Do Large-Firm Bargaining Models Amplify and Propagate Aggregate Productivity Shocks?

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Abstract

Labor search models struggle to generate empirically-plausible amplification and propagation of productivity shocks. Several authors have argued that allowing for heterogeneity in firm-level employment and employment growth helps. I study a model that can match a wide variety of cross-sectional and time series observations but show that it generates neither amplification nor propagation. In one specification, unemployment and vacancies are analytically constant as aggregate productivity varies, while in a second their behavior is indistinguishable from the Mortensen-Pissarides benchmark. An important contribution of the paper is analytical tractability. Despite allowing for both transitory and permanent idiosyncratic heterogeneity in firm productivity as well as for aggregate productivity shocks, the model can be solved nearly in closed form.

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1 Introduction

There is enormous heterogeneity in the U.S. economy across firms and establishments in wages, employment levels, and employment growth.[1] The benchmark Mortensen-Pissarides (MP) model of the interaction of workers and firms in a frictional labor market, however, abstracts from the notion of firm size: all firms have precisely one job.[2] Accordingly, the MP tradition cannot account for cross-sectional patterns of employment, wages, and worker and job flows. Is this omission important for an economist seeking to understand the cyclical behavior of purely aggregate variables such as the unemployment rate or the vacancy-unemployment ratio, or is it an acceptable short cut?

More specifically, the MP model has two well-known difficulties in accounting for the cyclical behavior of key aggregate labor market variables. First, if the surplus a worker enjoys from employment relative to unemployment is plausibly large, model-generated time series for unemployment and for the vacancy-unemployment ratio are not nearly as volatile as U.S. data (Shimer, 2005, 2010; Costain and Reiter, 2008; Hagedorn and Manovskii, 2008). Second, while the response of the vacancy-unemployment ratio to productivity shocks is hump-shaped in the data, the model lacks a propagation mechanism: the vacancy-unemployment ratio is a jump variable (Fujita and Ramey, 2007; Fujita, 2011). One might hope—and several recent papers have argued (Acemoglu and Hawkins, 2011; Elsby and Michaels, 2011; Fujita and Nakajima, 2009)—that allowing for firms that can hire multiple workers and exhibit decreasing returns to labor will help with one or both of these issues. A non-degenerate distribution of employment across firms will respond sluggishly to aggregate shocks and might provide an important propagation mechanism. Firms' endogenous choice of how many workers to hire might generate low marginal product jobs that will quickly be destroyed when aggregate productivity falls; this might amplify the employment response to aggregate productivity shocks.

In this paper, however, I show that under natural assumptions an economist interested only in aggregate variables can abstract from firm heterogeneity. I study a model of a frictional labor market that allows for rich heterogeneity in employment and employment growth across establishments, and show that it can be calibrated to match the cross-sectional distributions of these variables in U.S. data. However, when it comes to accounting for the response of unemployment and of the vacancy-unemployment ratio to shocks to labor productivity, the model exhibits the familiar properties of the MP model.

I establish this main result in two forms. The first is analytical. I follow Shimer (2010) and assume that recruiting is a time-intensive activity and that preferences take a balanced-growth form so that income and substitution effects of productivity shocks on labor supply cancel. In this case,

[1] Axtell (2001) and Davis, Haltiwanger, Jarmin, and Miranda (2006) report the distribution of employment levels, while Haltiwanger, Jarmin, and Miranda (2010) report that young establishments increase employment faster. Oi and Idson (1999) report that larger firms pay higher wages. Moscarini and Postel-Vinay (2009) argue that large and small firms exhibit different cyclical employment growth patterns.

[2] The standard MP model is due to Mortensen (1982), Pissarides (1985), and Mortensen and Pissarides (1994) and is summarized in Pissarides (2000). An alternative to the single job per firm assumption allows for production technologies which exhibit constant returns to labor, in which case the distribution of employment and employment growth across firms is indeterminate and irrelevant for aggregate variables. See Pissarides (2000).

unemployment and the vacancy-unemployment ratio do not vary with labor productivity, just as Shimer (2010) shows for the MP model.

I then make the assumption that is more customary in the labor search literature in which recruiting is goods-intensive and households have linear preferences over consumption. Now the unemployment rate and the vacancy-unemployment ratio do vary with aggregate productivity; however, the model does not have additional explanatory power relative to the MP benchmark, analogously to the results of Shimer (2005) and Hagedorn and Manovskii (2008). Unemployment is not very volatile unless the employment surplus is small, and the vacancy-unemployment ratio is a jump variable that directly inherits the properties of the stochastic process for labor productivity without any additional propagation.

In summary, the omission of productive heterogeneity of firms in the MP model does not by itself account for the failure of that model to amplify or propagate productivity shocks. Even though in my model the firm-size distribution is a state variable and responds only slowly to aggregate shocks, it generates neither propagation nor amplification.

An important contribution of the paper is that despite the rich microeconomic heterogeneity—I allow for firms to have heterogeneous productivity with both transitory and permanent components, as well as for an aggregate productivity shock—the model is nearly solvable in closed form. Conditional on a guess for the relationship between aggregate labor productivity and the equilibrium vacancy-unemployment ratio, solving the model essentially reduces to solving a system of linear equations. In particular, I do not need approximate aggregation techniques (Krusell and Smith, 1998) nor the requisite assumption that economic agents are boundedly rational in order to compute the equilibrium. The only caveat for this result is that the equilibrium I compute does not apply if it is possible for shocks to aggregate productivity to be sufficiently negative that firm entry ceases completely for some time, a possibility that seems irrelevant if the model is calibrated to match the empirical behavior of establishment entry in the US economy.

I now explain some of the key features of the model. The technology operated by firms exhibits decreasing returns to labor; in a fuller model these decreasing returns might arise from the presence of fixed factors such as physical capital, from imperfect substitutability for consumers of the goods produced by different firms, or from managerial span-of-control as in Lucas (1978). Firms are heterogeneous in productivity, and this heterogeneity can include both permanent and transitory components. The stochastic nature of firm-level productivity allows the model to account for the empirical cross-sectional patterns of firm employment levels and growth rates.

Matching between firms and workers in the labor market is frictional, and the search process is random. Firms post vacancies in order to hire workers, and the vacancy-posting technology exhibits constant returns at the firm level both in terms of cost and of the number of workers contacted. However, as in the MP model, when the aggregate vacancy-unemployment ratio is high, the yield of contacted workers per vacancy declines, so that a firm needs to post more vacancies in order to hire the same number of workers. The fact that hiring workers is costly (although firing them is not) means that firms do not always alter their employment levels in response to productivity shocks, and allows for the firm-size distribution to respond only sluggishly to aggregate productivity shocks. (However, as already explained, a key contribution of the paper is to show that despite the persistent response of the firm size distribution to productivity shocks, this does not propagate to sluggish adjustment of market tightness.)

Wages are determined according to the generalization of Nash bargaining to the case of many-to-one matching proposed in the labor context by Bertola and Caballero (1994), Stole and Zwiebel (1996a,b), and Smith (1999). In the Stole and Zwiebel framework, a firm bargains with each worker as if she were marginal; the outside option of the worker is unemployment while that of the firm is to recommence the bargaining process with one fewer worker. This is by now the benchmark for wage determination in the context of random-search models with multi-worker firms.[3] Although it is well-known that the Stole and Zwiebel bargaining process does not generate a constrained efficient outcome (firms hire inefficiently many workers so as to drive down the wages of their inframarginal colleagues), this inefficiency does not drive the results presented in this paper.[4]

I am not the first to study a model of a frictional labor market in which firms employ multiple workers and operate production technologies with decreasing returns to labor. Recent contributions along these lines have been made by Acemoglu and Hawkins (2011), Elsby and Michaels (2011), Fujita and Nakajima (2009), Kaas and Kircher (2011), and Schaal (2010).[5] I defer a more detailed discussion of these papers to Section 5; however, it is worth explaining here the reason why several of these papers find qualitatively different results for amplification and propagation than I do. The key assumptions concern the technologies for creating new firms and new jobs at existing firms. I assume a constant returns to scale vacancy-posting technology. In both Acemoglu and Hawkins (2011) and Kaas and Kircher (2011), on the other hand, a firm that wishes to hire twice as many new workers must pay more than twice the cost, and this convex hiring cost generates a sluggish aggregate response to productivity shocks as firms spread out vacancy creation over time.[6] I also assume that new firms can be created at constant marginal cost, unlike Elsby and Michaels (2011) and Fujita and Nakajima (2009) who do not allow for entry. Thus, in their work, vacancy creation takes place only on the intensive margin (additional hiring by incumbent firms), while I also allow for the extensive margin (entry of new firms). Thus, in their models, new hiring in response to a positive productivity shock must entirely be made by incumbent firms, but because of the decreasing returns to labor, this generates a qualitatively different response of aggregate variables

[3] The Stole and Zwiebel framework has been applied to study contracting and technological adoption (Acemoglu, Antràs, and Helpman, 2007), wage determination (Roys, 2010), the interaction of product market regulation and the labor market (Felbermayr and Prat, 2007; Delacroix and Samaniego, 2009; Ebell and Haefke, 2009), and the labor market effects of changes in trade patterns (Coşar, Guner, and Tybout, 2010; Helpman and Itskhoki, 2010).

[4] More precisely, I show in Hawkins (2011) that the dynamics of both aggregate variables like unemployment and the vacancy-unemployment ratio and of idiosyncratic variables like firm-level employment in the model presented in the current paper are precisely identical to those in the constrained efficient allocation of an alternative economy that differs only in the average level of productivity.

[5] The contribution of Cooper, Haltiwanger, and Willis (2007) is similar to that of Elsby and Michaels (2011) with the additional assumption that firms have all the bargaining power. See also Veracierto (2008, 2009) for related work in the tradition of Hopenhayn and Rogerson (1993).

[6] Convex hiring costs already provide a propagation mechanism in the benchmark MP framework (Yashiv, 2006).

to productivity shocks to that in the present model.[7] Thus, the assumption on entry is key for the adjustment of the economy to aggregate shocks, a point that goes back at least to Lucas (1967).

The structure of the remainder of the paper is as follows. In Section 2 I study the version of the model with balanced growth preferences and a time-intensive vacancy-posting technology; in this version productivity shocks do not induce fluctuations in unemployment. Section 3 modifies the benchmark model to introduce linear preferences over consumption and goods-intensive vacancy-posting. In Section 4 I undertake a quantitative analysis of the model in Section 3, and relate its findings to those of the MP model. Section 5 relates the model and the results of the paper to the literature, and discusses possible extensions. Section 6 concludes.

2 Model with Balanced Growth Preferences

In this section I introduce the benchmark model and prove that there is an equilibrium in which consumption and wages are proportional to current productivity and unemployment is constant. I then provide an algorithm for computing this equilibrium.

Time is discrete, $t = 0, 1, 2, \ldots$. There are four types of agents in the economy, a representative household containing a continuum of measure 1 of workers and continua of three types of firms, production firms, recruitment firms, and construction firms. Only some production firms are active at any time. For the sake of brevity, the term 'firm' in the sequel refers to production firms unless otherwise specified. Each period, the following events occur.

- Nature draws aggregate productivity and, for each active production firm, idiosyncratic productivity. These values are publicly observable.
- Incumbent production firms choose how many of their existing workers to fire.
- Entry of new production firms occurs and their idiosyncratic productivity is realized.
- Incumbent and new entrant production firms engage recruitment firms in order to hire unemployed workers; meetings between recruitment firms and unemployed workers occur.
- Production occurs and wages are paid.
- Some firms are exogenously destroyed.

I now discuss the economic agents and each of the listed events in more detail.

2.1 Productivity and Production Firms

There is a large continuum of production firms. Production firms are ultimately owned by the representative household, via a mutual fund which holds a diversified portfolio of all production firms. Any profits earned by the production firms are rebated to the household as dividends.

[7] The model studied in this paper is closely related to that of Elsby and Michaels in other respects.

At the beginning of each period, active production firms learn the new realizations of the aggregate and idiosyncratic productivity shocks. I therefore commence by describing the assumptions on the productivity shock process. Productivity shocks affect production firms only. I assume that when the aggregate productivity shock is p and its own idiosyncratic productivity shock is π, a production firm with n employees produces output per period of $y(n; p, \pi) = p\pi y(n)$, where $y(n)$ is strictly increasing, strictly concave, and satisfies the usual Inada conditions.

To simplify notation and to enable me to provide a constructive algorithm to calculate equilibrium, I assume that the aggregate productivity shock p takes only finitely many possible values, $z_1 < z_2 < \ldots < z_{m_p}$. Denote by $p^t = (p_0, p_1, \ldots, p_t)$ the history of aggregate productivity shocks. I assume complete markets for claims on the consumption good contingent on aggregate variables, so it is helpful to denote by $s^t = (s_0, s_1, \ldots, s_t)$ the history of the economy, and by $P(s^t)$ the period-0 probability of history s^t being realized through period t. In addition to the aggregate productivity shock p_t, s_t incorporates all payoff-relevant variables such as the distribution of employment across firms, prices and wages, and the consumption, asset holdings and labor supply of households. In fact I will study an equilibrium in which the history of aggregate productivity (along with period-0 initial conditions) is a sufficient statistic for the state of the economy, but I do not wish to assume this structure from the outset and therefore keep track of s^t and not merely p^t.

Denote by $\pi_l^t = (\pi_{0l}, \pi_{1l}, \ldots, \pi_{tl})$ the history of idiosyncratic productivity shocks for an arbitrary production firm (indexed by l). I assume that the idiosyncratic productivity shock process is iid across firms (conditional on the history of idiosyncratic productivity shocks), orthogonal to the aggregate productivity shock, and the same for entrants and incumbents.[8] If a firm becomes active during period $\tau \geq 0$ after history s^τ, I write $\pi_{tl} = 0$ for $t < \tau$, and $\tilde{\Pi}(\pi^t; s^\tau)$ for the probability, before the realization of the period-τ value of the idiosyncratic productivity shock, of the history π^t through date t. Note that by assumption $\tilde{\Pi}(\pi^t; s^\tau) = 0$ if $\pi_\sigma > 0$ for any $\sigma < \tau$. The distribution of idiosyncratic productivity shocks that can be drawn by new entrants (that is, the set of probabilities $\tilde{\Pi}((0, \ldots, 0, \zeta_j); s^\tau)$ for $j = 1, 2, \ldots, m_\pi$) could potentially be different than the marginal distribution of the new idiosyncratic shock in period t for earlier entrants or for firms already active at period 0. I denote this distribution by $F_\pi^e(\pi)$, and assume that it is independent of the period τ and history s^τ.[9]

The idiosyncratic shock takes only finitely many values, $\zeta_1 < \zeta_2 < \ldots < \zeta_{m_\pi}$, and follows

[8] Nothing material is altered by relaxing any of these assumptions except for the notational burden. For example, if entrants have a different productivity process to incumbents after the period of entry, then this may act like an aggregate productivity shock, but it is inconvenient to analyze because I then have to write separate notation for initially-active firms and for entrants even after the period of entry.

I do not assume that the idiosyncratic shock has a unique ergodic distribution. In the numerical implementation in Section 4 I assume that it takes the form of a firm fixed effect multiplied by a purely transitory component.

[9] One possible assumption on $F_\pi^e(\cdot)$, following Mortensen and Pissarides (1994), is that potential entrants know their productivity before entering, so that only firms with the maximum productivity π_{m_π} will enter in equilibrium. An alternative assumption is that potential entrants face the same distribution of idiosyncratic shocks as incumbents. A third possibility, used in the quantitative implementation of the model in Section 4, is that F_π^e is concentrated on low values of π, so that entrants increase in expected productivity over time. This is consistent with evidence reported by Lee and Mukoyama (2011) from the Annual Survey of Manufactures of the U.S. Census Bureau that new entrant plants have only 60% as many employees as continuing plants.

a first-order Markov process which is identical across firms (including for initially inactive firms once they become active).[10] Write $\Pi(\pi^t)$ for the probability, before the realization of the period-0 idiosyncratic productivity shocks, of the history π^t through date t for any initially-active firm, independently across firms. The first-order Markov assumption implies that there is a function $\mu(\cdot,\cdot)$ such that

$$\frac{\Pi((\pi^{t-1},\pi_t,\pi_{t+1}))}{\Pi((\pi^{t-1},\pi_t))} = \frac{\tilde{\Pi}((\pi^{t-1},\pi_t,\pi_{t+1}),s^\tau)}{\tilde{\Pi}((\pi^{t-1},\pi_t),s^\tau)} = \mu(\pi_t,\pi_{t+1}). \tag{1}$$

Note that I allow for arbitrary history dependence in the aggregate productivity process, although in the calibrated model I will assume that the aggregate productivity shock processes, like that for idiosyncratic shocks, is first-order Markov.

The first choice firms face each period is whether to lay off workers. There is no firing cost.[11]

Next, each inactive firm has the option of becoming active by buying a factory. Factories are sold by construction firms (described in Section 2.2 below) for a price of $k^f(s^t)$ units of the history-s^t consumption good. After buying a factory, a newly-active firm observes its idiosyncratic productivity shock. (Inactive production firms who do not enter simply wait for the next period.)

After factories have been purchased, incumbent and new entrant production firms alike pay recruitment firms in order to hire unemployed workers. The opportunity to match with an unemployed worker can be purchased from recruitment firms (to be described in Section 2.3 below) for a price of $k^r(s^t)$ units of the history-s^t consumption good. Next, bargaining (described in Section 2.5 below) commences between the firm and its employees, incumbent and newly-recruited. Denote by $w(s^t,a(s^t);\pi^t,n)$ the bargained wage (in units of the history-s^t consumption good). Following this, production occurs and wages are paid.

