Increasing Effort through Softening Incentives in Contests

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Abstract

When designing incentives for heterogeneous agents facing competition there is a conflicting interaction: as the more able are incentivized the less able are disincentivized. I label the former the "incentive effect" and the latter the "discouragement effect." Such adverse interaction becomes severe in the face of participants having convex costs of effort or capacity constraints, larger contests, contestants with similar levels of ability, and contest designers with concave benefit over participant effort. Indeed, in such a world, the "discouragement effect" dominates the "incentive effect," prescribing the optimal incentives to be flat or possibly even inverted. That is, providing greater benefit to the lesser able can elicit more total effort than having greater benefit awarded to the most able.

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1 Introduction

Designing optimal incentives within employment relationships has been an important and well looked after topic. A fundamental lesson from this work has been the prescription of sharp incentives within the firm. Consider the canonical principal (employer) and agent (employee) model. Assuming risk neutrality of both parties the trivial solution is to "sell the store" to the employee, yielding first best from maximally powered incentives. But when employees are different the problem becomes more complex.

Instead, we will argue, interacting sharp, competitive incentives with heterogeneous ability can in fact destroy effort. In particular, the less able are less likely to win and thus "give up." We dub this the "discouragement effect." Meanwhile, the most able do increase their efforts when facing sharper incentives, which we dub the "incentive effect." However, the interaction of these two effects can become so severe the "discouragement" effect dominates the "incentive effect," prescribing soft incentives.

The intuition for shifting the top performer's bonus to the lesser performers is actually quite simple: we lose some effort from the most able, which are most likely to receive the first place bonus; however, we receive increased effort from all the rest as a result of their more likely earned second place bonus being larger. If this increased effort overcomes the most able's lessoned effort, total effort is increased.

We naturally have in mind broader applications of softened incentives than personnel economics. In fact, whenever we encounter people or firms competing for a prize or prizes, our results will often apply. Indeed, many economic settings can be cast as a contest. Whether firms are competing for business or to avoid a regulator, whether sales people are competing for bonuses, nonprofits are vying for donors' dollars or even politicians seeking election, we have multiple agents seeking after prizes. Therefore, for most of our analysis, we will refer to competing employees more generally as contestants or participants. The firm or beneficiary of the agents' effort is simply called the designer of the contest.

We study the contest design problem through the (incomplete information) all-pay contest framework. We will also relate the complete information contest with the incomplete setting; though the latter is more likely to be witnessed in practice, and is thus our focus. Indeed, we only require an ε of uncertainty over types to yield our incomplete information results.

We are not the first to suggest it might be optimal to offer a second prize over a winner-takes-all scheme. Moldovanu and Sela (2001), hereafter MS, in their seminal paper find a designer ought to offer some fraction of the total prize to second place if the contestant's cost function has the "right" curvature. In particular, if the curvature of the cost function is convex "enough" the result follows. Contrarily, they find, if the cost function is linear or concave, a winner-takes-all scheme dominates.

My paper begins by generalizing MS's analysis by removing their restriction on monotonic prize ordering. We create a new mechanism dubbed the Generalized Second Prize Contest to allow non monotonic prize allocations. We can then provide conditions of when equal prizes, or even a larger second prize, is optimal, in addition to asking when we want to offer any fraction of a second prize. We can then also determine for the case of an *indivisible* prize if it is best to reward a single prize to first or second place. This new mechanism weakly dominates MS's mechanism in terms of total revenue generated from offering two prizes.

We next study a new class of contests—linear contestant costs with capacity constraints–to extend our results and address four questions that MS do not consider. First, what is the best prize division when the degree of contestant heterogeneity changes? One can imagine some settings where there is a group of mostly very skilled contestants, while some other settings witness contestants with a wide distribution of ability. The previous intuition from MS's results and our "incentive" and "discouragement" effect would suggest when skill is widely distributed, we want to offer more of a second prize. However, this intuition is wrong. In fact, it is actually when competition is the fiercest—when contestants have similar levels of ability—that it is critical to soften incentives by flattening the prize distribution. To my knowledge, this is the first paper to explore the role of skill heterogeneity in optimal (incomplete information) contest design.

Second, we explore how a designer should construct a contest if she values effort across contestants in a manner other than perfect substitutes. One can imagine a setting where workers have complementary work inputs. Another example is the designer that has the goal of contestant proficiency. Under these types of settings there is a nice parallel to our earlier results: when the curvature of the designer's benefit function over effort is *concave*, it is best to offer more of a second prize, *even* when contestant cost functions are linear with no constraints.

Third, we provide some crisp empirical predictions. It is often difficult to observe cost functions in data, yet alone to measure their curvature. However, with our new class of contests, we are able to provide some sharp predictions that do not rely on cost function curvature and instead are based on more readily observable factors. Finally, we show how our predictions and comparative statics also apply to an English auction.

2 Related Literature

A thorough review of providing incentives within the firm can be found in Prendergast (1999). The end result is much of the theoretical literature has prescribed sharp employee incentives. However, as he points out, the empirical literature, at best, finds mixed support that such high powered schemes are witnessed in practice. In short, it seems firms should generally be offering very sharp incentives, but they do not. We argue this inconsistency is reconciled by accounting for the *interaction* effect of incentives and competition. To this end, we provide some empirical predictions than can then be taken to the data anew.

The contest literature, meanwhile, can be divided into three main strands. One of the first, and earliest strands was initiated by Tullock (1980). He set the problem up as players having a chance of winning a contest as a function of a particular contestant's effort vis-a-vis all other contestants' effort level. