The Impact of Competition on Prices with Numerous Firms*

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Abstract

We show how extreme value theory (EVT) can be a useful tool in basic price theory. We derive a widely-applicable formula relating equilibrium prices to the level of competition in a variety of models. When the number of firms is large, markups are proportional to $1/\left(nF'\left[F^{-1}\left(1-1/n\right)\right]\right)$, where F is the random utility noise distribution, and n is the number of firms. This implies that the crucial element for predictions about prices is the noise distribution, in a manner that is independent of the many other details. The elasticity of the markup with respect to the number of firms is shown to be the EVT tail exponent of the distribution for preference shocks and in many cases is quite insensitive to the number of firms. For example, for the Gaussian case, asymptotic markups are proportional to $1/\sqrt{\ln n}$, implying a zero asymptotic elasticity of the markup with respect to the number of firms. Thus competition only exerts weak pressure on prices. Besides this basic price-theoretic issue, we consider applications to behavioral economics (as competition does not correct the price effects of the noise perceived by consumers, and increases the "noise" supplied by firms) and macroeconomics (where we obtain endogenous markups).

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1 Introduction

It is natural to assume that economic agents make choices that reflect both deterministic factors and noise (Luce 1959, McFadden 1981). The presence of noise is motivated by two mutually compatible micro-foundations: Noise reflects true preference variation unobserved by the econometrician and may reflect evaluation errors by the consumer. Noise is routinely included in models of consumer choice. To what extent specific results depend on the choice of the noise distribution remains an open question. Drawing from extreme value theory (EVT), we develop tools to analyze the impact of the noise on equilibrium prices in three important cases of random utility models: Hart (1985), Perloff and Salop (1985), and Sattinger (1984). Expressions for equilibrium markups have previously been derived for these models. The existing markup formulae, however, include integrals that are generally analytically intractable. Only for a few specific distributions are closed-form solutions available. Drawing from extreme value theory, the current paper solves this tractability problem and provides a simple, useful formula for markups. Our analysis also reveals important robust features about markups and relates these markups to limit pricing, i.e. Bertrand competition with heterogeneous firms.

Previous analysis of random-utility models has focused on a small number of special cases in which markups turn out to be either unresponsive to competition or highly responsive to competition. For instance, consider the Perloff-Salop model and assume that noise has an exponential density or a logit (i.e. Gumbel) density. In this case, markups converge to a strictly positive value as n, the number of competing firms, goes to infinity. Hence, asymptotic markups have zero elasticity with respect to n (Perloff and Salop 1985, Anderson et al. 1992). By contrast, when noise is uniformly distributed, markups are proportional to $1/n$, so markups have a unit elasticity and hence a strong negative relationship with n (Perloff and Salop 1985).

All three of these illustrative distributions — exponential, logit, and uniform — are appealing for their analytic tractability rather than their realism. In comparison to the Gaussian distribution, the exponential and logit cases have relatively fat tails while the uniform case has no tails. We like to know how prices respond to competition when the noise follows more general distributions, particularly distributions that are considered to be empirically realistic.

We use extreme value theory (abbreviated as EVT) to analyze general noise distributions. In each of the random-utility models mentioned above we show that markups are asymptotically proportional to $1/\left(nF'\left[F^{-1}\left(1-1/n\right)\right]\right)$, where F is the distribution function for noise.

Moreover, we show that this markup turns out to be almost equivalent to the markup obtained under limit pricing. The markup is asymptotically proportional (and often equal) to the expected gap between the highest draw and second highest draw in a sample of n draws. These results hold for virtually all commonly-used noise distributions.

This "detail-independence" for the value of the markup is surprising. Each of the Hart, Perloff and Salop, and Sattinger models differ in a host of specifications.[1] Yet, the models lead asymptotically to the same value of the markup (up to a scaling constant).

We pay particular attention to the Gaussian case because it is a leading approximation of natural phenomena. For the Gaussian case we show that asymptotic markups are proportional to $1/\sqrt{\ln n}$, where n is the number of competing firms. This formula implies that mark-ups fall slowly as n rises. Moreover, the elasticity of the markup with respect to n converges to 0. Hence, the Gaussian case behaves much more like the exponential and logit cases than like the uniform case. Rising competition in an environment with a Gaussian noise distribution only produces weak downward pressure on prices.

Another insight that arises from our analysis is that for "heavy-tailed" distributions (including subexponential distributions like the log-normal and power-law distributions like the Pareto distribution), mark-ups *increase* as the number of competing firms increase.

More generally, if n is large we find that the elasticity of the markup with respect to the number of firms equals the EVT tail exponent of the distribution; a magnitude that is easy to calculate. These results demonstrate an intimate relationship between the economic logic of competition in large economies and the mathematics of EVT. We conclude that markups are robust in large economies, since markups have a zero asymptotic elasticity for many empirically realistic noise distributions. Only noise distributions with extremely thin tails (like the uniform distribution) and very heavy tails (like the Pareto), have markups with elasticities different from zero.

Moving away from the specifics of random demand models, the tools that we develop allow us to calculate the asymptotic behavior of integrals for a general class of functions $h(x)$, of the form

$$\int h(x)\, f^k(x) F(x)^n dx dx, \tag{1}$$

where $k \geq 1$, which appear in a very large class of economic situations, some of which we will review later. For instance, these cover the expected value of a function of the maximum of

[1] For instance, in the Perloff-Salop model consumers need to buy one unit of the good. In the Sattinger model, they allocate a fixed dollar amount to the good. The Hart model does not impose either constraint.

n random variables, or the gap between the maximum and the second largest value of those random variables. Using EVT we are able to derive robust approximations of this integral for a large n.

Extremes and EVT-related techniques are important in many parts of economics. Most well known is McFadden's (1981) foundation of the logit specification for discrete choice with random demand as set out in Luce (1959) by means of the Gumbel extreme value distribution. This specification has been used widely in the analysis of product differentiation, regional economics, geography and trade, see e.g. Anderson et al. (1992), Dagsvik (1994), Dagsvik and Karlstrom (2005), Ibragimov and Walden (2010), and Armstrong (2012). In international trade, Eaton and Kortum (2002) and Bernard et al. (2003), used the aggregation properties of the Fréchet extreme value distribution to analyze international trade at the producer level (see also Chaney 2008). Acemoglu, Chernozhukov and Yildiz (2006, 2009) show the importance of the tail specification for learning. In macroeconomics, Gabaix (2011) shows the importance of tails of the firm size distribution to understand macroeconomic fluctuations; these extremes in the size distribution of firms can emerge from random growth (Gabaix 1999, Luttmer 2006) or the network structure of the buyer-supplier relations amongst firms (Acemoglu et al. (2012)). Jones (2005) models the distribution of innovative ideas and analyzes the impact of this distribution on the bias of technical change. In applied finance, much use has been made of EVT in risk management and systemic risk analysis; see e.g. Jansen and De Vries (1991) for an early contribution and Ibragimov, Jaffee and Walden (2009, 2011) for recent examples. In an application to the economics of health care, Garber, Jones and Romer (2006) discuss how the distribution of benefits generated by medical innovations relates to the optimal scheme for incentivizing innovation. More recent are applications in the theory of auctions, see Hong and Shum (2004). We suspect that the techniques we develop here could be useful in the above setups. As an example, Gabaix and Landier (2008) use some of this paper's results to analyze the upper tail of the distribution of CEO talents.

In summary, our paper makes three contributions. The first is methodological: We draw from extreme value theory to develop a core mathematical result and asymptotic approximation that is useful in a variety of economic contexts where extremes and tails matter (e.g. price theory, auctions).

The second contribution is to basic price theory: We show which features are important for prices, and which are not, in markets with many suppliers. Specifically, the tail of the noise distribution (as captured by the tail exponent) is the crucial determinant of prices, whereas the details of the demand-side modelling (e.g. Perloff-Salop versus Sattinger) do not matter much asymptotically, or not at all. We analyze to what extent competition puts pressure on prices as in the analysis of limit pricing, i.e. Bertrand competition with heterogeneous firms and auctions. As many common noise distributions have a tail index of zero, our results suggest that in many real-world contexts, competition has have little effect on prices, once we go beyond a very small number of firms

The third contribution is to understanding of endogenous markups in behavioral economics and macroeconomics. In a behavioral economics context, if we interpret the magnitude of the noise as "confusion" supplied by firms, we find that competition, for a given amount of noise, does little to dampen markups; and when confusion is endogenous, competition increases the amount of noise supplied by firms, in a way we make quantitatively precise. In a macroeconomic equilibrium context we study how the endogenous preference shocks provide an interesting contrast to the usual exogenously postulated markup shocks, with different predictions relating the type of shocks and the number of differentiated goods.

The paper proceeds as follows. Section 2 provides a summary of the economic models we analyze. Section 3 presents our core mathematical result. Section 4 develops the implications for the three models we consider. Section 5 presents extensions and applications of our core models, and demonstrates that our mathematical techniques and economic insights are robust to such extensions. Section 6 concludes. The proofs are relegated to the Appendices.

2 Economic Models

In the following three subsections we introduce three models of monopolistic competition under oligopoly. The three models differ in the specification of the consumer's utility function, but all share the feature that the representative consumer's preference for a given firm's good is represented by a random taste shock.

Each model features a representative consumer, and an exogenously specified number n of firms. The timing of each model is as follows:

1. Firms simultaneously set prices;

2. Random taste shocks are realized;

3. Consumers make purchase decisions;

4. Profits are realized.

The key economic object of interest is the price markup in a symmetric equilibrium, which we derive by solving the first-order condition for each firm's profit maximization problem. The firm i's profit function is given by

$$\pi_i = (p_i - c)D\left(p_1, ..., p_n; i\right)$$

where $D\left(p_1, ..., p_n; i\right)$ is the demand function for firm i given the price vector $\left(p_1, ..., p_n\right)$ of the n goods, and where c is the marginal cost of production. The first order condition for profit maximization implies the following equilibrium markup in a symmetric equilibrium

$$p - c = -\frac{D\left(p, p; n\right)}{D_1\left(p, p; n\right)}. \tag{2}$$

Here p is the symmetric equilibrium price, $D\left(p, p'; n\right)$ denotes the demand function for a firm that sets price p when there are n goods and all other firms set price p', and $D_1\left(p, p'; n\right) \equiv \partial D\left(p, p'; n\right)/\partial p$. Denote the markup $p - c$ in a symmetric equilibrium with n firms as μ_n.

The evaluation of this markup expression for the three models of monopolistic competition gives rise to different integral problems, which we address in Section 4 (using tools developed in Section 3.) We briefly discuss the features of each model, and list the resulting expressions for price markups in terms of integrals. The derivation of these expressions from the first-order conditions is relegated to Appendix B, and we verify the second-order conditions in Appendix D.

2.1 Perloff-Salop (1985): Linear Random Utility

In the Perloff-Salop model, a particular consumer can purchase exactly one unit of the differentiated good. The consumer receives net utility $X_i - p_i$ by purchasing the good of firm i, where X_i is a random taste shock, i.i.d. across firms and consumers, and p_i is the price charged by firm i. Thus the consumer chooses to purchase the good that maximizes $X_i - p_i$. In a symmetric-price equilibrium, the demand function of firm i is the probability that the consumer's surplus at firm i, $X_i - p_i$, exceeds the consumer's surplus at all other firms,

$$D\left(p_1, ..., p_n; i\right) = \mathbb{P}(X_i - p_i \geq \max_{j \neq i}\left\{X_j - p_j\right\}) = \mathbb{P}(X_i \geq \max_{j \neq i}\left\{X_j\right\}). \tag{3}$$

Let M_n denote the max $\{X_1, ..., X_n\}$. Evaluation of (2) gives the following markup expression for the symmetric equilibrium of the Perloff-Salop model:

$$\mu_n^{PS} = \frac{1 - F(M_{n-1})}{-f(M_{n-1})} = \frac{1}{n\,(n-1)\int f^2(x)F^{n-2}(x)\,dx}. \tag{4}$$

Here F is the distribution function and f is the corresponding density of X_i.

2.2 Sattinger (1984): Multiplicative Random Utility

Sattinger (1984) analyzes the case of multiplicative random demand. There are two types of goods. One is a composite good purchased from an industry with homogenous output, and the other is obtained from a monopolistically competitive (MC) market with n differentiated producers. The consumer has utility function

$$U = Z^{1-\theta}[\sum_{i=1}^{n} A_i Q_i]^{\theta}, \tag{5}$$

where Z is the quantity of the composite good, $A_i = \exp(X_i)$ is the random taste shock, and Q_i is the quantity consumed of good i. The X_i are i.i.d. across consumers and firms and have distribution function F. The consumer faces the budget constraint $y = qZ + \sum_i p_i Q_i$ where y is the consumer's endowment, q is the price of the composite good and p_i is the price of good i.

One shows that the demand function of firm i is

$$D\,(p_1, ..., p_n; i) = \frac{\theta y}{p_i}\mathbb{P}\left(\frac{e^{X_i}}{p_i} \geq \max_{j \neq i} \frac{e^{X_j}}{p_j}\right). \tag{6}$$

Evaluation of (2), see Appendix B, gives the following markup expression for the symmetric equilibrium of the Sattinger model:

$$\mu_n^{Satt} = \frac{c}{n\,(n-1)\int f^2(x)F^{n-2}(x)\,dx}. \tag{7}$$

Note that the markup expressions for the Perloff-Salop and Sattinger models are almost identical, except for the marginal costs factor c from (2). Thus

$$\mu_n^{Satt} = c \cdot \mu_n^{PS}. \tag{8}$$

2.3 Hart (1985): Power Utility

Hart (1985) analyzes a model of monopolistic competition where both the quantity and the dollar amount spent depend on the prospective utility of the good purchased. In comparison, in the Perloff-Salop model from section 2.1, the quantity demanded is fixed; while in the Sattinger model (section 2.2), dollar expenditure is also fixed. The Hart model thus allows us to study the impact of competition in a slightly richer economic context than the previous models of monopolistic competition. Interestingly, with a particular choice of noise distribution, the Hart (1985) model generates the same demand function, see our Proposition 5, as the traditional Dixit-Stiglitz (1979) model, but within a random utility framework.

In Hart's model, the consumer's payoff function is:

$$U = \sum_{i=1}^{n} \left[\frac{\psi + 1}{\psi} \left(A_i Q_i \right)^{\psi/(\psi+1)} - p_i Q_i \right]. \tag{9}$$

where i is the index of the consumed good, $A_i = e^{X_i}$ is the associated random taste shock, Q_i is the consumed quantity and p_i is the unit price of good i. The X_i are i.i.d. across consumers and firms and have distribution function F; note that this specification allows for negative realizations of X_i. Hart shows that the demand function for firm i is

$$D\left(p_1, ..., p_n; i\right) = \mathbb{E}\left[\frac{e^{\psi X_i}}{p_i^{1+\psi}} 1_{\left\{ e^{X_i}/p_i \geq \max_{j \neq i} e^{X_j}/p_j \right\}} \right]. \tag{10}$$

Evaluation of (2), see Appendix B, gives the following markup expression for the symmetric equilibrium of the Hart model:

$$\mu_n^{Hart} = c \left(\psi + (n-1) \frac{\int e^{\psi x} f^2(x) F^{n-2}(x) \, dx}{\int e^{\psi x} f(x) F^{n-1}(x) \, dx} \right)^{-1}. \tag{11}$$

Note that, by comparing (6) with (10) and (7) with (11):

$$D^{Hart}\left(p_1, ..., p_n; i\right)\Big|_{\psi=0} = D^{Satt}\left(p_1, ..., p_n; i\right) / \left(\theta y\right),$$
$$\mu_n^{Hart}\Big|_{\psi=0} = \mu_n^{Satt},$$

where D^{Hart} is the demand function in the Hart model and D^{Satt} is the demand function in the Sattinger model. In the special case $\psi = 0$, the Hart model generates the same demand functions and markups as the Sattinger model.

3 Extreme Value Theory and Related Results

Solving for the symmetric equilibrium outcome in the models discussed above, for distribution function F, requires the evaluation of integrals of the form

$$\int x^j e^{\psi x} f^k(x) F(x)^{n-l} dx \tag{12}$$

where $k, l \geq 1$ and $j, \psi \geq 0$. For large n, such integrals mainly depend on the tail of the distribution F, which suggests that techniques from Extreme Value Theory (EVT) may be applied. (See de Haan (1970), Resnick (1987), and Embrechts *et al.* (1997) for an introduction to EVT.)

This section develops the mathematical tools that we will use to asymptotically evaluate (12). Section 3.1 states a number of technical assumptions and introduces notation. Section 3.2 presents the main mathematical results. Proofs are relegated to Appendix A.

3.1 Preliminaries

First, we introduce a few useful objects. Define

$$M_n \equiv \max_{i=1,\ldots,n} X_i,$$

to be the maximum of n independent random variables X_i with distribution F. Also, define the counter-cumulative distribution function $\overline{F}(x) \equiv 1 - F(x)$.[2] We are particularly interested in the connection between M_n and $\overline{F}^{-1}(1/n)$; informally in analogy with the empirical distribution function, one may think of $\overline{F}^{-1}(1/n)$ as the "typical" value of M_n. In fact, the key to our analysis is to formalize this relationship between $\overline{F}^{-1}(1/n)$ and M_n for large n.

Our analysis is restricted to what we call *well-behaved* distributions:

Definition 1 *Let F be a distribution function with support on (w_l, w_u). Let $f = F'$ be the corresponding density function. We say F is well-behaved iff f is differentiable in a*

[2] Strictly speaking, we abuse notation in cases where F is not strictly increasing by using $\overline{F}^{-1}(t)$ to denote $\overline{F}^{\leftarrow}(t) = F^{\leftarrow}(1-t)$, where $F^{\leftarrow}(t) = \inf\{x \in (w_l, w_u) : F(x) \geq t\}$ is the generalized inverse of F (Embrechts et al. 1997, p.130). This is for expositional convenience; our results hold with the generalized inverse as well.

neighborhood of w_u, $\lim_{x\to w_u} \overline{F}/f = a$ *exists with* $a \in [0,\infty]$, *and*

$$\gamma = \lim_{x\to w_u} d\left(\frac{\overline{F}(x)}{f(x)}\right)/dx \tag{13}$$

exists and is finite. The γ *is called the tail index of* F.

Being well-behaved imposes a restriction on the right tail of F. The case $\gamma < 0$ consists of thin-tailed distributions with right-bounded support such as the uniform distribution. The case $\gamma = 0$ consists of distributions with tails of intermediate thickness. A wide range of economically interesting distributions fall within this domain, ranging from the relatively thin-tailed Gaussian distribution to the relatively thick-tailed lognormal distribution, as well as other distributions in between, such as the exponential distribution. The case $\gamma > 0$ consists of fat-tailed distributions such as Pareto's power-law and the Fréchet distributions.

Being well-behaved in the sense of Definition 1 is not a particularly strong restriction. It is satisfied by most distributions of interest, and is easy to verify. Condition (13) is well-known in the EVT literature as a second-order von Mises condition; for example, (13) is a slightly stronger version of the assumption found in Pickands (1986). Table 1 lists several well-behaved densities f, the tail index γ of the associated distribution F, and corresponding values for $\overline{F}^{-1}(1/n)$ and $nf\left(\overline{F}^{-1}(1/n)\right)$ (which will be useful for our analysis). Note that tail fatness is increasing in γ.

Definition 1 ensures that the right tail of F behaves appropriately. To ensure that the integral (12) does not diverge, we also impose some restriction on the rest of F. The following notation will simplify the exposition of our results.

Definition 2 *Let* $j : \mathbb{R} \to \mathbb{R}$ *have support on* (w_l, w_u). *The function* $j(x)$ *is* $[w_l, w_u)$*-integrable iff*

$$\int_{w_l}^{w} |j(x)|\, dx < \infty$$

for all $w \in (w_l, w_u)$.

For example, in Theorem 2 we require that f^2 be $[w_l, w_u)$-integrable. Verification of this condition is typically straightforward; it is useful to note, for example, that $f^2(x)$ is $[w_l, w_u)$-integrable if $f = F'$ is uniformly bounded.

Finally, the following definition of regular variation will be useful.

