Real Business Cycles with a Human Capital Investment Sector and Endogenous Growth: Persistence, Volatility and Labor Puzzles[*]

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Abstract

The household sector produces human capital investment sector, which is subject to shocks along with the goods sector, whereby the shock causes growth to temporarily rise, but permanent income levels to rise permanently. This causes consumption to move more with respect to income because permanent income is fluctuating by more than in exogenous growth *RBC* models. This helps solve the central *RBC* consumption-output puzzle while capturing US data's output growth persistence, with hump-shaped impulse responses; hump-shaped physical capital investment impulse responses; and Gali's (1999) negative impulse response of labour supply plus hours volatility. Intuitively the identical two-sector productivity shock causes Rybczynski (1955) and Stolper and Samuelson (1941) effects that release leisure time and initially raise the relative price of human capital investment so as to favor it at first over goods production, with this reversing as the cycle progresses.

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1 Introduction

Traditional real business cycle (RBC) models have long been criticized for their lack of an interior propagation mechanism to spread the effect of a shock over time, starting with Cogley and Nason (1995) and Rotemberg and Woodford (1996). The dynamics of output predicted by a standard exogenous growth business cycle model tend to closely resemble the exogenous TFP innovations, so that the shock has to be highly autocorrelated. Still, related to this, Cogley and Nason (p. 492) summarize two stylized facts about the dynamics of US GNP that prototypical RBC models are unable to match:

> "GNP growth is positively autocorrelated over short horizons and has weak and possibly insignificant negative autocorrelation over longer horizons. Second, GNP appears to have an important trend-reverting component that has a hump-shaped impulse response function."

In basic RBC models that only rely on physical capital accumulation and intertemporal substitution to spread shocks over time, another problem is that the output and investment growth are often negatively and insignificantly autocorrelated over all horizons and output and investment usually have only monotonically decreasing impulse response curves following a positive technology shock. This is fixed for example by in the Boldrin et al. (2001) hallmark paper that keeps exogenous growth and adds a second sector for the adjustment cost of physical capital, combined with habit persistence. However Stokey (2010) extends Lucas's (1988) two-sector human capital endogenous growth model to explain development and notes that

> "human capital accumulation takes resources away from production, reducing consumption in the short run. In addition, human capital accumulation is necessarily slow. Thus, while it eventually leads to higher technology inflows, the process is prolonged."

Labour supply volatility also tends to be low relative to US data: the one-sector standard RBC model in King and Rebelo (1999) predicts the volatility of labour supply to be about a half of that of output, compared to data with labour supply fluctuating nearly as much as does output. Adding external labor margins with exogenous growth helps on this,[1] but Gali (1999) emphasizes that RBC models still cannot reproduce the empirical finding that labour supply decreases after a positive goods sector productivity shock, as described in Gali and Hammour (1991). Many approaches within exogenous growth have been taken to combat Gali's important critique, such as Chari et al. (2008) criticism using data generated with technology and "labor wedge" shocks, with additional feedback on the Chari et al. (2008) approach for example from Christiano and Davis (2006).

Here we demonstrate that all of these dynamic features can be reproduced by taking a standard RBC model as extended to Lucas (1988) endogenous growth, with a "household" or "home" sector, except that the home sector produces human capital investment instead of a separate good that enters the utility function.[2] Now there is an endogenous growth balanced growth path (BGP) equilibrium, and cyclical growth facts can also be explained unlike standard models. The human capital investment does not directly add to utility, but rather affects the effective wage through a trade-off going back at least to Becker (1975). We let the productivity shock be identical across both goods and human capital investment sectors, as a first baseline model, as if it were a single aggregate productivity shock as in the Jones et al. (2005) one-sector model. Unlike a typical TFP shock or as in Jones et al., this aggregate shock causes a temporary goods sector productivity shock, plus a permanent shock to the level of human capital and output, through a temporary effect on productivity in the human capital investment sector. The effect on levels of consumption and output as the growth rate of human capital gets shocked upwards, leads to a resolution of the salient facts mentioned. The "internal propogation mechanism" is simply that the aggregate productivity shock causes reallocation across sectors with the goods output gradually rising,

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[1] Hansen (1985) has an indivisible labour supply, Rogerson (1988) an external labor margin, Burnside and Eichenbaum (1996) a factor-hoarding model, and Wen (1998) habit formation in leisure.

[2] See the seminal papers of Greenwood and Hercowitz (1991) and Benhabib et al. (1991), updated for example by Rupert et al. (2000).

goods sector labor at first falling, and physical capital investment having its expected hump shape.

The shock results can be explained in standard international trade terms: upon the economy-wide shock impacting, the agent wants to expand output but finds a shortage of human capital. This causes the relative price of human capital investment relative to physical capital investment to impulse up. This is a Stolper and Samuelson (1941) effect in which they describe that if for whatever reason (here it is the TFP shock to both sectors) the relative price on output for one sector increases (here it is the human capital investment sector), then resources will shift towards that sector. This happens initially: time and capital shifts to the human capital sector initially. This causes a Gali type of short run decrease in labor in the goods sector as resources shift to the human capital sector. But then more human capital is produced, resulting in a Rybczynski (1955) effect. This is that there is an increase in the supply of a factor (human capital), with the result that the sector intensive in that factor (this being the human capital investment sector) sees its relative price fall.[3] And indeed the relative price of human capital investment impulses upwards and then falls steadily as more human capital is increased. The falling relative price of human capital investment relative to physical capital investment causes resources to then in the business cycle frequency to shift back towards the goods sector. This is because the relative price of goods to the human capital investment sector is the same as the relative price of physical capital relative to human capital, since goods are costlessly turned into physical capital investment. As a result the normal upturn in the goods sector then occurs at the business cycle frequency, while including Gali's impulse down in labor in the goods sector initially upon the economy wide TFP shock hitting.

The closest paper to ours may be Benhabib, Perli and Sakellaris (2005) who model multiple sectors including two physical capital sectors, use an identical economy-wide productivity shock, explain output growth evidence

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[3] Rybczynksi (1955): "Our conclusion is that an increase in the quantity of one factor will always lead to a worsening in the terms of trade, or the relative price, of the commodity using relatively much of that factor." Less leisure use increases the usage rate of human capital in productive activity, effectively increasing its supply.

and include a Gali type of negative labor impulse from a positive TFP shock. In contrast, such features are absent in a one sector economy with human capital such as Jones et al. (2005) because there is no adjustment cost of producing human capital. Therefore the relative price of investment in either human or physical capital is always identical. In our more dissaggregated model with a separate human capital sector as our second capital sector, the inter-sectoral reallocation induced by the human capital investment sector creates a concave economy-wide production possibility frontier, as discussed in Mulligan and Sala-I-Martin (1993;section IIIb). In related research, Perli and Sakellaris (1998) has a constant elasticity of substitution aggregator of skilled and unskilled labour, without a balanced growth path equilibrium or shocks to the human capital investment sector. Maffezzoli (2000) has an extended two-country model relative to ours, with spillovers and trade. DeJong and Ingram (2001) include human capital investment as a second sector and their empirical findings and those of Dellas and Sakellaris (2003) both suggest significant substitution between skills acquisition or higher education and competing labour market activities over business cycle frequencies, which add support to the approach of our paper.

Sections 2 and 3 set out the model, its equilibrium, and a postwar US data based calibration. Section 4 shows numerical results, with Section 5 conducting sensitivity analysis. Section 6 compares the results to Jones et al. (2005), using four variants of the two-sector model, including one case that nests the one sector model. Sections 7 concludes.

2 Model environment

2.1 The model

The representative agent maximizes the expected sum of discounted utility derived from a stream of consumptions and leisure, denoted by C_t and L_t at time t. With $A > 0$, and $\sigma > 0$, the time t utility is given by

$$U(C_t, L_t) = \frac{(C_t L_t^A)^{1-\sigma} - 1}{1-\sigma},$$

which satisfies necessary conditions for existence of a balanced growth path equilibrium (King et al., 1988). The representative agent is confined by a time endowment constraint for every period t, where N_t is the fraction spent in goods production, and M_t in human capital investment production:

$$N_t + M_t + L_t = 1 \tag{1}$$

The laws of motions of physical capital K_t and human capital H_t, with δ_k and δ_h denoting the assumed constant depreciation rates, and I_{kt} and I_{ht} denoting investment in physical and human capital, are

$$I_{kt} = K_{t+1} - (1 - \delta_k)K_t \tag{2}$$

$$I_{ht} = H_{t+1} - (1 - \delta_h)H_t \tag{3}$$

Denote by Y_t the real goods output that corresponds to the notion of GDP; A_g is a positive factor productivity parameter; Z_t is a productivity shock described below; K_t is the physical capital stock that has been accumulated by the beginning of period t; V_t is the share of the physical capital stock being used in the goods sector; V_tK_t is the amount of capital used in goods production. H_t is stock of human capital at the beginning of period t; N_t denotes the share of time used in goods production; N_tH_t represents the "effective labour input", or more simply the amount of human capital used. And $\phi_1 \in [0, 1]$ is share of physical capital in the production function:

$$Y_t = F(V_tK_t, N_tH_t) = A_g Z_t (V_tK_t)^{\phi_1} (N_tH_t)^{1-\phi_1} \tag{4}$$

The technology shock to physical sector is assumed to evolve according to a stationary autoregressive process, described in log form as:

$$\log Z_{t+1} = \rho_z \log Z_t + \varepsilon^z_{t+1}.$$

The innovations ε^z_{t+1} is a sequence of independently and identically distributed normal random variables with mean zero and variance σ^2_z.

Human capital is reproducible in a separate sector as in Lucas (1988) and Uzawa (1965). Social activities in the real economy that typically are thought of as corresponding to this sector include formal education, job trainings and, some argue, elements such as health care. Production of human capital investment also is constant return to scale in terms of physical and human capital inputs. I_{ht} denotes the new human capital produced in this period; $A_h > 0$ is the productivity parameter for the human capital sector; S_t represents the productivity shock to human sector; $1 - V_t$ is the remaining fraction of physical capital allocated to the human capital investment sector; M_t denotes the fraction of human capital used in production; and ϕ_2 is the rental share of physical capital in the value of the human capital investment output:

$$I_{ht} = G\left[(1 - V_t)K_t, M_t H_t\right] = A_h S_t \left[(1 - V_t)K_t\right]^{\phi_2} (M_t H_t)^{1-\phi_2}. \tag{5}$$

The productivity shock to human capital sector in general takes the form

$$\log S_{t+1} = \rho_s \log S_t + \varepsilon^s_{t+1},$$

where the innovations ε^s_{t+1} is a sequence of independently and identically distributed normal random variables with mean zero and variance σ_s^2. However this shock is collapsed to the goods sector productivity shock identically except in Section 5 on sensitivity analysis.

With no externalities, the competitive equilibrium of the economy coincides with the result of the social planner problem, which is stated as:

$$\begin{array}{ll} \underset{C_t, V_t, L_t, N_t, M_t, H_{t+1}, K_{t+1}}{MAX} & E_0 \sum\limits_{t=0}^{\infty} \beta^t \frac{(C_t L_t^A)^{1-\sigma} - 1}{1-\sigma} \\ s.t. \quad (4), (5), (1), (2), (3) & \end{array} \tag{6}$$

2.2 Equilibrium

Definition 1 *A general equilibrium of this model is a set of contingent plans* $\{C_t, K_{t+1}, H_{t+1}, V_t, L_t, N_t, M_t\}$ *that solve the central planer's maximization problem (6) for some initial endowment* $\{K_0, H_0\}$ *and exogenous stochastic technology processes* $\{Z_t, S_t\}$, *with initial conditions* $\{Z_0, S_0\}$.

