The Optimal Design of Rewards in Contests*

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Abstract

Using contests to generate innovation has and is widely used. Such contests often involve offering a prize that depends upon the accomplishment (effort). Using an all-pay auction as a model of a contest, we determine the optimal reward for inducing innovation. In a symmetric environment, we find that the reward should be set to $c(x)/(c'(x)-\beta)$ where c is the cost of producing an innovation of level x and β is the weight attached by the designer to the sum of efforts. In an asymmetric environment with two firms, we find that it is optimal to set different rewards for each firm. There are cases where this can be replicated by a single reward that depends upon accomplishments of both contestants.

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1 Introduction

Using contests to generate innovation has been around for hundreds of years. In the 1700s, the Longitude prize of £20,000 offered by the British Parliament induced John Harrison to invent the marine chronometer (see Sobel, 1996). More recently, the Ansari X-prize was a ten-million-dollar competition created to jump-start the space tourism industry by attracting the attention of the most talented entrepreneurs and rocket experts in the world.[1] Such R&D contests are an example of a competition in which all contestants, including those that do not win any reward (prize), incur costs as a result of their efforts but only the winner gets the reward. In such contests, the designer may often offer smaller prizes for lesser achievements. In fact, while the full longitude prize was given for determining longitude within 30 nautical miles, £10,000 was given for a method for determining longitude within 60 miles, and £15,000 for a method within 40 nautical miles. Another example with smaller prizes is where Netflix offers a prize for improving their movie recommendation system.[2] This prize increases if the improvement is more than 10%.[3]

[1] See www.xprize.org for details.

[2] See www.netflixprize.com.

[3] Other interesting examples include the Methuselah Mouse Prize (see www.mprize.org) for creating a long-lived mouse. If the prize money is z, the oldest previous mouse lived x years and someone creates a mouse that lives $y > x$ years, then they would receive $z \cdot y/(x + y)$. There was also the Schneider trophy (see Eves, 2001) created to inspire

We model a contest as an all-pay auction. When the prize depends upon the result, this is equivalent to having a bid-dependent reward. Such environments have been analyzed before both positively, studying the equilibrium behavior properties and normatively, determining what are optimal contest designs. Environments with complete information have been analyzed from a positive point of view in Kaplan et al. (2003) and Siegel (2009, 2010), the normative point of view was analyzed in Che and Gale (2003) and Fu et al. (2011). Environments with incomplete information were studied from a positive point of view in Kaplan et al. (2002), the normative point of view was investigated in Moldovanu and Sela (2002) and Chen et al. (2008). Similar research was carried out for rent-seeking contests, Nitzan (1994) provided a positive analysis, Franke et al. (2009) provided a normative analysis. Konrad (2009) provides an excellent survey of equilibrium and optimal design in contests.

In this paper, we provide further normative analysis for environments with complete information. We look at the optimal rewards under complete information when the designer cares about both the largest effort and the sum of the efforts by the participants. The designer wishes to maximize this expression net of the rewards paid out. We determine the designer's optimal bid-dependent reward structure to acheive this goal as a function of costs in both symmetric and asymmetric environments.

Interestingly, the solution under symmetry when the designer cares only about the highest effort produces equivalent behavior to that in Che and

aviation design. There was a competition between the fastest seaplanes held 11 times between 1913 and 1931. Each victory won a smaller prize and the full prize of 70,000 Franc prize would be given if the same club won three times in a row. When this happened by an English group (won by a forerunner to the Spitfire), the competition ceased.

Gale (2003) where the firms compete by choosing both effort and price. In our paper, the solution under asymmetry is similar to that under symmetry except the rewards are a firm specific function that firm's costs. One may consider this problematic in the sense that the designer must know which firm is which and bias the contest in favor one of the firms. We address this issue by describing settings where this firm specific reward structure can be replaced by a reward (to the winner) that depends upon both of the firms' efforts. In our setting, we consider a richer class of contests than considered by Che and Gale (2003) and as a result, in some cases, the optimal contest generates higher surplus for the designer than their solution of handicapping one firm.