Table 1: **Properties of Common Densities**

The noise has density f, and tail index γ given by (13); $\overline{F}^{-1}(1/n)$ is the approximate location of the maximum of n samples of the noise. Distributions are listed in order of increasing tail fatness whenever possible.

	f	γ	$nf\left(\overline{F}^{-1}(1/n)\right)$	$\overline{F}^{-1}(1/n)$
Uniform	$1, \quad x \in [-1,0]$	-1	n	$-\frac{1}{n}$
Bounded Power Law	$\alpha(-x)^{\alpha-1}, \alpha \geq 1, x \in [-1,0]$	$-1/\alpha$	$\alpha n^{1/\alpha}$	$-n^{-1/\alpha}$
Weibull	$\alpha(-x)^{\alpha-1}e^{-(-x)^{\alpha}}, \alpha \geq 1, x < 0$	$-1/\alpha$	$\alpha n^{1/\alpha}$	$\sim -n^{-1/\alpha}$
Gaussian	$(2\pi)^{-1/2}e^{-x^2/2}$	0	$\sim \sqrt{2\ln n}$	$\sim \sqrt{2\ln n}$
Rootzen Class	$\kappa\lambda\phi x^{a+\phi-1}e^{-x^{\phi}}, \quad x > 0, \phi > 1$	0	$\sim \phi\lambda^{1/\phi}(\ln n)^{1-1/\phi}$	$\sim (\ln n)^{1/\phi}$
Gumbel	$\exp(-e^{-x} - x)$	0	~ 1	$\sim \ln n$
Exponential	$e^{-x}, \quad x > 0$	0	1	$\ln n$
Log-normal	$(2\pi)^{-1/2}x^{-1}e^{-\left(\log^2 x\right)/2}, x > 0$	0	$\sim \frac{\sqrt{2\ln n}}{\overline{F}^{-1}(1/n)}$	$\sim e^{\sqrt{2\ln n}}$
Power law	$\alpha x^{-\alpha-1}, \quad \alpha > 0, x \geq 1$	$1/\alpha$	$\alpha n^{-1/\alpha}$	$n^{1/\alpha}$
Fréchet	$\alpha x^{-\alpha-1}e^{-x^{-\alpha}}, \quad \alpha > 0, x \geq 0$	$1/\alpha$	$\alpha n^{-1/\alpha}$	$\sim n^{1/\alpha}$

Definition 3 *A function $h : \mathbb{R}^+ \to \mathbb{R}$ is regularly varying at ∞ with index ρ if h is strictly positive in a neighborhood of ∞, and*

$$\forall \lambda > 0, \lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \lambda^{\rho}. \tag{14}$$

We indicate this by writing $h \in RV_{\rho}^{\infty}$.

Analogously, we say that $h : \mathbb{R}^+ \to \mathbb{R}$ is regularly varying at zero with index ρ if, $\forall \lambda > 0, \lim_{x\to 0} h(\lambda x)/h(x) = \lambda^{\rho}$, and denote this by $h \in RV_{\rho}^{0}$. Intuitively, a regularly varying function $h(x)$ with index ρ behaves like x^{ρ} as x goes to the appropriate limit. For instance, x^{ρ} and $x^{\rho}|\ln x|$ are regularly varying (with index ρ) at both 0 and ∞. Much of our analysis will require the concept of regular variation; specifically, we will require that certain transformations of the noise distribution F be regularly varying. In the case $\rho = 0$, we say that h is *slowly varying* (for example $\ln x$ varies slowly at infinity and zero).

Finally, following the notation of Definition 1, define

$$w_l = \inf\{x : F(x) > 0\} \quad \text{and} \quad w_u = \sup\{x : F(x) < 1\} \tag{15}$$

to be the lower and upper bounds of the support of F, respectively.

3.2 Core Mathematical Result

Our core mathematical result documents an asymptotic relationship between M_n and $\overline{F}^{-1}(1/n)$.

Theorem 1 *Let F be a differentiable CDF with support on (w_l, w_u) and $f = F'$, and assume that F is strictly increasing in a left neighborhood of w_u. Let $G : (w_l, w_u) \to \mathbb{R}$ be a strictly positive function in a left neighborhood of w_u. Suppose that $\widehat{G}(t) \equiv G\left(\overline{F}^{-1}(t)\right) \in RV_\rho^0$ with $\rho > -1$, and that $\left|\widehat{G}(t)\right|$ is integrable on $t \in (\bar{t}, 1)$ for all $\bar{t} \in (0,1)$ (or, equivalently, $G(x) f(x)$ is $[w_l, w_u)$-integrable in the sense of definition 2). Then, for $n \to \infty$*

$$\mathbb{E}[G(M_n)] = \int_{w_l}^{w_u} nG(x) f(x) F(x)^{n-1} dx \sim G\left(\overline{F}^{-1}(1/n)\right)\Gamma(\rho+1) \tag{16}$$

where M_n is the largest realization of n i.i.d. random variables with CDF F.

The intuition for equation (16) is as follows. By definition of M_n, if X is distributed as F and if M_n and X are independent, then $\mathbb{P}[X > M_n] = 1/(n+1)$; that is, $\mathbb{E}\left[\overline{F}(M_n)\right] = 1/(n+1) \approx 1/n$. Consequently, we might conjecture (via heroic commutation of the expectations operator) that

$$\mathbb{E}[M_n] \approx \overline{F}^{-1}\left(\frac{1}{n}\right) \tag{17}$$

and that $\mathbb{E}[G(M_n)] \approx G(\mathbb{E}[M_n]) \approx G\left(\overline{F}^{-1}(1/n)\right)$.

It turns out that this heuristic argument gives us the correct approximation, up to a correction factor $\Gamma(\rho+1)$. [3]

We next present an intermediate result that is technically undemanding but will allow us to apply Theorem 1 to expressions of the form (12).

[3] To understand the correction factor, start with the linear case $G(x) = x$, in which case the theorem gives $E[M_n] \sim \overline{F}^{-1}(1/n)\Gamma(-\gamma+1)$. Then the correction factor arises because the distribution of the maximum is $F^n(x)$, not $F(x)$. For distributions with an exponential type tail, $\gamma = 0$ and no correction is required. For distributions with a power type tail and finite mean, $\gamma \in (0,1)$, an upward correction is needed. To provide some intuition for this, consider the $\log(-\log \mathbb{P}\{M_n \leq t\})$, and where the distribution F is either Gumbel or Fréchet, see Table 1. In case of the Gumbel one finds $\log n - t$, while the Fréchet gives $\log n - \alpha \log t$. Take n and t large. In the Gumbel case n plays a minor role, while in the case of the distribution n and t are of similar order of magnitude, so that n affects the distribution and its moments. More generally, if $G(x)$ is not linear, the tail behavior of $G(x)$ interacts with the tail behavior of $F(x)$. Both functions then determine ρ in the correction factor as indicated in the theorem. For example, take $G(x) = x^m$ and $F(x) = 1 - x^{-\alpha}$, $m < \alpha$, then $\mathbb{E}[(M_n)^m] \simeq n^{m/\alpha}\Gamma(1 - m/\alpha)$.

Lemma 1 *Let F be well-behaved with tail index γ. Then*

1. $f\left(\overline{F}^{-1}(t)\right) \in RV^0_{\gamma+1}$.

2. *If* $w_u = \infty$*, then* $\overline{F}^{-1}(t) \in RV^0_{-\gamma}$*. If* $w_u < \infty$*, then* $w_u - \overline{F}^{-1}(t) \in RV^0_{-\gamma}$.

3. *If a is finite, then* $e^{\overline{F}^{-1}(t)} \in RV^0_{-a}$.

Lemma 1 ensures that when F is well-behaved, (12) satisfies the conditions imposed in Theorem 1 for a wide range of parameter values. The following proposition is then an immediate implication of Theorem 1 and Lemma 1.

Proposition 1 *Let F be well behaved with tail index γ. Let $j, \psi \geq 0$, $k \geq 1$ and let $x^j e^{\psi x} f^k(x)$ be $[w_l, w_u)$-integrable. If $j > 0$, assume that $w_u > 0$. If $\psi = 0$, we can treat $\psi a = 0$ in the following expressions. If $(k-j-1)\gamma - \psi a + k > 0$, then as $n \to \infty$,*

$$\int_{w_l}^{w_u} x^j e^{\psi x} f^k(x) F(x)^{n-l} dx$$
$$\sim \begin{cases} n^{-1}\left(\overline{F}^{-1}(1/n)\right)^j e^{\psi \overline{F}^{-1}(1/n)} f^{k-1}\left(\overline{F}^{-1}(1/n)\right) \Gamma\left((k-j-1)\gamma - \psi a + k\right) : w_u = \infty \\ n^{-1} w_u^j e^{\psi w_u} f^{k-1}\left(\overline{F}^{-1}(1/n)\right) \Gamma\left((k-1)\gamma + k\right) : w_u < \infty \end{cases}.$$

Proposition 1 allows us to approximate (12) for well-behaved distributions[4]. The parameter restriction $(k-j-1)\gamma - \psi a + k > 0$ is necessary to ensure that (12) does not diverge. For our purposes, this restriction is rather mild, as we will see when we apply Proposition 1 in the subsequent sections. One notable exception is that when $\psi > 0$, we cannot analyze heavy-tailed distributions (which have fatter-than-exponential tails) such as the lognormal distribution; for these distributions, $a = \infty$.

Here we define a distribution to be heavy-tailed if $e^{\lambda x}\overline{F}(x) \to \infty$ as $x \to \infty$ for all $\lambda > 0$. To see why $a = \infty$ in this case, note that $\lim_{x\to\infty} \overline{F}(x)/f(x) = \infty$ implies $-\frac{d}{dx}\log \overline{F}(x) = o(1)$ as $x \to \infty$, so $-\log \overline{F}(x) = o(x)$ and $e^{-\lambda x} = o\left(\overline{F}(x)\right)$ for all λ.

4 Asymptotic Markups

This section applies our newly-developed mathematical tools to the integral problems raised in Section 2. Section 4.1 derives asymptotic expressions for the PS, Sattinger, and Hart markups

[4]For antecedents to this result, see Resnick (1971) or Maller and Resnick (1984).

and price elasticities. Section 4.2 discusses the implications of these findings: specifically, how the choice of noise distribution determines the relationship between competition and prices. Section 4.3 briefly discusses some implications for consumer surplus under random demand frameworks.

4.1 Asymptotic Expressions for Markups

Taking Proposition 1 and substituting into (4), (7) and (11), we immediately obtain asymptotic approximations for equilibrium markups for each of the Perloff-Salop, Sattinger and Hart models.

Theorem 2 *Assume that F is well-behaved, and that $f^2(x)$ is $[w_l, w_u)$-integrable. For the Perloff-Salop and Sattinger models, assume that $-1.45 \leq \gamma \leq 0.64$.*[5] *For the Hart model with parameter ψ, assume that $-1 < \gamma \leq 0$; if $\gamma = 0$, we further require that $1 - \psi a > 0$.*
Then the symmetric equilibrium markups in the Perloff-Salop, Sattinger and Hart models are asymptotically

$$\mu_n^{PS} = \mu_n^{Satt}/c \sim \mu_n^{Hart}/c \sim \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)\Gamma(\gamma+2)}. \tag{18}$$

Theorem 2 treats the Hart model for the case where taste shocks have weakly thinner tails than the exponential distribution. There is no such restriction for the Perloff-Salop and Sattinger models; we are able to obtain markup expressions for fat-tailed taste shocks as well. The proof of Theorem 2 is in Appendix C.

Theorem 2 delivers the perhaps unexpected result that the three models generate asymptotically equal (up to a multiplicative constant) markups. Hence, they exhibit a sort of "detail-independence": equilibrium markups do not depend on the details of the model of competition. This logic underlying this phenomenon will be developed in section 4.4. The markup under a limit pricing model of competition is also asymptotic to our markups in Theorem 2. Hence, the key ingredient in the modeling is the specification of the noise distribution, rather than the details of the particular oligopoly model.

The key mathematical objects in Theorem 2, γ and $f\left(\overline{F}^{-1}(1/n)\right)$, are easy to calculate for most distributions of interest. Table 1 lists $nf\left(\overline{F}^{-1}(1/n)\right)$ and γ for commonly used distributions, from which the asymptotic markup may immediately be calculated.

[5] This is the range over which the second order condition holds; the first order condition holds whenever $\gamma > -2$.

The following proposition shows that γ has a concrete economic implication. The tail parameter γ in (13) is the asymptotic elasticity of the markup with respect to the number of firms. In other words, the markup behaves locally as $\mu \sim kn^{\gamma}$. We interpret n as a continuous variable in the expression of the markup.

Proposition 2 *Assume that the conditions in Theorem 2 hold. Further, assume that* $\log F(x) f^2(x)$ *is* $[w_l, w_u)$*-integrable. Then the asymptotic elasticity of the Perloff-Salop, Sattinger and Hart markups with respect to the number of firms* n *is:*

$$\lim_{n\to\infty} \frac{n}{\mu_n}\frac{d\mu_n}{dn} = \gamma.$$

For taste shocks with distributions fatter than the uniform ($\gamma > -1$), Proposition 2 shows that the mark-up falls more slowly than $1/n$. The proof is relegated to Appendix C.

Finally, the following fact, which is easily verified using Lemma 3 part 6, may often be useful to simplify calculations further. As $n \to \infty$,

$$\frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)} \sim \begin{cases} \gamma\overline{F}^{-1}(1/n), & \gamma > 0 \\ -\gamma(w_u - \overline{F}^{-1}(1/n)), & \gamma < 0 \end{cases}.$$

4.2 Applications to Markup and Industry Equilibrium

We discuss the economic implications of the industry equilibrium. We will use μ_n to denote the Perloff-Salop markup (with n firms) while keeping in mind that, in virtue of Theorems 2 and 4, the Hart, Perloff-Salop and Sattinger markups are asymptotically equal up to a constant multiplicative factor. This allows us to unify the discussion for all four models.

To analyze the impact of competition on markups, we examine the equilibrium markup for various noise distributions. Table 2 shows how markups change as competition intensifies.

The distributions in Table 2 are generally presented in increasing order of fatness of the tails. For the uniform distribution, which has the thinnest tails, the markup is proportional to $1/n$. This is the same equilibrium markup generated by the Cournot model. However the uniform cum Cournot case is unrepresentative of the general picture. Table 2 implies that markups scale with n^{γ}. For the distributions reported in Table 2, γ is bounded below by -1, so the uniform distribution is an extreme case.

In the Perloff-Salop and Sattinger cases, for the distributions with the fattest tails, the

Table 2: **Asymptotic Expressions for Markups**

This table lists asymptotic markups (under symmetric equilibrium) for various noise distributions as a function of the number of firms n. The column f describes any parameter restrictions on the density f. Distributions are listed in order of increasing tail fatness. The $\mu^{PS}, \mu^{Satt}, \mu^{Hart}$ are respectively markups under the Perloff-Salop, Sattinger and Hart models. Asymptotic approximations are calculated using Theorem 2 except where the markup can be exactly evaluated. Note that μ is asymptotically equal for all three models for large n. Note that the Hart markup is not defined for distributions fatter than the exponential.

	f	$\mu_n^{PS} = \mu_n^{Satt}/c$	μ_n^{Hart}/c	$\lim_{n\to\infty} \mu_n$
Uniform	$1, \quad x \in [-1, 0]$	$1/n$	$\sim 1/n$	0
Bounded Power Law	$\alpha\,(-x)^{\alpha-1}$ $\alpha > 0, x \in [-1, 0]$	$\frac{\Gamma(1-1/\alpha+n)}{\alpha\Gamma(2-1/\alpha)\Gamma(1+n)} \sim \frac{n^{-1/\alpha}}{\alpha\Gamma(2-1/\alpha)}$	$\sim \frac{n^{-1/\alpha}}{\alpha\Gamma(2-1/\alpha)}$	0
Weibull	$\alpha\,(-x)^{\alpha-1}\, e^{-(-x)^{\alpha}}$ $\alpha \geq 1, x < 0$	$\frac{1}{\alpha\Gamma(2-1/\alpha)} \frac{n^{1-1/\alpha}}{n-1} \sim \frac{n^{-1/\alpha}}{\alpha\Gamma(2-1/\alpha)}$	$\sim \frac{n^{-1/\alpha}}{\alpha\Gamma(2-1/\alpha)}$	0
Gaussian	$(2\pi)^{-1/2}\, e^{-x^2/2}$	$\sim (2\log n)^{-1/2}$		0
Rootzen class, $\phi > 1$	$\kappa\lambda\phi x^{a+\phi-1} e^{-x^{\phi}}$	$\sim \frac{1}{\phi\lambda^{1/\phi}}\,(\log n)^{1/\phi-1}$		0
Gumbel	$\exp(-e^{-x} - x)$	$\frac{n}{n-1}$	~ 1	1
Exponential	$e^{-x}, \quad x > 0$	1		1
Rootzen Gamma $(\tau < 1)$	$\tau x^{\tau-1} e^{-x^{\tau}}$ $x > 0, \tau < 1$	$\sim \frac{1}{\tau}\,(\log n)^{1/\tau-1}$	–	∞
Log-normal	$\frac{\exp(-2^{-1}\log^2 x)}{x\sqrt{2\pi}}$ $x > 0$	$\sim \frac{1}{\sqrt{2\ln n}} e^{\sqrt{2\ln n}}$	–	∞
Power law	$\alpha x^{-\alpha-1}$ $\alpha > 1, x \geq 1$	$\frac{\Gamma(1+1/\alpha+n)}{\alpha\Gamma(2+1/\alpha)\Gamma(1+n)} \sim \frac{n^{1/\alpha}}{\alpha\Gamma(2+1/\alpha)}$	–	∞
Fréchet	$\alpha x^{-\alpha-1} e^{-x^{-\alpha}}$ $\alpha > 1, x \geq 0$	$\frac{1}{\alpha\Gamma(2+1/\alpha)} \frac{n^{1+1/\alpha}}{n-1} \sim \frac{n^{1/\alpha}}{\alpha\Gamma(2+1/\alpha)}$	–	∞

markups paradoxically *rise* as the number of competitors *increases*.[6] Intuitively, for fat-tailed noise, as n increases, the difference between the best draw and the second-best draw, which is proportional to $nf\left(\overline{F}^{-1}(1/n)\right)$, increases with n (see section 4.4 below). However, even though markups rise with n, profits per firm go to zero (keeping market size constant) since firm prices scale with n^{γ} but sales volume per firm is proportional to $1/n$ in the Perloff-Salop case and $1/n^{1+\gamma}$ in the Sattinger case.

This phenomenon whereby prices rise with more intense competition has recently attracted some attention. Chen and Riordan (2008) present a model where markups rise when competition goes from one to two firms. As in our paper, this is because consumers can become less price-sensitive when there are more firms.[7] In the context of a broad analysis of pass-through, Weyl and Fabinger (2012) clarifies this effect by showing that the right-hand side of our mark-up formula (18) falls with n for log-concave distributions, and rises with n for log-convex distributions. The result can also be verified by taking the derivative of our asymptotic formula with respect to n. Weyl and Fabinger show how in the log-concave case pricing strategies are strategic complements, while in the log-convex case, they are strategic substitutes.[8] Their analysis is qualitative and general, or is quantitative and specific.

Thin-tailed distributions (e.g. uniform) and fat-tailed distributions (e.g. power-laws) are the extreme cases in Table 2. Most of the distributional cases imply that competition typically has remarkably *little* impact on markups. For instance with Gaussian noise, the markup μ_n is proportional to $1/\sqrt{\ln n}$, and the elasticity of the markup with respect to n is asymptotically zero. So μ_n converges to zero, but this convergence proceeds at a glacial pace. Indeed, the elasticity of the markup with respect to n converges to zero.

To illustrate the slow convergence, we calculate μ_n when noise is Gaussian for a series of values of n. Table 3 shows that in the models we study and with Gaussian noise, a highly competitive industry with $n = 1,000,000$ firms will retain a third of the markup of a highly concentrated industry with only $n = 10$ competitors. We also compare markups in our monopolistic competition models to those in the Cournot model, which features markups proportional to $1/n$ and a markup elasticity w.r.t. n of -1 (note that this is equal to markups in the Perloff-Salop model with uniformly distributed noise.)

[6]No symmetric price equilibrium can be calculated in the Hart model for these distributions, because each firm would face infinite demand.