Definition 2 *A deterministic balanced growth path equilibrium of this model is a set of paths* $\{\overline{C}_t, \overline{K}_{t+1}, \overline{H}_{t+1}, \overline{V}_t, \overline{L}_t, \overline{N}_t, \overline{M}_t\}$ *that solve the central planer's* *maximization problem (6) for some initial endowment* $\{K_0, H_0\}$ *and exogenous technology parameters* $\{\overline{Z}_t = 1, \overline{S}_t = 1\}$, *such that* $\{\overline{C}_t, \overline{K}_{t+1}, \overline{H}_{t+1}\}$ *grow at a common trend, and* $\{\overline{V}_t, \overline{L}_t, \overline{N}_t, \overline{M}_t\}$ *are constant.*

For existence and uniqueness of the deterministic BGP equilibrium, note that the maximization problem is nonconcave, because the human capital stock has asymmetric effects on different uses of time: it enhances productive time but not leisure, allowing for potentially multiple steady states.[4] There may be multiple steady states but in Appendix A, uniqueness of the steady state is shown to be reduced down to the uniqueness of a single variable, the balanced growth rate. Numerical checks on the calibrations, with robustness to sensitivity analysis, finds that there is always a unique internal steady state so that leisure time on balanced growth path is between 0 and 1 (See Ben-Gad, 2007).

Also the usual sufficient second order conditions guaranteeing optimality do not apply, in that the Arrow (1968) condition is not met generically and the Mangassarian (1966) condition is not met at least for the particular calibration. However, Ladron-De-Guevara et al. (1999) show in a similar endogenous growth model with leisure that stable steady states with non-complex roots correspond to optimal solutions (theorem 3.1 p. 614 and in their appendix). In the baseline calibration here, and in various alternative specifications, the dynamics of the state-like variable $\left(\frac{K}{H}\right)$ near the unique steady state is stable with non-complex roots, with the implication that the first order conditions should correspond to a maximum.

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[4] To see this, rewrite agents' utility function as: $U = \frac{(C_t(L_tH_t)^A H_t^{-A})^{1-\sigma}-1}{1-\sigma}$. The objective function loses the property of joint concavity because of the term H_t^{-A}.

2.3 Equilibrium Dynamics

With λ_t and μ_t the co-state variables to physical and human capital respectively, such that the first-order conditions are

$$\begin{aligned} U_{1,t} &= \lambda_t,\ U_{2,t} = \lambda_t F_{2,t} H_t,\ U_{2,t} = \mu_t H_{2,t} H_t,\ \lambda_t F_{1,t} = \mu_t H_{1,t}, \\ \lambda_t &= E_t \beta \left(\lambda_{t+1} V_{t+1} F_{1,t+1} + \mu_{t+1}(1 - V_{t+1}) H_{1,t+1} + 1 - \delta_k \right), \\ \mu_t &= E_t \beta \left(\mu_{t+1} M_{t+1} H_{2,t+1} + \lambda_{t+1} N_{t+1} F_{2,t+1} + 1 - \delta_h \right). \end{aligned}$$

Define $P_t \equiv \frac{\mu_t}{\lambda_t}$ as the relative price of human capital in terms of physical capital. Note that since physical capital and goods output are perfect substitutes (output can be turned into new physical capital without cost) then P_t is also the price of the human capital investment sector relative to the goods sector. Also denote by r_t and W_t the own marginal productivity conditions of physical and human capital such that $r_t \equiv F_{1,t}$ and $W_t \equiv F_{2,t}$. The first order conditions can be stated as

$$\frac{AC_t}{L_t} = W_t H_t \tag{7}$$

$$\frac{1-\phi_1}{\phi_1} \frac{V_t K_t}{N_t H_t} = \frac{1-\phi_2}{\phi_2} \frac{(1-V_t) K_t}{M_t H_t} \tag{8}$$

$$P_t = \frac{Z_t}{S_t} \frac{A_g}{A_h} \left(\frac{\phi_1}{\phi_2} \right)^{\phi_2} \left(\frac{1-\phi_1}{1-\phi_2} \right)^{1-\phi_2} \frac{V_t K_t}{N_t H_t}^{\phi_1 - \phi_2} \tag{9}$$

$$1 = E_t \beta \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{L_{t+1}}{L_t} \right)^{A(1-\sigma)} (1 + r_{t+1} - \delta_k) \right] \tag{10}$$

$$1 = E_t \beta \left[\left(\frac{P_{t+1}}{P_t} \right) \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{L_{t+1}}{L_t} \right)^{A(1-\sigma)} \left(1 + (1 - L_{t+1}) \frac{W_{t+1}}{P_{t+1}} - \delta_h \right) \right] \tag{11}$$

Equation (7) sets the marginal rate of substitution between consumption and leisure equal to the relative price of leisure; (8) equates weighted factor intensities across sectors; (9) expresses the relative price of human capital as a function of the factor intensity in the goods sector in general, but shows it is exogenously determined if $\phi_1 = \phi_2$. Equations (10) and (11) are intertemporal capital efficiency, or "arbitrage", for human and physical capital, where the "capacity utilization" factor of human capital is one minus leisure, $(1 - L_{t+1})$, and it affects both the dynamics and the growth rate[5]. The dynamics of the model are summarized by two complementary sets of conditions: static equilibrium conditions that govern intratemporal resources allocations (equations (7), (8) and (9)) and dynamic conditions that determine investment decisions (equations (10) and (11)).

By equations (1), (8) and (9),

$$\frac{V_t (1 - N_t - L_t)}{N_t (1 - V_t)} = \frac{\phi_1 (1 - \phi_2)}{\phi_2 (1 - \phi_1)}, \tag{12}$$

if an aggregate productivity shock causes decreases in both leisure L_t and labor in the goods sector, N_t, then it must be also that the share of physical capital in the goods sector, V_t, also must fall in order for the righthandside of equation (12) to remain constant. This is in fact what impulse responses show to be the case in the relevant section below. However this demonstrates the sense in which leisure produces a type of asymmetry that drives dynamic results in the sense of the Rybczynski (1955) increase in a factor.

The relative price ends up rising initially, and falling later, as a result of the productivity shock, and this gives a full general equilibrium basis in a change in exogenous processes for the Stolper and Samuelson (1941) theorem. Unlike an unspecified reason for the relative price to rise, which Stolper and Samuelson (1941) say is not important to specify in their footnote 3, here the aggregate productivity shock causes the subsequent price and marginal product changes as tempered at the same time by a Rybczynski effect through the decrease in leisure.

Consider that capital factor rewards by equations (8) and (9),can be derived analytically as functions of P_t:

$$\begin{aligned} r_t &= S_t^{\frac{\phi_1 - 1}{\phi_1 - \phi_2}} Z_t^{\frac{1-\phi_2}{\phi_1 - \phi_2}} \Psi_r P_t^{\frac{\phi_1 - 1}{\phi_1 - \phi_2}}, \\ \Psi_r &= \phi_1 A_h^{\frac{\phi_1 - 1}{\phi_1 - \phi_2}} A_g^{\frac{1-\phi_2}{\phi_1 - \phi_2}} \left(\frac{\phi_2}{\phi_1}\right)^{\frac{\phi_2(\phi_1 - 1)}{\phi_1 - \phi_2}} \left(\frac{1-\phi_2}{1-\phi_1}\right)^{\frac{(1-\phi_2)(\phi_1 - 1)}{\phi_1 - \phi_2}}; \end{aligned} \tag{13}$$

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[5] In contrast, Collard (1999) allows human capital to enter utility function directly by specifying a momentary utility function similar to $\frac{(C(LH)^A)^{1-\sigma}-1}{1-\sigma}$. Human capital is then fully utilized such that its net return is $\frac{W}{P} - \delta_h$. In our results, human capital capacity utilization plays a key role in that it affects the steady state growth rate: $\left[\left(1 + (1 - L)\frac{W}{P} - \delta_h\right) / (1 + \rho)\right]^{1/\sigma} - 1$.

$$
\begin{aligned}
W_t &= S_t^{\frac{\phi_1}{\phi_1-\phi_2}} Z_t^{\frac{-\phi_2}{\phi_1-\phi_2}} \Psi_w P_t^{\frac{\phi_1}{\phi_1-\phi_2}}, \qquad (14) \\
\Psi_w &= (1-\phi_1) A_h^{\frac{\phi_1}{\phi_1-\phi_2}} A_g^{\frac{-\phi_2}{\phi_1-\phi_2}} \left(\frac{\phi_2}{\phi_1}\right)^{\frac{\phi_1\phi_2}{\phi_1-\phi_2}} \left(\frac{1-\phi_2}{1-\phi_1}\right)^{\frac{\phi_1(1-\phi_2)}{\phi_1-\phi_2}}.
\end{aligned}
$$

Proposition 3 *The sign of the derivative of r_t and W_t with respect to P_t depends only on the factor intensity ranking.*

Proof. Given the assumption that human capital investment is relatively more human capital intensive than goods production, so that $\phi_1 > \phi_2$, then by equations (13) and (14), $r_t'(P_t) < 0$ and $W_t'(P_t) > 0$. ■

Corollary 4 *An increase in the price of human capital relative to physical capital, given unchanged relative productivity parameters between sectors, increases the reward to human capital while decreasing the reward to physical capital.*

Proof. Consider that from equations (13) and (14),

$$
\hat{W}_t - \hat{r}_t = \frac{\hat{P}_t - \left(\hat{Z}_t - \hat{S}_t\right)}{\phi_1 - \phi_2} \qquad (15)
$$

with "ˆ" denoting the variable's percentage deviation from its corresponding steady state value. With $\phi_1 > \phi_2$, and identical shocks such that $\hat{Z}_t = \hat{S}_t$, an upswing in P_t causes $\hat{W}_t - \hat{r}_t$ to increase, where W_t rises and r_t falls. ■

This is a general equilibrium form of the Stolper and Samuelson (1941) theorem: in a two-sector production model, an increase in the relative price of output of one sector rewards relatively more the factor that is used more intensively in this sector.