While in this paper we phrase the problem as designing a research contest, our analysis is applicable to many other scenarios that have such a winner-take-all form. For instance, many races offer prizes to the winners that depend upon time. Also, in a contest to receive a promotion at a company, the firm may set the salary increase with the promotion conditional on the worker's performance. This paper would suggest how to structure these rewards.

Our paper is proceeds as follows. In Section 2, we introduce the general environment with the optimal rewards for symmetric case. Afterwards, in Section 3, we allow for asymmetry between firms. Finally, in Section 4, we present the concluding remarks.

2 Symmetric Environment

2.1 Model

A buyer (designer) desires an innovation. There are n firms that have potential to innovate. Firms can create an innovation of value x (to the designer) for a cost $c(x)$. This value x includes external benefits generated by the contest.[4] We assume $c(0) \geq 0$, $c' \geq 0$, $c'' > 0$, and is common knowledge.[5] Furthermore, we assume there exists an x such that $x > c(x)$. The buyer can design a contest where the reward depends upon the bid of the firm. He does so by choosing a reward function $R(x)$ that depends upon the winning bid (it could be constant). We assume that R must be continuous with $R(0) \geq 0$.[6] The buyer wishes to maximize:

$$E[\max\{x_1, ..., x_n\} + \beta \sum_{i=1}^{n} x_i - R(\max\{x_1, ..., x_n\})].$$

At this point we would like to further motivate our study of contests rather than other mechanisms: One alternative could be to run a Vickrey auction where the firms compete by offering potential innovations and then the winning firm would create the innovation promised. Another could be making a take-it-or-leave-it-offer to a single firm (or multiple firms when

[4] We assume that the designer has the potential to capture all the external benefits accrued to the winner with a contract signed before the contest (such as with the show Pop Idol).

[5] While we assume the designer knows c, we also assume that c is not verifiable in court.

[6] The assumption of continuity of R is natural, since even a discontinuous reward function is equivalent to a continuous reward function with a minimum amount of noise. Consider the case that each x_i has a noise ε that affects the final result. (For instance, there could be a slight wind in the 100 m dash.) In this case, the actual reward would be $\widetilde{R}(x_i) \equiv E[R(x_i + \varepsilon)]$ and is continuous.

$\beta > 0$). Our reasons are as follows. First, there may be external benefits (publicity) for both the designer and winning firm for running a contest.[7] Second, as mentioned in Scotchmer (2004, chapter 2), without a contest, there is a hold-up type problem when the ex-post payment depends upon the firm delivering a future innovation of a specific quality.[8] Finally, in practice, contests are commonly used in a plethora of economic activities, while Scotchmer (2004, chapter 2) points out that to her knowledge (and ours) that a Vickrey auction has never been used in procuring an innovation. Thus, studying the optimal contest is a worthwhile endeavor.

2.2 Analysis

As long as there exists an x such that $R(x) > c(x)$, there is no pure strategy equilibrium.[9] In such a case, however, there will be a symmetric mixed-strategy equilibrium where each firm chooses x according to a cumulative, atomless (except possibly at 0) distribution F.

Proposition 1 *In the optimal design, the buyer sets $R(x) = c(x)/(c'(x)-\beta)$. This generates a surplus of*

$$\frac{n}{n-1}\int_{\max\{c'^{-1}(\beta),0\}}^{c'^{-1}(1)} (xc'(x) - c(x))\frac{c''(x)}{(c'(x)-\beta)^{\frac{n-2}{n-1}}}dx.$$

[7] In 1959, Feynmann offered a prize for the development of a small motor and reducing written text that could fit the encyclopedia on a pin. The innovations themselves were useless, but the challenge provided inspiration for nanotechnology.

[8] It is reasonable to assume that the designer can commit to paying a prize (for instance by setting up a foundation) and avoid a hold-up problem the other way.

[9] When this condition does not hold, the pure-strategy equilibrium has no firm entering and the buyer earning zero surplus.