The last event in each period is that a fraction δ of active production firms are destroyed at the end of each period. These firms are removed from the economy (there is no scrap value) and replaced by new inactive firms. Their employees begin the following period unemployed.[12]

In summary, the objective of a firm that begins period 0 active and with n_{-1} employees, when the history of the economy is s^0 and that of its own idiosyncratic shock is π^0, is to maximize

$$\begin{aligned} J(n_{-1};s^0,\pi^0) = \sum_{t=0}^{\infty}\sum_{s^t}\sum_{\pi^t} q_0(s^t)(1-\delta)^t\Pi(\pi^t)\Big[& -h_t(n_{-1},s^t,\pi^t)k^r(s^t) + p_t\pi_t y(n(n_{-1},s^t,\pi^t)) \\ & - n(n_{-1},s^t,\pi^t)w(s^t,a(s^t);\pi^t,n(n_{-1},s^t,\pi^t))\Big]. \end{aligned} \tag{2}$$

Here $q_0(s^t)$ denotes the Arrow-Debreu price of goods in period t after history s^t (the numeraire is

[10] This assumption simplifies notation by allowing for a unified treatment of all firms, regardless of age.

[11] A per-worker firing cost $\tau > 0$ can be incorporated in the model; however, the bargaining equation (27) needs to be replaced by $\eta\left[J_n(w(n;s^t,\pi^t),n;s^t,\pi^t)+\tau\right] = (1-\eta)S(w(n;s^t,\pi^t),n;s^t,\pi^t)c(s^t)$ so that the firing cost is incorporated in the firm's outside option in bargaining. The main results of the paper are unchanged in this case.

[12] Because there is no operating cost beyond wage payments, if all the realizations of aggregate and idiosyncratic productivity are nonnegative then no firm ever wishes to exit endogenously. Allowing for a period-by-period operating cost and therefore for endogenous exit is possible at the cost of increasing the notational burden; the main results of the paper carry over unchanged.

the date-0, history s^0 good). The law of motion for the firm's employment can be written

$$n(n_{-1}, s^t, \pi^t) = n(n_{-1}, s^{t-1}, \pi^{t-1}) - f(n_{-1}, s^t, \pi^t) + h_t(n_{-1}, s^t, \pi^t), \tag{3}$$

where $f(n_{-1}, s^t, \pi^t) \geq 0$ denotes the firm's choice of how many workers to fire in period t after history (s^t, π^t) and $h_t(n_{-1}, s^t, \pi^t) \geq 0$ denotes the corresponding hiring choice. (Since no information is revealed between the decision on firing and that on recruitment, it is immediate that at most one of these two variables will be non-zero at any time for any production firm.)

It is convenient to write the firm's problem in recursive notation. To do so, denote by $\hat{J}(n_0; s^t, \pi_t)$ the value of a firm which arrives with n_0 workers at the beginning of period t, before firing and recruitment occur but after the aggregate and idiosyncratic productivity shocks are revealed, and by $J(n; s^t, \pi^t)$ the value of a firm which has n workers after the recruitment season is over in period t. These values are to be measured in terms of date t, history s^t goods. Then

$$\hat{J}(n; s^t, \pi^t) = \max_{f,h \geq 0} \left[-k^r(s^t)h + J(n - f + h; s^t, \pi^t) \right] \tag{4}$$

and

$$J(n; s^t, \pi^t) = p_t \pi_t y(n) - nw(n; s^t, \pi^t) + (1 - \delta) \sum_{s^{t+1}|s^t} q_t(s^{t+1}) \sum_{\pi^{t+1}} \mu(\pi_t, \pi_{t+1}) \hat{J}(n; s^{t+1}, \pi^{t+1}). \tag{5}$$

Here $q_t(s^{t+1})$ denotes the period-t price of a unit of consumption goods at date $t+1$. I write $f(n; s^t, \pi^t)$ and $h(n; s^t, \pi^t)$ for the optimal firing and recruitment policies of the firm, and

$$n'(n; s^t, \pi^t) = n - f(n; s^t, \pi^t) + h(n; s^t, \pi^t) \tag{6}$$

for the law of motion for the firm's employment.

The objective of a firm that begins period 0 inactive but becomes active after history s^τ is very similar to that just described for active firms. I describe it in the recursive formulation directly. The firm first pays $k^f(s^\tau)$ units of the history-s^τ good to buy its factory. After this but before it has learned its idiosyncratic shock, its value $J^e(s^\tau)$ is given by $\sum_j F^e(\pi_j)\hat{J}(0; s^t, (0, \ldots, 0, \pi_j))$. There is free entry for production firms, which requires that

$$J^e(s^t) = \sum_j F^e(\pi_j)\hat{J}(0; s^t, (0, \ldots, 0, \pi_j) \leq k^f(p^t) \tag{7}$$

with equality if a positive amount of entry occurs.

Last, since it will be needed below, introduce the notation $G(n; s^t, \pi_t)$ for the cumulative distribution (conditional on history s^t and current idiosyncratic productivity shock π_t) of the number of firms with employment less than or equal to n during the production phase of period t.

2.2 Construction Firms

An inactive production firm needs to purchase a factory from a construction firm in order to enter. The construction firm sector consists of many firms each of which operates a linear technology which uses κ units of labor services to produce a single factory. These firms are not subject to search frictions, and rent their labor each period in a competitive market. Because the construction of factories is finished before the labor market operates, any worker in the economy is able to work in the construction sector during any period in addition to any other activities that she may be engaged in (such as work in the production sector). Denote by $w^f(s^t)$ the wage paid by a construction firm; then the assumption of perfect competition ensures that the price of a factory satisfies

$$k^f(s^t) = \kappa w^f(s^t). \tag{8}$$

2.3 Recruitment Firms

Production firms need to purchase the opportunity to meet with new workers from recruitment firms. The recruitment firm sector consists of many firms, each of which operates a technology that produces $\frac{1}{\gamma} v m(\theta)$ recruitment opportunities if it employs v recruiters and if the aggregate ratio of recruiters to unemployed workers looking for jobs is θ. The probability that a particular unemployed worker is matched to a particular recruiting firm is proportional to the number of recruiters employed by that recruiting firm. As in Shimer (2010), this matching technology represents a matching function that exhibits constant returns to scale in recruiters and unemployed workers jointly. I assume that m is continuous and nonincreasing, with $\lim_{\theta\to 0} m(\theta) = +\infty$. Denote by $f(\theta) = \theta m(\theta)$ the probability that an unemployed worker successfully meets a recruitment firm. I assume that a law of large numbers applies, so that although matching is stochastic from the perspective of an unemployed worker, it is deterministic from the point of view of recruitment firms. I assume that $\lim_{\theta\to 0} f(\theta) = 0$ and that $f(\cdot)$ is continuous and strictly increasing.

As in the construction sector, the workers employed as recruiters by recruitment firms are rented each period in a competitive labor market which is not affected by search frictions. Denote by $w^r(s^t)$ the wage paid by a recruitment firm. I assume that the recruitment sector is perfectly competitive, so that the price of a recruitment opportunity is

$$k^r(s^t) = \gamma w^r(s^t)/m(\theta(s^t)). \tag{9}$$

Because the recruitment sector operates after the construction sector is closed and before the production sector operates, any worker in the economy can work in the recruitment sector independently of other activities she may undertake.

The unemployment pool able to be hired in the current period includes those exogenously separated at the end of the previous period, as well as those fired from their firms at the beginning of the current period. Thus, some unemployment spells have a length of zero.

2.4 Households

There is a representative household which contains a continuum of measure 1 of members. Each individual member of the household maximizes expected utility and lives for ever. Each member of the household has time-separable preferences over histories of consumption and labor supply, with the period felicity function given by

$$\log c - e^f \phi^f - e^r \phi^r - e^y \phi^y. \tag{10}$$

Here e^f and e^r are the worker's labor supply respectively in the construction and recruitment sectors. e^f and e^r take nonnegative real values. e^y, on the other hand, takes the value 1 if the worker works in the production sector and 0 otherwise.[13]

The discount factor for future felicity is β. Following Merz (1995), there is complete insurance within the household: because of the separability in (10), the household allocates consumption among its members equally, so as to equalize the marginal utility of consumption across them. The household also instructs its members on whether to work in the recruitment sector or the construction sector. However, it cannot influence the number of its workers that are employed in the production sector, which is determined from last period's employment in the production sector according to the choices of production firms for firing, entry, and recruitment. In summary, the household acts as if it maximizes the utility function

$$\sum_{t=0}^{\infty} \beta^t P(s^t) \left[\log c(s^t) - \phi^f e^f(s^t) - \phi^r e^r(s^t) - \phi^y e^y(s^t)\right],$$

where $e^f(s^t)$, $e^r(s^t)$, and $e^y(s^t)$ are the average labor supply of household members respectively to the construction, recruitment, and production sectors during period t after history s^t. $e^y(s^t)$ can be interpreted as the fraction of household members working in the production sector.

The household faces the budget constraint that the expected present discounted value of consumption less labor income and dividends must equal initial assets:

$$a_0 = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t q_0(s^t) \left(c(s^t) - e^f(s^t) w^f(s^t) - e^r(s^t) w^r(s^t) - e^y(s^t) \mathbf{E} w(s^t) - d(s^t)\right).$$

Here $\mathbf{E}w(s^t)$ denotes the average wage earned by each member of the household in the production sector after history s^t. The household takes the evolution of $\mathbf{E}w(s^t)$ and $e^y(s^t)$ as given.[14] The

[13] That is, there is an intensive margin of labor supply for the individual worker in the construction and recruitment sectors, but only an extensive margin in the production sector. Note that because these activities are not mutually exclusive, there are no additional restrictions on the values of e^f, e^r, and e^y.

[14] To define $\mathbf{E}w(s^t)$ more precisely, I need to assume that wages depend only on a firm's employment n and current idiosyncratic productivity shock π_t and on the history s^t. This will be established in Lemma 1 below. In this case the total wages earned by the household's members employed in the production sector satisfy

$$e^y(s^t) \mathbf{E} w(s^t) = \sum_{s^t} \sum_{\pi^t} \int_0^{\infty} n w(n; z_i, \zeta_j)\, dG(n; s^t, \zeta_j).$$

household owns, via a mutual fund, shares in the market portfolio of firms; $d(s^t)$ denotes the dividends received from this portfolio.

The household's problem can be written in recursive notation. Denote by $V(s^t, a)$ the household's utility after history s^t if its assets are a. During period t the household chooses employment in the construction and recruitment sectors along with state-contingent asset holdings for period $t+1$, so that its value can be represented by

$$V(s^t, a) = \max_{a(s^{t+1}), e^c, e^r} \left[\log c - \phi^f e^f - \phi^r e^r - \phi^y e^y(s^t) + \beta \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} V(s^{t+1}), a(s^{t+1}) \right] \quad (11)$$

subject to the constraints that $e^c, e^r \in [0, 1]$, to the intertemporal budget constraint

$$c = a + e^f w^f(s^t) + e^r w^r(s^t) + e^y(s^t)\mathbf{E}w(s^t) - \sum_{s^{t+1}|s^t} q_t(s^{t+1})a(s^{t+1}), \quad (12)$$

and taking the evolution of $e^y(\cdot)$ and $\mathbf{E}w(\cdot)$ as given.

Assume that parameters are such that there is a positive amount of entry and recruiting in every period. Because of the linear disutility, the household's labor supply to these sectors can take a positive finite value only if

$$\phi^f = \frac{w^f(s^t)}{c(s^t)} \qquad \text{and} \qquad \phi^r = \frac{w^r(s^t)}{c(s^t)}. \quad (13)$$

where $c(s^t) = c(s^t, a(s^t))$ is the equilibrium value of the representative household's consumption after history s^t.

As is standard, combining the envelope condition for current assets and the first-order condition for next period's assets yields the asset pricing equation

$$q_t(s^{t+1}) = \beta \frac{P(s^{t+1})c(s^t)}{P(s^t)c(s^{t+1})}; \quad (14)$$

the particular functional form of the pricing kernel arises because of the assumption of logarithmic utility of consumption. Suppressed in the notation for the household's value function is the fact that the household's labor income in the production sector depends on the distribution of employment across firms and on the wages paid by each firm.

2.5 Wage determination

I assume that wages are set by the appropriate version of the bargaining protocol of Stole and Zwiebel (1996a,b). In order to set up the equation which characterizes the bargained wage, I need to write expressions for the marginal benefit to the firm of employing a marginal worker and to the household of having a marginal member employed at a firm.

2.5.1 Marginal value of employment for the firm

First, consider the firm. Denote by $\hat{J}_n(n; s^t, \pi^t)$ and $J_n(n; s^t, \pi^t)$ the marginal benefit to the firm of employing a worker respectively at the beginning of period t (after learning the new aggregate and idiosyncratic productivity shocks but before firing and recruitment occur) and at the beginning of the production phase of period t (before production and wage payments occur, but after the recruitment phase is closed). This notation and timing is consistent with that introduced in (4) and (5). Accordingly, I can differentiate equation (5) to obtain that

$$J_n(n; s^t, \pi^t) = \frac{\partial}{\partial n}\left[p_t \pi_t y(n) - n w(n; s^t, \pi^t)\right] + (1-\delta) \sum_{s^{t+1}|s^t} q_t(s^{t+1}) \sum_{\pi^{t+1}} \mu(\pi_t, \pi_{t+1}) \hat{J}_n(n; s^{t+1}, \pi^{t+1}). \tag{15}$$

Next, note that if $\hat{J}(n; s^t, \pi^t)$ is strictly concave in n, then the optimal hiring and firing behavior of the firm takes a particularly simple form because of the constant marginal costs of recruitment and of firing workers (in fact, the latter cost is zero). There exist two values $n^*(s^t, \pi^t) < \bar{n}(s^t, \pi^t)$, respectively the firm's *hiring target* and *firing target*. Any firm which entered the period with employment $n > n(s^t, \pi^t)$ will fire $n - \bar{n}(s^t, \pi^t)$ workers after learning (s^t, π^t) and reduce its employment to $\bar{n}(s^t, \pi^t)$. Such a firm will not recruit. On the other hand, if $n < n^*(s^t, \pi^t)$, the firm does not fire workers and instead immediately recruits $n^*(s^t, \pi^t) - n$ workers. I will assume for now that the firm's hiring and firing policies take this form, and will verify in Lemma 2 that indeed the value function is strictly concave in the relevant range.

The firing target is characterized by the condition that the marginal value to the firm during the production phase of having an additional worker equal zero. Because excess workers can be fired at zero cost, for any $n \geq \bar{n}(s^t, \pi^t)$,

$$\hat{J}_n(n; s^t, \pi^t) = \hat{J}_n(\bar{n}(s^t, \pi^t); s^t, \pi^t) = J_n(\bar{n}(s^t, \pi^t); s^t, \pi^t) = 0. \tag{16}$$

Analogously, the hiring target is characterized by the condition that the marginal value to the firm during the production phase equal the cost of recruiting an additional worker, $k^r(s^t)$. Because workers can be recruited at constant marginal cost, for any $n \leq n^*(s^t, \pi^t)$,

$$\hat{J}_n(n; s^t, \pi^t) = \hat{J}_n(n^*(s^t, \pi^t); s^t, \pi^t) = J_n(\bar{n}^*(s^t, \pi^t); s^t, \pi^t) = k^r(s^t). \tag{17}$$

Finally, any firm whose initial employment lies strictly between the hiring and firing target neither hires nor fires. It immediately follows that for such a firm,

$$\hat{J}_n(n; s^t, \pi^t) = J_n(n; s^t, \pi^t). \tag{18}$$

(16), (17), and (18) imply that the marginal benefit to the firm from employing a worker is always the same at the beginning of period t and during the hiring phase. Substitute into (15) to see that

$$J_n(n;s^t,\pi^t) = \frac{\partial}{\partial n}\left[p_t\pi_t y(n) - nw(n;s^t,\pi^t)\right] + (1-\delta)\sum_{s^{t+1}|s^t} q_t(s^{t+1})\sum_{\pi^{t+1}}\mu(\pi_t,\pi_{t+1})J_n(n;s^{t+1},\pi^{t+1}). \tag{19}$$

Finally, define $\tilde{J}_n(w,n;s^t,\pi^t)$ to be the marginal value to the firm during the production phase of period t of a worker who will be paid during period t only an arbitrary wage w rather than the regular bargained wage $w(n;s^t,\pi^t)$ to be described below. That is,

$$\tilde{J}_n(w,n;s^t,\pi^t) = -(w - w(n;s^t,\pi^t)) + J_n(n;s^t,\pi^t), \tag{20}$$

since the worker is the same as any other worker except for her different period-t wage.

2.5.2 Marginal value of employment for the household

Now consider the marginal value to the household of having a worker employed.[15] Analogously to the notation used for the firm, denote by $\hat{V}_n(n;s^t,\pi^t)$ the marginal value to the household of having a worker employed in a firm with idiosyncratic shock history π^t and employment n at the beginning of period t, after the aggregate and idiosyncratic productivity shocks have been realized but before firing, recruitment, or production have occurred. Denote by $V_n(n;s^t,\pi^t)$ the marginal value to the household of having a worker employed in such a firm after firing and recruitment but before production and wage payments occur. Denote by $\hat{V}^u(s^t)$ the marginal value to the household of an unemployed worker before the labor market closes in period t. Finally, define

$$x(s^t) = \hat{V}^u(s^t) - \beta\sum_{s^{t+1}|s^t}\frac{P(s^{t+1})}{P(s^t)}\hat{V}^u(s^{t+1}). \tag{21}$$

$x(s^t)$ can be interpreted as the 'rental value' of an unemployed worker for a single period. If $\hat{V}^u(\cdot)$ is independent of the aggregate history, then $x(\cdot) = (1-\beta)\hat{V}^u(\cdot)$. More generally (21) accounts for changes over time in the value of an unemployed worker due to changes in job-finding probabilities and in wages of employed workers.