[7]See also Bénabou and Gertner (1993), Carlin (2009), Bulow and Klemperer (2002) and Rosenthal (1980) for perverse competitive effects generated by different microfoundations.

[8]For more background, see Bulow, Geanakoplos and Klemperer (1985).

Table 3: **Markups with Gaussian Noise and Uniform Noise**

Markups are calculated for (i) the symmetric equilibrium of the Perloff-Salop model for Gaussian noise and (ii) under Cournot Competition, for various values of the number of firms n. Note that markups in the Sattinger model and the Hart model are asymptotically equal, up to a constant cost factor c, to markups in the Perloff-Salop model. The number of firms in the market is n. Markups are normalized to equal one when $n = 10$.

n	Markup with Gaussian noise	Markup under Cournot Competition
10	1	1
100	0.61	0.1
$1,000$	0.47	0.01
$10,000$	0.40	0.001
$100,000$	0.35	0.0001
$1,000,000$	0.32	0.00001

More generally, in cases with moderate fatness, such as the Gumbel (i.e. logit), exponential, and log-normal densities, the markup again shows little (or no) response to changes in n. Nevertheless, the markups become unbounded for the lognormal distribution. Finally, the case of Bounded Power Law noise shows that an infinite support is not necessary for our results. In this case the markup is proportional to $n^{-1/\alpha}$ and markup decay remains slow for large α.

In practical terms, these results imply that in markets with noise we should not necessarily expect increased competition to dramatically reduce markups. The mutual fund industry may exemplify such stickiness.[9] Currently 10,000 mutual funds are available in the U.S. and many of these funds offer similar portfolios. Even in a narrow class of homogenous products, such as medium capitalization value stocks or S&P 500 index funds, it is normal to find 100 or more competing funds (Hortacsu and Syverson 2004). Despite the large number of competitors in such sub-markets, mutual funds still charge high annual fees, often more than 1% of assets under management. Most interestingly, these fees have not fallen as the number of homogeneous competing funds has increased by a factor of 10 over the past several decades.[10]

[9] Carlin(2009) makes a similar point in a model of price complexity with boundedly rational consumers.

[10] Finally, we add a note about the well-known behavior of markups when the noise is multiplied by a constant. Consider random shock X and its linear transform $X' = \sigma X + k$. A higher σ means that there is a higher standard deviation of the noise, while k is simply a shift. Calling $\mu_n = \mu_n(1, 0)$ the markup under X

4.3 Consumer Surplus

Sometimes the random utility framework is criticized as generating too high a value for consumer surplus. Indeed, if the distribution is unbounded, the total surplus goes to ∞ as the number of firms increases. Our analytical results allow us to examine this criticism. For brevity, we restrict ourself to the Perloff-Salop case, with unbounded distributions and $\gamma \geq 0$. Expected gross surplus is $\mathbb{E}[M_n]$, where M_n is the highest of n draws. Theorem 1 shows that $\mathbb{E}[M_n] \sim \Gamma(2-\gamma)\overline{F}^{-1}(1/n)$ for $\gamma \geq 0$. For all the distributions that we study except the unbounded power law case, $\overline{F}^{-1}(1/n)$ rises only slowly with n. Hence, even for unbounded distributions, and large numbers of producers, consumer surplus can be quite small. For example, for the case of Gaussian noise when consumer preferences have a standard deviation of \$1, $\overline{F}^{-1}(1/n) \sim \sqrt{2\ln n}$. So, with a million toothpaste producers, consumer surplus averages only \$5.25 per tube. Hence, in many instances, the framework — even with unbounded distributions — does not generate counterfactual predictions about consumer surplus or counterfactual predictions about the prices that cartels would set.

4.4 Limit Pricing Model Interpretation

We now derive equilibrium markups for an alternative model of oligopolistic competition, and show that it produces markups that are asymptotically equal to those from the Perloff-Salop, Sattinger, and Hart models. Importantly, we explain how the same logic underlies the equilibrium markups for all of these models. This generates an simple but useful interpretation of our economic results from Section 4. As an aside, we demonstrate an equivalence between our results and the mathematics of second-price auctions in Section 5.4.

This model is sometimes called "limit pricing", and has proved very useful in trade and macroeconomics (e.g. Bernard et al. 2003, see also Auer and Chaney 2009). Each firm i draws a quality shock X_i, then sets a price p_i. (This is in contrast with the models of Section 2, where prices are set *before* taste shocks are observed.) The representative consumer needs to consume one unit of the good, and picks the firm with the largest of $X_i - p_i$. As before, call $M_n = \max_{i=1\ldots n} X_i$ the maximum draw of the n qualities, and S_n the second-largest draw. In the competitive equilibrium, the firm with the highest quality, M_n, gets all the market share,

and $\mu_n(\sigma, k)$ the markup under the distribution of X', we have $\mu_n^{PS}(\sigma, k)' = \sigma\mu_n^{PS}$ i.e., the markup is simply multiplied by σ, while k does not matter for the markup (it is the difference in qualities that matters, not their absolute level). Likewise, $\mu_n^{Satt}(\sigma, k)' = \sigma\mu_n^{Satt}$, and $\mu^{Hart}(\psi, \sigma) = \sigma\mu^{Hart}(\psi\sigma, 1)$. So asymptotically, the markup is simply multiplied by σ.

and sets a price $p = c + M_n - S_n$. This is just enough to take all the market away from the firm with the second-highest quality. So, the markup is $\mu_n^{LP} = M_n - S_n$.

The next Proposition analyzes the average behavior of that Limit Pricing markup.

Proposition 3 *Let F be well-behaved with tail index $\gamma < 1$, and assume that $xf(x)$ is $[w_l, w_u)$-integrable. Call M_n and S_n, respectively, the largest and second largest realizations of n i.i.d. random variables with CDF F. Then limit pricing markup is $\mu_n^{LP} = M_n - S_n$, and*

$$\mathbb{E}\left[\mu_n^{LP}\right] \sim_{n\to\infty} \frac{\Gamma(1-\gamma)}{nf\left(\overline{F}^{-1}(1/n)\right)}. \tag{19}$$

We see that the markup is asymptotic to the markup in the other three random utility models, derived in Theorem 2. This strengthens the result that the behavior in the markup is in many ways independent of the details of the modelling of competition. There is also an intuitive interpretation for Proposition 3, which clarifies the general economics of competition with a large number of firms.

We observed that $\mathbb{E}\left[\overline{F}(M_n)\right] \simeq 1/(n+1)$, which suggested that M_n will be close to $\overline{F}^{-1}(1/n)$. Similarly, with S_n the second-highest draw, $\mathbb{E}\left[\overline{F}(S_n)\right] \simeq 2/(n+1)$. So it is likely that $S_{n-1} \approx \overline{F}^{-1}(2/n)$. So

$$\begin{aligned}
\mathbb{E}\left[\mu_n^{LP}\right] &\approx M_n - S_n \approx \overline{F}^{-1}(1/n) - \overline{F}^{-1}(2/n) = \overline{F}^{-1}(1/n) - \overline{F}^{-1}(1/n + 1/n) \\
&\approx -\left.\frac{d\overline{F}^{-1}(x)}{dx}\right|_{x=1/n} \cdot \frac{1}{n} \text{ by Taylor expansion} \\
&= \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)}.
\end{aligned}$$

Proposition 1 shows that the heuristic argument generates the right approximation for the distributions when $\gamma = 0$ (e.g. Gaussian, logit (Gumbel), exponential, and lognormal), and that the approximation remains accurate up to a corrective constant in the other cases. So an informal intuition for the Perloff-Salop, Sattinger and Hart models is as follows. To set its optimum price, a firm conditions on its getting the largest draw, then evaluates the likely draw of the second highest firm, and engages in limit pricing, where it charges a markup equal to the difference between its draw and the next highest draw.

5 Extensions and Applications

This section discusses two extensions of our basic models and an application to macro. Section 5.1 endogenizes the degree of product differentiation between firms, and in doing so demonstrates a connection between our results and the functional form of the Dixit-Stiglitz (1977) demand function. The Dixit-Stiglitz specification is highly popular in macroeconomics. We develop a simple macroeconomic framework in Section 5.2 to demonstrate how the random demand specification may be used in place of the common Dixit-Stiglitz specification. In Section 5.3, we enrich the Perloff-Salop model and show that (i) our mathematical methods can be applied to richer oligopoly models that incorporate complicated assumptions about consumer preferences (beyond the standard models that we previously introduced), and that (ii) our economic insights about the "noise-dependence" of equilibrium markups remain under such additional assumptions.

5.1 Endogenous Product Differentiation or Noise

So far, the models we analyze have assumed that the standard deviation of the noise term exogenous. This section relaxes this assumption and allows firms to choose the degree of differentiation (in the traditional interpretation), or the degree of "confusion" (in a behavioral interpretation). As a bonus, we show that with Gumbel-distributed noise, the Hart model with endogenous differentiation produces the familiar Dixit-Stiglitz (1997) demand function.

Assume that firms can choose the degree to which their own product is differentiated from the rest of the market; specifically, assume that each firm i can choose σ_i at a cost $c(\sigma_i)$ so that the firm's demand shock is $X_i = \sigma_i X$, where X has CDF F. The game then has the following timing:

1. firms simultaneously choose (p, σ).

2. random taste shocks are realized.

3. consumers make purchase decisions.

4. profits are realized.

Firm i's profit function is given by

$$\pi((p_i, \sigma_i), (p, \sigma); n) = (p_i - c(\sigma_i)) D((p_i, \sigma_i), (p, \sigma); n)$$

in step 1, where $D((p_i,\sigma_i),(p,\sigma);n)$ is the demand for good i when the firm chooses (p_i,σ_i) and the remaining $n-1$ firms choose (p,σ). Each firm i then chooses (p_i,σ_i) to maximize $\pi((p_i,\sigma_i),(p,\sigma);n)$; the symmetric equilibrium is then characterized by

$$(p,\sigma) = \arg\max_{(p',\sigma')} \pi((p',\sigma'),(p,\sigma);n).$$

Our techniques allow us to analyze the symmetric equilibrium outcome of this game, for each of the Perloff-Salop, Sattinger and Hart models.

Proposition 4 *Consider the Perloff-Salop, Sattinger and Hart models where firms simultaneously choose p and σ, under the same assumptions as Theorem 2. Assume that $w_u > 0$.*[11] *Further, in the Perloff-Salop and Sattinger cases, assume that $xf^2(x)\,dx$ is $[w_l,w_u)$-integrable, and that $c'>0, c''>0, \lim_{t\to\infty} c'(t)=\infty$. In the Hart case, assume that $c'>0, (\ln c)''>0, \lim_{t\to\infty}(\ln c(t))'=\infty$.*
Then the equilibrium outcome with n firms is asymptotically, as $n\to\infty$

$$\mu_n^{PS}(\sigma_n) = \frac{\mu_n^{Satt}(\sigma_n)}{c^{Satt}(\sigma_n)} \sim \frac{\mu_n^{Hart}(\sigma_n)}{c^{Hart}(\sigma_n)} \sim \frac{\sigma_n}{nf\left(F^{-1}\left(1-\frac{1}{n}\right)\right)\Gamma(\gamma+2)},$$
$$c^{PS'}(\sigma_n) = \frac{c^{Satt'}(\sigma_n)}{c^{Satt}(\sigma_n)} \sim \frac{c^{Hart'}(\sigma_n)}{c^{Hart}(\sigma_n)} \sim \begin{cases} \overline{F}^{-1}(1/n) : w_u<\infty \\ \frac{\overline{F}^{-1}(1/n)}{\Gamma(\gamma+2)} : w_u=\infty \end{cases}.$$

That is, at the symmetric equilibrium, the normalized marginal cost of σ ($c'(\sigma_n)$ in the Perloff-Salop case and $c'(\sigma_n)/c(\sigma_n)$ in the Sattinger and Hart cases) asymptotically equals $\overline{F}^{-1}\left(\frac{1}{n}\right)$, up to a corrective constant. In particular, it goes closer to the upper bound of the distribution as the number of firms increases. In other terms, more firms create more product differentiation (in the optimistic traditional interpretation), or more confusion (in the pessimistic behavioral interpretation), in a way quantified by Proposition 4. We note that this potentially perfect effect of competition of the supply of noise is an important effect, see e.g. Carlin (2009), Ellison and Ellison (2009), and Spiegler (2006).

We can use the limit pricing heuristic from Section 4.4 to obtain an intuition for this result. Again, consider the Perloff-Salop case. Since the firm engages in limit pricing, it can charge

[11] This assumption that the largest possible realization of X_i is positive (possibly infinite) makes the firm's problem economically sensible. If, on the contrary, $w_u \le 0$, then each realization of X_i would be negative with probability 1. In that case, increasing σ would *reduce* the attractiveness of the firm's product to the consumer. To eliminate this possibility, we assume $w_u > 0$.

a markup of $\sigma M_n - \sigma^* S_n$ where σ is the firm's product differentiation choice and σ^* is the choice of all other firms, which we take as given. The marginal value of an additional unit of noise σ is thus $M_n \simeq \overline{F}^{-1}\left(\frac{1}{n}\right)$.

Interestingly, the Hart model with Gumbel-distributed noise generates the familiar demand function from Dixit-Stiglitz (1977), as the following proposition shows.[12]

Proposition 5 *Let X_i be Gumbel distributed with parameter ϕ: $F(x) = \exp\left(-e^{-x/\phi}\right)$. Then in the Hart model, demand for good i equals*

$$D(p_1, ..., p_n; i) = \Gamma(1 - \phi\psi\sigma^*) \frac{p_i^{-(1+1/(\phi\sigma^*))}}{\left(\sum_{i=1}^n p_j^{-1/(\phi\sigma^*)}\right)^{1-\phi\psi\sigma^*}}$$

where σ^* is the symmetric equilibrium choice of σ.

This result may be of independent interest. For example, it suggests that our framework may be used to model Dixit-Stiglitz demand functions with endogenous elasticity.

5.2 A Macroeconomic Framework with Random Demand

To model pricing power macro economists typically utilize the monopolistically competitive differentiated goods specification of Dixit and Stiglitz (1977) with a large number of goods. Shocks to the demand side are often modeled by shocking the coefficient of substitution in the Dixit-Stiglitz specification; see e.g. Woodford (2003, ch. 6), Smets and Wouters (2003) and Gali et al. (2012). This practice is criticized by Chari, Kehoe and McGrattan (2009), who argue that such shocks are not structural. To meet this criticism, we investigate the implications of a random demand specification. As we show below, demand shocks in the random demand approach are taken into account by the firms when setting prices, rather than treating these exogenously. Here we take an extreme view of demand shocks and model these as a taste of the entire population for a specific item from the set of differentiated goods (one year everybody desires a BlackBerry, the next year the iPhone).

To be able to demonstrate the implications of random demand for macro, we develop two macro models. One is based on the traditional Dixit-Stiglitz (DS) specification, the other is based on the Random Demand (RD) specification. The two models only differ with respect

[12] Anderson et al (1992, pp. 85-90) derive this result for the case $\psi = 0$.

to utility function. The Dixit-Stiglitz specification with endogenous labor supply is

$$U = Z^{1-\theta} \left[\frac{1}{n} \sum_{i=1}^{n} Q_i^{1/(1+\tau)} \right]^{\theta(1+\tau)} - \frac{1}{1+\eta} L^{1+\eta}, \tag{20}$$

where Z is the composite good, the Q_i are the differentiated goods and L is labor. The substitution coefficient τ is constrained to $\tau \in (0, \infty)$, which implies concavity; $\theta \in (0, 1)$.

The Random Demand model is based on Sattinger's (1984) utility function (5) amended with the same disutility of labor as in the DS specification

$$U = Z^{1-\theta} \left[\sum_{i=1}^{n} \exp(X_i) Q_i \right]^{\theta} - \frac{1}{1+\eta} L^{1+\eta}. \tag{21}$$

In this setup the taste shock affects all consumers equally, i.e.the demand shocks X_i are identical across consumers.

The supply side technologies are linear:

$$Z = BN \text{ and } Q_i = AN_i, \tag{22}$$

where A and B are the labor productivity coefficients while N and N_i are the respective labor demands. Note that A and B also capture the supply side productivity shocks. Perfect competition in the composite goods market implies that prices equal the per unit labor costs. The differentiated goods producer exploits his direct pricing power, but ignores his pricing effect on the price index of the differentiated goods and the consumer income. For the Random Demand case, the markup is μ_n from (7); in the Dixit-Stiglitz specification, the markup is τ. Note that the markup factors τ and μ_n can take on similar values, cf. Table 2. The models' solutions from the first order conditions is given in the Appendix C.

By dividing (60) by (59), and similarly dividing (57) by (55) from the solution, we calculate the respective labor productivities for the competitive good:

Lemma 2 *The labor productivities under the Dixit-Stiglitz and Random Demand specifications are, respectively,*

$$Q_j/L = \frac{\theta A}{1 + (1-\theta)\tau} \text{ and } Q_i/L = \frac{\theta A}{1 + (1-\theta)\mu_n}. \tag{23}$$

The following is an immediate implication:

Corollary 1 *The labor productivity in the Dixit-Stiglitz specification is a mixture of demand and supply shocks, while in the Random Demand specification there are only supply shocks.*

The main difference between the two demand specifications stems from the way in which demand shocks impinge on the macro variables. Woodford (2003, ch. 6), Smets and Wouters (2003) and Gali et al. (2012) generate demand shocks by shocking τ. This is different from the demand shock that arises from the random utility concept. Letting τ be random produces a time varying markup factor. The markup factor μ_n in the case of random utility, though, is not random, see (7). Only the amount demanded is random as X_i is part of the demand function. The deterministic markup in case of the random demand model can be explained by the fact that the uncertainty is anticipated on the supply side and 'disappears as an expectation'.

In the Random Demand expression from (23), the number of competitors n plays a role through the markup μ_n. In the case of the Dixit-Stiglitz specification, however, n does not enter as τ is exogenous. Consider the implications for the goods ratios Q/Z:

Proposition 6 *In the Dixit-Stiglitz specification, the goods ratio Q_j/Z does not depend on n. In the Random Demand case, if the distribution of the fashion shock is bounded or has exponential like tails, then Q_i/Z (approximately) equals the ratio of the expenditure shares $\theta/(1-\theta)$ times the ratio of the productivity shocks A/B. But in the case that the preference shocks have fat tails, the goods ratio $Q_i/Z \to 0$.*

Proof of Proposition 6 Combining (60) and (61), and (57) and (58) yields for respectively the DS and RD specifications

$$\frac{Q_j}{Z} = \frac{\theta}{(1-\theta)(1+\tau)} \frac{A}{B} \text{ and } \frac{Q_i}{Z} = \frac{\theta}{(1-\theta)(1+\mu_n)} \frac{A}{B}. \tag{24}$$

Then use Table 2 to plug in the details for μ_n depending on the type of distribution. With $\gamma \geq 0$ and a unbounded, $\lim_{n\to\infty}(Q_i/Z) = \lim_{n\to\infty}(1/\mu_n) = 0$. ■

Thus in the case that the preference shocks have fat tails and with numerous competitors, the differentiated good becomes unimportant relative to the competitive good.

5.3 Enriched Linear Random Utility (ELRU)

In this section we add two features, drawn from recent random demand models (see, for example, Berry, Levinsohn and Pakes 1995), to the Perloff-Salop model: an outside option good, and stochastic consumer price sensitivity. We call this enriched model *Enriched Linear Random Utlity* (ELRU). We will show that the essential insights we obtain in Section 4 are maintained in this setting. As before, we go through the modeling assumptions, then introduce the mathematical machinery before applying it to the equilibrium markup problem. The proofs are in Appendices C and D.

5.3.1 Model Setup: ELRU

There are n firms each producing a monopolistically competitive good, and a consumer who chooses either to purchase exactly one unit of the good from one firm, or to take his outside option. The consumer's utility from consuming firm i's good is

$$u_i = -\beta p_i + X_i, \qquad u_0 = \epsilon_0,$$

where p_i is the price of good i (set by firm i), $\beta \geq 0$ is a "taste for money", $\epsilon_0 \geq 0$ is the value of the consumer's outside option, and X_i is the random taste shock associated with good i. Each of $X_1, ..., X_n$ are identically distributed and independent of each other and (β, ϵ_0). The ϵ_0 may not be independent of β. Each X_i has CDF F. The joint distribution of (β, ϵ_0) is denoted by $H(.,.)$ and has a density $h(.,.)$; some times we use vector notation $y = (\beta, \epsilon_0)$ and will just write $H(y)$ and $h(y)$ respectively.

