Stability and Competitive Equilibrium in Trading Networks*

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December 31, 2010

Abstract

We introduce a model in which agents in a network can trade via bilateral contracts. We find that when continuous transfers are allowed and utilities are quasilinear, the full substitutability of preferences is sufficient to guarantee the existence of stable outcomes for any underlying network structure. Furthermore, the set of stable outcomes is essentially equivalent to the set of competitive equilibria, and all stable outcomes are in the core and are efficient. In contrast, for any domain of preferences strictly larger than that of full substitutability, the existence of competitive equilibria and stable outcomes cannot be guaranteed.

JEL classification: C78; C62; C71; D85; L14

Keywords: Matching; Networks; Stability; Competitive Equilibrium; Core; Efficiency; Submodularity; Lattice; Substitutes

*We thank Peter Cramton, Çağatay Kayı, Paul Milgrom, Al Roth, and Glen Weyl for helpful comments and suggestions. Hatfield and Nichifor appreciate the hospitality of Harvard Business School, which hosted them during parts of this research. Kominers thanks the National Science Foundation and the Yahoo! Key Scientific Challenges Program for support. Nichifor thanks the Netherlands Organisation for Scientific Research (NWO) for support under grant VIDI-452-06-013. Westkamp thanks the German Research Foundation (DFG) for support.

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1 Introduction

We study a model in which a network of agents can trade indivisible goods or services via bilateral contracts. We find that if all agents' preferences are *fully substitutable*, then in the presence of transferable utility, stable outcomes and competitive equilibria are guaranteed to exist and are efficient. Moreover, in that case, the sets of competitive equilibria and stable outcomes are in a sense equivalent. These results apply to general trading networks and do not require any assumptions on the network structure, such as two-sidedness or acyclicity. At the same time, they depend critically on the full substitutability of preferences and the presence of a numeraire.

In our model, contracts specify a buyer, a seller, provision of a good or service, and a monetary transfer. An agent may be involved in some contracts as a seller, and in other contracts as a buyer; our framework is therefore suitable for studying production economies in which some agents can turn inputs into outputs (e.g., iron ore into steel) at some cost. Agents' preferences are defined by cardinal utility functions over sets of contracts and are quasilinear with respect to the numeraire. We say that preferences are *fully substitutable* if contracts are substitutes for each other in a generalized sense, i.e., whenever an agent gains a new purchase opportunity, he becomes both less willing to make other purchases and more willing to make sales, and whenever he gains a new sales opportunity, he becomes both less willing to make other sales and more willing to make purchases. We formally show that this intuitive notion of substitutability, which has appeared in the literature on matching in vertical networks (Ostrovsky, 2008; Hatfield and Kominers, 2010b; Westkamp, 2010), is equivalent to the gross substitutes and complements condition introduced in the literature on competitive equilibrium in settings in which indivisible objects are allocated to consumers (Sun and Yang, 2006, 2009). It is also equivalent to the assumption of submodularity of the indirect utility function (Gul and Stacchetti, 1999; Ausubel and Milgrom, 2002).[1]

Our main results are as follows. When preferences are fully substitutable, competitive equilibria are guaranteed to exist. Moreover, there is a simple and natural mapping from the set of competitive equilibria to the set of stable outcomes. Thus, fully substitutable preferences are also sufficient for the existence of stable outcomes. Conversely, full substitutability is in a sense necessary to guarantee the existence of

[1]For definitions of the classical notions of substitutability in two-sided settings with and without transfers, see Kelso and Crawford (1982), Roth (1984), Gul and Stacchetti (1999), and Hatfield and Milgrom (2005).

stable outcomes (and thus also of competitive equilibria): if any agent's preferences are not fully substitutable, then fully substitutable preferences can be found for other agents such that no stable outcome exists.[2]

While competitive equilibria always induce stable outcomes regardless of agents' preferences, stable outcomes need not correspond to competitive equilibria in general. However, in the presence of fully substitutable preferences, all stable outcomes do correspond to competitive equilibria. This result is analogous to a similar finding of Kelso and Crawford (1982) for two-sided many-to-one matching markets, but it requires a more involved proof. In the setting of Kelso and Crawford (1982), one can construct "missing" prices for unrealized trades simply by considering those trades one by one, because in that setting, each worker can be employed by at most one firm. In our setting, that simple procedure would not work, because each agent can be involved in multiple trades.

We characterize the set of competitive equilibria, showing that the set of competitive equilibrium price vectors forms a lattice, generalizing the results of Gul and Stacchetti (1999) and Sun and Yang (2006). These results are also closely related to the "polarization of interests" results in classical matching theory, which show that the set of stable outcomes forms a lattice (see, e.g., Roth, 1984; Hatfield and Milgrom, 2005; and Ostrovsky, 2008).

Finally, we consider the relationship between stability and several other solution concepts. Generalizing the results of Shapley and Shubik (1971) and Sotomayor (2007), we show that all stable outcomes are in the core. We then consider the notion of strong group stability and show that in our setting, in contrast to the results of Echenique and Oviedo (2006) and Klaus and Walzl (2009) for matching settings with-

[2]In the setting of two-sided many-to-one matching with transfers, Kelso and Crawford (1982) show that substitutability is sufficient for the existence of stable outcomes and competitive equilibria; Gul and Stacchetti (1999) and Hatfield and Kojima (2008) prove corresponding necessity results.

In a setting in which two types of indivisible objects need to be allocated to consumers, Sun and Yang (2006) show that competitive equilibria are guaranteed to exist if consumers view objects of the same type as substitutes and view objects of different types as complements (see also Section 5.3).

Sufficiency and necessity of fully substitutable preferences also obtains in settings of many-to-many matching with and without contracts (Roth (1984), Echenique and Oviedo (2006), Klaus and Walzl (2009), and Hatfield and Kominers (2010a), prove sufficiency results; Hatfield and Kojima (2008) and Hatfield and Kominers (2010a), prove necessity results) and in matching in vertical networks (Ostrovsky (2008), and Hatfield and Kominers (2010b), prove sufficiency; Hatfield and Kominers (2010b), prove necessity). Substitutable preferences are sufficient for the existence of a stable outcome in the setting of many-to-one matching with contracts (Hatfield and Milgrom, 2005) but are not necessary (Hatfield and Kojima, 2008, 2010).

out transfers, the set of stable outcomes is in fact equal to the set of strongly group stable outcomes. We also consider chain stability, extending the definition of Ostrovsky (2008); while chain stability is logically weaker than stability, we show that chain stability is equivalent to stability when agents' preferences are fully substitutable.[3]

The remainder of this paper is organized as follows. In Section 2, we formalize our model. In Section 3, we discuss the notion of full substitutability in detail and prove a result on the equivalence of various alternative definitions. In Section 4, we present our main results. In Section 5, we consider the relationships between competitive equilibria, stable outcomes, and other solution concepts. We conclude in Section 6.

2 Model

There is a finite set I of *agents* in the economy. These agents can participate in bilateral *trades*. Each trade ω is associated with a *seller* $s(\omega) \in I$ and a *buyer* $b(\omega) \in I$, $b(\omega) \neq s(\omega)$. The set of possible trades, Ω, is finite and exogenously given. The set Ω may contain multiple trades that have the same buyer and the same seller. For instance, a worker (seller) may be hired by a firm (buyer) in a variety of different capacities with a variety of job conditions and characteristics, and each possible type of job may be represented by a different trade. One firm may sell multiple units of a good (or several different goods) to another firm, and each unit may be represented by a separate trade. We also allow Ω to contain trades ω_1 and ω_2 such that $s(\omega_1) = b(\omega_2)$ and $s(\omega_2) = b(\omega_1)$.

It is convenient to think of a trade as representing the nonpecuniary aspects of a transaction between a seller and a buyer (although in principle it could include some "financial" terms and conditions as well). The purely financial aspect of a trade ω is represented by a price p_ω; the complete vector of prices for all trades in the economy is denoted by $p \in \mathbb{R}^{|\Omega|}$. Formally, a *contract* x is a pair (ω, p_ω), with $\omega \in \Omega$ denoting the trade and $p_\omega \in \mathbb{R}$ denoting the price at which the trade occurs. The set of available contracts is $X \equiv \Omega \times \mathbb{R}$. For any set of contracts Y, we denote by $\tau(Y)$ the set of trades involved in contracts in Y: $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\}$.

For a contract $x = (\omega, p_\omega)$, we will let $s(x) \equiv s(\omega)$ and $b(x) \equiv b(\omega)$, the seller and the buyer associated with the trade ω of contract x. Consider any set of contracts $Y \subseteq X$. We denote by $Y_{\to i}$ the set of "upstream" contracts for i in Y, that is, the set

[3] Hatfield and Kominers (2010b) prove an analogous result for the setting of Ostrovsky (2008).

of contracts in Y in which agent i is the buyer: $Y_{\to i} \equiv \{y \in Y : i = b(y)\}$. Similarly, we denote by $Y_{i\to}$ the set of "downstream" contracts for i in Y, that is, the set of contracts in Y in which agent i is the seller: $Y_{i\to} \equiv \{y \in Y : i = s(y)\}$. We denote by Y_i the set of contracts in Y in which agent i is involved either as the buyer or the seller: $Y_i \equiv Y_{\to i} \cup Y_{i\to}$. We let $a(Y) \equiv \bigcup_{y\in Y}\{b(y), s(y)\}$ denote the set of agents involved in contracts in Y as buyers or sellers. We use analogous notation to denote the subsets of trades associated with some agent i for sets of trades $\Psi \subseteq \Omega$.

We say that the set of contracts Y is *feasible* if there is no trade ω and prices $p_\omega \neq \hat{p}_\omega$ such that both contracts (ω, p_ω) and $(\omega, \hat{p}_\omega)$ are in Y; i.e., a set of contracts is feasible if each trade is associated with at most one contract in that set. An *outcome* $A \subseteq X$ is a feasible set of contracts.[4] Thus, an outcome specifies which trades get formed, and what the associated prices are, but does not specify prices for trades that do not take place. An *arrangement* is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ is a set of trades and $p \in \mathbb{R}^{|\Omega|}$ is a vector of prices for all trades in the economy. For any arrangement $[\Psi; p]$ we denote by $\kappa([\Psi; p]) \equiv \cup_{\psi\in\Psi}\{(\psi, p_\psi)\}$, the set of contracts induced by the arrangement. Note that $\kappa([\Psi; p])$ is an outcome, and that $\tau(\kappa([\Psi; p])) = \Psi$.

2.1 Preferences

Each agent i has a valuation function u_i over sets of trades $\Psi \subseteq \Omega_i$; we extend u_i to Ω as follows: $u_i(\Psi) \equiv u_i(\Psi_i)$ for any $\Psi \subseteq \Omega$. The valuation u_i gives rise to a quasilinear utility function U_i over sets of trades and the associated transfers. We formalize this in two different ways. First, for any feasible set of contracts Y, we say that

$$U_i(Y) \equiv u_i(\tau(Y)) + \sum_{(\omega,p_\omega)\in Y_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in Y_{\to i}} p_\omega.$$

Second, for any arrangement $[\Psi; p]$, we say that

$$U_i([\Psi; p]) \equiv u_i(\Psi) + \sum_{\psi\in\Psi_{i\to}} p_\psi - \sum_{\psi\in\Psi_{\to i}} p_\psi.$$

Note that, by construction, $U_i([\Psi; p]) = U_i(\kappa([\Psi; p]))$.

[4]In the literature on matching with contracts, the term "allocation" has been used to refer to a set of contracts. Unfortunately, the term "allocation" is also used in the competitive equilibrium literature to denote an assignment of goods, without specifying transfers. For this reason, to avoid confusion, we use the term "outcome" to refer to a feasible set of contracts.

We allow $u_i(\Psi)$ to take the value $-\infty$ for some sets of trades Ψ in order to incorporate various technological constraints. However, we also assume that for all i, the outside option is finite: $u_i(\varnothing) \in \mathbb{R}$. That is, no agent is "forced" to sign any contracts at extremely unfavorable prices: he always has an outside option of completely withdrawing from the market at some potentially high but finite price.

Many of our results also depend critically on the assumption that preferences are fully substitutable. We present and discuss this assumption in Section 3.

The utility function U_i gives rise to both demand and choice correspondences. The *choice correspondence* of agent i from the set of contracts $Y \subseteq X$ is defined as the collection of the sets of contracts maximizing agent i's utility:

$$C_i(Y) \equiv \underset{Z \subseteq Y_i;\ Z \text{ is feasible}}{\operatorname{argmax}} U_i(Z).$$

The *demand correspondence* of agent i given a price vector $p \in \mathbb{R}^{|\Omega|}$ is defined as the collection of the sets of trades maximizing agent i's utility under prices p:

$$D_i(p) \equiv \underset{\Psi \subseteq \Omega_i}{\operatorname{argmax}} U_i([\Psi; p]).$$

Note that while the demand correspondence always contains at least one (possibly empty) set of trades, the choice correspondence may be empty-valued (e.g., if Y consists of all contracts with prices strictly between 0 and 1). If the set Y is finite, then the choice correspondence is also guaranteed to contain at least one set of contracts.

2.2 Stability and Competitive Equilibrium

The main solution concepts that we study are *stability* and *competitive equilibrium.* Both concepts specify which trades are formed and what the associated transfers are. Competitive equilibria also specify prices for trades that are not formed.

Definition 1. *An outcome A is* stable *if it is*

1. Individually rational: $A_i \in C_i(A)$ *for all i;*
2. Unblocked: *There is no feasible nonempty* blocking set $Z \subseteq X$ *such that*
   (a) $Z \cap A = \varnothing$, *and*
   (b) *for all $i \in a(Z)$, for all of i's choices $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.*

Individual rationality requires that no agent can become strictly better off by dropping some of the contracts that he is involved in. This is a standard requirement in the matching literature. The second condition states that when presented with a stable outcome A, one cannot propose a new set of contracts such that all the agents involved in these new contracts would strictly prefer to form all of them (and possibly drop some of their existing contracts in A) instead of forming only some of them (or none). This requirement is a natural adaptation of the stability condition of Hatfield and Kominers (2010b) to the current setting. We discuss the relationship between our notion of stability and those notions that have appeared in the previous literature, such as *chain stability*, in Section 5.

Our second solution concept is competitive equilibrium.

Definition 2. *An arrangement* $[\Psi; p]$ *is a* competitive equilibrium *if for all* $i \in I$,

$$\Psi_i \in D_i(p).$$

This is the standard notion of competitive equilibrium, adapted to the current setting: market-clearing is "built in," because each trade in Ψ carries with it the corresponding buyer and seller, and the condition is simply that each agent is optimizing given market prices. Note that we only require weak optimality—at a competitive equilibrium, an agent may be indifferent between the assigned bundle and some other bundle of trades.

