Multilateral Matching*

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Abstract

We introduce a matching model in which agents engage in joint ventures via multilateral contracts. This approach allows us to consider production complementarities previously outside the scope of matching theory. We show analogues of the first and second welfare theorems, and, when agents' utilities are concave in venture participation, show that competitive equilibria exist, correspond to stable outcomes, and yield core outcomes. Competitive equilibria exist in our setting even when externalities are present.

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1 Introduction

Multiparty business enterprises take a variety of forms: manufacturing requires complementary inputs for production; consumer products firms coordinate advertising campaigns across multiple publishers, in order to ensure that each consumer is exposed to multiple advertisements[1]; information technology firms collaborate on joint research ventures; cinema productions' actors are concerned with the identities of their costars and directors. In all of these settings, agents' preferences exhibit a form of complementarity: the willingness of two agents to contract with each other may be contingent on those agents' abilities to contract with third parties.

A natural equilibrium notion for multiparty contracting settings is matching-theoretic *stability*, the requirement that no set of agents can profitably recontract. Unfortunately, when agents contract over discrete goods or services, stable outcomes do not necessarily exist in the presence of complementarities across contracts. Consequently, standard matching theory rules out all forms of contractual complementarity, and thus can not be used to study multiparty enterprises.

This paper introduces a novel matching model with transferable utility in which sets of two or more agents may enter into multilateral contracts. Certain forms of complementarity can be expressed through such contracts; in particular, our model embeds a large class of economies with production complementarities. Our key insight is that stable multilateral contracting outcomes do exist when agents contract over continuously divisible quantities, so long as agents' valuations over production and consumption are concave.[2,3] Furthermore, when agents' utilities are concave, stable outcomes directly correspond to competitive equilibria. Conversely, competitive equilibria induce outcomes that are strongly group stable

[1]We thank Preston McAfee for suggesting this example, which is particularly relevant in the sale of Internet display advertisements.

[2]The assumption of concavity is natural in settings with decreasing returns to scale and scope. However, it is violated in settings with fixed costs or increasing returns to scale.

[3]Conversely, we also show a maximal domain result: If any one agent's valuation is not concave, then competitive equilibria can not be guaranteed.

and in the core.[4,5] Analogues of the first and second welfare theorems hold as well, showing in particular that stable outcomes (and competitive equilibria) are efficient. While our basic model disallows contractual externalities, competitive equilibria continue to exist even when such externalities are introduced, although they may not be efficient.

Previous work in matching theory has required (either explicitly or implicitly) that agents interact via bilateral contractual relationships[6]; in medical labor markets, medical students "sell" their services to hospitals (Roth and Peranson (1999)), and in school choice applications, schools "sell" their services to students (Abdulkadiroğlu et al. (2005a,b, 2009)). The restriction to bilateral contracts was material in the previous work as, in order to guarantee the existence of equilibria, agents were required to view contracts as substitutes (see Hatfield et al. (2011) and references contained therein). Meanwhile, it is well-known that equilibria may not exist in discrete matching models with multilateral contracting (see Alkan (1988) and Chapter 2 of Roth and Sotomayor (1990)); we avoid these difficulties by developing a matching model in which contract participation may be varied continuously.

The presence of continuously divisible contracts makes our underlying model similar to models of general equilibrium (Arrow and Debreu (1954); Mas-Colell (1990); Mas-Colell et al. (1995)). However, unlike in general equilibrium theory, we consider production (and other) relationships that are agent-specific: in our framework, a set of agents may share a nonpublic production technology, and that technology may require inputs from specific agents, such as

[4]Note that the corresponsdence between stable outcomes, core outcomes, and competitive equilibria justifies our attention to the competitive equilibrium solution concept, despite the presence of personalized prices in our setting.

[5]Hatfield et al. (2011) obtain analogous results in a setting distinct from ours, in which agents trade via discrete, bilateral contracts. It is known that analogous results do not hold in matching settings without transfers (Echenique and Oviedo (2006); Klaus and Walzl (2009)).

[6]For example, bilateral structure is imposed on relationships in the models of Gale and Shapley (1962), Crawford and Knoer (1981), Kelso and Crawford (1982), Roth (1984), Hatfield and Milgrom (2005), Echenique and Oviedo (2006), Ostrovsky (2008), Hatfield and Kominers (forthcoming), and Hatfield et al. (2011).

human capital.[7,8] Notwithstanding, we do not strictly extend general equilibrium theory, as we impose the requirement that agents' utilities be quasilinear in a numeraire.

The remainder of this paper is organized as follows. In the next section, we illustrate our model with a simple example (concrete production). In Section 3, we present our model in generality. We prove welfare theorems and existence results for competitive equilbria in Section 4; we then analyze the relationship between competitive equilibria, stable outcomes, and the core in Section 5. In Section 6, we present an application: economies with production complementarities embed naturally into the multilateral matching framework. We then extend the multilateral matching framework to include contractual externalities in Section 7. We conclude in Section 8. All proofs are presented in Appendix C.

2 An Illustrative Example

We illustrate our approach with a concrete example, using multilateral matching to model ready-mix concrete production.[9,10] Ready-mix concrete is produced by mixing three complementary inputs—cement, gravel, and sand—in proportions of approximately 1:2:2.[11] All three of these inputs are expensive to transport because of their weights; thus, each of these goods is only sold locally through relationship-specific contracts that incorporate the

[7]For production processes with complementary inputs, it is possible to model a multilateral contract as a collection of bilateral contracts (and in principle use more classical general equilibrium arguments). However, for settings with externalities across contractual partners, such as joint research ventures and entertainment production, multilateral contracting can not be reduced to a model with only bilateral contracting. (To see why entertainment production requires multilateral contracting, note that actors contract with studios, but face externalities derived from the studio's choices of other actors for a given production.)

[8]Note that our framework is conceptually distinct from the setting of the clubs literature (Ellickson et al. (1999, 2001)): In our work, we impose no structure on the set of agents, but require that joint venture participation levels are divisible, while in the clubs literature, participation in a club is a binary decision, but markets are required to be large (and agents are required to be of distinct types).

[9]While in principle this example can be studied using only bilateral contracts (as noted in Footnote 7) using the multilateral matching framework greatly simplifies the analysis. Note also that this example does not use the full generality of our framework—multilateral matching can be used to study economies with externalities across contractual parties, which can not be embedded into bilateral contracting models.

[10]In addition to exemplifying production complementarities which can be studied using multilateral matching, the concrete market has engendered a significant literature in industrial organization; see the work of Syverson (2004, 2008) and Collard-Wexler (2009).

[11]We simplify the discussion by omitting other ingredients, such as water and additives.

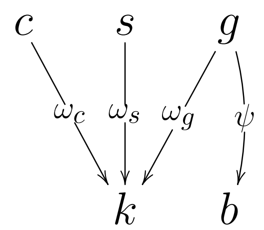

Figure 1: The example economy.

transport costs (Syverson (2008)).

The presence of input complementarities renders concrete production outside the scope of previous matching models. Indeed, previous work has required input substitutability in order to guarantee equilibrium existence (Gul and Stacchetti (1999), Hatfield et al. (2011)). As we illustrate, requiring continuous production adjustment (instead of allowing discrete adjustment as in the previous literature) enables us to relax the substitutability requirement and study industries, such as concrete production, with input complementarities.

It is natural to model the supply structure of a concrete producer k as requiring bilateral relationships ω_c, ω_g, and ω_s for the sale of cubic yards of cement, gravel, and sand, respectively, with suppliers c, g, and s. The gravel supplier g also has an outside option, ψ, to sell to another buyer, b. This economy structure is depicted in Figure 1.

Assuming constant marginal costs of cement and gravel production, and an increasing marginal cost of sand production, we assume the following supplier *valuation functions*:

$$\begin{aligned} v^c(r_{\omega_c}) &= -80 r_{\omega_c} \\ v^g(r_{\omega_g}, r_\psi) &= -25(r_{\omega_g} + r_\psi) \\ v^s(r_{\omega_s}) &= -5 r_{\omega_s} - \frac{1}{16} r_{\omega_s}^2, \end{aligned}$$

where r_χ denotes the number of cubic yards of the good associated with χ delivered.

We assume that concrete is produced with increasing marginal cost by k, and the demand of b is bounded above. We also make the simplifying assumption that concrete production

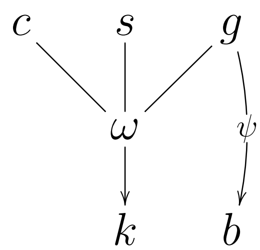

Figure 2: The example economy, reinterpreted as multilateral matching.

requires cement, gravel, and sand in exact 1:2:2 proportions.[12] This gives rise to valuations of the following form:

$$v^k(r_{\omega_c}, r_{\omega_g}, r_{\omega_s}) = 60 \min\left\{\frac{1}{5}r_{\omega_c}, \frac{2}{5}r_{\omega_g}, \frac{2}{5}r_{\omega_s}\right\} - \frac{7}{100}\left(\min\left\{\frac{1}{5}r_{\omega_c}, \frac{2}{5}r_{\omega_g}, \frac{2}{5}r_{\omega_s}\right\}\right)^2$$

$$v^b(r_\psi) = 32 \min\{r_\psi, 50\}.$$

The concavity of the valuation function of k with respect to the amount of concrete produced arises from the fact that k faces an (assumed) downward-sloping demand curve for concrete.[13]

Given the (fixed) proportionality in concrete production, k will never buy disproportionate amounts of cement, gravel, and sand. Thus, we may study the contracting decision of k from the perspective of total concrete production. We represent this by a single *multilateral venture* ω which denotes the production of one cubic yard of concrete using cement, gravel, and sand, as pictured in Figure 2. With this reparameterization, agents' utilities take the

[12]The assumption of exact proportionality is not necessary but simplifies the mathematical exposition.
[13]Alternatively, the same fucntional form could arise from increasing marginal costs of production.

following form:

$$v^c(r_\omega) = -16r_\omega$$
$$v^g(r_\omega, r_\psi) = -10r_\omega - 25r_\psi$$
$$v^s(r_\omega) = -2r_\omega - \frac{1}{100}r_\omega^2$$
$$v^k(r_\omega) = 60r_\omega - \frac{7}{100}r_\omega^2$$
$$v^b(r_\psi) = 32\min\{r_\psi, 50\}.$$

Since relationships are multilateral, the transfer prices corresponding to a relationship must define payments among all parties to the venture (instead of a single transfer from buyer to seller); hence the transfer prices associated with the venture ω are represented by a vector p_ω such that

$$p^k_\omega + p^c_\omega + p^g_\omega + p^s_\omega = 0.$$

Similarly, $p^b_\psi + p^g_\psi = 0$. Agents' utilities are assumed to be quasilinear in transfers. Given this formulation of prices and utilities, our definition of *competitive equilibrium* is natural: A competitive equilibrium consists of an allocation $r = (r_\omega, r_\psi)$ and a price matrix p such that r is utility-maximizing for every agent given p.

We now construct a competitive equilibrium of our economy; generalizations of this construction show that a competitive equilibrium exists for arbitrary concave valuations (Theorem 3). Our adaptatation of the first welfare theorem to this environment (Theorem 1) shows that all competitive equilibria are efficient in our model, and so we begin by identifying the efficient allocation. Aggregate welfare is given by

$$v^c(r_\omega) + v^g(r_\omega, r_\psi) + v^s(r_\omega) + v^k(r_\omega) + v^b(r_\psi);$$

this is maximized at $(\hat{r}_\omega, \hat{r}_\psi) = (200, 50)$. We now construct a price matrix to support this allocation in competitive equilibrium, demonstrating the second welfare theorem in our environment (Theorem 2). Competitive equilibrium pricing must render $\hat{r}$ individually optimal for each agent; hence we set transfer prices associated with the multilateral venture ω equal to the marginal utility of each agent for an additional unit of production at the efficient allocation; a simple computation shows that $(p^k_\omega, p^c_\omega, p^g_\omega, p^s_\omega) = (32, -16, -6, -10)$. These prices are guaranteed to sum to zero by the fact that $\hat{r}$ is efficient, from whence it follows that the social marginal utility of adjusting r_ω must vanish. Similarly, we have that $(p^b_\psi, p^g_\psi) = (25, -25)$.[14,15] Note that $\hat{r}$ and p together comprise the unique competitive equilibrium in this model.[16]

Furthermore, the competitive equilibrium above is *stable* in the matching-theoretic sense: No firm desires to unilaterally drop any venture $\chi \in \{\psi, \omega\}$ and associated transfer payments, and no set of firms wish to renegotiate venture participation levels and transfers. This fact can be shown directly by computation, or as a special case of our Theorem 7.

3 Model

In this section we introduce our general model of multilateral matching. As we demonstrate in Section 6, a large class of economies with production complementarities may be embedded into the multilateral matching framework; this class includes the economy discussed in the previous section.

There is a finite set I of *agents*, and a finite set Ω of *ventures*. Each venture $\omega \in \Omega$ is associated with a set of at least two agents $a(\omega) \subseteq I$; there may be several ventures associated with the same set of agents.[17] For $\Psi \subseteq \Omega$, we denote by $a(\Psi) \equiv \cup_{\psi \in \Psi} a(\psi)$ the set of agents associated with ventures in Ψ, and denote by $\Psi_i \equiv \{\psi \in \Psi : i \in a(\psi)\}$ the set

[14] Since the valuation function of b is not differentiable, subgradient calculations are needed in the computation of p_ψ; for details, see the proof of Theorem 2.

[15] This price matrix corresponds to prices of 80, 15, and 25 per cubic yard for cement, gravel, and sand, respectively.

[16] In our general model, the competitive equilibrium is always unique when all valuation functions are continuously differentiable and strictly concave.

[17] Mathematically, the set of agents and the set of ventures define a multi-hypergraph, where each agent is a node of the graph and each venture is a hyperedge; a hyperedge generalizes the notion of an edge to allow for an arbitrary number of endpoints, instead of just two.

of ventures in Ψ associated with agent i.

A venture may represent production (of a good such as concrete, as in Section 2), a joint research program, or any other multi-agent endeavor for which participation is continuously adjustable. The possibility of multiple ventures between a given set of agents allows us to encode production processes that do not require fixed input proportions.

We denote by $r_\omega \in [0, r_\omega^{\max}]$ the chosen allocation of investment in venture $\omega \in \Omega$ by the agents in $a(\omega)$; for instance, as in our example in Section 2, if the venture ω is between a supplier of cement, a supplier of gravel, a supplier of sand, and a producer of concrete, r_ω may paramaterize the number of cubic yards of concrete produced. As the notation suggests, we assume that participation in each venture $\omega \in \Omega$ is bounded by some finite bound $r_\omega^{\max} \in \mathbb{R}_{\geq 0}$.

Each agent $i \in I$ has a continuous valuation function $v^i(r)$ over ventures, where the vector $r \equiv (r_\omega)_{\omega \in \Omega}$ is an *allocation* which indicates the investment in each venture $\omega \in \Omega$. Many of our results rely on the assumption that the valuation functions v^i are *concave* in venture participation; this assumption is natural when firms face capacity constraints or when their production technologies exhibit decreasing returns to scale.[18] We assume that v^i is unaffected by ventures to which i is not a party, i.e., $v^i(r_\omega, r_{-\omega}) = v^i(\tilde{r}_\omega, r_{-\omega})$ for all ω such that $i \notin a(\omega)$. We relax this assumption in Section 7 in order to consider contracting externalities.

As illustrated in Section 2, the definitions presented—multilateral ventures and valuation functions—allow us to model production processes with fixed proportions. In fact, these defintions are quite flexible. In addition to proportional production, they can used, for example, to model a Cobb-Douglas production function: To see this, let $I = \{i, j, k\}$ and $\Omega = \{\psi, \omega\}$ with $a(\psi) = \{i, j\}$ and $a(\omega) = \{i, k\}$. If $v^i(r) = (r_\psi)^{\mathfrak{a}}(r_\omega)^{\mathfrak{b}}$ where $\mathfrak{a}, \mathfrak{b} \in [0, 1]$ and $\mathfrak{a} + \mathfrak{b} \leq 1$, then the production technology used by agent i has Cobb-Douglas form.

[18]Unfortunately, the concavity of agents' valuations may depend upon the specification of the venture set Ω. The issue of how contractual language interacts with agents' preferences arises throughout matching theory (see Hatfield and Kominers (2010)).

A venture $\omega \in \Omega$ only represents the nonpecuniary aspects of a transaction between the members of $a(\omega)$. The purely financial aspects of venture ω are represented by a vector p_ω, where p^i_ω is the transfer *price* per unit of the venture that agent i pays in order for the venture ω to transact; this transfer price may be negative if i receives compensation from the other agents in the venture. For any agent $j \notin a(\omega)$, we use the convention that $p^j_\omega \equiv 0$. Furthermore, ventures in and of themselves do not create or use the numeraire; hence $\sum_{i \in I} p^i_\omega = 0$ for all $\omega \in \Omega$. We denote by $p \equiv (p^i_\omega)_{i \in I, \omega \in \Omega}$ the matrix for which p^i_ω is the per-unit transfer from agent i when he engages in venture ω.