Observe that $\hat{V}_n(n;s^t,\pi^t) = V_n(n;s^t,\pi^t)$. This is obvious if the firm does not fire workers. If the firm enters period t with more workers than its firing target then, although some workers are fired, it follows from the condition characterizing the firing target (16), together with the bargaining equation (26) below, that these workers were generating zero surplus in any case.

I can then write the following recursive expression for the marginal value for the household of having a worker employed at a firm in state (s^t,π^t) and which has n employees after recruitment

[15] The measure of workers in the household is fixed, so more formally, the values defined in this paragraph are Lagrange multipliers on the appropriate constraints in the household's problem.

is completed and before the production stage has commenced.[16]

$$
\begin{aligned}
V_n(n;s^t,\pi^t) = {} & \frac{w(n;s^t,\pi^t)}{c(s^t)} - \phi^y + \beta\delta \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \hat{V}^u(s^{t+1}) \\
& + \beta(1-\delta) \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \sum_{\pi_{t+1}} \mu(\pi_t,\pi_{t+1}) V_n(n'(n;s^{t+1},\pi^{t+1});s^{t+1},\pi^{t+1}).
\end{aligned} \tag{22}
$$

The period-t wage $w(n;s^t,\pi^t)$ is valued at the household's marginal utility of consumption; ϕ_y is the disutility of labor. Recall from (6) that $n'(n;s^{t+1},\pi^{t+1})$ denotes the firm's policy for the state-contingent evolution of its employment.

Next, subtract $\hat{V}^u(s^t)$ from both sides of (22) and rearrange to see that the net surplus to the household from having an employed worker in period t relative to an unemployed worker,

$$
S(n;s^t,\pi^t) = V_n(n;s^t,\pi^t) - \hat{V}^u(s^t), \tag{23}
$$

satisfies

$$
\begin{aligned}
S(n;s^t,\pi^t) = {} & \frac{w(n;s^t,\pi^t)}{c(s^t)} - \phi^y - x(s^t) \\
& + \beta(1-\delta) \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \sum_{\pi_{t+1}} \mu(\pi_t,\pi_{t+1}) S(n'(n;s^{t+1},\pi^{t+1});s^{t+1},\pi^{t+1})
\end{aligned} \tag{24}
$$

Finally, define $\tilde{S}(w,n;s^t,\pi^t)$ to be the surplus relative to unemployment for the household of having a worker employed at a firm with idiosyncratic shock history π^t and employment n if the wage during period t only is equal to w and not to the wage $w(n;s^t,\pi^t)$ that would be selected by the wage bargaining mechanism to be described below. (In periods after period t, it is assumed that the wage returns to the bargained value.) That is,

$$
\tilde{S}(w,n;s^t,\pi^t) = \frac{w - w(n;s^t,\pi^t)}{c(s^t)} + S(n;s^t,\pi^t), \tag{25}
$$

since the household values the alteration in wage during period t according to its marginal utility of consumption; after period t, the wage returns to the bargained level, so that all other components of the value to the household of such a worker are given by (22).

[16] A reader familiar with the analysis of Shimer (2010)—see his equation 2.39—might wonder why equation (22) does not incorporate any terms relating to the probability that an unemployed worker finds a job except implicitly through $\hat{V}^u(\cdot)$. This is because of a difference of timing assumptions; he assumes that a worker unemployed at the beginning of a period cannot work in the production sector until the following period, whereas I assume that employment starts during the current period. My assumption generates a simple formula for the bargained wage.

2.5.3 Bargaining Equation and Wages

I assume that firms bargain with all their workers following Stole and Zwiebel (1996a,b). The wage w to be paid by a firm with n workers in period t after history (s^t, π^t) is

$$w(n; s^t, \pi^t) = \arg\max_w \tilde{S}(w, n; s^t, \pi^t)^\eta \tilde{J}_n(w, n; s^t, \pi^t)^{1-\eta}. \tag{26}$$

Here $\eta \in (0, 1)$, the bargaining power of workers, is a structural parameter of the model. Intuitively, firms and workers bargain over the marginal surplus generated by an employment relationship. The firm's outside option is to produce with one fewer worker, while the worker's outside option is unemployment. The firm bargains individually with its entire workforce, treating each as marginal.[17]

From (20), $\tilde{J}_n(w(n; s^t, \pi^t), n; s^t, \pi^t) = J_n(n; s^t, \pi^t)$ and $\frac{\partial}{\partial w}\tilde{J}_n(w(n; s^t, \pi^t), n; s^t, \pi^t) = -1$. From (25), $\tilde{S}(w(n; s^t, \pi^t), n; s^t, \pi^t) = S(n; s^t, \pi^t)$ and $\frac{\partial}{\partial w}\tilde{S}(w(n; s^t, \pi^t), n; s^t, \pi^t) = \frac{1}{c(s^t)}$. It follows that the solution to (26) satisfies[18]

$$\eta J_n(w(n; s^t, \pi^t), n; s^t, \pi^t) = (1-\eta) S(w(n; s^t, \pi^t), n; s^t, \pi^t) c(s^t). \tag{27}$$

Equation (27) holds for all (n, s^t, π^t). Substituting into (19) and (22) and using the asset pricing equation (14) establishes immediately that

$$(1-\eta)\left[w(n; s^t, \pi^t) - (\phi^y + x(s^t))c(s^t)\right] = \eta \frac{\partial}{\partial n}\left[p_t \pi_t y(n) - n w(n; s^t, \pi^t)\right]. \tag{28}$$

The power of equation (28) is that it establishes that wages depend on aggregate conditions only via the consumption of the representative household, $c(s^t)$, the value of an unemployed worker, $x(s^t)$, and the current aggregate productivity shock, p_t. Equation (28) can be solved through the use of the appropriate integrating factor; the general solution is

$$w(n; s^t, \pi^t) = (1-\eta)(\phi^y + x(s^t))c(s^t) + n^{-\frac{1}{\eta}}\left[C + p_t \pi_t \int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu)\, d\nu\right]$$

where C is a constant of integration. However, for the total wage bill of a firm to remain finite as its employment level n becomes small requires that $C = 0$. This proves the following Lemma.

[17] Notice that interpreting (23) as a surplus over an outside option in the bargaining equation (26) involves a problematic timing assumption. Should a worker reject the firm's offer, her outside option is $\hat{V}^u(s^t)$, which is available only if the labor market remains open for her to look for another job. For incumbent workers already employed by the firm the previous period, this is plausible if bargaining takes place before the recruitment phase of the period. For workers newly hired in the current period, the timing assumption is more problematic: it must be that, although hired in this period's labor market, the worker can return to that labor market and search again for a job this period should bargaining fail. This artificial assumption is necessary to generate a convenient solution for the bargained wage in the discrete-time environment considered here and would not be an issue in continuous time.

[18] Because a worker's assets affect her marginal utility of consumption, (27) suggests there should be an incentive for a household to increase its saving so as to increase bargained wages. I implicitly assume here that this is not the case: firms bargain as if facing a worker whose household has the same wealth as the representative household. This can be justified if the worker's wealth is not observable to the firm. An alternative justification is that the firm employs workers from many different households, but bargains simultaneously with all its workers. In either case a single household cannot change the wage its workers receive by saving more.

Lemma 1. *The wage paid by a production firm with n workers and idiosyncratic shock history π^t in period t after history s^t is given by*

$$w(n; s^t, \pi^t) = (1-\eta)(\phi^y + x(s^t))c(s^t) + p_t \pi_t n^{-\frac{1}{\eta}} \int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu)\, d\nu. \tag{29}$$

I assume henceforth that the value of η and the form of the production function are such that the integral on the right side of (29) is finite, with

$$\lim_{n\to 0^+} n^{\frac{\eta-1}{\eta}} \int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu)\, d\nu = 0. \tag{30}$$

In the case that $y(n; p, \pi) = p\pi n^\alpha$ is Cobb-Douglas, a necessary and sufficient condition to ensure this is that $\frac{1}{\eta} + \alpha > 1$, so that $\nu^{\frac{1-\eta}{\eta}} y'(\nu) = A\nu^{\frac{1}{\eta}+\alpha-2}$ is integrable near $\nu = 0$. Since $\eta < 1$ and $\alpha > 0$, the condition is always satisfied in this case, which will be the case used in the calibrated model. In this case the integral on the right side of (29) can be calculated in closed form, establishing that

$$w(n; s^t, \pi^t) = (1-\eta)(\phi^y + x(s^t))c(s^t) + \frac{\alpha\eta}{\alpha\eta + 1 - \eta} p_t \pi_t n^{\alpha-1}. \tag{31}$$

More generally, (29) can be rewritten as

$$w(n; s^t, \pi^t) = (1-\eta)(\phi^y + x(s^t))c(s^t) + \eta \frac{\int_0^n \nu^{\frac{1-\eta}{\eta}} p_t \pi_t y'(\nu)\, d\nu}{\int_0^n \nu^{\frac{1-\eta}{\eta}}\, d\nu}.$$

Wages are a weighted average of two terms. The first term is the disutility of labor (measured in terms of the consumption good by adjusting using the marginal utility of consumption). The second term is itself a weighted average of the inframarginal products of workers at smaller firms than n. Stole and Zwiebel (1996a) explain how such a term arises in a static model; intuitively, inframarginal products are relevant for the firm's outside option, which is that it bargains with a smaller workforce if the marginal worker were to refuse its wage offer.

The functional form for wages given by (29) guarantees that the the firm's flow profit function, $y(n; p, \pi) - nw(n; s^t, \pi^t)$ has the standard properties of a neoclassical production function, which justifies assumptions made above in deriving the nature of optimal firing and recruitment policies.

Lemma 2. *The function*

$$\tilde{y}(n) = y(n) - n^{\frac{\eta-1}{\eta}} \int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu)\, d\nu \tag{32}$$

is always nonnegative, strictly increasing, strictly concave and satisfies the Inada conditions $\lim_{n\to 0} \tilde{y}'(n) = +\infty$ *and* $\lim_{n\to+\infty} \tilde{y}'(n) = 0$.

The proof of Lemma 2 is in the Appendix.

The bargaining equation provides a particularly simple characterization of $x(s^t)$, the rental value of an unemployed worker. The Bellman equation for an unemployed worker is

$$\hat{V}^u(s^t) = f(\bar{\theta})\mathbf{E}V_n(n; s^t, \pi^t) + (1 - f(\bar{\theta}))\beta\hat{V}^u(s^{t+1}) \tag{33}$$

where the expectation is taken over all hiring firms in proportion to their recruitment purchases. However, at any firm which recruits a nonzero number of workers, equation (27) and the characterization of optimal hiring in Section 2.5.1 imply that

$$V_n(n; s^t, \pi^t) = \hat{V}^u(s^t) + \frac{\eta}{1-\eta}\frac{J_n(n; s^t, \pi^t)}{c(s^t)} = \hat{V}^u(s^t) + \frac{\eta}{1-\eta}\frac{k^r(s^t)}{c(s^t)}.$$

(9) and (13) imply that $k^r(s^t) = \gamma\phi^r c(s^t)/m(\theta(s^t))$. Substitute into (33) and use the relation $f(\theta) = \theta m(\theta)$ to deduce that

$$x(s^t) = \hat{V}^u(s^t) - \beta\mathbf{E}\hat{V}^u(s^{t+1}) = \frac{\theta(s^t)\eta\gamma\phi^r}{(1 - f(\theta(s^t)))(1-\eta)}. \tag{34}$$

2.6 Market Clearing

Closing the model requires imposing market clearing conditions. Two of these have already been mentioned: interior solutions for labor supply in the construction and recruiting sectors require that the wages in those sectors satisfy (13). In addition to this, the market for the consumption good must clears, so that the representative household's consumption equals production by firms:

$$c(s^t) = \sum_{s^t}\sum_{\pi^t}\int_0^\infty p_t\pi_t y(n)\,dG(n; s^t, \pi^t), \tag{35}$$

I need to impose several consistency conditions. First, the evolution of the firm size distribution must be consistent with the firing and recruitment decisions of firms, so that for incumbent firms

$$G(n; s^t, \pi^t) = (1-\delta)\int_{T(n;s^t,\pi^t)} dG(\nu; s^{t-1}, \pi^{t-1}) \tag{36}$$

where $T(n; s^t, \pi^t) = \{\nu \,|\, n \geq \nu - f(\nu; s^t, \pi^t) + h(\nu; s^t, \pi^t)\}$ is the set of employment values at the beginning of period t which lead the firm to have employment at most n during the production stage of period t when the realized productivity shocks are s^t and π^t. For entrants,

$$G(n; s^t, (0, \ldots, 0, \zeta_j)) = \begin{cases} 0 & n < n'(0; s^t, (0, \ldots, 0, \zeta_j)) \\ n^e(s^t)F^e_\pi(\zeta_j) & \text{otherwise.} \end{cases} \tag{37}$$

Here $n^e(s^t)$ is the measure of entrant firms following history s^t.

Second, the ratio of recruiters to unemployed workers must equal $\theta(s^t)$; that is,

$$\theta(s^t) = \frac{e^r(s^t)}{u(s^t)}, \tag{38}$$

where $u(s^t)$, the number of unemployed workers after the firing stage of period t, satisfies

$$u(s^t) = u(s^{t-1})(1 - f(\theta(s^{t-1}))) + \delta(1 - u(s^{t-1})(1 - f(\theta(s^{t-1})))) \\ + (1-\delta)\sum_{\pi^t}\int f(n; s^t, \pi^t)\mu(\pi_{t-1}, \pi_t)\, dG(n; s^{t-1}, \pi^{t-1}). \tag{39}$$

The explanation of this equation is that unemployed workers arise from three sources, those who were unemployed in the previous period and failed to find a job in the production sector (which occurred with probability $1 - f(\theta(s^{t-1}))$, those who were employed in the production sector during the previous period but whose firm was exogenously destroyed (with probability δ), and those who were fired from the production sector at the beginning of the current period.

Last, employment in the construction sector must be consistent with construction firms' output of factories, which in turn must equal entrant firms' demand,

$$e^f(s^t) = \kappa n^e(s^t); \tag{40}$$

employment in the recruitment sector must be consistent with recruitment firms' production of recruitment opportunities, which in turn must be consistent with production firms' hiring policies,

$$\frac{1}{\gamma}e^r(s^t)m(\theta(s^t)) = (1-\delta)\sum_{\pi^t} h(n; s^t, \pi^t)\int dG(n; s^{t-1}, \pi^{t-1}); \tag{41}$$

and labor supply to the production sector must be consistent with employment at active firms,

$$e^y(s^t) = \sum_{s^t}\sum_{\pi^t}\int_0^\infty n\, dG(n; s^t, \pi^t). \tag{42}$$

2.7 Equilibrium

I am now ready to define equilibrium.

Definition 1. An *equilibrium* is a set of (aggregate history-dependent) sequences for consumption $c(s^t)$, asset holding $a(s^t)$, labor supply in the construction, recruitment, and production sectors $e^f(s^t)$, $e^r(s^t)$, and $e^y(s^t)$, unemployment $u(s^t)$, the recruiter-unemployment ratio $\theta(s^t)$, entry by new production firms $n^e(s^t)$, wages in the construction and recruitment sectors $w^f(s^t)$ and $w^r(s^t)$, prices of factories and of recruitment opportunities $k^f(s^t)$ and $k^r(s^t)$, and date-0 prices of history-s^t consumption goods $q_0(s^t)$ together with a set of (aggregate and idiosyncratic history-dependent) sequences for firing, recruitment, and target employment by production firms $f(n; s^t, \pi^t)$, $h(n; s^t, \pi^t)$, and $n'(n; s^t, \pi^t)$, wages paid by production firms $w(n; s^t, \pi)$, and the distribution of production firms by employment level $G(n; s^t, \pi^t)$, such that

- the household chooses consumption, savings, and labor supply to the construction and recruitment sectors to maximize the right side of (11), so that (13) and (14) hold;

- the household's budget constraint (12) holds;
- construction firms choose factory production and recruitment firms choose recruitment opportunity production optimally subject to perfect competition, so that (8) and (9) hold;
- wages in the production sector are determined by (29), and firing and recruitment are chosen to maximize profits, so that $f(n; s^t, \pi^t)$ and $h(n; s^t, \pi^t)$ maximize the right side of (4);
- the free entry condition (7) holds, and entry is always non-negative, and strictly positive only if (7) holds with equality;
- markets clear for consumption goods, factories, and recruitment opportunities, so that (35), (40), and (41) hold;
- labor supply by the household to the production sector is consistent with labor demand by production firms, so that (42) holds;
- unemployment evolves according to (39);
- individual production firm sizes evolve according to (3) and the firm size distribution evolves according to (36) and (37); and
- $\theta(s^t)$ is consistent with the recruitment activities of firms and the number of unemployed workers, according to (38)

The definition of equilibrium looks unwieldy, since the interaction between aggregate variables and the idiosyncratic shock process appears intractable. Fortunately, because of the balanced growth preferences of households, an equilibrium with a very simple structure exists. In this equilibrium, unemployment, labor supply to the construction and recruitment sectors and employment in the production sector, the entry of production firms, and the distribution of production firm sizes are constant over time and independent of the history of aggregate productivity p^t. Consumption, wages, and prices all move proportionally to current aggregate productivity p_t.