3 Full Substitutability

Kelso and Crawford (1982) introduced the gross substitutes condition (GS) in the context of a two-sided many-to-one matching market between firms and workers. The (GS) condition requires that an increase in the salary of some worker cannot cause a firm to drop other initially employed workers whose salaries did not change. This condition explicitly deals with indifferences, by placing restrictions on the behavior of the multi-valued demand function when workers' salaries change. Kelso and Crawford use (GS) to establish the existence of a core outcome for many-to-one matching models in which prices can be either discrete or continuous. Substitutability conditions similar to (GS) have been shown to be sufficient for the existence of stable outcomes in a number of other two-sided settings (Roth and Sotomayor (1990) describe these results).

Ausubel and Milgrom (2002) offered a convenient alternative definition of (GS) for a setting with continuous prices, in which demand is not guaranteed to be single-valued: goods are (gross) substitutes if the demand for each one is nondecreasing in the prices of others when attention is restricted only to the vectors of prices at which demand is single-valued. Additionally, Ausubel and Milgrom (2002) showed that (GS) is equivalent to submodularity of the indirect utility function.

In a setting with discrete contract sets, Ostrovsky (2008) introduced a combination of two substitutability assumptions: same-side substitutability (SSS) and cross-side complementarity (CSC). These assumptions impose two constraints: First, when an agent's opportunity set on one side of the market expands, that agent does not choose any options previously rejected from that side of the market. Second, when an agent's opportunity set on one side of the market expands, that agent does not reject any options previously chosen from the other side of the market. Ostrovsky (2008) and Hatfield and Kominers (2010b) show that under (SSS) and (CSC) a stable outcome always exists if the contractual set contains no cycles. In his setting, indifferences are assumed away, and thus the conditions do not need to explicitly address multi-valued choice functions. Hatfield and Kominers (2010b) also show that (SSS) and (CSC) are together equivalent to the assumption of quasisubmodularity of the indirect utility function—an adaptation of submodularity to the discrete setting.

Independently, Sun and Yang (2006) introduced the gross substitutability and complementarity (GSC) condition for a setting with continuous transfers in which indivisible objects are allocated to consumers. This condition requires that objects can be divided into two groups such that objects in the same group are substitutes, and objects in different groups are complements. Sun and Yang (2006) show that under (GSC), there always exists a competitive equilibrium in this setting. The Sun and Yang (2006) definition explicitly deals with indifferences and multi-valued demand functions. Sun and Yang (2009) show that (GSC) is equivalent to the condition that the indirect utility function is submodular.

In this section, we introduce the notion of *full substitutability*[5] for the current setting: When presented with additional contractual options to purchase, an agent both rejects any previously rejected purchase options, and continues to choose any

[5]Since preferences are quasilinear in our settting, there is no distinction between gross and net substitutes. Therefore, we drop the "gross" specification.

previously chosen sale options. Analogously, when presented with additional contractual options to sell, an agent both rejects any previously rejected sale options, and continues to choose any previously chosen purchase options.

We introduce several alternative definitions, adapting the ones mentioned above to our setting, and show that all of them are equivalent. For convenience, in this section, we use the approach of Ausubel and Milgrom (2002) and restrict attention to sets and vectors of prices for which choices and demands are single-valued. In Appendix A, we consider definitions that explicitly deal with indifferences and multi-valued correspondences and show that they are equivalent to the ones in this section.

3.1 Definitions of Full Substitutability

First, we define full substitutability in the language of sets and choices, adapting and merging (SSS) and (CSC).

Definition 3. *Agent i's preferences are* choice-language fully substitutable *(CFS) if:*

1. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{\to i} - Y^*_{\to i}) \subseteq (Z_{\to i} - Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*
2. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{i\to} - Y^*_{i\to}) \subseteq (Z_{i\to} - Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.*

In other words, the choice correspondence C_i is fully substitutable if, when attention is restricted to sets for which C_i is single-valued, when the set of options available to i on one side expands, i rejects a larger set of contracts on that side (SSS), and selects a larger set of contracts on the other side (CSC).

Our second definition uses the language of prices and demands, and goes back to the gross substitutes and complements condition (GSC).

Definition 4. *Agent i's preferences are* demand-language fully substitutable *(DFS) if:*

1. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;*

2. *for all price vectors* $p, p' \in \mathbb{R}^{|\Omega|}$ *such that* $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{\to i}$, *and* $p_\omega \leq p'_\omega$ *for all* $\omega \in \Omega_{i\to}$, *for the unique* $\Psi \in D_i(p)$ *and* $\Psi' \in D_i(p')$, *we have* $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ *and* $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

The demand correspondence D_i is fully substitutable if, when attention is restricted to prices for which demands are single-valued, a decrease in the price of some inputs for agent i leads to the decrease in his demand for other inputs and to an increase in his supply of outputs, and an increase in the price of some outputs leads to the decrease in his supply of other outputs and an increase in his demand for inputs.

Our third definition is essentially a reformulation of Definition 4, using a convenient vector notation due to Hatfield and Kominers (2010b). For each agent i, for any set of trades $\Psi \subseteq \Omega_i$, define the *(generalized) indicator function* $e(\Psi) \in \{-1, 0, 1\}^{|\Omega_i|}$ to be the vector with component $e_\omega(\Psi) = 1$ for each upstream trade $\omega \in \Psi_{\to i}$, $e_\omega(\Psi) = -1$ for each downstream trade $\omega \in \Psi_{i\to}$, and $e_\omega(\Psi) = 0$ for each trade $\omega \notin \Psi$. The interpretation of $e(\Psi)$ is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in Ψ, and "buys" a strictly negative amount if he is the seller of such a trade.

Definition 5. *Agent i's preferences are* indicator-language fully substitutable *(IFS) if for all price vectors* $p, p' \in \mathbb{R}^{|\Omega|}$ *such that* $|D_i(p)| = |D_i(p')| = 1$ *and* $p \leq p'$, *for the unique* $\Psi \in D_i(p)$ *and* $\Psi' \in D_i(p')$ *we have* $e_\omega(\Psi) \leq e_\omega(\Psi')$ *for each* $\omega \in \Omega_i$ *such that* $p_\omega = p'_\omega$.

This definition clarifies the reason for the term "full substitutability": an agent is more willing to "demand" a trade (i.e., keep an object that he could potentially sell, or buy an object that he does not initially own) if prices of other trades increase.

Our final definition concerns neither the choice correspondences nor the demand correspondences, but agents' indirect utility functions. It is a generalization of the earlier submodularity assumption on the indirect utility of an agent (Ausubel and Milgrom, 2002).

For price vectors $p, p' \in \mathbb{R}^{|\Omega|}$, let the *join of* p *and* p', denoted $p \vee p'$, be defined as the pointwise maximum of p and p'; let the *meet of* p *and* p', denoted $p \wedge p'$, be defined as the pointwise minimum of p and p'.

Definition 6. *The* indirect utility function *of agent i,*

$$V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} U_i([\Psi; p]),$$

is submodular *if, for all price vectors* $p, p' \in \mathbb{R}^{|\Omega|}$, $V_i(p \wedge p') + V_i(p \vee p') \leq V_i(p) + V_i(p')$.

3.2 Equivalence of the Definitions

The main result of this section shows that the three definitions of full substitutability above are equivalent, and are also equivalent to the assumption of submodularity of the indirect utility function.[6] Subsequently, we will typically use the term *full substitutability* in place of the notations (CFS), (DFS), and (IFS).

Theorem 1. *(CFS), (DFS), and (IFS) are all equivalent, and hold if and only if the indirect utility function is submodular.*

We present the proof of Theorem 1 in Appendix A, along with several alternative characterizations of full substitutability. In Appendix A, we also show that in quasilinear settings, full substitutability implies the Laws of Aggregate Supply and Demand (Hatfield and Kominers, 2010b), extending an analogous result for the (GS) condition (Hatfield and Milgrom, 2005). We use these additional characterizations and properties to prove some of the main results in the paper.

4 Main Results

We first show that competitive equilibria exist when preferences are fully substitutable and characterize the structure of the set of competitive equilibria. We then show that full substitutability implies that the set of competitive equilibria essentially coincides with the set of stable outcomes. Finally, we show that if preferences are not fully substitutable, then stable outcomes and competitive equilibria need not exist. With the exception of the proof of Theorem 7, which is given in the Online Appendix, the proofs of all the results of this and the subsequent sections are presented in Appendix B.

4.1 The Existence of Competitive Equilibria

Theorem 2. *Suppose that agents' preferences are fully substitutable. Then, there exists a competitive equilibrium.*

[6] One can also give an equivalent definition of full substitutability in terms of $M^\natural$-concavity, analogous to the equivalent definitions of (GS) of Reijnierse et al. (2002) and Fujishige and Yang (2003).

The main idea in the proof of Theorem 2 is to associate to the original market a two-sided many-to-one matching market with transferable utility, in which each agent corresponds to a "firm" and each trade corresponds to a "worker." The valuation of firm i for hiring a set of workers $\Psi \subseteq \Omega_i$ in the associated two-sided market is given by

$$v_i(\Psi) \equiv u_i\left(\Psi_{\to i} \cup (\Omega - \Psi)_{i\to}\right). \tag{1}$$

Intuitively, we think of the firm as employing all of the workers associated with trades that the firm buys and with trades that the firm *does not* sell. We show that v_i satisfies the gross substitutes condition (GS) of Kelso and Crawford (1982) as long as u_i is fully substitutable. Workers strictly prefer to work rather than being unemployed, and their utilities are monotonically increasing in wages. Worker ω has a strict preference for being employed by $s(\omega)$ and $b(\omega)$ rather than some other firm $i \in I - \{s(\omega), b(\omega)\}$. With these definitions, we have constructed a two-sided market of the type studied by Kelso and Crawford (1982). In such markets, a competitive equilibrium in which no worker is unemployed is guaranteed to exist.

We then transform this competitive equilibrium back into a set of trades and prices for the original economy as follows: Let ω be formed in the original economy if the worker ω was hired by $b(\omega)$ in the associated market and do not let it be formed if ω was hired by $s(\omega)$ (our construction ensures that these are the only two possibilities). We use the wages in the associated market as prices in the original market. We thus obtain a competitive equilibrium of the original economy: Given the prices generated, a trade ω is demanded by its buyer if and only if it is also demanded by its seller (i.e., *not* demanded by the seller in the associated market).

This construction also provides an algorithm for finding a competitive equilibrium. For instance, once we have transformed the original economy into an associated market, we can use an ascending auction for workers to find the minimal-price competitive equilibrium of the associated market; we may then map that competitive equilibrium back to a competitive equilibrium of the original economy.

One technical complication that we need to address in the proof is that the modified valuation function in Equation (1) may in principle be unbounded and take the value $-\infty$ for sets of trades that are not feasible. To deal with this issue, we further modify the valuation function by bounding it in a way that preserves full substitutability.

4.2 The Properties of Competitive Equilibria

We now consider the structure of the set of competitive equilibria. We first note an analogue of the first welfare theorem in our economy.

Theorem 3. *Suppose* $[\Psi; p]$ *is a competitive equilibrium. Then* Ψ *is an* efficient *set of trades, i.e.,* $\sum_{i \in I} u_i(\Psi) \geq \sum_{i \in I} u_i(\Psi')$ *for any* $\Psi' \subseteq \Omega$.

The proof of this result is standard and follows simply from observing that in a competitive equilibrium, each agent is maximizing his utility under prices p, and when the utilities are added up across all agents, prices cancel out, leaving only the sum of agents' valuations.

Our next result can be viewed as a strong version of the second welfare theorem for our setting, providing a converse to Theorem 3: For any efficient set of trades Ψ and any competitive equilibrium price vector p, the arrangement $[\Psi; p]$ is a competitive equilibrium.[7] Also, the set of competitive equilibrium price vectors forms a lattice.

Theorem 4. *Suppose that agents' preferences are fully substitutable. Then the set of competitive equilibrium price vectors is a lattice. Furthermore, for any element of this lattice* p *and any efficient set of trades* Ψ*,* $[\Psi; p]$ *is a competitive equilibrium.*

The lattice structure of the set of competitive equilibrium prices is analogous to the lattice structure of the set of stable outcomes for economies without transferable utility (see, e.g., Hatfield and Milgrom, 2005). In those models, there is a buyer-optimal and a seller-optimal stable outcome. In our model, the lattice of equilibrium prices may in principle be unbounded. If the lattice is bounded,[8] then there are lowest-price and highest-price competitive equilibria.

4.3 The Relationship between Competitive Equilibria and Stable Outcomes

We now show how the sets of stable outcomes and competitive equilibria are related. First, we show that for every competitive equilibrium $[\Psi; p]$, the associated outcome $\kappa([\Psi; p])$ is stable.

[7] Generically, the efficient set of trades is unique, and then this statement follows immediately from Theorem 3. We show that it also holds when there are multiple efficient sets of trades.

[8] E.g., if all valuations u_i are bounded.

Theorem 5. *Suppose* $[\Psi; p]$ *is a competitive equilibrium. Then* $\kappa([\Psi; p])$ *is stable.*

If for some competitive equilibrium $[\Psi; p]$ the outcome $\kappa([\Psi; p])$ is not stable, then either it is not individually rational or it is blocked. If it is not individually rational for some agent i, then $\kappa([\Psi; p])_i \notin C_i(\kappa([\Psi; p]))$. Hence, $\Psi_i \notin D_i(p)$, and so $[\Psi; p]$ can not be a competitive equilibrium. If $\kappa([\Psi; p])$ admits a blocking set Z, then all the agents with contracts in Z are strictly better off after the deviation. Hence, there must exist some agent i with a contract in Z such that i is strictly better off choosing an element of $C_i(Z \cup \kappa([\Psi; p]))$ given the original price vector p; hence, $\Psi_i \notin D_i(p)$ and so $[\Psi; p]$ is not a competitive equilibrium. Note that this result does not rely on substitutability.

However, it is not generally true that all stable outcomes correspond to competitive equilibria. Consider the following example.

Example 1. Let $\Omega = \{\chi, \psi\}$, $I = \{i, j\}$, $s(\psi) = s(\chi) = i$, $b(\psi) = b(\chi) = j$, and

$$u_i(\varnothing) = 0,\ u_i(\{\chi, \psi\}) = u_i(\{\chi\}) = u_i(\{\psi\}) = -4,$$
$$u_j(\varnothing) = 0,\ u_j(\{\chi, \psi\}) = u_j(\{\chi\}) = u_j(\{\psi\}) = 3.$$

In this case, $\varnothing$ is stable. Since $\varnothing$ is the only efficient set of trades, by Theorem 4 any competitive equilibrium is of the form $[\varnothing; p]$. However, the preferences of agent i imply that $p_\chi + p_\psi \leq 4$, as otherwise i will choose to sell at least one of ψ or χ. Moreover, the preferences of agent j imply that $p_\chi, p_\psi \geq 3$, as otherwise j will buy at least one of ψ or χ. Clearly, all three inequalities can not jointly hold. Hence, while $\varnothing$ is stable, there is no corresponding competitive equilibrium.

The key issue is that an outcome A need only specify prices for the trades in $\tau(A)$ while a competitive equilibrium, by contrast, must specify prices for all trades (including those trades that do not transact). Hence, it may be possible, as in Example 1, for an outcome A to be stable, but, because of complementarities in preferences, it may be impossible to find a price vector for trades outside of $\tau(A)$ such that $\tau(A_i)$ is in fact the optimal set of contracts for every agent i. Note that in Example 1 the preferences of agent j are fully substitutable but those of agent i are not.

If, however, the preferences of all agents are fully substitutable, then for any stable outcome A we can in fact find a supporting set of prices p such that $[\tau(A); p]$ is a competitive equilibrium and the prices of trades that transact are the same as in the stable outcome.

Theorem 6. *Suppose that agents' preferences are fully substitutable and A is a stable outcome. Then there exists a price vector $p \in \mathbb{R}^{|\Omega|}$ such that $[\tau(A); p]$ is a competitive equilibrium and if $(\omega, \bar{p}_\omega) \in A$, then $p_\omega = \bar{p}_\omega$.*

To construct a competitive equilibrium from a stable outcome A, we need to find appropriate prices for the trades that are not part of the stable outcome, i.e., trades $\omega \in \Omega - \tau(A)$. To do this, we construct a modified market in which the set of trades available is $\Omega - \tau(A)$ and the valuation of each player i for a set of trades $\Psi \subseteq \Omega - \tau(A)$ is equal to the highest value he can attain by combining the trades in Ψ with various subsets of A_i. We show that the corresponding utility of each player i is fully substitutable, and thus the modified market has a competitive equilibrium. Furthermore, at least one such equilibrium has be of the form $[\varnothing; \hat{p}]$ for some vector $\hat{p} \in \mathbb{R}^{|\Omega-\tau(A)|}$—we show that otherwise, there exists a nonempty set blocking A. Assigning prices $\hat{p}$ to the trades that are not part of A, we obtain a competitive equilibrium of the original economy.

4.4 The Necessity of Full Substitutability for Existence

We now show that if the preferences of any one agent are not fully substitutable, then stable outcomes need not exist. In fact, in that case we can construct *simple* preferences for other agents such that no stable outcome exists.

Definition 7. $\psi, \omega \in \Omega_i$ *are*

1. Independent *if* $u_i(\{\psi, \omega\} \cup \Phi) - u_i(\{\omega\} \cup \Phi) = u_i(\{\psi\} \cup \Phi) - u_i(\Phi)$ *for all* $\Phi \subseteq \Omega_i - \{\psi, \omega\}$.
2. Incompatible *if* $\psi, \omega \in \Omega_{\to i}$ *or* $\psi, \omega \in \Omega_{i\to}$ *and* $u_i(\{\psi, \omega\} \cup \Phi) - u(\Phi) = -\infty$ *for all* $\Phi \subseteq \Omega_i - \{\psi, \omega\}$.
3. Dependent *if* $\psi \in \Omega_{\to i}$, $\omega \in \Omega_{i\to}$ *and either* $u_i(\{\psi\} \cup \Phi) - u(\Phi) = -\infty$ *or* $u_i(\{\omega\} \cup \Phi) - u(\Phi) = -\infty$ *for all* $\Phi \subseteq \Omega_i - \{\psi, \omega\}$.

Preferences of agent i are simple *if for all $\psi, \omega \in \Omega_i$, ψ and ω are either independent, incompatible, or dependent.*

Two trades ψ and ω are independent for i if the marginal utility i obtains from performing ψ does not affect the marginal utility that i obtains from performing ω. In contrast, the trades ψ and ω are incompatible for i if i is unable to perform ψ and ω simultaneously; for instance, if ψ and ω both denote the transfer of a particular object, but to different individuals, then $u_i(\{\psi, \omega\}) = -\infty$. Finally, the trades ψ and ω are dependent for i if i can perform one of them only while performing the other; for instance, if ψ denotes the transfer from $s(\psi)$ to i of a necessary input of a production process, and ω denotes the transfer of the output of that process from i to $b(\omega)$, then $u_i(\{\omega\}) = -\infty$.

Simple preferences play a role similar to that of *unit-demand preferences*, used in the Gul and Stacchetti (1999) result characterizing the maximal domain for the existence of competitive equilibria in exchange economies. However, in our setting we must allow an individual agent to act as a set of unit-demand consumers, as each contract specifies both the buyer and the seller, and the violation of substitutability may only occur for an agent i when he holds multiple contracts with another agent.

Our necessity result requires sufficient "richness" of the set of trades. Specifically, we require that the set of trades Ω is *exhaustive*, i.e., that for each $i \neq j \in I$ there exist $\omega_i, \omega_j \in \Omega$ such that $b(\omega_i) = s(\omega_j) = i$ and $b(\omega_j) = s(\omega_i) = j$.

Theorem 7. *Suppose that there exist at least four agents and that the set of trades is exhaustive. Then if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no stable outcome exists.*[9]

To understand the result, consider the following example.

Example 2. Agent i is just a buyer, and has perfectly complementary preferences over the contracts ψ and ω, and is not interested in other contracts, i.e., $u_i(\{\psi, \omega\}) = 1$ and $u_i(\{\psi\}) = u_i(\{\omega\}) = u_i(\varnothing) = 0$.

Suppose that $s(\psi)$ and $s(\omega)$ also have contracts $\hat{\psi}$ and $\hat{\omega}$ (where $s(\hat{\psi}) = s(\psi)$ and

[9]The proof of this result also shows that for two-sided markets with transferable utility if any agent's preferences are not fully substitutable, then if there exists at least one other agent on the same side of the market, simple preferences can be constructed such that no stable outcome exists.

$s(\hat{\omega}) = s(\omega)$) with another agent j, and let the valuations of agents $j \neq i$ be given by:

$$u_{s(\psi)}(\{\hat{\psi}\}) = u_{s(\psi)}(\{\psi\}) = u_{s(\psi)}(\varnothing) = 0, \qquad u_{s(\psi)}(\{\psi, \hat{\psi}\}) = -\infty,$$
$$u_{s(\omega)}(\{\hat{\omega}\}) = u_{s(\omega)}(\{\omega\}) = u_{s(\omega)}(\varnothing) = 0, \qquad u_{s(\omega)}(\{\omega, \hat{\omega}\}) = -\infty,$$
$$u_j(\{\hat{\psi}, \hat{\omega}\}) = u_j(\{\hat{\psi}\}) = u_j(\{\hat{\omega}\}) = \frac{3}{4}, \qquad u_j(\varnothing) = 0.$$

Then in any stable outcome $s(\psi)$ will sell at most one of ψ and $\hat{\psi}$, and similarly for $s(\omega)$. It can not be that $\{\psi, \omega\}$ is part of a stable outcome, as their total price is at most 1, meaning at least one of them has a price at most $\frac{1}{2}$; suppose it is ω—we then have that $\{(\hat{\omega}, \frac{5}{8})\}$ is a blocking set. It also can not be the case that $\{(\hat{\psi}, p_{\hat{\psi}})\}$ or $\{(\hat{\omega}, p_{\hat{\omega}})\}$ is stable: in the former case, $p_{\hat{\psi}}$ must be less than $\frac{3}{4}$, in which case $\{(\psi, \frac{7}{8}), (\omega, \frac{1}{16})\}$ is a blocking set. A symmetric construction holds for the latter case.

The proof of Theorem 7, presented in the Online Appendix to this paper, essentially generalizes Example 2.

Since a stable outcome does not necessarily exist when preferences are not fully substitutable, and (for any preferences) all competitive equilibria generate stable outcomes (by Theorem 5), Theorem 7 immediately implies the following corollary.

Corollary 1. *Suppose that there exist at least four agents and that the set of trades is exhaustive. Then, if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no competitive equilibrium exists.*

5 Relation with Other Solution Concepts

In this section, we describe the relationships between competitive equilibrium, stability, and other solution concepts that have played important roles in the previous literature.

5.1 The Core and Strong Group Stability

We start by introducing a classical solution concept: the *core*.

Definition 8. *An outcome A is in the* core *if it is* core unblocked*: there does not exist a set of contracts Z such that, for all $i \in a(Z)$, $U_i(Z) > U_i(A)$.*

The definition of the core differs from that of stability in two ways. First, a core block requires all the agents with contracts in the blocking set drop their contracts with other agents; this is a more stringent restriction than that of stability, which allows agents with contracts in the blocking set to retain previous relationships. Second, a core block does not require that $Z_i \in C_i(Z \cup A)$ for all $i \in a(Z)$; rather, it requires only the less stringent condition that $U_i(Z) > U_i(A)$.

We now introduce *strong group stability*, which is more restrictive than both core and stability.

Definition 9. *An outcome A is* strongly group stable *if it is*

1. *Individually rational;*
2. Strongly unblocked: *There does not exist a nonempty feasible $Z \subseteq X$ such that*
   (a) $Z \cap A = \varnothing$, *and*
   (b) *for all $i \in a(Z)$, there exists a $Y^i \subseteq Z \cup A$ such that $Z \subseteq Y^i$ and $U_i(Y^i) > U_i(A)$.*

Strong group stability is a more stringent notion than both stability and core as, when considering a block Z, agents may retain previously held contracts (as in the defintion of stability, but not in the definition of the core), and the new set of contracts for each agent need only be an improvement, not optimal (as in the definition of the core, but not the definition of stability).

We call this notion strong group stability as it is more stringent than both strong stability (introduced by Hatfield and Kominers, 2010a) and group stability (introduced by Roth and Sotomayor, 1990, and extended to the setting of many-to-many matching by Konishi and Ünver, 2006): Strong group stability is more stringent than strong stability as strong stability imposes the additional requirement on blocking sets that each Y^i be individually rational. Strong group stability is more stringent than group stability as group stability imposes the additional requirement on blocking sets that if $y \in Y^{b(y)}$, then $y \in Y^{s(y)}$, i.e., that the deviating agents agreed on which contracts from the original allocation would be kept after the deviation. Strong stability and group stability are themselves strengthenings of the notion of setwise stability, introduced by Echenique and Oviedo (2006) and Klaus and Walzl (2009), which imposes both of the above requirements on blocking sets.[10]

Given these definitions, the following result is immediate.

[10] The notion of setwise stability used in these works is slightly stronger than the definition of

Theorem 8. *Any strongly group stable outcome is stable and in the core. Furthermore, any core outcome is efficient.*

Without additional assumptions on preferences, no additional structure need be present. In particular, it may be that both stable and core outcomes exist for a given set of preferences, but that no outcome is both stable and core.

Example 3. There are two agents, i and j, and two trades, ψ and ω, where $s(\psi) = s(\omega) = i$ and $b(\psi) = b(\omega) = j$. Agents' valuations are:

Ψ	$\varnothing$	$\{\psi\}$	$\{\omega\}$	$\{\psi,\omega\}$
$u_i(\Psi)$	0	-2	-2	-6
$u_j(\Psi)$	0	0	0	7

.

The set of core outcomes is given by $\{\{(\psi, p_\psi), (\omega, p_\omega)\} : 6 \le p_\psi + p_\omega \le 7\}$. However, the unique stable outcome is $\varnothing$: Note that any outcome of the form $\{(\psi, p_\psi)\}$ or $\{(\omega, p_\omega)\}$ is clearly not individually rational, and any outcome of the form $\{(\psi, p_\psi), (\omega, p_\omega)\}$ can only be individually rational if $p_\psi, p_\omega \ge 4$ and $p_\psi + p_\omega \le 7$, which can not simultaneously all hold.

Example C.1 of Appendix C demonstrates that even an outcome that is both stable and in the core need not be strongly group stable.

However, even when preferences are fully substitutable, the relationship between strong group stability and stability is unclear. For models without transferable utility (see e.g., Sotomayor, 1999; Echenique and Oviedo, 2006; Klaus and Walzl, 2009; Hatfield and Kominers, 2010a; and Westkamp, 2010), strong group stability is strictly more restrictive than stability. However, with quasilinear utility and fully substitutable preferences, these equilibrium notions coincide.

Theorem 9. *If preferences are fully substitutable and A is a stable outcome, then A is strongly group stable and in the core. Moreover, for any core outcome A, there exists a stable outcome $\hat{A}$ such that $\tau(A) = \tau(\hat{A})$.*

However, even for fully substitutable preferences, the core may be strictly larger than the set of stable outcomes.

setwise stability introduced by Sotomayor (1999); see Klaus and Walzl (2009) for a discussion of the subtle differences between these two definitions.

Example 4. Consider again the setting of Example 3, but take preferences to be:

Ψ	$\varnothing$	$\{\psi\}$	$\{\omega\}$	$\{\psi, \omega\}$
$u_i(\Psi)$	0	0	0	-3
$u_j(\Psi)$	0	5	5	9

In this case, $\{(\psi, 2), (\omega, 2)\}$ is a core outcome, but is not individually rational for agent i; he will choose to drop one of the two contracts. We therefore see that the set of imputed utilities of a core outcome may not correspond to the set of imputed utilities for any stable outcome: in this example, the payoff of agent i in any stable outcome is at least 3, while it is only 1 in the core outcome.

5.2 Chain Stability

The stability concept used in this paper also appears substantively different and noticeably stronger than the chain stability concept used in Ostrovsky (2008): the former (Definition 1) requires robustness to all blocking sets, while the latter requires robustness only to very specific blocking sets—chains of contracts. However, Hatfield and Kominers (2010b) show that in the presence of fully substitutable preferences, in the setting of Ostrovsky (2008) these conditions are in fact equivalent. In this section, we prove an analogous result for the current setting with continuous prices.

