1.44	1.14	0.22	0.28	0.15	0.15
ValMom	0.94	0.55	-0.33	-0.33	-0.22	-0.22
Idio. Volatility	0.66	-0.39	0.04	-0.10	-0.15	-0.13
Momentum	1.35	0.56	-0.24	-0.07	0.05	0.04
Long Run Rev.	0.47	0.82	0.06	-0.04	-0.07	-0.03
Beta Arbitrage	0.48	0.25	0.33	0.20	0.37	0.24
	max. sq. SR		PC factors' max. squared SR			
All anomalies	2.00	0.34	1.05	1.08	1.25	1.28
$\chi^2$ -pval. for zero pricing errors		(0.00)	(0.00)	(0.00)	(0.00)	(0.01)
<i>For comparison:</i>						
25 SZ/BM (ex level factor)	2.29	0.13	0.41	0.53	0.53	0.76
$\chi^2$ -pval. for zero pricing errors		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
SMB and HML	0.37					

intercepts or alphas, from these regressions. The raw mean excess return (in percent per month) is shown in the first column, alphas for specifications with an increasing number of PC factors in the second to sixth column. With just the first PC (PC1) as a single factor, the SDF does not fit well. Alphas reach magnitudes up to 1.14 percent per month. Adding PC2 and PC3 to the factor model drastically shrinks the pricing errors. With five factors, the maximum alpha is 0.30.

The bottom panel reports the (ex post) maximum squared SR of the anomaly portfolios (2.00) and the maximum squared SR of the PC factors. With five factors, the highest-SR combination of the factors achieves a squared SR of 1.25. This is still considerably below the maximum squared SR of the anomaly portfolios and the  $p$ -values from a  $\chi^2$ -test of the zero-pricing error null hypothesis rejects at a high level of confidence. However, it is important to realize that this pricing performance of the PC1-5 factor model is actually better than the performance of the Fama-French factor model in pricing the  $5 \times 5$  size-B/M portfolios—which is typically regarded as a success. As the Table shows, the maximum squared SR of the  $5 \times 5$  size-B/M portfolios after removing the level factor (to make it comparable to the long-short anomaly portfolios that don't have much exposure to a level factor) is 2.29, i.e., higher than for the anomaly portfolios. But the squared SR of SMB and HML (which are the two relevant remaining factors after removing the level factor exposure from the portfolio returns) is only 0.37. As the Table shows, PC1-2, a combination of the first two PCs of the size-B/M portfolios (ex level factor), has a squared SR of 0.41 and gets slightly closer to the mean-variance frontier than the Fama-French factors. While the PC factor models and the Fama-French factor model are statistically rejected at a high level of confidence, the fact that the Fama-French model is typically viewed as successful in explaining the size-B/M portfolio returns suggests that one should also view the PC1-2 factor model as successful. In terms of the distance to the mean-variance frontier, the PC1-5 factor for the anomalies in the upper panel is even better at explaining the cross-section of anomaly returns than the Fama-French model in explaining the size-B/M portfolio returns.

Overall, this analysis shows that one can construct reduced-form factor models simply from the principal components of the return covariance matrix. There is nothing special, for example, about the construction of the Fama-French factors. Intended or not, the Fama-French factors are similar

to the first three PCs of the size-B/M portfolios and they perform similarly well in explaining the cross-section of average returns of those portfolios.

We have maintained so far that expected returns must line up with the first few principal components, otherwise high-SR opportunities would arise. We now provide empirical support for this assertion. We do so by asking, counterfactually, what the maximum SR of the test assets would be if expected returns did not line up, as they do in the data, with the first few (high-eigenvalue) PCs, but were instead also correlated with the higher-order PCs. To do this, we go back to equation (4). We assume that  $\mu_i$  is correlated with  $K$  PCs, while the correlation with the remaining PCs is exactly zero. For simplicity of exposition, we further assume that all non-zero correlations are equal. Since the sum of all squared correlations must add up to one, each squared correlation is then  $1/K$ . From (4) it is clear that the lowest possible SDF volatility arises if the  $K$  PCs with non-zero correlation with  $\mu_i$  are the first  $K$  with the highest eigenvalues. Thus, we have

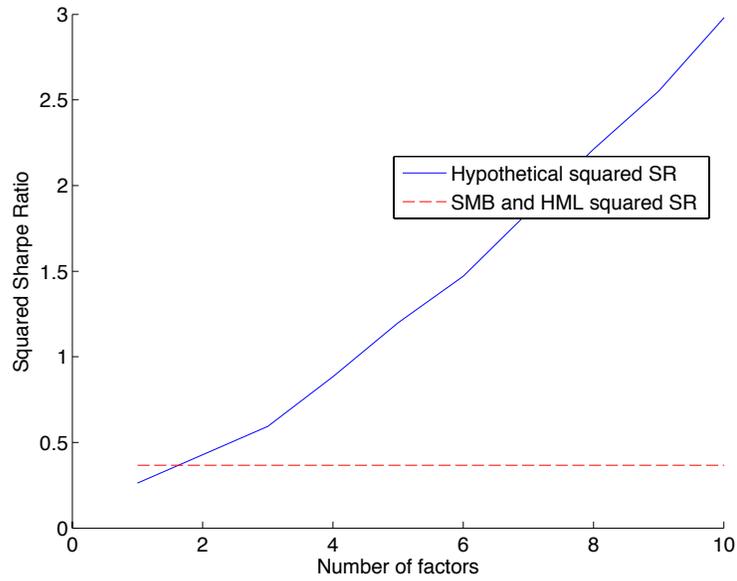
$$\text{Var}(M) \geq \frac{\mu_m^2}{\sigma_m^2} + \frac{N}{K} \text{Var}(\mu_i) \sum_{k=2}^K \frac{1}{\lambda_k}. \quad (5)$$

We now use the principal components extracted from the empirical covariance matrix of our test assets to calculate the bound (5) for different values of  $K$ .

Figure ?? presents the results. Panel (a) shows the counterfactual squared SR for the anomaly long-short portfolios. If expected returns of these portfolios lined up equally with the first two PCs but not the higher-order ones, the squared SR would be around 0.40, close to the squared SR of the Fama-French factors, which is plotted as the dashed line in the figure for comparison. If expected returns lined up instead equally with the first 10 PCs, the squared SR would be 3.

Panel (b) shows a similar analysis for the  $5 \times 5$  size-B/M portfolios. Here the counterfactual squared SR increase even more with  $K$ . If expected returns lined up equally with the first two PCs, the squared SR would be approximately equal to the squared SR of SMB and HML, consistent with the earlier analysis in Table 2. However, if expected returns were correlated equally with the first 10 PCs, the squared SR would reach more than 5.

(a) Anomaly long-short strategies



(b)  $5 \times 5$  Size-B/M portfolios (in excess of level factor)

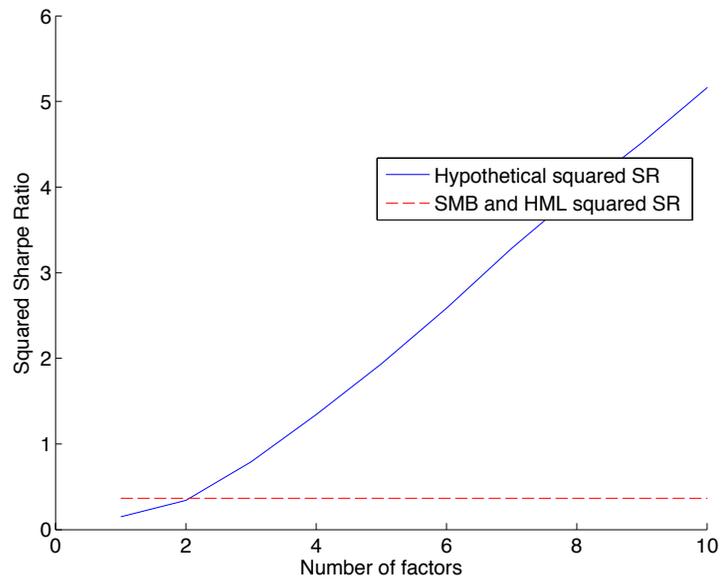


Figure 2: Hypothetical Sharpe Ratios if expected returns line up with first  $K$  (high-eigenvalue) principal components.

### 3.2 Characteristics vs. covariances: In-sample and out-of-sample

Daniel and Titman (1997) and Brennan, Chordia, and Subrahmanyam (1998) propose tests that look for expected return variation that is correlated with firm *characteristics* (e.g., B/M), but not with reduced-form factor model *covariances*. Framed in reference to our analysis above, this would mean looking for cross-sectional variation in expected returns that is orthogonal to the first few PCs—which implies that it must be variation that lines up with some of the higher-order PCs. The underlying presumption behind these tests is that “irrational” pricing effects should manifest themselves as mispricings that are orthogonal to covariances with the first few PCs.

From the evidence in Table 2 that the ex-post squared SR obtainable from the first few PCs falls short, by a substantial margin, of the ex-post squared SR of the test assets, one might be tempted to conclude that (i) there is actually convincing evidence for mispricing orthogonal to factor covariances, and (ii) that therefore the approach of looking for mispricings unrelated to factor covariances is a useful way to test behavioral asset pricing models. After all, at least ex-post, average returns appear to line up with the characteristics orthogonal to factor covariances.

We think that this conclusion would not be warranted. First, there is certainly substantial sampling error in the ex-post squared SR. Of course, the  $\chi^2$ -test in Table 2 takes the sampling error into account and still rejects the low-dimensional factor models. However, there are additional reasons to suspect that high ex-post SR are not robust indicators of persistent near-arbitrage opportunities. Data-snooping biases can overstate the in-sample SR. Short-lived near-arbitrage opportunities might exist for a while, without being a robust, persistent feature of the cross-section of expected returns.