In equilibrium, the rate of return to physical capital equals the rate of return to human capital plus some form of "capital gain" of human capital investment, along with differentiated covariance risk effects. From equations

(10) and (11),

$$\begin{aligned} & E_t\left[1+r_{t+1}-\delta_k\right] \\ = \; & E_t\left[\frac{P_{t+1}}{P_t}\left(1+\left(1-L_{t+1}\right)\frac{W_{t+1}}{P_{t+1}}-\delta_h\right)\right] \\ & -\frac{Cov_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{L_{t+1}}{L_t}\right)^{A(1-\sigma)},\left(1+r_{t+1}-\delta_k\right)\right]}{E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{L_{t+1}}{L_t}\right)^{A(1-\sigma)}\right]} \\ & +\frac{Cov_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{L_{t+1}}{L_t}\right)^{A(1-\sigma)},\frac{P_{t+1}}{P_t}\left(\left(1-L_{t+1}\right)\frac{W_{t+1}}{P_{t+1}}+1-\delta_h\right)\right]}{E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{L_{t+1}}{L_t}\right)^{A(1-\sigma)}\right]} \end{aligned}$$

This "no-arbitrage" condition suggests how the adjustment process is stable when human capital investment sector is more human capital intensive than the goods sector (i.e. $\phi_1 > \phi_2$). For simplicity of presentation, note that with equal covariance terms, and letting $\delta_h = \delta_h \equiv \delta$, and $L_{t+1} = 0$ for the moment, then equation (15) allows the no-arbitrage equation to reduce to

$$\hat{P}_t = \left(1+\frac{r}{1+r-\delta}\frac{1-(\phi_1-\phi_2)}{\phi_1-\phi_2}\right)E_t\hat{P}_{t+1},$$

where r is the steady state value for r_{t+1}. The coefficient $1+\frac{r}{1+r-\delta}\frac{1-(\phi_1-\phi_2)}{\phi_1-\phi_2}$ is greater than 1 *iff* $\phi_1 > \phi_2$ given normal parameters ranges so that $r-\delta > 0$.[6]

On impact of a positive aggregate shock, the relative price of human capital investment increases initially, as physical capital investment can made without cost from goods output while human capital investment requires time and so is relatively scarce. This induces resources flow from the goods sector to the human capital investment sector. However the increase in leisure, which reinforces the magnitude of the increase in the human capital investment, also pressures down the relative price P_t and in subsequent periods, this price decreases and the direction of the inter-sectoral resource transfer reverses. As the effect of the shock dies out in subsequent periods, labour flows back slowly to the goods sector due to an optimal spreading of the inter-sectoral adjustment cost across periods.

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[6] Similar results are found in Barro and Sala-I-Martin (1995) and Bond et al. (1996).

The shock also gives an inverse relation between productivity and market employment: when productivity increases, labor is shifted from the goods sector to the human capital investment sector, even as output in the goods sector expands.[7] Rather than encouraging goods sector employment, higher productivity results in an initially lower employment rate (excluding the household sector). Market output still increases mildly because the positive technology effect dominates the negative effect induced by the outflow of labor, and of physical capital.

2.4 Normalization

The characterization of the equilibria of similar two-sector endogenous growth models is in Caballe and Santos (1993) and Bond et al. (1996), without leisure, and in Ladron et al. (1997) with leisure. Due to nonstationarity of steady state, the standard log-linearization method does not apply directly. However, if those growing variables are transformed to have stationary distributions, one can linearize the model in the neighborhood of the stationary transformation. To compute the impulse response function of output, nonstationary variables can be normalized by discounting at the rate of their common constant *BGP* growth rate γ, which is independent with the initial resource endowments:

$$c_t \equiv \frac{C_t}{(1+\gamma)^t},\ k_t \equiv \frac{K_t}{(1+\gamma)^t},\ h_t \equiv \frac{H_t}{(1+\gamma)^t}$$

In the nonstochastic version of the transformed model, all variables will converge to and continue to stay on a particular *BGP* once the initial values for the physical and human capital are given, with no indeterminacy of *BGP* once the initial resource endowment is fixed.

For simulations of this stochastic growth model, however, a new *BGP*, in general, will be triggered when a shock occurs to the economy. In other

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[7] Gali and Hammour (1991, p.15) suggest "Recessions have a 'cleaning-up' effect that causes less productive jobs to be closed down. This can happen either because those jobs become unprofitable, or because recessions provide an excuse for firms to close them down in the context of formal or informal worker-firms arrangements. As a consequence, the average productivity of jobs will rise."

words, the non-stationary variables do not converge back to the previous BGP after even a temporary shock. Normalization by a deterministic trend is only valid to attain impulse response functions that capture the reactions of variables after only one shock, rather than repeated shocks. The normalization method used to simulate the model is to divide all growing variables by the current stock of human capital such that variables in ratios are constant along nonstochastic BGP:

$$c_t \equiv \frac{C_t}{H_t}, k_t \equiv \frac{K_t}{H_t}, \gamma_{ht+1} \equiv \frac{H_{t+1}}{H_t}.$$

For details on solving the model numerically under the two different scaling methods, please refer to Appendix B.

3 Calibration

Gomme and Rupert (2007) detail a calibration for business cycle statistics using models with a second "household" sector. The calibration is therefore made close to that of Gomme and Rupert except where Perli and Sakellaris and others provide estimates used for the human capital sector specifically, here as a special case of household production. The data set is US quarterly from 1954 to 2004, as provided by Gomme and Rupert; Appendix C provides a detailed data description. All parameters are on a quarterly basis unless stated otherwise.

Table 1 presents the calibrated parameters and target values of variables. Utility is assume to be log, with a 1.55 leisure preference weight, and a unitary coefficient of relative risk aversion; the physical capital share in the goods sector is 0.36, as is standard; the time preference discount factor is 0.986. For the period, the US GDP, aggregate consumption and investment, on average, grew roughly at a common rate 0.42% per quarter, providing the targeted balanced growth path growth rate. The depreciate rate of physical capital is 0.20, to match in the steady state the empirical physical capital investment to output ratio of approximately 25.3%.

Early results by Jorgenson and Fraumeni (1989) suggest an annual depreciation rate of human capital between 1% and 3%; Jones et al. (2005)

estimate a lower bound for this at about 1.5%, while using an intermediate value at 2.5% yearly, which corresponds to about 0.625% quarterly; DeJong and Ingram (2001) estimate 0.5% per quarter. We follow this latter estimate and use it for the baseline case.

Labour supply is targeted at 0.3, with Jones et al. (2005) having a low-end value of 0.17,and Gomme and Ruppert (2007) at 0.255. Leisure is 0.54 compared to 0.505 in Gomme and Ruppert, and human capital investment time is 0.16 compared to 0.24 of household time in Gomme and Ruppert.

Perli and Sakellaris (1998) assume the human capital investment sector in theory has its real economy counterparts in two social activities: education and on-the-job training, similar to Becker (1975). Using data from Jorgenson and Fraumeni (1989), they calculate the contribution of physical capital to educational output at 8%, with labour's share at 92%. For job training, they assume the same technology as for goods production, arriving at a weighted average of the share of physical capital in human capital investment between 11% and 17%. We use the lower bound of 0.11for the baseline calibration.[8]

The technology shock to the goods sector is calibrated in typical fashion given the well-known difficulty in separating out human capital. The resulting autocorrelation coefficient of $\log Z_t$ recovered from Solow residuals is about 0.95 and the variance of innovation is about 0.0007, a result close to Perli and Sakellaris (1998). The technology shock to the human capital investment sector is assumed identical to the shock to physical sector, every period, so there is in essence just one aggregate shock affecting both sectors, as in Jones et al. (2005). Separate sectoral shocks are allowed in the section below on sensitivity analysis. The scale parameter associated to physical sector, A_g, is normalized to one and A_h is set equal to 0.0461.

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[8] See also Einarsson and Marquis (1999).

Baseline Calibration of Parameters		
β	Subjective discount factor	0.986
σ	Coefficient of relative risk aversion	1
A	Weight of leisure in preference	1.55
ϕ_1	Share of physical capital in physical sector	0.36
ϕ_2	Share of physical capital in human sector	0.11
δ_k	Depreciation rate of physical capital	0.02
δ_h	Depreciation rate of human capital	0.005
A_g	Scale parameter for goods sector	1
A_h	Scale parameter of human sector	0.0461
$\rho_z = \rho_s$	Persistence parameter of shock	0.95
$\sigma_z^2 = \sigma_s^2$	Variance of innovation	0.0007
Target Values of Variables		
γ	BGP growth rate	0.0042
r	Steady state real interest rate	0.0185
A_h	Scale parameter of human sector	0.0461
N	Steady state working time	0.3
M	Steady state learning time	0.16
L	Steady state leisure time	0.54
A	Weight of leisure in preference	1.55
$\frac{C}{Y}$	Steady state consumption-output ratio	0.75
$\frac{I_k}{Y}$	Steady state physical investment-output ratio	0.25
V	Steady state share of physical capital in goods sector	0.89

Table 1: Calibration of the two-sector SEG model

4 Cyclic Simulation Results

4.1 Impulse response functions

Figure 1 shows the impulse response functions for an equal technology shock to both sectors simultaneously for selected nonstationary variables. Similar to data, the reaction of consumption and output is small on impact and continues to increase in subsequent periods, while investment shows a hump and human capital rises initially and then declines monotonically. The initially small reaction of output on impact is the joint effect of the relative price and Rybczynski (1955) effects, pushing factors towards the human capital sector even as goods productivity rises. In subsequent periods, the flow back of factors towards the goods sector starts to reinforce the now-fading of the goods sector productivity shock, so as to sustain the long-lasting expansion in output. The hump response of physical capital investment emerges since it is the difference between output rising faster than consumption. The responses of these variables do not resemble the goods sector technology shock itself, indicating a sense in which the human capital sector causes an "internal propogation mechanism".

Figure 2 shows the positive aggregate shock effect on selected stationary variables, which return to the initial equilibrium after the transitory shock. Leisure decreases on impact due to higher productivity in the productive use of time; working hours decrease and learning time increases. The decline of working hours on impact is consistent with the empirical finding of Gali (1999), who identifies a negative correlation between productivity and working hours using VAR evidence. Therefore the observed decline in working hours in face of higher labour productivity is consistent with this RBC model. The "physical capital allocation" refers to the variable V_t, the share of physical capital in the goods sector, which declines are resources flow to the human capital investment sector.