First, we need to define the notion of chain stability. It is more general than that in the earlier work, because in the current setting trading cycles are allowed, and the definition needs to accommodate that. If trading cycles are prohibited (i.e., there is an upstream–downstream partial ordering of agents), then the definition is essentially the same as in Ostrovsky (2008), adjusted only for the presence of continuous prices.

Definition 10. *A non-empty set of contracts Z is a* chain *if its elements can be arranged in some order $y^1, \ldots, y^{|Z|}$ such that $s(y^{\ell+1}) = b(y^\ell)$ for all $\ell < |Z|$.*

Note that under this definition, the buyer in contract $y^{|Z|}$ is allowed to be the seller in contract y^1 (in which case the chain becomes a cycle), and also the same agent can be involved in the chain multiple times (because there is no enforced upstream–downstream partial ordering).

Definition 11. *An outcome A is* chain stable *if it is*

1. Individually rational*;*

2. Not blocked by a chain*: There does not exist a chain $Z \subseteq X$ such that*

(a) $Z \cap A = \varnothing$*, and*

(b) for all agents $i \in a(Z)$*, for all* $Y \in C_i (Z \cup A)$*, we have* $Z_i \subseteq Y$*.*

This definition is weaker than the defintion of stability (Definition 1), and so it is immediate that any stable outcome is chain stable for any preferences. The converse, for fully substitutable preferences, is a direct corollary of the following, more general result, which shows that any blocking set of an arbitrary outcome A can be "decomposed" into blocking chains.

Theorem 10. *Suppose agents' preferences are fully substitutable, and consider any outcome* A *that is blocked by some nonempty set* Z*. Then for some* $\overline{m} \geq 1$*, we can partition the set* Z *into a collection of* $\overline{m}$ *chains* W^m *such that* $Z = \cup_{m=1}^{\overline{m}} W^m$*,* A *is blocked by* W^1*, and for any* $m \leq \overline{m} - 1$*, the set of contracts* $A \cup W^1 \cup \cdots \cup W^m$ *is blocked by chain* W^{m+1}*.*

Corollary 2. *If agents' preferences are fully substitutable, then any chain stable outcome* A *is stable.*

Full substitutability is critical for these results. Without it, chain stability is strictly weaker than stability. For instance, consider Example 2 above. There is no stable outcome in that example, yet $\{(\hat{\omega}, 0)\}$ is chain stable: any set that blocks $\{(\hat{\omega}, 0)\}$ includes two contracts of the form (ψ, p_ψ) and (ω, p_ω), and so the blocking set does not constitute a blocking chain.

5.3 Competitive Equilibria without Personalized Prices

The notion of competitive equilibrium studied in this paper considers trades as the basic unit of analysis; a price vector specifies one price for each trade. E.g., if agent i has one object to sell, a competitive equilibrium price vector generally specifies a different price for each possible buyer, allowing for personalized pricing. This is in contrast to the notions of competitive equilibrium that assign a single price to each object (see, e.g., Gul and Stacchetti, 1999, and Sun and Yang, 2006). We now introduce a condition on utilities under which it is convenient to study non-personalized pricing.

Definition 12. *Consider an arbitrary agent* i*. The trades in some set* $\Psi \subseteq \Omega_i$ *are*

1. mutually incompatible for i *if for all* $\Xi \subseteq \Omega_i$ *such that* $|\Xi \cap \Psi| \geq 2$, $u_i(\Xi) = -\infty$.
2. perfect substitutes for i *if for all* $\Xi \subseteq \Omega_i - \Psi$ *and all* $\omega, \omega' \in \Psi$, $u_i(\Xi \cup \{\omega\}) = u_i(\Xi \cup \{\omega'\})$.

Theorem 11. *Let agents' preferences be fully substitutable. Suppose for agent* i*, trades in* $\Psi \subseteq \Omega_i$ *are mutually incompatible and perfect substitutes and let* $[\Xi; p]$ *be an arbitrary competitive equilibrium.*

(a) If $\Psi \subseteq \Omega_{i\to}$*, let* $\overline{p} = \max_{\psi \in \Psi} p_\psi$ *and define* q *by* $q_\psi = \overline{p}$ *for all* $\psi \in \Psi$ *and* $q_\psi = p_\psi$ *for all* $\psi \in \Omega - \Psi$*. Then,* $[\Xi; q]$ *is a competitive equilibrium.*
(b) If $\Psi \subseteq \Omega_{\to i}$*, let* $\underline{p} = \min_{\psi \in \Psi} p_\psi$ *and define* q *by* $q_\psi = \underline{p}$ *for all* $\psi \in \Psi$ *and* $q_\psi = p_\psi$ *for all* $\psi \in \Omega - \Psi$*. Then,* $[\Xi; q]$ *is a competitive equilibrium.*

Note that since the preferences of agent i are fully substitutable, a trade $\omega \in \Omega_{i\to}$ cannot be a perfect substitute for a trade $\omega' \in \Omega_{\to i}$. Hence, the two cases in the theorem are exhaustive.

This result allows us to embed the model of Sun and Yang (2006) as a special case of our model. In their model, a finite set S of indivisible objects needs to be allocated to a finite set I of agents with quasilinear utility. Objects are partitioned into two groups, S_1 and S_2. For each agent i, preferences satisfy the (GSC) condition: Objects in the same group are substitutes and two objects belonging to different groups are complements. To embed this model into our setting, one can view each object in S_1 as an agent who can sell goods to agents in I, and each object in S_2 as an agent who can buy goods from agents in I. Each agent in $S = S_1 \cup S_2$ only cares about prices and can not buy from/sell to more than one agent in I. For this embedding (GSC) is equivalent to full substitutability and all our results apply immediately. Since trades are mutually incompatible and perfect substitutes for every agent in S, we can apply the procedure outlined in Theorem 11 to any competitive equilibrium (with personalized prices) in order to obtain a competitive equilibrium without personalized prices. In particular, our Theorem 2 implies the existence result in Sun and Yang (2006).[11]

[11] Note that our network setting is strictly more general, as it cannot be embedded in the setting of Sun and Yang (2006). E.g., consider a simple market with three agents i, j, and k, where i can sell trades to both j and k, j can sell trades only to k, k cannot sell trades to anyone, and all agents' preferences are fully substitutable. In this market, it is impossible to separate trades into two groups S_1 and S_2 in such a way that every agent views trades in one group as substitutes and views trades in different groups as complements.

6 Conclusion

We have introduced a general model in which a network of agents can trade via bilateral contracts. In this setting, when continuous transfers are allowed and agents' preferences are quasilinear, full substitutability of preferences is sufficient and necessary for the guaranteed existence of stable outcomes. Furthermore, full substitutability implies that the set of stable outcomes is equivalent to the set of competitive equilibria, and that all stable outcomes are strongly group stable, in the core, and efficient.

Previous results have shown that for discrete contractual sets over which agents' preferences are fully substitutable, supply chain structure (i.e., the acyclicity of the contractual set) is sufficient to guarantee the existence of stable outcomes. Moreover, both full substitutability and supply chain structure are necessary to ensure the existence of stable outcomes if the contractual set is otherwise fully general. We show that, for contractual sets that allow for continuous transfers, in the presence of quasilinearity, supply chain structure is not necessary for the existence of stable outcomes, although full substitutability is. It is an open question why the presence of a continuous numeraire can replace the assumption of a supply chain structure in ensuring the existence of stable outcomes. Additionally, the extent to which our results depend on the assumption of quasilinearity is not known: while many of our proofs rely on quasilinearity, analogues of our results may hold without this assumption.

Appendix A: Full Substitutability

In this Appendix, we show the equivalence of eight definitions of full substitutability. Four of these definitions were presented in Section 3. Below we introduce four other definitions, all of which deal directly with indifferences in agents' preferences. These definitions are more convenient to work with in some settings, and also correspond directly to those introduced in the prior literature. After proving the equivalence of these definitions, we introduce the Laws of Aggregate Supply and Demand for the current setting and show that quasilinearity and full substitutability imply those laws. This result is subsequently used in the proof of Theorem 10, and is also of independent interest.

A.1 Equivalent Definitions

The first two definitions are analogues of Definition 3, explicitly considering indifferences in preferences. The first one states what happens when an agent's set of options expands, and the second one states what happens when it shrinks.

Definition A.1. *The preferences of agent i are* choice-language expansion fully substitutable *(CEFS) if:*

1. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that $(Y_{\to i} - Y^*_{\to i}) \subseteq (Z_{\to i} - Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*
2. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that $(Y_{i\to} - Y^*_{i\to}) \subseteq (Z_{i\to} - Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.*

Definition A.2. *The preferences of agent i are* choice-language contraction fully substitutable *(CCFS) if:*

1. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for every $Z^* \in C_i(Z)$ there exists $Y^* \in C_i(Y)$ such that $(Y_{\to i} - Y^*_{\to i}) \subseteq (Z_{\to i} - Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*
2. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for every $Z^* \in C_i(Z)$ there exists $Y^* \in C_i(Y)$ such that $(Y_{i\to} - Y^*_{i\to}) \subseteq (Z_{i\to} - Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.*

Note that we use Y as the "starting set" in (CEFS) and Z as the "starting set" in (CCFS) to make the two notions of choice-language full substitutability more easily comparable. Furthermore, note that in Case 1 of (CEFS) and (CCFS), requiring $Y_{\to i} - Y^*_{\to i} \subseteq Z_{\to i} - Z^*_{\to i}$ is equivalent to requiring that $Z^* \cap Y_{\to i} \subseteq Y^*$, and similarly, in Case 2, requiring $Y_{i\to} - Y^*_{i\to} \subseteq Z_{i\to} - Z^*_{i\to}$ is equivalent to requiring that $Z^* \cap Y_{i\to} \subseteq Y^*$.

The next two definitions are analogues of Definition 4.

Definition A.3. *The preferences of agent i are* demand-language expansion fully substitutable *(DEFS) if:*

1. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;*

2. *for all price vectors* $p, p' \in \mathbb{R}^{|\Omega|}$ *such that* $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{\rightarrow i}$ *and* $p_\omega \leq p'_\omega$ *for all* $\omega \in \Omega_{i\rightarrow}$*, for every* $\Psi \in D_i(p)$ *there exists* $\Psi' \in D_i(p')$ *such that* $\{\omega \in \Psi'_{i\rightarrow} : p_\omega = p'_\omega\} \subseteq \Psi_{i\rightarrow}$ *and* $\Psi_{\rightarrow i} \subseteq \Psi'_{\rightarrow i}$*.*

Definition A.4. *The preferences of agent* i *are* demand-language contraction fully substitutable *(DCFS) if:*

1. *for all price vectors* $p, p' \in \mathbb{R}^{|\Omega|}$ *such that* $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{i\rightarrow}$ *and* $p_\omega \geq p'_\omega$ *for all* $\omega \in \Omega_{\rightarrow i}$*, for every* $\Psi' \in D_i(p')$ *there exists* $\Psi \in D_i(p)$ *such that* $\{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \subseteq \Psi_{\rightarrow i}$ *and* $\Psi_{i\rightarrow} \subseteq \Psi'_{i\rightarrow}$*;*
2. *for all price vectors* $p, p' \in \mathbb{R}^{|\Omega|}$ *such that* $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{\rightarrow i}$ *and* $p_\omega \leq p'_\omega$ *for all* $\omega \in \Omega_{i\rightarrow}$*, for every* $\Psi' \in D_i(p')$ *there exists* $\Psi \in D_i(p)$ *such that* $\{\omega \in \Psi'_{i\rightarrow} : p_\omega = p'_\omega\} \subseteq \Psi_{i\rightarrow}$ *and* $\Psi_{\rightarrow i} \subseteq \Psi'_{\rightarrow i}$*.*

Note that we use p as the "starting price vector" in (DEFS) and p' as the "starting price vector" in (DCFS). Also, in Case 1 of (DEFS) and (DCFS), requiring $\{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \subseteq \Psi_{\rightarrow i}$ is equivalent to requiring that $\{\omega \in (\Omega_{\rightarrow i} - \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{\rightarrow i} - \Psi'$, and similarly, in Case 2, requiring $\{\omega \in \Psi'_{i\rightarrow} : p_\omega = p'_\omega\} \subseteq \Psi_{i\rightarrow}$ is equivalent to requiring that $\{\omega \in (\Omega_{i\rightarrow} - \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{i\rightarrow} - \Psi'$.

A.2 The Equivalence Result

Theorem A.1. *(CFS), (DFS), (IFS), (CEFS), (CCFS), (DEFS), and (DCFS) are all equivalent, and hold if and only if the indirect utility function is submodular.*

Proof. It is immediate that (CEFS) and (CCFS) each imply (CFS). Below we show that (CFS) $\Rightarrow$ (DFS), (DFS) $\Rightarrow$ (DEFS), (DFS) $\Rightarrow$ (DCFS), (DEFS) $\Rightarrow$ (CEFS), and (DCFS) $\Rightarrow$ (CCFS), thus proving the equivalence of these six definitions. We then show that these definitions are equivalent to (IFS) and the submodularity of the indirect utility function.

(CFS) $\Rightarrow$ (DFS)

We first show that Case 1 of (DFS) implies Case 1 of (CFS). For any agent i and price vector $p \in \mathbb{R}^{|\Omega|}$, let $X_i(p) \equiv \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{\rightarrow i},\ \hat{p}_\omega \geq p_\omega\} \cup \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{i\rightarrow},\ \hat{p}_\omega \leq p_\omega\}$, in essence denoting the set of contracts available to agent i under prices p.

Let price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, and $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\rightarrow i}$. Let $Y = X_i(p)$ and $Z = X_i(p')$. Clearly,

$Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Furthermore, it is immediate that $\Psi \in D_i(p)$ if and only if $\kappa([\Psi; p]) \in C_i(Y)$, and similarly, $\Psi' \in D_i(p')$ if and only if $\kappa([\Psi'; p']) \in C_i(Z)$. In particular, we have $|C_i(Y)| = |C_i(Z)| = 1$ and can thus apply (CFS) to the sets Y and Z.

Take the unique $\Psi \in D_i(p)$, let $Y^* = \kappa([\Psi, p])$, and note that $Y^* \in C_i(Y)$. By (CFS), the unique $Z^* \in C_i(Z)$ satisfies $Y_{\to i} - Y^*_{\to i} \subseteq Z_{\to i} - Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$. Let $\Psi' = \tau(Z^*)$ and note that $\Psi' \in D_i(p')$. We show that Ψ' satisfies the conditions in Case 1 of Definition 4.