To shed light on this robustness issue, we perform pseudo-out-of-sample analyses. We split our sample period in two halves, and we treat the first half as our in-sample period, and the second half as our out-of-sample period. We start with a univariate perspective with the 15 anomaly long-short portfolios. Figure 3 plots the in-sample squared SR in the first subperiod on the x-axis and the ratio of out-of-sample to in-sample squared SR on the y-axis. The figure shows that there is generally a substantial deterioration of SR. Out-of-sample SR are, on average, less than half as

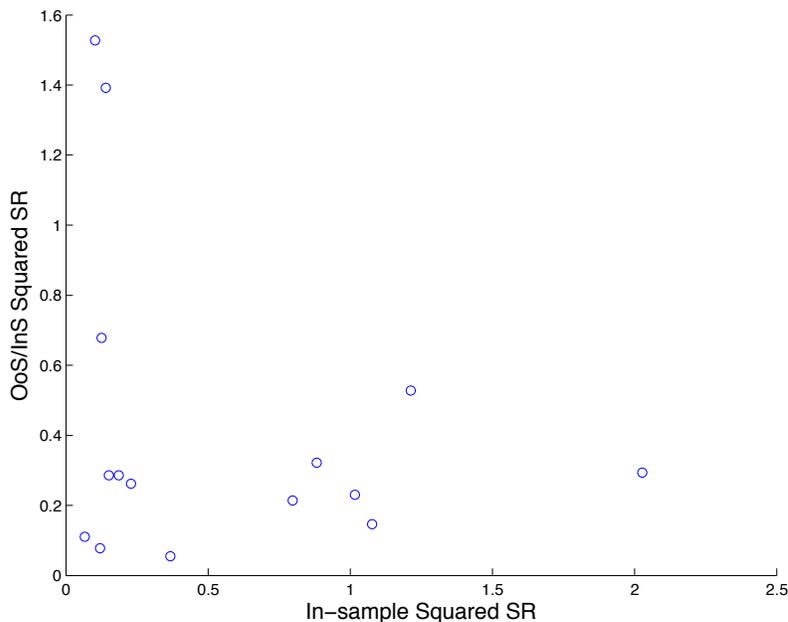


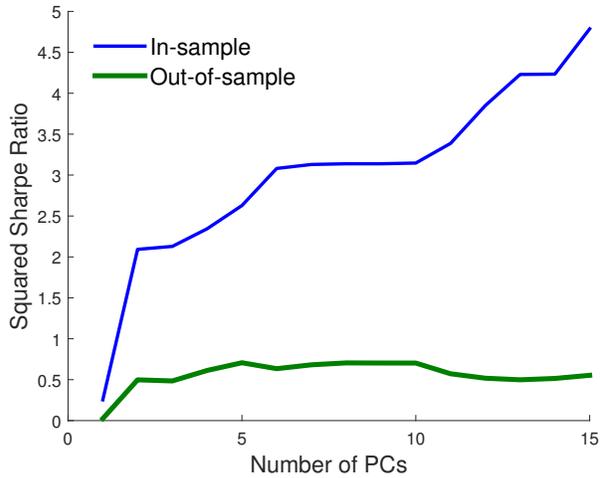
Figure 3: In-sample and out-of-sample squared Sharpe Ratios of anomaly long-short strategies. The sample period is split into two halves. In-sample squared SR are those in the first subperiod. Out-of-sample SR are those in the second sub-period. The ratio of out-of-sample to in-sample SR is plotted on the y-axis. The in-sample squared SR on the x-axis is annualized.

big as the in-sample SR and almost all of them are lower in the out-of-sample period. Furthermore, the strategies that hold up best are those that have relatively low in-sample SR. This is one first indication that high in-sample SR do not readily lead to high out-of-sample SR.

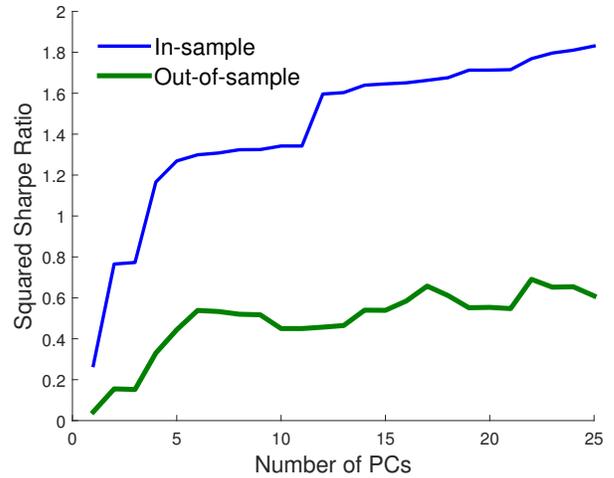
This finding is related to recent work by McLean and Pontiff (2014) that examines the true out-of-sample performance of a large number of cross-sectional return predictors that appeared in the academic literature in recent decades. They find a substantial decay in returns from the researchers' in-sample period to the out-of-sample period after the publication of the academic study. Most relevant for our purposes is their finding that the predictors with higher in-sample  $t$ -statistics are the ones that experience the biggest decay.<sup>5</sup>

In Figure 4, panel (a), we consider all anomaly portfolios jointly. Focusing first on the in-sample period in the first half of the sample, we look at the maximum squared SR that can be obtained

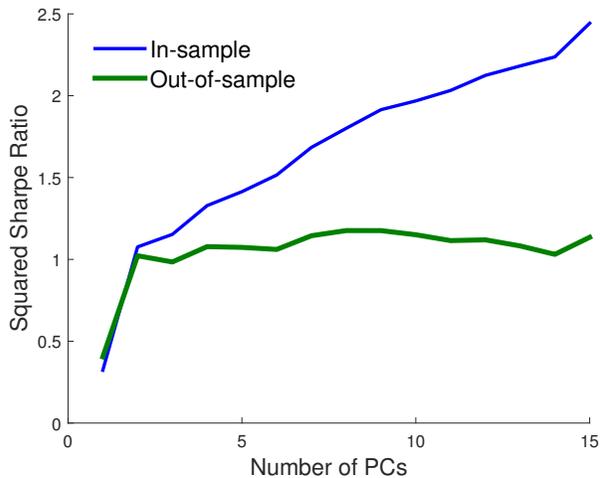
<sup>5</sup>In private correspondence, Jeff Pontiff provided us with estimation results showing that a stronger decay is also present for predictors with high in-sample SR. We thank Jeff for sending us those results.



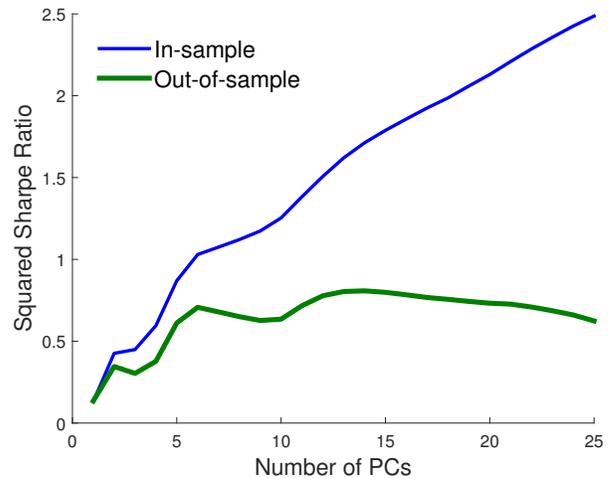
(a) Anomaly long-short portfolios (sample split)



(b) Fama-French 25 Portfolios (sample split)



(c) Anomaly long-short portfolios (bootstrap)



(d) Fama-French 25 Portfolios (bootstrap)

Figure 4: In-sample and out-of-sample maximum squared Sharpe Ratios (annualized) of first  $K$  principal components. In panels (a) and (b) the sample period is split into two halves. We extract PCs in the first sub-period and calculate SR-maximizing combination of first  $K$  PCs in first subperiod. We then apply the portfolio weights implied by this combination in the out-of-sample period (second sub-period). In panels (c) and (d) we randomly sample (without replacement) half of the returns to extract PCs and calculate SR-maximizing combination of first  $K$  PCs in the subsample. We then apply the portfolio weights implied by this combination in the out-of-sample period (remainder of the data). The procedure is repeated 1,000 times; average squared SRs are shown.

from a combination of the first  $K$  principal components. The blue line in the figure plots the result. With  $K = 2$ , the maximum squared SR is around 2, but raising  $K$  further raises the squared SR close to 5 for  $K = 15$ . However, out of sample, the picture looks different. For each  $K$ , we now take the asset weights that yield the maximum SR from the first  $K$  PCs in the first subperiod, and we apply these weights to returns from the second subperiod. The green line in the figure shows the result. Not surprisingly, overall SR are lower out of sample. Most importantly, it makes no difference whether one picks  $K = 2$  or  $K = 15$ —the out-of-sample squared SR is about the same. Hence, while the higher-order PCs add substantially to the squared SR in sample, they provide no incremental improvement of the SR in the out-of-sample period. Whatever these higher-order PCs were picking up in the in-sample period is not a robust feature of the cross-section of expected return that persists out of sample. In panel (b) we repeat the same analysis for the  $5 \times 5$  size-B/M portfolios and their PC factors. The results are similar.

In Figure 4, panels (c) and (d), we perform a bootstrap estimation. First, we randomly sample (without replacement) half of the returns to extract PCs and calculate SR-maximizing combination of first  $K$  PCs in the subsample. We then apply the portfolio weights implied by this combination in the out-of-sample period (remainder of the data). The procedure is repeated 1,000 times; average squared SRs are shown. Panel (c) shows the results for anomaly portfolios. In panel (d) we repeat the same analysis for the  $5 \times 5$  size-B/M portfolios and their PC factors. Similarly to our findings that used a sample split, we show that the higher-order PCs provide essentially no incremental improvement of the SR in the out-of-sample period.

In summary, the empirical evidence suggests that reduced-form factor models with a few principal component factors provide a good approximation of the SDF, as one would expect if near-arbitrage opportunities do not exist. However, as we discuss in the rest of the paper, this fact tells us little about the “rationality” of investors and the degree to which “behavioral” effects influence asset prices.

## 4 Factor pricing in economies with sentiment investors

We now proceed to show that mere absence of near-arbitrage opportunities has limited economic content. We model a multi-asset market in which fully rational risk averse investors (arbitrageurs) trade with investors whose asset demands are driven by distorted beliefs (sentiment investors).

Consider an IID economy with discrete time  $t = 0, 1, 2, \dots$ . There are  $N$  stocks in the economy indexed by  $i = 1, \dots, N$ . The supply of each stock is normalized to  $1/N$  shares. A risk-free bond is available in perfectly elastic supply at an interest rate of  $R_F > 0$ . Stock  $i$  earns time- $t$  dividends  $D_{it}$  per share. Collect the individual-stock dividends in the column vector  $D_t$ . We assume that  $D_t \sim \mathcal{N}(0, \Gamma)$ .

We assume that the covariance matrix of asset cash flows  $\Gamma$  features a few dominant factors that drive most of the stocks' covariances. Consider its eigenvalue decomposition  $\Gamma = Q\Lambda Q'$ . Assume that the first PC is a level factor, with identical constant value for each element of the corresponding eigenvector  $q_1 = \iota N^{-1/2}$ . Then, the variance of returns on the market portfolio becomes

$$\sigma_m^2 = \text{Var}(R_{m,t+1}) = N^{-2} \iota' q_1 q_1' \iota \lambda_1 = N^{-1} \lambda_1.$$

All other principal components, by construction, are long-short portfolios, i.e.,  $\iota' q_k = 0$  for  $k > 1$ .