Figure 3 shows how the aggregate productivity shock affects the marginal input products versus the relative sectoral output price, of human capital investment to goods output. It demonstrates that the Stolper and Samuelson (1941) effect of equation (15) holds in this general equilibrium. The relative

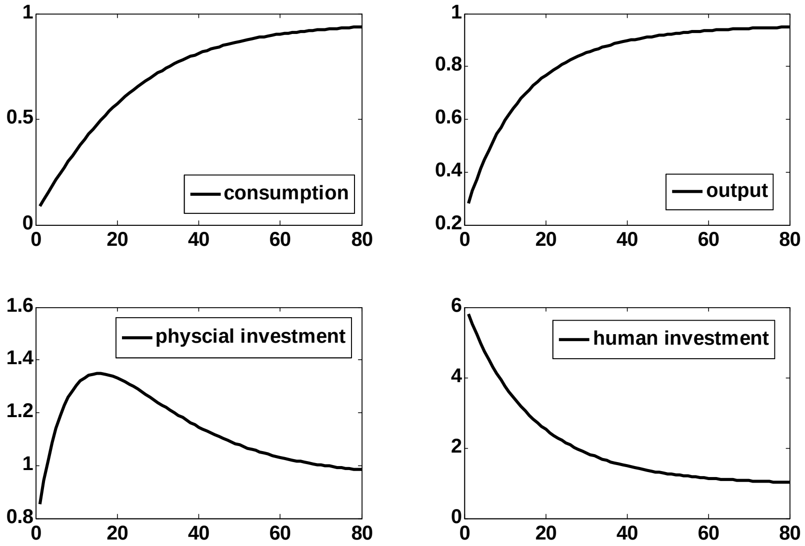

Figure 1: Impulse response functions to technology shock

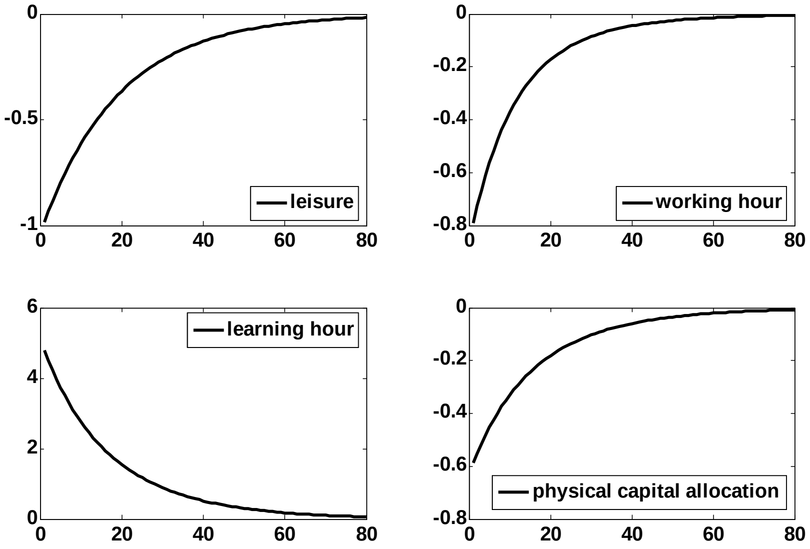

Figure 2: Impulse response functions to technology shock

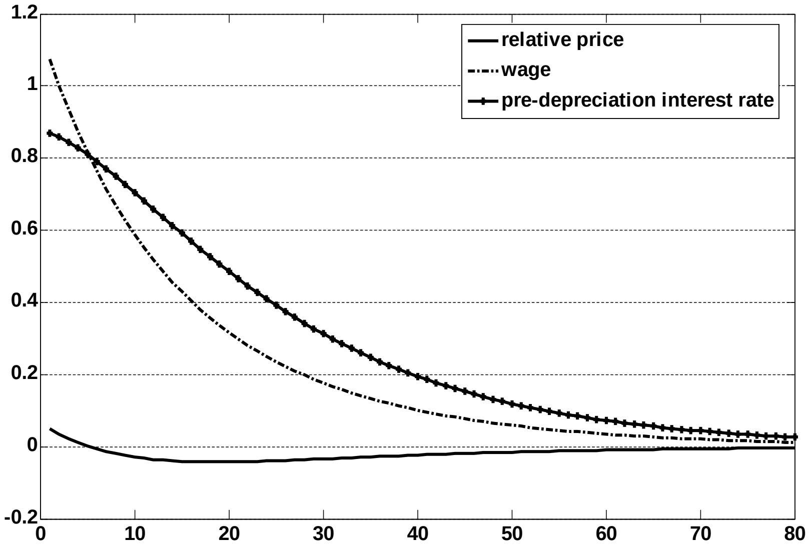

Figure 3: Impulse response functions to technology shock: Stolper-Samuelson effect

price of human capital P_t is higher than its steady state value when $\hat{W}_t$, the top-most curve at period 1, lies above $\hat{r}_t$. And P_t, the bottom curve, rises initially above 0, but then falls below 0 as $\hat{W}_t$ falls below $\hat{r}_t$.

4.2 Persistence and volatility

Table 4 reports the statistics computed from US data and a simulated sample of 30,000 periods. The data set to calculate moment statistics is the same as what is used for calibration. Due to the endogenous growth component of the model, the nonstochastic steady state of the model economy is growing at an endogenously determined rate. The nonstationarity of steady state means it is not possible to compute standard volatility statistics. An alternative approach is to calculate moment statistics of growth rates of variables, which by construction have stationary distributions along the *BGP*, as in Jones et al. (2005).

The third column of Table 4 shows that US data suggests the consumption growth rate fluctuates about half as much as output growth, which in turn fluctuates about half that of investment growth. And the endogenous growth model fits these volatilities quite well although, with some under-prediction. The model's volatility of output growth is 0.82 compared to 1.14 in the data; simulated consumption growth volatility is 0.43 compared to 0.52 in the data; and the simulated investment growth volatility is 2.23 compared to 2.38 in the data. The endogenous growth model shows resolution of the labour volatility puzzle; in Table 4, the model's simulated volatility of hours is 0.54, well matching its empirical counterpart of 0.52.

The fourth to sixth columns of Table 4 show a weakness of the model, in general terms of too much higher order persistence. The model replicates the autocorrelation properties of output growth data strikingly well in the first order autocorrelation coefficient, producing exactly the same degree of first-order persistence as observed in the data. However, higher order autocorrelation coefficients in the data fall to zero more quickly than those simulated from the model.

Note that the autocorrelation coefficient of investment growth is usually not reported in business cycle research. But in the data, the growth rate of investment is autocorrelated at an even higher degree than those of output and consumption (0.38 compared to 0.29 and 0.24). Conventional *RBC* models fail to reproduce the persistence of investment growth for the same reason as they fail for output growth, yet the model captures some of this persistence, matching it in the third-order term. For working hours, the model does well but generates somewhat less persistence than in the data.

With regards to the contemporaneous correlations between output growth and other variables and the lead-and-lag pattern, in general, the model fits the data well. Consumption and investment growth are pro-cyclical both in the model and data. However, in the data, labour supply is only slightly negatively correlation with output growth, at -0.07, but the model predicts labour supply to be more strongly counter-cyclical, at -0.73. In addition, output growth is mildly and positively correlated with next period labour supply in the data (0.01) while the model predicts a negative value for the same correlation (-0.67).

		Standard Deviation				Correlation with output growth			
		2_6	6_32	32_200	2_200	2_6	6_32	32_200	2_200
		High freq	Med freq	Low freq	Med t cycle	High freq	Med freq	Low freq	Med t cycle
gr Y	data	0.82	0.72	0.32	1.15	1.00	1.00	1.00	1.00
	sim	0.57	0.42	0.38	0.80	1.00	1.00	1.00	1.00
gr C	data	0.39	0.29	0.20	0.53	0.24	0.77	0.81	0.48
	sim	0.16	0.15	0.27	0.35	0.99	0.91	0.88	0.85
gr I	data	1.60	1.55	0.74	2.38	0.57	0.94	0.68	0.73
	sim	1.78	1.26	0.88	2.35	1.00	0.99	0.90	0.97
N	data	0.42	1.58	3.14	3.57	0.17	-0.20	-0.02	-0.04
	sim	1.17	2.50	4.02	4.88	-0.84	-0.70	-0.96	-0.71
M	data	?	?	?	?	?	?	?	?
	sim	7.01	15.21	27.89	32.55	0.84	0.67	0.92	0.66

Table 2: Business Cycle Statistics by frequencies

Here γ_x is the growth rate of variable x; N is fraction of time spent working in the goods sector; $\sigma(x)$ measures variable's percentage deviation from the mean; $\rho(x,y)$ is the correlation coefficient of variables x and y. The model predicts $\gamma_Y, \gamma_C, \gamma_{I_k}$ and N to have stationary distributions along BGP. Therefore, US aggregate data on Y, C, I_k are logged and first-differenced and data on working hours is in levels. Unit root tests on the data suggest that the logged and first differenced series of output, consumption and physical investment are stationary, but not the level of per-capita working hours. The variability of per-capita working hours is therefore normalized by the mean and measured by $\frac{\sigma(N)}{E(N)}$.

Here γ_x is the growth rate of variable x; N is fraction of time spent working in the goods sector; $\sigma(x)$ measures variable's percentage deviation from the mean; $\rho(x,y)$ is the correlation coefficient of variables x and y. The model predicts $\gamma_Y, \gamma_C, \gamma_{I_k}$ and N to have stationary distributions along BGP. Therefore, US aggregate data on Y, C, I_k are logged and first-differenced and data on working hours is in levels. Unit root tests on the data

		Standard Deviation				Correlation with output growth			
		2_6	6_32	32_200	2_200	2_6	6_32	32_200	2_200
		High freq	Med freq	Low freq	Med t cycle	High freq	Med freq	Low freq	Med t cycle
gr Y	data	0.82	0.72	0.32	1.15	1.00	1.00	1.00	1.00
	sim	0.77	0.57	0.53	1.09	1.00	1.00	1.00	1.00
gr C	data	0.39	0.29	0.20	0.53	0.24	0.77	0.81	0.48
	sim	0.02	0.09	0.33	0.34	-0.25	0.70	0.92	0.51
gr I	data	1.60	1.55	0.74	2.38	0.57	0.94	0.68	0.73
	sim	3.07	2.08	1.25	3.91	1.00	1.00	0.95	0.98
N	data	0.42	1.58	3.14	3.57	0.17	-0.20	-0.02	-0.04
	sim	2.27	4.74	6.74	8.55	0.70	-0.27	-0.85	-0.27
M	data	?	?	?	?	?	?	?	?
	sim	10.20	21.64	36.67	43.79	-0.69	0.33	0.94	0.35

Table 3: Business Cycle Statistics by frequencies (smaller human capita lshock)

				$\rho\left(x_t, x_{t-j}\right)$			$\rho\left(\gamma_{Y_t}, x_{t+j}\right)$				
x_t		$\sigma\left(x_t\right)$	$j =$	1	2	3	-2	-1	0	1	2
γ_{Y_t}	data	1.14		0.29	0.16	0.03	0.16	0.29	1	0.29	0.16
	model	0.82		0.29	0.27	0.25	0.27	0.29	1	0.29	0.27
γ_{C_t}	data	0.52		0.24	0.14	0.19	0.20	0.37	0.49	0.27	0.16
	model	0.43		0.78	0.75	0.73	0.35	0.38	0.83	0.50	0.49
$\gamma_{I_{kt}}$	data	2.38		0.38	0.24	0.11	0.17	0.39	0.75	0.41	0.24
	model	2.23		0.14	0.11	0.10	0.19	0.21	0.96	0.15	0.11
N_t	data	5.52		0.99	0.96	0.93	-0.22	-0.18	-0.07	0.01	0.07
	model	5.54		0.92	0.85	0.73	-0.37	-0.40	-0.73	-0.67	-0.62

Table 4: Business cycle statistics for baseline calibration

	$x =$	Persistence of x γ_Y	γ_C	γ_{I_k}	N	Volatility of x γ_Y	γ_C	γ_{I_k}	N
US data		0.29	0.24	0.39	0.99	1.14	0.52	2.38	5.52
Baseline ($\rho = 0.95$)		0.29	0.77	0.14	0.92	0.82	0.43	2.36	5.44
$\rho = 0.90, (\delta_h = 0.015)$		0.63	0.70	0.64	0.82	0.63	0.38	1.43	4.21
$\rho = 0.85, (\delta_h = 0.015)$		0.31	0.63	0.24	0.77	0.69	0.30	1.81	3.41
$\rho = 0.80, (\delta_h = 0.015)$		0.16	0.61	0.09	0.73	0.75	0.26	2.16	2.93

Table 5: Changes in the aggregate shock autocorrelation

suggest that the logged and first differenced series of output, consumption and physical investment are stationary, but not the level of per-capita working hours. The variability of per-capita working hours is therefore normalized by the mean and measured by $\frac{\sigma(N)}{E(N)}$.