Proof. The designer's expected surplus can be rewritten as

$$\int (x - R(x))\, dF^n + \beta \cdot n \int x dF.$$

Similar to Kaplan et al. (2003), the firms will have zero expected profits. Since it is a mixed strategy equilibrium, the firms must be indifferent over all x in the support of F. Hence,

$$F(x)^{n-1} R(x) - c(x) = 0.$$

By integrating we get:

$$\int F(x)^{n-1} R(x) dF - \int c(x) dF = 0 \Longrightarrow$$
$$\int R(x) dF^n = n \int c(x) dF.$$

Substituting this into the designer's objective yields

$$\int x dF^n + \beta \cdot n \int x dF - n \int c(x) dF =$$
$$n \int \left(x F^{n-1} + \beta \cdot x - c(x) \right) dF.$$

We can now do a change of variables so that we are integrating over F (rather than x).

$$n \int_0^1 \left(x(F) F^{n-1} + \beta \cdot x(F) - c(x(F)) \right) dF.$$

Now we can independently choose our $x(F)$ to maximize the integrand. Thus, we get $F^{n-1} + \beta = c'(x(F))$ or $F(x)^{n-1} + \beta = c'(x)$. From the zero profit equation of the firm, $F(x)^{n-1}R(x) - c(x) = 0$, the optimal reward is $R(x) = c(x)/(c'(x) - \beta)$. This is true for whenever $c'(x) - \beta > 0$. No one would choose x such that $c'(x) - \beta < 0$ since in equilibrium, there would be no chance of winning. Hence, as long as reward is finite, it will not affect behavior. The expression for the surplus is generated by substitution (for example, $F(x) = (c'(x) - \beta)^{1/(n-1)}$, so $\frac{dF(x)}{dx} = \frac{1}{n-1}(c'(x) - \beta)^{1/(n-1)-1}c''(x))$. Note the lower limit of the integral yielding the surplus is 0 if there is no x such that $c'(x) - \beta = 0$. ■

Remark 1 *While we have thus talked about a single reward for the winner, if there is an $x > 0$ such that $c'(x) - \beta = 0$, the optimal reward would involve infinite rewards. This can be avoided since a designer can implement the optimal reward structure with any two functions L and R such that the winner receives R and the losers receive L where $(c'(x) - \beta)^{n-1}R(x) + (1 - (c'(x) - \beta)^{n-1})L(x) = c(x)$ and $L(x) \leq c(x)$ for all $x \leq x^*$ and $L(x) = c(x^*)$ for $x \geq x^*$ where $c'(x^*) = \beta$.*

Example 1 *$n = 2$, $\beta = 1$, $c(x) = x^2$. With just a reward we have $R(x) = x^2/(2x - 1)$ (for $x > 1/2$ and 0 elsewhere). In such a case $x^* = 1/2$. We can also have $L(x) = c(x) \cdot 2x = 2x^3$ for $x < 1/2$ and $L(x) = 1/4$ for $x \geq 1/2$. We then must have $((2x - 1)R(x) + (2 - 2x)/4 = x^2$ or $R(x) = (x^2 + x/2 - 1/2)/(2x - 1) = (x + 1)/2$.*

2.3 Comparison to Che and Gale (2003)

In Che and Gale (2003), a buyer also wishes to acquire an innovation that can be of varying quality. There, the buyer designs a competition where firms expend effort to innovate where a higher effort results in a higher quality of innovation. After innovating each firm specifies a price to the buyer. The buyer then chooses the firm offering the highest surplus (quality minus price). The winning firm receives payment while all firms bear the cost of their sunk effort. In this setup, the buyer cares only about the firm offering the largest surplus and derives no surplus from effort put forth by the other firms, i.e., $\beta = 0$.