An allocation r along with a price matrix p together define an *arrangement* $[r; p]$. The *utilty function* $u^i([r;p])$ of an agent i is quasilinear over ventures and transfer prices, hence it can be expressed in the form

$$u^i([r;p]) \equiv v^i(r) - p^i \cdot r.$$

Given prices p, we define the *demand correspondence* $D^i(p)$ for agent i as

$$D^i(p) \equiv \underset{0 \le r \le r^{\max}}{\arg\max}\, u^i([r;p]).$$

Since any two allocations which differ only on ventures to which i is not a party provide the same payoff to i, $u^i([r;p])$ does not depend on the size of r_ω for any $\omega \in \Omega - \Omega_i$. Hence, the demand correspondence $D^i(p)$ has the feature that if $(r_{\Omega_i}, r_{\Omega - \Omega_i}) \in D^i(p)$, then $(r_{\Omega_i}, \check{r}_{\Omega - \Omega_i}) \in D^i(p)$ for all $\check{r}_{\Omega - \Omega_i}$ such that $0 \le \check{r}_{\Omega - \Omega_i} \le r^{\max}_{\Omega - \Omega_i}$. We adopt this somewhat unintuitive convention—which typically makes $D^i(p)$ very large—so that we may define the natural demand correspondence for the entire economy as

$$D(p) \equiv \bigcap_{i \in I} D^i(p),$$

which exactly characterizes the levels of investment in each of the (joint) ventures at which all agents' demands are satsified given prices p.

A *contract* x is comprised of a venture $\omega \in \Omega$, a size of that venture $r_\omega \in [0, r_\omega^{\max}]$, and a transfer vector $s_\omega \in \mathbb{R}^{|I|}$ (where we set $s_\omega^j = 0$ for all $j \notin a(\omega)$, maintaining the convention that agents do not receive transfers for ventures to which they are not associated). We study contracts which specify transfers s_ω (instead of per-unit prices p_ω) in order to maintain consistency with the previous literature (e.g., Hatfield et al. (2011)); transfers are generally related to per-unit prices by the formula $s_\omega = r_\omega p_\omega$.

The set of all contracts is

$$X \equiv \left\{ (\omega, r_\omega, s_\omega) \in \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{R}^{|I|} : r_\omega \leq r_\omega^{\max},\ s_\omega^i = 0 \text{ for } i \notin a(\omega), \text{ and } \sum_{i \in I} s_\omega^i = 0 \right\}.$$

For $x = (\omega, r_\omega, s_\omega) \in X$, we let $\tau(x) \equiv \omega$; for $Y \subseteq X$ we let $\tau(Y) \equiv \cup_{y \in Y} \{\tau(y)\}$. Analogously to the notation for ventures, for a contract $x \in X$ we let $a(x) \equiv a(\tau(x))$ and for $Y \subseteq X$ we let $a(Y) \equiv a(\tau(Y))$. Similarly, $Y_i \equiv \{y \in Y : i \in a(y)\}$. We define $\kappa([r;p])$ to be the set of contracts that implement the arrangement $[r;p]$, i.e.

$$\kappa([r;p]) \equiv \{(\omega, \check{r}_\omega, \check{s}_\omega) \in X : \check{r}_\omega = r_\omega > 0 \text{ and } \check{s}_\omega = r_\omega p_\omega\}.$$

A set of contracts $Y \subseteq X$ is an *outcome* if it describes a well-defined participation and pricing plan, i.e. if for any $(\omega, r_\omega, s_\omega), (\omega', \tilde{r}_{\omega'}, \tilde{s}_{\omega'}) \in Y$ such that $(\omega, r_\omega, s_\omega) \neq (\omega', \tilde{r}_{\omega'}, \tilde{s}_{\omega'})$, we have that $\omega \neq \omega'$.[19] For instance, for any arrangement $[r;p]$, the set of contracts $\kappa([r;p])$ is an outcome. For a given outcome Y, we let $\rho(Y)$, defined by

$$\rho_\omega(Y) \equiv \begin{cases} r_\omega & (\omega, r_\omega, s_\omega) \in Y \\ 0 & \text{otherwise,} \end{cases}$$

[19]Without loss of generality, we also impose the requirement that outcomes not include contracts of the form $(\omega, 0, s_\omega)$.

denote the associated allocation vector of venture sizes. Similarly, we let $\pi(Y)$, where

$$\pi^j_\omega(Y) \equiv \begin{cases} \frac{s^j_\omega}{r_\omega} & (\omega, r_\omega, s_\omega) \in Y \\ 0 & \text{otherwise,} \end{cases}$$

denote the matrix of per-unit transfer prices associated to Y. The *utility* from an outcome Y for agent i is then given by

$$u^i(Y) \equiv v^i(\rho(Y)) - \pi^i(Y) \cdot \rho(Y).$$

The *choice correspondence* of agent i is given by

$$C^i(Y) \equiv \underset{\text{outcomes } Z \subseteq Y_i}{\arg\max}\; u^i(Z).$$

4 Competitive Equilibria

We first introduce the *competitive equilibrium* solution concept.

Definition. A *competitive equilibrium* is an arrangement $[r;p]$ such that $r \in D(p)$.

The statement that the arrangement $[r;p]$ is a competitive equilibrium incorporates both individual optimality and market clearing. Individual optimality holds in competitive equilibrium, as each agent i demands the allocation r given the prices p. Furthermore, markets clear in competitive equilibrium, as if an agent $i \in a(\omega)$ demands r_ω at competitive equilibrium prices p, each other agent $j \in a(\omega)$ demands r_ω at those prices.

4.1 Welfare Theorems for Multilateral Matching

In our setting, we obtain results on the relationship between efficient allocations and competitive equilibria that are analogous to the first and second welfare theorems of general equilibrium theory. However, because our setting allows for arbitrarily large transfers of the numeraire, the standard Pareto optimality condition is replaced by (global) efficiency.

An allocation $\hat{r}$ is *efficient* if

$$\hat{r} \in \underset{0 \le r \le r^{\max}}{\arg\max} \sum_{i \in I} v^i(r),$$

i.e., if it maximizes social surplus. Our "First Welfare Theorem" indicates that any competitive equilibrium allocation is efficient.

Theorem 1. *For any competitive equilibrium* $[r; p]$*, the allocation* r *is efficient.*

The proof of Theorem 1 uses standard techniques: For any competitive equilibrium $[r; p]$, suppose that some other allocation $\hat{r}$ delivers strictly greater social surplus than r does. Then, since $\sum_{i \in I} p^i_\omega = 0$ for all $\omega \in \Omega$,

$$\sum_{i \in I} (v^i(r) - p^i \cdot r) = \sum_{i \in I} v^i(r) < \sum_{i \in I} v^i(\hat{r}) = \sum_{i \in I} (v^i(\hat{r}) - p^i \cdot \hat{r}). \tag{1}$$

However, the inequality (1) can only hold if there exists an agent j such that

$$v^j(r) - p^j \cdot r < v^j(\hat{r}) - p^j \cdot \hat{r}.$$

But then $r \notin D^j(p)$.

Our "Second Welfare Theorem" gives a partial converse to Theorem 1.

Theorem 2. *Suppose that agents' valuation functions are concave. Then, for any efficient allocation* r*, there exist prices* p *such that* $[r; p]$ *is a competitive equilibrium.*

While the result of Theorem 2 is familiar, the proof, unlike in general equilibrium settings, relies on arguments from differential algebra. The logic is especially transparent in the case that agents' valuation functions are differentiable: In this case, for an efficient allocation r, let $p^i_\omega \equiv \frac{\partial}{\partial r_\omega} v^i(r)$. It follows from the linearity of the differential operator and the fact that

r is globally optimal that, for all $\omega \in \Omega$,

$$\sum_{i \in I} p^i_\omega = \sum_{i \in I} \frac{\partial}{\partial r_\omega} v^i(r) = \frac{\partial}{\partial r_\omega} \sum_{i \in I} v^i(r) = 0,$$

hence p is a valid price matrix. Furthermore, as each v^i is concave, by the construction of p we have that

$$r \in D^i(p)$$

for each $i \in I$. It then follows immediately that $[r; p]$ is a competitive equilibrium.

4.2 Existence of Competitive Equilibria

An immediate consequence of Theorem 2 is that a competitive equilibrium exists in our setting whenever agents' valuation functions are concave.

Theorem 3. *Suppose that agents' valuation functions are concave. Then there exists a competitive equilibrium. If the agents' valuation functions are strictly concave and continuously differentiable, then there exists a unique competitive equilibrium.*

Concavity of agents' valuation functions and the boundedness of the allocation space imply the existence of an efficient allocation $\hat{r}$. Theorem 2 then shows that there exist prices p such that $[\hat{r}; p]$ is a competitive equilibrium. Note that, as preferences are quasilinear in the numeraire, this argument does not require the fixed-point methods used in general equilibrium theory. The proofs of Theorems 2 and 3 therefore imply a simple algorithm for computing competitive equilibria in our setting.

Our next result shows that the conditions of Theorem 3 are tight—the domain of concave valuations is the maximal domain for which competitive equilibria are guaranteed to exist.