5 Sensitivity analysis

This section presents tests on the robustness of the results obtained previously regarding business cycle persistence and cyclical moments to alternative specifications of exogenous parameters. For a first alternative specification, consider lowering the autocorrelation coefficient of the aggregate productivity shock down from 0.95. Combining this variation with an increase in the human capital depreciation rate from 0.005 up to 0.015, there is some ability to decrease the aggregate shock autocorrelation downwards and still retain a similar ability to match the business cycle data, a somewhat striking feature.

Table 3 shows for example that with $\rho_z = \rho_s \equiv \rho = 0.85$ and $\delta_h = 0.015$, the match of output growth persistence is still good; the match with consumption and investment growth persistence improve, while the labor persistence falls below the data level. And the volatility of the growth of these variables falls further down from the data levels. A decrease from $\rho = 0.95$ to 0.85 is a significant decrease in the persistence built into the shock process, made possible by the additional human capital sector.

While identical shocks to both sectors appear necessary in experiments to generate the reasonable results thus far presented, one modest deviation from identical shocks is presented next through different correlation coefficients

	Persistence of x				Volatility of x			
$x =$	γ_Y	γ_C	γ_{I_k}	γ_N	γ_Y	γ_C	γ_{I_k}	N
US data	0.29	0.24	0.39	0.99	1.14	0.52	2.38	5.52
Baseline ($\rho_{zs} = 1$)	0.29	0.77	0.14	0.92	0.82	0.43	2.36	5.44
$\rho_{zs} = 0.995$	0.16	0.71	0.05	0.92	1.08	0.44	3.33	5.62
$\rho_{zs} = 0.99$	0.11	0.68	0.01	0.92	1.27	0.46	4.03	5.81
$\rho_{zs} = 0.95$	0.03	0.49	-0.04	0.90	2.29	0.60	7.66	7.00
$\rho_{zs} = 0.9$	-0.02	0.35	-0.05	0.89	3.16	0.73	10.67	8.09
$\rho_{zs} = 0.7$	-0.04	0.24	-0.06	0.87	5.33	1.11	18.14	11.62

Table 6: Business cycle stastistics for sector-specific shocks

of the shock innovations. A generalized representation of exogenous forces in the two-sector model is to represent sector-specific shocks as a vector autoregressive process:

$$\begin{bmatrix} \log Z_{t+1} \\ \log S_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_z & 0 \\ 0 & \rho_s \end{bmatrix} \begin{bmatrix} \log Z_t \\ \log S_t \end{bmatrix} + \begin{bmatrix} \varepsilon^z_{t+1} \\ \varepsilon^s_{t+1} \end{bmatrix}$$

where ε^z_{t+1} and ε^s_{t+1} are i.i.d. disturbances to $\log Z_{t+1}$ and $\log S_{t+1}$ respectively. Assuming 0 elements in the upper-right and lower-left positions in the autocorrelation coefficient matrix implies no technology diffusion across sectors. The variance-covariance matrix of the disturbances is:

$$V \begin{bmatrix} \varepsilon^z_{t+1} \\ \varepsilon^s_{t+1} \end{bmatrix} = \begin{bmatrix} \sigma^2_z & \sigma_{zs} \\ \sigma_{zs} & \sigma^2_s \end{bmatrix}$$

where $\sigma_{zs} = \rho_{zs}\sigma_z\sigma_s$, and ρ_{zs} is the correlation coefficient of ε^z_t and ε^s_t. Still assuming that Z_t and S_t have the same specifications of $\rho_z = \rho_s \equiv \rho$ and $\sigma^2_z = \sigma^2_s$, realizations of Z_t and S_t can be different if a departure is made from the baseline assumption that $\rho_{zs} = 1$. Table 6 displays the model's simulated persistence and volatility for different values of ρ_{zs}. It emerges that as ρ_{zs} falls, output and investment growth persistence fall, consumption growth persistence rises and then falls, and labor growth falls only slightly. The high values of ρ_{zs} may be justified, for example, with inventions such as the internet improving productivity economy-wide. Overall, the baseline of 1 appears the perform best.

	Persistence of x				Volatility of x			
$x =$	γ_Y	γ_C	γ_{I_k}	N	γ_Y	γ_C	γ_{I_k}	N
US data	0.29	0.24	0.39	0.99	1.14	0.52	2.38	5.52
Baseline ($\phi_2 = 0.11$)	0.29	0.77	0.14	0.92	0.82	0.43	2.36	5.44
$\phi_2 = 0.03$	0.14	0.19	0.12	0.93	0.99	0.71	1.97	6.77
$\phi_2 = 0.05$	0.20	0.31	0.16	0.93	0.92	0.60	1.98	6.50
$\phi_2 = 0.07$	0.26	0.46	0.17	0.93	0.85	0.52	2.02	6.26
$\phi_2 = 0.09$	0.27	0.62	0.15	0.92	0.84	0.47	2.17	5.95
$\phi_2 = 0.13$	0.28	0.91	0.10	0.92	0.86	0.40	2.62	5.11
$\phi_2 = 0.15$	0.23	0.96	0.07	0.92	0.91	0.38	3.00	4.46
$\phi_2 = 0.17$	0.18	0.98	0.05	0.92	1.01	0.37	3.55	4.06

Table 7: Sensitivity of physical capital share in human sector

Three other sets of sensitivity analysis are presented, for variations in the share of physical capital in human sector (ϕ_2), the rate of depreciation of human capital (δ_h) and the coefficient of relative risk aversion (σ) , in Tables 7, 8, and 9. For example, Jones et al. (2005) emphasize the importance of the coefficient of relative risk aversion.

Table 7 shows the baseline is still probably the best specification for ϕ_2, although trade-offs between better persistence and better volatility results are apparent. Table 8 for changes in δ_h,expressed in quarterly units, correspond to a yearly range between 1% and 6%. As δ_h gets bigger, growth rates of output, consumption and physical capital investment all becomes more autocorrelated, indicating a higher degree of persistence. For instance, autocorrelation coefficient of output growth is as high as 0.94 when δ_h is 0.015. This suggests that increasing the depreciation rate of human capital produces greater persistence of the model's variables. For volatility, growth rates of output and physical investment fluctuate less while consumption growth fluctuates more as δ_h increases. The volatility of labour supply does not seem to be affected by δ_h.

For Table 9, the model generates little persistence when σ rises up to 1.5. Changes in σ slightly affect labor supply growth persistence, but have a large impact on the volatilities of variables, in a non-monotonic fashion except for labor.

		Persistence of x				Volatility of x			
	$x =$	γ_Y	γ_C	γ_{I_k}	N	γ_Y	γ_C	γ_{I_k}	N
US data		0.29	0.24	0.39	0.99	1.14	0.52	2.38	5.52
Baseline ($\delta_h = 0.005$)		0.29	0.77	0.14	0.92	0.82	0.43	2.36	5.44
$\delta_h = 0.0025$		0.18	0.75	0.06	0.93	0.92	0.39	2.88	5.53
$\delta_h = 0.0075$		0.44	0.78	0.29	0.91	0.76	0.44	1.89	5.38
$\delta_h = 0.0100$		0.66	0.79	0.60	0.90	0.72	0.49	1.56	5.58
$\delta_h = 0.0125$		0.84	0.80	0.85	0.89	0.69	0.51	1.37	5.52
$\delta_h = 0.0150$		0.94	0.81	0.89	0.88	0.71	0.54	1.47	5.60

Table 8: Sensitivity of human capital depreciation rate

		Persistence of x				Volatility of x			
	$x =$	γ_Y	γ_C	γ_{I_k}	N	γ_Y	γ_C	γ_{I_k}	N
US data		0.29	0.24	0.39	0.99	1.14	0.52	2.38	5.52
Baseline ($\sigma = 1$)		0.29	0.77	0.14	0.92	0.82	0.43	2.36	5.44
$\sigma = 0.6$		-0.07	-0.07	0.08	0.81	40.68	49.33	19.02	66.52
$\sigma = 0.7$		0.02	0.05	0.13	0.91	6.42	6.32	7.40	24.20
$\sigma = 0.8$		0.20	0.11	0.93	0.91	1.50	1.86	1.61	10.59
$\sigma = 0.9$		0.96	0.49	0.36	0.91	0.54	0.62	1.86	7.00
$\sigma = 1.1$		0.12	0.25	0.07	0.93	1.13	0.65	2.66	4.66
$\sigma = 1.2$		0.07	0.12	0.04	0.93	1.36	0.87	2.90	4.14
$\sigma = 1.3$		0.04	0.07	0.03	0.93	1.51	1.02	3.04	3.57
$\sigma = 1.4$		0.03	0.05	0.02	0.94	1.62	1.15	3.14	3.42
$\sigma = 1.5$		0.01	0.03	0.00	0.94	1.70	1.24	3.21	3.17
$\sigma = 2.0$		-0.01	0.00	-0.02	0.94	1.95	1.53	3.43	2.48

Table 9: Sensitivity of coefficient of relative risk aversion

6 Comparison with One Sector Human Capital Model

6.1 Timing of responses

This section compares the paper's baseline model to such a one sector *RBC* model with human capital, to a standard *RBC* model without human capital, and to an alternative baseline model except with the depreciation rates equal on physical and human capital rather than having the physical capital depreciation rate higher as in the standard model. More specifically, in case 1, goods production and human capital investment production are treated symmetrically, produced by an identical technology and with equal capital depreciation rates, and so correspond to a standard one-sector *RBC* model ($\phi_1 = \phi_2,\ \delta_k = \delta_h$). In case 2, human capital investment is assumed to be produced by the same technology producing goods, but with a slower depreciation rate for human capital relative to physical capital ($\phi_1 = \phi_2, \delta_k > \delta_h$); the parameterized model in this case is essentially the same as the one-sector model in Jones et al. (2005). In case 3, the second sector for the production of human capital investment is more human capital intensive than goods production, but depreciation rates on physical and human capital are equal ($\phi_1 > \phi_2, \delta_k = \delta_h$). Finally, in case 4 is the baseline model, with the second sector for producing human capital investment more human capital intensive than goods production and with human capital depreciating at a slower rate than does physical capital ($\phi_1 > \phi_2, \delta_k > \delta_h$).

Figure 5 shows the responses to a positive aggregate technology shock, for the four cases described above, for working hours and human capital investment time hours, and in Figure 6 for consumption and output. Except for the first case, working hour and learning time move in opposite directions following the productivity shock, as is consistence with the empirics in Dellas and Sakellaris (2003), of significant substitution between education and competing labour activities over business cycles. However only cases 3 and 4 show the initial drop in working hours as in Gali (1999).