Definition 2. A *stationary equilibrium* is an equilibrium with the property that there exist constant values $\bar{c}$, $\bar{a}$, $\bar{e}^f$, $\bar{e}^r$, $\bar{e}^y$, $\bar{u}$, $\bar{n}^e$, $\bar{w}^f$, $\bar{w}^r$, $\bar{k}^f$, $\bar{k}^r$, and functions $\bar{f}, \bar{h}, \bar{n}', \bar{w}, \bar{G} : [0, \infty) \times \{\zeta_1, \ldots, \zeta_{m_\pi}\} \to [0, \infty)$, with $G(\cdot, \zeta_j)$ nondecreasing for $1 \leq j \leq m_\pi$, such that for all t, s^t, n, and π^t,

- $e^f(s^t) = \bar{e}^f$, $e^r(s^t) = \bar{e}^r$, $e^y(s^t) = \bar{e}^y$, $u(s^t) = \bar{u}$, $n^e(s^t) = \bar{n}^e$, $f(n; s^t, \pi^t) = \bar{f}(n, \pi_t)$, $h(n; s^t, \pi^t) = \bar{h}(n, \pi_t)$, $n'(n; s^t, \pi^t) = \bar{n}'(n, \pi_t)$, and $G(n; s^t, \pi_t) = \bar{G}(n, \pi_t)$;
- $c(s^t) = p_t\bar{c}$, $a(s^t) = p_t\bar{a}$, $w^f(s^t) = p_t\bar{w}^f$, $w^r(s^t) = p_t\bar{w}^r$, $k^f(s^t) = p_t\bar{k}^f$, $k^r(s^t) = p_t\bar{k}^r$, and $w(n; s^t, \pi^t) = p_t\bar{w}(n, \pi_t)$.

Proposition 1. *A stationary equilibrium exists.*

The proof of Proposition 1 is in the Appendix. The structure of the proof is as follows. First, normalize the value function for firms by dividing through by current aggregate productivity. Under the assumption that the equilibrium is stationary, the functional form of wages given by (29) shows that the firm's normalized value depends only on current idiosyncratic productivity and current employment. Market clearing shows that the cost of a factory and of recruiting a worker depend only on the level of consumption, $\bar{c}$, and on the recruiter-unemployment ratio $\bar{\theta}$. Given $\bar{c}$, there is a unique $\bar{\theta}$ consistent with free entry; then by continuity I can also find a value for $\bar{c}$ that is consistent with goods market clearing. Walras' law then ensures the asset market clears.

A stationary equilibrium exists for the same reasons as in Shimer (2010). I assumed balanced growth preferences such that income and substitution effects cancel, as well as that that the entry and recruiting sectors are labor-intensive. When aggregate productivity is temporarily high, all else equal, active production firms would increase recruitment and inactive production firms would increase entry. However, the representative household is made richer by the positive aggregate shock, which it benefits from via an increase in wages in all three sectors of the economy.[19] This increases its desire to consume and enjoy leisure today, which, in general equilibrium, must lead to an increase in the interest rate. Under the assumed preferences, these two effects exactly cancel each other.

In generating the exact neutrality result, it is important not only that the utility function of the representative household takes the particular form it does (with logarithmic utility of consumption and additive separability), but also that the construction and recruitment sectors use only labor to generate new factories and new matching opportunities. This allows for a stationary allocation to be an equilibrium, since when consumption is proportional to aggregate productivity, the cost of creating a factory or a match is constant in terms of marginal utility. This would not be true if these sectors used the consumption good as an input. This is again analogous to Shimer (2010).

Finally, note that a stationary equilibrium only exists if the initial conditions of the economy are suitably chosen. In particular, the measure of production firms and the distribution of employment across them must take specific values; if there are too few production firms, intuitively, more should enter over time as the economy converges towards stationarity. A formal proof of the stability of the stationary equilibrium is beyond the scope of the current paper.

The most noteworthy implication of Proposition 1 is that a stationary equilibrium exists independently of the shock process for idiosyncratic productivity. Essentially any observations on the firm size distribution and on the stochastic process for the size of an individual firm can be rationalized by the model using an appropriate choice for the stochastic process for idiosyncratic productivity, μ.[20] Such data only have implications for the dynamics of aggregate variables if either

[19]One might think that the household would also benefit from higher dividend payments, but in a stationary equilibrium, the dividend payment $d(s^t) \equiv 0$ as payment of entry costs for new firms always exactly offsets profits from active firms.

[20]Intuitively, suppose that idiosyncratic productivity includes both a permanent and a transitory component. The variance of the transitory component can be chosen to match the cross-sectional dispersion in firm employment growth, and then conditionally on this, the variance of the permanent component can be chosen to match the cross-sectional dispersion in firm employment levels. In the quantitative analysis in Section 4 I perform such a calibration.

preferences are not consistent with balanced growth (so that income and substitution effects do not cancel), or if the entry and recruitment technologies do not use only labor as an input. To investigate the effect of such assumptions, in Section 3 I modify the model in two ways that are common in the search literature. There, I assume that workers are risk-neutral, so that the general equilibrium effect of productivity on interest rates is absent, so that an increase in productivity no longer drives up the disutility of labor, measured terms of consumption goods. I assume also that the entry and recruitment technologies use the consumption good as an input.

2.8 Constructive Solution Algorithm

The proof of Proposition 1 is not constructive, so it is useful to provide an algorithm for constructing the equilibrium allocation. In the proof of the Proposition, I show that the normalized Bellman equation for a firm in a stationary allocation (defined as $\hat{K}(n; s^t, \pi^t) = \hat{J}(n; s^t, \pi^t)/p_t$) in fact depends only on π_t and so can be written

$$\hat{K}(n,\pi) = \max_{n'} \left[-\bar{k}^r \max(n'-n,0) + \pi\tilde{y}(n') - n'(1-\eta)(\phi^y + \bar{x})\bar{c} + \beta(1-\delta)\sum_{\pi^{t+1}} \mu(\pi_t, \pi_{t+1})\hat{K}(n', \pi_{t+1}) \right], \tag{43}$$

where $\bar{k}^r = \gamma\phi^r\bar{c}/m(\bar{\theta})$, $\bar{x} = x(s^t)$ is defined in (21) and is constant in a stationary allocation, and $\tilde{y}(\cdot)$ is defined in (32). The hiring and firing targets characterized by the first-order conditions (16) and (17) also depend only on p_t:

$$\hat{K}_n(n^*(\pi), \pi) = K_n(n^*(\pi), \pi) = \bar{k}^r \qquad \text{and} \qquad \hat{K}_n(\bar{n}(\pi), \pi) = K_n(\bar{n}(\pi), \pi) = 0. \tag{44}$$

In addition, for $n < n^*(\pi)$,

$$\hat{K}(n,\pi) = -(n^*(\pi) - n)\bar{k}^r + K(n^*(\pi), \pi) \qquad \text{and} \qquad \hat{K}_n(n,\pi) = \bar{k}^r, \tag{45}$$

and for $n > \bar{n}(\pi)$,

$$\hat{K}(n,\pi) = K(\bar{n}(\pi), \pi) \qquad \text{and} \qquad \hat{K}_n(n,\pi) = 0. \tag{46}$$

Finally, for $n \in [n^*(\pi), \bar{n}(\pi)]$, of course, $\hat{K}(n,\pi) = K(n,\pi)$, since neither firing nor recruitment occurs. Thus in all cases, $\hat{K}_n(n,\pi) = K_n(n,\pi)$. This allows (43) to be rewritten as

$$K(n,\pi) = \begin{cases} -(n^*(\pi) - n)\bar{k}^r + K(n^*(\pi), \pi) & n \leq n^*(\pi) \\ \tilde{y}(n) - n(1-\eta)(\phi^y + \bar{x})\bar{c} + \beta(1-\delta)\sum_{\pi'} \mu(\pi, \pi')K(n, \pi') & n^*(p) \leq n \leq \bar{n}(\pi) \\ K(\bar{n}(\pi), \pi) & n \geq \bar{n}(\pi). \end{cases} \tag{47}$$

Differentiate (47) with respect to n and use this fact to observe that for $n \in [n^*(\pi), \bar{n}(\pi)]$,

$$K_n(n,\pi) = \begin{cases} \bar{k}^r & n \leq n^*(\pi) \\ \tilde{y}'(n) - (1-\eta)(\phi^y + \bar{x})\bar{c} + \beta(1-\delta)\sum_{\pi'} \mu(\pi,\pi')K_n(n,\pi') & n^*(p) \leq n \leq \bar{n}(\pi) \\ 0 & n \geq \bar{n}(\pi). \end{cases} \quad (48)$$

Now write $K_j'(n_l^*) = K_n(n^*(\zeta_l),\zeta_j)$ and $K_j'(\bar{n}_l) = K_n(\bar{n}(\zeta_l),\zeta_j)$. Substituting repeatedly into (48) gives $2m_\pi^2 + 2m_\pi$ linear equations in the $2m_\pi^2 + 2m_\pi$ variables $\left\{K_j'(n_l^*)\right\}_{jl} \cup \left\{K_j'(\bar{n}_l)\right\}_{jl} \cup \{\tilde{y}'(n_l^*)\}_l \cup \{\tilde{y}'(\bar{n}_l)\}_l$.[21] Given the value of the key endogenous variables $\bar{c}$ and $\bar{\theta}$ (which enter the value of $\bar{k}^r$), this system of linear equations depends only on the ordering of the set $L \equiv \{n_l^*\}_{jl} \cup \{\bar{n}_l\}_{jl}$, since this ordering determines which of the three cases applies in (48). If I knew the ordering of L, I could solve the equations and determine $\{\tilde{y}'(n_l^*)\}_l \cup \{\tilde{y}'(\bar{n}_l)\}_l$. Since $\tilde{y}'$ is strictly decreasing according to Lemma 2, this would determine $\{n_l^*\}_l \cup \{\bar{n}_l\}_l$. Conveniently, the solution to (47) and (48) for $\{n_l^*\}_l \cup \{\bar{n}_l\}_l$ is unique conditional on the rank order of the elements of L and the value of $\bar{c}$.[22] Next, substitute the resulting solution for the elements of L into (43) to obtain another system of $2m_\pi^2$ linear equations in $\{K_j(n_l^*)\}_{jl} \cup \{K_j(\bar{n}_l)\}_{jl}$. These must be consistent with the free entry condition, which takes the form

$$\sum_{j=1}^{m_\pi} F_\pi^e(\zeta_j)\hat{K}(0,\zeta_j) = \bar{k}^f. \quad (49)$$

This suggests the following algorithm for constructing a solution.

1. Guess values for $\bar{c}$ and $\bar{\theta}$.

2. Guess the rank order of the elements of L.

3. Solve the $2m_\pi^2 + 2m_\pi$ linear equations given by (48) for $\left\{K_j'(n_l^*)\right\}_{jl} \cup \left\{K_j'(\bar{n}_l)\right\}_{jl} \cup \{\tilde{y}'(n_l^*)\}_l \cup \{\tilde{y}'(\bar{n}_l)\}_l$.

4. Solve for $\{n_l^*\}_l \cup \{\bar{n}_l\}_l$. Verify whether the order of the solution is consistent with the guessed order from step 2. If not, guess a new order and return to step 3.

5. Solve the $2m_\pi^2$ equations given by (43) for $\{K_j(n_l^*)\}_{jl} \cup \{K_j(\bar{n}_l)\}_{jl}$.

6. Verify whether the free entry condition (49) holds with equality for every i. If not, adjust the guess for $\bar{\theta}$ and return to step 2.

7. Verify whether the goods market clears; if not, adjust the guess for $\bar{c}$ and return to step 1.

The only part of the algorithm that is not fully specified above is how to guess the rank order of the set L at step 2. Brute-force checking of all possibilities is not feasible: L has $2m_\pi$ elements,

[21] The 'additional' $2m_\pi$ equations come because for every j, both the first and second cases in (48) hold when $n = n^*(\zeta_j)$, and both the second and third cases hold when $n = \bar{n}(\zeta_j)$.

[22] I prove this statement as Lemma 4 in the Appendix.

so the number of possible orderings is $(2m_\pi)!$, which already exceeds 3.6×10^6 if $m_\pi = 5$. Some orderings can be ruled out *a priori* since necessarily $n_j^* < \bar{n}_j$ for each j, and, if the distribution of a firm's future idiosyncratic productivity is increasing in the sense of first-order stochastic dominance in today's productivity, also $n_{j_1}^* < n_{j_2}^*$ and $\bar{n}_{j_1} < \bar{n}_{j_2}$ whenever $j_1 < j_2$. However, this still leaves a large set of possible orderings for L. I use the following procedure to generate the correct order. The first time I reach step 2, I use an arbitrary ordering. Next, each time I need to make a new guess for the ordering at step 4, I use the ordering implied by the solution for $\{n_l^*\}_l \cup \{\bar{n}_l\}_l$ just generated using the old guess. In practice this process quickly converges. The dependence on $\bar{c}$ and $\bar{\theta}$ is smooth, so that standard numerical algorithms for solution-finding work well in steps 6 and 7.

3 Model with Linear Utility

The equilibrium of the model described in the previous section does not generate aggregate fluctuations. In this, it matches the findings of Shimer (2010) for an environment in which firms wish to hire only a single worker. However, in the search literature, it is more common to assume that workers have linear preferences over consumption and that the technologies for creating new firms and recruiting workers use the consumption good as inputs, rather than labor. In this section I incorporate these features into the model. This allows for aggregate fluctuations in employment, unemployment, and vacancy creation, but these fluctuations need not be any more persistent than in the standard MP model. In fact, if aggregate shocks are small, then I can use a modification of the algorithm described in Section 2.8 to construct an equilibrium in which labor market tightness is a jump variable, as in the usual equilibrium of the benchmark MP model.[23]

3.1 Model Description

Since the economic environment is similar to that described in Section 2, for the sake of brevity I here indicate only what is different. Anything not mentioned is unchanged from the earlier model.

First, construction firms now use κ units of the consumption good to produce a factory. Since the construction sector is perfectly competitive, this will also be the price of a factory.

Second, the inputs to the matching technology operated by recruitment firms are now vacancy advertising and unemployed workers. A recruitment firm which uses v units of advertising generates $vm(\theta)$ matches with unemployed workers, where the assumptions on $m(\theta)$ remain as in Section 2.3. A unit of advertising is generated at a cost of γ units of the consumption good. Because the recruitment sector is perfectly competitive, the price to a production firm to recruit a worker is $k^r = \gamma/m(\theta)$ units of the consumption good.

Workers have linear utility over consumption, so that (39) is replaced by the felicity function

$$c - e^y \phi^y. \tag{50}$$

[23] In the MP model, this equilibrium is unique in the continuous-time version of the model, although other equilibria can exist in discrete time (Pissarides, 2000; Shimer, 2005).

(Terms for labor supply in the construction and recruitment sectors are now absent since there is no demand for labor in these sectors.) Since the worker is risk-neutral, there is now no role for consumption insurance within the household; therefore it is sufficient to assume that the worker always consumes her income. Under this assumption, households make no economic decisions, since employment in the production sector is determined by the recruitment activities of production firms and not by the decisions of workers. This is just as in the MP model.

The asset pricing equation for the price at date t of history-s^{t+1} consumption goods is the risk-neutral equivalent to (14), that is,

$$q_t(s^{t+1}) = \beta \frac{P(s^{t+1})}{P(s^t)}. \tag{51}$$

Wage bargaining is still determined by (26), but the resulting form of the wage equation is slightly different. Routine manipulations establish that

Lemma 3. *The wage paid by a production firm in the linear utility model with n workers and idiosyncratic shock history π^t in period t after history s^t is given by*

$$w(n; s^t, \pi^t) = (1-\eta)(\phi^y + x(s^t)) + p_t \pi_t n^{-\frac{1}{\eta}} \int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu)\, d\nu. \tag{52}$$

Wages depend on aggregate conditions only through $x(s^t)$, the rental value of an unemployed worker, and p_t, the current aggregate productivity shock. This is simpler than under balanced growth preferences, since the consumption of the representative household no longer matters.

3.2 Equilibrium

The definition of an equilibrium in this environment is the same as Definition 1 modified as suggested by Section 3.1. (Specifically, (50), (51), and (52) replace their respective counterparts (39), (14), and (29) from the model with balanced growth preferences.) I therefore omit it.

Since income and substitution effects no longer cancel in the setting of the current section, no stationary equilibrium can now exist. A natural alternative conjecture for the structure of an equilibrium is that, analogously to results of Pissarides (2000) and Shimer (2005) for the MP model, labor market tightness θ_t may be a jump variable whose value depends only on the current value of aggregate productivity, p_t. In this section I investigate whether an equilibrium of this form exists.

Assume that the stochastic process for the aggregate productivity shock is a first-order Markov process with support on the m_p values $z_1, \ldots, z_{m_p}$. Write $\lambda(p_t, p_{t+1})$ for the probability that the period-$t+1$ aggregate productivity shock is p_{t+1}, conditional on the value of the period-t shock p_t. By assumption, θ depends only on p, so for each $i = 1, \ldots, m_p$ I can write θ_i for the value of θ associated with the productivity shock ζ_i.

If market tightness depends only on the aggregate productivity shock, then (34) shows that $x(s^t)$ depends only on the aggregate productivity shock also; I can therefore write x_i for the value of $x(s^t)$ when $p_t = z_i$. Also $k_r = \gamma/m(\theta)$, so I can likewise write k_i^r for the value of k_r when $p_t = z_i$.