There are two groups of investors in this economy. The first group comprises competitive rational arbitrageurs in measure  $1 - \theta$ . The representative arbitrageur has CARA utility with absolute risk aversion  $a$ . In this IID economy, the optimal strategy for the arbitrageur is to maximize next period wealth, i.e.,

$$\begin{aligned} \max_y E[-\exp(-aW_{t+1})] \\ \text{s.t. } W_{t+1} = (W_t - C_t)(1 + R_F) + y'R_{t+1}, \end{aligned}$$

where  $R_{t+1} \equiv P_{t+1} + D_{t+1} - (1 + r_F)P_t$  is a vector of excess dollar returns. Since prices are constant in this IID case, the covariance matrix of returns equals the covariance matrix of dividends,  $\Gamma$ . From

arbitrageurs’ first-order condition and their budget constraint, we obtain their asset demand

$$y_t = \frac{1}{a} \Gamma^{-1} E[R_{t+1}] \quad (6)$$

Sentiment investors, the second group, are present in measure  $\theta$ . Like arbitrageurs, they have CARA utility with absolute risk aversion  $a$  and they face a similar budget constraint, but they have an additional sentiment-driven component to their demand  $\delta$ . Their risky asset demand vector is

$$x_t = \frac{1}{a} \Gamma^{-1} E[R_{t+1}] + \delta. \quad (7)$$

where we assume that  $\delta' \iota = 0$ . The first term is the rational component of the demand, equivalent to the arbitrageur’s demand. The second term is the sentiment investors’ excess demand  $\delta$ , which is driven by investors’ behavioral biases or misperceptions of the true distribution of returns. This misperception is only cross-sectional; there is no misperception of the market portfolio return distribution since  $\delta' \iota = 0$ .

The sentiment investors “extra” demand due to the belief distortion is constrained to

$$\delta' \delta \leq 1. \quad (8)$$

Our motivation for this constraint is that sentiment investors are likely to be relatively unsophisticated. It would not be plausible to have them take highly levered bets or or make extensive use of short selling. Without constraints on  $\delta$ , they could end up taking extremely large long-short positions on some close-to-idiosyncratic risks that they perceive as mispriced. Arbitrageurs would be willing to take offsetting extreme positions to exploit the resulting near-arbitrage opportunities. To rule out this implausible scenario, we impose (8). By limiting the cross-sectional sum of squared belief deviations from rational weights in this way, the maximum deviation that we allow in an individual stock is, approximately, one that results in a portfolio weight of  $+/- 1$  in one stock and  $1/N - / + 1/N$  in all others.<sup>6</sup> Thus, the constraint still allows sentiment investors to have rather

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<sup>6</sup>In equilibrium, the rational investor with objective expectations would hold the market portfolio with weights  $1/N$ . Deviating to a weight of 1 in one stock and to zero in all the other  $N - 1$  stocks therefore implies a sum of

extreme portfolio tilts.

Market clearing,

$$\theta\delta + \frac{1}{a}\Gamma^{-1}E[R_{t+1}] = \frac{1}{N}\iota, \quad (9)$$

implies

$$E[R_{t+1}] - \mu_m\iota = -a\theta\Gamma\delta, \quad (10)$$

where  $\mu_m \equiv (1/N)\iota'E[R_{t+1}]$  and we used the fact that, due to the presence of the level factor,  $\iota$  is an eigenvector of  $\Gamma$  and so  $\Gamma^{-1}\iota = \frac{1}{\lambda_1}\iota = \frac{1}{N\sigma_m^2}\iota$ . Moreover, we used  $\mu_m = a\sigma_m^2$ . Then, after substituting into arbitrageurs optimal demand, we get

$$y = \frac{1}{N}\iota - \theta\delta. \quad (11)$$

As a consequence, we obtain the SDF,

$$\begin{aligned} M_{t+1} &= 1 - a(R - E[R])'y \\ &= 1 - a[R_{m,t+1} - \mu_m] + a(R_{t+1} - E[R_{t+1}])'\theta\delta, \end{aligned} \quad (12)$$

and the SDF variance,

$$\text{Var}(M) = a^2\sigma_m^2 + a^2\theta^2\delta'\Gamma\delta. \quad (13)$$

The effect of  $\delta$  on the factor structure and the volatility of the SDF depends on how  $\delta$  lines up with the PCs. To characterize the correlation of  $\delta$  with the PCs, we express  $\delta$  as a linear combination of PCs,

$$\delta = Q\beta, \quad (14)$$

with  $\beta_1 = 0$ . Without loss of generality we can assume that  $\beta \geq 0$ . Note that  $\delta'\delta = \beta'Q'Q\beta = \beta'\beta$  squared deviations of  $(1 - 1/N)^2 + (N - 1)/N^2 = 1 - 1/N \approx 1$  and exactly zero mean deviation.

so the constraint (8) can be expressed in terms of  $\beta$ :

$$\beta' \beta \leq 1. \quad (15)$$

#### 4.1 Dimensionality of the SDF

We combine (14) and (13) to obtain excess SDF variance, expressed, for comparison, as a fraction of the SDF variance accounted for by the market factor,

$$\begin{aligned} V(\beta) &\equiv \frac{\text{Var}(M) - a^2 \sigma_m^2}{a^2 \sigma_m^2} \\ &= \frac{\theta^2}{\sigma_m^2} \delta' \Gamma \delta \\ &= \kappa^2 \sum_{k=2}^N \beta_k^2 \lambda_k \end{aligned} \quad (16)$$

where  $\kappa \equiv \frac{\theta}{\sigma_m}$ . From equation (16) we see that SDF excess variance is linear in the PC eigenvalues, with weights  $\beta_k^2$ . For sentiment  $\delta$  to have a large impact on asset prices, it must line up primarily with the high-eigenvalue (volatile) principal components of asset returns. The constraint on components of  $\beta$ , eq. (15), implies that if  $\beta$  did line up with some of the low-eigenvalue PCs, the loadings on high-eigenvalue PCs would be substantially reduced and hence the variance of the SDF would be low. As a consequence, either the SDF can be approximated well by a low-dimensional factor model with the first few PCs as factors, or the SDF can't be volatile and hence Sharpe Ratios only very small.

We now proceed to assess this claim quantitatively. The share of excess SDF variance in (16) accounted for by the  $k$ -th PC is  $\frac{\beta_k^2 \lambda_k}{\beta' \Lambda \beta}$ . The reciprocal of the sum of squared shares,

$$d(\beta) \equiv \frac{\left(\beta' \Lambda \beta\right)^2}{\sum_{k=2}^N \beta_k^4 \lambda_k^2}, \quad (17)$$

can be used to assess the dimensionality of the SDF. If  $\delta$  loads on a single PC, the sum of squared shares tends to one,  $d(\beta) = 1$ , and the SDF is a two-factor SDF with the market and a single

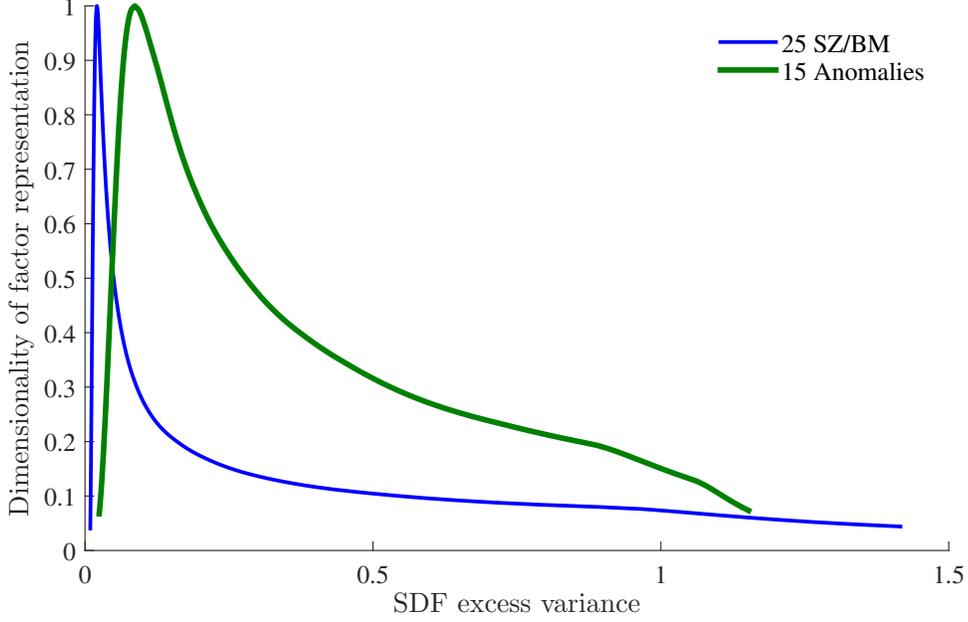


Figure 5: Dimensionality of SDF vs. SDF excess variance: The plot show the upper envelope of the set of all achievable combinations of SDF excess variance,  $V(h)$ , and SDF factor dimensionality scaled by the number of assets. The right tail of each curve can be interpreted as the highest achievable SDF excess variance for any given level of dimensionality of the SDF. The blue curve corresponds to  $5 \times 5$  size-B/M portfolios; the green curve is based on 15 anomaly long-short portfolios.

additional PC as factors. If, additionally,  $\delta$  loads on a principal component associated with high eigenvalue, the excess variance of the SDF in equation (16) is high. For example, if  $\beta_2 = 1$ ,  $\beta_{-2} = 0$ , then the sum equals  $\lambda_2$  (which is relatively large). If  $\beta_N = 1$ ,  $\beta_{-N} = 0$ , then it equals  $\lambda_N$  (which is tiny).

To get an SDF in which all PCs play an equal role so that a low dimensional factor model does not hold, set  $\beta_k \propto \lambda_k^{-0.5}$ . In this case, the measure of SDF dimensionality in equation (17) is maximized,  $d(\beta) = N - 1$ , and the share for each PC's SDF excess variance contribution is  $\frac{1}{N-1}$ .

Figure 5 illustrates this with data based on the covariance matrix of actual portfolios used as  $\Gamma$  and with  $\theta = 0.5$ . We consider two sets of portfolios: (i) 25 SZ/BM portfolios and (ii) 15 anomaly long-short portfolios. The plot shows the achievable combinations of SDF excess variance,  $V(\beta)$ , and SDF factor dimensionality scaled by the number of assets,  $d(\beta)/(N - 1)$ . For clarity we plot

only the upper envelope of the set generated by all  $\delta$  which satisfy the restriction (8).<sup>7</sup> The right tail of each curve can be interpreted as the highest achievable SDF excess variance for any given level of dimensionality of the SDF. The plot shows that a substantial SDF excess variance can be achieved only if the elements of  $\delta$  are highly correlated with a few high-eigenvalue PCs, which leads to a high SDF excess variance and a low-dimensional factor structure of the SDF (to the right in the plot).