The demand function for good i at price p (given that all other goods are priced at p') is the probability that the consumer's payoff for good i exceeds his payoff to all other goods, as well as the outside option:

$$D(p_1, ..., p_n; i) = \mathbb{P}\left(-\beta p_i + X_i \geq \max\left\{\max_{j \neq i}\{-\beta p_j + X_j\}, \epsilon_0\right\}\right).$$

If we set $\epsilon_0 = -\infty$ and β to be a constant, this simplifies to the Perloff-Salop model. Evaluation of (2) gives the following markup expression for the symmetric equilibrium of the ELRU model:

$$\mu_n^{ELRU} = \frac{\mathbb{E}\left[1 - F\left(\max\left\{M_{n-1}, \beta p + \epsilon_0\right\}\right)\right]}{\mathbb{E}\left[f\left(\max\left\{M_{n-1}, \beta p + \epsilon_0\right\}\right)\beta\right]}. \tag{25}$$

where $M_{n-1} = \max_{j=1,\ldots,n-1} X_j$ is the maximum of $n-1$ independent random variables X_j with distribution F.

5.3.2 Equilibrium Markup: ELRU

We limit the calculation of equilibrium markups to the case where teh distribution satisfies $\gamma = 0$ and $a < \infty$; that is, to distributions that are weakly thinner than the exponential. This restriction allows us to retain common distributions such as the Gumbel, Gaussian, and Exponential; put another way, distributions that produce equilibrium markups that are weakly decreasing with the degree n of competition. The main result of this section is that, with some mild assumptions on the distributions of β and ϵ_0, equilibrium markups are asymptotically equal (up to a factor $\mathbb{E}[\beta]$) to the Perloff-Salop, Sattinger and Hart markups. We separately consider the cases of bounded and unbounded support for the distribution $H(y)$.

The first result is for the case that $H(y)$ has bounded support and where the densities $f(x)$ are of the Rootzen (1987) type

$$f(x) \sim \kappa\lambda\phi x^{\phi+\nu-1} \exp\left(-\lambda x^{\phi}\right),\ \kappa > 0,\ \lambda > 0,\ \phi \geq 1,\ \nu \in \mathbb{R}. \tag{26}$$

It is a simple calculation to check that for this class $\gamma = 0$ and $a = 0$ whenever $\phi > 1$. Note that the classic tail expansion of the normal distribution $1 - \Phi(x) \sim \phi(x)/x$ fits the Rootzen class (38) when we set $\phi = 2$, $\lambda = 1/2$, $\kappa = 1/\sqrt{2\pi}$ and $a = -1$.

Theorem 3 *Assume that $f(x)$ is of the Rootzen type as defined in (26). Assume that the density function $h(y)$ has bounded support. Then the symmetric equilibrium markup in the ELRU model is asymptotically*

$$\mu_n^{ELRU} \sim \frac{1/\mathbb{E}[B]}{\phi\lambda^{1/\phi} (\ln n)^{1-1/\phi}}. \tag{27}$$

The second result treats the case when the density $h(y)$ has unbounded support. It requires the assumption that $h(y)$ is multivariate regularly varying:[13]

Definition 4 *A multivariate density function $h : (\mathbb{R}^+)^k \to \mathbb{R}$ on a random vector Θ is regularly varying at ∞ with index ρ if the density h_v of every linear combination $v \cdot \Theta \in \mathbb{R}$ of Θ is regularly varying:*

$$\forall\lambda > 0, \lim_{x\to\infty} \frac{h_v(\lambda x)}{h_v(x)} = \lambda^{\rho}. \tag{28}$$

[13] This condition is not difficult to satisfy. For example, it is satisfied if each element of y is independent and regularly varying. See Basrak, Davis, and Mikosch (2002) for more details and alternative characterizations of multivariate regular variation.

Theorem 4 *Assume that F is well-behaved with tail index $\gamma = 0$ and $a < \infty$, that $f(x)$ is non-increasing for $x \geq 0$, and that $f^2(x)$ is $[w_l, w_u)$-integrable. Assume that the density function $h(y)$ is multivariate regularly varying at infinity with $w_u = \infty$, that $\mathbb{E}\left[\beta^{2+\delta}\right] < \infty$ for some $\delta > 0$ and that $\epsilon_0 \geq 0$. Then the symmetric equilibrium markup in the ELRU model is asymptotically*

$$\mu_n^{ELRU} \sim \frac{1/\mathbb{E}[\beta]}{nf\left(F^{-1}(1-1/n)\right)}. \tag{29}$$

Note that in each of the Hart, Perloff-Salop, and Sattinger models, the marginal utility of money equals 1, which corresponds to the case $\beta \equiv 1$ in the ELRU model.

5.4 Auctions

Consider an second-price auction with a single good and n bidders where each bidder i privately values the good at X_i, which is i.i.d. with CDF F. It is well-known that if F is strictly increasing on (w_l, w_u), then the equilibrium outcome of this auction is that each bidder makes a bid equal to his private valuation; the bidder with the highest valuation (M_n) wins and pays the second-highest valuation (S_n). Thus the expected revenue in a second price auction equals $\mathbb{E}[S_n]$, and the expected surplus for the winner in a second price auction equals $\mathbb{E}[M_n - S_n]$. We can apply Theorem 1 and Lemma 1 to obtain asymptotic approximations for both of these expressions.[14] Since this is closely related to the results for limit pricing model, we state the following without proof.

Proposition 7 *Let F be well-behaved with tail index $\gamma < 1$, and assume that $xf(x)$ is $[w_l, w_u)$-integrable. Then in a second-price auction where valuations are i.i.d. as F, the expected revenue to the seller, $\mathbb{E}[S_n]$, is*

$$\mathbb{E}[S_n] \sim_{n\to\infty} \overline{F}^{-1}(1/n)\,\Gamma(2-\gamma) \quad \text{if } w_u = \infty,$$
$$\mathbb{E}[S_n] = w_u - \left(w_u - \overline{F}^{-1}(1/n)\right)\Gamma(2-\gamma) + o\left(w_u - \overline{F}^{-1}(1/n)\right) \quad \text{if } w_u < \infty,$$

and the expected surplus for the winner of the auction is:

$$\mathbb{E}[M_n - S_n] \sim_{n\to\infty} \frac{\Gamma(1-\gamma)}{nf\left(\overline{F}^{-1}(1/n)\right)}. \tag{30}$$

[14]Result (30) appeared in Caserta (2002, Prop. 4.1) in the case $\gamma \neq 0$. Caserta does not have the key argument of the proof, with the integration by parts.

6 Conclusion

The choice of noise distribution in existing models of consumer choice is generally dictated by tractability concerns. This raises the question of how sensitive the implications of these models are to assumptions about the noise distributions. The increasing popularity of structural estimation techniques makes this question particularly important. We make progress on this issue in two ways.

First, we demonstrate a tractable technique for modelling general noise distributions. We have characterized equilibrium markups in markets where product differentiation is generated by idiosyncratic taste shocks. We managed to calculate equilibrium markups for general noise distributions in various examples of monopolistic competition models. We demonstrate that our approach is applicable even to richer consumer choice models such as that incorporate various technically challenging assumptions.

Second, our results reveal a somewhat surprising "detail-independence" of the behavior of price markups, which are asymptotically identical (up to a constant factor) for all four models. For a wide range of commonly used distributions, including the canonical case of Gaussian noise, we show that the elasticity of markups to the number of firms is asymptotically zero, so that markups are relatively insensitive to the degree of competition.

7 Appendix A: Proofs

First, to clarify notation: denote $f_n \sim g_n$ if $f_n/g_n \to 1$, $f_n = o(g_n)$ if $f_n/g_n \to 0$ and $f_n = O(g_n)$ if there exists $M > 0$ and $n' \geq 1$ such that for all $n \geq n', |f_n| \leq M |g_n|$. We start by collecting some useful facts about regular variation, see Resnick (1987) or Bingham et al. (1989).

Lemma 3 *1. If $g(t) \in RV_a^0$, then $\lim_{t\to 0} g(xt)/g(t) = x^a$ holds locally uniformly (with respect to x) on $(0, \infty)$.*

2. *If $\lim_{x\to 0} h(x)/s(x) = 1$, $\lim_{x\to 0} s(x) = 0$ and $g(x) \in RV_\rho^0$, then $g(h(x)) \sim g(s(x))$.*

3. *If $g(t) \in RV_a^0$ and $h(t) \in RV_b^0$, then $g(t)h(t) \in RV_{a+b}^0$.*

4. *If $g(t) \in RV_a^0$, $h(t) \in RV_b^0$ and $\lim_{t\to 0} h(t) = 0$, then $g \circ h(t) \in RV_{ab}^0$.*

5. *If $g(t) \in RV_a^0$ and non-decreasing, then $g^{-1}(t) \in RV_{a^{-1}}^0$ if $\lim_{t\to 0} g(t) = 0$.*

6. *Let $U \in RV_\rho^0$. If $\rho > -1$ (or $\rho = -1$ and $\int_0^x U(t)\,dt < \infty$), then $\int_0^x U(t)\,dt \in RV_{\rho+1}^0$ and*

$$\lim_{x\to 0} \frac{xU(x)}{\int_0^x U(t)\,dt} = \rho + 1.$$

If $\rho \leq -1$, then for $\overline{x} > 0$, $\int_x^{\overline{x}} U(t)\,dt \in RV_{\rho+1}^0$ and

$$\lim_{x\to 0} \frac{xU(x)}{\int_x^{\overline{x}} U(t)\,dt} = -\rho - 1.$$

7. *If $\lim_{t\to\infty} tj'(t)/j(t) = \rho$, then $j \in RV_\rho^\infty$. Similarly, if $\lim_{t\to 0} tj'(t)/j(t) = \rho$, then $j \in RV_\rho^0$.*

8. *If $g \in RV_\rho^\infty$ and $\varepsilon > 0$, then $g(t) = o(t^{\rho+\varepsilon})$ and $t^{\rho-\varepsilon} = o(g(t))$ as $t \to \infty$; and if $g \in RV_\rho^0$ and $\varepsilon > 0$, then $g(t) = o(t^{\rho-\varepsilon})$ and $t^{\rho+\varepsilon} = o(g(t))$ as $t \to 0$.*

Proof

1. Follows upon inversion from Resnick (1987, Prop. 0.5).

2. This fact follows from the observation that for $\frac{g(s(x))}{g(h(x))} = \frac{g(\frac{s(x)}{h(x)}h(x))}{g(h(x))} \sim \left(\frac{s(x)}{h(x)}\right)^{\rho} \to_{x\to 0} 1$ where we can take the limit as $x \to 0$ because of Lemma 3(1). Going into more detail, choose $\delta(\cdot)$ such that $\lim_{t\to 0}\delta(t) = 0$ and $|s(t')/h(t') - 1| < \delta(t)$ for $t' < t$. Such $\delta(\cdot)$ exists by our assumptions on s and h. Choose $\varepsilon(\cdot,\cdot)$ such that $\lim_{t\to 0}\varepsilon(t,\delta) = \lim_{\delta\to 0}\varepsilon(t,\delta) = 0$ and $|g(xt')/g(t') - x^{\rho}| < \varepsilon(t,\delta)$ for $x \in (1-\delta, 1+\delta)$ and $t' < t$. Lemma 3(1) ensures that such $\varepsilon(\cdot,\cdot)$ exists. Then

$$|g(s(t'))/g(h(t')) - 1| = \left|g\left(\frac{s(t')}{h(t')}h(t')\right)/g(h(t')) - 1\right| < \varepsilon(h(t'), \delta(t)) + \rho O(\delta(t))$$

for $t' < t$. Since the RHS goes to zero as $t \to 0$, the result follows.

3. Since $\lim_{t\to 0}\frac{g(xt)}{g(t)} = x^a$ and $\lim_{t\to 0}\frac{h(xt)}{h(t)} = x^b$, we have $\lim_{t\to 0}\frac{g(xt)h(xt)}{g(t)h(t)} = x^{a+b}$.

4. Follows upon inversion from Resnick (1987, Prop. 0.8, iv).

5. Follows upon inversion from Resnick (1987, Prop. 0.8, v).

6. Both parts follow upon inversion from Resnick (1987, Th. 0.6, a).

7. Follows from Resnick (1987, Prop. 0.7) and by inversion.

8. Directly by Resnick (1987, Prop. 0.8, ii) and upon inversion. ■

Our proof of Theorem 1 depends critically on the following result.

Theorem 5 *(Karamata's Tauberian Theorem) Assume $U : (0,\infty) \to [0,\infty)$ is weakly increasing, $U(x) = 0$ for $x < 0$, and assume $\int_0^{\infty} e^{-sx}dU(x) < \infty$ for all sufficiently large s. With $\alpha \geq 0$, $U(x) \in RV_{\alpha}^{0}$ if and only if*

$$\int_0^{\infty} e^{-sx}dU(x) \sim_{s\to\infty} U(1/s)\Gamma(\alpha+1).$$

For a proof, see Bingham et al. (1987, pp.38, Th. 1.7.1') or Feller (1972, XIII.5, Th. 1) for another version of Karamata's Tauberian theorem.

Proof of Theorem 1

Assume for now that $G(x) \geq 0$ for all $x \in (w_l, w_u)$, and show later that this assumption can be relaxed. Differentiation of $\mathbb{P}(M_n \leq x) = F^n(x)$ gives the density of M_n: $f_n(x) =$

$nf(x)F^{n-1}(x)$. Using the change of variable $x = \overline{F}^{-1}(t)$ and observing that $d\overline{F}^{-1}(t)/dt = -1/f\left(\overline{F}^{-1}(t)\right)$

$$\begin{aligned}\mathbb{E}\left[G\left(M_n\right)\right] &= \int_{w_l}^{w_u} G(x)nf(x)F^{n-1}(x)dx \\ &= n\int_{w_l}^{w_u} G(x)F^{n-1}(x)\left(f(x)dx\right) \\ &= n\int_0^1 G(\overline{F}^{-1}(t))[F(\overline{F}^{-1}(t))]^{n-1}dt \\ &= n\int_0^1 \widehat{G}(t)\left(1-t\right)^{n-1}dt.\end{aligned}$$

We next use the change in variables $x = -\ln(1-t)$, so $t = 1-e^{-x}$, $dt = e^{-x}dx$, and so

$$\mathbb{E}\left[G\left(M_n\right)\right] = n\int_0^\infty \widehat{G}\left(1-e^{-x}\right)e^{-x}e^{-n'x}dx$$

where $n' = n-1$.

Next, define $h(x) = \widehat{G}(1-e^{-x})e^{-x}$, and $\eta(x) = \int_0^x h(y)\,dy$. Since $\widehat{G}$ is regularly varying at zero with index $\rho > -1$, Lemma 3(8) implies that $\int_0^s \left|\widehat{G}(t)\right| dt < \infty$ for sufficiently small s. This, with the assumptions $G(t) \geq 0$ and $\int_s^1 \left|\widehat{G}(t)\right| dt < \infty$ for all $s \in (0,1)$, ensure that $\mu(x) = \int_0^{1-e^{-x}} \widehat{G}(t)\,dt$ is finite and non-decreasing on $[0,\infty)$. By Lemma 3(2), $h(x) \sim_{x\to 0} \widehat{G}(x)$. So $h \in RV_\rho^0$, and by Lemma 3(6)

$$\begin{aligned}\eta(x) &= \int_0^x h(y)\,dy \\ &\sim_{x\to 0} \frac{1}{1+\rho}h(x)x \\ &\sim_{x\to 0} \frac{1}{1+\rho}\widehat{G}(x)x.\end{aligned}$$

Therefore $\eta(x) \in RV_{\rho+1}^0$.

Noting our assumption that $\rho + 1 > 0$, we can now apply Karamata's Theorem 5 in combination with the last expression to obtain

$$\begin{aligned}\int_0^\infty e^{-n'x} d\eta(x) &\sim_{n'\to\infty} \eta(1/n')\,\Gamma(2+\rho) \\ &\sim_{n'\to\infty} \frac{1}{1+\rho}\widehat{G}(1/n')\,(n')^{-1}\,\Gamma(2+\rho) \\ &\sim_{n\to\infty} \widehat{G}(1/n)\,n^{-1}\Gamma(1+\rho)\,.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}\left[G(M_n)\right] &= n\int_0^\infty e^{-n'x} d\eta(x) \\ &\sim n\widehat{G}(1/n)\,n^{-1}\Gamma(1+\rho) = G(\overline{F}^{-1}(1/n))\Gamma(1+\rho)\end{aligned}$$

holds when $G(x) \geq 0$ for all $x \in (w_l, w_u)$.

Now relax the assumption that $G(x) \geq 0$ for all $x \in (w_l, w_u)$. Choose $\bar{t} \in (0,1)$ such that $G(t) > 0$ for $t \in \left[0, \bar{t}\right]$. The assumption that $G(\cdot)$ is strictly positive in a left neighborhood of w_u ensures that such $\bar{t}$ exists. Thus we can write

$$\mathbb{E}\left[G(M_n)\right] = n\int_0^{\bar{t}} \widehat{G}(t)\,(1-t)^{n-1}\,dt + n\int_{\bar{t}}^1 \widehat{G}(t)\,(1-t)^{n-1}\,dt$$

Consider $\widetilde{G} : (0,1) \to \mathbb{R}$ defined by

$$\widetilde{G}(t) \equiv \begin{cases} \widehat{G}(t) : t \leq \bar{t} \\ 0 : t > \bar{t} \end{cases}.$$

It is easy to check that $\widetilde{G}$ satisfies the conditions of the theorem and additionally is weakly positive everywhere on (w_l, w_u). The argument above shows that as $1/n \to 0$

$$n\int_0^{\bar{t}} \widehat{G}(t)\,(1-t)^{n-1}\,dt = n\int_0^1 \widetilde{G}(t)\,(1-t)^{n-1}\,dt \sim \widetilde{G}(1/n)\,\Gamma(1+\rho) \sim \widehat{G}(1/n)\,\Gamma(1+\rho)\,. \tag{31}$$

To complete the proof we demonstrate that as $n \to \infty$

$$\left| \int_{\bar{t}}^{1} \widehat{G}(t)(1-t)^{n-1}\, dt \right| = o\left(\int_{0}^{\bar{t}} \widehat{G}(t)(1-t)^{n-1}\, dt \right).$$

First, by (31) for $n \to \infty$

$$\int_{0}^{\bar{t}} \widehat{G}(t)(1-t)^{n-1}\, dt \sim n^{-1}\widehat{G}(1/n)\,\Gamma(1+\rho) \in RV_{-\rho-1}^{\infty}.$$

Lemma 3(8) implies that $\int_{0}^{\bar{t}} \widehat{G}(t)(1-t)^{n-1}\, dt > n^{-\rho-1-\varepsilon}$ for sufficiently large n and given some $\varepsilon > 0$. Also,

$$\begin{aligned}
\left| \int_{\bar{t}}^{1} \widehat{G}(t)(1-t)^{n-1}\, dt \right| &\leq \int_{\bar{t}}^{1} \left|\widehat{G}(t)\right| (1-t)^{n-1}\, dt \\
&\leq \left(1-\bar{t}\right)^{n-1} \int_{\bar{t}}^{1} \left|\widehat{G}(t)\right| dt \\
&\leq \left(1-\bar{t}\right)^{n-1} \int_{0}^{1} \left|\widehat{G}(t)\right| dt.
\end{aligned}$$

By assumption $\int_{s}^{1} \left|\widehat{G}(t)\right| dt < \infty$ for all $s \in (0,1)$, therefore

$$\frac{\left| \int_{\bar{t}}^{1} \widehat{G}(t)(1-t)^{n-1}\, dt \right|}{\int_{0}^{\bar{t}} \widehat{G}(t)(1-t)^{n-1}\, dt} \leq \frac{\left(1-\bar{t}\right)^{n-1} \int_{0}^{1} \left|\widehat{G}(t)\right| dt}{n^{-\rho-1-\varepsilon}} = o(1) \text{ as } n \to \infty.$$

This completes the proof. ■

Proof of Lemma 1.