(52) now shows that wages depend only on a firm's current employment, current idiosyncratic productivity shock, and on the current aggregate shock. Denote by $J_{ij}(n)$ the Bellman value of a firm with n employees when aggregate productivity is $p = z_i$ and idiosyncratic productivity is $\pi = \zeta_j$. The Bellman equation for a firm can be written analogously to (43), as

$$J_{ij}(n) = \max_{n'} \left[-k_i^r \max(n'-n,0) + z_i \zeta_j \tilde{y}(n') - n'(1-\eta)(\phi^y + x_i) + \beta(1-\delta) \sum_{i'=1}^{m_p} \sum_{j'=1}^{m_\pi} \lambda(z_i, z_{i'}) \mu(\zeta_j, \zeta_{j'}) J_{i'j'}(n') \right]. \tag{53}$$

The associated Bellman operator is a contraction, so that there is a unique solution (conditional on the values of $(\theta_1, \ldots, \theta_{m_p})$) for $J_\cdot(\cdot)$. Arguing as in Section 2.8, the firm's hiring and firing behavior are characterized by hiring targets n_{ij}^* and firing targets $\bar{n}_{ij}$ that depend only on the current aggregate and idiosyncratic productivity states. I can write the analog of (47), that

$$J_{ij}(n) = \begin{cases} -(n_{ij}^* - n)k_i^r + J_{ij}(n_{ij}^*) & n \le n_{ij}^* \\ z_i \zeta_j \tilde{y}(n) - n(1-\eta)(\phi^y + \bar{x}_i) + \beta(1-\delta) \sum_{i'=1}^{m_p} \sum_{j'=1}^{m_\pi} \lambda(z_i, z_{i'}) \mu(\zeta_j, \zeta_{j'}) J_{i'j'}(n) & n_{ij}^* \le n \le \bar{n}_{ij} \\ J_{ij}(\bar{n}_{ij}) & n \ge \bar{n}_{ij}. \end{cases} \tag{54}$$

Differentiation produces the analog of (48),

$$J'_{ij}(n) = \begin{cases} k_i^r & n \le n_{ij}^* \\ z_i \zeta_j \tilde{y}'(n) - (1-\eta)(\phi^y + \bar{x}_i) + \beta(1-\delta) \sum_{i'=1}^{m_p} \sum_{j'=1}^{m_\pi} \lambda(z_i, z_{i'}) \mu(\zeta_j, \zeta_{j'}) J'_{i'j'}(n) & n_{ij}^* \le n \le \bar{n}_{ij} \\ 0 & n \ge \bar{n}_{ij}. \end{cases} \tag{55}$$

Arguing as in the proof of Proposition 1, I can readily establish that a solution to (54) and (55), together with the free-entry condition (7), exists. This solution will be an equilibrium provided only that the stochastic process for entry of new production firms that is implied takes only nonnegative values. This can be guaranteed provided the stochastic process for aggregate productivity is not too volatile. More formally, I can state the following Proposition.

Proposition 2. *If the volatility of aggregate productivity is not too large and if the transition matrix $\lambda(p_t, p_{t+1})$ is strictly diagonally dominant, then there exists an equilibrium in the model with linear utility in which market tightness depends only on contemporaneous aggregate productivity.*

I can proceed analogously to the argument in Section 2.8 to provide a constructive algorithm to calculate the equilibrium. Suppose that the values of market tightness, θ_i are known for each i. Using the matching function, I can calculate the marginal recruiting cost $k_i^r = \gamma/m(\theta_i)$. For i, $k \in \{1, 2, \ldots, m_p\}$ and j, $l \in \{1, 2, \ldots, m_\pi\}$, substitute $n = n_{kl}^*$ and $n = \bar{n}_{kl}$ into equations (54) and (55). This generates a system of $2m_p^2 m_\pi^2 + 2m_p m_\pi$ linear equations in the $2m_p^2 m_\pi^2 + 2m_p m_\pi$

variables $\{J'_{ij}(n^*_{kl})\}_{i,j,k,l}$, $\{J'_{ij}(\bar{n}_{kl})\}_{i,j,k,l}$, $\{{n^*_{kl}}^{\alpha-1}\}_{k,l}$, and $\{\bar{n}^{\alpha-1}_{kl}\}_{k,l}$.[24] However, in order to know which line of (54) and (55) to use, I need to know the rank order of the terms of the tuple $L \equiv (n^*_{11}, n^*_{12}, \ldots, n^*_{1m_\pi}, n^*_{21}, \ldots, n^*_{2m_\pi}, \ldots, n^*_{m_p m_\pi}, \bar{n}_{11}, \bar{n}_{12}, \ldots, \bar{n}_{1m_\pi}, \bar{n}_{21}, \ldots, \bar{n}_{2m_\pi}, \ldots, \bar{n}_{m_p m_\pi})$. The following obvious generalization of the solution algorithm from Section 2.8 then applies.

1. Guess values for market tightness θ_i for each $i = 1, 2, \ldots, m_p$; calculate $\{k^r_i\}_i$.
2. Guess the rank order of the elements of L.
3. Solve the $2m_p^2 m_\pi^2 + 2m_p m_\pi$ equations given by (55) for $\{J'_{ij}(n^*_{kl})\}_{i,j,k,l}$, $\{J'_{ij}(\bar{n}_{kl})\}_{i,j,k,l}$, $\{{n^*_{kl}}^{\alpha-1}\}_{k,l}$, and $\{\bar{n}^{\alpha-1}_{kl}\}_{k,l}$.
4. Verify whether the order of the solution for $\{n^*_{ij}\}_{ij} \cup \{\bar{n}_{ij}\}_{ij}$ is consistent with the guessed order from step 2. If not, guess a new order and return to step 3.
5. Solve the $2m_p^2 m_\pi^2$ equations given by (54) for $\{J_{ij}(n^*_{kl})\}_{i,j,k,l}$ and $\{J_{ij}(\bar{n}_{kl})\}_{i,j,k,l}$.
6. Verify whether the free entry condition holds with equality for every i. If not, guess a new set of market tightnesses $\{\theta_i\}_i$ and return to step 1.

Again, the algorithm converges rapidly to the correct ordering for L, and the resulting reduced-form equations for the values θ_i can readily be solved by standard nonlinear equation-solving techniques.

4 Quantitative Analysis

I now turn to a quantitative analysis of the properties of the model. Since we observe fluctuations in unemployment and in the vacancy-unemployment ratio, I restrict my discussion to the model with linear preferences and goods-intensive recruiting outlined in Section 3. (The implications for the cross-sectional properties of the equilibrium are the same in either version of the model.) I first describe the calibration strategy, and show that the model is able to match the cross-sectional patterns of employment and employment growth. I then examine the cyclical properties of the model, and compare them to those of the benchmark MP model.

4.1 Calibration Strategy

The calibration strategy for the cross-section of the model is similar to that used in Elsby and Michaels (2011), hereafter EM, which facilitates comparison with their results. I calibrate parameters largely to match moments pertaining to the steady state, and use the cyclical behavior of the model as a test of its fit. The model period is monthly, with discount factor $\beta = 0.96^{1/12} = 0.9966$. I normalize the units of output by setting ϕ^y, the consumption equivalent of the disutility of labor in the production sector, to one. Several calibration targets are standard in the labor

[24] There are $m_p^2 m_\pi^2$ (i, j, k, l)-tuples, each of which generates two equations (one using n^*_{kl} and one using $\bar{n}_{kl}$). An additional two equations arise from tuples of the form (i, j, i, j), where two cases in (55) both provide valid equations.

search and matching literature. I assume that the matching function takes the Cobb-Douglas form $m(\theta) = Z\theta^{-\zeta}$, and set $\zeta = 0.6$, following Petrongolo and Pissarides (2001) and EM. I then choose Z so the worker's probability of finding a job within a month, $f(\theta) = \theta m(\theta)$, takes the value 0.45 when the vacancy-unemployment ratio θ equals 0.72.[25] This requires setting $Z = 0.515$.

I set the cost of vacancy posting, following Silva and Toledo (2009) and EM, so that the cost of a recruitment opportunity, $\gamma/m(\theta)$, is 42% of the average monthly wage. I then choose the bargaining power of workers to be $\eta = 0.5$. Intuitively, η maps monotonically to the replacement ratio (that is, the ratio of the disutility of labor, ϕ^h—or equivalently, the value of leisure—to the average wage of employed workers). To see the intuition, imagine that all workers earned the same wage w and that the separation rate s was constant across workers. In that case, the two steady-state Bellman equations for a worker unemployed at the beginning of the labor market phase in a particular period and for an employed worker would respectively be given by

$$V^u = f(\theta)V^e + (1 - f(\theta))\beta V^u \tag{56}$$

$$V^e = w - \phi^h + \beta s V^u + \beta(1 - s)V^e. \tag{57}$$

The version of (34) appropriate for linear utility implies in steady state that

$$(1 - \beta)V^u = \frac{\eta}{1 - \eta} \cdot \frac{\theta}{1 - f(\theta)} \cdot \frac{\gamma}{m(\theta)}. \tag{58}$$

Equations (56), (57), and (58) can be rearranged to yield that

$$\frac{\phi^h}{w} = 1 - \frac{\eta}{1 - \eta} \frac{1 - \beta(1 - f(\theta))(1 - s)}{1 - f(\theta)} \frac{\gamma/m(\theta)}{w}.$$

Since $f(\theta)$, s, and $(\gamma/m(\theta))/w$ are all calibration targets, it follows that η is montonically related to the replacement ratio. In this simplified model, $\eta = 0.5$ corresponds to a replacement ratio of 0.640. In the full model, dispersion in wages and separation rates makes this calculation not exactly correct; the actual replacement ratio turns out to be 0.629.

Because the choice of $\eta = 0.5$ is arbitrary, and because the replacement ratio is a crucial parameter in the benchmark MP model, I also investigate the effect of choosing $\eta = 0.25$, corresponding to a high replacement ratio of 0.880.

Following EM, I assume that idiosyncratic productivity π takes the form $\pi = \pi^f \pi^x$ where π^f is a fixed effect drawn at entry and π^x is a mean-reverting transitory component. In this representation, the process for π^x governs the distribution of employment growth; for this π^f is irrelevant.[26] I follow

[25] This value follows Pissarides (2009) and EM, but since I do not use data on vacancies as a calibration target, it is simply a normalization. To see this, observe that for $x > 0$, multiplying the scale parameter in the matching function, Z, by $x^{\zeta-1}$ and dividing the vacancy-posting cost parameter γ by x leads to an equilibrium which differs only in that the equilibrium value of θ is multiplied by x and the equilibrium value of $m(\theta)$ is divided by x. This leaves the matching rate for workers, $f(\theta) = \theta m(\theta)$, the cost of hiring for firms $\frac{\gamma}{m(\theta)}$ unchanged.

[26] To see that π^f is irrelevant for the distribution of employment growth, suppose that two firms are born at the same time, one with permanent shock $x > 1$ times the other, and both draw the same sequence of realizations of π^x. Let the optimal sequence of employment choices by the firm with the lower value of π^f by $\hat{n}_t$. Then it is easy

EM again by assuming that the stochastic process for π^x is such that with probability λ, $\pi^x_{t+1} = \pi^x_t$, while with probability $1-\lambda$, π^x_{t+1} is drawn independently of π^x_t (as well as of the aggregate state and of the productivity of other firms) from a Pareto(m_x, k_x) distribution.[27] I choose m_x to normalize the expected value of π^x to one. Finally, I assume that the initial draw for the transitory component of idiosyncratic productivity takes the lowest possible value. This ensures that young firms are not unrealistically large, and that expected employment growth is positive for young firms, consistent with evidence from Haltiwanger, Jarmin, and Miranda (2010).

Because the distribution of π^f is irrelevant for the cross-sectional distribution of log employment growth and for the cyclical properties of aggregate variables, it can be chosen so as to match properties of the cross-sectional distribution of employment. I follow EM in assuming the distribution of π^f is Pareto(m_f, k_f). I choose m_f to match a minimum firm size of 1 worker, and set $k_f = (1-\alpha)$ so that the upper tail of the employment distribution satisfies a power law with exponent $k^f/(1-\alpha) = 1$, consistent with Zipf's law (Axtell, 2001).[28]

I assume that exogenous firm destruction occurs with a monthly probability of 1 percent, but only for those firms in the bottom 9.4 percent of the transitory idiosyncratic productivity distribution.[29] This implies that around 1.05 percent of firms are destroyed annually, a value that is on the low end of the range suggested by the results in Haltiwanger, Jarmin, and Miranda (2010). Note that because all entrant firms begin with the lowest possible value of π^x, this ensures that firm exit falls with firm age.

A key parameter value is α, the returns-to-scale parameter in the production function. The appropriate value depends on the source of decreasing returns in the model. It is standard to take $\alpha = 0.67$, the labor share, but this is not an appropriate motivation when wages are bargained rather than determined competitively and in addition implausible since even if wages were competitive, this parameterization would logically require capital to be fixed for the entire life of the firm. Thinking of decreasing returns as arising from Lucas (1978) span-of-control suggests a much higher value for α. I consider an intermediate value of $\alpha = 0.8$.

I choose the entry cost K so that entering firms make zero profits in the steady state.

There are two parameters remaining to calibrate, namely k_x and λ, respectively governing the dispersion of the transitory idiosyncratic productivity shock and the probability with which this shock changes its value. I use these moments to match two moments reported by EM from the Longitudinal Business Database for 1992-2005, namely that each year 37.2% of firms do not change their employment, while the standard deviation of (log) employment growth is 0.416. As an overidentifying test of the model, I also target by choice of these two parameters a total monthly

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to verify that second firm always optimally sets its employment equal to $x^{\frac{1}{1-\alpha}}\hat{n}_t$. It follows that changes in log employment are identical across the two firms. Note also that both firms will pay the same wages, according to (31). This property is also present in EM but their solution method using approximate aggregation obscures it.

[27] The density of a Pareto(m_x, k_x) distribution is $G(\pi) = k_x m_x^{k_x} \pi^{k_x+1}$ on $[m_x, \infty)$ and zero elsewhere.

[28] Under Zipf's law the moments of firm-level employment are not finite. Truncating the distribution at an arbitrary upper limit resolves this problem and does not affect any of the results I report below.

[29] Since I discretize the probability distribution for π^x using a 31-point grid, this corresponds to the lowest six grid points.

separation rate (due to both the endogenous firing by firms receiving adverse productivity shocks and the exogenous job destruction at closing firms) of 3.12%, consistent with Shimer (2005) as well as EM.

Moving to the stochastic model, I need to parameterize the stochastic process for aggregate productivity, p_t. I use the method suggested by Stokey (2009, Section 2.8) to approximate an Ornstein-Uhlenbeck process. Specifically, I allow for $\log p_t$ to take $2M+1$ possible values, on the equally-spaced grid $(-Md, -(M-1)d, \ldots, Md)$ for $d > 0$. If $\log p_t = \kappa d$ for $\kappa \in \{-M, -M-1, \ldots, M\}$, then $\log p_{t+1}$ takes the value $\log p_t$ with probability $1 - 2M\mu$, $\log p_t - d$ with probability $(M+\kappa)\mu$, and $\log p_t + d$ with probability $(M-\kappa)\mu$; here $\mu \in (0, (2M)^{-1})$ is a parameter. I set $M = 2$, and choose d and μ so that in a long time series, model-generated GDP (aggregated to quarterly frequency and considered as a log deviation from the sample mean) matches the empirical volatility and autocorrelation of US GDP per capita from 1947Q1 through 2010Q4 (log deviation from an HP trend with smoothing parameter 1600).

Finally, I also target the volatility of establishment openings. I obtain seasonally-adjusted data on establishment openings between 1992Q3 and 2010Q2 from the Business Employment Dynamics database of the Bureau of Labor Statistics (BLS), detrend, and regress on detrended GDP per capita. The point estimate for the elasticity of establishment openings with respect to GDP per capita is 1.03, and is statistically significant.[30] To match this, I need to assume that the cost of entry varies with aggregate productivity, with elasticity $\xi = 0.0294$. (If $\xi = 0$, then entry is too volatile in the model).

The resulting parameter values are reported in Table 1. The model is nonlinear and the productivity distribution is discrete, so not all of the calibration targets can be fit precisely. Table 2 shows the fit of the benchmark model.

4.2 Results

4.2.1 Cross-Sectional Properties

I first discuss some properties of the behavior of the model without aggregate productivity shocks. To do this, I fix the aggregate component of productivity at its steady-state value.

Figure 1 shows a representative example of the hiring and firing behavior of a single firm. The blue line indicates the time path of the firm's number of employees, $n(t)$. The green and red lines respectively indicate the hiring target, n_j^*, and the firing target, $\bar{n}_j$, corresponding to the current idiosyncratic productivity state ζ_j at time t. As can be seen, whenever the hiring target increases

[30] Data on establishment openings predating 1992Q3 at sub-annual frequency is not publicly available. I use GDP per capita as a cyclical measure, rather than output per worker, because, as is well known, the detrended output per worker is much less highly correlated with detrended GDP per capita and much less highly negatively correlated with unemployment after 1990 than before 1990. If I use detrended output per worker as the cyclical indicator, establishment openings are countercyclical but the point estimate is small and not statistically significant. It is beyond the scope of the current paper to speculate on why the cyclical relationship between labor productivity and GDP per capita appears to have changed after 1990. This is a problem for the vast majority of the labor search literature using US data, but is usually hidden because in longer time series the period from 1948 to 1990 in which output per worker was highly positively correlated with GDP per capita drives the results.