Thus, either  $\delta$  lines up with high-eigenvalue PC, then a low-dimensional factor model holds, or it lines up with many of them, then a low-dimensional factor model does not hold. In the latter case, however, the arbitrageurs find it a low-risk opportunity to trade against the sentiment investor demands and undo their impact on prices. Hence, the effect on prices and SDF variance is small in this case.

## 4.2 Characteristics vs. covariances

A different way of looking at the dimensionality of the SDF is to ask how much of the cross-sectional variation in expected returns is explained by covariances with the first few PCs. If one views the belief distortion  $\delta$  as associated with certain stock characteristics, then this is the question posed in Daniel and Titman (1997) whether characteristics or covariances explain cross-sectional variation in expected returns.

To analyze this question, we look at how much of expected return variation in this model is explained by the first  $K$  PCs if we take the covariance matrix from empirically observed portfolio returns. Equilibrium expected returns in this model are given by (10) and hence cross-sectional variation in expected returns is

$$\begin{aligned} \frac{1}{N}(E[R_{t+1}] - \mu_{m\ell})'(E[R_{t+1}] - \mu_{m\ell}) &= a^2\theta^2\delta'\Gamma\Gamma\delta \\ &= a^2\theta^2\beta'\Lambda^2\beta. \end{aligned} \tag{18}$$

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<sup>7</sup>Please refer to the Appendix, section A.1.2 for more details on the construction of Figure 5.

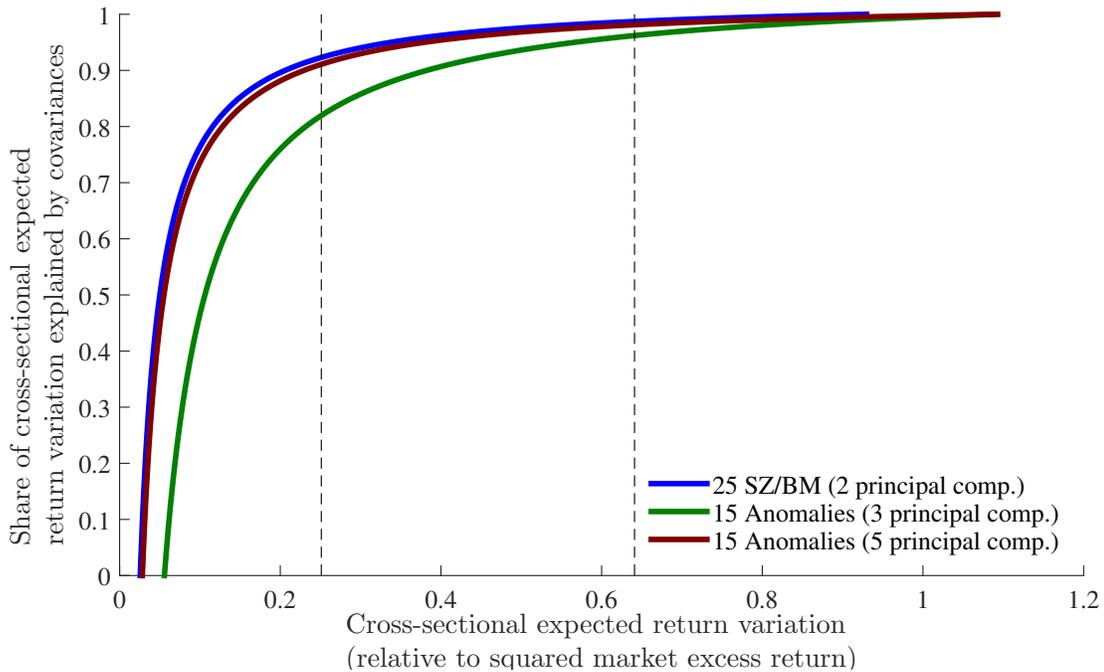


Figure 6: Characteristics vs. covariances: Cross-sectional variation in expected returns explained by first two principal components for  $5 \times 5$  size-B/M portfolios and 3 and 5 principal components for anomaly long-short portfolios. Dashed lines depict in-sample estimates of the ratio of cross-sectional variation in expected returns and the squared market excess return for two sets of portfolios.

The cross-sectional variation in expected returns explained by the first  $K$  PCs is

$$a^2 \theta^2 \sum_{k=2}^K \beta_k^2 \lambda_k^2. \quad (19)$$

We now plot the proportion of cross-sectional variation in expected returns explained by the first  $K$  principal components, i.e., the ratio of (19) to (18), and the ratio of (the upper bound of) cross-sectional variance in expected returns, (18), to squared expected excess market returns. We again set  $\theta = 0.5$ . For clarity we plot only the right envelope of the set generated by all  $\delta$  which satisfy the restriction (8).<sup>8</sup>

Figure 6 presents the results. As the figure shows, it is not possible to generate much cross-sectional variation in expected returns without having the first two principal components of size-B/M portfolios (in excess of the market return) and 3 principal components of 15 anomaly long-

<sup>8</sup>Please refer to the Appendix, section A.1.3 for more details on the construction of Figure 6.

short portfolios explain almost all the cross-sectional variation in expected returns of their respective portfolios. For comparison, the ratio of cross-sectional variation in expected returns and the squared market excess return is 0.25 for the  $5 \times 5$  size-B/M portfolios and 0.64 for anomaly long-short portfolios (depicted with dashed vertical lines on the plot).<sup>9</sup> At this level of cross-sectional variation in expected returns, virtually all expected return variation has to be aligned with loadings on the first few principal components.

Thus, despite the fact that all deviations from the CAPM in this model are due to belief distortions, a horse race between characteristics and covariances as in Daniel and Titman (1997) cannot discriminate between a rational and a sentiment-driven theory of the cross-section of expected returns. Covariances and expected returns are almost perfectly correlated in this model—if they weren't, near-arbitrage opportunities would arise, which is not consistent with the presence of some rational investors in the model.

### 4.3 Investment-based expected stock returns

So far our focus has been on the interpretation of empirical reduced-form factor models. There is a related literature that uses reduced-form specifications of the SDF in models of firm decisions with the goal of deriving predictions about the cross-section of stock returns. Our critique that reduced-form factor models have little to say about the beliefs and preferences of investors applies to these models, too.

The models in this literature feature firms that make optimal investment decisions. They produce the prediction that stock characteristics such as the book-to-market ratio, firm size, investment, and profitability should be correlated with expected returns. We discuss two classes of models. In the first one, firms continuously adjust investment, subject to adjustment costs. One recent example is Lin and Zhang (2013). In the second class, firms are presented with randomly arriving investment opportunities that differ in systematic risk. The firm can either take or reject an arriving project. A prominent example of a model of this kind is Berk, Green, and Naik (1999).

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<sup>9</sup>The anomaly portfolios are already long-short portfolios. So rather than looking at squared deviations from the mean of the average anomaly returns, we use the uncentered second moment of one-half the anomaly mean returns. Using the cross-sectional variance would yield almost the same result, though: 0.66.

Our focus is on the question whether these models have anything to say about the reason why investors price some stocks to have higher expected returns than others. These theories are often presented as *rational* theories of the cross-section of expected returns that are contrasted with *behavioral* theories in which investors are not fully rational.<sup>10</sup> However, a common feature of these models is that firms optimize taking as given a generic SDF that is not restricted any further. Existence of such a generic SDF requires nothing more than the absence of arbitrage opportunities. Thus, these models make essentially no assumption about investor preferences and beliefs. As a consequence, these models cannot deliver any conclusions about investor preferences or beliefs. As our analysis above shows, it is perfectly possible to have an economy in which all cross-sectional variation in expected returns is caused by sentiment, and yet an SDF not only exists, but it also has a low-dimensional structure in which the first few principal components drive SDF variation, similar to many popular reduced-form factor models. For this reason, models that focus on firm optimization, taking a generic SDF as given, cannot answer the question about investor rationality.

To illustrate, consider a model of firm investment similar to the one in Lin and Zhang (2013). Firms operate in an IID economy, and they take the SDF as given when making real investment decisions. At each point in time, a firm has a one-period investment opportunity. For an investment  $I_t$  the firm will make profit  $\Pi_{t+1}$  per unit invested. The firm faces quadratic adjustment costs and the investment fully depreciates after one period. The full depreciation assumption is not necessary for what we want to show, but it simplifies the exposition. To reduce clutter, we also drop the  $i$  subscripts for each firm.

Every period, the firm has the objective

$$\max_{I_t} -I_t - \frac{c}{2}I_t^2 + E[M_{t+1}\Pi_{t+1}I_t]. \quad (20)$$

The SDF that appears in this objective function is not restricted any further. Hence, the SDF could be, for example, the SDF (12) from our earlier example economy in which all cross-sectional

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<sup>10</sup>For example, Berk, Green, and Naik (1999), p. 1553, motivate their analysis by pointing to these competing explanations and commenting that “these competing explanations are difficult to evaluate without models that explicitly tie the characteristics of interest to risks and risk premia.”

variation in expected returns is due to sentiment. Taking this SDF as given, we get the firm's first-order condition

$$I_t = -\frac{1}{c} + E[M_{t+1}\Pi_{t+1}] \quad (21)$$

$$= -\frac{1}{c} + E[M_{t+1}] + E[\Pi_{t+1}] + \text{Cov}(M_{t+1}, \Pi_{t+1}). \quad (22)$$

Since the economy features IID shocks,  $I_t$  is constant over time, i.e., we can write  $I_t = I$ . The firm's cash flow net of (recurring) investment each period, is

$$D_{t+1} = I\Pi_{t+1} - \frac{c}{2}I^2 - I. \quad (23)$$

If we let  $\Pi_{t+1}$  be normally distributed, this fits into our earlier framework as the cash-flow generating process (with a slight modification to allow for a positive average cash flow and heterogeneous expected profitability across firms),

$$I = -\frac{1}{c} + E[M_{t+1}] + E[\Pi_{t+1}] + \frac{1}{I}\text{Cov}(M_{t+1}, D_{t+1}), \quad (24)$$

where  $M_{t+1}$  is the SDF (12) that reflects the sentiment investor demand.