Note that $Y_{\to i} - Y^*_{\to i} \subseteq Z_{\to i} - Z^*_{\to i}$ implies $\{\omega \in \Omega_{\to i} - \Psi_{\to i} : p_\omega = p'_\omega\} \subseteq \tau(Y_{\to i}) - \tau(Y^*_{\to i}) \subseteq \tau(Z_{\to i}) - \tau(Z^*_{\to i}) \subseteq \Omega_{\to i} - \Psi'_{\to i}$. Furthermore, $Y^*_{i\to} \subseteq Z^*_{i\to}$ and $p_\omega = p'_\omega$ for each $\omega \in \Omega_{i\to}$ imply $\Psi'_{i\to} \subseteq \Psi_{i\to}$.

The proof that Case 2 of (CFS) implies Case 2 of (DFS) is analogous.

(DFS) ⇒ (DEFS), (DFS) ⇒ (DCFS)

We first show that Case 1 of (DFS) implies Case 1 of (DEFS). Take two price vectors p, p' such that $p'_\omega \le p_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and fix an arbitrary $\Psi \in D_i(p)$. We need to show that there exists a set $\Psi' \in D_i(p')$ that satisfies the conditions of Case 1 of (DEFS).

As the statement is trivial when $D_i(p') = \{\Xi : \Xi \subset \Omega_i\}$, we assume the contrary. Furthermore, we assume that $D_i(p) \neq \{\Xi : \Xi \subset \Omega_i\}$; the arguments below are easily extended to the case where this assumption is not satisfied. In the following, let $\tilde{\Omega}_{\to i} = \{\omega \in \Omega_{\to i} : p'_\omega < p_\omega\}$. Let $\varepsilon_1 = V_i(p) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(p)} U_i([\Xi; p])$, $\varepsilon_2 = V_i(p') - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(p')} U_i([\Xi; p'])$, and $\varepsilon_3 = \min_{\omega \in \tilde{\Omega}_{\to i}} (p_\omega - p'_\omega)$. Let $\varepsilon = \frac{\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}}{3|\Omega_i|}$. Note that by construction, $\varepsilon > 0$.

We now define a price vector q^1 by

$$q^1_\omega = \begin{cases} p_\omega - \varepsilon & \omega \in \Omega_{i\to} - \Psi \text{ or } \omega \in \Psi_{\to i} \\ p_\omega + \varepsilon & \omega \in \Omega_{\to i} - \Psi \text{ or } \omega \in \Psi_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^1) = \{\Psi\}$. Now define q^2 by $q^2_\omega = q^1_\omega$ for all $\omega \in \Omega - \tilde{\Omega}_{\to i}$ and $q^2_\omega = p'_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. We claim that $D_i(q^2) \subseteq D_i(p')$. To see this, fix an arbitrary $\Phi \in D_i(p')$ and an arbitrary $\Xi \notin D_i(p')$. Then we must have

$$U([\Phi; q^2]) \geq U([\Phi; p']) - |\Phi|\varepsilon > U([\Xi; p']) + |\Xi|\varepsilon \geq U([\Xi; q^2]),$$

where the first and third inequalities follow directly from the definitions of q^2, and the second inequality follows from $(|\Xi| + |\Phi|)\varepsilon \leq 2|\Omega_i|\varepsilon < U([\Phi; p']) - U([\Xi; p'])$.

We will now show that the condition in Case 1 of Definition 4 is satisfied for any set of trades $\Psi' \in D_i(q^2)$. Take any such Ψ'. Similar to the above, we define $\delta_1 = V_i(q^1) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^1)} U_i([\Xi; q^1])$, $\delta_2 = V_i(q^2) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^2)} U_i([\Xi; q^2])$, and $\delta_3 = \min_{\omega \in \tilde{\Omega}_{\to i}} (q^1_\omega - p'_\omega)$. Let $\delta = \frac{\min\{\delta_1, \delta_2, \delta_3\}}{3|\Omega_i|}$, and define price vector q^3 as

$$q^3_\omega = \begin{cases} q^2_\omega - \delta & \omega \in \Omega_{i\to} - \Psi' \text{ or } \omega \in \Psi'_{\to i} \\ q^2_\omega + \delta & \omega \in \Omega_{\to i} - \Psi' \text{ or } \omega \in \Psi'_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^3) = \{\Psi'\}$. Now define q^4 by $q^4_\omega = q^3_\omega$ for all $\omega \in \Omega - \tilde{\Omega}_{\to i}$ and $q^4_\omega = q^1_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. Similar to the above, we can show that $D_i(q^4) \subseteq D_i(q^1)$, and therefore $D_i(q^4) = \{\Psi\}$. Since $q^3_\omega < q^4_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$ and $q^3_\omega = q^4_\omega$ for all $\omega \in \Omega - \tilde{\Omega}_{\to i}$, we can now apply Case 1 of (DFS) to conclude that Ψ' satisfies the condition in Case 1 of (DEFS).

The proofs that Case 2 of (DFS) implies Case 2 of (DEFS), and that (DFS) implies (DCFS) are completely analogous.

(DEFS) $\Rightarrow$ (CEFS), (DCFS) $\Rightarrow$ (CCFS)

We first prove Case 1 of (CEFS). Take agent i and any sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$. Define *usable* and *unusable* trades in Y as follows. Take trade $\omega \in Y_{i\to}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' > r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Similarly, take trade $\omega \in Y_{\to i}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' < r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Note that an unusable trade cannot be a part of any contract involved in any optimal choice in $C_i(Y)$. The definitions of trades usable and unusable in Z are completely analogous.

We now construct preliminary price vectors q and q' as follows. First, for every trade $\omega \notin \Omega_i$, $q_\omega = q'_\omega = 0$. Second, for every trade ω unusable in Y, $q_\omega = 0$, and for every trade ω unusable in Z, $q'_\omega = 0$. Next, for any trade $\omega \in \Omega_{i\to}$ usable in Y, $q_\omega = \max\{r : (\omega, r) \in Y\}$, and similarly, for any trade $\omega \in \Omega_{i\to}$ usable in Z, $q'_\omega = \max\{r : (\omega, r) \in Z\}$. Finally, for any trade $\omega \in \Omega_{\to i}$ usable in Y, $q_\omega = \min\{r : (\omega, r) \in Y\}$ and for any trade $\omega \in \Omega_{\to i}$ usable in Z, $q'_\omega = \min\{r : (\omega, r) \in Z\}$.

We now construct price vectors p and p'. First, for any trade $\omega \notin \Omega_i$, $p_\omega = p'_\omega = 0$. Second, for any trade $\omega \in \Omega_i$ that is usable in both Y and Z, let $p_\omega = q_\omega$ and let $p'_\omega = q'_\omega$. Finally, we need to set prices for trades unusable in Y or Z. We already noted that for any trade ω unusable in set Y, it has to be the case that ω is not involved in any contract in any optimal choice in $C_i(Y)$; and likewise, if ω is unusable in Z, then ω is not involved in any contract in any optimal choice in $C_i(Z)$. Thus, in forming prices p and p', we will need to assign to these trades prices that are so large (or small, depending on which side the trade is on) that the corresponding trades are not demanded by agent i.

Let Π be a very large number. For instance, let

$$\Delta_1 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i(\Omega_1; q) - U_i(\Omega_2; q)|,$$

$$\Delta_2 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i(\Omega_1; q') - U_i(\Omega_2; q')|,$$

and $\Pi = 1 + \Delta_1 + \Delta_2 + \max_{\omega \in \Omega_i} |q_\omega| + \max_{\omega \in \Omega_i} |q'_\omega|$. For all $\omega \in \Omega_{i\to}$ that are unusable in Y (and thus also in Z), let $p_\omega = p'_\omega = -\Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in both Y and Z, let $p_\omega = p'_\omega = \Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in Y but not in Z, let $p_\omega = \Pi$ and $p'_\omega = q'_\omega$. Finally, for all $\omega \in \Omega_{\to i}$ that are unusable in Z but not in Y, let $p_\omega = p'_\omega = q_\omega$. Note that for any such ω, since $Y \subset Z$, $(\omega, q_\omega) \in Z$; also, as ω is unusable in Z, there are no contracts involving ω in any optimal choice in $C_i(Z)$.

Now, $p'_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p'_\omega \le p_\omega$ for all $\omega \in \Omega_{\to i}$. Take any $Y^* \in C_i(Y)$, and let $\Psi = \tau(Y^*)$. By construction, $\Psi \in D_i(p)$. By (DEFS), there exists $\Psi' \in D_i(p')$ such that $\{\omega \in (\Omega_{\to i} - \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} - \Psi'_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Let $Z^* = \kappa([\Psi', p'])$. Again, by construction, $Z^* \in C_i(Z)$. We now show that this set of contracts satisfies the conditions in Case 1 of (CEFS).

First, take some $y \in Y_{\to i} - Y^*_{\to i}$ and suppose that contrary to what we want to show, $y \in Z^*_{\to i}$. The latter implies that $y = (\omega, p'_\omega)$ for some trade ω, which, in turn, implies that $p_\omega = p'_\omega$ (because $y = (\omega, p'_\omega) \in Y$ and, since $Y \subset Z$, $(\omega, r) \notin Y$ for any $r < p'_\omega$).

But then, by construction, $\{\omega \in (\Omega_{\to i} - \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} - \Psi'_{\to i}$, contradicting $y \in Z^*_{\to i}$. Second, since $Y^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi_{i\to}\}$, $Z^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi'_{i\to}\}$, and $\Psi_{i\to} \subseteq \Psi'_{i\to}$, it is immediate that $Y^*_{i\to} \subseteq Z^*_{i\to}$. This completes the proof that Case 1 of (DEFS) implies Case 1 of (CEFS).

The proofs that Case 2 of (DEFS) implies Case 2 of (CEFS) and that (DCFS) implies (CCFS) are completely analogous.

(DFS) $\Rightarrow$ (IFS)

Take two price vectors p, p' such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$. Let $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$ be the unique demanded sets. We have to show that $e_\omega(\Psi') \geq e_\omega(\Psi)$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

First, let p^1 be a price vector such that $p^1_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^1_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$. By (DCFS) there must exist a $\Psi^1 \in D_i(p^1)$ such that $\{\omega \in \Psi_{\to i} : p^1_\omega = p_\omega\} \subseteq \Psi^1$ and $\Psi^1_{i\to} \subseteq \Psi_{i\to}$. This immediately implies $e_\omega(\Psi^1) \geq e_\omega(\Psi)$ for all $\omega \in \Omega_i$ such that $p^1_\omega = p_\omega$. Now by (DEFS) we must have $\{\omega \in \Omega_{\to i} - \Psi^1 : p^1_\omega = p_\omega\} \subseteq \Omega - \Psi'$ and $\Psi^1_{\to i} \subseteq \Psi'_{\to i}$, implying $e_\omega(\Psi') \geq e_\omega(\Psi^1)$ for all $\omega \in \Omega_i$ such that $p'_\omega = p^1_\omega$. Combining this with the above, we obtain the desired statement.

(IFS) $\Rightarrow$ (DFS)

This follows immediately, because the price change conditions in both Cases 1 and 2 of (DFS) are special cases of the price change condition of (IFS).

Submodularity

The proof of the equivalence of the (DFS) condition and the submodularity of the indirect utility function is completely analogous to the proof of Theorem 2 of Sun and Yang (2009). We cannot apply Theorem 2 of Sun and Yang (2009) directly, because in addition to (DFS) ((DCFS) to be precise), Sun and Yang impose monotonicity and boundedness conditions on u_i. The proof of that theorem, however, does not rely on these additional assumptions. More specifically, it only relies on (i) (DCFS), (ii) the equivalence of (DCFS) and (DEFS), which we have shown above to hold in our setting as well, and (iii) the monotonicity of the *indirect* utility function, which always holds. $\square$

A.3 Laws of Aggregate Supply and Demand

An important property of fully substitutable preferences in two-sided quasilinear settings is the Law of Aggregate Demand (Hatfield and Milgrom, 2005). Its analogues for the current setting are the Laws of Aggregate Supply and Demand, introduced by Hatfield and Kominers (2010b). Below we show that in the current network setting with quasilinear utilities and continuous transfers, full substitutability implies that preferences satisfy these laws.

Definition A.5. *Agent i's preferences satisfy the* Law of Aggregate Demand *if for all finite sets of contracts Y and Y' such that $Y_{i\to} = Y'_{i\to}$ and $Y_{\to i} \subseteq Y'_{\to i}$, for any $W \in C_i(Y)$, there exists $W' \in C_i(Y')$ such that $|W'_{\to i}| - |W_{\to i}| \geq |W'_{i\to}| - |W_{i\to}|$.*

Agent i's preferences satisfy the Law of Aggregate Supply *if for all finite sets of contracts Y and Y' such that $Y_{i\to} \subseteq Y'_{i\to}$ and $Y_{\to i} = Y'_{\to i}$, for any $W \in C_i(Y)$, there exists $W' \in C_i(Y')$ such that $|W'_{i\to}| - |W_{i\to}| \geq |W'_{\to i}| - |W_{\to i}|$.*

Theorem A.2. *If the preferences of agent i are fully substitutable and quasilinear in the numeraire, then they satisfy the Laws of Aggregate Supply and Demand.*

Proof. We prove the Law of Aggregate Demand; the proof of the Law of Aggregate Supply is analogous.

Fix a fully substitutable valuation function u_i for agent i. First, take two finite sets of contracts Y and Y' such that $|C_i(Y)| = |C_i(Y')| = 1$, $Y_{i\to} = Y'_{i\to}$, and $Y_{\to i} \subseteq Y'_{\to i}$, and let $W \in C_i(Y)$, $W' \in C_i(Y')$. Define a modified valuation $\tilde{u}_i$ on $\tau(Y) \cup \tau(Y')$ for agent i by setting, for each $\Psi \subseteq \tau(Y) \cup \tau(Y')$,

$$\tilde{u}_i(\Psi) = u_i(\Psi_{\to i} \cup (\tau(Y) - \Psi)_{i\to}).$$

Let $\tilde{C}_i$ denote the associated choice correspondence. Using the same arguments as in the proof of Theorem 2 (step 2), one can show that $\tilde{u}_i$ satisfies the gross substitutes condition of Kelso and Crawford (1982). Furthermore, we must have $\tilde{C}_i(Y) = \{W_{\to i} \cup (Y - W)_{i\to}\}$ and $\tilde{C}_i(Y') = \{W'_{\to i} \cup (Y - W')_{i\to}\}$. Since we assume quasilinear utility, the law of aggregate demand applies to $\tilde{C}_i$ (Theorem 7 in Hatfield and Milgrom, 2005). Since $Y \subseteq Y'$, this implies $|W'_{\to i} \cup (Y - W')_{i\to}| \geq |W_{\to i} \cup (Y - W)_{i\to}|$. The last inequality is equivalent to $|W'_{\to i}| - |W_{\to i}| \geq |W'_{i\to}| - |W_{i\to}|$, which is precisely what we needed to show. The proof that the Law of Aggregate Demand for the case in which choice correspondences are single-valued implies the more general case in which they can be multi-valued is analogous to the proof of the implication (DFS)⇒(DEFS) in Theorem A.1. □

Appendix B: Proofs of Results in Sections 4 and 5

Proof of Theorem 2

The proof consists of four steps: (1) transforming the original valuations into bounded ones, (2) constructing a two-sided many-to-one matching market with transfers, based on the network market with bounded valuations, (3) picking a full-employment competitive equilibrium in the two-sided market, and (4) using that equilibrium to construct a competitive equilibrium in the original market.