In the simplest version of the Che-Gale model, each firm i chooses effort x_i, surplus s_i and price p_i to solve $\max_{x_i, s_i, p_i} \pi(s_i)p_i - c(x_i)$ s.t. $x_i - p_i = s_i$ (where π is the probability that the other firms choose a surplus lower than one's own). Substituting the constraint into the maximand, we get $\pi(x_i - p_i)p_i - c(x_i)$. The firm will optimize over the choices of x and p which implies (from the FOCs) $\pi'(s_i)p_i = \pi(s_i)$ and $\pi'(s_i)p_i = c'(x_i)$. Together, these imply $\pi(s_i) = c'(x_i)$. The zero profit condition of the firm implies that $\pi(s_i)p_i = c(x_i)$. Thus, $p_i = c(x_i)/c'(x_i)$. The behavior induced and payoffs are identical to our solution for the case when $\beta = 0$.

Intuitively, this works out to be the same since the firms in the Che and Gale model optimize over effort and price given a specific level of surplus offered. In our model, the designer optimizes the trade-off between value of the effort (to the designer) and its cost (to the firm) for a given probability of winning (note an effort is worthless to the designer if it is not the highest).

For the symmetric environment, each mechanism has its own benefits. The Che and Gale mechanism has the advantage that the designer does not need to know the cost function beforehand which our mechanism requires for determining the rewards. The Che and Gale mechanism has the disadvantage that off equilibrium, the buyer may have to purchase the inferior innovation since it offers him a lower price. This could be politically difficult and precludes the possibility of renegotiation on price.

2.4 Properties of the optimal reward

Remark 2 *The optimal reward function may assume many forms: increasing, decreasing, have both increasing and decreasing parts, or be constant.*

The remark is shown through a series of examples when $\beta = 0$ for simplicity. Such examples also exist for $\beta > 0$.

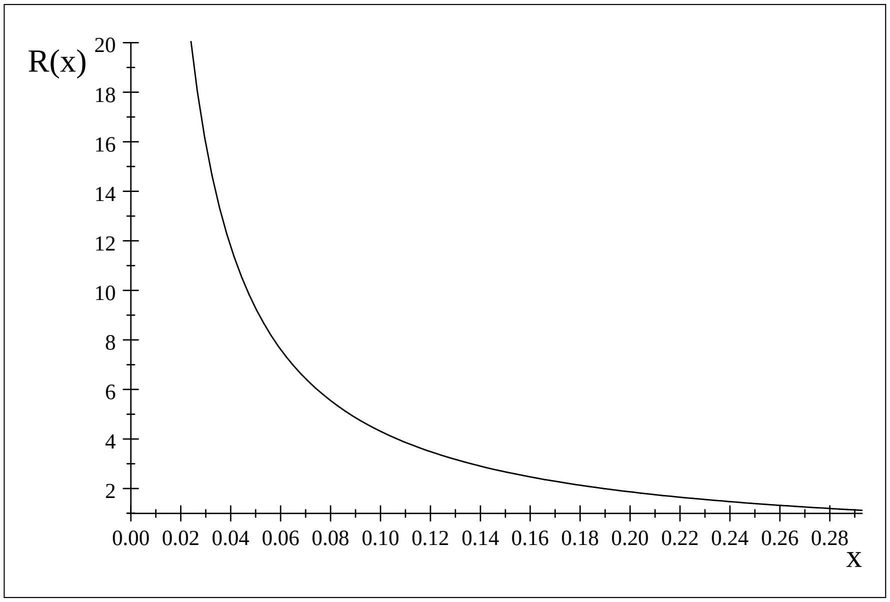
Example 2 *A strictly increasing reward function:* $n = 2$, $c(x) = x^{\alpha}$ *where* $\alpha > 1$.

For such a cost, the optimal reward is $R(x) = \frac{c(x)}{c'(x)} = \frac{x^{\alpha}}{\alpha x^{\alpha-1}} = \frac{x}{\alpha}$. This is strictly increasing in x. In equilibrium, the firms choose effort by using a cumulative distribution function $F(x) = c'(x) = \alpha x^{\alpha-1}$.

Example 3 *A strictly decreasing reward function:* $n = 2$, $c(x) = \frac{1}{1-x} - x$.