In all four cases, the consumption response is smooth, due to the intertem-

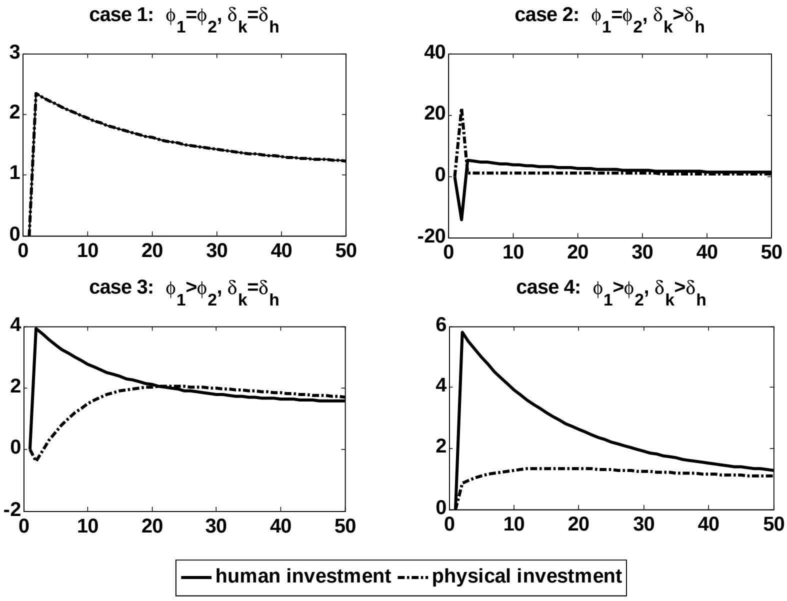

Figure 4: Comparing impulse response functions to technology shock: physical and human capital investments

poral substitution effect. However, only in cases 3 and 4, are the trajectories for output smoothly rising as in data. This indicates the role of cross-sector factor intensity disparity in generating output persistence. Similarly, the '$\vee$' shape response of working time in cases 3 and 4, in contrast to the '$\wedge$' shape response in cases 1 and 2, and in standard one-sector models, gives rise to the hump in the impulse response curve of output.

6.2 Persistence and volatility of some variants

Table 10 reports the moments statistics for the four cases. Overall, case 4 matches the empirical data best. In case 1, the traditional *RBC* model, the autocorrelation coefficients for output and investment growth are both very close to zero, showing a lack of persistence that is a well-known failing of traditional *RBC* models. Another major problem in case 1 is the too-low simulated working-hour volatility, also a well-known drawback of original *RBC* models. For case 2, growth rates of investment and output

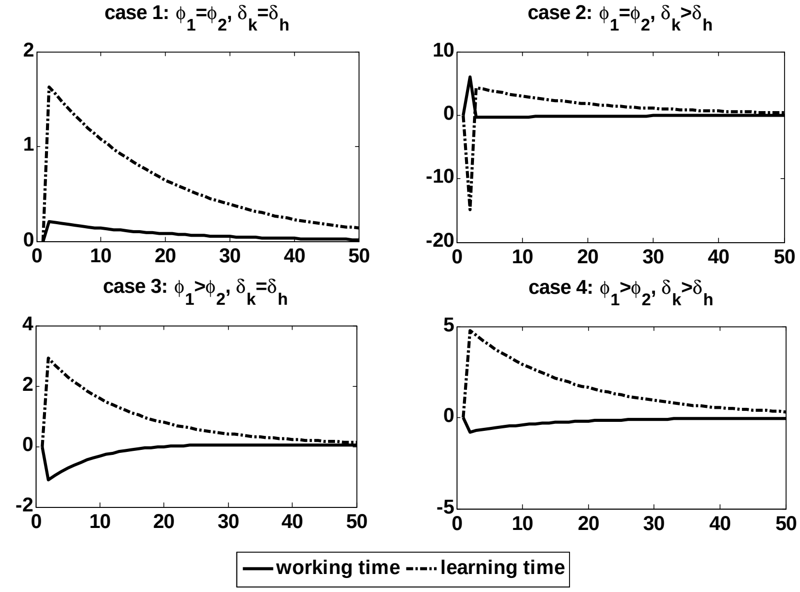

Figure 5: Comparing impulse response functions: working hours and learning time

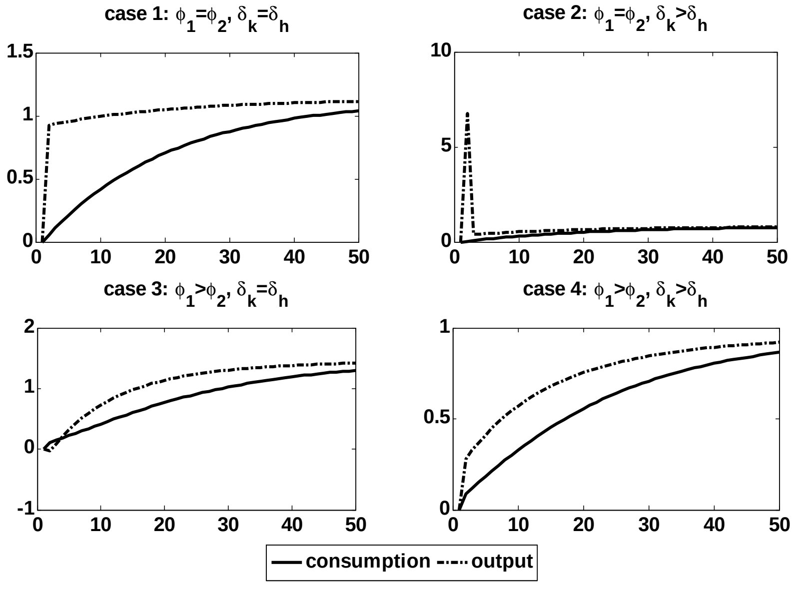

Figure 6: Comparing impulse response functions: consumption and output

are significantly negatively autocorrelated, in contrast to the data. These inconsistencies between model simulations and data indicate that asymmetric depreciate rates of capitals appear to be unable to match certain key persistence and moment statistics. Case 3 with different factor intensities across sectors appears successful in replicating moment statistics, but generates rather too much persistence. For example, output and consumption growth in the model are autocorrelated with coefficients of 0.86 and 0.81 respectively while in the data the counterparts are only 0.29 and 0.24. In case 4, human capital is assumed to depreciate at a slower rate of 0.005 per quarter. The results show that lowering the human capital depreciation rate reduces the simulated degree of persistence to a level closer to US observations.

Quantitatively, our results in our case 2 corresponding to Jones et al. are quite different, if qualitatively similar. This results because a direct comparison to Jones et al. (2005) using our model confronts several difficulties, involving data frequency and the definitions of "output" and "consumption". Jones et al. use yearly data frequency while we use a quarterly frequency. [9] This makes the annual investment-to-capital ratio, in general, four times as large as the quarterly counterpart. Therefore, yearly investment accounts for a bigger fraction of capital stock than when measured on quarterly basis, making volatility as measured by annual data significantly less than that measured by quarterly data. This explains the good performance of Jones et al. (2005) regarding volatility statistics, while our case 2 above does not find this.

7 Conclusion

Adding the human capital investment sector creates a key difference relative for example to the benchmark work of Jones et al (2005), in terms of the timing order of the responses of investments to physical and human capital to a technology shock. In the two-sector model here, people tend to increase

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[9] This frequency issue is also pointed out by Maury and Tripier (2003) who find that a version of the Jones et al. model on a quarterly basis does not perform as well as it does on a yearly frequency.

	data	case 1 $\phi_1 = \phi_2$ $\delta_k = \delta_h$	case 2 $\phi_1 = \phi_2$ $\delta_k > \delta_h$	case 3 $\phi_1 > \phi_2$ $\delta_k = \delta_h$	case 4 $\phi_1 > \phi_2$ $\delta_k > \delta_h$
$\sigma(\gamma_Y)$	1.14	2.51	25.09	0.86	0.82
$\sigma(\gamma_C)$	0.52	0.48	0.37	0.61	0.43
$\sigma(\gamma_{I_k})$	2.38	6.38	82.93	2.25	2.23
$\sigma(N)$	5.52	1.57	16.71	6.01	5.44
$\rho(\gamma_{Y_t}, \gamma_{Y_{t-1}})$	0.29	0.01	-0.50	0.86	0.29
$\rho(\gamma_{C_t}, \gamma_{C_{t-1}})$	0.24	0.95	0.95	0.81	0.78
$\rho(\gamma_{I_{kt}}, \gamma_{I_{kt-1}})$	0.38	-0.02	-0.49	0.48	0.14
$\rho(\gamma_{N_t}, \gamma_{N_{t-1}})$	0.99	0.95	-0.02	0.86	0.92
$\rho(\gamma_{Y_t}, \gamma_{C_t})$	0.49	0.35	0.02	0.68	0.83
$\rho(\gamma_{Y_t}, \gamma_{I_{kt}})$	0.75	0.99	1.00	0.87	0.96
$\rho(\gamma_{Y_t}, \gamma_{N_t})$	-0.07	0.35	0.75	-0.73	-0.73

Table 10: Comparing business cycle statistics for the variants

human capital stock immediately after a good shock and accumulate physical capital with a delay. Investments to the two capitals then adjust differently following an aggregate productivity shock, enabling the model to successfully reproduce the output growth and investment persistence, hump-shaped impulse responses for output and investment, greater labor volatility, and Gali's (1999) labor decrease after a positive productivity shock, so as to be broadly consistent with US data.

These results are explained intuitively in terms of sectoral reallocations as in international trade theory, in particular the Stolper and Samuelson (1941) theorem and the Rybczynski (1955) effect. Sensitivity analysis included examination of simulation results with respect to key parameter assumptions, as well as relaxing the baseline assumption that the sectoral shocks are an identical aggregate shock. When very high correlations are assumed between the sector-specific shocks, similar simulation properties result, with the implication that an identical aggregate productivity shock, as in Jones et al (2005), across both goods and human capital investment sectors best fits the data.

Potential extensions include experimentation with the magnitude to the shock to the goods and human capital sector, while keeping the shock otherwise identical, while also allowing for independent shocks. We are also considering matching a broader array of cyclic frequencies in a Comin and Gertler's (2006) fashion, such as including the Medium Term cycle. This includes trying to explain data at several frequencies besides the business cycle, including also the shorter run higher frequency, the longer run lower frequency and their all-inclusive "medium term" frequency.

In separate extended preliminary work, we appeasr to show that our model also solves the "excess sensitivity" and "excess smoothness" puzzles because a positive shock to human capital investment increases the permanent income of the consumer, rather than only the temporary income, in a fashion related to the shock to the second investment sector in Boldrin et al. (2001). Consumption rises more relative to goods output as a result of such a shock because permanent income rises when the endogenous growth rate is temporarily shocked upwards.