Theorem 4. *Suppose that the valuation function v^i of some agent i is not concave. Then there exist concave valuation functions for the other agents such that no competitive equilibrium exists.*

To demonstrate the intuition behind this result, consider the case where $I = \{i, j\}$, $\Omega = \{\omega\}$, $a(\omega) = \{i, j\}$, and $r_\omega^{\max} = 2$. Let

$$v^i(r_\omega) = r_\omega^2.$$

In this case, $v^i(r_\omega)$ is not concave at $r_\omega = 1$; in fact, $v^i(r_\omega)$ is globally convex. Let

$$v^j(r_\omega) = \begin{cases} 2011 r_\omega & r_\omega \leq 1 \\ 2011(2 - r_\omega) & 1 \leq r_\omega \leq r_\omega^{\max}, \end{cases}$$

which is globally concave. It is clear that the efficient allocation is $r_\omega = 1$. Hence, any competitive equilibrium must be of the form $[(1); (p_\omega)]$. However, for any price p_ω^i, we have that

$$D^i(p) \subseteq \{(0), (2)\}$$

because v^i is globally convex. Hence, no competitive equilibrium exists.

The intuition of the preceding example generalizes to prove Theorem 4: If there is a point at which the valuation function of agent i is not concave, then we construct concave valuation functions for the other agents so that the efficient allocation is at that point. Given that the utility function of agent i is quasilinear in the numeraire, there does not exist a price vector such that it is individually optimal for i to demand an allocation at which his valuation function is not concave. Thus there does not exist a price vector that induces i to demand the allocation that is efficient in the constructed economy. Hence, since by Theorem 1 all competitive equilibria are efficient, no competitive equilibrium exists.

4.3 Comparative Statics

We now prove an intuitive comparative static result: as an individual venture ψ becomes more valuable for the agents in $a(\psi)$, those agents will not choose to participate in ψ less than before.[20]

Theorem 5. *Consider a family of valuation functions $v^i(\cdot;\ell)$ parameterized by ℓ. Suppose that for all $i \in I$, v^i is strictly concave in r for all $\ell \in \mathbb{R}$ and is twice continuously differentiable in r and ℓ. Suppose additionally that for all $i \in I$, all $\ell \in \mathbb{R}$, and some $\psi \in \Omega$,*

$$\frac{\partial^2 v^i(r;\ell)}{\partial r_\psi \partial \ell} \geq 0 \text{ and}$$
$$\frac{\partial^2 v^i(r;\ell)}{\partial r_\omega \partial \ell} = 0 \text{ for all } \omega \in \Omega \text{ such that } \omega \neq \psi.$$

Let $[\hat{r}(\ell);\hat{p}(\ell)]$ be the unique competitive equilibrium in the economy (for the parameter ℓ) implied by Theorem 3. Then,

$$\frac{\partial \hat{r}_\psi(\ell)}{\partial \ell} \geq 0.$$

Note that under the conditions of Theorem 5, the efficient allocation $\hat{r}(\ell)$ is unique. As venture ψ becomes more profitable for the agents in $a(\psi)$, the conditions on r_ψ in the global optimization problem slacken. The implicit function theorem then shows that the efficient level of participation in ψ must increase. Since the competitive equilibrium allocation is efficient, the value of $\hat{r}_\psi$ must therefore also increase.

5 Cooperative Solution Concepts

5.1 Definitions

We now introduce the standard notion of *stability* from the matching literature.[21]

Definition. An outcome A is *stable* if it is:

1. *Individually rational*: for all $i \in I$, $A_i \in C^i(A)$.

[20] Note that we can not characterize how participation in any other venture ξ changes as ψ becomes more valuable, as ψ and ξ may act as either complements or substitutes.

[21] Note that unlike in classical matching theory, we must consider the possibility of indifference between two sets of contracts, and hence use the definition of Hatfield et al. (2011).

2. *Unblocked*: there does not exist a nonempty $Z \subseteq X - A$ such that for all $i \in a(Z)$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$.

Individual rationality of A requires that no agent i prefer to drop some of the contracts in A_i. Unblockedness of A requires that there not exist a new set of contracts Z such that all the agents in $a(Z)$ would strictly prefer to sign all the contracts in Z (and possibly drop some of their existing contracts in A) rather than only sign some (or none) of them.

Closely related to stability is the standard solution concept of cooperative game theory: the *core*.

Definition. An outcome A is in the *core* if it is *coalitionally unblocked*: there does not exist a nonempty $Z \subseteq X$ such that $u^i(Z) > u^i(A)$ for all $i \in a(Z)$.

The definition of the core differs from that of stability in two ways. First, coalitional unblockedness requires that all the agents in $a(Z)$ drop all of their contracts in $A - Z$; this is a more stringent restriction than that of stability, which allows agents in $a(Z)$ to retain previous relationships. Second, coalitional unblockedness does not require that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$ (for all $i \in a(Z)$); rather, it requires only that the weaker condition that $u_i(Z) > u_i(A)$.

Finally, we introduce *strong group stability*, first proposed by Hatfield et al. (2011), which is a stronger solution concept than both the core and stability.

Definition. An outcome A is *strongly group stable* if it is:

1. Individually rational.

2. *Strongly unblocked*: There does not exist a nonempty set $Z \subseteq X$ such that for all $i \in a(Z)$ there exists a $Y^i \subseteq Z \cup A$ such that $Z_i \subseteq Y^i$ and $u^i(Y^i) > u^i(A)$.

Strong group stability is more restrictive than the core—unlike coalitional unblockedness, strong unblockedness does not require agents to drop previous relationships. Additionally, strong group stability is more restrictive than stability, as strong unblockedness does not require that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$ (for all $i \in a(Z)$), as is required by unblockedness, but only that there exists a $Y^i \supseteq Z_i$ such that $u^i(Y^i) > u^i(A)$.[22]

5.2 The Relationship between Cooperative Solution Concepts

The following result is immediate from the definitions.

Theorem 6. *If an outcome Y is strongly group stable, then Y is stable and in the core. Furthermore, all core allocations are efficient.*

In general, there are no relationships between the cooperative solution concepts beyond those in Theorem 6 without additional assumptions on the valuation functions. In particular, suppose $\Omega = \{\psi, \omega\}$, $a(\omega) = a(\psi) = I = \{i, j\}$, and $r_\psi^{\max} = r_\omega^{\max} = 1$. Let the valuation functions of the two agents be given by

$$v^i(r) = 7 \min\{r_\psi, r_\omega\}$$
$$v^j(r) = -6 \min\{r_\psi, r_\omega\}.$$

The unique efficient allocation is $r = (1, 1)$. It follows that the core is given by

$$\{\{(\psi, r_\psi, (s_\psi^i, s_\psi^j)), (\omega, r_\omega, (s_\omega^i, s_\omega^j))\} \subseteq X : r_\psi = r_\omega = 1, 6 \le s_\psi^i + s_\omega^i \le 7\}.$$

However, no core outcome is stable. Suppose, without loss of generality, that $s_\psi^i \ge s_\omega^i$. Since $s_\psi^i + s_\omega^i \le 7$ we have that $s_\omega^i \le \frac{7}{2}$. Then agent j will choose to drop the contract $(\omega, 1, (s_\omega^i, s_\omega^j))$ as it costs him 6 (due to the cost of production) but gains him at most $\frac{7}{2}$ in transfer.

[22] This notion is called strong group stability as it is stronger than both strong stability and group stability. Strong stability (introduced by Hatfield and Kominers (2010)) also required that each Z_i be individually rational. Group stability (introduced by Roth and Sotomayor (1990) and extended to the setting of many-to-many matching by Konishi and Ünver (2006)) required that if $y \in Y^i$ for some $i \in a(Y)$, then $y \in Y^j$ for all $j \in a(y)$, i.e. that the deviating agents agreed on which contracts from the original allocation would be kept after deviation. See Hatfield et al. (2011) for a further discussion.

However, the outcome $\varnothing$ is stable. Consider a blocking set of the form $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi))\}$ or $\{(\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$; since no agent gains benefits or incurs costs from such a set of contracts, each agent is indifferent between this set of contracts and $\varnothing$. For a blocking set of the form $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi)), (\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$ we have that $s^i_\psi + s^i_\omega > -7\min\{r_\psi, r_\omega\}$ as i must choose both contracts; this implies that $s^j_\psi + s^j_\omega < 7\min\{r_\psi, r_\omega\}$. Suppose without loss of generality that $s^j_\psi \geq s^j_\omega$; then $s^j_\omega < \frac{7}{2}\min\{r_\psi, r_\omega\} < 6\min\{r_\psi, r_\omega\}$. Hence j strictly prefers $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi))\}$ to $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi)), (\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$ and so $\{(\psi, r_\psi, (s^i_\psi, s^j_\psi)), (\omega, r_\omega, (s^i_\omega, s^j_\omega))\}$ is not a blocking set.

This example illustrates that, in general, there is no logical relationship between stable and core outcomes.[23] Appendix A gives an example of an outcome that is both stable and core, but is not strongly group stable.

5.3 The Relationship between Stable Outcomes and Competitive Equilibria

We now show that every competitive equilibrium is associated with a stable outcome.