The optimal reward is $R(x) = \frac{c(x)}{c'(x)} = x + \frac{1}{2x} - \frac{3}{2(2-x)}$ which is strictly decreasing and positive for $0 \leq x < 1$.

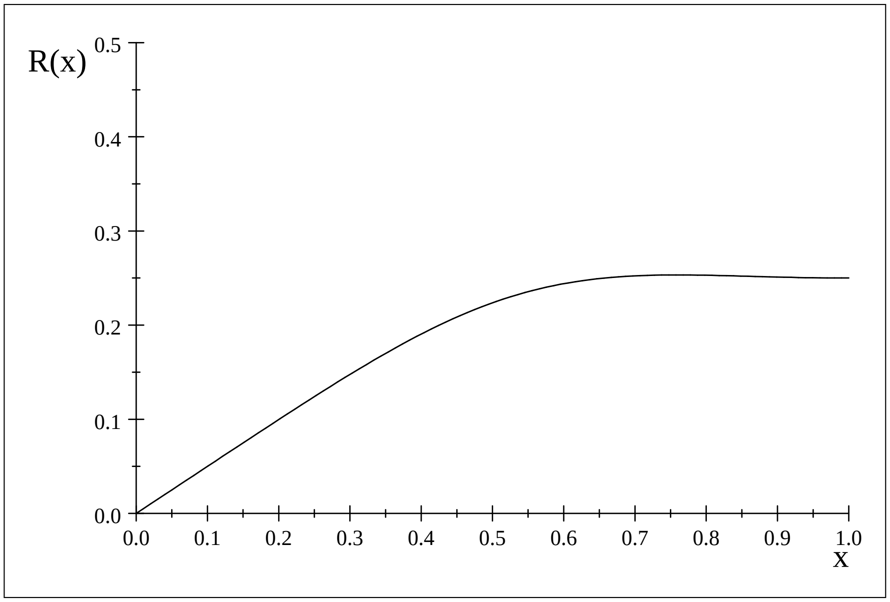
In equilibrium, the firms choose effort by using a cumulative distribution function $F(x) = c'(x) = \frac{1}{(1-x)^2} - 1$. Thus, each firm uses a mixed strategy on $[0, 0.2929]$. See Figure 1.



Example 2: Decreasing optimal reward.

Example 4 *An increasing and then decreasing reward function:* $c(x) = \frac{x^6+x^2}{8}$.

The optimal reward is $R(x) = \frac{c(x)}{c'(x)} = \frac{x^5+x}{6x^4+2}$. This increases until $x = 0.76$ and then decreases. See Figure 2.



Example 3: Increasing and decreasing optimal reward.

In equilibrium, the firms choose effort by using a cumulative distribution function $F(x) = c'(x) = \frac{6x^5+2x}{8}$ on $[0, 1]$.

Example 5 *A constant reward:* $n = 2$, $c(x) = e^{x/2}$.

The optimal reward is $R(x) = \frac{c(x)}{c'(x)} = 2$. In equilibrium, the firms choose effort by using a cumulative distribution function $F(x) = c'(x) = \frac{1}{2}e^{x/2}$ on $[0, 1.39]$. Notice that there is an atom of $1/2$ at zero. If one firm makes an ε effort, it has a $1/2$ chance of winning a reward of 2 and it costs the firm 1. Also note that we implicitly assume that a firm can stay out and not pay $c(0)$.

Remark 3 *The optimal $R(x)$ is constant if and only if there is a fixed cost and $c(x) = e^{x/r+\gamma} - \beta r$ where $r > 0$, $\ln(\beta r) \leq \gamma < \ln(r(1+\beta))$ and $R(x) = r$.*

Proof. Since $R(x) = \frac{c(x)}{c'(x)-\beta}$, if $R(x)$ is constant and equal to r, we have $\frac{c'(x)-\beta}{c(x)} = \frac{1}{r}$. This yields $c(x) = e^{x/r+\gamma} - \beta \cdot r$. Since $F(x)^{n-1} = c'(x) - \beta = c(x)/r$, we have $F(0) \geq 0$ if $e^{\gamma} - \beta r \geq 0$. Also, we must have $F(0) < 1$, which implies we must have $e^{\gamma} - \beta r < r$. These two inequalities yield the bounds on γ in the remark. ■