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A Uniqueness of steady state

Since proof of uniqueness of the steady state is viewed as infeasible in such numerically solved models, here the uniqueness of the steady state is demonstrated for the given calibration. Express the first order conditions and constraints of the two-sector model by the variables’ long-run values (variables with no time subscript denote their long-run values and A_g is normalized to unity):

$$\frac{AC}{(1-N-M)\,H} = (1-\phi_1)\left(\frac{VK}{NH}\right)^{\phi_1} \tag{16}$$

$$\frac{1-\phi_1}{\phi_1}\frac{VK}{NH} = \frac{1-\phi_2}{\phi_2}\frac{(1-V)\,K}{MH} \tag{17}$$

$$(1+\gamma)^{-\sigma} = \frac{1+\phi_1\left(\frac{VK}{NH}\right)^{\phi_1-1}-\delta_k}{1+\frac{1-\beta}{\beta}} \tag{18}$$

$$(1+\gamma)^{-\sigma} = \frac{1+(N+M)\,A_h\,(1-\phi_2)\left(\frac{(1-V)K}{MH}\right)^{\phi_2}-\delta_h}{1+\frac{1-\beta}{\beta}} \tag{19}$$

$$\gamma = \left(\frac{VK}{NH}\right)^{\phi_1-1} V - \delta_k - \frac{C}{K} \tag{20}$$

$$\gamma = A_h\left(\frac{(1-V)\,K}{MH}\right)^{\phi_2} M - \delta_h \tag{21}$$

Where γ is the balanced growth rate. Define $f_k \equiv \frac{VK}{NH}$ and $f_h \equiv \frac{(1-V)K}{MH}$. The simultaneous equation system can then be rearranged in 6 unknowns $(f_k, f_h, N, M, \gamma, \frac{C}{K})$:

$$\frac{A}{1-N-M}\frac{C}{K}\,(Nf_k+Mf_h) = (1-\phi_1)\,f_k^{\phi_1} \tag{22}$$

$$\frac{1-\phi_1}{\phi_1}f_k = \frac{1-\phi_2}{\phi_2}f_h \tag{23}$$

$$(1+\gamma)^{-\sigma} = \beta\left(1+\phi_1 f_k^{\phi_1-1}-\delta_k\right) \tag{24}$$

$$(1+\gamma)^{-\sigma} = \beta\left[1+(N+M)\,A_h\,(1-\phi_2)\,f_h^{\phi_2}-\delta_h\right] \tag{25}$$

$$\gamma = f_k^{\phi_1-1}\left(\frac{Nf_k}{Nf_k+Mf_h}\right) - \delta_k - \frac{C}{K} \tag{26}$$

$$\gamma = A_h f_h^{\phi_2} M - \delta_h \tag{27}$$

The exogenous information set is $(A, \beta, \sigma, \phi_1, \phi_2, \delta_k, \delta_h, A_h)$. Uniqueness of the solution to the above system of equations can be reduced down to the uniqueness of variable γ. To see this, one can solve for $f_k, f_h, N, M, \frac{C}{K}$ in terms of γ using equations 23 to 27 :

- from equation 24, $f_k = \left(\frac{\frac{(1+\gamma)^{-\sigma}}{\beta}-1+\delta_k}{\phi_1}\right)^{\frac{1}{\phi_1-1}}$

- from equation 23, $f_h = \left(\frac{(1-\phi_1)\phi_2}{(1-\phi_2)\phi_1}\right) f_k$
- from equation 25, $N + M = \frac{\frac{(1+\gamma)^{-\sigma}}{\beta} - 1 + \delta_h}{A_h (1-\phi_2)} f_h^{-\phi_2}$
- from equation 27, $M = \frac{\gamma+\delta_h}{A_h} f_h^{-\phi_2}$
- from equation 26, $\frac{C}{K} = f_k^{\phi_1 - 1} \left(\frac{N f_k}{N f_k + M f_h}\right) - \delta_k - \gamma$

Substitute all these into equation 22 to obtain a highly nonlinear function in γ: $\Theta(\gamma) = 0$. Then one can find the zeros of $\Theta(\gamma)$ for the baseline calibration of exogenous parameters: $A = 1.5455$, $\frac{1-\beta}{\beta} = 0.0142$, $\sigma = 1$, $\phi_1 = 0.36$, $\phi_2 = 0.11$, $\delta_k = 0.02$, $\delta_h = 0.005$, $A_h = 0.0461$. The numerical solution shows that there is only one internal solution that satisfies $0 < L < 1$:

$$\gamma^* = 0.0042,\ L^* = 0.542,\ N^* = 0.298,\ M^* = 0.160,\ \left(\frac{K}{H}\right)^* = 11.06$$

B Normalization

B.1 Deterministic discounting

Let $c_t \equiv \frac{C_t}{(1+\gamma)^t}$, $k_t \equiv \frac{K_t}{(1+\gamma)^t}$, $h_t \equiv \frac{H_t}{(1+\gamma)^t}$. The system consisting of equations (39) to (45) changes to:

$$\frac{A c_t}{1 - N_t - M_t} = (1-\phi_1) Z_t \left(\frac{V_t k_t}{N_t h_t}\right)^{\phi_1} h_t \tag{28}$$

$$\frac{(1-\phi_1) V_t}{\phi_1 N_t} = \frac{(1-\phi_2)(1-V_t)}{\phi_2 M_t} \tag{29}$$

$$P_t = \left(\frac{\phi_1}{\phi_2}\right)^{\phi_2} \left(\frac{1-\phi_1}{1-\phi_2}\right)^{1-\phi_2} \left(\frac{V_t k_t}{N_t h_t}\right)^{\phi_1 - \phi_2} \tag{30}$$

$$1 = E_t \beta \left\{ \begin{array}{l} \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} (1+\gamma)^{-\sigma} \left(\frac{1-N_{t+1}-M_{t+1}}{1-N_t-M_t}\right)^{A(1-\sigma)} \times \\ \left[\phi_1 Z_{t+1} \left(\frac{V_{t+1} k_{t+1}}{N_{t+1} h_{t+1}}\right)^{\phi_1 - 1} + 1 - \delta_k\right] \end{array} \right\} \tag{31}$$

$$1 = E_t \beta \left\{ \begin{array}{l} \left(\frac{P_{t+1}}{P_t}\right) \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} (1+\gamma)^{-\sigma} \left(\frac{1-N_{t+1}-M_{t+1}}{1-N_t-M_t}\right)^{A(1-\sigma)} \times \\ \left[(N_{t+1} + M_{t+1})(1-\phi_2) S_{t+1} \left(\frac{(1-V_{t+1}) k_{t+1}}{M_t h_{t+1}}\right)^{\phi_2} + 1 - \delta_h\right] \end{array} \right\} \tag{32}$$

$$c_t + (1+\gamma)\, k_{t+1} - (1-\delta_k)k_t = A_g Z_t (V_t k_t)^{\phi_1} (N_t h_t)^{1-\phi_1} \tag{33}$$

$$(1+\gamma)\, h_{t+1} - (1-\delta_h)h_t = A_h S_t \left[(1-V_t)k_t\right]^{\phi_2} (M_t h_t)^{1-\phi_2} \tag{34}$$

The system can then be log-linearized and expressed in percentage deviations:

$$0 = Ax_{t+1} + Bx_t + Dy_t + Fu_t \tag{35}$$

$$0 = E_t \left(Gx_{t+1} + Hx_t + Jy_{t+1} + Ly_t + Mu_{t+1}\right) \tag{36}$$

Where $y_t = \left[\hat{c}_t, \hat{V}_t, \hat{N}_t, \hat{M}_t, \hat{P}_t\right]'$, a vector collecting all control variables; and $x_t = \left[\hat{k}_t, \hat{h}_t\right]'$, containing two endogenous state variables; and $u_t = \left[\hat{Z}_t, \hat{S}_t\right]'$, containing exogenous state variables. The model is then solved by method of undetermined coefficients and the solution is characterized by two recursive equations:

$$x_{t+1} = Px_t + Qu_t \tag{37}$$

$$y_t = Rx_t + Su_t \tag{38}$$

P, Q, R and S satisfy the conditions listed in Appendix B.2. Responses of variables collected in y_t and x_t to innovations to u_t can then be calculated.

B.2 Stochastic discounting

The first order conditions of the two-sector model and the constraints are:

$$\frac{AC_t}{1-N_t-M_t} = (1-\phi_1)\, Z_t \left(\frac{V_t K_t}{N_t H_t}\right)^{\phi_1} H_t \tag{39}$$

$$\frac{(1-\phi_1)\, V_t}{\phi_1 N_t} = \frac{(1-\phi_2)\,(1-V_t)}{\phi_2 M_t} \tag{40}$$

$$P_t = \left(\frac{\phi_1}{\phi_2}\right)^{\phi_2} \left(\frac{1-\phi_1}{1-\phi_2}\right)^{1-\phi_2} \left(\frac{V_t K_t}{N_t H_t}\right)^{\phi_1-\phi_2} \tag{41}$$

$$1 = E_t \beta \left\{ \begin{array}{c} \left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \left(\frac{1-N_{t+1}-M_{t+1}}{1-N_t-M_t}\right)^{A(1-\sigma)} \times \\ \left[\phi_1 Z_{t+1} \left(\frac{V_{t+1}K_{t+1}}{N_{t+1}H_{t+1}}\right)^{\phi_1-1} + 1 - \delta_k\right] \end{array} \right\} \tag{42}$$

$$1 = E_t \beta \left\{ \begin{array}{c} \left(\frac{P_{t+1}}{P_t}\right) \left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \left(\frac{1-N_{t+1}-M_{t+1}}{1-N_t-M_t}\right)^{A(1-\sigma)} \times \\ \left[(N_{t+1}+M_{t+1})\,(1-\phi_2)\, S_{t+1} \left(\frac{(1-V_{t+1})K_{t+1}}{M_{t+1}H_{t+1}}\right)^{\phi_2} + 1 - \delta_h\right] \end{array} \right\} \tag{43}$$

$$C_t + K_{t+1} - (1-\delta_k)K_t = A_g Z_t (V_t K_t)^{\phi_1} (N_t H_t)^{1-\phi_1} \tag{44}$$

$$H_{t+1} - (1-\delta_h)H_t = A_h S_t \left[(1-V_t)K_t\right]^{\phi_2} (M_t H_t)^{1-\phi_2} \tag{45}$$

And Z_t and S_t are governed by an exogenous vector autoregressive process:

$$\begin{bmatrix} \log Z_{t+1} \\ \log S_{t+1} \end{bmatrix} = N \begin{bmatrix} \log Z_t \\ \log S_t \end{bmatrix} + \varepsilon_{t+1} \tag{46}$$

Where N is $\begin{bmatrix} \rho_z & 0 \\ 0 & \rho_s \end{bmatrix}$ and ε_t is $\begin{bmatrix} \varepsilon_t^z \\ \varepsilon_t^s \end{bmatrix}$.