Thus, a firm with high  $E[\Pi_{t+1}]$  (relative to other firms) must either have high investment or a strongly negative  $\text{Cov}(M_{t+1}, D_{t+1})$  (which implies a high expected return). Similarly, a firm with high  $I$  must either have high profitability or a not very strongly negative  $\text{Cov}(M_{t+1}, D_{t+1})$  (which implies a low expected return). Thus, together  $I$  and  $E[\Pi_{t+1}]$  should explain cross-sectional variation in  $\text{Cov}(M_{t+1}, D_{t+1})$  and hence in expected returns.

These relationships arise because firms align their investment decisions with the SDF and the expected return—which is their cost of capital—that they face in the market. From the viewpoint of the firm in this type of model, it is irrelevant whether cross-sectional variation in expected returns is caused by sentiment or not. The implications for firm investment and for the relation between expected returns, investment and profitability are observationally equivalent. Thus, the empirical evidence in Fama and French (2006), Hou, Xue, and Zhang (2014), Novy-Marx (2012)

that investment and profitability are related, cross-sectionally, to expected stock returns is to be expected in a model in which firms optimize. Moreover, as long as the firm optimizes, the Euler equation  $E[M_{t+1}R_{t+1}] = 1$  also holds for the firm’s investment return, as in Liu, Whited, and Zhang (2009), again irrespective of whether *investors* are rational or have distorted beliefs.

Testing whether empirical relationships between expected returns, investment, and profitability exist in the data is a test of the model of firm decision-making, but not of how investors price securities. Empirical evidence on this question does not help resolve the question about how to specify investor beliefs and preferences. Only models that make assumptions about investor beliefs and preferences—which result in restrictions on the SDF—can deliver testable predictions that could potentially help distinguish between competing models of how investors price assets.

Turning to the second class of models, we focus on the version of Berk, Green, and Naik (1999) (BGN) with constant interest rates, which is sufficient to produce the key predictions of their model. BGN assert the existence of a generic SDF  $M$  that is not restricted any further apart an auxiliary assumption that  $M$  is log-normal. Hence, this SDF could represent, for example, an SDF that arises in an economy in which sentiment causes all cross-sectional variation in expected returns, as in our earlier example economy. All of their conclusions about the relationships between expected returns, firms’ book-to-market ratios, and firm size would arise in this model irrespective of the specification of investor beliefs and preferences (rational, behavioral, or otherwise).

Firms in their model are presented with randomly arriving and dying investment projects that all have the same expected profitability and scale, but differ randomly in the covariance of their cash-flows shocks  $\varepsilon_i$  with the SDF. Projects with very negative  $\gamma_i = \text{Cov}(\varepsilon_i, M)$  have a high expected return, i.e., a high cost of capital, and are rejected. Ones with less negative  $\gamma_i$  are taken on by the firm. Again, it is important to keep in mind that  $\gamma_i$  is a covariance with a generic SDF. Other than the existence of such an SDF, nothing has been assumed that would imply that  $\gamma_i$  has to represent “rationally priced” risk. Each firm also has an (identical) stock of growth options from the future arrival of new investment projects. Since expected profitability is assumed to be constant in this model and since we work with the constant-interest rate version of their model, the value of these growth options is simply the value of a risk-free bond. At a given point in time,

the firm's return covariance with  $M$  is then determined by the number of projects,  $n_t$ , the firm has taken on in the past that are still alive (relative to constant stock of riskless growth options) and by the aggregated  $\gamma_i$  of the still-alive projects, which we denote  $\gamma_t$ . Since expected excess returns are equal to the negative of the covariance with  $M$ , it follows that

$$E[R_{t+1}] = f(n_t, \gamma_t) \tag{25}$$

for some function  $f(\cdot)$ . As BGN show (see their equation 45), this leads to a linear relationship between expected returns, the book-to-market ratio and market value,

$$E[R_{t+1}] = a_0 + a_1(B_t/M_t) + a_2(1/M_t), \tag{26}$$

where  $B_t/M_t$  depends positively on  $n_t$  (as having more ongoing projects reduce the weight on the riskless growth options) and positively on  $\gamma_i$  (as higher expected return lowers market value), while  $1/M_t$  depends negatively on  $n_t$  (as more projects taken on raise market value) and positively on  $\gamma_i$ .

Nowhere in this derivation is there any assumption that would restrict investor preferences and beliefs any further than asserting the existence of an SDF. Thus, if BGN's model of firm decision-making is correct, the conclusions that expected returns are linear in  $B/M$  and  $1/M$ , as in (26), would apply in any world in which an SDF exists, even if all cross-sectional variation in expected returns is caused by sentiment.

## 5 Factor pricing in economies with sentiment investors: Dynamic case

In this section we show that the observational equivalence between “behavioral” and “rational” asset pricing with regards to factor pricing also applies, albeit to a lesser degree, to partial equilibrium intertemporal capital asset pricing models (ICAPM) in the tradition of Merton (1973). To this extent, we specify and solve a dynamic model with time-varying investor sentiment.

We model the economy in discrete time and infinite horizon framework. Suppose there are  $N$  stocks,  $i = 1, \dots, N$ , with per-period dividends  $D_t \sim \mathcal{MVN}(0, \Gamma)$ . The supply of stocks is fixed,  $\iota \equiv (1, 1, \dots, 1)'$ . The risk-free one-period bond is in perfectly elastic supply at a constant interest rate of  $r_F$ . Define gross interest rate as  $R_F = 1 + r_F$ . Finally, we assume there exists a measure  $\theta$  of sentiment investors with exogenously specified demand

$$x_t = \iota + \delta \xi_t, \quad (27)$$

where  $\xi_{t+1} \sim \mathcal{N}(\hat{\xi}_t, \omega^2)$ , and the mean of  $\xi_{t+1}$  is given by

$$\hat{\xi}_t \equiv \mu + \phi \xi_t. \quad (28)$$

We assume  $\delta$  has a level component and a component orthogonal to the level component. Then, market clearing requires

$$y_t = \iota - \frac{\theta}{1 - \theta} \delta \xi_t. \quad (29)$$

Arbitrageurs maximize their life-time exponential utility

$$J_t(W_t, \xi_t) = \max_{(C_s, y_s), s \geq t} \text{E}_t \left[ - \sum_{s=t}^{\infty} \beta^s \exp(-\alpha C_s) \right], \quad (30)$$

where the maximization is subject to

$$W_{t+1} = (W_t - C_t)R_F + y_t' R_{t+1}, \quad (31)$$

where  $R_{t+1} \equiv P_{t+1} - R_F P_t + D_{t+1}$ .

We further define the market portfolio as  $R_{M,t+1} \equiv \iota' R_{t+1}$  and the arbitrageur's investment portfolio  $R_{A,t+1} = y_t' R_{t+1}$ . Theorem 1 below lists the main asset pricing implications of the model.

**Theorem 1** *Equilibrium prices that solve the problem in equations (27)-(31) are given by*

$$P_t = a_0 + a_1 \xi_t$$

and expected returns are

$$\begin{aligned} E_t(R_{t+1}) &= \gamma \text{Cov}_t(R_{t+1}, R_{A,t+1}) + \psi_t \text{Cov}_t(R_{t+1}, \xi_{t+1}) \\ &= \gamma \text{Cov}_t(R_{t+1}, R_{A,t+1}) + \hat{\psi}_t \text{Cov}_t(R_{t+1}, E_{t+1} R_{M,t+2}), \end{aligned} \quad (32)$$

where  $\psi_t = \omega^2 (b_1 + 2\hat{\xi}_t^2 b_2)$ ,  $\hat{\psi}_t = -\frac{\psi_t}{\iota' a_1 (R_F - \phi) \omega^2}$ , and  $a_1, a_2, b_1, b_2$  are constants provided in the Appendix A.2;  $\text{Cov}_t(R_{t+1}, R_{A,t+1}) = \text{Cov}_t(R_{t+1}, R_{M,t+1}) - \xi_t \left( \frac{\theta}{1-\theta} \right) \text{Cov}_t(R_{t+1}, R_{\delta,t+1})$ ,  $R_A$  is the return on arbitrageur's investment portfolio, and  $R_\delta$  is the return on the portfolio driven by "excess" demands of sentiment investors.

**Proof.** Refer to the Appendix A.2 for more details. ■

We now consider several cases of the sentiment investors' expected demand specification  $\xi_t$ . In particular, we will focus on different assumptions with regards to the mean of  $\xi_t$ ,  $\hat{\xi}_t$ , as given by equation (28).

**Case 1**  $\hat{\xi}_t$  is always zero,  $\hat{\xi}_t = 0$ .

In this case shocks to the demand of sentiment investors are IID and mean-zero and thus the demand fully reverts to zero in expectation next period. It then follows that  $\mu = \phi = 0$  and

$$E(R_{t+1}) = \gamma \text{Cov}(R_{t+1}, R_{M,t+1} - E_t R_{M,t+1}) + \frac{\gamma}{R_F} \text{Cov}(R_{t+1}, E_{t+1} R_{M,t+2})$$

Thus, we get an ICAPM similar to Campbell (1993, equation (23)). The degree of presence of sentiment traders does not show up directly, but it is indirectly in  $\text{Cov}(R_{t+1}, E_{t+1}[R_{M,t+2}])$ , because as  $\theta$  goes to zero, this covariance shrinks to zero. Alternatively, note that  $\text{Cov}(D_{t+1}, R_{M,t+1} - E_t[R_{M,t+1}]) = \gamma\Gamma\iota$  and so we can write

$$E[R_{t+1}] = \gamma\text{Cov}(D_{t+1}, R_{M,t+1} - E_t[R_{M,t+1}]) \quad (33)$$

This is a bad beta, good beta specification as in Campbell and Vuolteenaho (2004), but here with a zero risk premium for the “good” beta, i.e., the discount rate beta. The “good” beta disappears because the hedging demand due to time variation in expected returns goes in opposite direction to the discount rate component of the market return, and exactly cancels out when returns are iid (so that low returns today lead to immediate one-to-one increase in expected returns for the next period). Arbitrageurs therefore do not demand a risk premium for discount-rate beta exposure, because expected return variation only has transitory effects on their wealth. Only the cash-flow beta (“bad”) beta is compensated with a risk premium.