Remark 4 *When $\beta = 0$, multiplying the costs by a constant does not effect the optimal $R(x)$.*

One may intuitively think that doubling costs would entail an increase of the optimal rewards; however, since $R(x) = \frac{c(x)}{c'(x)}$, there is no change. This is due to the result that if cost is doubled, then it is optimal to have F doubled (a decrease in the effort). In order to induce this, $R(x)$ should stay the same. This is clearly not true when $\beta > 0$.

3 Asymmetric Environment

Now assume that there are two firms that differ by their cost functions $c_1(x)$, $c_2(x)$ where $c_1(x) \geq c_2(x)$. For now, assume that the designer can make a separate reward offer to either firm: $R_1(x)$ and $R_2(x)$. Assume that the buyer chooses rewards such that the equilibrium has both firms making a positive effort.

Under these assumptions, again there must be a mixed-strategy equilibrium which we denote by $F_1(x)$ and $F_2(x)$.

Lemma 1 *In the optimal design, firms make zero profits.*

Proof. The proof is by contradiction. Let us say that the two reward functions R_1 and R_2 are optimal and induce behaviour F_1 and F_2. Assume that the equilibrium is such that $R_1(x)F_2(x) - c_1(x) = 0$ and $R_2(x)F_1(x) - c_2(x) = \pi$. (Note that in equilibrium at least one must make zero profits.) Create an $\widehat{R}_2(x)$ as follows: $\widehat{R}_2(x) = R_2(x) - \frac{\pi}{F_1}$. This $\widehat{R}_2$ is less costly and induces the same equilibrium distribution functions. Hence, there is a contradiction to the initial assumption that R_1 and R_2 are optimal for the designer. ■

When profits are zero the objective of the designer can be written as the sum of total social welfare and the expected sum of efforts multiplied by β. Let us look at the case were there are cost functions $c_1(x)$ and $c_2(x)$. The objective function is then:

$$\left(\int xdF_1F_2 - \int c_1(x)dF_1 - \int c_2(x)dF_2\right) + \beta\left(\int xdF_1 + \int xF_2\right) = \int (xF_2 + \beta x - c_1(x))dF_1 + \int (xF_1 + \beta x - c_2(x))dF_2$$

The designer's problem is then

$$\max_{F_1,F_2} \quad \int (xF_2 + \beta x - c_1(x))dF_1 + \int (xF_1 + \beta x - c_2(x))dF_2$$
$$s.t. \text{ the supports of } F_1 \text{ and } F_2 \text{ coincide.}$$

Proposition 2 *If $c_1'^{-1}(1) = c_2'^{-1}(1)$ and $c_1'(0) = c_2'(0)$, then optimal design has the buyer set $R_i(x) = c_i(x)/(c_i'(x) - \beta)$.*

Proof. Let us do a change of variables to choose $x(F_1)$ and $F_2(F_1)$. Now the maximization problem becomes

$$\max_{x(F_1),F_2(F_1)} \int \left(x(F_1)F_2(F_1) + \beta x(F_1) - c_1(x(F_1)) + \left[x(F_1)F_1 + \beta x(F_1) - c_2(x(F_1))\right] F_2'(F_1)\right) dF_1.$$

Choosing $x()$ pointwise leads to the following FOC:

$$F_2(F_1) + \beta - c_1'(x(F_1)) + \left[F_1 + \beta - c_2'(x(F_1))\right] F_2'(F_1) = 0.$$

Choosing $F_2'(F_1)$ pointwise leads to the second FOC:

$$-\int_0^{F_1} x(\widetilde{F}_1)d\widetilde{F}_1 + x(F_1)F_1 + \beta x(F_1) - c_2(x(F_1)) = 0$$

Note that in order to do this last step, we have to use integration by parts to rewrite the integral $\int x(F_1)F_2(F_1)dF_1$ as $\int_0^{F_1} x(\widetilde{F}_1)d\widetilde{F}_1 F_2(F_1)\Big|_0^1 - \int \int_0^{F_1} x(\widetilde{F}_1)d\widetilde{F}_1 \cdot F_2'(F_1)dF_1$.