1. Note that $F\left(\overline{F}^{-1}(t)\right) = 1-t$ implies $f\left(\overline{F}^{-1}(t)\right)\left(\overline{F}^{-1}(t)\right)' = -1$. Let $x = \overline{F}^{-1}(t)$, $j(t) = f\left(\overline{F}^{-1}(t)\right)$. Then $tj'(t)/j(t) = -tf'\left(\overline{F}^{-1}(t)\right)/f^2\left(\overline{F}^{-1}(t)\right) = -\overline{F}(x)f'(x)/f^2(x) = \left(\overline{F}/f\right)'(x) + 1$, so $\lim_{t\to 0} tj'(t)/j(t) = \lim_{x\to F^{-1}(1)} \left(\overline{F}/f\right)'(x) + 1 = \gamma + 1$ by Definition 1. Lemma 3(7) then implies the desired result.

2. Note that $-\frac{d}{dt}\overline{F}^{-1}(t) = 1/f\left(\overline{F}^{-1}(t)\right) \in RV_{-\gamma-1}^{0}$. So if $w_u < \infty$ (which implies $\gamma \leq 0$;

see Embrechts et al., 1997) then Lemma 3(6) implies

$$\overline{F}^{-1}(0) - \overline{F}^{-1}(t) = \int_0^t 1/f\left(\overline{F}^{-1}(s)\right) ds \in RV^0_{-\gamma}.$$

If $w_u = \infty$ (which implies $\gamma \geq 0$) then Lemma 3(6) implies, for any choice of $\overline{t} > 0$, that

$$\overline{F}^{-1}(t) \sim \overline{F}^{-1}(t) - \overline{F}^{-1}(\overline{t}) = \int_t^{\overline{t}} 1/f\left(\overline{F}^{-1}(s)\right) ds \in RV^0_{-\gamma};$$

see also Embrechts et al. (1997, pp. 160).

3. We have

$$\frac{t\frac{d}{dt}e^{\overline{F}^{-1}(t)}}{e^{\overline{F}^{-1}(t)}} = \frac{-t}{f\left(\overline{F}^{-1}(t)\right)} = \frac{-\overline{F}(x)}{f(x)} \text{ for } x = \overline{F}^{-1}(t).$$

Lemma 3(7) then implies the desired result. ■

8 Appendix B: Details of Monopolistic Competition Models

This section provides details to the derivation of the markup expressions for the four monopolistic competition models. The subsequent appendix provides the proofs for the economic claims.

8.1 Perloff-Salop

Recall from (3) that in the Perloff-Salop model, the demand function for good i is the probability that difference between the demand shock and the price is maximized by good i:

$$\begin{aligned} D\left(p_1, ..., p_n; i\right) &= \mathbb{P}\left(X_i - p_i \geq \max_{j \neq i} X_j - p_j\right) \\ &= \mathbb{E}_{X_i}\left[\prod_{j \neq i} \mathbb{P}\left(x - p_i \geq X_j - p_j \mid X_i = x\right)\right] \\ &= \mathbb{E}_{X_i}\left[\prod_{j \neq i} F\left(x - p_i + p_j\right)\right] \\ &= \int_{w_l}^{w_u} f\left(x\right) \prod_{j \neq i} F\left(x - p_i + p_j\right) dx. \end{aligned}$$

Using $D\left(p_i, p; n\right)$ to denote the demand for good i at price p_i when all other firms set price p and using $D_1\left(p_i, p; n\right)$ to denote $\partial D\left(p_i, p; n\right) / \partial p_i$, we may calculate

$$\begin{aligned} D\left(p_i, p; n\right) &= \int_{w_l}^{w_u} f\left(x\right) F^{n-1}\left(x - p_i + p\right) dx \\ D_1\left(p_i, p; n\right) &= -\left(n-1\right) \int_{w_l}^{w_u} f\left(x\right) f\left(x - p_i + p\right) F^{n-2}\left(x - p_i + p\right) dx. \end{aligned}$$

Note that in a symmetric equilibrium

$$\begin{aligned} D(p, p; n) &= \int_{w_l}^{w_u} f(x) F^{n-1}(x)\, dx = 1/n, \\ D_1(p, p; n) &= -\left(n-1\right) \int_{w_l}^{w_u} f^2(x) F^{n-2}(x)\, dx. \end{aligned}$$

It follows that

$$\mu_n^{PS} = -\frac{D(p, p; n)}{D_1(p, p; n)} = \frac{1}{n\left(n-1\right) \int_{w_l}^{w_u} f^2(x) F^{n-2}(x)\, dx}.$$

To interpret the Perloff-Salop markup equation, use the notation M_{n-1} (the largest of the $n-1$ noise realizations: $M_{n-1} \equiv \max_{j \in \{1,\ldots,n\}, j \neq i} X_j$). Then, $D\left(p, p; n\right) = \mathbb{P}\left(X_i > M_{n-1}\right)$, so

$$D\left(p, p; n\right) = \mathbb{E}\left[\overline{F}\left(M_{n-1}\right)\right]. \tag{32}$$

This formulation emphasizes that the demand for good i is driven by the properties of the right-hand tail of the cumulative distribution function $\bar{F}$, as M_{n-1} is likely to be large.

8.2 Sattinger

Under the utility specification (5), goods from the monopolistically competitive (MC) market are perfect substitutes. The consumer optimizes by buying only one monopolistically competitive good; the good i which maximizes e^{X_i}/p_i. The consumer's utility function is thus Cobb-Douglas in the composite good and the chosen MC good; it is then easy to show that the consumer spends fraction θ of his income on the chosen MC good. Without loss of generality, normalize the consumer's endowment y to equal $1/\theta$, so that the consumer always spends 1 unit of income on the MC good.

The demand function of firm i is the probability that the good i has a higher attraction-price ratio than all other goods, multiplied by the purchased quantity $1/p_i$ of the chosen good i; so

$$\begin{aligned} D(p_1, ..., p_n; i) &= \frac{1}{p_i}\mathbb{P}\left(\frac{e^{X_i}}{p_i} = \max_{j=1,...,n} \frac{e^{X_j}}{p_j}\right) \\ &= \frac{1}{p_i}\mathbb{P}\left(X_i - \ln p_i = \max_{j=1,...,n} X_j - \ln p_j\right). \end{aligned} \tag{33}$$

We may rewrite this expression as

$$D(p_1, ..., p_n; i) = \frac{1}{p_i}\int f(x) \prod_{j\neq i} F(x - \ln p_i + \ln p_j)\, dx.$$

Proceeding as in the case of the Perloff-Salop model, we get

$$\begin{aligned} D(p_i, p; n) &= \frac{1}{p_i}\int_{w_l}^{w_u} f(x)\, F^{n-1}(x - \ln p_i + \ln p)\, dx, \\ D_1(p_i, p; n) &= -\frac{1}{p_i^2}\int_{w_l}^{w_u} f(x)\, F^{n-1}(x - \ln p_i + \ln p)\, dx \\ &\quad - \frac{(n-1)}{p_i^2}\int_{w_l}^{w_u} f(x)\, f(x - \ln p_i + \ln p)\, F^{n-2}(x - \ln p_i + \ln p)\, dx \end{aligned}$$

In a symmetric equilibrium

$$D(p,p;n) = \int_{w_l}^{w_u} f(x)F^{n-1}(x)\,dx = \frac{1}{pn},$$
$$D_1(p,p;n) = -\frac{1}{p^2}\left(\frac{1}{n} + (n-1)\int_{w_l}^{w_u} f^2(x)F^{n-2}(x)\,dx\right)$$

After some simple manipulations, it follows that the Sattinger markup in symmetric equilibrium is

$$\mu_n^{Satt} = -\frac{D(p,p;n)}{D_1(p_i,p;n)} = \frac{c}{n\,(n-1)\int_{w_l}^{w_u} f^2(x)F^{n-2}(x)\,dx}.$$

8.3 Hart

Recall that the consumer's objective is to choose quantities to maximize:

$$\max_{i=1\ldots n}\max_{Q_i\geq 0} U = \sum_{i=1}^{n}\left[\frac{\psi+1}{\psi}\left(e^{X_i}Q_i\right)^{\psi/(\psi+1)} - p_iQ_i\right]. \tag{34}$$

As in the Sattinger case, it is clear that because goods are perfect substitutes, the consumer will purchase only from one firm, which we denote by i. The first-order condition of the consumer's problem is then

$$0 = \frac{d}{dQ_i}\left[\frac{\psi+1}{\psi}\left(e^{X_i}Q_i\right)^{\psi/(\psi+1)} - p_iQ_i\right] = e^{X_i\psi/(\psi+1)}Q_i^{-1/(\psi+1)} - p_i$$

which gives us the optimal quantity for the chosen good i: $Q_i = e_i^{X_i\psi}/p_i^{1+\psi}$, and the total net utility is:

$$V_i = \frac{\psi+1}{\psi}\left(e^{X_i}Q_i\right)^{\psi/(\psi+1)} - p_iQ_i$$
$$= \left(\frac{\psi+1}{\psi} - 1\right)p_iQ_i = \frac{1}{\psi}p_ie_i^{X_i\psi}/p_i^{1+\psi} = \frac{1}{\psi}\left(\frac{e_i^{X_i}}{p_i}\right)^{\psi}$$

The consumer chooses the good that maximizes his net utility, i.e. $\arg\max_i \left(e^{X_i}/p_i\right)$. We may then calculate the demand function for good i as

$$D(p_1, ..., p_n; i) = \mathbb{E}\left[\frac{e^{\psi X_i}}{p_i^{1+\psi}} I_{\left\{e^{X_i}/p_i = \max_{j=1,\ldots,n} e^{X_i}/p_j\right\}}\right] \tag{35}$$

$$= \mathbb{E}\left[\frac{e^{\psi X_i}}{p_i^{1+\psi}} I_{\{X_i - \ln p_i = \max_{j=1,\ldots,n} X_j - \ln p_j\}}\right] \tag{36}$$

where $I\{\cdot\}$ is the indicator function. Writing out the expectation and differentiating gives

$$\begin{aligned} D\left(p_i, p; n\right) &= \frac{1}{p_i^{1+\psi}} \int_{w_l}^{w_u} e^{\psi x} f\left(x\right) F^{n-1}\left(x - \ln p_i + \ln p\right) dx, \\ D_1\left(p_i, p; n\right) &= -\frac{1+\psi}{p_i^{2+\psi}} \int_{w_l}^{w_u} e^{\psi x} f\left(x\right) F^{n-1}\left(x - \ln p_i + \ln p\right) dx \\ &\quad - \frac{n-1}{p_i^{2+\psi}} \int_{w_l}^{w_u} e^{\psi x} f\left(x\right) f\left(x - \ln p_i + \ln p\right) F^{n-2}\left(x - \ln p_i + \ln p\right) dx. \end{aligned}$$

In a symmetric equilibrium

$$\begin{aligned} D(p, p; n) &= \frac{1}{p^{1+\psi}} \int_{w_l}^{w_u} e^{\psi x} f\left(x\right) F^{n-1}\left(x\right) dx \\ D_1(p, p; n) &= -\frac{1}{p^{2+\psi}} \left((1+\psi) \int_{w_l}^{w_u} e^{\psi x} f\left(x\right) F^{n-1}\left(x\right) dx + (n-1) \int_{w_l}^{w_u} e^{\psi x} f^2\left(x\right) F^{n-2}\left(x\right) dx \right). \end{aligned}$$

With some simple calculations, it follows that the Hart markup in symmetric equilibrium is[15]

$$\begin{aligned} \mu_n^{Hart} &= -\frac{D(p, p; n)}{D_1(p, p; n)} \\ &= c\left(\psi + (n-1) \frac{\int e^{\psi x} f^2\left(x\right) F^{n-2}\left(x\right)}{\int e^{\psi x} f\left(x\right) F^{n-1}\left(x\right) dx}\right)^{-1}. \end{aligned}$$

[15] We can also describe a production problem with the same solution as the Hart model. A firm uses two inputs, Q (capital) and L (labor), in production. When the firm chooses technology $i \in \{1, ..., n\}$ and input quantities Q and L, the production function is $Y = \left(\exp\left(X_i\right) Q\right)^a L^b$. Let X_i be a random technology shock that is i.i.d. as F across technologies. The profit function is $\pi = Y - tQ - wL$, so that the output price is normalized at unity while capital and labor have marginal cost t.and w respectively. Solving the first order conditions for profit maximization, we get demand for $Q = c(w) t^{-1-\frac{a}{1-a-b}} \exp\left(\frac{a}{1-a-b} X\right)$ where $c(w) = a^{1+\frac{a}{1-a-b}} b^{\frac{b}{1-a-b}} w^{-\frac{b}{1-a-b}}$.Thus demand for capital Q now is like in (10), with $\psi = \frac{a}{1-a-b}$. Assuming exogenous cost of labor w and an MC market for capital, we obtain the same pricing problem for technology as in the Hart demand model.

9 Appendix C: Proofs of Economic Claims

We give the proofs to the economic implications using the main Theorem 1.

9.1 Asymptotic Markups

This part derives the asymptotic markups for the three different pricing models.

9.1.1 Perloff-Salop, Sattinger and Hart Models

Proof of Theorem 2.

The Perloff-Salop and Sattinger cases follow immediately from Proposition 1; we will omit those calculations and focus on the Hart case. Applying Proposition 1 to (11), we immediately infer that

$$\frac{\mu_n^{Hart}}{c} \sim \frac{1}{\psi + nf\left(\overline{F}^{-1}(1/n)\right)\frac{\Gamma(\gamma+2-\psi a)}{\Gamma(1-\psi a)}}$$

under the conditions of the theorem. We will use the fact that $anf\left(\overline{F}^{-1}(1/n)\right) \sim 1$, which holds because

$$\lim_{n\to\infty} \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)} = \lim_{x\to w_u} \frac{\overline{F}(x)}{f(x)} = a.$$

Consider first the case where $a = 0$. Then $nf\left(\overline{F}^{-1}(1/n)\right) \to \infty$, and the expression simplifies to

$$\frac{\mu_n^{Hart}}{c} \sim \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)\left[\frac{\psi}{nf\left(\overline{F}^{-1}(1/n)\right)} + \frac{\Gamma(\gamma+2-\psi a)}{\Gamma(1-\psi a)}\right]} \sim \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)\Gamma(\gamma+2)}.$$

Next, consider the case $0 < a < \infty$, which implies $\gamma = 0$. We have

$$\begin{aligned}
\frac{\mu_n^{Hart}}{c} &\sim \frac{1}{\psi + nf\left(\overline{F}^{-1}(1/n)\right)\frac{\Gamma(2-\psi a)}{\Gamma(1-\psi a)}} = \frac{1}{\psi + nf\left(\overline{F}^{-1}(1/n)\right)(1-\psi a)} \\
&= \frac{1}{\psi\left(1 - anf\left(\overline{F}^{-1}(1/n)\right)\right) + nf\left(\overline{F}^{-1}(1/n)\right)} \\
&\sim \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)} = \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)\Gamma(2+\gamma)}
\end{aligned}$$

when $\gamma = 0$. ■

9.2 Economic Implications

Proof of Proposition 2.

First we treat the Perloff-Salop case. Treating n as continuous, we have

$$\frac{n}{\mu_n^{PS}} \frac{d\mu_n^{PS}}{dn} = -\left(\frac{2n-1}{n-1} + \frac{n \int f^2(x) F^{n-2}(x) \log F(x)\, dx}{\int f^2(x) F^{n-2}(x)\, dx} \right).$$

Noting that $-\log(1-x) \sim x \in RV_1^0$, applying Theorem 1 to $G(x) \equiv \frac{f(x)}{F(x)} \log F(x)$, using Lemma 3(3), we obtain

$$\int f^2(x) F^{n-2}(x) \log F(x) dx \sim -n^{-2} f\left(\overline{F}^{-1}(1/n)\right) \Gamma(3+\gamma).$$

Together with Theorem 2, it follows that

$$\frac{n}{\mu_n} \frac{d\mu_n}{dn} = -\left(2 - \frac{n^{-2} n f\left(\overline{F}^{-1}(1/n)\right) \Gamma(3+\gamma)}{n^{-2} n f\left(\overline{F}^{-1}(1/n)\right) \Gamma(2+\gamma)} + o(1) \right) = \gamma + o(1).$$

Note that this also proves our claim for the Sattinger case. Next we treat the Hart case. Notationally, let

$$J_{\psi,n}(k,l) = \int e^{\psi x} f^k(x) F^{n-k}(x) (\log F(x))^l\, dx.$$

Then we have

$$\mu_n^{Hart} = c \left(\psi + (n-1) \frac{\int e^{\psi x} f^2(x) F^{n-2}(x)\, dx}{\int e^{\psi x} f(x) F^{n-1}(x)\, dx} \right)^{-1}, \text{ so}$$

$$\frac{n}{\mu_n^{Hart}} \frac{d\mu_n^{Hart}}{dn} = -n \frac{\left(\frac{J_{\psi,n}(2,0)}{J_{\psi,n}(1,0)} + (n-1) \frac{J_{\psi,n}(1,0) J_{\psi,n}(2,1) - J_{\psi,n}(2,0) J_{\psi,n}(1,1)}{J_{\psi,n}(1,0)^2} \right)}{\left(\psi + (n-1) \frac{J_{\psi,n}(2,0)}{J_{\psi,n}(1,0)} \right)}.$$

Again, using the methods from Proposition 1, we may show that

$$J_{\psi,n}(k,l) \sim \begin{cases} (-1)^l n^{-l-1} f^{k-1}\left(\overline{F}^{-1}(1/n)\right) e^{\psi \overline{F}^{-1}(1/n)} \Gamma\left((\gamma+1)(k-1)+l+1\right) : \gamma < 0 \\ (-1)^l n^{-l-1} f^{k-1}\left(\overline{F}^{-1}(1/n)\right) e^{\psi \overline{F}^{-1}(1/n)} \Gamma\left(k+l-\psi a\right) : \gamma = 0 \end{cases}.$$

In the case $a = 0$, we may then verify that

$$\frac{n}{\mu_n^{Hart}} \frac{d\mu_n^{Hart}}{dn} = -\frac{2\Gamma(\gamma+2) - \Gamma(\gamma+3)}{\Gamma(\gamma+2)} + o(1) = \gamma + o(1).$$

In the case $0 < a < \infty$ (which implies $\gamma = 0$; we do not consider $\gamma > 0$), we may verify that

$$\lim_{n\to\infty} \frac{n}{\mu_n^{Hart}} \frac{d\mu_n^{Hart}}{dn} = -(1-\psi a)\left[(1-\psi a) - (1-\psi a)\right] = 0. \blacksquare$$

Proof of Proposition 3

We first show a lemma that links differences between the two top order statistics to the behavior of the top tail statistics, and hence allows us to apply our general results.

Lemma 4 *Call M_n and S_n, respectively, the largest and second largest realizations of n i.i.d. random variables with CDF F and density $f = F'$, and G a function such that $\int G(x) f(x) dx < \infty$, $\lim_{x\to F^{-1}(0)} G(x) F(x) = \lim_{x\to F^{-1}(1)} G(x) \overline{F}(x) = 0$. Then:*

$$\mathbb{E}\left[G(M_n) - G(S_n)\right] = \mathbb{E}\left[\frac{G'(M_n)\overline{F}(M_n)}{f(M_n)}\right] \tag{37}$$

Proof: Recall that the density of M_n is $nf(x) F^{n-1}(x)$, and the density of S_n is

$$n(n-1) f(x) \overline{F}(x) F^{n-2}(x).$$

So

$$\begin{aligned}
\mathbb{E}\left[G(S_n)\right] &= \int n(n-1) G(x) f(x) \overline{F}(x) F^{n-2}(x) dx \\
&= n\left[G(x)\overline{F}(x) F^{n-1}(x)\right]_{F^{-1}(0)}^{F^{-1}(1)} - \int n\left(G(x)\overline{F}(x)\right)' F^{n-1}(x) dx \\
&= 0 + \int nG(x) f(x) F^{n-1}(x) dx - \int n\frac{G'(x)\overline{F}(x)}{f(x)} f(x) F^{n-1}(x) dx \\
&= \mathbb{E}\left[G(M_n)\right] - \mathbb{E}\left[\frac{G'(M_n)\overline{F}(M_n)}{f(M_n)}\right]
\end{aligned}$$

From this lemma, the proof follows for $G(x) = x$. As $f(\overline{F}^{-1}(t)) \in RV_{1+\gamma}^0$, $t/f(\overline{F}^{-1}(t)) \in RV_{-\gamma}^0$, and we may apply Theorem 1 to obtain the desired result. $\blacksquare$

Proof of Proposition 4

First, some notation: $\pi\left(\left(p,\sigma\right),\left(p^{*},\sigma^{*}\right);n\right)$ denotes the profit function of a firm that chooses $\left(p,\sigma\right)$ when the remaining $n-1$ firms choose $\left(p^{*},\sigma^{*}\right)$. Also, $\pi\left(p,\sigma;n\right)$ denotes the profit function of a firm when all n firms choose $\left(p,\sigma\right)$.