Step 1: We first transform a fully substitutable but potentially unbounded from below valuation function u_i into a fully substitutable and bounded valuation function $\hat{u}_i$. For this purpose, we now introduce a very high price Π. Specifically, for each agent i, let $\overline{u}_i$ be the highest possible absolute value of the utility of agent i from a combination of trades, i.e., $\overline{u}_i = \max_{\{\Psi\in\Omega_i:|u_i(\Psi)|<\infty\}} |u_i(\Psi)|$. Then set $\Pi = 2\sum_i \overline{u}_i + 1$. Consider the following *modified economy*. Assume that for every trade, the buyer of that trade can always purchase a perfect substitute for that trade for Π and the seller of that trade can always produce this trade at the cost of Π with no inputs needed. Formally, for each agent i, for a set of trades $\Psi \subseteq \Omega_i$, let

$$\hat{u}_i(\Psi) = \max_{\Psi'\subseteq\Psi} \left[u_i(\Psi') - \Pi \cdot |\Psi - \Psi'|\right].$$

For the economy with valuations $\hat{u}_i$, let $\hat{U}_i$ denote agent i's utility function and let $\hat{D}_i$ denote the modified demand correspondence. Note that by the choice of Π, for any $\Psi \subseteq \Omega_i$, $\overline{u}_i \geq \hat{u}_i(\Psi) \geq \max\{u_i(\varnothing) - \Pi \cdot |\Psi|, u_i(\Psi)\}$, and that $\hat{u}_i(\Psi) = u_i(\Psi)$ whenever $u_i(\Psi) \neq -\infty$. We will use these facts throughout the proof.

The rest of Step 1 consists of proving the following lemma.

Lemma B.1. *Utility function $\hat{U}_i$ is fully substitutable.*

Proof. We will first prove the following auxiliary statement. Take any fully substitutable valuation function u_i. Take any trade $\phi \in \Omega_{i\to}$. Consider a modified valuation function u_i^ϕ:

$$u_i^\phi(\Psi) = \max[u_i(\Psi), u_i(\Psi - \phi) - \Pi].$$

I.e., this valuation function allows (but does not require) agent i to pay Π instead of forming one particular trade, ϕ. Then valuation function u_i^ϕ is also fully substitutable.

To see this, consider utility U_i^ϕ and demand D_i^ϕ corresponding to valuation u_i^ϕ. We will show that D_i^ϕ satisfies (IFS). Fix two price vectors p and p' such that $p \leq p'$ and $|D_i^\phi(p)| = |D_i^\phi(p')| = 1$. Take $\Psi \in D_i^\phi(p)$ and $\Psi' \in D_i^\phi(p')$. We need to show that for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$, $e_\omega(\Psi) \leq e_\omega(\Psi')$.

Let price vector q coincide with p on all trades other than ϕ, and set $q_\phi = \min\{p_\phi, \Pi\}$. Note that if $p_\phi < \Pi$, then $p = q$ and $D_i^\phi(p) = D_i(p)$. If $p_\phi > \Pi$, then under utility U_i^ϕ, agent i always wants to form trade ϕ at price p_ϕ, and the only decision is whether to "buy it out" or not at the cost Π; i.e., the agent's effective demand is the same as under price vector q. Thus, $D_i^\phi(p) = \{\Xi \cup \{\phi\} : \Xi \in D_i(q)\}$. Finally, if $p_\phi = \Pi$, then $p = q$ and $D_i^\phi(p) = D_i(p) \cup \{\Xi \cup \{\phi\} : \Xi \in D_i(p)\}$. Construct price vector q' corresponding to p' analogously.

Now, if $p_\phi \leq p'_\phi < \Pi$, then $D_i^\phi(p) = D_i(p)$, $D_i^\phi(p') = D_i(p')$, and thus $e_\omega(\Psi) \leq e_\omega(\Psi')$ follows directly from (IFS) for demand D_i.

If $\Pi \leq p_\phi \leq p'_\phi$, then (since we assumed that D_i^ϕ was single-valued at p and p') it has to be the case that D_i is single-valued at the corresponding price vectors q and q'. Let $\Xi \in D_i(q)$ and $\Xi' \in D_i(q')$. Then $\Psi = \Xi \cup \{\phi\}$, $\Psi' = \Xi' \cup \{\phi\}$, and the statement follows from (IFS) for demand D_i, because $q \leq q'$.

Finally, if $p_\phi < \Pi \leq p'_\phi$, then $p = q$, Ψ is the unique element in $D_i(p)$, and Ψ' is equal to $\Xi' \cup \{\phi\}$, where Ξ' is the unique element in $D_i(q')$. Then for $\omega \neq \phi$, the statement follows from (IFS) for demand D_i, because $p \leq q'$. For $\omega = \phi$, the statement does not need to be checked, because $p_\phi < p'_\phi$.

Thus, valuation function u_i^ϕ is fully substitutable. The proof for the case $\phi \in \Omega_{\rightarrow i}$ is completely analogous.

To complete the proof of the lemma, it is now enough to note that valuation function $\hat{u}_i(\Psi) = \max_{\Psi' \subseteq \Psi} [u_i(\Psi') - \Pi \cdot |\Psi - \Psi'|]$ can be obtained from the original valuation u_i by allowing agent i to buy out all of his trades, one by one, and since each such transformation preserves substitutability, $\hat{u}_i$ is substitutable as well. $\square$

Step 2: We now transform the modified economy with bounded and fully substitutable valuations $\hat{u}_i$ into an associated two-sided many-to-one matching market with transfers, which will satisfy the assumptions of Kelso and Crawford (1982; subsequently KC). The set of firms in this market is I, and the set of workers is Ω.

Worker ω can be matched to at most one firm. His utility is defined as follows. If he is matched to firm $i \in \{s(\omega), b(\omega)\}$, then his utility is equal to the monetary transfer that he receives from that firm, i.e., his salary $p_{i,\omega}$, which can in principle be negative. If he is matched to any other firm i, his utility is equal to $-\Pi - 1 + p_{i,\omega}$, where Π is as defined in Step 1 and $p_{i,\omega}$ is the salary firm i pays him. If worker ω remains unmatched, his utility is equal to $-2\Pi - 2$.

Firm i can be matched to any set of workers, but only its matches to workers $\omega \in \Omega_i$ have an impact on its valuation. Formally, firm i's valuation from hiring a set of workers $\Psi \subseteq \Omega$ is given by

$$\tilde{u}_i(\Psi) = \hat{u}_i\left(\Psi_{\to i} \cup (\Omega - \Psi)_{i\to}\right) - \hat{u}_i\left(\Omega_{i\to}\right),$$

where the second term in the difference is simply a constant, which ensures that $\tilde{u}_i(\varnothing) = 0$ and thus valuation function $\tilde{u}_i$ satisfies assumption (NFL) of KC. Hiring a set of workers $\Psi \subseteq \Omega$ when the salary vector is $p \in \mathbb{R}^{|I|\times|\Omega|}$ yields i a utility of

$$\tilde{U}_i([\Psi; p]) \equiv \tilde{u}_i(\Psi_i) - \sum_{\omega\in\Psi} p_{i,\omega}.$$

The associated demand correspondence is denoted by $\tilde{D}_i$.

Assumption (MP) of KC requires that any firm's change in valuation from adding a worker, ω, to any set of other workers is at least as large as the lowest salary worker ω would be willing to accept from the firm when his only alternative is to remain unmatched. This assumption is also satisfied in our market: A worker's utility from remaining unmatched is $-2\Pi-2$, while his valuation, excluding salary, from matching with any firm is at least $-\Pi-1$, and so he would strictly prefer to work for any firm for negative salary $-\Pi$ instead of remaining unmatched. At the same time, the change in valuation of any firm i from adding worker ω to a set of workers Ψ is equal to $\tilde{u}_i(\Psi \cup \{\omega\}) - \tilde{u}_i(\Psi) \geq -\bar{u}_i - \bar{u}_i > -\Pi$, and thus every firm i would also always strictly prefer to hire worker ω for the negative salary $-\Pi$.

Finally, we show that i's preferences in this market satisfy the gross substitutes (GS) condition of KC. Take two salary vectors $p, p' \in \mathbb{R}^{|I|\times|\Omega|}$ such that $p \leq p'$ and $|\tilde{D}_i(p)| = |\tilde{D}_i(p')| = 1$. Let $\Psi \in \tilde{D}_i(p)$ and $\Psi' \in \tilde{D}_i(p')$. Denote by $q = (p_{i,\omega})_{\omega\in\Omega}$ and

$q' = (p'_{i,\omega})_{\omega \in \Omega}$ the vectors of salaries that i faces under p and p', respectively. Note that $\Psi \in \tilde{D}_i(p)$ if and only if $(\Psi_{\to i} \cup (\Omega - \Psi)_{i\to}) \in \hat{D}_i(q)$ and $\Psi' \in \tilde{D}_i(p')$ if and only if $(\Psi'_{\to i} \cup (\Omega - \Psi')_{i\to}) \in \hat{D}_i(q')$. In particular, $|\hat{D}_i(q)| = |\hat{D}_i(q')| = 1$. Since $q \leq q'$ and demand $\hat{D}_i$ is fully substitutable, (IFS) implies that for any $\omega \in \Psi_{\to i}$ such that $q_\omega = q'_\omega$, we have $\omega \in \Psi'_{\to i}$, and for any $\omega \notin (\Omega_{i\to} - \Psi_{i\to})$ such that $q_\omega = q'_\omega$, we have $\omega \notin (\Omega_{i\to} - \Psi'_{i\to})$. In other words, for every $\omega \in \Psi$ such that $q_\omega = q'_\omega$, we have $\omega \in \Psi'$, and thus the (GS) condition is satisfied for all salary vectors for which demand $\tilde{D}_i$ is single-valued. By the implication (DFS)$\Rightarrow$(DCFS) of Theorem A.1, it follows that (GS) is satisfied for all salary vectors.

Step 3: By the results of KC (Theorem 2 and the discussion in Section 2), there exists a full-employment competitive equilibrium of the two-sided market constructed in Step 2. Take one such equilibrium, and for every ω and i, let $\mu(\omega)$ denote the firm matched to ω in this equilibrium and let $r_{i,\omega}$ denote equilibrium salary of ω at i.

Note that in this equilibrium, it must be the case that every worker ω is matched to either $b(\omega)$ or $s(\omega)$. Indeed, suppose ω is matched to some other firm $i \notin \{b(\omega), s(\omega)\}$. Since by definition, for any $\Psi \subset \Omega$, $\tilde{u}_i(\Psi \cup \{\omega\}) - \tilde{u}_i(\Psi) = 0$, it must be the case that $r_{i,\omega} \leq 0$. Then, for worker ω to weakly prefer to work for i rather than $b(\omega)$, it must be the case that $r_{b(\omega),\omega} \leq -\Pi - 1$. But at that salary, firm $b(\omega)$ strictly prefers to hire ω, contradicting the assumption that ω is not matched to $b(\omega)$ in this equilibrium.

Note also that if $\mu(\omega) = b(\omega)$, then $r_{b(\omega),\omega} \geq r_{s(\omega),\omega}$, and if $\mu(\omega) = s(\omega)$, then $r_{s(\omega),\omega} \geq r_{b(\omega),\omega}$ (otherwise, worker ω would strictly prefer to change his employer). Now, define prices $p_{i,\omega}$ as follows: if $i \neq b(\omega)$ and $i \neq s(\omega)$, then $p_{i,\omega} = r_{i,\omega}$. Otherwise, $p_{i,\omega} = \max\{r_{b(\omega),\omega}, r_{s(\omega),\omega}\}$. Note that matching μ and associated prices $p_{i,\omega}$ also constitute a competitive equilibrium of the two-sided market.

Step 4: We can now construct a competitive equilibrium for the original economy. Let $p^* \in \mathbb{R}^{|\Omega|}$ be defined as $p^*_\omega \equiv p_{\mu(\omega),\omega}$ for each $\omega \in \Omega$, i.e., the salary that ω actually receives in the equilibrium of the two-sided market. Let $\Psi^* \equiv \{\omega \in \Omega : \mu(\omega) = b(\omega)\}$, i.e., the set of trades/workers who in the equilibrium of the two-sided market are matched to their buyers (and thus not matched to their sellers!).