	Value	Target		Value	Target
β	0.997	Annual discount factor 0.96	ϕ^y	1	Normalization
ζ	0.6	See text	Z	0.515	Monthly job-finding prob. 0.45
δ	0.01	Annual exit prob. 2.37%	γ	0.415	Recruiting cost 42% of wage
η	0.5	Replacement ratio 0.629	λ	0.887	Annual inaction prob. 37%
m_x	0.757	Normalization: $E\pi^x = 1$	k_x	4.11	Std. dev. $\Delta \log$ employment 0.416
m_f	2.60	Min. establishment size = 1	k_f	5	Zipf's law
α	0.67	See text	K	13.6	See text
M	2	See text	d	0.0161	Variance of GDP per capita
μ	0.0463	Autocorr. of GDP per capita	ξ	0.0294	Volatility of entry

Table 1. Parameterization

Moment	Target	Model
Standard deviation of $\Delta \log$ employment	0.416	0.351
Annual inaction probability	0.372	0.425
Recruiting cost / wage	0.420	0.416
Monthly separation probability	0.0312	0.0361
Standard deviation GDP per capita	0.0169	0.0169
Autocorrelation of GDP per capita	0.842	0.828
Regression coefficient of entry on GDP per capita	1.030	1.030

Table 2. Model fit

to a level that exceeds the size of the firm in the preceding period, hiring occurs to the new value of the hiring target. Similarly, when the firing target falls below the current size of the firm, firing occurs so as to keep the current size no greater than that target. Otherwise, if current employment lies between the two targets, the firm does not change its employment level.

The right panel of Figure 1 shows a histogram for annual employment growth rates g_t. The growth rate g_t is defined as in Davis and Haltiwanger (1990, 1992) by $g_t = \frac{n_t - n_{t-1}}{\frac{1}{2}(n_t + n_{t-1})}$. This shows the substantial measure of firms which do not change employment, together with the large dispersion in employment growth, both of which were targeted by the calibration.[31]

Figure 2 gives two different visualizations of the cross-sectional distribution of firm sizes in the steady state. The left panel shows the firm size distribution conditional on the value of the permanent component of the idiosyncratic shock π^f.[32] The distribution is plotted for the smallest value of π^f, so that the minimum realized employment level is 1. Substantial dispersion arises from the highly-dispersed idiosyncratic shock. The right panel of Figure 2 shows the unconditional

[31] The peak at zero in the Figure differs from the calibration target because it incorporates small nonzero employment growth rates as well as firms exhibiting true inaction.

[32] Recall that two firms with different values of π^f have employment paths that differ only by a constant log increment, so that the empirical counterpart to this measure in a panel data set with a long enough time series dimension is the distribution of employment conditional on average employment. This is not the same as the limiting ergodic distribution for the size of a firm that never gets destroyed and so lives for ever; this is because firms are smaller at birth than thereafter. Firms with log employment below 0.5 are almost all young.

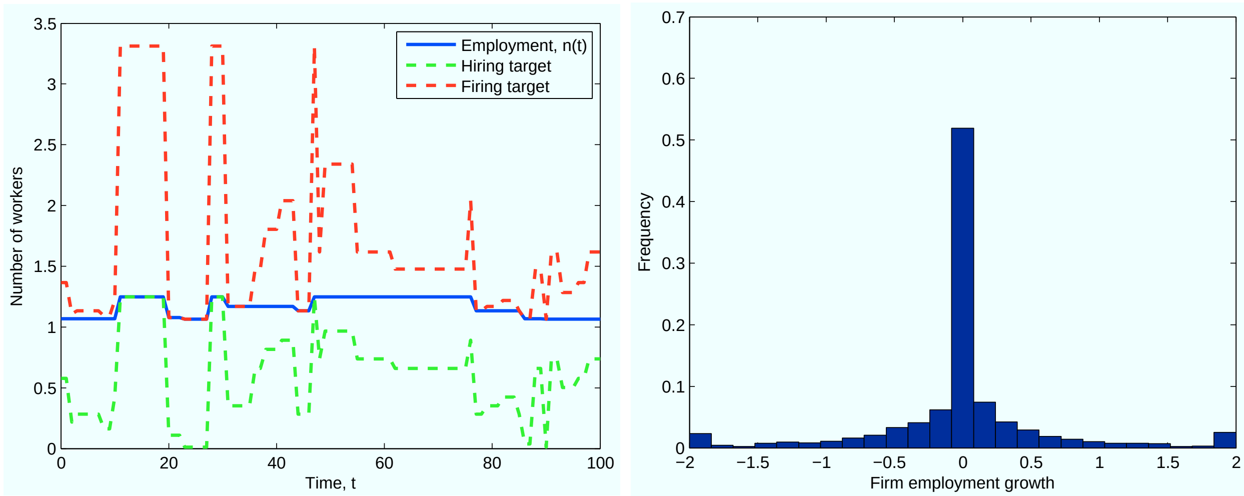

Figure 1. Left panel: Example firm size dynamics. Right panel: Histogram of firm employment growth.

distribution of firm size. Because this distribution obeys Zipf's law in the tail, I plot the log of firm size against the logarithm of one minus the cumulative distribution function. In the left part of the distribution, the distribution does not obey a power law because very small firms must have both a low value for the transitory component of idiosyncratic productivity as well as a low value for the permanent component, so that transitory fluctuations in firm size are more important here.

The model is consistent with a positive cross-sectional correlation between firm size and wages, as Oi and Idson (1999) report for US data. After controlling for the permanent component of productivity, in data generated by the model I regress log wages ($\log w_{it}$) on log employment ($\log n_{it}$) and a constant; the resulting regression relationship is

$$\log w_{it} = 0.385 + 0.062 \log n_{it} + \varepsilon_{it}$$

with $R^2 = 0.357$ and $\mathrm{Var}(\varepsilon_{it}) = 0.00177$.[33] The correlation arises entirely from a positive relationship between wages and employment conditional on the permanent component of the idiosyncratic productivity shock; as already mentioned, a firm with higher permanent productivity hires more workers than, but pays the same wages as, a firm with lower permanent productivity. [34]

The model is also consistent with the empirical positive relationship between wages and firm growth rates. In model-generated data, I regress log wages on firm growth, $g_{it} = \frac{n_{it}-n_{i,t-1}}{\frac{1}{2}(n_{it}-n_{i,t-1})}$, and obtain that

$$\log w_{it} = 0.435 + 0.042 g_{it} + \varepsilon_{it}$$

with $R^2 = 0.137$ and $\mathrm{Var}(\varepsilon_{it}) = 0.00238$. This positive correlation between wages and firm growth

[33] I do not report standard errors because there is no sampling error: the firm size distribution is calculated exactly. The positive relationship is robust to the inclusion of higher moments of log employment.

[34] Accordingly, if I do not control for permanent productivity, the relationship is much weaker:

$$\log w_{it} = 0.408 + 0.015 \log n_{it} + \varepsilon_{it},$$

with $R^2 = 0.090$ and $\mathrm{Var}(\varepsilon_{it}) = 0.00257$.

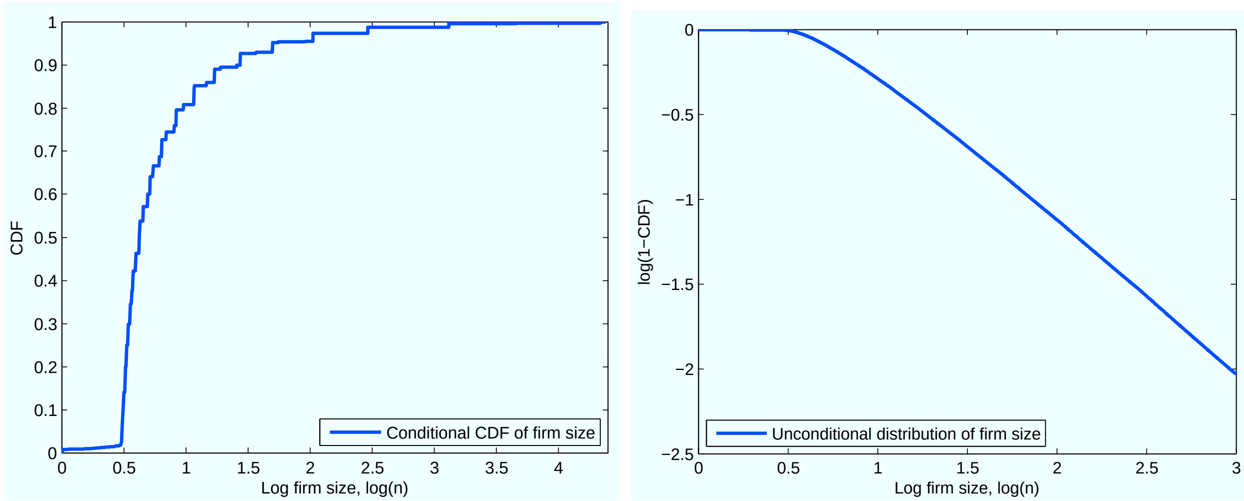

Figure 2. Left panel:
CDF of firm size conditional on permanent component of idiosyncratic productivity.
Right panel: $\log(1 - \text{Unconditional CDF})$ of firm size.

might seem to be counterintuitive, since the wage-setting assumption implies that conditional on productivity, firms with higher employment pay lower wages because the marginal product of the last worker is lower. However, this observation ignores the endogeneity of employment. If there were no labor market friction, all firms would hire and fire up to the point that the bargained wage was equated across firms (and in fact equal to the worker's outside option, as in Stole and Zwiebel, 1996a,b). In the presence of costly recruiting, firms that receive a positive productivity shock do not hire enough workers to drive the wage so far down, so that such firms pay higher wages, at least conditional on employment.[35]

Finally, the model is also consistent with a further stylized feature of the employment growth distribution. Haltiwanger, Jarmin, and Miranda (2010) report that young businesses and startups create a disproportionate fraction of jobs in the US economy. Figure 3 shows the model's predictions for employment growth by firm age. As is apparent, the model replicates this stylized fact successfully: young firms hire many employees at entry as they grow to their target size. Moreover, a firm that initially draws a low value of the transitory component of idiosyncratic productivity expects to increase in size in the future as it draws higher values. As the firm ages, its employment level is more likely to lie in the inaction region, and its growth rate declines. In addition, the difference between employment growth at continuing firms and employment growth at all firms falls with the age of the cohort of firms; this is because firm exit rates fall with firm age, as mentioned above. In addition, because firm size is largely driven by the permanent component of idiosyncratic productivity, the model is also consistent with the finding that there is no systematic relationship between firm sizes and growth rates, also reported by Haltiwanger, Jarmin, and Miranda.

[35]Because I assume that entrant firms draw from the same transitory productivity distribution as incumbents, this mechanism renders the model inconsistent with the fact that younger firms on average pay lower wages (Davis and Haltiwanger, 1991). This could easily be rectified by assuming that the productivity distribution for entrants is dominated (in the sense of first-order stochastic dominance) by that for incumbents.

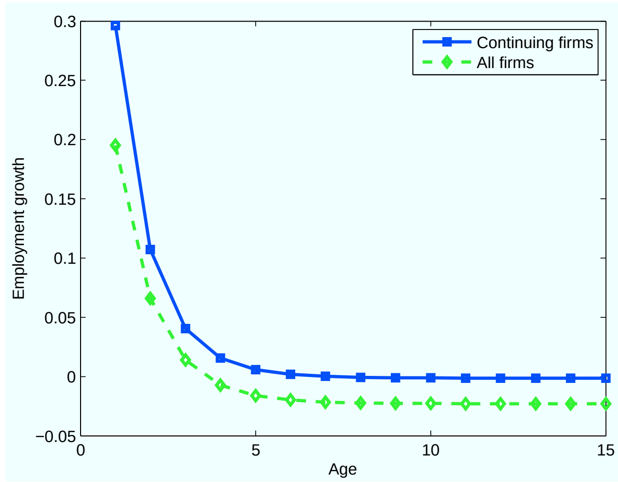

Figure 3. The relationship between net employment growth and firm age.

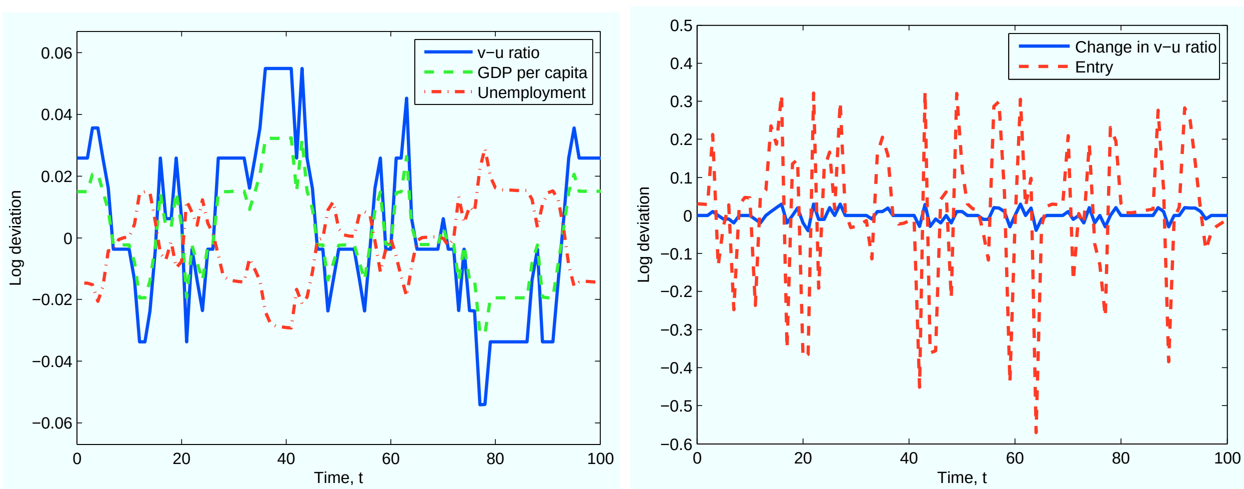

Figure 4. Benchmark calibration.
Left panel: dynamics of market tightness, GDP per capita, and unemployment.
Right panel: dynamics of first-differenced market tightness and entry.

4.2.2 Business Cycle Properties

I now turn to describing the business-cycle features of the model. The left panel of Figure 4 shows a sample path for the dynamics of market tightness, GDP per capita, and unemployment. The first feature to note is that, as established theoretically, the vacancy-unemployment ratio θ is a jump variable whose value is affected only by contemporaneous productivity. Although unemployment is not a jump variable—it exhibits transitional dynamics following changes in aggregate productivity—it is extremely highly correlated with the v-u ratio, with a negative correlation with absolute value 0.951 in the benchmark parameterization. GDP per capita in theory exhibits slightly more complex dynamics, which arise not only from movements of workers between unemployment and employment but also from movement of workers across firms; however, in practice this variable is extremely highly correlated with labor market tightness (correlation greater than 0.999).

U.S. Data		u	v	v/u	e	p	y
Standard deviation		0.136	0.140	0.268	0.035	0.011	0.013
Quarterly autocorrelation		0.923	0.873	0.888	0.037	0.771	0.885
	u	1	-0.925	-0.960	-0.460	0.494	-0.911
	v		1	0.990	0.433	-0.365	0.882
Correlation	v/u			1	0.433	-0.435	0.893
matrix	e				1	-0.097	0.494
	p					1	-0.136
	y						1

Notes: The (seasonally adjusted) unemployment rate u is reported by the BLS using Current Population Survey (CPS) data. Vacancies in the non-farm business sector, v, are obtained from the BLS' Job Openings and Labor Turnover Survey. Entry e is seasonally-adjusted number of opening establishments for the total private sector from the BED. Labor productivity p is measured as seasonally adjusted output per hour in the non-farm business sector, constructed by the BLS from the National Income and Product Accounts (NIPA). GDP per capita y comes from NIPA Table 7.1. All variables are reported as log deviations from an HP trend with smoothing parameters 1600.

Table 3. Business cycle statistics: U.S. data, 2000Q1-2010Q3

The right panel of Figure 4 exhibits the behavior of firm entry. For comparison, also shown is the first difference of the logarithm of the v-u ratio. The very high correlation of these two series is immediately apparent: firm entry spikes upwards on the arrival of a positive aggregate productivity shock, and downwards on the arrival of a negative shock. Following a positive shock, vacancies have to jump so that the vacancy-unemployment ratio jumps to its new value while unemployment does not immediately change. The fraction of this increase in vacancies that arises on the extensive margin (vacancies posted by new entrant firms) as opposed to the intensive margin (vacancies posted by incumbents) is governed by the parameter ξ, the elasticity of the entry cost with respect to aggregate productivity. Raising ξ would reduce the magnitude of the spikes in entry, and would reduce the volatility of entry (in the model, the standard deviation of log entry is 0.101, while in the Business Employment Dynamics database for 1992Q3-2010Q2, the analogous value is 0.034). However, this would come at the cost of worsening the model's fit to the target used to calibrate ξ, namely the elasticity of entry with respect to changes in GDP per capita. Improving the empirical performance of the model along this dimension could be accomplished by introducing a 'time-to-build' feature along the lines of Fujita and Ramey (2007); this might also help with making the vacancy-unemployment ratio respond with a lag to productivity shocks, as in their paper, although the implication is not clear since in my model incumbent firms can also respond to a shock by posting more vacancies. A further examination of this modification is beyond the scope of the current paper.

Table 3 reports standard business cycle statistics for the U.S. economy for the period 2000Q1 through 2010Q3. The short time period is because JOLTS data on vacancies are only available commencing in 2000. In this period, labor productivity is positively correlated with unemployment and negatively correlated with GDP per capita, vacancies and the v-u ratio, so that some terms of the correlation matrix have signs that are unexpected to those more familiar with statistics for longer sample periods.[36]

To investigate the model's ability to match these patterns, I simulate data from the calibrated model. I assume that initially the economy begins with the non-stochastic steady-state joint distribution of firms over employment and idiosyncratic productivity, and with aggregate productivity equal to its mean. I then simulate a time series of 123000 months of data. Each month I draw a new aggregate productivity shock p_t, and allow the sizes of existing firms to evolve according to the optimal hiring and firing policies associated with p_t and with their new draws of idiosyncratic productivity. I then solve for entry so that total vacancies are consistent with the (known) value of the v-u ratio θ_t consistent with aggregate productivity p_t. I aggregate to quarterly frequency, and discard the first 2000 quarters of data to alleviate any dependence on initial conditions. Each subsequent 39 quarters gives me a single sample from the model. For each such sample, I calculate the log deviation from its sample mean for each variable of interest. I then calculate the same statistics as reported in Table 3.[37] Finally, I repeat this procedure for each of the resulting 1000 samples. In Table 4 I then report the mean and standard deviation for each business cycle statistic of interest for the 1000 data points so generated.[38]

The results reported in Table 4 show that the model matches the data very well in some dimensions. There is a clear Beveridge curve, indicated by the strong negative correlation between unemployment and vacancies, although the negative correlation in the data is stronger still. Vacancies and establishment entry are positively correlated, and while all other business cycle variables are quite positively autocorrelated, entry is almost serially uncorrelated. Entry is more volatile than GDP per capita.