Perloff-Salop Case

Call σ^{*} and p^{*} the equilibrium choices of the other firms:

$$\begin{aligned}\pi\left(\left(p,\sigma\right),\left(p^{*},\sigma^{*}\right);n\right) &= \left(p-c\left(\sigma\right)\right)\mathbb{P}\left(\sigma X_{1}-p\geq\max_{j\neq i}\sigma^{*}X_{j}-p^{*}\right)\\ &= \left(p-c\left(\sigma\right)\right)\mathbb{P}\left(\frac{\sigma}{\sigma^{*}}X_{i}+\frac{p^{*}-p}{\sigma^{*}}\geq\max_{j\neq i}X_{j}\right)\\ &= \left(p-c\left(\sigma\right)\right)\int f\left(x\right)F^{n-1}\left(\frac{\sigma}{\sigma^{*}}x+\frac{p^{*}-p}{\sigma^{*}}\right)dx.\end{aligned}$$

The first-order conditions for profit maximization are as follows. Differentiating with respect to p yields

$$p-c\left(\sigma\right)=\frac{\int f\left(x\right)F^{n-1}\left(x\right)dx}{\frac{1}{\sigma}\left(n-1\right)\int f^{2}\left(x\right)F^{n-2}\left(x\right)dx}$$

and differentiating with respect to σ gives

$$c'\left(\sigma\right)\int f\left(x\right)F^{n-1}\left(x\right)dx=\left(n-1\right)\left(p-c\left(\sigma\right)\right)\int xf^{2}\left(x\right)F^{n-2}\left(x\right)dx\frac{1}{\sigma}.$$

Some manipulation reveals

$$c'\left(\sigma\right)=\frac{\int xf^{2}\left(x\right)F^{n-2}\left(x\right)dx}{\int f^{2}\left(x\right)F^{n-2}\left(x\right)dx}.$$

Now we consider two cases: $w_{u}<\infty$ and $w_{u}=\infty$. If $w_{u}<\infty$, then

$$\frac{\int xf^{2}\left(x\right)F^{n-2}\left(x\right)dx}{\int f^{2}\left(x\right)F^{n-2}\left(x\right)dx}=\frac{n^{-1}w_{u}f\left(\overline{F}^{-1}\left(1/n\right)\right)\Gamma\left(\gamma+2\right)}{n^{-1}f\left(\overline{F}^{-1}\left(1/n\right)\right)\Gamma\left(\gamma+2\right)}+o\left(1\right)=w_{u}+o\left(1\right).$$

If $w_{u}=\infty$ then

$$\frac{\int xf^{2}\left(x\right)F^{n-2}\left(x\right)dx}{\int f^{2}\left(x\right)F^{n-2}\left(x\right)dx}\sim\frac{n^{-1}\overline{F}^{-1}\left(1/n\right)f\left(\overline{F}^{-1}\left(1/n\right)\right)\Gamma\left(2\right)}{n^{-1}f\left(\overline{F}^{-1}\left(1/n\right)\right)\Gamma\left(\gamma+2\right)}\sim\frac{\overline{F}^{-1}\left(1/n\right)}{\Gamma\left(\gamma+2\right)}.$$

Sattinger Case

We have

$$\pi\left(\left(p,\sigma\right),\left(p^{*},\sigma^{*}\right);n\right) = \frac{p-c\left(\sigma\right)}{p}\mathbb{P}\left(\frac{e^{\sigma X_i}}{p} \geq \max_{j\neq i}\frac{e^{\sigma^{*}X_j}}{p^{*}}\right)$$
$$= \frac{p-c\left(\sigma\right)}{p}\int f\left(x\right)F^{n-1}\left(\frac{\sigma}{\sigma^{*}}x+\frac{\log p^{*}-\log p}{\sigma^{*}}\right)dx$$

so the first-order conditions for profit maximization become

$$0=\pi_2\left(p,\sigma;n\right) = -\frac{c'\left(\sigma\right)}{p}\int f\left(x\right)F^{n-1}\left(x\right)dx+\frac{p-c\left(\sigma\right)}{\sigma p}\left(n-1\right)\int xf^{2}\left(x\right)F^{n-2}\left(x\right)dx$$

and

$$0=\pi_1\left(p,\sigma;n\right)=\frac{c\left(\sigma\right)}{p^{2}}\int f\left(x\right)F^{n-1}\left(x\right)dx-\frac{p-c\left(\sigma\right)}{\sigma p^{2}}\left(n-1\right)\int f^{2}\left(x\right)F^{n-2}\left(x\right)dx$$

Rearranging, we get

$$\frac{p-c\left(\sigma\right)}{c\left(\sigma\right)}=\frac{\sigma}{n\left(n-1\right)\int f^{2}\left(x\right)F^{n-2}\left(x\right)dx}$$

and

$$c'\left(\sigma\right)=\frac{\frac{p-c(\sigma)}{\sigma p}\left(n-1\right)\int xf^{2}\left(x\right)F^{n-2}\left(x\right)dx}{\int f\left(x\right)F^{n-1}\left(x\right)dx},$$

so

$$\frac{c'\left(\sigma\right)}{c\left(\sigma\right)}=\frac{\int xf^{2}\left(x\right)F^{n-2}\left(x\right)dx}{\int f^{2}\left(x\right)F^{n-2}\left(x\right)dx}$$
$$=\begin{cases}\overline{F}^{-1}\left(1/n\right)+o\left(\overline{F}^{-1}\left(1/n\right)\right)=w_u+o\left(1\right):w_u<\infty\\ \frac{\overline{F}^{-1}(1/n)+o\left(\overline{F}^{-1}(1/n)\right)}{\Gamma(\gamma+2)}:w_u=\infty\end{cases}$$

as calculated in the Perloff-Salop case.

Hart Case

We have

$$\begin{aligned}
\pi\left(\left(p,\sigma\right),\left(p^{*},\sigma^{*}\right);n\right) &= \left(p-c\left(\sigma\right)\right)\mathbb{E}\left[\frac{e^{\psi\sigma X_i}}{p^{1+\psi}}I_{\left\{\frac{e^{\sigma X_i}}{p}\geq\max_{j\neq i}\frac{e^{\sigma^{*}X_j}}{p^{*}}\right\}}\right] \\
&= \left(p-c\left(\sigma\right)\right)\mathbb{E}\left[\frac{e^{\psi\sigma X_i}}{p^{1+\psi}}I_{\left\{\frac{\sigma}{\sigma^{*}}X_i+\frac{\log p^{*}-\log p}{\sigma^{*}}=\max_{j\neq i}X_j\right\}}\right] \\
&= \left(p-c\left(\sigma\right)\right)\int\frac{e^{\psi\sigma x}}{p^{1+\psi}}f\left(x\right)F^{n-1}\left(\frac{\sigma}{\sigma^{*}}x+\frac{\log p^{*}-\log p}{\sigma^{*}}\right)dx
\end{aligned}$$

so the first-order conditions for profit maximization become

$$0=\pi_2\left(p,\sigma;n\right)=-c'\left(\sigma\right)\int\frac{e^{\psi\sigma x}}{p^{1+\psi}}f\left(x\right)F^{n-1}\left(x\right)dx+\left(p-c\left(\sigma\right)\right)\left\{\begin{array}{c}\int\psi x\frac{e^{\psi\sigma x}}{p^{1+\psi}}f\left(x\right)F^{n-1}\left(x\right)dx\\+\frac{n-1}{\sigma}\int x\frac{e^{\psi\sigma x}}{p^{1+\psi}}f^{2}\left(x\right)F^{n-2}\left(x\right)dx\end{array}\right\}$$

and

$$0=\pi_1\left(p,\sigma;n\right)=\int\frac{e^{\psi\sigma x}}{p^{1+\psi}}f\left(x\right)F^{n-1}\left(x\right)dx-\left(p-c\left(\sigma\right)\right)\left\{\begin{array}{c}\left(1+\psi\right)\int\frac{e^{\psi\sigma x}}{p^{2+\psi}}f\left(x\right)F^{n-1}\left(x\right)dx\\+\frac{n-1}{\sigma}\int\frac{e^{\psi\sigma x}}{p^{2+\psi}}f^{2}\left(x\right)F^{n-2}\left(x\right)dx\end{array}\right\}$$

so

$$\frac{p-c\left(\sigma\right)}{c\left(\sigma\right)}=\frac{\int e^{\psi\sigma x}f\left(x\right)F^{n-1}\left(x\right)dx}{\psi\int e^{\psi\sigma x}f\left(x\right)F^{n-1}\left(x\right)dx+\frac{n-1}{\sigma}\int e^{\psi\sigma x}f^{2}\left(x\right)F^{n-2}\left(x\right)dx}$$

and

$$\begin{aligned}
\frac{c'\left(\sigma\right)}{c\left(\sigma\right)} &= \frac{p-c\left(\sigma\right)}{c\left(\sigma\right)}\frac{\int\psi xe^{\psi\sigma x}f\left(x\right)F^{n-1}\left(x\right)dx+\frac{n-1}{\sigma}\int xe^{\psi\sigma x}f^{2}\left(x\right)F^{n-2}\left(x\right)dx}{\int e^{\psi\sigma x}f\left(x\right)F^{n-1}\left(x\right)dx} \\
&= \frac{\psi\int xe^{\psi\sigma x}f\left(x\right)F^{n-1}\left(x\right)dx+\frac{n-1}{\sigma}\int xe^{\psi\sigma x}f^{2}\left(x\right)F^{n-2}\left(x\right)dx}{\psi\int e^{\psi\sigma x}f\left(x\right)F^{n-1}\left(x\right)dx+\frac{n-1}{\sigma}\int e^{\psi\sigma x}f^{2}\left(x\right)F^{n-2}\left(x\right)dx}.
\end{aligned}$$

Now we consider two cases: $w_u<\infty$ and $w_u=\infty$. If $w_u<\infty$, then (noting that $a=0$ in this case)

$$\frac{c'(\sigma)}{c(\sigma)} = \frac{\psi \int x e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int x e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}{\psi \int e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}$$

$$= \frac{\psi n^{-1} w_u e^{\sigma\psi w_u} \Gamma(1) + \frac{1}{\sigma} w_u e^{\sigma\psi w_u} f\left(\overline{F}^{-1}(1/n)\right) \Gamma(\gamma + k)}{\psi n^{-1} e^{\sigma\psi w_u} \Gamma(1) + \frac{1}{\sigma} e^{\sigma\psi w_u} f\left(\overline{F}^{-1}(1/n)\right) \Gamma(\gamma + k)} + o(1)$$

$$= w_u + o(1).$$

If $w_u = \infty$, then noting that $\gamma = 0$,

$$\frac{c'(\sigma)}{c(\sigma)} = \frac{\psi \int x e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int x e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}{\psi \int e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}$$

$$\sim \frac{\psi n^{-1} \overline{F}^{-1}(1/n)\, e^{\sigma\psi \overline{F}^{-1}(1/n)} \Gamma(1 - \psi a) + \frac{1}{\sigma} \psi \overline{F}^{-1}(1/n)\, f\left(\overline{F}^{-1}(1/n)\right) e^{\sigma\psi \overline{F}^{-1}(1/n)} \Gamma(2 - \psi a)}{\psi n^{-1} e^{\sigma\psi \overline{F}^{-1}(1/n)} \Gamma(1 - \psi a) + \frac{1}{\sigma} \psi f\left(\overline{F}^{-1}(1/n)\right) e^{\sigma\psi \overline{F}^{-1}(1/n)} \Gamma(2 + \gamma - \psi a)}$$

$$= \overline{F}^{-1}(1/n).$$

Proof of Proposition 5

Recall that $F(x) = \exp\left(-e^{-x/\phi}\right)$ and $f(x) = \frac{1}{\phi} \exp\left(-\frac{x}{\phi} - e^{-x/\phi}\right)$. Starting with (10), some calculations reveal

$$D(p_1, ..., p_n; \sigma^*) = \frac{1}{p_i^{1+\psi}} \int_{w_l}^{w_u} e^{\psi\sigma^* x} f(x) \prod_{j \neq i} F\left(x + \frac{\ln p_i - \ln p_j}{\sigma^*}\right) dx$$

$$= \frac{1}{\phi p_i^{1+\psi}} \int_{-\infty}^{\infty} \exp\left(x\left(\sigma^*\psi - \frac{1}{\phi}\right) - \sum_{j=1}^{n} \left(\frac{p_i}{p_j}\right)^{1/(\phi\sigma^*)} e^{-x/\phi}\right) dx$$

$$= \Gamma(1 - \phi\psi\sigma^*) \frac{p_i^{-(1+1/(\phi\sigma^*))}}{\left(\sum_{j=1}^{n} p_j^{-1/(\phi\sigma^*)}\right)^{1-\phi\psi\sigma^*}}. \blacksquare$$

9.3 Results relating to Enhanced Linear Random Utility

Proof of Theorem 3.

Recall that the demand function for good i at price p (given the prices p_j of all other goods) is the probability that the consumer's payoff for good i exceeds his payoff to all other goods, as well as the outside option:

$$D\left(p_1, ..., p_n; i\right) = \mathbb{P}\left(-\beta p_i + X_i \geq \max\left\{\max_{j \neq i}\left\{-\beta p_j + X_j\right\}, \epsilon_0\right\}\right).$$

We start the analysis for the case that the densities f are of the Rootzen type (26). Consider the symmetric pricing equilibrium $p_j = p$. Define the convolution $\beta p + \epsilon_0 = Q$ and denote its distribution function and density respectively by $K(.)$ and $k(.)$. Suppose that the support of K is bounded below by 0 and bounded above by $\overline{q}(p) < \infty$, say. Moreover, assume that the distribution $F(.)$ has a wider support such that $F(\overline{q}(p)) < 1$, i.e. has mass beyond $\overline{q}(p)$. Then the demand for the i-th differentiated good in the symmetric equilibrium reads

$$\begin{aligned} D(p_i, p_j) &= \mathbb{P}\left\{X_i - \beta p_i \geq \left(\bigvee_{j \neq i}\left[X_j - \beta p_j\right]\right) \vee \epsilon_0\right\} \\ &= \mathbb{P}\left\{X_i \geq \bigvee_{j \neq i} X_j \vee \left[\epsilon_0 + \beta p\right]\right\} \\ &= \mathbb{P}\left\{X_i \geq \bigvee_{j \neq i} X_j \vee Q\right\} \\ &= \mathbb{E}_Q\left[\mathbb{P}\left\{X_i \geq \bigvee_{j \neq i} X_j \vee q\right\} \middle| Q = q\right] \\ &= \mathbb{E}_Q\left[\int_q^\infty F^{n-1}(s) f(s) ds\right] \\ &= \frac{1}{n} - \frac{1}{n}\mathbb{E}_Q\left[F^n(q)\right] \\ &= \frac{1}{n} + o(\frac{1}{n}). \end{aligned}$$

To obtain the partial derivative $\partial D(p_i, p_j)/\partial p_i$ note that we can alternatively express demand in a symmetric equilibrium in terms of the distribution of X_i

$$
\begin{aligned}
D(p_i,p_j)\big|_{p_i=p_j=p} &= \mathbb{P}\left\{X_i-\beta p_i \geq \left(\bigvee_{j\neq i}[X_j-\beta p_j]\right)\vee \epsilon_0\right\}\Bigg|_{p_i=p_j=p} \\
&= \mathbb{P}\left\{X_i-\beta p \geq \left(\bigvee_{j\neq i}[X_j-\beta p]\right)\vee \epsilon_0\right\} \\
&= \mathbb{P}\left\{X_i \geq \bigvee_{j\neq i} X_j \vee [\epsilon_0+\beta p]\right\} \\
&= 1-\mathbb{P}\left\{X_i \leq M_{n-1}\vee[\epsilon_0+\beta p]\right\} \\
&= 1-\mathbb{E}_Q\mathbb{E}_{M_{n-1}}\left[\mathbb{P}\left\{X_i\leq m_{n-1}\vee q\right\}\big| M_{n-1}=m_{n-1},\epsilon_0+\beta p=q\right] \\
&= 1-\mathbb{E}_Q\mathbb{E}_{M_{n-1}}\left[F\left(m_{n-1}\vee q\right)\right].
\end{aligned}
$$

This facilitates the differentiation with respect p_i at a symmetric equilibrium

$$
\begin{aligned}
\frac{\partial D(p_i,p_j)}{\partial p_i}\bigg|_{p_i=p_j=p} &= \frac{\partial\left\{1-\mathbb{E}_{M_{n-1}}\mathbb{E}_Y\left[F\left(m_{n-1}\vee\epsilon_0+\beta p\right)\right]\right\}}{\partial p_i} \\
&= -\mathbb{E}_{\beta,\epsilon_0}\mathbb{E}_{M_{n-1}}\left[f\left(m_{n-1}\vee q\right)*\beta\right] \\
&= -\mathbb{E}_{\beta,\epsilon_0}\left[\beta\int_q^\infty (n-1)\,f(s)F^{n-2}(s)f(s)ds\right]
\end{aligned}
$$

We like to use the Theorem 1. To this end we first need the asymptotic inverse of the Rootzen distributions

$$
\overline{F}\left(x\right)=1-F\left(x\right)\sim\kappa x^{\nu}\exp\left(-\lambda x^{\phi}\right),\ \kappa>0,\ \lambda>0,\ \phi\geq 1,\ \nu\in\mathbb{R} \tag{38}
$$

The inverse of the asymptotic upper tail of (38) reads

$$
\overline{F}^{-1}\left(y\right)\sim\left(\frac{1}{\lambda}\right)^{1/\phi}\left[\ln\left(\frac{\kappa\lambda^{-\nu/\phi}}{y}\times\left[\ln\left(\frac{\kappa\lambda^{-\nu/\phi}}{y}\right)\right]^{\nu/\phi}\right)\right]^{1/\phi}
$$

for y close to zero, see Li (2008). We need to ensure that $f\left(\overline{F}^{-1}\left(y\right)\right)\in RV_{\rho}^{0}$ with $\rho>-1$. To check this, we first simplify notation. Write shorthand $A=\kappa\lambda^{-\nu/\phi}$. Then for y close to zero,

$$f\left(\overline{F}^{-1}(y)\right) \sim \kappa \left\{ \lambda\phi \left[\left(\frac{1}{\lambda}\right)^{1/\phi} \left[\ln\left(\frac{A}{y} \times \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right)\right]^{1/\phi}\right]^{\phi} - \nu \right\} \times$$
$$\left[\left(\frac{1}{\lambda}\right)^{1/\phi} \left[\ln\left(\frac{A}{y} \times \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right)\right]^{1/\phi}\right]^{\nu-1} \times$$
$$\exp\left(-\lambda \left[\left(\frac{1}{\lambda}\right)^{1/\phi} \left[\ln\left(\frac{A}{y} \times \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right)\right]^{1/\phi}\right]^{\phi} \right)$$
$$= \kappa \left\{ \phi \ln\left(\frac{A}{y} \times \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right) - \nu \right\} \left(\frac{1}{\lambda}\right)^{(\nu-1)/\phi} \left[\ln\left(\frac{A}{y} \times \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right)\right]^{(\nu-1)/\phi} \times$$
$$\exp\left(-\ln\left(\frac{A}{y} \times \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right)\right)$$
$$= y\lambda^{1/\phi} \left[\ln\frac{A}{y}\right]^{-\nu/\phi} \left[\ln\left(\frac{A}{y} \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right)\right]^{(\nu-1)/\phi} \left\{ \phi \ln\left(\frac{A}{y} \left[\ln\frac{A}{y}\right]^{\nu/\phi}\right) - \nu \right\}.$$

Taking ratios to investigate the regular variation property and writing $t = 1/y$

$$\lim_{y\downarrow 0} \frac{f\left(\overline{F}^{-1}(xy)\right)}{f\left(\overline{F}^{-1}(y)\right)} = \lim_{t\to\infty} \frac{f\left(\overline{F}^{-1}(x/t)\right)}{f\left(\overline{F}^{-1}(1/t)\right)}$$
$$= x \lim_{t\to\infty} \frac{\left[\ln \frac{At}{x}\right]^{-\nu/\phi} \left[\ln\left(\frac{At}{x}\left[\ln\frac{At}{x}\right]^{\nu/\phi}\right)\right]^{(\nu-1)/\phi} \left\{\phi \ln\left(\frac{At}{x}\left[\ln\frac{At}{x}\right]^{\nu/\phi}\right) - \nu\right\}}{\left[\ln At\right]^{-\nu/\phi} \left[\ln\left(At\left[\ln At\right]^{\nu/\phi}\right)\right]^{(\nu-1)/\phi} \left\{\phi \ln\left(At\left[\ln At\right]^{\nu/\phi}\right) - \nu\right\}}$$
$$= x.$$

Hence, $f\left(\overline{F}^{-1}(y)\right) \in RV_1^0$, so that $\rho > -1$. Define the function $G(s)$ from the Theorem 1 as the density $f(s)$ from (26)

$$G(s) = \kappa\lambda\phi x^{\phi+\nu-1} \exp\left(-\lambda x^{\phi}\right).$$

Thus $G(x)$ is positive, moreover $\widehat{G}(s)$ is integrable on $(0, 1)$. Lastly, recall that q is bounded.