Theorem 7. *Suppose that $[r;p]$ is a competitive equilibrium. Then, $\kappa([r;p])$ is (strongly group) stable and in the core.*

The proof of Theorem 7 is similar to, but more technical than, the proof of Theorem 1 sketched in Section 4.2. If $\kappa([r;p])$ is not individually rational, then $\kappa([r;p])_i \notin C^i(\kappa([r;p]))$, which implies that $r \notin D^i(p)$, so $[r;p]$ is not a competitive equilibrium. If $\kappa([r;p])$ is not strongly unblocked, then there is a set Z such that for all $i \in a(Z)$, there exists $Y^i \supseteq Z_i$ such that $u^i(Y^i) > u^i(\kappa([r;p]))$. Summing over individuals, and using the fact that $\pi^i(\kappa([r;p])) \cdot \rho(\kappa([r;p])) = p^i \cdot \rho(\kappa([r;p]))$, we obtain

$$\sum_{i \in a(Z)} v^i(\rho(Y^i)) - \pi^i(Y^i) \cdot \rho(Y^i) > \sum_{i \in a(Z)} v^i(\rho(\kappa([r;p]))) - p^i \cdot \rho(\kappa([r;p])).$$

[23]Moreover, since no outcome in this example is both stable and in the core, no strongly stable outcome exists.

Since transfers among agents in $a(Z)$ sum to 0, we have that $\sum_{i \in a(Z)} \pi^i_\omega(Y^i) = 0 = \sum_{i \in a(Z)} p^i_\omega$ for each $\omega \in \tau(Z)$. Hence,

$$\sum_{i \in a(Z)} v^i(\rho(Y^i)) - p^i \cdot \rho(Y^i) > \sum_{i \in a(Z)} v^i(\rho(\kappa([r; p]))) - p^i \cdot \rho(\kappa([r; p])).$$

But then there must exist $j \in a(Z)$ such that $u^j(\kappa([\rho(Y^j); p])) > u^j(\kappa([r; p]))$, and hence

$$r = \rho(\kappa([r; p])) \notin D^j(p),$$

implying that $[r; p]$ is not a competitive equilibrium.

An immediate corollary of Theorems 3 and 7 is the existence of strongly group stable outcomes for concave valuation functions.

Corollary 1. *Suppose that agents' valuation functions are concave. Then a (strongly group) stable outcome exists.*

The converse of Theorem 7 is not true: not all (strongly group) stable outcomes correspond to competitive equilibria. Consider the case where there are two agents i and j, and two ventures ψ and ω. Suppose that

$$v^i(r) = -4 \max\{r_\psi, r_\omega\}$$
$$v^j(r) = 3 \max\{r_\psi, r_\omega\}.$$

Then $\varnothing$ is a (strongly group) stable outcome; however, no competitive equilibrium exists. As Theorem 1 shows, every competitive equilibrium is efficient, hence any competitive equilibrium must be of the form $[(0, 0); p]$. For any price matrix p, we must have that

$$\min\{p^j_\psi, p^j_\omega\} \geq 3$$

as otherwise agent j will demand positive amounts of ψ or ω. This implies that

$$p^i_\psi + p^i_\omega \leq -6,$$

as $p^j_\psi = -p^i_\psi$ and $p^j_\omega = -p^i_\omega$. Then agent i will demand positive amounts of both ψ and ω; hence no prices will support $r = (0, 0)$ as a competitive equilibrium.

The example above relies on the fact that the valuation function of j is not concave. Our next two results show that the lack of concavity is essential for the example: when all agents have concave valuation functions, every stable outcome corresponds to an efficient allocation, and hence induces a competitive equilibrium.

Theorem 8. *Suppose that agents' valuation functions are concave. Then, for any stable outcome A, the allocation $\rho(A)$ is efficient.*

When all agents' valuation functions are concave, at any outcome A corresponding to an inefficient allocation $r = \rho(A)$, either A is not individually rational or there exists a venture ψ such that the total welfare of the agents in $a(\psi)$ can be increased by adjusting r_ψ to some other value $\tilde{r}_\psi$. The agents in $a(\psi)$ can then choose transfers $\tilde{s}_\psi$ so as to share the surplus from adjusting r_ψ to $\tilde{r}_\psi$. By construction, then, it follows that $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\}$ blocks A. Thus, if A is stable, then $\rho(A)$ is efficient.

Combining Theorems 2 and 8, we immediately obtain the following corollary.

Corollary 2. *Suppose that agents' valuation functions are concave. Then, for any stable outcome A, there exists a price matrix p such that the arrangement $[\rho(A); p]$ is a competitive equilibrium.*

An analogous result holds for core outcomes, since those outcomes are efficient by Theorem 6.

Corollary 3. *Suppose that agents' valuation functions are concave. Then, for any core outcome A, there exists a price vector p such that the arrangement $[\rho(A); p]$ is a competitive equilibrium.*

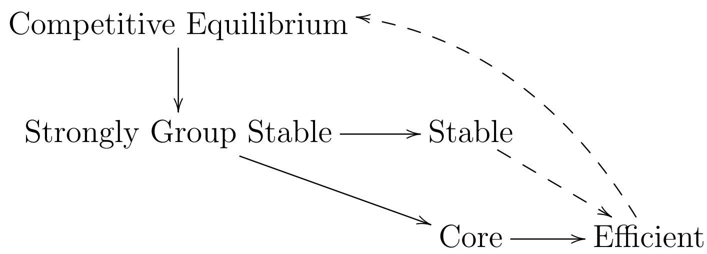

Figure 3: The Relationship Between the Solution Concepts

Note that Corollaries 2 and 3 imply that the underlying allocations of stable and core outcomes can be supported in competitive equilibrium, but do not imply any relationship between the supporting prices and the transfers associated with the original outcomes. These results are analagous to results in general equilibrium theory, where the set of utilities induced by core outcomes is also, in general, larger than the set of utilities induced by competitive equilibria.

We summarize the relationship between the various solution concepts in Figure 3. Solid lines indicate relationships which hold in general; dashed lines represent relationships that hold in the presence of concave valuations.

6 Application: Production Economies with Complementary Inputs

We now demonstrate how our model can be applied to economies where production requires complementary inputs. In Appendix B, we show how the example presented in Section 2 can be described as such an economy.

Consider an economy with a set of agents I in which each agent $i \in I$ has an initial endowment $e^i_{\mathsf{g}} \geq 0$ of each good g; the set of all goods is denoted G. Available to the agents are a set of production processes Ω. Each $\omega \in \Omega$ is represented by a matrix in $\mathbb{R}^{|I|\times|\mathsf{G}|}$, where the value ω^i_{g} indicates that i obtains ω^i_{g} units of good g per unit of process ω executed. If $\omega^i_{\mathsf{g}} > 0$, then good g is an output of the process for agent i; if $\omega^i_{\mathsf{g}} < 0$, then good g is an input of the process for agent i. Note that these processes need not result in the creation or destruction of goods: For example, the process

$$\psi^i_{\mathsf{g}} = \begin{cases} -1 & i = j \text{ and } \mathsf{g} = \mathsf{x} \\ 1 & i = k \text{ and } \mathsf{g} = \mathsf{x} \\ 0 & \text{otherwise} \end{cases}$$

denotes the transfer of one unit of good x from agent j to agent k. In contrast, a linear production process of the form

$$\chi^i_{\mathsf{g}} = \begin{cases} -1 & i = j \text{ and } \mathsf{g} = \mathsf{x} \\ -2 & i = k \text{ and } \mathsf{g} = \mathsf{y} \\ 1 & i = h \text{ and } \mathsf{g} = \mathsf{z} \\ 0 & \text{otherwise} \end{cases}$$

denotes the production of one unit of good z by agent h using one unit of good x from agent j and two units of good y from agent k.[24] We denote by $r_\omega \le r^{\max}_\omega$ the quantity of engagement in process ω; for instance, $r_\psi = 2$ (where ψ is as defined above) indicates the transfer of two units of good x from i to j.

The final consumption of agent i is given by a vector c^i, where

$$c^i_{\mathsf{g}}(r) = e^i_{\mathsf{g}} + \sum_{\omega \in \Omega} r_\omega \omega^i_{\mathsf{g}}.$$

Each agent $i \in I$ has a continuous valuation function over consumption, denoted $\dot{v}^i(c^i)$. Note that since the production processes we have specified are linear, all production costs

[24]Note that while each process specifies the exact amount of each input good to be provided by each agent, it is possible that two processes χ and ω might have the same net inputs and outputs (i.e., $\sum_{i \in I} \omega^i_{\mathsf{g}} = \sum_{i \in I} \chi^i_{\mathsf{g}}$ for all $\mathsf{g} \in \mathsf{G}$) but differ in the idenities of the providers of the inputs and recipients of the outputs (i.e., $\omega^i_{\mathsf{g}} \neq \chi^i_{\mathsf{g}}$ for at least one $i \in I$ and $\mathsf{g} \in \mathsf{G}$). Similarly, mulitple processes may produce the same quantity of good g for i, but use different mixtures of inputs.

are implicitly embedded into agents' valuations.[25] Thus, the valuations $\dot{v}^i$ are concave when production exhibits nonincreasing returns to scale and scope, and agents receive diminishing marginal utility from final consumption.