However, there are three dimensions in which the model does not do a particularly job of replicating the data, and all of these are revealing. The first is the excess volatility of entry in the model, discussed above. The second, which is the main focus of the current paper, is that the model does not generate volatile unemployment and vacancies. The moment most readers will probably focus on most is the ratio of the standard deviation of the v-u ratio to that of productivity. This ratio is around 1.75 in Shimer's model, and around 1.87 in mine. However, recall that I calibrated the replacement ratio to around 0.63, higher than Shimer's value of 0.4. Using the back-of-the-envelope elasticity formulae provided on page 36 of Shimer's paper, one can see that this change alone would tend to increase the elasticity of unemployment with respect to productivity shocks in the MP model by a factor of around $(1 - 0.4)/(1 - 0.63) \approx 1.62$. Observe that $1.62 \times 1.75 \approx 2.84 > 1.87$, so that when this difference is controlled for, my model actually generates *less* volatility in the v-u ratio than the benchmark MP model! (Of course, there is no distinction between average and

[36]Note also that during this period, which mostly consists of years from the Great Moderation, GDP per capita was slightly less volatile and more strongly autocorrelated than over the period from 1947Q1 through 2010Q4 used to generate the calibration targets reported in Table 2.

[37]Note that I do not remove an HP trend from the (stationary) model-generated data to avoid introducing spurious autocorrelation arising from the introduction of a time trend arising from the filtering process. Statistics calculated using HP filtering are available on request, and do not differ significantly.

[38]Because Table 4 reports statistics for short time series of model-generated data, the volatilities of business-cycle variables are slightly lower than those that would obtain in longer samples, and therefore in particular the volatility of GDP per capita does not exactly match the correlation target reported in Table 2.

Model		u	v	v/u	e	p	y
Standard deviation		0.012	0.015	0.026	0.098	0.014	0.015
		(0.003)	(0.003)	(0.005)	(0.014)	(0.003)	(0.003)
Quarterly autocorrelation		0.791	0.607	0.731	0.036	0.731	0.736
		(0.075)	(0.133)	(0.097)	(0.141)	(0.097)	(0.096)
Correlation matrix	u	1	-0.838	-0.948	0.161	-0.948	-0.952
			(0.060)	(0.023)	(0.063)	(0.023)	(0.021)
	v		1	0.968	0.345	0.968	0.965
				(0.011)	(0.113)	(0.010)	(0.012)
	v/u			1	0.126	1.000	1.000
					(0.093)	(0.000)	(0.000)
	e				1	0.126	0.115
						(0.093)	(0.091)
	p					1	1.000
							(0.000)
	y						1

Table 4. Business cycle statistics: model

marginal product in Shimer's model, so there is no reason to expect the two models to produce similar results here; still, the complete absence of amplification is striking.)

The other main class of differences between the model and the data arise because during the sample period I consider, starting in 2000, empirical labor productivity is weakly negatively correlated with GDP per capita. Since labor productivity is the only driving force in the model, on the other hand, there it is almost perfectly positively correlated with GDP per capita. In addition, labor productivity is empirically positively correlated with unemployment and negatively correlated with vacancies and the v-u ratio, while the opposite is true in the model. If the results for labor productivity are ignored and GDP per capita is taken as the cyclical indicator in both the data and the model, the results correspond much more closely: GDP per capita in both cases is strongly positively correlated with vacancies and the v-u ratio and strongly negatively correlated with unemployment. This shortcoming of the model is shared with almost all models in the labor search literature which take labor productivity as the driving force of business cycles. (For example, comparing Table 4 with Table 3 of Shimer (2005) shows that my model behaves very similarly to the benchmark MP model along these dimensions, and is therefore much more consistent with U.S. data in the period from 1951 to 2003 reported in Shimer's Table 1.) It is beyond the scope of the paper to account for the change in the empirical behavior of labor productivity after the mid-1980s (Galí and Gambetti, 2009; Stiroh, 2009; Galí and van Rens, 2010; Hagedorn and Manovskii, 2011).[39]

[39]One possible reason for the negative correlation of labor productivity with GDP per capita is composition bias (low-marginal product workers may move from employment to non-employment during recessions). Another is financial frictions, which drive a wedge between the marginal benefit to a firm of employing an additional worker and the only component of this present in the model, namely the value of the marginal output.

	u	v	v/u	e	p	y
Benchmark	0.014	0.016	0.029	0.236	0.016	0.017
$\alpha = 0.80$	0.015	0.015	0.029	0.146	0.016	0.017
$\eta = 0.25$	0.033	0.040	0.072	0.187	0.014	0.017

Table 5. Standard deviations of endogenous variables under alternative parameterizations

Finally, note that I essentially imposed the values of the returns-to-scale parameter in the production function and of the bargaining power of workers arbitrarily. Therefore, as a robustness check, I separately consider the effects of reparameterizing the model with less decreasing returns to scale ($\alpha = 0.80$), or with a lower value of workers' bargaining power ($\eta = 0.25$, corresponding to a replacement ratio of 0.840). In each case I recalibrate four parameters that relate to the cross-section (δ, γ, k_x, and λ) and three relating to the time-series behavior of the model (μ, d, and ξ) so as to match the same targets listed in Table 2. Table 5 reports the results of this exercise.

Neither parameterization changes the behavior of the model substantially. When $\eta = 0.25$, unemployment, vacancies, and the vacancy-unemployment ratio are slightly more volatile than in the benchmark model, while if $\alpha = 0.80$, the standard deviation of unemployment, vacancies, and the v-u ratio are almost unchanged from the benchmark although the volatility of entry falls slightly. Autocorrelations and cross-correlations (not reported) are barely changed from Table 4.Again, these changes in volatility are no more than would be suggested by Shimer's (2005) back-of-the-envelope calculation given the replacement ratio.

5 Discussion

As noted in the Introduction, the question of whether a firm-size distribution that responds sluggishly to shocks provides a channel for propagating or amplifying aggregate shocks has not been unanimously resolved in the literature to date. The current paper therefore casts in starker relief what features of the models of Acemoglu and Hawkins (2011), Elsby and Michaels (2011), Fujita and Nakajima (2009) and Kaas and Kircher (2011) are important for generating propagation and amplification. (I hereafter refer to these papers as AH, EM, FN, and KK respectively.)

The first important difference among the five papers is the nature of returns to scale in the recruiting technology. AH and KK both assume that a firm that wishes to recruit progressively more workers in a given period faces a convex cost schedule for doing so, while the current paper, along with EM and FN, assumes a constant returns vacancy-posting technology. The propagation found by AH and by KK, but not by the current paper, appears to arise from this feature of the model. In response to an aggregate shock, incumbent firms may change their size target. However, arriving at the new size target instantaneously is very costly, so that at the level of the individual firm, there are slow adjustment dynamics. This generates sluggish aggregate dynamics for the vacancy-unemployment ratio. By contrast, in the model of the current paper, there is no such sluggishness at the level of the individual firm. Moreover, the changing composition of hiring at the intensive and extensive margin over the cycle provides a margin by which the desire of incumbent firms to hire en masse in response to a shock can be accommodated. The resulting aggregate dynamics therefore display no persistence in the vacancy-unemployment ratio.

Propagation arises in EM and in FN for a different reason. There, the vacancy-posting technology does exhibit constant returns, but the crucial difference with the current paper is that the number of firms is fixed. Consider the response to a negative productivity shock. In the model of the current paper, entry falls sharply on the arrival of the shock, with vacancy-posting by incumbents rising to take up some of the slack. Wages fall as incumbent firms increase in size, so their marginal product decreases. Once wages have fallen, entry by new firms returns nearly to its pre-shock level. In the absence of an entry margin, the response is different, however. Suppose that the vacancy-unemployment ratio were to fall instantly to the post-shock quasi-steady state level. Then, because unemployment rises during the transition to the new quasi-steady state, vacancies would also have to fall on impact and then rise thereafter. However, on impact of the shock, wages fall as the outside option of being unemployed falls with the job-finding rate and with the fall in aggregate productivity, so that incumbent firms instead wish to increase vacancy posting at the arrival of the shock. This inconsistency means that the fall in vacancy-posting on impact must be more gradual, and this leads to persistence.

The crucial role of how precisely the entry process is modeled for the cyclical properties of models with frictional labor markets applies also beyond the context of multi-worker firms. Coles and Moghaddasi (2011) study a model in which entrepreneurs can only post vacancies after a time-consuming construction period has elapsed. They show that this allows for larger and more persistent fluctuations in unemployment and in market tightness than in the benchmark Mortensen-Pissarides model. This is consistent with the relatively volatile and persistent fluctuations in employment and market tightness in EM and FJ relative to the current paper, to AH, and to KK.

Another difference between EM and FN and the three other papers mentioned is that those authors find substantial amplification of productivity shocks. Of course, some of the difference in this respect arises because of the surplus of average labor productivity over the opportunity cost of employment in these two papers. (The replacement ratio in EM is around 0.85, and around 0.83 in the benchmark calibration of FN.) However, as the sensitivity analysis in this paper made clear, such a replacement ratio is not sufficient to generate large fluctuations in employment by itself. The more important assumption that drives the results of those two papers is the absence of an extensive margin of adjustment. This works to generate increased amplification, due to the role of the wage bargaining assumption. Following a positive productivity shock, not only do firms wish to hire more workers simply because of the direct increase in profitability due to the shock, but also because the rise in wages for all workers is offset as larger firms pay lower wages conditional on productivity. As the average size of firms increases following the shock, the rise in wages due to the productivity shock is partially offset. This dampening effect on the elasticity of wages due to the productivity shock acts as would an exogenously sticky wage, increasing the response of employment to the shock. By contrast, in the current model, the adjustment to a positive productivity shock occurs largely on the extensive margin: more establishments enter, and the size of incumbent firms barely changes. Thus, this second feedback channel is absent.

6 Conclusion

The key contributions of this paper are twofold. The first is theoretical. I solve, nearly in closed form, a large-firm bargaining model in the presence of a rich structure of cross-sectional heterogeneity. The second is more quantitative. I show that in this class of models, the ability to match cross-sectional features of the employment and employment growth distribution has no implications for the response of the model to aggregate shocks. Indeed, if preferences take a balanced-growth form and recruiting is time-intensive, then no matter what the cross-sectional features of the model in steady state, there will be no fluctuations in unemployment in response to an aggregate productivity shock! In the more familiar case with linear preferences and goods-intensive recruiting, the responses of the key aggregate variables described by the MP model are not significantly different in my model. The presence of large firms and of a rich structure of cross-sectional heterogeneity need not amplify or propagate business-cycle shocks. To propagate shocks, it appears that either convex costs of recruiting or shutting down the extensive margin of adjustment to shocks is required. To amplify shocks seems to require that the surplus from employment be small, something which is unsurprising since the work of Shimer (2005) and Hagedorn and Manovskii (2008).

A significant success not just of the current paper, but also of the quantitatively-oriented contributions of Elsby and Michaels (2011) and Kaas and Kircher (2011) is to match the empirical cross-sectional patterns of employment and employment growth well. However, a concern about all three papers is that the calibrational strategy that underlies this success is not well-disciplined: in the calibration, the productivity distribution is essentially a residual. A natural direction for future research in theoretical modeling is to determine to what extent the results of the three papers are dependent on the adjustment cost structure (what would happen in the presence of a fixed cost of adjusting the labor force?) A natural direction for empirical research is to attempt to ground the productivity distribution more directly in the data. Finally, since the key contribution of the current paper is to underline the crucial role of the processes for the entry of new firms and for vacancy creation for the aggregate cyclical properties of multi-worker search models, it would also be very useful to understand more about the empirical properties of these processes. All are necessary for a better understanding of the role of heterogeneous firms in the aggregate labor market.

A Omitted Proofs

This Appendix contains statements of some definitions and results that are not formalized in the text, together with all technical proofs. Some proofs are not yet complete in this preliminary version of the paper.

Proof of Lemma 2. Define

$$T(n) = \frac{1}{\eta} n^{-\frac{1}{\eta}} \int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu)\, d\nu = \frac{\int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu)\, d\nu}{\int_0^n \nu^{\frac{1-\eta}{\eta}}\, d\nu}. \tag{59}$$

From the definition of $\tilde{y}(\cdot)$ in (32), it is apparent that $\tilde{y}(n) = y(n) - nT(n)$. Differentiating establishes also that $\tilde{y}'(n) = (1-\eta)T(n)$ and $\tilde{y}''(n) = \frac{1-\eta}{\eta n}[y'(n) - T(n)]$. Thus, to establish the desired claims about $\tilde{y}(\cdot)$, it is sufficient to show that $y'(n) < T(n) < \frac{y(n)}{n}$ for all n. This establishes directly that $\tilde{y}(n)$ is strictly positive and that $\tilde{y}''(\cdot)$ is strictly negative. That $\tilde{y}'(n) > 0$ follows *a fortiori* because $y'(n) > 0$. The two Inada conditions also follow: as $n \to 0^+$, $T(n) > y'(n) \to +\infty$, and as $n \to +\infty$, $0 < T(n) < \frac{y(n)}{n} \to 0$.

To see that $T(n) > y'(n)$, note that the last expression in (59) shows that $T(n)$ is a weighted average of inframarginal products $y'(\nu)$ for $\nu \in (0, n)$. Since $y(\cdot)$ is strictly concave, $y'(\nu)$ strictly exceeds $y'(n)$ whenever $0 \leq \nu < n$ since $y'(\cdot)$ is a strictly decreasing function; that is, $T(n) > y'(n)$ for all n.

To see that $T(n) < y(n)/n$, write

$$y(n)/n = \frac{\int_0^n y'(\nu)\, d\nu}{\int_0^n d\nu}. \tag{60}$$

Comparing the last expression in (59) with (60), it is apparent that both $y(n)/n$ and $T(n)$ are weighted averages of $y(\nu)$ for $\nu \in (0, n)$. However, the weighting function $\nu \mapsto \nu^{(1-\eta)/\eta}$ in (59) is strictly increasing in ν, while that in (60) is a constant function. Because the integrand $y'(\nu)$ is strictly decreasing in ν, the weighting function that puts relatively higher weights on lower values of ν, that is, the constant function, yields a higher weighted average. This completes the proof. $\square$

Proof of Proposition 1. I show that amongst the allocations which satisfy the hypotheses of Proposition 1 (that is, stationary allocations), I can find one which is an equilibrium. The proof consists of several parts. First, I normalize the problem of a production firm in a stationary allocation to make it independent of aggregate productivity. Second, I construct values of the key endogenous variable $\bar{c}$ and $\bar{\theta}$ that are consistent with equilibrium in the labor market, equilibrium in the goods market, and free entry for production firms. Finally, I show that there are values for the remaining endogenous variables consistent with equilibrium.

Therefore, consider an arbitrary stationary allocation. Observe first that $x(s^t)$ is constant since $\theta(s^t) \equiv \bar{\theta}$, so that from (34) it follows that in a stationary allocation,

$$x(s^t) \equiv \bar{x} = \frac{\bar{\theta}\eta\gamma\phi^r}{(1 - f(\bar{\theta}))(1-\eta)} \tag{61}$$

is a function only of $\bar{\theta}$ and parameters. Note that $\bar{x}$ is strictly increasing in $\bar{\theta}$.