**Case 2** If  $\hat{\xi}_t$  is a non-zero constant,  $\hat{\xi}_t = \mu$ .

In this case shocks to the demand of sentiment investors are IID but not mean-zero. This feature introduces a persistent skew to the portfolio held by arbitrageurs relative to the market portfolio. They hold more of stocks that sentiment investors dislike and relatively less (or short) of stocks that sentiment investors hold in excess of their market weights. In this case  $\phi = 0$  and thus:

$$\begin{aligned} E_t(R_{t+1}) &= \gamma\text{Cov}_t(R_{t+1}, R_{A,t+1}) + \psi\text{Cov}_t(R_{t+1}, \xi_{t+1}) \\ &= \gamma\text{Cov}_t(R_{t+1}, R_{A,t+1}) + \hat{\psi}\text{Cov}_t(R_{t+1}, E_{t+1}R_{M,t+2}), \end{aligned} \quad (34)$$

where  $\psi = \omega^2(b_1 + 2\mu^2b_2)$ ,  $\hat{\psi} = -\frac{\psi}{\iota' a_1 R_F \omega^2}$ . This equation is similar to (32) in Theorem 1, but

prices of risk are now constant. We can therefore easily condition the model down:

$$E(R_{t+1}) = \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - E_t R_{A,t+1}) + \psi \text{Cov}(R_{t+1}, \xi_{t+1}) \quad (35)$$

$$= \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - E_t R_{A,t+1}) + \hat{\psi} \text{Cov}(R_{t+1}, E_{t+1} R_{M,t+2}), \quad (36)$$

where

$$\begin{aligned} \text{Cov}(R_{t+1}, R_{A,t+1} - E_t R_{A,t+1}) &= \text{Cov}(R_{t+1}, R_{M,t+1} - E_t R_{M,t+1}) \\ &\quad - \mu \left( \frac{\theta}{1 - \theta} \right) \text{Cov}(R_{t+1}, R_{\delta,t+1} - E_t R_{\delta,t+1}). \end{aligned}$$

The main difference between Cases 1 and 2 is that in the equation (36) of Case 2 the “market” return is replaced by the arbitrageur’s total wealth portfolio, reflecting the fact that rational investors (arbitrageurs) have to deviate from the market portfolio to offset the sentiment investors’ excess demand. This specification is a generalization of the static model in section 4 to the case when investor sentiment is time-varying. The effect on prices and risk premia comes from two channels: (i) contemporaneous demands that line up with eigenvectors associated with high eigenvalues of the returns covariance matrix are that risky to trade against away and thus result in excess returns on stocks that are in relatively low demand by sentiment investors, consistent with equation (10) in section 4; and (ii) intertemporal hedging results in high risk premium on stocks that pay well when expected returns on the market portfolio are high (sentiment demand is low), consistent with the ICAPM reasoning in Campbell (1993) and Campbell and Vuolteenaho (2004).

Equation (35) is convenient for empirical estimation given that we have an empirical proxy for the sentiment investor flow vector  $\xi_t$ .

**Case 3**  $\hat{\xi}_t$  is an  $AR(1)$  process,  $\hat{\xi}_t = \mu + \phi \xi_t$ .

This is the most general case of the sentiment investors’ excess demand specification. We find that

in this case the price of discount rate risk is time-varying. Unconditionally we get:

$$\begin{aligned} E(R_{t+1}) &= \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - E_t R_{A,t+1}) + \bar{\psi} \text{Cov}(R_{t+1}, \xi_{t+1} - E_t \xi_{t+1}) \\ &= \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - E_t R_{A,t+1}) + \tilde{\psi} \text{Cov}(R_{t+1}, E_{t+1} R_{M,t+2} - E_t R_{M,t+2}) \end{aligned}$$

where  $\bar{\psi} = \omega^2 (b_1 + 2b_2 E[\hat{\xi}_t^2])$ ,  $\tilde{\psi} = -\frac{\bar{\psi}}{i' a_1 (R_F - \phi) \omega^2}$ .

Comparing this to equation (34) we can see that the price of the discount-rate risk is magnified by AR(1) specification when  $\phi > 0$ .

In summary, the analysis shows that time-varying investor sentiment can give rise to an ICAPM-like SDF. As in our static model in the previous section, this model is “behavioral” and “risk-based” at the same time. Deviations from the static CAPM are caused by sentiment, but from the viewpoint of the arbitrageurs, time-varying sentiment generates hedging demands, because it makes the arbitrageurs’ investment opportunities time-varying. When evaluating how aggressively to accommodate sentiment investor demand in a particular stock, arbitrageurs consider the covariance of a stock’s return with the sentiment-driven investment opportunity state variable. As a result, expected returns reflect this state-variable risk.

## 6 Conclusions

Reduced-form factor models are useful to provide a parsimonious summary of the cross-section of asset returns. Yet, their success or failure in explaining the cross-section of asset returns does not help to answer the question whether asset pricing is “rational.” As we have shown, even if all cross-sectional variation in expected returns is driven by belief distortions on the part of some investors, a low-dimensional SDF with the first few principal components of returns as factors should still explain asset prices. This only requires that near-arbitrage opportunities are absent. For the same reason, tests that look for stock characteristics capture expected return variation in the cross-section that is orthogonal to common factor covariances are unlikely to be of much help in answering that question either. Therefore, tests of reduced-form factor models cannot shed light on questions regarding the “rationality” of investors.

In fact, the framing of the question concerning investor “rationality” is unhelpfully imprecise in the first place. The arbitrageurs in our model are rational. From their viewpoint, expected returns are consistent with the risk premia that they require as compensation for tilting their portfolio weights away from the market portfolio. But it is the sentiment investor demand that arbitrageurs accommodate which causes these risk premia. Thus, there is no dichotomy between “risk-based” and “behavioral” asset pricing in this model.

The only path to a better understanding of investor beliefs is to develop and test structural asset pricing models with specific assumptions about investor beliefs and preferences that deliver predictions about the factors that should be in the SDF and the probability distribution under which this SDF prices assets. While we discussed these issues in the context of equity markets research, similar conclusions apply to reduced-form no-arbitrage models in bond and currency market research.

The recognition that factor covariances should explain cross-sectional variation in expected returns even in a model of sentiment-driven asset prices should also be useful for the development of models that meet a challenge pointed out by Cochrane (2011):

*Behavioral ideas—narrow framing, salience of recent experience, and so forth—are good*

*at generating anomalous prices and mean returns in individual assets or small groups. They do not easily generate this kind of coordinated movement across all assets that looks just like a rise in risk premium. Nor do they naturally generate covariance. For example, “extrapolation” generates the slight autocorrelation in returns that lies behind momentum. But why should all the momentum stocks then rise and fall together the next month, just as if they are exposed to a pervasive, systematic risk?*

The answer to this question could be that some components of sentiment-driven asset demands are aligned with covariances with important common factors, some are orthogonal to these factor covariances. Trading by arbitrageurs eliminates the effects of the orthogonal asset demand components, but those that are correlated with common factor exposures survive because arbitrageurs are not willing to accommodate these demands without compensation for the factor risk exposure.

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## A Appendix

### A.1 Absence of near-arbitrage

#### A.1.1 SDF Variance

Define

$$\omega_m = \frac{1}{\sqrt{N}}q_1 \quad (37)$$

$$\mu_m = \omega'_m \mu \quad (38)$$

$$\sigma_m^2 = \omega'_m \Omega \omega_m \quad (39)$$

The last definition implies that  $\sigma_m^2 = \frac{\lambda_1}{N}$ .

Then

$$\text{Var}(M) = \frac{\mu_m^2}{\sigma_m^2} + (\mu - \mu_m)' Q_z \Lambda_z^{-1} Q'_z (\mu - \mu_m) \quad (40)$$

Let

$$\omega_k = \frac{1}{\sqrt{N}}q_k \quad (41)$$

$$\mu_k = \omega'_k \mu \quad (42)$$

$$\sigma_k^2 = \omega'_k \Omega \omega_k \quad (43)$$

The last definition implies that  $\sigma_k^2 = \frac{\lambda_k}{N}$ .  $\sigma_k^2$  is decreasing from second to higher-order PCs proportional to eigenvalue. We refer to  $R_k$  as the return on the zero-investment portfolio associated with the  $k$ -th principal component.

Then

$$\text{Var}(M) = \mu_m^2 / \sigma_m^2 + \sum_{k=2}^N N^2 \frac{\text{Cov}(\mu_i, q_{ki})^2}{\lambda_k} \quad (44)$$

$$= \mu_m^2 / \sigma_m^2 + \text{Var}(\mu_i) \sum_{k=2}^N \frac{\text{Corr}(\mu_i, q_{ki})^2}{\sigma_k^2} \quad (45)$$

Covariance is a cross-sectional covariance, and for the second line we used the fact that  $q_{ki}$  is mean zero and has variance  $N^{-1}$ . The sum of the squared correlations is equal to one. But the sum weighted by the inverse  $\sigma_k^2$  depends on which of the PCs  $\mu$  lines up with. If it lines up with high  $\sigma_k^2$  PCs then the sum is much lower than if it lines up with low  $\sigma_k^2$  PCs. Thus, if expected returns line up with low-eigenvalue PCs, then we get much higher SR.

### A.1.2 Dimensionality of SDF vs. SDF excess variance

This subsection provides additional detail on the construction of the Figure 5.

For each set of portfolios we solve for the left and right tails of their corresponding curves in the following way. The left tail is solved for by choosing a vector of  $\beta$  to minimize the equation (16) subject to (15) and (17) for all values of dimensionality in  $[0, N - 1]$ . We scale dimensionality by  $N - 1$  to normalize for the number of portfolios. Each point on the plot, therefore, corresponds to an SDF excess variance associated with a particular solution vector  $\beta$  for a given level of dimensionality  $d(h)$ .

Similarly, we solve for the right tail by maximizing the equation (16) subject to (15) and (17). Combining the solutions of sequences of maximization and minimization problems on one plot produces a smooth curve as shown in Figure 5. A curve is an upper envelope of all achievable of the set generated by all  $\beta$ . The right tail of each curve shows the maximum achievable SDF excess variance for any given level of dimensionality.

In total we solve 500 standalone constrained optimization problems for 25 SZ/BM portfolios (250 for the left tail and 250 for the right tail) and 300 optimization problems (150 for both left and right tails) for 15 anomaly long-short portfolios. Figure 5 entirely comprises of the plotted

solutions to these optimization problems comprise (linearly interpolated).

### A.1.3 Characteristics vs. covariances

This subsection provides additional detail on the construction of the Figure 6.

Figure 6 plots the right envelope of the set generated of all  $\beta$  that satisfy restriction (15). To construct this right envelope we put all weight of  $\beta$  onto two eigenvectors: the eigenvector associated with the highest eigenvalue (1-st eigenvector) and the  $(K + 1)$ -th eigenvector, i.e. the eigenvector associated with the highest principal component from the remainder of  $N - K$  principal components not used in equation (19)). We then vary weights on these two components in a way that satisfies (15). For each set of weights, we compute the ratio of (19) to (18), and the ratio of (the upper bound of) cross-sectional variance in expected returns, (18), to squared expected excess market returns.

Each curve in Figure 6 corresponds to a set of pairs of those two quantities. We plot three curves in total: one for 25 SZ/BM portfolios, which we explain with two principal components, and two curves for 15 anomaly long-short portfolios (for  $K = 3$  and  $K = 5$ ).