The economy just described may be reinterpreted as a multilateral matching economy with agent set I, venture set Ω, and valuation functions

$$v^i(r) = \dot{v}^i(c^i(r)).$$

It is clear that v^i is concave if $\dot{v}^i$ is, as then, for all $\mathfrak{a} \in [0,1]$,

$$\begin{aligned}
\mathfrak{a}v^i(r) + (1-\mathfrak{a})v^i(\tilde{r}) &= \mathfrak{a}\dot{v}^i(c^i(r)) + (1-\mathfrak{a})\dot{v}^i(c^i(\tilde{r})) \\
&\geq \dot{v}^i(\mathfrak{a}c^i(r) + (1-\mathfrak{a})c^i(\tilde{r})) \\
&= \dot{v}^i(c^i(\mathfrak{a}r + (1-\mathfrak{a})\tilde{r})) \\
&= v^i(\mathfrak{a}r + (1-\mathfrak{a})\tilde{r}),
\end{aligned}$$

where the inequality follows from the concavity of $\dot{v}^i$ and the subsequent equality follows from the linearity of c^i. Thus agents' valuations are concave whenever their underlying preferences over goods are concave.

The preceding discussion shows that multilateral matching encompasses a large class of economies with production complementarites. Unlike general equilbrium theory, the multilateral matching framework allows us to model economies with agent-specific production, i.e. production that relies on technologies available only to certain agents.[26]

[25] For illustration, consider an economy with a single good g, which can be produced by the agent $i \in I$ via process ω (using inputs from other agents). When using process ω, agent i incurs a convex cost of production $\mathfrak{c}(c^i_{\mathsf{g}})$. After production, i receives linear utility from consuming good g. The valuation $\dot{v}^i$ then takes the form $\dot{v}^i(c^i) = c^i_{\mathsf{g}} - \mathfrak{c}(c^i_{\mathsf{g}})$.

Note that this convention also allows us to consider the case where an agent can produce the same product at two different factories (with appropriate inputs); we specify the set of goods G to include a good for the product of each factory, so that we may model utility from total consumption alongside convex costs of production at each factory.

[26] Agent-specificity may be material, for instance, in economies with intellectual property rights or implicit knowledge gained from learning-by-doing.

The class of economies with production complementarities includes standard examples from manufacturing, such as the assembly of automobiles and computers. Additionally, this class encompasses economies in which production requires many complementary inputs but the value of the output is uncertain; real world examples of such economies include the oil and gas industries.[27] By contrast, the formation of joint research ventures between firms is not adequately modeled in a production economy setting; nonetheless, it is apparent that research venture formation may be modeled using multilateral matching.[28]

7 Extension: Markets with Externalities

In this section, we incorporate externalites into our model by relaxing the assumption that $v^i(r_\omega, r_{-\omega}) = v^i(\tilde{r}_\omega, r_{-\omega})$ for all $\omega \in \Omega$ such that $i \notin a(\omega)$. For clarity, throughout this section we express the valuation function v^i of agent i as $v^i(r_{\Omega_i}; r_{\Omega - \Omega_i})$, to highlight the fact that i treats participation in ventures to which he is not a party as exogenous. Abusing terminology slightly, we will say that $v^i(r_{\Omega_i}; r_{\Omega - \Omega_i})$ is *concave* if it is concave in the venture participation r_{Ω_i} of agent i for all $r_{\Omega - \Omega_i}$. Note that we allow for arbitrary externalities so long as each $v^i(r)$ is continuous in $r = (r_{\Omega_i}, r_{\Omega - \Omega_i})$.

We must now consider demand functions of the form

$$\bar{D}^i(p; \tilde{r}) \equiv \underset{0 \le r \le r^{\max}}{\arg\max} [v^i(r_{\Omega_i}; \tilde{r}_{\Omega - \Omega_i}) - p^i \cdot r],$$

where the additional input $\tilde{r}$ highlights the dependence of the demand of agent i on the venture participation of other agents, $\tilde{r}_{\Omega - \Omega_i}$. As in the case without externalities, the demand correspondence $\bar{D}^i(p; \tilde{r})$ has the feature that if $(r_{\Omega_i}, r_{\Omega - \Omega_i}) \in \bar{D}^i(p; \tilde{r})$, then $(r_{\Omega_i}, \check{r}_{\Omega - \Omega_i}) \in$

[27]To incorporate such economies, it suffices that the valuation functions of agents with uncertain outcomes incorporate an expectation operator.

[28]As we remarked in Footnote 7, examples such as joint research ventures show the true strength of our framework; while economies with production complementarities may be modeled using only bilateral contracting, economic activities which exhibit externalities across contractual partners require the full generality of multilateral contracting.

$\bar{D}^i(p;\tilde{r})$ for all $\check{r}_{\Omega-\Omega_i}$ such that $0 \le \check{r}_{\Omega-\Omega_i} \le r^{\max}_{\Omega-\Omega_i}$; this allows us to define the demand correspondence for the entire economy by

$$\bar{D}(p;\tilde{r}) \equiv \bigcap_{i \in I} \bar{D}^i(p;\tilde{r}).$$

In this context a *competitive equilibrium* is an arrangement $[r;p]$ such that $r \in \bar{D}(p;r)$.

Our next theorem shows that competitive equilibria exist when agents' valuation functions are concave—even in the presence of externalities.

Theorem 9. *Suppose that agents' valuation functions $v^i(r_{\Omega_i}; r_{\Omega-\Omega_i})$ are concave (in r_{Ω_i}). Then a competitive equilibrium exists.*

Unlike the proof of Theorem 3, the proof of Theorem 9 relies on fixed-point methods. In particular, we use Kakutani's fixed point theorem to show that

$$F(\tilde{r}) \equiv \underset{0 \le r \le r^{\max}}{\arg\max} \sum_{i \in I} v^i(r_{\Omega_i}; \tilde{r}_{\Omega-\Omega_i})$$

has a fixed point $\hat{r}$. Arguments analogous to the the proof of Theorem 2 show that there exist prices that support $\hat{r}$ in competitive equilibrium. Note, however, that competitive equilibria in the presence of externalities are generally not efficient.

While this approach allows us to find competitive equilibria in settings with externalities, the added generality comes at a cost: we must use Kakutani's fixed point theorem rather than the differential (and easily computable) method used to prove Theorem 3.

Stable outcomes correspond to competitive equilibria in the presence of externalities if agents, when considering whether to choose to contracts in a blocking set Z, assume that no other contracts will change.[29] If, however, agents are able to accurately predict that contracts in $Z - Z_i$ will transact, then competitive equilibria may not correspond to stable outcomes.[30]

[29] It is clear that if $[r;p]$ is a competitive equilibrium, then $\kappa([r;p])_i$ is individually rational for all $i \in I$.

[30] This distinction in the stability of competitive equilibrium outcomes is analogous to the distinction

To see the difference between the two stability notions, suppose that $I = \{h, i, j, k\}, \Omega = \{\psi, \omega\}$ where $a(\psi) = \{i, j\}$, $a(\omega) = \{h, k\}$, and $r^{\max} = (1, 1)$. Let

$$v^i(r_\psi; r_\omega) = -r_\psi, \qquad v^h(r_\omega; r_\psi) = -r_\omega,$$
$$v^j(r_\psi; r_\omega) = 3r_\psi r_\omega, \qquad v^k(r_\omega; r_\psi) = 3r_\omega r_\psi.$$

These valuations may be interpeted as indicating that i and h sell raw materials to j and k respectively, and that there is only a market for j's product if k sells its product and vice versa. In this setting, there are two competitive equilibrium allocations: $(0, 0)$ and $(1, 1)$. For the allocation $(0, 0)$, the only supporting price matrix is 0; each pair ($\{i, j\}$ and $\{h, k\}$) is unwilling to begin production without the other pair doing so as well, so the set $Z = \{(\psi, 1, (2, -1, 0, 0)), (\omega, 1, (0, 0, 2, -1))\}$ blocks $\varnothing$ if and only if every agent expects all of the other agents to choose their contracts in Z.[31]

8 Conclusion

Our work shows that matching theory can incorporate certain forms of complementarity so long as contracts are continuously divisible. In that case, when agents' valuation functions are concave, competitive equilibria exist, correspond to (strongly group) stable outcomes, and yield core outcomes. Analogues of the first and second welfare theorems hold as well. Even in the presence of externalities, competitive equilibria exist so long as agents' valuations are concave. Further work is needed, however, to identify the correct notion of stability for matching models with externalities and characterize the relationship between that stability concept and the concept of competitive equilibrium.