We can now apply the main Theorem 1 to obtain an asymptotic expression for the markup in the ELRU setting. At given values β and ϵ_0:

$$\left.\frac{\partial D(p,p)}{\partial p_i}\right|_{\epsilon_0+\beta p=q} = -\left(n-1\right)\beta\int_q^{\infty} f(s)F^{n-2}(s)f(s)ds$$
$$= -\left(n-1\right)\beta\int_q^{\infty} G(s)F^{n-2}(s)f(s)ds$$
$$\sim -\beta G\left(F^{-1}\left(1-\frac{1}{n-1}\right)\right)\Gamma\left(2\right)$$
$$\sim -\beta\phi\lambda^{1/\phi}\frac{1}{n}\left(\ln n\right)^{1-1/\phi}.$$

Note that the lower bound in the integral disappears, since what matters in the main Theorem 1 is the behavior of G at the upper end of the support of F. Next we take the stochastic nature of β and ϵ_0 into account. After substitution

$$-\mathbb{E}_{\beta,\epsilon_0}\left[\beta\int_q^{\infty}\left(n-1\right)f(s)F^{n-2}(s)f(s)ds\right] = -\left[\phi\lambda^{1/\phi}\frac{1}{n}\left(\ln n\right)^{1-1/\phi}\right]\mathbb{E}[\beta].$$

It follows that

$$\mu_n^{ELRU} \sim \frac{1/\mathbb{E}[\beta]}{\phi\lambda^{1/\phi}\left(\ln n\right)^{1-1/\phi}}.\blacksquare$$

Next, we turn to the case where the distribution of (β,ϵ_0) is unbounded and varies regularly at infinity. The following results are used in the proof of Theorem 4, which we demonstrate at the end of this section.

Lemma 5 *Under the assumptions of Theorem 4 and assuming* $w_u=\infty$ *for simplicity,*

$$D\left(p,p\right)\sim_{n\to\infty}\frac{1}{n}.$$

Proof of Lemma 5

Some notation: we use $x\vee y$ to denote $\max\left\{x,y\right\}$. Let $H_p\left(x\right)$ be the distribution function and (as mentioned previously) $h_p\left(x\right)$ be the density function of $\beta p+\epsilon_0$. Abusing notation, denote $H_p = \beta p+\epsilon_0$. Note that the assumption that the density $h\left(\beta,\epsilon_0\right)$ is multivariate regularly varying at infinity implies that $h_p\left(x\right)$ is regularly varying at infinity for all p.

Obviously, M_{n-1} is independent of $Q=\epsilon_0+\beta p$ and therefore

$$D(p,p)=\mathbb{P}[X_i\geq\bigvee_{j\neq i}X_j\vee Q]=\int_{-\infty}^{\infty}F^{n-1}(s)H(s)dF(s).$$

Let

$$\phi = \int_{-\infty}^{\infty} H(s)dF(s) = \mathbb{P}[Q \leq X_i]]$$

and

$$J(x) = \int_{-\infty}^{x} H(s)dF(s)/\phi.$$

Then from Resnick (1971, pp. 204) we have

$$nD(p,p) = \phi n \int_{-\infty}^{\infty} F^{n-1}(s)dH(s) \to \ell,$$

if and only if

$$\lim_{x\to\infty} \frac{1-J(x)}{\overline{F}(x)} = \frac{\ell}{\phi}. \tag{39}$$

For (39) to hold we require

$$\lim_{x\to\infty} \frac{\int_x^{\infty} H(s)dF(s)}{\overline{F}(x)} = \ell.$$

But then we see $\ell = 1$ since

$$H(x)\frac{\overline{F}(x)}{\overline{F}(x)} \leq \frac{\int_x^{\infty} H(s)dF(s)}{\overline{F}(x)} \leq 1 \cdot \frac{\overline{F}(x)}{\overline{F}(x)},$$

and $H(s) \to 1$ as $s \to \infty$. ■

We now develop the asymptotic form for $D_1(p,p)$. It will be convenient to use the function

$$Z(x) = -\log F(x).$$

Note that

$$f\left(Z^{-1}(1/t)\right) = f\left(\overline{F}^{-1}\left(1-e^{-1/t}\right)\right) \sim f\left(\overline{F}^{-1}\left(\frac{1}{t}\right)\right)$$

by Lemma 3(ii).

Before proceeding, we first introduce an intermediate lemma that follows immediately from Proposition 1 and Lemma 1.1.

Lemma 6 *Assume that the conditions of Lemma 5 are satisfied. Assume also that F is well-behaved with tail index $\gamma = 0$, and that $f^2(x)$ is $[w_l, w_u)$-integrable. Then for any $r \geq 1$,*

$$\mathbb{E}\left[f\left(M_n\right)^r\right] \sim_{n\to\infty} \Gamma\left(1+r\right)\left(f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)\right)^r \in RV_{-r}^{\infty} \tag{40}$$

Lemma 7 *Let F be well-behaved with $\gamma = 0$ and $a < \infty$. Assume $r \geq 1$, $s \geq 1$ and $\mathbb{E}[\beta^s] < \infty$. Then*

$$(\mathbb{E}[(f(Q))^r 1_{M_n \leq Q}]) \leq o\left(f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)^r\right), \quad n \to \infty, \tag{41}$$

which implies

$$\mathbb{E}[(f(Q)) 1_{M_n \leq Q}\beta] \leq (\mathbb{E}[(f(Q))^r 1_{M_n \leq Q}])^{1/r} \mathbb{E}[\beta^s]^{1/s} = o\left(f\left(\overline{F}^{-1}\frac{1}{n}\right)\right). \tag{42}$$

Proof of Lemma 7 Let

$$\hat{V}(n) = \mathbb{E}[(f(Q))^r 1_{M_n \leq Q}]$$

and

$$V(\frac{1}{n}) = \int_0^{1/n} \left(f(Z^{-1}(y))\right)^r h\left(Z^{-1}(y)\right) d\left(-Z^{-1}(y)\right) = \int_0^{1/n} \left(f(Z^{-1}(y))\right)^{r-1} h(Z^{-1}(y)) e^{-y} dy$$
$$= \int_n^{\infty} \left(f(Z^{-1}(1/y))\right)^{r-1} h(Z^{-1}(1/y)) e^{-1/y} \frac{dy}{y^2}.$$

Note that Karamata's Tauberian theorem (Bingham, Goldie and Teugels 1989, Theorem 1.7.1', p. 38) implies $\hat{V}(n) = o\left(f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)^r\right)$ iff $V(1/n) = o\left(f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)^r\right)$. Write

$$\begin{aligned}
V(1/n) \leq& \sup_{y \geq n} h(Z^{-1}(y)) \int_n^{\infty} \left(f(Z^{-1}(1/y))\right)^{r-1} e^{-1/y} \frac{dy}{y^2} \\
\sim& (const) \sup_{y \geq n} h(Z^{-1}(1/y)) \left(f\left(Z^{-1}(1/n)\right)\right)^{r-1} / n \\
=& (const) \sup_{y \geq n} h(Z^{-1}(1/y)) \left(f\left(Z^{-1}(1/n)\right)\right)^r / \left(nf\left(Z^{-1}(1/n)\right)\right) \\
=& o\left(f\left(Z^{-1}(1/n)\right)\right)^r.
\end{aligned}$$

We then apply Holder's inequality with appropriate choice of r and s to obtain

$$\mathbb{E}[(f(Q)) 1_{M_n \leq Q}\beta] \leq (\mathbb{E}[(f(Q))^r 1_{M_n \leq Q}])^{1/r} \mathbb{E}[\beta^s]^{1/s} = o\left(f\left(\overline{F}^{-1}\frac{1}{n}\right)\right). \tag{43}$$

Proposition 8 *Let F be well-behaved with $\gamma = 0$ and suppose $h^{-1}(Z^{-1}(1/y))$ is regularly* *varying and*

$$\lim_{t\to\infty}\sup_{s\geq t} h(s)\frac{\overline{F}(t)}{f\,(t)} = 0, \tag{44}$$

and that for some $\delta > 0$, $\mathbb{E}\left[\beta^{2+\delta}\right] < \infty$. *Then*

$$-D_1(p,p;n) = \mathbb{E}\left(f(M_n \vee Q)\beta\right) \sim \mathbb{E}(\beta) f(\overline{F}^{-1}\left(\frac{1}{n}\right)).$$

Proof of Proposition 8 We have

$$\begin{aligned}-D_1(p,p;n) =& \mathbb{E}\left[f(M_n \vee Q)\beta\right] = \mathbb{E}\left[f(M_n)\beta 1_{M_n > Q}\right] + \mathbb{E}\left[f(Q)\beta 1_{M_n \leq Q}\right]\\ =& \mathbb{E}\left[f(M_n)\beta\right] - \mathbb{E}\left[f(M_n)\beta 1_{M_n\leq Q}\right] + \mathbb{E}\left[f(Q)\beta 1_{M_n\leq Q}\right].\end{aligned}$$

Lemma 6 shows that the first term

$$\mathbb{E}\left[f\left(M_n\right)\beta\right] \sim_{n\to\infty} f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)\mathbb{E}\left[\beta\right]. \tag{45}$$

Lemma 7 shows that the last term

$$\mathbb{E}\left[\left(f(Q)\right)1_{M_n\leq Q}\beta\right] = o\left(f\left(\overline{F}^{-1}\frac{1}{n}\right)\right). \tag{46}$$

For the middle term note that, again applying Holder's inequality for $r \geq 1$, $s \geq 1$ and $r^{-1} + s^{-1} = 1$,

$$\begin{aligned}\mathbb{E}\left[f(M_n)\beta 1_{M_n\leq Q}\right] &\leq \left(\mathbb{E}\left[f(M_n)\beta\right]^r\right)^{1/r}\mathbb{P}[M_n\leq Q]^{1/s}\\ &= \left(\mathbb{E}\left[f(M_n)\right]^r\right)^{1/r} E\left[\beta^r\right]^{1/r}\mathbb{P}[M_n\leq Q]^{1/s}\\ &\sim \left(f(\overline{F}^{-1}\left(\frac{1}{n}\right))^r\right)^{1/r} E\left[\beta^r\right]^{1/r}\mathbb{P}[M_n\leq Q]^{1/s}\\ &= f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)o(1).\end{aligned}$$

Corollary 2 *Under the assumptions of Proposition 8 and assuming* $c \leq p(n) \leq M$ *for some*

M and for all n, if $h_p(x) = 0$ for $x < 0$ and if $f(x)$ is non-increasing for $x \geq 0$,

$$-D_1(p(n), p(n); n) = \mathbb{E}(f(M_n \vee Q)\beta) \sim \mathbb{E}(\beta) f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right),$$

$$D(p(n), p(n); n) \sim \frac{1}{n}.$$

Proof of Lemma 2

We have, from the monotonicity of $D(p, p; n)$ in p,

$$D(M, M; n) \leq D(p(n), p(n); n) \leq D(c, c; n).$$

Since the extremes are asymptotic to $1/n$, so is $D(p(n), p(n); n)$. Similarly, the monotonicity of $D_1(p, p; n)$ in p implies that $-D_1(p(n), p(n); n) \sim \mathbb{E}(\beta) f\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)$.

Proof of Theorem 4

First, note that the condition

$$\lim_{t \to \infty} \sup_{s \geq t} h(s) \frac{\overline{F}(t)}{f(t)} = 0, \tag{47}$$

in Proposition 8 is satisfied whenever $\gamma = 0$ and $a < \infty$. Let $q(p, n) = p - c + D(p, p; n)/D_1(p, p; n)$. We seek $p(n)$ such that $q(p(n), n) = 0$. Clearly, $q(c, n) < 0$ for all n. Let M be large and independent of n. Then $q(M, n) = M - c + D(M, M)/D_1(M, M)$. From Proposition 8,

$$\lim_{n \to \infty} (-D(M, M)/D_1(M, M)) \sim \frac{1}{nf\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)\mathbb{E}[\beta]}.$$

Since $a < \infty$,

$$\frac{1}{nf\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)\mathbb{E}[\beta]} = O(1),$$

So $q(M, n) > 0$ for large n and M. Thus a solution exists on the interval $[0, M]$, for large

n. This verifies the assumption $p(n) < M$. We may then apply Corollary 2 to obtain

$$
\begin{aligned}
p(n) - c &= -\frac{D(p(n), p(n); n)}{D_1(p(n), p(n); n)} \\
&\sim \frac{1}{n\mathbb{E}[f(M_{n-1})]\,\mathbb{E}[\beta]} \\
&\sim \frac{1}{nf\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right)\mathbb{E}[\beta]}. \blacksquare
\end{aligned}
$$

9.4 Macroeconomic Framework

We elaborate on the first order conditions for the Random Demand specification. Utility (21) is to be maximized subject to the consumer budget constraint

$$
wL + \Pi(Q) = qZ + \frac{1}{n}\sum_{i=1}^{n} p_i Q_i \tag{48}
$$

and where w is the wage rate and q, p_i are the goods prices, while $\Pi(Q)$ are the profits received from the differentiated goods sector. The number n equals the number of goods for which demand is strictly positive, i.e. $n = \sum_i \chi_{Q_i > 0}$. Suppose that

$$
\frac{e^{X_i}}{p_i} \geq \max\left\{\frac{e^{X_j}}{p_j}\right\}, \text{ for } j = 1, ..., n.
$$

Optimality requires

$$
\begin{gathered}
Q_j = 0, \text{ for } j \neq i. \\
(1-\theta) Z^{-\theta}\left[e^{X_i}Q_i\right]^{\theta} - \lambda q = 0, \\
\theta Z^{1-\theta}\left[e^{X_i}Q_i\right]^{\theta-1} e^{X_i} - \lambda p_i = 0, \\
-L^{\eta} + \lambda w = 0
\end{gathered}
$$

and

$$
wL + \Pi(Q) = qZ + p_i Q_i.
$$

Manipulating these conditions yields by dividing the first two conditions

$$
p_i Q_i = \frac{\theta}{1-\theta} qZ.
$$

This together with the budget constraint implies

$$Z = (1-\theta)\,\frac{wL + \Pi(Q)}{q} \tag{49}$$

and

$$Q_i = \theta \frac{wL + \Pi(Q)}{p_i}. \tag{50}$$

Furthermore, from the first three first order conditions

$$\begin{aligned} q/w &= (1-\theta)\, Z^{-\theta} \left[e^{X_i} Q_i\right]^{\theta} L^{-\eta} \\ &= (1-\theta) \left(\frac{1-\theta}{\theta}\frac{p_i}{q} Q_i\right)^{-\theta} \left[e^{X_i} Q_i\right]^{\theta} L^{-\eta} \end{aligned}$$

so that

$$L = \left(\frac{w}{q^{1-\theta} p_i^{\theta}}\,(1-\theta)^{1-\theta}\,\theta^{\theta}\right)^{1/\eta} e^{(\theta/\eta) X_i}.$$

The well known consumer first order conditions for the Dixit-Stiglitz case imply the same demand functions for Z and Q_i as in (49) and (50) and are left to the reader. The only difference is that in the Dixit-Stiglitz case there is demand for all differentiated goods.

Use that in the symmetric equilibrium all prices for the differentiated goods will be equal $p_i = p$. Conditional on $e^{X_i}/p_i \geq \max_j \left[e^{X_j}/p_j\right]$, we get for both specifications

$$Q_i = \theta \frac{wL + \Pi(Q)}{p} \tag{51}$$

and

$$Z = (1-\theta)\,\frac{wL + \Pi(Q)}{q}. \tag{52}$$

These demand functions reflect the expenditure shares inherent to the Cobb-Douglas type utility function. The first order conditions imply for the RD case

$$L = \left((1-\theta)^{1-\theta}\,\theta^{\theta} \frac{w}{q^{1-\theta} p_i^{\theta}}\right)^{1/\eta} e^{(\theta/\eta) X_i}, \tag{53}$$

whereas labor supply in the DS case reads

$$L = \left((1-\theta)^{1-\theta}\,\theta^{\theta} \frac{w}{q^{1-\theta} p^{\theta}}\right)^{1/\eta}. \tag{54}$$

On the supply side, the Ricardian technologies for the two types of goods are

$$Z = BN \text{ and } Q_i = AN_i.$$

Here A and B are the labor productivity coefficients while N and N_i are the respective labor demands. Perfect competition in the composite goods market implies that prices equal the per unit labor costs $q = w/B$.

The differentiated goods producer exploits his direct pricing power, but ignores his pricing effect on the price index of the differentiated goods and the consumer income $wL + \Pi(Q)$. For the Random Demand case, the markup is μ_n from (7), so that by (2) $p_i = (1 + \mu_n)\, w/A$ and where $c = w/A$. Substitute this and and $q = w/B$ into (53) to determine the equilibrium labor supply as a function of the markup factor conditional on the specific demand shock

$$L = \left(\theta^{\theta}\left(1-\theta\right)^{1-\theta} A^{\theta} B^{1-\theta}\right)^{1/\eta}\left(\frac{1}{1+\mu_n}\right)^{\theta/\eta} e^{(\theta/\eta)X_i} = \varphi\left(A, B\right)\left(\frac{1}{1+\mu_n}\right)^{\theta/\eta} e^{(\theta/\eta)X_i}, \quad (55)$$

say, and where $\varphi\left(A, B\right)$ is a composite of supply shocks A and B. Combine the differentiated product sector profits $(p - w/A)Q_i$ with the demand Q_i from (51) to get

$$\Pi(Q) = \frac{\theta\mu_n}{1 + (1-\theta)\,\mu_n} wL \quad (56)$$

Finally, combining (56) with (55) yields the unconditional per capita macro demand for the differentiated good

$$Q_i = \frac{\theta}{1 + (1-\theta)\,\mu_n} A\varphi\left(A, B\right)\left(\frac{1}{1+\mu_n}\right)^{\theta/\eta} e^{(\theta/\eta)X_i} \quad (57)$$

and

$$Q_j = 0,\ \forall j \neq i$$

Similarly, the demand for the competitive good is

$$Z = \frac{(1-\theta)\,(1+\mu_n)}{1 + (1-\theta)\,\mu_n} B\varphi\left(A, B\right)\left(\frac{1}{1+\mu_n}\right)^{\theta/\eta} e^{(\theta/\eta)X_i}. \quad (58)$$

In the Dixit-Stiglitz specification, the pricing power requires setting prices in proportion to unit labor costs and the markup factor τ, so that $p_i = (1+\tau)\, w/A$. Labor supply then follows from (54)

$$L=\varphi\left(A,B\right)\left(\frac{1}{1+\tau}\right)^{\theta/\eta}. \tag{59}$$

Analogous to the Random Demand case, the demand for the differentiated good is

$$Q_j=\frac{\theta}{1+\left(1-\theta\right)\tau}A\varphi\left(A,B\right)\left(\frac{1}{1+\tau}\right)^{\theta/\eta},\ \forall j \tag{60}$$

and for the competitive good

$$Z=\frac{\left(1-\theta\right)\left(1+\tau\right)}{1+\left(1-\theta\right)\tau}B\varphi\left(A,B\right)\left(\frac{1}{1+\tau}\right)^{\theta/\eta}. \tag{61}$$

To determine the price level, a simple quantity type relation $M=wL$ suffices. This determines wages w and prices p_i, q in terms of the quantity of money M.