We now claim that $[\Psi^*; p^*]$ is a competitive equilibrium of the network economy with bounded valuations $\hat{u}_i$. Take any set of trades $\Psi \in \Omega_i$. We will show that $\hat{U}_i([\Psi^*; p^*]) \geq \hat{U}_i([\Psi; p^*])$. By construction, for any $\omega \in \Omega_{\to i}$, $\omega \in \Psi^*$ if and only if $i = \mu(\omega)$, and for any $\omega \in \Omega_{i\to}$, $\omega \in \Psi^*$ if and only if $i \neq \mu(\omega)$. Thus, in the equilibrium of the two-sided market, firm i is matched to the set of workers $\Psi^*_{\to i} \cup (\Omega_{i\to} - \Psi^*_{i\to})$, which implies that

$$\tilde{u}_i(\Psi^*_{\to i} \cup (\Omega_{i\to} - \Psi^*_{i\to})) - \sum_{\omega\in\Psi^*_{\to i}} p_{i,\omega} - \sum_{\omega\in(\Omega_{i\to}-\Psi^*_{i\to})} p_{i,\omega}$$
$$\geq \tilde{u}_i(\Psi_{\to i} \cup (\Omega_{i\to} - \Psi_{i\to})) - \sum_{\omega\in\Psi_{\to i}} p_{i,\omega} - \sum_{\omega\in(\Omega_{i\to}-\Psi_{i\to})} p_{i,\omega}. \quad (2)$$

Using the definition of $\tilde{u}_i$ and the fact that for any set $\Phi \subseteq \Omega_{i\to}$, $\sum_{\omega\in(\Omega_{i\to}-\Phi)} p_{i,\omega} = \left(\sum_{\omega\in\Omega_{i\to}} p_{i,\omega}\right) - \left(\sum_{\omega\in\Phi} p_{i,\omega}\right)$, we can rewrite the inequality (2) as

$$\hat{u}_i(\Psi^*_{\to i} \cup \Psi^*_{i\to}) - \sum_{\omega\in\Psi^*_{\to i}} p_{i,\omega} + \sum_{\omega\in\Psi^*_{i\to}} p_{i,\omega} \geq \hat{u}_i(\Psi_{\to i} \cup \Psi_{i\to}) - \sum_{\omega\in\Psi_{\to i}} p_{i,\omega} + \sum_{\omega\in\Psi_{i\to}} p_{i,\omega},$$

which in turn can be rewritten as

$$\hat{U}_i([\Psi^*; p^*]) = \hat{u}_i(\Psi^*_i) - \sum_{\omega\in\Psi^*_{\to i}} p^*_\omega + \sum_{\omega\in\Psi^*_{i\to}} p^*_\omega \geq \hat{u}_i(\Psi) - \sum_{\omega\in\Psi_{\to i}} p^*_\omega + \sum_{\omega\in\Psi_{i\to}} p^*_\omega = \hat{U}([\Psi; p^*]).$$

We now show that $[\Psi^*; p^*]$ is an equilibrium of the original economy with valuations u_i. Suppose to the contrary that there exists an agent i and a set of trades $\Xi \subset \Omega_i$, such that $U_i([\Xi; p^*]) > U_i([\Psi^*; p^*])$. Since $\hat{U}_i([\Xi; p^*]) \leq \hat{U}_i([\Psi^*; p^*])$, and by the construction of $\hat{u}_i$, $\hat{U}_i([\Xi; p^*]) \geq U_i([\Xi; p^*])$, it follows that $\hat{U}_i([\Psi^*; p^*]) > U_i([\Psi^*; p^*])$. This, in turn, implies that for some nonempty set $\Phi \subseteq \Psi^*_i$, we have $\hat{u}_i(\Psi^*_i) = u_i(\Psi^*_i - \Phi) - \Pi \cdot |\Phi| \leq \overline{u}_i - \Pi$. This implies that $\sum_{j\in I} \hat{u}_j(\Psi^*) = \hat{u}_i(\Psi^*) + \sum_{j\neq i} \hat{u}_j(\Psi^*) \leq \overline{u}_i - \Pi + \sum_{j\neq i} \overline{u}_j = \sum_{j\in I} \overline{u}_j - \Pi = -\sum_{j\in I} \overline{u}_i - 1 < \sum_{j\in I} u_j(\varnothing)$, contradicting Theorem 3. (The proof of Theorem 3 is entirely self-contained.)

Proof of Theorem 3

If $[\Psi; p]$ is a competitive equilibrium, then for any $\Xi \subseteq \Omega$, we have

$$u_i(\Psi) + \sum_{\omega\in\Psi_{i\to}} p_\omega - \sum_{\omega\in\Psi_{\to i}} p_\omega = U_i([\Psi; p]) \geq U_i([\Xi; p]) = u_i(\Xi) + \sum_{\omega\in\Xi_{i\to}} p_\omega - \sum_{\omega\in\Xi_{\to i}} p_\omega$$

for every $i \in I$. By summing these inequalities over all $i \in I$, we get

$$\sum_{i \in I} u_i(\Psi) \geq \sum_{i \in I} u_i(\Xi).$$

Proof of Theorem 4

Interchangeability of Price Vectors

First, we prove the latter half of Theorem 4, using an approach analogous to that Gul and Stacchetti (1999) used to prove their Lemma 6. Suppose $[\Xi; p]$ is a competitive equilibrium and $\Psi \subseteq \Omega$ is an efficient set of trades. Since Ψ is efficient, we have

$$\begin{aligned}\sum_{i \in I}\left[u_i(\Psi) + \sum_{\omega \in \Psi_{i\to}} p_\omega - \sum_{\omega \in \Psi_{\to i}} p_\omega\right] &= \sum_{i \in I} U_i([\Psi; p]) \\ &\geq \sum_{i \in I} U_i([\Xi; p]) = \sum_{i \in I}\left[u_i(\Xi) + \sum_{\omega \in \Xi_{i\to}} p_\omega - \sum_{\omega \in \Xi_{\to i}} p_\omega\right].\end{aligned} \tag{3}$$

In light of the fact that $[\Xi; p]$ is a competitive equilibrium, so that

$$\begin{aligned}u_i(\Xi) + \sum_{\omega \in \Xi_{i\to}} p_\omega - \sum_{\omega \in \Xi_{\to i}} p_\omega &= U_i([\Xi; p]) \\ &\geq U_i([\Psi; p]) = u_i(\Psi) + \sum_{\omega \in \Psi_{i\to}} p_\omega - \sum_{\omega \in \Psi_{\to i}} p_\omega\end{aligned}$$

for every $i \in I$, we see that (3) can hold only if $U_i([\Xi; p]) = U_i([\Psi; p])$ for every $i \in I$. Hence (as $\Xi \in D_i(p)$ for all $i \in I$) we have $\Psi_i \in D_i(p)$ for all $i \in I$, and therefore $[\Psi; p]$ is a competitive equilibrium.

Lattice Structure

The proof is analogous to the proof of Theorem 3 of Sun and Yang (2009), applied to the network setting. Given a price vector p, let $V(p) \equiv \sum_{i \in I} V_i(p)$. Let $\Psi^* \subseteq \Omega$ be any efficient set of trades and let $U^* = \sum_{i \in I} u_i(\Psi^*)$. Note that for any competitive equilibrium price vector p^*, $V(p^*) = U^*$.

We first prove an analogue of Lemma 1 of Sun and Yang (2009).

Lemma B.2. *A price vector p' is a competitive equilibrium price vector if and only if $p' \in \operatorname{argmin}_p V(p)$.*

Proof. To prove the first implication of the lemma, we let p' be a competitive equilibrium price vector and let p be an arbitrary price vector. For each agent i, consider some arbitrary $\Psi^i \in D_i(p)$. By construction, we have

$$\begin{aligned} V(p) = \sum_{i \in I} V_i(p) &= \sum_{i \in I} \left[u_i(\Psi^i) + \sum_{\omega \in \Psi^i_{i \to}} p_\omega - \sum_{\omega \in \Psi^i_{\to i}} p_\omega \right] \\ &\geq \sum_{i \in I} \left[u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \to}} p_\omega - \sum_{\omega \in \Psi^*_{\to i}} p_\omega \right] \\ &= \sum_{i \in I} u_i(\Psi^*) = U^* = V(p'), \end{aligned}$$

where the inequality follows from utility maximization.

Now, to prove the other implication of the lemma, let p' be any price vector that minimizes V (and thus satisfies $V(p') = U^*$). We claim that $[\Psi^*; p']$ is a competitive equilibrium. To see this, note that the definition of V_i implies that

$$V_i(p') \geq u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \to}} p'_\omega - \sum_{\omega \in \Psi^*_{\to i}} p'_\omega. \tag{4}$$

Summing (4) across $i \in I$ gives

$$\sum_{i \in I} V_i(p') \geq \sum_{i \in I} \left[u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \to}} p'_\omega - \sum_{\omega \in \Psi^*_{\to i}} p'_\omega \right] = \sum_{i \in I} u_i(\Psi^*) = U^*, \tag{5}$$

with equality holding exactly when (4) holds with equality for every i. If (4) were strict for any i, we would obtain $V(p') > U^*$ from (5), contradicting the assumption that p' minimizes V and thus satisfies $V(p') = U^*$. Thus, for all $i \in I$, equality holds in (4), and thus $[\Psi^*; p]$ is a competitive equilibrium. □

Now, suppose p and q are two competitive equilibrium price vectors, and let $p \wedge q$ and $p \vee q$ denote their meet and join, respectively. Note that

$$\begin{aligned} 2U^* &\leq V(p \wedge q) + V(p \vee q) \\ &\leq V(p) + V(q) = 2U^*, \end{aligned}$$

where the first inequality follows because (by Lemma B.2) U^* is the minimal value of V, the second follows from the submodularity of V (which holds because by Theorem 1, V_i is submodular for every $i \in I$), and the equality follows from Lemma B.2, because p and q are competitive equilibrium price vectors. Since we also know that $V(p \wedge q) \geq U^*$ and $V(p \vee q) \geq U^*$, it has to be the case that $V(p \wedge q) = V(p \vee q) = U^*$, and so by Lemma B.2, $p \wedge q$ and $p \vee q$ are competitive equilibrium price vectors.

Proof of Theorem 5

Let $A \equiv \kappa([\Psi; p])$. Suppose A is not stable; then either it is not individually rational or there exists a blocking set.

If A is not individually rational, then $A_i \notin C_i(A)$ for some $i \in I$. Hence, $A_i \notin \operatorname{argmax}_{Z \subseteq A_i} U_i(Z)$, and therefore $\tau(A_i) = \Psi_i \notin D_i(p)$, contradicting the assumption that $[\Psi; p]$ is a competitive equilibrium.

Suppose now that there exists a set Z blocking A, and let $J = a(Z)$ be the set of agents involved in contracts in Z. For any trade ω involved in a contract in Z, let $\tilde{p}_\omega$ be the price for which $(\omega, \tilde{p}_\omega) \in Z$. For each $j \in J$, pick a set $Y^j \in C_j(Z \cup A)$. As Z blocks A, (by definition) we have $Z_j \subseteq Y^j$. Since $Z \cap A = \varnothing$, and for all $Y \in C_j(Z \cup A)$ we have that $Z_j \subseteq Y$ and $A_j \notin C_j(Z \cup A)$. Hence, for all $j \in J$,

$$U_j(A) < U_j(Y^j) = \left[\begin{array}{c} u_j(\tau(Y^j)) + \\ \sum_{\omega \in \tau(Z)_{j\rightarrow}} \tilde{p}_\omega - \sum_{\omega \in \tau(Z)_{\rightarrow j}} \tilde{p}_\omega + \\ \sum_{\omega \in \tau(Y^j - Z)_{j\rightarrow}} p_\omega - \sum_{\omega \in \tau(Y^j - Z)_{\rightarrow j}} p_\omega \end{array} \right].$$

Summing these inequalities over all $j \in J$, we have

$$\begin{aligned} \sum_{j \in J} U_j(A) &< \sum_{j \in J} \left[\begin{array}{c} u_j(\tau(Y^j)) + \\ \sum_{\omega \in \tau(Y^j - Z)_{j\rightarrow}} p_\omega - \sum_{\omega \in \tau(Y^j - Z)_{\rightarrow j}} p_\omega \end{array} \right] \\ &= \sum_{j \in J} \left[\begin{array}{c} u_j(\tau(Y^j)) + \\ \sum_{\omega \in \tau(Z)_{j\rightarrow}} p_\omega - \sum_{\omega \in \tau(Z)_{\rightarrow j}} p_\omega + \\ \sum_{\omega \in \tau(Y^j - Z)_{j\rightarrow}} p_\omega - \sum_{\omega \in \tau(Y^j - Z)_{\rightarrow j}} p_\omega \end{array} \right] \\ &= \sum_{j \in J} \left[\begin{array}{c} u_j(\tau(Y^j)) + \\ \sum_{\omega \in \tau(Y^j)_{j\rightarrow}} p_\omega - \sum_{\omega \in \tau(Y^j)_{\rightarrow j}} p_\omega \end{array} \right] = \sum_{j \in J} U_j(Y^j), \end{aligned}$$

where we repeatedly apply the fact that for every trade ω in $\tau(Z)$, the price (first $\tilde{p}_\omega$

and then p_ω) of this trade is added exactly once and subtracted exactly once in the summation over all agents.

Now, the preceding inequality says that the sum of the utilities of agents in J given prices p would be strictly higher if each $j \in J$ chose Y^j instead of A_j. It therefore must be the case that for some $j \in J$, we have $U_j([\tau(Y^j); p]) > U_j([A; p])$. It follows that $A_j \notin D_j(p)$, and therefore $[\Psi; p]$ cannot be a competitive equilibrium.

Proof of Theorem 6

Consider a stable set $A \subseteq X$. For every agent $i \in I$, define a modified valuation function $\hat{u}_i$, on sets of trades $\Psi \subseteq \Omega - \tau(A)$:

$$\hat{u}_i(\Psi) = \max_{\Xi \subseteq A_i} \left[u_i(\Psi \cup \tau(\Xi)) + \sum_{(\omega, \bar{p}_\omega) \in \Xi_{i\to}} \bar{p}_\omega - \sum_{(\omega, \bar{p}_\omega) \in \Xi_{\to i}} \bar{p}_\omega \right].$$

In other words, the modified valuation $\hat{u}_i(\Psi)$ of Ψ is equal to the maximal value attainable by agent i by combining the trades in Ψ_i with various subsets of A_i. We denote the utility function associated to $\hat{u}_i$ by $\hat{U}_i$. Since the original utilities were fully substitutable, and thus the choice correspondences C_i satisfied (CEFS), the choice correspondences $\hat{C}_i$ for utility functions $\hat{U}_i$ also satisfy (CEFS) and thus every $\hat{U}_i$ is also fully substitutable.

Now, consider a modified economy for the set of agents I: The set of trades is $\Omega - \tau(A)$, and utilities are given by $\hat{U}$. If there is a competitive equilibrium of the modified economy of the form $[\varnothing; \hat{p}_{\Omega - \tau(A)}]$, i.e., involving no trades, then we are done. We combine the prices in this competitive equilibrium with the prices in A to obtain the price vector p as

$$p_\omega = \begin{cases} \bar{p}_\omega & (\omega, \bar{p}) \in A \\ \hat{p}_\omega & \text{otherwise.} \end{cases}$$

It is clear that $[\tau(A); p]$ is a competitive equilibrium of the original economy: since $\varnothing \in \hat{D}_i(\hat{p})$ for every i, no agent strictly prefers to add trades not in $\tau(A)$, and by the individual rationality of A, no agent strictly prefers to drop any trades in $\tau(A)$.

Now suppose there is no competitive equilibrium of this modified economy in which no trades occur. By Theorem 2, this economy has at least one competitive equilibrium $[\hat{\Psi}; \hat{p}]$. By Theorems 3 and 4, we know that $\hat{\Psi}$ is efficient and $\varnothing$ is not.