Previous matching models have obtained conclusions similar to ours—existence and correspondence results for competitive equilibria and stable outcomes (in the presence of quasi-

between Cournot and consistent conjectures (Bresnahan (1981)) equilibria in oligopoly theory.

[31]When writing transfer vectors, we list transfers in the alphabetical order of agents.

linear utility). However, these results have depended crucially on the presence of (full) preference substitutability, which rules out complementarities of the types encoded in our model's multilateral contracts. The key distinction between the prior work and our model is in the structure of the contractual space: whereas previous models have typically allowed agents to contract over discrete participation levels, we require instead that agents be allowed to continuously adjust participation. Our work therefore reveals a tradeoff between modeling assumptions: when contract participation levels are discrete, complementarities must be assumed away, while when they are continuous, some complementarities can be incorporated.

Assuming contractual divisibility seems reasonable in a number of industrial settings, such as chemical synthesis, assembly of durable goods, and automobile manufacturing (Fox (2008, 2010)). It also seems appropriate in the context of online advertising, where billions of impressions are sold. Multilateral matching models allow us to understand the market outcomes in these settings; they may also prove useful for both empirical work and market design applications in settings with complementarities.

Meanwhile, divisibility may not be a reasonable assumption for markets (such as that for large-scale construction) where each individual product is unique and of a discretely specified size. In those markets, other analytical tools are needed: perhaps "large market" effects will facilitate analysis as they have in the setting of matching with couples (Kojima et al. (2010); Ashlagi et al. (2011)).

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A Example Omitted from the Main Text

We present an example of an outcome that is in the core and stable, but is not strongly group stable. Let $I = \{i, j\}$, $\Omega = \{\chi, \psi, \omega\}$, and $a(\chi) = a(\psi) = a(\omega) = I$, $r_\chi^{\max} = r_\psi^{\max} = r_\omega^{\max} = 1$ and let

$$v^i(r) = -2r_\psi - 2r_\omega - 5\min\{r_\psi, r_\omega\} - 11\min\{r_\chi, r_\psi, r_\omega\},$$
$$v^j(r) = 2r_\chi + r_\psi + r_\omega + 11\min\{r_\chi, r_\psi, r_\omega\}.$$

Any outcome of the form $A = \{(\chi, 1, (-q, q))\}$ such that $0 \leq q \leq 2$ is both stable and in the core. However, it is not strongly group stable, as it is not strongly unblocked—to see this, take $Z = \{(\psi, 1, (-6, 6)), (\omega, 1, (-6, 6))\}$.

B The Illustrative Example Revisited

In this appendix, we provide the underlying production economy for the example of Section 2, under the additional assumption that $(r_\omega^{\max}, r_\psi^{\max}) = (400, 210)$. Let $I = \{c, s, g, k, b\}$, $\mathsf{G} = \{\mathsf{c}, \mathsf{g}, \mathsf{s}, \mathsf{k}\}$, and $\Omega = \{\omega, \psi\}$ where the production processes are defined by

$$\omega_{\mathsf{h}}^i = \begin{cases} -\frac{1}{5} & i = c \text{ and } \mathsf{h} = \mathsf{c} \\ -\frac{2}{5} & i = g \text{ and } \mathsf{h} = \mathsf{g} \\ -\frac{2}{5} & i = s \text{ and } \mathsf{h} = \mathsf{s} \\ 1 & i = k \text{ and } \mathsf{h} = \mathsf{k} \\ 0 & \text{otherwise,} \end{cases} \qquad \psi_{\mathsf{h}}^i = \begin{cases} -1 & i = g \text{ and } \mathsf{h} = \mathsf{g} \\ 1 & i = b \text{ and } \mathsf{h} = \mathsf{g} \\ 0 & \text{otherwise,} \end{cases}$$

consumption valuations are given by

$$\begin{aligned} \dot{v}^c(c^c) &= 80 c_{\mathsf{c}}^c, \\ \dot{v}^g(c^g) &= 25 c_{\mathsf{g}}^g, \\ \dot{v}^s(c^s) &= 30 c_{\mathsf{s}}^s - \frac{1}{16}(c_{\mathsf{s}}^s)^2, \\ \dot{v}^k(c^k) &= 60 c_{\mathsf{k}}^k - \frac{7}{100}(c_{\mathsf{k}}^k)^2, \\ \dot{v}^b(c^b) &= 32 \max\{c_{\mathsf{g}}^b, 50\}, \end{aligned}$$

and initial endowments are

$$e^c = (80, 0, 0, 0),$$

$$e^g = (0, 210, 0, 0),$$

$$e^s = (0, 0, 200, 0),$$

$$e^k = (0, 0, 0, 0),$$

$$e^b = (0, 0, 0, 0).$$

(We use the convention that the elements of vector $e^i = (e^i_{\mathsf{c}}, e^i_{\mathsf{g}}, e^i_{\mathsf{s}}, e^i_{\mathsf{k}})$ are given in the order cement, gravel, sand, concrete.) It is clear that the production economy thus illustrated yields the multilateral matching economy presented in Section 2.

C Proofs of Theorems

Proof of Theorem 1

Consider any competitive equilibrium $[r; p]$. Theorem 7 shows that $\kappa([r; p])$ is strongly group stable, hence by Theorem 6 it is in the core and efficient.

Proof of Theorem 2

We consider any efficient $\hat{r}$. By definition, $\hat{r}$ is a solution to the problem

$$\underset{0 \leq r \leq r^{\max}}{\arg\max} \sum_{i \in I} v^i(r). \tag{2}$$

For each agent $i \in I$, denote by $\partial v^i(r)$ the subgradient of v^i at r. Since the v^i are all continuous, $\partial v^i(r)$ is nonempty for all r.

If $p^i \in \partial v^i(\hat{r})$, then $\hat{r}$ is a solution to

$$\underset{0 \leq r \leq r^{\max}}{\arg\max} (v^i(r) - p^i \cdot r)$$

as $v^i(r)$ is concave. Thus, to show the result it suffices to show that for each $i \in I$ there exists $p^i \in \partial v^i(\hat{r})$ such that the matrix p is a valid price matrix (i.e. so that $\sum_{i \in I} p^i_\omega = 0$ for all $\omega \in \Omega$). But this is immediate: Since $\hat{r}$ maximizes (2), we must have[32]

$$0 \in \partial \sum_{i \in I} v^i(\hat{r}) = \sum_{i \in I} \partial v^i(\hat{r});$$

it follows that there exist $p^i \in \partial v^i(\hat{r})$ such that $\sum_{i \in I} p^i = 0$.

[32] Here, for sets $A \subseteq \mathbb{R}^{|\Omega|}$ and $B \subseteq \mathbb{R}^{|\Omega|}$, we denote by $A + B$ the *sumset*

$$A + B = \{a + b : a \in A, b \in B\}.$$

Proof of Theorem 3

Let $\hat{r}$ be a solution to (2); such a solution is guaranteed to exist as the v^i are all continuous and the domain of the maximization problem (2) is compact. The allocation $\hat{r}$ is efficient and hence, by Theorem 2, there exist prices p such that $[\hat{r}; p]$ is a competitive equilibrium.

The uniqueness of the competitive equilibrium in the case where the v^i are strictly concave and continuously differentiable is immediate: Strict concavity implies that there exists a unique $\hat{r}$ solving (2). Furthermore, when the valuation functions v^i are continuously differentiable, the subgradients ∂v^i are single-valued, and hence yield a unique price matrix p in the proof of Theorem 2.

Proof of Theorem 4

We suppose that the function $v^i(r)$ is not concave at the point $\tilde{r} \in \times_{\omega \in \Omega}[0, r_\omega^{\max}]$.[33] For each $j \neq i$, we set

$$v^j(r) = -\mathfrak{m}\|r_{\Omega_j} - \tilde{r}_{\Omega_j}\|,$$

where $\|\cdot\|$ is the Euclidean norm and $\mathfrak{m} \in \mathbb{R}_{\geq 0}$ is sufficiently large that $\tilde{r}$ is the unique solution to the global maximization problem (2).

By construction, $\tilde{r}$ is the only efficient allocation. However, there do not exist prices p for which $[\tilde{r}; p]$ is a competitive equilibrium. Indeed, for any choice of p^i we have

$$\tilde{r} \notin \underset{0 \leq r \leq r^{\max}}{\arg\max}(v^i(r) - p^i \cdot r),$$

as v^i is not concave at $\tilde{r}$. Hence, $\tilde{r} \notin D^i(p) \supseteq D(p)$ for any p. It then follows from Theorem 1 that no arrangement can be a competitive equilibrium.

[33]Note that this implies that there is least one agent in addition to i who shares participation in some venture in Ω_i.