Let us now write the second FOC by writing F in terms of x:

$$\begin{aligned} \int_0^x F_1(\widetilde{x})d\widetilde{x} - xF_1(x) + xF_1(x) + \beta x - c_2(x) &= 0 \\ \int_0^x F_1(\widetilde{x})d\widetilde{x} + \beta x - c_2(x) &= 0 \\ F_1(x) &= c_2'(x) - \beta \end{aligned}$$

Substituting this into the first FOC yields $F_2(x) = c_1'(x) - \beta$. Using the indifference conditions of the firms yields the optimal reward functions. The conditions $c_1'^{-1}(1) = c_2'^{-1}(1)$ and $c_1'(0) = c_2'(0)$ ensures that the supports coincide. ■

Example 6 $\beta = 0$, $c_1(x) = \frac{x^a}{a}$, $c_2(x) = \frac{x^b}{b}$ *(where* $a, b > 1$*). We have*

$R_1(x_1) = \frac{x_1}{a}$ *and* $R_2(x_2) = \frac{x_2}{b}$ *where* $F_1(x_1) = x_1^{b-1}$ *and* $F_2(x_2) = x_2^{a-1}$.

Notice that such a reward structure requires that the designer not only knows which firm has which cost function, but is also able to openly discriminate against one of the firms. Such favoritism could be problematic politically. It would be much easier and more elegant if there could be a single reward function. We, hence, proceed to try and construct a reward function that depends not only on one's own effort but also on that of the other firm and which in expectation replicates, in equilibrium, the two separate reward functions.

Remark 5 *The optimal design can sometimes be implemented by a single reward function that depends upon both efforts.*

Proof. We wish to create a reward function $R(x_h, x_l)$ This reward function represents the reward paid to the firm with the highest effort and depends upon both the high and low effort levels x_h and x_l. The expectation of this reward function should yield the individual expected reward functions, namely,

$$\begin{aligned} \frac{\int_0^{x_h} R(x_h, x_l) F_2'(x_l) dx_l}{F_2(x_h)} &= R_1(x_h), \\ \frac{\int_0^{x_h} R(x_h, x_l) F_1'(x_l) dx_l}{F_1(x_h)} &= R_2(x_h). \end{aligned}$$

Rewriting yields

$$\begin{aligned} \int_0^{x_h} R(x_h, x_l) F_2'(x_l) dx_l &= R_1(x_h) F_2(x_h), \\ \int_0^{x_h} R(x_h, x_l) F_1'(x_l) dx_l &= R_2(x_h) F_1(x_h). \end{aligned}$$

Substituting the functions used in our example yields

$$\begin{aligned}\int_0^{x_h} R(x_h, x_l) x_l^{a-2} dx_l &= \frac{1}{a(a-1)} x_h^a, \\ \int_0^{x_h} R(x_h, x_l) x_l^{b-2} dx_l &= \frac{1}{b(b-1)} x_h^b.\end{aligned}$$

The solution to these two equations is $R(x_h, x_l) = \frac{1}{a+b-1} x_h^{\frac{ab}{a+b-1}} x_l^{1-\frac{ab}{a+b-1}}$. ■

Note that for the example in the above proof the exponent on x_h is always positive and the exponent on x_l is always less than 1 and could be negative. We can also compute the expected profit for the above example which is $\int_0^1 (xc_1'(x) - c_1(x))c_2''(x)dx + \int_0^1 (xc_2'(x) - c_2(x))c_1''(x)dx =$
$1 - \frac{1}{a} - \frac{1}{b} + \frac{1}{a+b-1}$.