## A.2 Dynamic Model

Bellman equation

$$J_t(W_t, \xi_t) = \max_{C_t, y_t} \{-\beta^t \exp(-\alpha C_t) + E_t [J_{t+1}(W_{t+1}, \xi_{t+1})]\} \quad (46)$$

Guess

$$P_t = a_0 + a_1 \xi_t \quad (47)$$

$$J_t(W_t, \xi_t) = -\beta^t \exp(-\gamma W_t - b_0 - b_1 \xi_t - b_2 \xi_t^2) \quad (48)$$

where  $a_0$  and  $a_1$  are vectors of constants and  $\gamma$ ,  $b_0$ ,  $b_1$ , and  $b_2$  are scalars. Note that, based on this

guess,

$$R_{t+1} = D_{t+1} + a_1(\xi_{t+1} - \xi_t) - r_F a_0 - r_F a_1 \xi_t \quad (49)$$

and hence

$$\mathbf{E}_t[R_{t+1}] = -r_F a_0 - R_F a_1 \xi_t + a_1 \hat{\xi}_t \quad (50)$$

$$\mathbf{E}_t[W_{t+1}] = (W_t - C_t)R_F - y'_t(r_F a_0 + R_F a_1 \xi_t - a_1 \hat{\xi}_t) \quad (51)$$

$$\text{Var}_t(W_{t+1}) = y'_t(\Gamma + a_1 a'_1 \omega^2) y_t \quad (52)$$

$$\text{Cov}_t(W_{t+1}, \xi_{t+1}) = y'_t a_1 \omega^2 \quad (53)$$

$$\mathbf{E}_t(\xi_{t+1}^2) = \omega^2 + \hat{\xi}_t^2 \quad (54)$$

$$\text{Cov}_t(\xi_{t+1}, \xi_{t+1}^2) = 2\omega^2 \hat{\xi}_t \quad (55)$$

$$\text{Cov}_t(W_{t+1}, \xi_{t+1}^2) = 2\omega^2 \hat{\xi}_t y'_t a_1 \quad (56)$$

$$\text{Var}_t(\xi_{t+1}^2) = 2\omega^4 + 4\hat{\xi}_t^2 \omega^2 \quad (57)$$

where we used the following derivations:

$$\mathbf{E}_t(\xi_{t+1}^2) = \text{Var}_t[\xi_{t+1}] + [\mathbf{E}_t(\xi_{t+1})]^2 = \omega^2 + \hat{\xi}_t^2$$

$$\begin{aligned} \text{Cov}_t(\xi_{t+1}, \xi_{t+1}^2) &= \mathbf{E}_t \left[ (\xi_{t+1} - \hat{\xi}_t) (\xi_{t+1}^2 - \hat{\xi}_t^2 - \omega^2) \right] \\ &= \mathbf{E}_t \left[ (\xi_{t+1} - \hat{\xi}_t)^3 \right] + \mathbf{E}_t \left[ (\xi_{t+1} - \hat{\xi}_t) (2\xi_{t+1} \hat{\xi}_t - 2\hat{\xi}_t^2 - \omega^2) \right] \\ &= 2\omega^2 \hat{\xi}_t \end{aligned}$$

$$\begin{aligned}
\text{Var}_t(\xi_{t+1}^2) &= \text{Var}_t \left[ (\xi_{t+1} - \hat{\xi}_t)^2 + 2\hat{\xi}_t (\xi_{t+1} - \hat{\xi}_t) \right] \\
&= \omega^4 \text{Var}_t \left[ \left( \frac{\xi_{t+1} - \hat{\xi}_t}{\omega} \right)^2 \right] + 4\hat{\xi}_t^2 \omega^2 \\
&= 2\omega^4 + 4\hat{\xi}_t^2 \omega^2
\end{aligned}$$

where the second line follows because the third moment of a mean zero normally-distributed random variable is zero, and the last line follows as the variance of  $\chi^2(1)$  distribution.

Then

$$\begin{aligned}
\mathbb{E}_t [J_{t+1}(W_{t+1}, \xi_{t+1})] &= -\beta^{t+1} \exp \left( -\gamma \mathbb{E}_t[W_{t+1}] - b_0 - b_1 \hat{\xi}_t - b_2 (\omega^2 + \hat{\xi}_t^2) \right. \\
&\quad + \frac{1}{2} \gamma^2 \text{Var}_t(W_{t+1}) + \frac{1}{2} b_1^2 \omega^2 + \frac{1}{2} b_2^2 \text{Var}_t(\xi_{t+1}^2) \\
&\quad + \gamma b_1 \text{Cov}_t(W_{t+1}, \xi_{t+1}) + \gamma b_2 \text{Cov}_t(W_{t+1}, \xi_{t+1}^2) \\
&\quad \left. + b_1 b_2 \text{Cov}_t(\xi_{t+1}, \xi_{t+1}^2) \right) \tag{58}
\end{aligned}$$

First-order condition for  $y_t$ ,

$$0 = \gamma(r_F a_0 + R_F a_1 \xi_t - a_1 \hat{\xi}_t) + \gamma^2 (\Gamma + a_1 a_1' \omega^2) y_t + \gamma b_1 a_1 \omega^2 + 2\gamma b_2 a_1 \omega^2 \hat{\xi}_t \tag{59}$$

which we can solve for

$$y_t = -\frac{1}{\gamma} (\Gamma + a_1 a_1' \omega^2)^{-1} (r_F a_0 + R_F a_1 \xi_t - a_1 \hat{\xi}_t + b_1 a_1 \omega^2 + 2b_2 a_1 \omega^2 \hat{\xi}_t) \tag{60}$$

Plugging this solution into the market clearing condition, and rearranging, we get

$$-\gamma (\Gamma + a_1 a_1' \omega^2) \left( \iota - \frac{\theta}{1-\theta} \delta \xi_t \right) - b_1 a_1 \omega^2 - 2b_2 a_1 \omega^2 \hat{\xi}_t = r_F a_0 + R_F a_1 \xi_t - a_1 \hat{\xi}_t \tag{61}$$

Since the MC has to hold for any value of  $\xi_t$ , we can apply the method of undetermined coefficients

and get

$$a_0 = -\frac{\gamma}{r_F}(\Gamma + a_1 a_1' \omega^2) \iota - \frac{b_1}{r_F} a_1 \omega^2 - 2 \frac{b_2}{r_F} a_1 \omega^2 \mu + \frac{1}{r_F} a_1 \mu \quad (62)$$

$$a_1 = \frac{\gamma}{R_F - \phi + 2b_2 \phi \omega^2} (\Gamma + a_1 a_1' \omega^2) \frac{\theta}{1 - \theta} \delta \quad (63)$$

From the latter equation, we obtain

$$a_1 \left( 1 - a_1' \delta \omega^2 \frac{\theta}{1 - \theta} \frac{\gamma}{R_F - \phi + 2b_2 \phi \omega^2} \right) = \frac{\gamma}{R_F - \phi + 2b_2 \phi \omega^2} \Gamma \delta \frac{\theta}{1 - \theta} \quad (64)$$

Pre-multiplying with  $\delta$  we get a quadratic equation in  $a_1' \delta$ ,

$$c_1 (a_1' \delta)^2 - (a_1' \delta) + c_2 = 0 \quad (65)$$

which we can solve for the positive solution (that we will not need below, though).

We can rewrite (60) as

$$\underbrace{y_t' (\Gamma + a_1 a_1' \omega^2) y_t \gamma}_{\text{Var}_t(W_{t+1})} + y_t' \underbrace{(r_F a_0 + R_F a_1 \xi_t - a_1 \hat{\xi}_t)}_{-E_t[R_{t+1}]} + b_1 \underbrace{(y_t' a_1 \omega^2)}_{\text{Cov}_t(W_{t+1}, \xi_{t+1})} + b_2 \underbrace{(2\omega^2 \hat{\xi}_t y_t' a_1)}_{\text{Cov}_t(W_{t+1}, \xi_{t+1}^2)} = 0 \quad (66)$$

Substituting in variance and covariance from (52) and (53), multiplying through by  $\gamma$ , and subtracting one-half the variance term on both sides,

$$\begin{aligned} & \frac{1}{2} \gamma^2 \text{Var}_t(W_{t+1}) + \gamma y_t' (r_F a_0 + R_F a_1 \xi_t - a_1 \hat{\xi}_t) + \gamma b_1 \text{Cov}_t(W_{t+1}, \xi_{t+1}) + \gamma b_2 \text{Cov}_t(W_{t+1}, \xi_{t+1}^2) \\ & = -\frac{1}{2} \text{Var}_t(W_{t+1}) \gamma^2 \\ & = -\frac{1}{2} \left( \iota - \frac{\theta}{1 - \theta} \delta \xi_t \right)' (\Gamma + a_1 a_1' \omega^2) \left( \iota - \frac{\theta}{1 - \theta} \delta \xi_t \right) \gamma^2 \end{aligned} \quad (67)$$

We can write the right-hand side as

$$\lambda_0 + \lambda_1 \xi_t + \lambda_2 \xi_t^2 \quad (68)$$

where

$$\lambda_1 = \iota' (\Gamma + a_1 a_1' \omega^2) \delta \frac{\theta}{1-\theta} \gamma^2 \quad (69)$$

$$\lambda_2 = \left( \frac{\theta}{1-\theta} \right)^2 \delta' (\Gamma + a_1 a_1' \omega^2) \delta \quad (70)$$

Now, going back to (58), we can write

$$\mathbb{E}_t [J_{t+1}(W_{t+1}, \xi_{t+1})] = -\beta^{t+1} \exp \left( -\gamma(W_t - C_t)R_F + \hat{\lambda}_0 + \hat{\lambda}_1 \xi_t + \hat{\lambda}_2 \xi_t^2 \right)$$

where

$$\hat{\lambda}_0 = \lambda_0 - b_0 - b_1 \mu - b_2 (\omega^2 + \mu^2) + \frac{1}{2} b_1^2 \omega^2 + b_2^2 \omega^2 (\omega^2 + 2\mu^2) + 2b_1 b_2 \omega^2 \mu$$

$$\hat{\lambda}_1 = \lambda_1 - b_1 \phi - 2b_2 \mu \phi + 4b_2^2 \mu \phi \omega^2 + 2b_1 b_2 \phi \omega^2$$

$$\hat{\lambda}_2 = \lambda_2 - b_2 \phi^2 + 2b_2^2 \omega^2 \phi^2.$$

Now we evaluate the first-order condition for consumption

$$U'(C_t) = \frac{\partial \mathbb{E}_t [J_{t+1}(W_{t+1}, \xi_{t+1})]}{\partial C_t} \quad (71)$$