10 Appendix D: Second-Order Conditions for Profit Maximization

Recall that the profit function $\pi\left(p_i,p\right)$ for firm i when it sets price p_i and all other firms set price p is

$$\pi\left(p_i,p\right)=(p_i-c)D\left(p_i,p\right)-K. \tag{62}$$

So far, we have analyzed the first-order condition for profit maximization, $\pi_1\left(p,p;n\right)=0$, which is necessary but not sufficient to ensure equilibrium. Anderson et al. (1992) show (Prop. 6.5, p.171 and Prop. 6.9, p.184) that symmetric price equilibria exist in the Perloff-Salop, Sattinger and Hart models when f is log-concave. Thus in these cases (62) defines the unique symmetric price equilibrium. However, their results do not cover distributions where f is not log-concave. We are unable to derive global conditions for existence of equilibrium in these cases. Instead, we verify in this appendix that the markups we study satisfy the second-order conditions for profit-maximization.

10.1 Perloff-Salop, Sattinger and Hart Models

The following three propositions show that the symmetric equilibrium markup expression (2) which we use in our calculations also satisfies the second-order condition for profit maximization, $\pi_{11}(p,p;n) < 0$. It is useful to note that, via simple calculations, the second order condition is

$$\pi_{11}(p,p;n) = 2D_1(p,p;n) - \frac{D(p,p;n)}{D_1(p,p;n)} D_{11}(p,p;n) < 0. \tag{63}$$

Proposition 9 *Assume that F satisfies the conditions for Theorem 2, that $f^3(x)$ is $[w_l, w_u)$-integrable, and that*

$$-4\Gamma(\gamma+2)^2 + \Gamma(2\gamma+3) < 0,$$

which holds for $-1.45 < \gamma < 0.64$. Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Perloff-Salop model.

Note that this covers all distributions with thin ($-1 \leq \gamma \leq 0$) and medium fat tails ($\gamma = 0$), and all the heavy tailed distributions with a finite variance, i.e. $\gamma \in (0, 1/2]$.

Proposition 10 *Assume that F satisfies the conditions for Theorem 2, that $f^3(x)$ is $[w_l, w_u)$-integrable, and that either $\gamma > 0$ or*

$$-4\Gamma(\gamma+2)^2 + \Gamma(2\gamma+3) < 0,$$

which holds for $-1.45 < \gamma \leq 0$. Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Sattinger model.

Proposition 11 *Assume that the conditions for Theorem 2 are satisfied, and that $e^{\psi x} f^3(x)$ is $[w_l, w_u)$-integrable. Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Hart model.*

Proof of Proposition 9

We use $U_n = \overline{F}^{-1}(1/n)$ as a shortcut notation in several of the proofs below. Note, from Appendix B, that

$$D(p_i, p) = \int f(x) F^{n-1}(x + p - p_i)\, dx \text{ and}$$
$$D_1(p_i, p) = -(n-1) \int f(x) f(x + p - p_i) F^{n-2}(x + p - p_i)\, dx,$$

from which we may calculate

$$D_{11}(p,p)=\frac{(n-1)(n-2)}{2}\int f^3(x)F^{n-3}(x)\,dx+\frac{n-1}{2}f^2(x)F^{n-2}(x)\Big|_{-\infty}^{\infty}$$

where the last term on the RHS vanishes. So, applying Proposition 1,

$$\begin{aligned}\pi_{11}(p,p;n)&=2D_1(p,p;n)-\frac{D(p,p;n)}{D_1(p,p;n)}D_{11}(p,p;n)\\&=-2(n-1)\int f^2(x)F^{n-2}(x)\,dx+\frac{\frac{(n-1)(n-2)}{2}\int f^3(x)F^{n-3}(x)\,dx}{n(n-1)\int f^2(x)F^{n-2}(x)\,dx}\\&=-2(n-1)\int f^2(x)F^{n-2}(x)\,dx+\frac{(n-2)\int f^3(x)F^{n-3}(x)\,dx}{2n\int f^2(x)F^{n-2}(x)\,dx}\\&\sim-2f(U_n)\Gamma(\gamma+2)+\frac{f(U_n)\Gamma(2\gamma+3)}{2\Gamma(\gamma+2)}\\&=\frac{f(U_n)}{2\Gamma(\gamma+2)}\left(-4\Gamma(\gamma+2)^2+\Gamma(2\gamma+3)\right).\end{aligned}$$

since we can easily verify numerically that $-4\Gamma(\gamma+2)^2+\Gamma(2\gamma+3)<0$ for $-1.45<\gamma\leq 0$, it follows that

$$\pi_{11}(p,p;n)<0 \text{ for } \gamma\in[-1.45,0.64]\,.\blacksquare$$

Proof of Proposition 10

Without loss of generality, let $\theta y=1$. Then, from Appendix B,

$$\begin{aligned}D(p_i,p)&=\frac{1}{p_i}\int f(x)F^{n-1}(x+\ln p-\ln p_i)\,dx \text{ and}\\D_1(p_i,p)&=-\frac{1}{p_i^2}\int f(x)F^{n-1}(x+\ln p-\ln p_i)\,dx\\&\quad-\frac{n-1}{p_i^2}\int f(x)f(x+\ln p-\ln p_i)F^{n-2}(x+\ln p-\ln p_i)\,dx,\end{aligned}$$

from which we may calculate

$$\begin{aligned}D_{11}(p,p)&=\frac{2}{p^3}\int f(x)F^{n-1}(x)\,dx+3\frac{n-1}{p^3}\int f^2(x)F^{n-2}(x)\,dx\\&\quad+\frac{(n-1)(n-2)}{2p^3}\int f^3(x)F^{n-3}(x)\,dx+\frac{n-1}{2p^3}\left[f^2(x)F^{n-2}(x)\right]_{-\infty}^{\infty}\end{aligned}$$

where the last term on the RHS vanishes. We may then substitute our expressions for $D(p,p;n), D_1(p,p;n), D_{11}(p,p;n)$ into (63) and apply Proposition 1. The asymptotic expression simplifies to

$$\begin{aligned}
&\pi_{11}(p,p;n) = 2D_1(p,p;n) - \frac{D(p,p;n)}{D_1(p,p;n)} D_{11}(p,p;n) \\
&= -\frac{2}{p^2}\left(\int f(x) F^{n-1}(x)\ dx + (n-1)\int f^2(x) F^{n-2}(x)\ dx\right) \\
&+ \frac{\begin{pmatrix} 2\int f(x) F^{n-1}(x)\ dx + 3(n-1)\int f^2(x) F^{n-2}(x)\ dx \\ +\frac{(n-1)(n-2)}{2}\int f^3(x) F^{n-3}(x)\ dx \end{pmatrix}}{p^2 n\left(\int f(x) F^{n-1}(x)\ dx + (n-1)\int f^2(x) F^{n-2}(x)\ dx\right)} \\
&= \frac{p^{-2}}{n}\begin{pmatrix} -2\left(1 + nf(U_n)\Gamma(\gamma+2)\right) + o(nf(U_n)) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))} \end{pmatrix}
\end{aligned}$$

In the case $nf(U_n) = o(1)$, which implies $\gamma \geq 0$ and $f(w_u) = 0$, we get

$$\begin{aligned}
&\pi_{11}(p,p;n) = \frac{p^{-2}}{n}\begin{pmatrix} -2\left(1 + nf(U_n)\Gamma(\gamma+2)\right) + o(nf(U_n)) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+\frac{1}{2}(nf(w_u))^2+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))} \end{pmatrix} \\
&= \frac{p^{-2}}{n}\begin{pmatrix} -2\left(1 + nf(U_n)\Gamma(\gamma+2)\right) + \frac{2+3nf(U_n)\Gamma(\gamma+2)}{1+nf(U_n)\Gamma(\gamma+2)} \\ +o(nf(U_n)) \end{pmatrix} \\
&= \frac{p^{-2}}{n}\begin{pmatrix} -\frac{nf(U_n)\Gamma(\gamma+2)}{1+nf(U_n)\Gamma(\gamma+2)} \\ +o(nf(U_n)) \end{pmatrix} \\
&< 0.
\end{aligned}$$

In the case $\lim_{n\to\infty} nf(U_n) \in (0,\infty)$, which implies $\gamma = 0$, we get

$$\begin{aligned}
&\pi_{11}(p,p;n) = \frac{p^{-2}}{n}\begin{pmatrix} -2\left(1 + nf(U_n)\Gamma(\gamma+2)\right) + o(nf(U_n)) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+\frac{1}{2}(nf(w_u))^2+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))} \end{pmatrix} \\
&= \frac{p^{-2}}{n}\left(-2\left(1 + nf(U_n)\right) + \frac{2 + 3nf(U_n) + (nf(U_n))^2}{1 + nf(U_n)} + o(nf(U_n))\right) \\
&= \frac{p^{-2}}{n}\left(-nf(U_n) + o(nf(U_n))\right) \\
&< 0.
\end{aligned}$$

In the case $\lim_{n\to\infty} nf(U_n) = \infty$, which implies $\gamma \leq 0$, we get

$$\begin{aligned}
\pi_{11}(p,p;n) &= \frac{p^{-2}}{n}\begin{pmatrix} -2\left(1+nf\left(U_n\right)\Gamma\left(\gamma+2\right)\right)+o\left(nf\left(U_n\right)\right) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))}\end{pmatrix} \\
&= \frac{p^{-2}}{n}\begin{pmatrix} -2\left(nf\left(U_n\right)\Gamma\left(\gamma+2\right)\right)+o\left(nf\left(U_n\right)\right) \\ +\frac{\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+o(nf(U_n))^2}{nf(U_n)\Gamma(\gamma+2)+o(nf(U_n))}\end{pmatrix} \\
&= p^{-2}f\left(U_n\right)\left(-2\Gamma\left(\gamma+2\right)+\frac{1}{2}\frac{\Gamma\left(2\gamma+3\right)}{\Gamma\left(\gamma+2\right)}\right);
\end{aligned}$$

since we can easily verify numerically that $-2\Gamma\left(\gamma+2\right)+\frac{1}{2}\frac{\Gamma(2\gamma+3)+1}{\Gamma(\gamma+2)} < 0$ for $-1.45 < \gamma \leq 0$, it follows that

$$\pi_{11}(p,p;n) < 0 \text{ for } \gamma \in [-1.45, 0]. \blacksquare$$

Proof of Proposition 11

Note that in the Hart case, we are restricted to $\gamma \in [-1,0]$. We have, from Appendix B,

$$D(p_i,p) = \frac{1}{p_i^{1+\psi}}\int e^{\psi x} f(x) F^{n-1}(x+\ln p-\ln p_i)\,dx \text{ and}$$

$$D_1(p_i,p) = -\frac{1}{p_i^{2+\psi}}\left\{\begin{array}{c} (1+\psi)\int e^{\psi x} f(x) F^{n-1}(x+\ln p-\ln p_i)\,dx \\ +(n-1)\int e^{\psi x} f(x) f(x+\ln p-\ln p_i) F^{n-2}(x+\ln p-\ln p_i)\,dx \end{array}\right\},$$

from which we may calculate

$$D_{11}(p,p) = \frac{1}{p^{3+\psi}}\left\{\begin{array}{c} (1+\psi)(2+\psi)\int e^{\psi x} f(x) F^{n-1}(x)\,dx \\ +3\left(1+\frac{\psi}{2}\right)(n-1)\int e^{\psi x} f^2(x) F^{n-2}(x)\,dx \\ +\frac{1}{2}(n-1)(n-2)\int e^{\psi x} f^3(x) F^{n-3}(x)\,dx \end{array}\right\}.$$

We may then substitute our expressions for $D(p,p;n), D_1(p,p;n), D_{11}(p,p;n)$ into (63) and apply Proposition 1. This gives us

$$2D_1(p,p;n) - \frac{D(p,p;n)}{D_1(p,p;n)}D_{11}(p,p;n) = \frac{e^{\psi U_n}}{p_i^{2+\psi}}(A+B),$$

where

$$A \sim -2\left(1+\psi\right)\Gamma\left(1-a\psi\right) - 2nf\left(U_n\right)\Gamma\left(\gamma+2-a\psi\right), \text{ and}$$

$$B \sim \Gamma\left(1-a\psi\right)\frac{\left\{\begin{array}{c}\left(1+\psi\right)\left(2+\psi\right)\Gamma\left(1-a\psi\right)\\ +3\left(1+\frac{\psi}{2}\right)nf\left(U_n\right)\Gamma\left(\gamma+2-a\psi\right)\\ +\frac{1}{2}\left(nf\left(U_n\right)\right)^2\Gamma\left(2\gamma+3-a\psi\right)\end{array}\right\}}{\left(1+\psi\right)\Gamma\left(1-a\psi\right)+nf\left(U_n\right)\Gamma\left(\gamma+2-a\psi\right)}$$

After some tedious but straightforward calculations: if $a = 0$, then $nf\left(U_n\right) \to_{n\to\infty} \infty$, and the asymptotic expression simplifies to

$$\begin{aligned}&\pi_{11}\left(p,p;n\right)\\ &\sim \frac{e^{\psi U_n}}{p_i^{2+\psi}}nf\left(U_n\right)\left(-2\Gamma\left(\gamma+2\right)+\frac{\Gamma\left(2\gamma+3\right)}{2\Gamma\left(\gamma+2\right)}\right)\\ &< 0 \text{ for } \gamma \in [-1,0]\end{aligned}$$

Since we can verify that $-2\Gamma\left(\gamma+2\right)+\frac{\Gamma(2\gamma+3)}{2\Gamma(\gamma+2)} < 0$ for $\gamma \in [-1,0]$, our claim holds in the case $a = 0$.

If $0 < a < \infty$, then $\gamma = 0$, $nU_n \to 1/a$ and the asymptotic expression simplifies to

$$\pi_{11}\left(p,p;n\right) \sim \frac{e^{\psi U_n}}{p_i^{2+\psi}}\Gamma\left(1-a\psi\right)\left(-2\left(1+1/a\right)+\frac{\left\{\begin{array}{c}\left(1+\psi\right)\left(2+\psi\right)+3\left(1+\psi/2\right)\left(1/a-\psi\right)\\ +\frac{1}{2}\left(2/a-\psi\right)\left(1/a-\psi\right)\end{array}\right\}}{1+1/a}\right)$$

$$= -\frac{e^{\psi U_n}}{p_i^{2+\psi}}\frac{\Gamma\left(1-a\psi\right)}{a} < 0. \blacksquare$$

10.2 ELRU Model

Finally, we check the second order condition for the ELRU model in the case that the distribution $F(x)$ is of the Rootzen type.

Proposition 12 *Assume that the conditions for Theorem 27 are satisfied. Suppose, moreover, that the distribution for the "taste for money" is such that the variance is smaller than the mean, i.e. $V[B] < E[B]$. Then the second-order condition for profit maximization is satisfied* *in the symmetric equilibrium of the ELRU model.*

Proof of Proposition 12 First condition on β, ϵ_0 and hence $q = \beta p + \epsilon_0$ having a fixed value. Differentiation gives

$$\begin{aligned}
\frac{\partial^2 D(p,p)}{\partial p_i^2} &= -\beta^2 \int_q^\infty \frac{\partial f(s)}{\partial s} \left[(n-1)\, F^{n-2}(s) f(s)\right] ds \\
&\sim -\beta^2 \int_q^\infty \left[-\kappa\lambda^2\phi^2 x^{2\phi+\nu-2} \exp\left(-\lambda x^\phi\right)\right] \left[(n-1)\, F^{n-2}(s) f(s)\right] ds.
\end{aligned}$$

Moreover, for s close to zero

$$f'\left(\overline{F}^{-1}(s)\right) \sim -\phi^2\lambda^{2/\phi} s \left[\ln\frac{A}{s}\right]^{-a/\phi} \left[\ln\left(\frac{A}{s}\left[\ln\frac{A}{s}\right]^{a/\phi}\right)\right]^{2+(\nu-2)/\phi}.$$

From this the regular variation at zero of $f'\left(\overline{F}^{-1}(y)\right)$ follows:

$$\begin{aligned}
\lim_{y\downarrow 0} \frac{f'\left(\overline{F}^{-1}(xy)\right)}{f'\left(\overline{F}^{-1}(y)\right)} &= \lim_{t\to\infty} \frac{f'\left(\overline{F}^{-1}(x/t)\right)}{f'\left(\overline{F}^{-1}(1/t)\right)} \\
&= x \lim_{t\to\infty} \frac{\left[\ln\frac{At}{x}\right]^{-\nu/\phi}\left[\ln\left(\frac{At}{x}\left[\ln\frac{At}{x}\right]^{\nu/\phi}\right)\right]^{2+(\nu-2)/\phi}}{\left[\ln At\right]^{-\nu/\phi}\left[\ln\left(At\left[\ln At\right]^{a/\phi}\right)\right]^{2+(\nu-2)/\phi}} \\
&= x \lim_{t\to\infty} \left(1 - \frac{\ln x}{\ln At}\right)^{-\nu/\phi} \left\{\frac{1 - \frac{\ln x}{\ln At} + \frac{\nu}{\phi}\frac{\ln(\ln At)}{\ln At}}{1 + \frac{\nu}{\phi}\frac{\ln(\ln At)}{\ln At}}\right\}^{2+(\nu-2)/\phi} \\
&= x.
\end{aligned}$$

Hence, $f'\left(\overline{F}^{-1}(y)\right) \in RV_1^0$. So that by the main theorem

$$\begin{aligned}
\frac{\partial^2 D(p,p)}{\partial p_i^2} &\sim -b^2 f'\left(\overline{F}^{-1}\left(\frac{1}{n}\right)\right) \\
&\sim b^2\phi^2\lambda^{2/\phi}\frac{1}{n}\left[\ln A + \ln n\right]^{-\nu/\phi}\left\{\ln A + \ln n + \frac{\nu}{\phi}\ln\left(\ln A + \ln n\right)\right\}^{2+(\nu-2)/\phi} \\
&\sim b^2\phi^2\lambda^{2/\phi}\frac{1}{n}\left(\ln n\right)^{2-2/\phi}.
\end{aligned}$$

Now integrate out over the random taste for money and outside option. This gives

$$\frac{\partial^2 D(p,p)}{\partial p_i^2} = \phi^2 \lambda^{2/\phi} \frac{1}{n} (\ln n)^{2-2/\phi} E[B^2]$$

The second order condition becomes

$$\begin{aligned}
2D_1 - \frac{D}{D_1} D_{11} &\simeq -2\phi\lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} E[B] \\
&\quad + \frac{1/n}{\phi\lambda^{1/\phi}\frac{1}{n}(\ln n)^{1-1/\phi} E[B]} \phi^2 \lambda^{2/\phi} \frac{1}{n} (\ln n)^{2-2/\phi} E[B^2] \\
&= -2\phi\lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} E[B] + \phi\lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \frac{E[B^2]}{E[B]} \\
&= -\phi\lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \frac{1}{E[B]} \left\{2E[B]^2 - E[B^2]\right\} \\
&= -\phi\lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \frac{E[B] - V[B]}{E[B]}.
\end{aligned}$$

Note that the SOC is satisfied if the variance of the taste for money is smaller than the mean. Note that if the taste for money were non-random, then the SOC is certainly satisfied as in this case

$$2D_1 - \frac{D}{D_1} D_{11} \simeq -\phi\lambda^{1/\phi} \frac{1}{n} (\ln n)^{1-1/\phi} \beta < 0. \blacksquare$$

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