3.1 Comparison to Che and Gale (2003)

Che and Gale allow the buyer to handicap the stronger firm by limiting the price the firm can charge. Now the firm's problem is $\max_{x_i, s_i, p_i} \pi_i(s_i)p_i - c_i(x_i)$ s.t. $x_i - p_i = s_i$ and $p_i \leq p_i^*$. Without the constraint binding, as before $\pi_i(s_i) = c_i'(x_i)$ and $p_i = c_i(x_i)/c_i'(x_i)$. Once the constraint binds, $\pi_i(s_i) = (u_i + c_i(s + p^*))/p^*$ where u_i is the profit of firm i. The buyer is able to choose p^* in order to limit the profit of this firm. The profit is determined by the maximum surplus the other firm can offer which equals $s_j^* = \max_x x - c_j(x)$. If p_i^* is binding, then $u_i = p_i^* - c_i(s_j^* + p_i^*)$. If one wishes to set u_i to zero, we have $p_i^* = c_i(s_j^* + p_i^*)$

Example 7 *The Che and Gale (2003) mechanism when* $c_1(x) = \frac{2}{3}x^{\frac{3}{2}}$, $c_2(x) = \frac{1}{2}x^2$.

For the weak buyer $p_j = c_j(x_j)/c'_j(x_j) = 2x_j/3$. Since $s_j = x_j - p_j$, we have $s_j = x_j/3$. Since $\pi_j(s_i) = c'_j(x_i)$, we have $\pi_j(s) = (3 \cdot s)^{1/2}$. Likewise for the strong buyer, when p^* is not binding, we have $\pi_i(s) = 2s$. Using the probability of winning π_i, we can determine the strategy G_i of each player:

$$
\begin{aligned}
G_1(s) &= (3 \cdot s)^{1/2} \\
G_2(s) &= \begin{cases} 2 \cdot s & \text{if } s < p, \\ \frac{(s+p)^2}{2 \cdot p} & \text{if } s > p \text{ where } p = \frac{2-\sqrt{3}}{3}. \end{cases}
\end{aligned}
$$

We can now compute the expected profit:

$$
\begin{aligned}
\int_0^{1/3} s \cdot d(G_1 \cdot G_2) = \int_0^p s \cdot d(G_1 \cdot 2s) + \int_p^{1/3} s \cdot d(G_1 \cdot \frac{(s+p)^2}{2 \cdot p}) &= \\
\int_0^p 3\,(3)^{1/2} \cdot s^{3/2} ds + \int_p^{1/3} \frac{(3)^{1/2}}{2} s \cdot d((s)^{1/2} \cdot \frac{(s+p)^2}{p}) &= \\
\left(173 - 76\sqrt{2} + 20\sqrt{3} + 44\sqrt{6}\right) \frac{1}{945} &= \\
0.220041.
\end{aligned}
$$

Using the mechanism in this paper, the expected profit is $1 - \frac{1}{a} - \frac{1}{b} + \frac{1}{a+b-1} = \frac{7}{30} = 0.23333$, which is higher.

Note that this finding does not contradict those in the Che and Gale (2003), since our mechanism uses bid-dependent rewards which are not feasible in their environment and added flexibility is an advantage. Furthermore, we avoid directly handicapping one of the firms by using a combined reward function. This allows the handicapping indirectly through the behaviour of the other firm that handicaps it.

4 Conclusion

We have examined the optimal design of rewards in a contest with complete information. We find a simple rule for setting the optimal rewards in the symmetric case. This allows the designer to simply choose the best design and pay the winner according to the prespecified reward. With asymmetry, it is optimal to have different firms receive different rewards. We show it might be possible, for some environments, to replicate this with a common joint reward function that depends upon both efforts. This design method yielded "better outcomes" then previously used mechanisms.

Further research is needed to examine the effect of changing the number of firms. Several open issues remain for the asymmetric environment case: What are general conditions under which it is possible to create a joint reward function? What is the best design, when the optimal reward functions do not share the same support? Finally, it is of interest to see what the optimal reward function would be under additional constraints, for instance, if one were limited to offering the same reward to both firms where this reward could only depend upon the highest effort.

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