Proof of Theorem 5

From Theorem 1, we know that

$$\hat{r}(\ell) = \arg\max_{0 \le r \le r^{\max}} \sum_{i \in I} v^i(r; \ell). \tag{3}$$

Taking first-order conditions of the constrained maximization problem (3) with respect to r_ω for all $\omega \in \Omega$, we have

$$\sum_{i \in I} \frac{\partial v^i(\hat{r}(\ell); \ell)}{\partial r_\omega} + \lambda_\omega - \mu_\omega = 0$$

along with the constraint conditions

$$\lambda_\omega(0 - r_\omega) = 0,$$
$$\mu_\omega(r_\omega - r_\omega^{\max}) = 0.$$

From the implicit function theorem, we have

$$\frac{\partial \hat{r}(\ell)}{\partial \ell} = -\mathbf{H}^{-1} \frac{\partial}{\partial \ell} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j \in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_{\omega^1}} \\ \vdots \\ \sum_{j \in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_{\omega^{|\Omega|-1}}} \\ \sum_{i \in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_\psi} \end{pmatrix},$$

where $\mathbf{H}$ is the bordered Hessian of our constrained maximization problem and we have denoted $\Omega = \{\omega^1, \ldots, \omega^{|\Omega|-1}, \psi\}$. Hence,

$$\frac{\partial \hat{r}(\ell)}{\partial \ell} = -\mathbf{H}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i \in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_\psi} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \frac{\partial \hat{r}(\ell)}{\partial \ell} = -\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \mathbf{H}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i \in I} \frac{\partial v^i(\hat{r}(\ell);\ell)}{\partial r_\psi} \end{pmatrix}.$$

It follows immediately that $\frac{\partial \hat{r}_\psi(\ell)}{\partial \ell} \geq 0$, as $\mathbf{H}$ is negative semidefinite.

Proof of Theorem 7

Since $[r;p]$ is a competitive equilibrium, we have that for all $i \in I$,

$$\begin{aligned} r \in D^i(p) &= \arg\max_{0 \leq r \leq r^{\max}} (v^i(r) - p^i \cdot r), \\ \kappa([r;p])_i &\in \arg\max_{Z \subseteq Y_i} (v^i(Z) - p^i \cdot \rho(Z)), \\ \kappa([r;p])_i &\in C^i(\kappa([r;p])_i), \end{aligned}$$

where $Y \equiv \{(\omega, \tilde{r}_\omega, \tilde{s}_\omega) \in X : \omega \in \Omega, \tilde{r}_\omega \in [0, r^{\max}_\omega], \tilde{s}_\omega = p_\omega \cdot \tilde{r}_\omega\}$. The last line follows as $\kappa([r;p]) \subseteq Y$. Hence, $\kappa([r;p])$ is individually rational.

Now suppose that $\kappa([r;p])$ is not strongly unblocked, and let Z be a set such that for all $i \in a(Z)$ there exists a $Y^i \subseteq Z \cup \kappa([r;p])$ such that $Z_i \subseteq Y^i$ and $u^i(Y^i) > u^i(\kappa([r;p]))$. For each $i \in a(Z)$, fix a $Y^i \in C^i(Z \cup \kappa([r;p]))$ such that $Z_i \subseteq Y^i$. For all $i \in a(Z)$, we have that

$$u^i(Y^i) > u^i(\kappa([r;p]))$$
$$v^i(\rho(Y^i)) - \pi^i(Y^i) \cdot \rho(Y^i) > v^i(\rho(\kappa([r;p]))) - \pi^i(\kappa([r;p])) \cdot \rho(\kappa([r;p])). \tag{4}$$

Summing (4) over agents $i \in a(Z)$, we obtain

$$\sum_{i \in a(Z)} \big(v^i(\rho(Y^i)) - \pi^i(Y^i) \cdot \rho(Y^i)\big) > \sum_{i \in a(Z)} \big(v^i(\rho(\kappa([r;p]))) - \pi^i(\kappa([r;p])) \cdot \rho(\kappa([r;p]))\big),$$
$$\sum_{i \in a(Z)} \big(v^i(\rho(Y^i)) - p^i \cdot \rho(Y^i)\big) > \sum_{i \in a(Z)} \big(v^i(\rho(\kappa([r;p]))) - p^i \cdot \rho(\kappa([r;p]))\big), \tag{5}$$

where the second inequality follows as

1. $\pi^i(\kappa([r;p])) \cdot \rho(\kappa([r;p])) = p^i \cdot \rho(\kappa([r;p]))$,

2. if $(\omega, \hat{r}_\omega, \hat{s}_\omega) \in Z$, then $a(\omega) \subseteq a(Z)$; hence $\sum_{i \in a(Z)} \pi^i_\omega(Y^i) = 0 = \sum_{i \in a(Z)} p^i_\omega$ as $(\omega, \hat{r}_\omega, \hat{s}_\omega) \in Y^i$ for all $i \in a(Z)$, and

3. if $(\omega, \hat{r}_\omega, \hat{s}_\omega) \in Y^i - Z$ for some $i \in a(Z)$, then $\hat{s}_\omega = p^i_\omega \cdot r_\omega$.

But the inequality (5) implies that for at least one $j \in a(Z)$,

$$v^j(\rho(Y^j)) - p^j \cdot \rho(Y^j) > v^j(\rho(\kappa([r;p]))) - p^j \cdot \rho(\kappa([r;p]))$$
$$v^j(\rho(Y^j)) - p^j \cdot \rho(Y^j) > v^j(r) - p^j \cdot r$$

so that $r \notin D^j(p)$ and, hence, $[r;p]$ is not a competitive equilibrium.

Thus, $\kappa([r;p])$ is strongly unblocked and, hence, is strongly group stable. That $\kappa([r;p])$ is stable and in the core follows from Theorem 6.

Proof of Theorem 8

Consider any stable outcome A. Let $r = \rho(A)$ and $p = \pi(A)$. Let $s^i_\omega = p^i_\omega \cdot r_\omega$ be the transfer payment to agent i as part of the contract $(\omega, r_\omega, s_\omega) \in A$. Suppose that r is not efficient. Then as the v^i are concave, we know that $0 \notin \partial \sum_{i \in I} v^i(r)$. It follows that there exists $\psi \in \Omega$ such that $0 \neq \check{r}_\psi$ for all $\check{r} \in \partial \sum_{i \in I} v^i(r)$. Choose some $\mathring{r} \in \partial \sum_{i \in I} v^i(r)$, let

$$\check{r}_\omega = \begin{cases} \mathring{r}_\psi & \omega = \psi \\ 0 & \text{otherwise,} \end{cases}$$

and let $\tilde{r} \equiv r + \epsilon \check{r}$, with $\epsilon \neq 0$ chosen so that $\sum_{i \in I} v^i(\tilde{r}) > \sum_{i \in I} v^i(r)$ and $0 \leq \tilde{r} \leq r^{\max}$.[34]

Now consider the set $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\}$ where

$$\tilde{s}^j_\psi \equiv \begin{cases} s^j_\psi - (v^j(\tilde{r}) - v^j(r)) + \frac{\sum_{i \in a(\psi)} [v^i(\tilde{r}) - v^i(r)]}{|a(\psi)|} & j \in a(\psi) \\ 0 & \text{otherwise.} \end{cases}$$

Each agent $j \in a(\psi)$ strictly prefers $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\} \cup (A - \{(\psi, r_\psi, s_\psi)\})$ to A. It follows that $(\psi, \tilde{r}_\psi, \tilde{s}_\psi) \in Y$ for each $Y \in C^j(\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\} \cup A)$, and so $\{(\psi, \tilde{r}_\psi, \tilde{s}_\psi)\}$ blocks A.

Proof of Corollary 2

Given Theorem 8, the result follows immediately from Theorem 2, as for any efficient allocation r, we can find prices p such that $[r; p]$ is a competitive equilibrium.

Proof of Theorem 9

We let

$$F(\tilde{r}) \equiv \underset{0 \leq r \leq r^{\max}}{\arg\max} \sum_{i \in I} v^i(r_{\Omega_i}; \tilde{r}_{\Omega - \Omega_i}).$$

[34] Note that since $0 \notin \partial \sum_{i \in I} v^i(r)$, r is not a global maximum, and in particular since $0 \notin [\partial \sum_{i \in I} v^i(r)]_\psi$, there exists some ϵ such that $\sum_{i \in I} v^i(r) < \sum_{i \in I} v^i(\tilde{r})$; it is clear that $\tilde{r}_\psi \in [0, r^{\max}_\psi]$.

Note that by the Theorem of the Maximum, F is non-empty, compact-valued, and upper hemicontinuous. As $\times_{\omega \in \Omega}[0, r_\omega^{\max}]$ is non-empty, compact, and convex, Kakutani's fixed point theorem implies that there exists an $\hat{r}$ such that $F(\hat{r}) = \hat{r}$.

An argument exactly analogous to the proof of Theorem 2 then shows that there exists a price matrix p such that

$$\hat{r} \in \bar{D}^i(p; \hat{r}),$$

hence $[\hat{r}; p]$ is a competitive equilibrium.