It follows that
$$\frac{\sum_{i\in I}\hat{u}_i(\hat{\Psi})-\sum_{i\in I}\hat{u}_i(\varnothing)}{2|\Omega_i|}>0;$$
we denote this value by δ.

Now, consider a second modification of the valuation functions:

$$\tilde{u}_i(\Psi)=\hat{u}_i(\Psi)-\delta|\Psi_i|.$$

We show next that utility functions $\tilde{U}_i$ corresponding to $\tilde{u}_i$ are fully substitutable. Take agent i. Take any two price vectors p' and p''. Construct a new price vector $\tilde{p}'$ as follows. For every trade $\omega\in\Omega-\tau(A)$, $\tilde{p}'_\omega=p_\omega+\delta$ if $b(\omega)=i$, $\tilde{p}'_\omega=p_\omega-\delta$ if $s(\omega)=i$, and $\tilde{p}'_\omega=0$ if $\omega\notin\Omega_i$. Construct price vector $\tilde{p}''$ analogously, starting with p''. Note that for any set of trades $\Psi\subset\Omega-\tau(A)$, we have $\tilde{U}_i([\Psi;p'])=\hat{U}_i([\Psi;\tilde{p}'])$ and $\tilde{U}_i([\Psi;p''])=\hat{U}_i([\Psi;\tilde{p}''])$, and therefore, for the corresponding indirect utility functions, we have $\tilde{V}_i(p')=\hat{V}_i(\tilde{p}')$ and $\tilde{V}_i(p'')=\hat{V}_i(\tilde{p}'')$.

Now, by the submodularity of $\hat{V}_i$, we have

$$\hat{V}_i(\tilde{p}'\wedge\tilde{p}'')+\hat{V}_i(\tilde{p}'\vee\tilde{p}'')\le\hat{V}_i(\tilde{p}')+\hat{V}_i(\tilde{p}''),$$

and therefore
$$\tilde{V}_i(p'\wedge p'')+\tilde{V}_i(p'\vee p'')\le\tilde{V}_i(p')+\tilde{V}_i(p'').$$

Hence, $\tilde{V}_i$ is submodular, and therefore $\tilde{U}_i$ is fully substitutable.

Now, by our choice of δ, $\sum_{i\in I}\tilde{u}_i(\hat{\Psi})>\sum_{i\in I}\tilde{u}_i(\varnothing)$. Thus, $\varnothing$ is not efficient under the valuations $\tilde{u}$ and therefore cannot be supported in a competitive equilibrium under those valuations. Take any competitive equilibrium $[\tilde{\Psi},q]$ of the economy with agents I, trades $\Omega-\tau(A)$, and utilities $\tilde{U}$. We know that $\tilde{\Psi}\neq\varnothing$. Moreover, since $\tilde{\Psi}\in\tilde{D}_i(q)$ for every i (where $\tilde{D}$ is the demand correspondence induced by $\tilde{U}$), we know that $\tilde{U}([\tilde{\Psi};q])\ge\tilde{U}([\Phi;q])$ for any $\Phi\subsetneq\tilde{\Psi}$, which in turn implies $\hat{U}([\tilde{\Psi};q])>\hat{U}([\Phi;q])$. This, in turn, implies that for all i, in the original economy with trades Ω and utility functions U_i, the set of trades $\{(\psi,q_\psi):\psi\in\tilde{\Psi}_i\}$ is a subset of every $Y\in C_i(A\cup\{(\psi,q_\psi):\psi\in\tilde{\Psi}_i\})$. Thus, $\{(\psi,q_\psi):\psi\in\tilde{\Psi}_i\}$ is a blocking set for A, contradicting the assumption that A is stable.

Proof of Theorem 9

Suppose A is a stable outcome that is not strongly group stable. Let Z be a set that blocks A. By the second part of the proof of Theorem 5, there is no vector of prices p such that $[\tau(A)\,;p]$ is a competitive equilibrium. This contradicts Theorem 6. To see that for any core outcome A there is a stable outcome $\hat{A}$ such that $\tau(A) = \tau(\hat{A})$, note that by Theorem 8, every core outcome induces an efficient set of trades. By Theorem 4, we can find a competitive equilibrium corresponding to any efficient set of trades. Finally, by Theorem 5, the competitive equilibrium induces a stable outcome.

Proof of Theorem 10

We prove the following result, which implies Theorem 10 by a natural inductive argument:

Lemma B.3. *For any feasible outcome A blocked by a nonempty set Z, if Z is not itself a chain, then there exists a nonempty chain $W \subsetneq Z$ such that set A is blocked by $Z - W$ and set $A \cup (Z - W)$ is blocked by W.*

Proof. The second part of the statement, that $A \cup (Z - W)$ is blocked by W, is true for any $W \subseteq Z$: Set Z blocks A, and thus for every agent i involved in W (and hence in Z), every choice of i from $A \cup Z = (A \cup (Z - W)) \cup W$ contains every contract in Z_i, and thus contains every contract in $W_i \subset Z_i$. So we only need to prove the first part of the statement, that we can "remove" some chain W from Z so that the remaining set $Z - W$ still blocks A.

Also, without loss of generality, we can assume that set A is empty: As in the proof of Theorem 6, consider a modified economy with the set of available trades equal to $\Omega - \tau(A)$ and agents' valuations over subsets Ψ of this set given by $\hat{u}_i(\Psi) = \max_{\Xi \subseteq A_i} \left[u_i(\Psi \cup \tau(\Xi)) + \sum_{(\omega,p_\omega)\in\Xi_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in\Xi_{\to i}} p_\omega\right]$. Agents' preferences in this modified economy are fully substitutable, and blocking of outcome A by any set of contracts Y in the original economy is equivalent to blocking of the empty set of contracts by the same set Y in the modified economy. Note that by definition, set Y blocks the empty set if and only if for every i involved in Y, the unique optimal choice of agent i from Y_i is equal to Y_i itself.

Consider now any contract $y \in Z$, and let $y^0 = y$. We will algorithmically "grow" a chain

$$W^{\ell_s,\ell_b} \equiv \{y^{-\ell_s}, \ldots, y^{\ell_b}\}$$

by applying (generally in both directions from y^0) the iterative procedure below, starting with $\ell_s = \ell_b = 0$ and $W^{0,0} = \{y^0\}$. We will ensure that at every step of the procedure, the following four conditions hold for every agent i. If $i \neq b(y^{\ell_b})$, $i \neq s(y^{\ell_s})$, then $C_i(Z - W^{\ell_s,\ell_b}) = \{(Z - W^{\ell_s,\ell_b})_i\}$. If $i = b(y^{\ell_b}) \neq s(y^{\ell_s})$, then $C_i((Z - W^{\ell_s,\ell_b}) \cup \{y^{\ell_b}\}) = \{(Z - W^{\ell_s,\ell_b})_i \cup \{y^{\ell_b}\}\}$, and if $i = s(y^{\ell_s}) \neq b(y^{\ell_b})$, then $C_i((Z - W^{\ell_s,\ell_b}) \cup \{y^{\ell_s}\}) = \{(Z - W^{\ell_s,\ell_b})_i \cup \{y^{\ell_s}\}\}$. Finally, if $i = s(y^{\ell_s}) = b(y^{\ell_b})$, then $C_i((Z - W^{\ell_s,\ell_b}) \cup \{y^{\ell_b}, y^{\ell_s}\}) = \{(Z - W^{\ell_s,\ell_b})_i \cup \{y^{\ell_b}, y^{\ell_s}\}\}$. Clearly, these conditions are satisfied in the beginning. Our chain-growing algorithm has two main steps:

Buyer Step: Let $j = b(y^{\ell_b}) \neq s(y^{\ell_s})$. If $C_j(Z - W^{\ell_s,\ell_b}) = \{(Z - W^{\ell_s,\ell_b})_j\}$, stop. Otherwise, recall that $C_j((Z - W^{\ell_s,\ell_b}) \cup \{y^{\ell_b}\}) = \{(Z - W^{\ell_s,\ell_b})_j \cup \{y^{\ell_b}\}\}$. This implies (by full substitutability) that each $Y \in C_j(Z - W^{\ell_s,\ell_b})$ must contain $(Z - W^{\ell_s,\ell_b})_{\to j}$ as a subset, and also (by the Law of Aggregate Demand; see Section A.3) that each such Y excludes at most one contract in $(Z - W^{\ell_s,\ell_b})_{j\to}$. Pick any such Y that excludes exactly one contract, and denote that excluded contract by y^{ℓ_b+1}. Note that by construction, the unique optimal choice of j from $(Z - W^{\ell_s,\ell_b+1})$ is $(Z - W^{\ell_s,\ell_b+1})_j$, and the four conditions are satisfied for W^{ℓ_s,ℓ_b+1}.

Seller Step: Let $k = s(y^{\ell_s}) \neq b(y^{\ell_b})$. Consider its optimal choices from $Z - W^{\ell_s,\ell_b}$. If the unique optimal choice is $(Z - W^{\ell_s,\ell_b})_k$, stop. Otherwise, by analogy with the Buyer Step, pick contract y^{ℓ_s-1}.

As long as $s(y^{\ell_s}) \neq b(y^{\ell_b})$, we sequentially iterate these two steps for W^{ℓ_s,ℓ_b}, decrementing ℓ_s and incrementing ℓ_b with each iteration.

Over the course of this process, it may happen that $s(y^{\ell_s}) = b(y^{\ell_b}) \equiv h$. If $C_h(Z - W^{\ell_s,\ell_b}) = \{(Z - W^{\ell_s,\ell_b})_h\}$, we are done—the chain W^{ℓ_s,ℓ_b} suffices for the result. Otherwise, note that as before, by full substitutability and the Laws of Aggregate Supply and Demand, each $Y \in C_h(Z - W^{\ell_s,\ell_b})$ excludes at most one upstream contract and at most one downstream contract from $(Z - W^{\ell_s,\ell_b})_h$. Take any such excluded contract y, and suppose it is a downstream contract for agent h (the argument if it is an upstream contract for h is completely analogous). Let $y^{\ell_b+1} = y$. Each $Y \in C_h((Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})$ is, by construction, also in $C_h(Z - W^{\ell_s,\ell_b})$. Each such Y contains $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{h\to}$ as a subset and excludes at most one contract from $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{\to h}$. If it so happens that each Y contains $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{\to h}$ as a subset, then we continue with the Buyer Step on W^{ℓ_s,ℓ_b+1}. Otherwise, we select any excluded contract in $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{\to h}$, take it to be y^{ℓ_s-1}, and continue with the Buyer Step on W^{ℓ_s-1,ℓ_b+1}.

Since set Z is finite, this algorithm must terminate, resulting in some chain W^{ℓ_s,ℓ_b}. At every iteration, we ensured that for each agent $i \notin \{b(y_b^\ell), s(y_s^\ell)\}$, $C_i(Z - W^{\ell_s,\ell_b}) = \{(Z - W^{\ell_s,\ell_b})_i\}$. The algorithm's stopping conditions ensure that the same equality also holds for $i \in \{b(y_b^\ell), s(y_s^\ell)\}$. Thus, the empty set is blocked by $Z - W^{\ell_s,\ell_b}$. □

Proof of Theorem 11

(a) We show first that $\Xi_i \in D_i(q)$. In the following let $\overline{p} = \max_{\omega\in\Psi} p_\omega$ and $\xi \in \Psi$ be any trade such that $p_\xi = \overline{p}$. Note that $|\Xi_i \cap \Psi| \in \{0, 1\}$ due to mutual incompatibility, and if $\omega \in \Xi_i \cap \Psi$ then $p_\omega = \overline{p}$: Otherwise, perfect substitutability would imply $U_i([\Xi - \{\omega\} \cup \{\xi\}; p]) > U_i([\Xi; p])$, contradicting the assumption that $[\Xi; p]$ is a competitive equilibrium.

This implies that prices for trades in Ξ_i have not been changed in going from p to q and in particular implies that $U_i([\Xi_i; p]) = U_i([\Xi_i; q])$. Since prices for trades in $\Omega - \Psi$ have also not been changed, we must have $U_i([\Xi_i; q]) = U_i([\Xi_i; p]) \geq U_i([\Phi; p]) = U_i([\Phi; q])$ for all $\Phi \subseteq \Omega - \Psi$. Now take any set $\Phi \subseteq \Omega - \Psi$ and any $\omega \in \Psi$. By perfect substitutability, $U_i([\Phi \cup \{\omega\}; q]) \leq U_i([\Phi \cup \{\xi\}; q]) = U_i([\Phi \cup \{\xi\}; p]) \leq U_i([\Xi_i; p]) = U_i([\Xi_i; q])$. Hence, $\Xi_i \in D_i(q)$.

Now consider an arbitrary agent $j \neq i$. If $\Xi_j \cap \Psi = \{\omega\}$, we must have $p_\omega = \overline{p}$, implying $U_j([\Xi_j; q]) = U_j([\Xi_j; p])$. If $\Xi_j \cap \Psi = \emptyset$ the last statement is evidently true as well. Let $\Phi \subseteq \Omega_j$ be arbitrary and note that $U_j([\Phi; q]) \leq U_j([\Phi; p])$ since trades in $\Phi \cap \Psi_{\to j}$ have become weakly more expensive. Since $U_j([\Xi_j; q]) = U_j([\Xi_j; p]) \geq U_j([\Phi; p])$, we obtain $\Xi_j \in D_j(q)$. This completes the proof.

(b) Analogous to part (a).

Appendix C: Example Omitted from Section 5.1

In this appendix, we provide an example of an outcome that is stable and in the core, but is not strongly group stable.

Example C.1. Let $I = \{i, j\}$, $\Omega = \{\chi, \psi, \omega\}$, and $s(\chi) = s(\psi) = s(\omega) = i$ and $b(\chi) = b(\psi) = b(\omega) = j$. Furthermore, let agents' valuations be given by:

Ψ	$\varnothing$	$\{\chi\}$	$\{\psi\}$	$\{\omega\}$	$\{\chi,\psi\}$	$\{\chi,\omega\}$	$\{\psi,\omega\}$	$\{\chi,\psi,\omega\}$
$u_i(\Psi)$	0	0	-2	-2	-2	-2	-9	-20 .
$u_j(\Psi)$	0	2	1	1	3	3	2	15

In this case, any outcome of the form $\{(\chi, p_\chi)\}$ such that $0 \leq p_\chi \leq 2$ is both stable and in the core. At the same time, any such outcome is not strongly group stable, as $\{(\psi, 6), (\omega, 6)\}$ constitutes a block.

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