The system that consists of seven equations in terms of seven endogenous variables $(C_t, K_{t+1}, H_{t+1}, V_t, N_t, M_t, P_t)$ is non-stationary because C_t, K_t and H_t are growing in steady-state. To achieve stationarity, define new variables in the following way: $c_t \equiv \frac{C_t}{H_t}$, $k_t \equiv \frac{K_t}{H_t}$, $\gamma_{ht+1} \equiv \frac{H_{t+1}}{H_t}$, where γ_{ht} is the gross growth rate of human capital stock. Rewrite the system in terms of stationary variables:

$$\frac{Ac_t}{1-N_t-M_t} = (1-\phi_1) Z_t \left(\frac{V_t k_t}{N_t}\right)^{\phi_1} \tag{47}$$

$$\frac{(1-\phi_1) V_t}{\phi_1 N_t} = \frac{(1-\phi_2)(1-V_t)}{\phi_2 M_t} \tag{48}$$

$$P_t = \frac{Z_t}{S_t} \left(\frac{\phi_1}{\phi_2}\right)^{\phi_2} \left(\frac{1-\phi_1}{1-\phi_2}\right)^{1-\phi_2} \left(\frac{V_t k_t}{N_t}\right)^{\phi_1-\phi_2} \tag{49}$$

$$1 = E_t\beta \left\{ \begin{array}{c} \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \gamma_{ht+1}^{-\sigma} \left(\frac{1-N_{t+1}-M_{t+1}}{1-N_t-M_t}\right)^{A(1-\sigma)} \times \\ \left[\phi_1 Z_{t+1} \left(\frac{V_{t+1}k_{t+1}}{N_{t+1}}\right)^{\phi_1-1} + 1 - \delta_k\right] \end{array} \right\} \tag{50}$$

$$1 = E_t\beta \left\{ \begin{array}{c} \left(\frac{P_{t+1}}{P_t}\right) \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \gamma_{ht+1}^{-\sigma} \left(\frac{1-N_{t+1}-M_{t+1}}{1-N_t-M_t}\right)^{A(1-\sigma)} \times \\ \left[(N_{t+1}+M_{t+1})(1-\phi_2) S_{t+1} \left(\frac{(1-V_{t+1})k_{t+1}}{M_t}\right)^{\phi_2} + 1 - \delta_h\right] \end{array} \right\} \tag{51}$$

$$c_t + k_{t+1}\gamma_{ht+1} - (1-\delta_k)k_t = A_g Z_t (V_t k_t)^{\phi_1} N_t^{1-\phi_1} \tag{52}$$

$$\gamma_{ht+1} - 1 + \delta_h = A_h S_t \left[(1-V_t)k_t\right]^{\phi_2} M_t^{1-\phi_2} \tag{53}$$

The next step is to rewrite these equations in steady state and calibrate the model to fit targeted variables given the steady state constraints are binding. The log-linearization method is now applicable to this transformed system. First, apply the first-order Taylor expansion for each individual equation around the steady state. Although this is a straightforward exercise, it is awkward to display all linearized equations due to the length of some equations. To summarize, the linearized system involves seven difference equations in seven variables: $\hat{c}_t, \hat{V}_t, \hat{N}_t, \hat{M}_t, \hat{P}_t, \hat{k}_t, \hat{\gamma}_{ht}$.

Variables expressed in the form of ratios over human capital stock need to be transformed into first differences. The method to do this is shown through an example of consumption. Recall that $c_t \equiv \frac{C_t}{H_t}$, so the growth rate of aggregate consumption can be calculated as below:

$$\begin{aligned}
\gamma_{ct+1} &= \log C_{t+1} - \log C_t \\
&= \log c_{t+1} - \log c_t + \log H_{t+1} - \log H_t \\
&= (\log c_{t+1} - \log c) - (\log c_t - \log c) + \log \frac{H_{t+1}}{H_t} \\
&= \hat{c}_{t+1} - \hat{c}_t + \big(\log \gamma_{ht+1} - \log \gamma_h\big) + \log \gamma_h \\
&= \hat{c}_{t+1} - \hat{c}_t + \hat{\gamma}_{ht+1} + \log \gamma_h
\end{aligned}$$

Where c and γ_h are steady-state values of c_t and γ_{ht}. Growth rates of other variables can be derived similarly. The model is solved using Uhlig's (1999) toolbox.

B.2.1 For Referee: Solution Methodology Details

Next, condense the system in vector form with distinction made between deterministic equations and expectational equations. To simplify notation, let $y_t = \left[\hat{c}_t, \hat{V}_t, \hat{N}_t, \hat{M}_t, \hat{P}_t\right]'$, a vector collecting all control variables; and $x_t = \left[\hat{k}_t, \hat{\gamma}_{ht}\right]'$, containing two endogenous state variables[10]; and $u_t = \left[\hat{Z}_t, \hat{S}_t\right]'$, containing exogenous state variables. Thus, the system is reorganized as follows:

$$\begin{aligned}
0 &= Ax_{t+1} + Bx_t + Dy_t + Fu_t && (54) \\
0 &= E_t\left(Gx_{t+1} + Hx_t + Jy_{t+1} + Ly_t + Mu_{t+1}\right) && (55)
\end{aligned}$$

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[10] Although $\hat{\gamma}_{ht}$ is named an endogenous state variable here, the policy function does not depend on this variable. This is because γ_{ht} is not present in the system of equations from 47 to 53 (only γ_{ht+1} exists). Therefore, the only effective state variable is $\hat{k}_t$.

Where A, B, F are 5×2 matrices; D is a 5×5 matrix; G, H, M are 2×2 matrices; and J, L are 2×5 matrices. Equation (54) summarizes five deterministic equations and equation (55) represents two expectational equations. Elements in $A, B, D, F, G, H, J, L, M$ are given numerically by the values of exogenous parameters and the steady state solution of the model. As before, represent the solution to this system by two equilibrium recursive law of motions:

$$x_{t+1} = Px_t + Qu_t \tag{56}$$

$$y_t = Rx_t + Su_t \tag{57}$$

Where P and Q are 2×2 matrices and R and S are 5×2 matrices. Substituting the two recursive equations back into equation (54) and (55) and equating coefficient matrices associated to x_t and u_t to zero lead to four simultaneous matrix equations in P, Q, R and S. Solving these matrix equations will complete characterizing the solution. According to Uhlig (1999),

- P satisfies the matrix quadratic equation

  $$0 = -JD^{-1}AP^2 + \left(G - JD^{-1}B - LD^{-1}A\right)P + H - LD^{-1}B \tag{58}$$

  Notice that since there are two endogenous state variables ($\hat{k}_t$ and $\hat{\gamma}_{ht}$) in this case, P is a 2×2 matrix, other than a scalar in the one-sector RBC model. Hence, solving for P requires solving this matrix quadratic equation. Again, a necessary condition for this quadratic equation to make sense is matrix D is nonsingular.

- R is given by

  $$R = -D^{-1}(AP + B) \tag{59}$$

- Q satisfies

  $$\left(-N' \otimes JD^{-1}A + I_2 \otimes \left(JR + G - LD^{-1}A\right)\right) Vec(Q) = \tag{60}$$

  $$Vec\left(\left(JD^{-1}F - M\right)N + LD^{-1}\right) \tag{61}$$

  Where $Vec(\cdot)$ is column-wise vectorization; $\otimes$ is Kronecker product; I_2 is identity matrix of size 2×2.

- S is given by

$$S = -D^{-1}(AQ + F) \tag{62}$$

The crucial part in deriving the solution is to solve the matrix quadratic equation in (58). To have a stationary recursive solution, one should pick up the solution for P whose eigenvalues are both smaller than one. Once P is solved, the rest of the solution is not hard to derive.

C Data Description, Summary Statistics

The data set covering from the first quarter of 1954 to the first quarter of 2004 is downloadable from *http://clevelandfed.org/research/Models/rbc/index.cfm*. According to Gomme and Rupert (2007), output (Y) is measured by real per capita GDP less real per capita Gross Housing Product. They argue that income in home sector should be removed when calculating market output using NIPA data set. The price deflator is constructed by dividing nominal expenditures on nondurables and services by real expenditures. Population is measured by civilians aged 16 and over. Consumption (C) is measured by real personal expenditures on nondurables and services less Gross Housing Product. Gomme and Ruppert report four types of investments: market investment to nonresidential structures, market investment to equipment and software, household investment to residential products and household investment to nondurables. Investment (I) here corresponds to the simple sum of these four types of investments. Working hours (N) is measured as per capita market time. Figure 7 depicts growth rates of output, consumption and investment over the periods from 1954.1 to 2004.1. Several observations are reflected in this picture:

1. Output growth fluctuates more than consumption growth; investment growth fluctuates more than output growth.
2. Consumption growth and investment growth are strongly procyclical.
3. Economy fluctuates substantially less after 1980s.

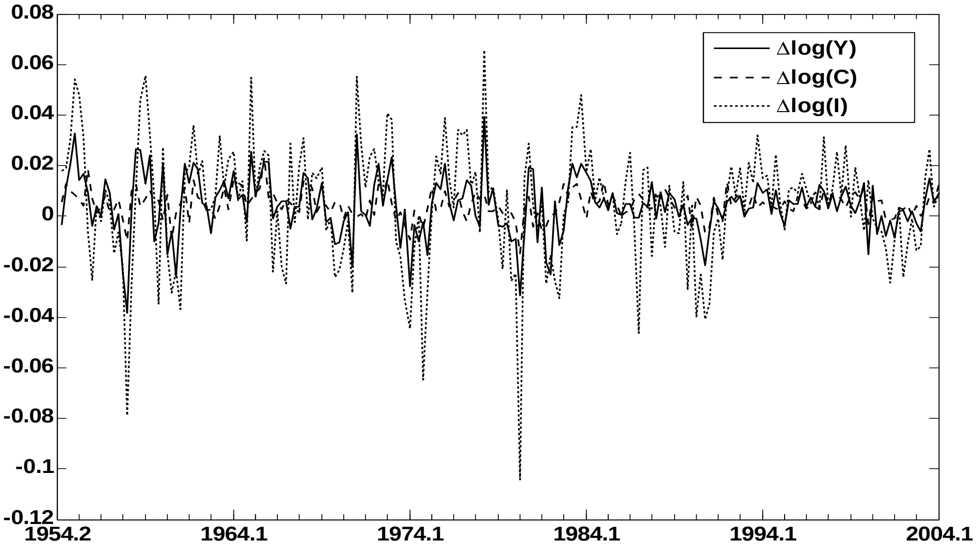

Figure 7: Plot of US data from 1954.1 to 2004.1

Table 11[11] summarizes the observed business cycle properties numerically. The first panel of table 11 shows that output, consumption and investment grow at similar rate over time. This is in line with the balanced growth path hypothesis. The second panel reflects the relative order of variabilities of main macro variables in figure 7. The third panel shows that the growth rates of variables are all positively autocorrelated. The last panel confirms that consumption and investment growth rates are procyclical and working hours are slightly countercyclical.

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[11] The second moment results are actually the standard deviation of the net growth rate multiplied by 100. For example, the standard deviation of the net output growth ($\Delta \log Y$) is 0.0114. Since standard deviation of the net growth rate equals that of the gross growth rate, this number (when multiplied by 100) can be interpreted as the percentage deviation of gross output growth from its mean.

Mean

$E\left(\Delta \log Y\right)$	$E\left(\Delta \log C\right)$	$E\left(\Delta \log I\right)$	$\frac{E(N)}{E(N)}$
0.0042	0.0047	0.0045	1

Fluctuation

$\sigma\left(\Delta \log Y\right)$	$\sigma\left(\Delta \log C\right)$	$\sigma\left(\Delta \log I\right)$	$\frac{\sigma(N)}{E(N)}$
1.14	0.52	2.38	5.6

Autocorrelation

$\rho\left(\Delta \log Y_t, \Delta \log Y_{t-1}\right)$	$\rho\left(\Delta \log C_t, \Delta \log C_{t-1}\right)$	$\rho\left(\Delta \log I_t, \Delta \log I_{t-1}\right)$	$\rho\left(N_t, N_{t-1}\right)$
0.29	0.24	0.39	0.98

Cross-correlation

$\rho\left(\Delta \log Y_t, \Delta \log Y_t\right)$	$\rho\left(\Delta \log Y_t, \Delta \log C_t\right)$	$\rho\left(\Delta \log Y_t, \Delta \log I_t\right)$	$\rho\left(\Delta \log Y_t, N_t\right)$
1	0.49	0.75	-0.07

Table 11: Business cycle statistics in US data from 1954.1 to 2004.1