After taking logs,

$$\log \alpha - \alpha C_t = \log(\beta \gamma R_F) - \gamma R_F (W_t - C_t) + \hat{\lambda}_0 + \hat{\lambda}_1 \xi_t + \hat{\lambda}_2 \xi_t^2$$

which we can solve for

$$C_t = \frac{\gamma R_F}{\alpha + \gamma R_F} W_t + \frac{1}{\alpha + \gamma R_F} \left[ \log \left( \frac{\alpha}{\beta \gamma R_F} \right) - \hat{\lambda}_0 - \hat{\lambda}_1 \xi_t - \hat{\lambda}_2 \xi_t^2 \right]$$

Substituting this solution into the Bellman equation, and using the fact that  $1 - \gamma R_F / (\alpha +$

$\gamma R_F) = \alpha/(\alpha + \gamma R_F)$  to calculate, we get

$$\begin{aligned}
& \exp(-\gamma W_t - b_0 - b_1 \xi_t - b_2 \xi_t^2) \\
&= \exp(-\alpha C_t) + \beta \exp[-\gamma(W_t - C_t)R_F + \hat{\lambda}_0 + \hat{\lambda}_1 \xi_t + \hat{\lambda}_2 \xi_t^2] \\
&= \exp\left(-\frac{\alpha \gamma R_F}{\alpha + \gamma R_F} W_t\right) \exp\left\{-\frac{\alpha}{\alpha + \gamma R_F} \log\left(\frac{\alpha}{\beta \gamma R_F}\right)\right\} \exp\left\{\frac{\alpha}{\alpha + \gamma R_F} (\hat{\lambda}_0 + \hat{\lambda}_1 \xi_t + \hat{\lambda}_2 \xi_t^2)\right\} \\
&\quad + \beta \exp\left(-\frac{\alpha \gamma R_F}{\alpha + \gamma R_F} W_t\right) \exp\left\{\frac{\gamma R_F}{\alpha + \gamma R_F} \log\left(\frac{\alpha}{\beta \gamma R_F}\right)\right\} \exp\left\{\frac{\alpha}{\alpha + \gamma R_F} (\hat{\lambda}_0 + \hat{\lambda}_1 \xi_t + \hat{\lambda}_2 \xi_t^2)\right\} \\
&= \left(\frac{\alpha}{\beta \gamma R_F}\right)^{-\frac{\alpha}{\alpha + \gamma R_F}} \left(1 + \beta \left(\frac{\alpha}{\beta \gamma R_F}\right)\right) \exp\left\{\frac{\alpha}{\alpha + \gamma R_F} \hat{\lambda}_0\right\} \\
&\quad \times \exp\left(-\frac{\alpha \gamma R_F}{\alpha + \gamma R_F} W_t\right) \exp\left\{\frac{\alpha}{\alpha + \gamma R_F} (\hat{\lambda}_1 \xi_t + \hat{\lambda}_2 \xi_t^2)\right\} \tag{72}
\end{aligned}$$

Comparing coefficients, we get

$$\gamma = \frac{\alpha \gamma R_F}{\alpha + \gamma R_F} \quad \text{i.e.,} \quad \gamma = \alpha \frac{r_F}{R_F} \tag{73}$$

$$b_1 = -\frac{\alpha}{\alpha + \gamma R_F} \hat{\lambda}_1 \quad \text{i.e.,} \quad b_1 = -\frac{\hat{\lambda}_1}{R_F} \tag{74}$$

$$b_2 = -\frac{\alpha}{\alpha + \gamma R_F} \hat{\lambda}_2 \quad \text{i.e.,} \quad b_2 = -\frac{\hat{\lambda}_2}{R_F} \tag{75}$$

and one can solve along similar lines for  $b_0$ .

**Asset pricing.** Having solved for these coefficients, we can now look at asset pricing. Rewrite equation (66) as

$$E_t(R_{t+1}) = \gamma (\Gamma + a_1 a_1' \omega^2) y_t + \omega^2 b_1 a_1 + 2\omega^2 \hat{\xi}_t b_2 a_1 \tag{76}$$

**Case 1.**  $\xi_t$  is a non-zero constant,  $\hat{\xi}_t = \mu$ . It follows that  $\phi = 0$ ,  $a_1 = \frac{\gamma}{R_F} (\Gamma + a_1 a_1' \omega^2) \frac{\theta}{1-\theta} \delta$ ,  $b_1 = -\frac{1}{R_F} \lambda_1 = -\gamma' a_1$ , and  $b_2 = -\frac{1}{\gamma} \frac{\theta}{1-\theta} \delta' a_1$ . Furthermore,  $\text{Cov}(R_{t+1}, E_{t+1}[R_{M,t+2}]) = -a_1 a_1' \iota R_F \omega^2$ ,

$\text{Cov}(R_{t+1}, \mathbf{E}_{t+1}[R_{\delta,t+2}]) = -a_1 a_1' \delta R_F \omega^2$  and so (76) yields:

$$\begin{aligned} \mathbf{E}_t(R_{t+1}) &= \gamma \text{Cov}_t(R_{t+1}, R_{A,t+1}) + \frac{\gamma}{R_F} \text{Cov}_t(R_{t+1}, \mathbf{E}_{t+1}R_{M,t+2}) \\ &\quad + \frac{2\theta\mu^2}{\gamma(1-\theta)R_F} \text{Cov}_t(R_{t+1}, \mathbf{E}_{t+1}R_{\delta,t+2}) \end{aligned} \quad (77)$$

where  $\gamma = \alpha \frac{r_F}{R_F}$ ,  $\text{Cov}_t(R_{t+1}, R_{A,t+1}) = \text{Cov}_t(R_{t+1}, R_{M,t+1}) - \xi_t \left( \frac{\theta}{1-\theta} \right) \text{Cov}_t(R_{t+1}, R_{\delta,t+1})$ ,  $R_A$  is the return on arbitrageur's investment portfolio, and  $R_\delta$  is the return on long-short portfolio driven by demands of sentiment investors.

We can rewrite (77) with two terms only:

$$\begin{aligned} \mathbf{E}_t(R_{t+1}) &= \gamma \text{Cov}_t(R_{t+1}, R_{A,t+1}) + \psi \text{Cov}_t(R_{t+1}, \xi_{t+1}) \\ &= \gamma \text{Cov}_t(R_{t+1}, R_{A,t+1}) + \hat{\psi} \text{Cov}_t(R_{t+1}, \mathbf{E}_{t+1}R_{M,t+2}) \end{aligned}$$

where  $\psi = \omega^2 (b_1 + 2\mu^2 b_2)$ ,  $\hat{\psi} = -\frac{\psi}{a_1' R_F \omega^2}$ .

Taking expectations of both sides gives

$$\begin{aligned} \mathbf{E}(R_{t+1}) &= \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - \mathbf{E}_t R_{A,t+1}) + \psi \text{Cov}(R_{t+1}, \xi_{t+1}) \\ &= \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - \mathbf{E}_t R_{A,t+1}) + \hat{\psi} \text{Cov}(R_{t+1}, \mathbf{E}_{t+1}R_{M,t+2}) \end{aligned} \quad (78)$$

where

$$\begin{aligned} \text{Cov}(R_{t+1}, R_{A,t+1} - \mathbf{E}_t R_{A,t+1}) &= \text{Cov}(R_{t+1}, R_{M,t+1} - \mathbf{E}_t R_{M,t+1}) \\ &\quad - \mu \left( \frac{\theta}{1-\theta} \right) \text{Cov}(R_{t+1}, R_{\delta,t+1} - \mathbf{E}_t R_{\delta,t+1}) \end{aligned}$$

Equation (78) is convenient for empirical estimation given that we have an empirical proxy for the sentiment investor flow vector  $\xi_t$ .

**Case 2.**  $\xi_t$  is zero,  $\hat{\xi}_t = 0$ . It follows that  $\mu = \phi = 0$ ,

$$E(R_{t+1}) = \gamma \text{Cov}(R_{t+1}, R_{M,t+1} - E_t R_{M,t+1}) + \frac{\gamma}{R_F} \text{Cov}(R_{t+1}, E_{t+1} R_{M,t+2})$$

Thus, we get an ICAPM similar to Campbell (1993, equation (23)). The degree of presence of sentiment traders does not show up directly, but it is indirectly in  $\text{Cov}(R_{t+1}, E_{t+1}[R_{M,t+2}])$ , because as  $\theta$  goes to zero, this covariance shrinks to zero. Alternatively, note that  $\text{Cov}(D_{t+1}, R_{M,t+1} - E_t[R_{M,t+1}]) = \gamma \Gamma \iota$  and so we can write

$$E[R_{t+1}] = \gamma \text{Cov}(D_{t+1}, R_{M,t+1} - E_t[R_{M,t+1}]) \quad (79)$$

This is a bad beta, good beta specification as in Campbell and Vuolteenaho (2004), but here with a zero risk premium for the “good” beta, i.e., the discount rate beta.

**Case 3.**  $\xi_t$  is  $\mathbf{AR}(1)$ ,  $\hat{\xi}_t = \mu + \phi \xi_t$ . Similarly to Case 1, we can derive the following equation

$$\begin{aligned} E_t(R_{t+1}) &= \gamma \text{Cov}_t(R_{t+1}, R_{A,t+1}) + \psi_t \text{Cov}_t(R_{t+1}, \xi_{t+1}) \\ &= \gamma \text{Cov}_t(R_{t+1}, R_{A,t+1}) + \hat{\psi}_t \text{Cov}_t(R_{t+1}, E_{t+1} R_{M,t+2}) \end{aligned}$$

where  $\psi_t = \omega^2 (b_1 + 2\hat{\xi}_t^2 b_2)$ ,  $\hat{\psi}_t = -\frac{\psi_t}{\iota' a_1 (R_F - \phi) \omega^2}$ .

Note that in this case the price of discount rate risk is time-varying. Unconditionally we get:

$$\begin{aligned} E(R_{t+1}) &= \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - E_t R_{A,t+1}) + \bar{\psi} \text{Cov}(R_{t+1}, \xi_{t+1} - E_t \xi_{t+1}) \\ &= \gamma \text{Cov}(R_{t+1}, R_{A,t+1} - E_t R_{A,t+1}) + \tilde{\psi} \text{Cov}(R_{t+1}, E_{t+1} R_{M,t+2} - E_t R_{M,t+2}) \end{aligned}$$

where  $\bar{\psi} = \omega^2 (b_1 + 2b_2 E[\hat{\xi}_t^2])$ ,  $\tilde{\psi} = -\frac{\bar{\psi}}{\iota' a_1 (R_F - \phi) \omega^2}$ .