Next, define normalized value functions for firms by dividing equations (4) and (5) by current aggregate productivity p_t, so that the normalized value of a firm after learning the realizations of productivity but before firing or recruitment is

$$\hat{K}(n; s^t, \pi^t) = \max_{f,h \geq 0} \left[-\frac{k^r(s^t)}{p_t} h + K(n - f + h; s^t, \pi^t) \right] \tag{62}$$

and the value after firing and recruitment but before production or wage payments is

$$K(n;s^t,\pi^t) = \pi_t y(n) - n\frac{w(n;s^t,\pi^t)}{p_t} + (1-\delta)\sum_{s^{t+1}|s^t} q_t(s^{t+1})\frac{p_{t+1}}{p_t}\sum_{\pi^{t+1}}\mu(\pi_t,\pi_{t+1})\hat{K}(n;s^{t+1},\pi^{t+1}). \tag{63}$$

By assumption of stationarity, $k^r(s^t) = \bar{k}^r p_t$. Also, from (29),

$$\frac{w(n;s^t,\pi^t)}{p_t} = (1-\eta)(\phi^y + \bar{x})\bar{c} + n^{-\frac{1}{\eta}}\int_0^n \nu^{\frac{1-\eta}{\eta}}\pi_j y'(\nu)\,d\nu. \tag{64}$$

That is, the ratio of wages to aggregate productivity depends only on current employment and idiosyncratic productivity at the firm. Denote this quantity (the right side of (64)) as $\bar{w}(n,\pi_t)$ hereafter. Finally, (14), together with stationarity, implies that

$$q_t(s^{t+1})\frac{p_{t+1}}{p_t} = \beta\frac{P(s^{t+1})c(s^t)}{P(s^t)c(s^{t+1})}\frac{p_{t+1}}{p_t} = \beta\frac{P(s^{t+1})}{P(s^t)}.$$

Substituting all these observations into (62) and (63) establishes that

$$\hat{K}(n;s^t,\pi^t) = \max_{f,h\geq 0}\left[-\bar{k}^r h + K(n-f+h;s^t,\pi^t)\right] \tag{65}$$

and

$$K(n;s^t,\pi^t) = \pi_t y(n) - n\bar{w}(n,\pi_t) + \beta(1-\delta)\sum_{s^{t+1}|s^t}\frac{P(s^{t+1})}{P(s^t)}\sum_{\pi^{t+1}}\mu(\pi_t,\pi_{t+1})\hat{K}(n;s^{t+1},\pi^{t+1}). \tag{66}$$

Now, suppose the solution for $\hat{K}$ and for K is in fact independent of the history s^t. In this case the only terms involving s^t are the weights $\frac{P(s^{t+1})}{P(s^t)}$ on the right side of (66); however, these weights and the entire summation are superfluous since the summand is constant with respect to s^{t+1}. That is, $\hat{K}$ and K are a history-independent solution to (65) and (66) if and only if they satisfy

$$\hat{K}(n,\pi_t) = \max_{f,h\geq 0}\left[-\bar{k}^r h + K(n-f+h,\pi_t)\right] \tag{67}$$

and

$$K(n,\pi_t) = \pi_t y(n) - n\bar{w}(n,\pi_t) + \beta(1-\delta)\sum_{\pi_{t+1}}\mu(\pi_t,\pi_{t+1})\hat{K}(n,\pi_{t+1}). \tag{68}$$

(Here I abuse notation and drop redundant arguments of the two value functions.) (67) and (68) can be combined as the single equation

$$\hat{K}(n,\pi) = \max_{n'}\left[-\bar{k}^r\max(n'-n,0) + \pi\tilde{y}(n') - n'(1-\eta)(\phi^y+\bar{x})\bar{c} + \beta(1-\delta)\sum_{\pi^{t+1}}\mu(\pi_t,\pi_{t+1})\hat{K}(n',\pi_{t+1})\right] \tag{69}$$

where $\tilde{y}(n)$ was defined in (32). The discussion there established the strict concavity of $\tilde{y}(\cdot)$. Also, Blackwell's sufficient conditions apply, so that there is a unique solution to (69) (and therefore to (67) and (68). This solution then solves (65) and (66), and therefore generates a solution to (4) and (5). (It is unique under weak assumptions on the stationarity of the aggregate productivity process, but formalizing this is unnecessary for the current argument.)

Because $\tilde{y}(\cdot)$ is a neoclassical production function, (69) can be viewed as characterizing the problem of a firm with this production function that, in a competitive labor market, maximizes its value subject to paying all its employees a wage of $(1-\eta)(\phi^y + \bar{x})\bar{c}$ and subject to a hiring cost of $\bar{k}^r$. The free entry condition in this modified economy is

$$\sum_{j=1}^{m_\pi} F_\pi^e(\zeta_j)\hat{K}(0,\zeta_j) = \bar{k}^f. \tag{70}$$

In a stationary allocation, (8), (9), and (13) imply that $\bar{k}^f = \kappa\phi^f\bar{c}$ and $\bar{k}^r = \gamma\phi^r\bar{c}/m(\bar{\theta})$.

I now show that there are values of $\bar{c}$ and $\bar{\theta}$ consistent with market clearing in the labor and goods markets and with free entry.

The problem of a neoclassical firm solving (69) is well-understood. The value of such a firm is continuously decreasing in $\bar{\theta}$ and in $\bar{c}$. (To prove the first of these claims more formally, parameterize the solution for different candidate values of $\bar{\theta}$ by $\hat{K}(n,\pi;\bar{\theta})$, and note that if $\bar{\theta}_1 > \bar{\theta}_2$, then $\bar{k}^r(\bar{\theta}_1) > \bar{k}^r(\bar{\theta}_2)$ (it is more expensive to recruit when the recruiter to unemployment ratio is high) and $\bar{x}(\bar{\theta}_1) > \bar{x}(\bar{\theta}_2)$ (wages are higher). A production firm in an economy with $\bar{\theta} = \bar{\theta}_1$ can always follow the policy that is optimal in an economy with $\bar{\theta} = \bar{\theta}_2$. This will achieve a higher value since the cost of the recruiting the firm does is reduced, as are wages. Further optimizing recruitment strategies can only increase the firm's value. Moreover, this increase in the value occurs continuously, in the sense that there is no $\bar{\theta}_1$ at which the value of an entrant, which is given by $\sum_{j=1}^{m_\pi} F_\pi^e(\zeta_j)\hat{K}(0,\zeta_j)$, increases discretely. This is because if $\bar{\theta}_2 \to \bar{\theta}_1$, a firm in the economy with $\bar{\theta} = \bar{\theta}_2$ could also follow the firing and recruitment policies that are optimal in the $\bar{\theta} = \bar{\theta}_1$ economy. A similar argument establishes the other claim.)

Next, the largest value of $\bar{c}$ that is consistent with the free entry condition is obtained when $\bar{\theta} \to 0$, so that $m(\bar{\theta}) \to +\infty$ and $\bar{k}^r \to 0$. In this case, recruitment has zero cost from the perspective of the firm, so that there are no costs of adjusting the firm's labor force, and it sets employment during the production phase of the economy always at the static optimum, which satisfies

$$\tilde{y}'(n) = (1-\eta)\phi^y\bar{c} \tag{71}$$

since $\bar{x} \to 0$ in this case. Because $\tilde{y}(\cdot)$ is a neoclassical production function by Lemma 2, it follows there is an finite upper bound $\bar{c}_1$ which is the largest value consistent with the free entry condition for the firm, and then only if $\bar{\theta} = 0$. On the other hand, as $\bar{c} \to 0^+$, in order to be consistent with the free entry condition, we must have $\bar{\theta} \to +\infty$ so that recruiting becomes expensive, since if $\bar{\theta}$ remains bounded, so does $\bar{x}$, so that the firm faces no other costs of operation as wages $(1-\eta)(\phi^y + \bar{x})\bar{c}$ vanish. Therefore if I write $\psi(\bar{c})$ for the value of $\bar{\theta}$ consistent with (70), then $\psi : (0, \bar{c}_1) \to [0, \infty)$ is a strictly decreasing continuous function with $\lim_{\bar{c}\to 0^+} \psi(\bar{c}) = +\infty$ and $\psi(\bar{c}_1) = 0$.

Next, define a function $\xi : (0, \bar{c}_1] \to [0, \infty)$ as the level of output consistent with firm behavior and labor market clearing in an economy where consumption is $\bar{c}$ and the recruiter-unemployment ratio is $\psi(\bar{c})$. To define this more precisely, first denote by $\bar{G}(n,\zeta_j) \in [0,1]$ the probability a firm has employment n and idiosyncratic shock ζ_j during the production phase of a period. Denote by $g(\bar{c})$ the measure of active firms (after entry and before exogenous destruction within any period) that is consistent with labor market clearing in this environment. $g(\bar{c})$ is uniquely determined by the job-finding probability of workers, $f(\bar{\theta}) = f(\psi(\bar{c}))$, together with the firing behavior of firms, determined as part of the solution to the firm's problem, and the distribution $\bar{G}(\cdot)$.[40] Then total

[40]More precisely, after the recruitment phase of a period, total employment is

$$\bar{e}^f = g(\bar{c}) \sum_{j=1}^{m_\pi} \int_0^\infty n\, dF(n;\zeta_j).$$

output by production firms is

$$\xi(\bar{c}) = g(\bar{c}) \sum_{j=1}^{m_\pi} \int_0^n y(n)\, dF(n, \zeta_j). \tag{72}$$

Since all output is consumed (some after being paid as wages directly, some paid via construction or recruitment firms which make zero profits and operate technologies which use labor as the only input), a necessary condition for equilibrium is that $\xi(\bar{c}) = \bar{c}$. It is apparent that ξ is continuous. Therefore it suffices to show that if $\bar{c} \to 0$ then $\xi(\bar{c}) > \bar{c}$ and if $\bar{c} \to \bar{c}_1$ then $\xi(\bar{c}) < \bar{c}$; the result then follows by the intermediate value theorem. But if $\bar{c} \to 0$, then $\psi(\bar{c}) \to +\infty$, so that workers' job-finding probability $f(\psi(\bar{c}))$ is strictly positive, and therefore so is output. If $\bar{c} \to \bar{c}_1$, then $\psi(\bar{c}) \to 0$, so that workers' job-finding probability $f(\psi(\bar{c})) \to 0$, so that the fraction of employed workers becomes small; moreover, since in this case firm's employment policy is given by (71), employment per firm does not vanish, so that output per employed worker remains bounded. Thus output converges to zero as $\bar{c} \to \bar{c}_1$. This completes the proof of the existence of a $(\bar{c}, \bar{\theta})$ pair that is consistent with clearing the labor and goods markets and with firm free entry.

Construction of the rest of the equilibrium is now routine. In the argument above, the number of entering firms and the optimal hiring and firing policies of firms was constructed; the amount of entry and recruitment is then readily constructed, as is unemployment and the firm size distribution. The normalized cost of a factory is $\bar{k}^f = \kappa \phi^f \bar{c}$ and the normalized cost of hiring a worker is $\bar{k}^r = \gamma \phi^r \bar{c}/m(\bar{\theta})$. The normalized wage in the construction sector is $\bar{w}^f = \phi^f \bar{c}$ from (8), and the normalized wage in the recruitment sector is $\bar{w}^r = \phi^r \bar{c}$ from (9). The normalized wage in the production sector, $\bar{w}(n, \pi)$ is given by the right side of (64). By construction, all firms optimize, and households choose optimally their labor supply to the construction and recruitment sectors. The free entry condition holds with equality. Clearance in the goods market implies that the household chooses consumption and savings optimally. That is, the allocation so constructed is an equilibrium. This completes the proof. □

Lemma 4. *The solution to (48) is unique conditional on the ordering of L and the value of $\bar{k}^r$.*

Proof. The result follows from the structure of the equations (47) and (48). First suppose that for some l, the value of n_l^* is known and consider the set of m_π equations given by (48) for $j = 1, 2, \ldots m_\pi$ with $n = n_l^*$, and assuming that the second case always applies (so that a firm with n_l^* workers never hires or fires). The coefficient matrix for this set of linear equations in $\left\{K_j'(n_l^*)\right\}_j$ is invertible. It also implies that for each j and each j', $\hat{K}_{j'}(n_l^*)$ is increasing in $\pi_j \tilde{y}'(n_l^*) - (1-\eta)(\phi^y + \bar{x})\bar{c}$, and therefore strictly decreasing in n_l^*. For any j for which $n_l^* \notin [n_j^*, \bar{n}_j]$, then replace the equation for

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$\delta \bar{e}^f$ workers lose their jobs to exogenous destruction, and

$$\bar{f} = (1-\delta) g(\bar{c}) \sum_{j=1}^{m_\pi} \sum_{j'=1}^{m_\pi} \int_0^\infty f(n, \zeta_{j'}) \mu(\zeta_j, \zeta_{j'})\, dF(n; \zeta_j)$$

to firing by continuing firms (here $f(n, \pi)$ denotes the number of workers fired by a firm with employment n and idiosyncratic shock π). Therefore a measure of $\delta \bar{e}^f + \bar{f}$ workers lose their jobs and join the existing $1 - \bar{e}^f$ workers who were not employed in the production phase of the previous period as unemployed, so that unemployment at the beginning of the recruitment phase is

$$\bar{u} = 1 - \bar{e}^f + \delta \bar{e}^f + \bar{f}.$$

Stationarity requires that this equal the measure of workers who find jobs, which is $f(\bar{\theta})\bar{u}$. Equating this to the expression for $\bar{u}$ gives a linear equation in which all terms but one are multiplied by $g(\bar{c})$, which uniquely determines $g(\bar{c})$.

$\hat{K}_j'(n_l^*)$ assuming no hiring and firing with the correct equation arising from (48). This equation is of the form $\hat{K}_j'(n_l^*) = s$ for some constant s, and so adducing it does not affect the full rank of the coefficient matrix, nor the property that $S_{j'}(n_l^*)$ is strictly decreasing in n_l^* for any j' such that $n_l^* \in [n_{j'}^*, \bar{n}_{j'}]$. Repeating this step as many times as necessary establishes that there is a unique solution for $\left\{\hat{K}_j(n_l^*)\right\}_j$, with the same comparative static property in n_l^* just mentioned. In particular, note that the value of $\hat{K}_l'(n_l^*)$ is decreasing in n_l^*. Also, note that this solution depends only on the rank order of L and the value of n_l^*. One can then proceed to use (48) to obtain the additional equation, so far unused, that $K_l'(n_l^*) = (1-\eta)(\phi^y + \bar{x})\bar{c}$, which is independent of n_l^*; thus it follows that there is also a unique solution for n_l^*. A similar argument then applies to establish that there is also a unique solution for $\left\{\hat{K}_j'(\bar{n}_l)\right\}_j$ and for $\bar{n}_l$. Repeating this argument for all l establishes the claimed uniqueness property. $\square$

Proof of Lemma 3. The firm's problem is unchanged, so that its marginal surplus $J_n(n; s^t, \pi^t)$ from employing a worker at the equilibrium wage $w(n; s^t, \pi^t)$ still satisfies (15), and the marginal value from employing a worker whose current-period wage is w and not $w(n; s^t, \pi^t)$ still satisfies (20). However, the marginal surplus over unemployment of employment at the equilibrium wage to the household is now given by a version of (24) modified to account for the linear utility of consumption,

$$\begin{aligned} S(n; s^t, \pi^t) &= w(n; s^t, \pi^t) - \phi^y - x(s^t) \\ &\quad + \beta(1-\delta) \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \sum_{\pi_{t+1}} \mu(\pi_t, \pi_{t+1}) S(n'(n; s^{t+1}, \pi^{t+1}); s^{t+1}, \pi^{t+1}) \end{aligned} \tag{73}$$

and the marginal value of employment at an arbitrary wage w is the analogous modification of (25), namely

$$\tilde{S}(w, n; s^t, \pi^t) = w - w(n; s^t, \pi^t) + S(n; s^t, \pi^t). \tag{74}$$

The first-order condition for the solution to (26) therefore takes the form that

$$\eta J_n(w(n; s^t, \pi^t), n; s^t, \pi^t) = (1-\eta) S(w(n; s^t, \pi^t), n; s^t, \pi^t), \tag{75}$$

or

$$(1-\eta)\left[w(n; s^t, \pi^t) - \phi^y - x(s^t)\right] = \eta \frac{\partial}{\partial n}\left[p_t \pi_t y(n) - n w(n; s^t, \pi^t)\right]. \tag{76}$$

As in the proof, integrating and setting the constant of integration to zero so that wages remain bounded in small firms completes the proof. $\square$

Proof of Proposition 2. The argument is analogous to that given in the proof of Proposition 1. The only difficulty is to show that values $\{\theta_i\}_i$ can be found so that the free-entry condition (7) holds for all aggregate states. To see this, first consider the simpler case where aggregate productivity is non-stochastic, so that $p_t \equiv \bar{p}$ is constant. In this case I seek to show that there exists an equilibrium in which θ is constant also. According to (34), this requires that $x = \frac{\theta\eta\gamma\phi^r}{(1-f(\theta))(1-\eta)}$ and $k^r = \gamma/m(\theta)$ are also constant. Specializing (53) to this case and dropping superfluous notation for aggregate productivity shows that

$$J_j(n) = \max_{n'} \left[-k^r \max(n'-n, 0) + p\pi_j \tilde{y}(n') - n'(1-\eta)(\phi^y + x) + \beta(1-\delta) \sum_{j'=1}^{m_\pi} \mu(\pi_j, \pi_{j'}) J_{j'}(n')\right]. \tag{77}$$

The right side of (77) is strictly decreasing and continuous in θ, since both x and k^r are continuous and strictly increasing in θ while the other terms do not depend on θ. Moreover, as $\theta \to 0^+$, so that $x, k^r \to 0^+$, it follows that $\sum_{j=1}^{m_\pi} F_\pi^e(\zeta_j) J_j(0) > k^f$. On the other hand, if $\theta \to +\infty$, then wages become arbitrarily large so that the optimal policy for the firm is to fire its entire workforce. Thus $J_j(n) \to 0$ for all j and all n, so that $\sum_{j=1}^{m_\pi} F_\pi^e(\zeta_j) J_j(0) \to 0 < k^f$. The intermediate value theorem then implies that there exists a θ such that (7) holds, and from the monotonicity in θ of the right side of (77) it follows that this solution is unique.

When aggregate productivity is again stochastic, existence of a solution for $\{\theta_i\}_i$ follows as in the proof of Proposition 5 of Kaas and Kircher (2011), since the transition matrix $\lambda(p_t, p_{t+1})$ is strictly diagonally dominant, $J_{ij}(n)$ is differentiable in θ, and all elements of the Jacobian $dJ_{ij}(n)/d\theta_{i'}$ are non-positive. Existence (and uniqueness) of a solution to (54), (55), and (7) then follows from the implicit function theorem in a neighborhood of the non-stochastic solution calculated in the previous paragraph. The only other condition that needs to be checked to ensure that the resulting solution is an equilibrium is that entry should never be negative; however, in the non-stochastic steady state, entry must be constant and strictly positive so as to keep employment constant. Entry after any sequence of realized productivity shocks depends continuously on $\{z_1, \ldots, z_{m_p}\}$, so it follows by continuity that entry is also always positive provided the aggregate shocks are not too large. □

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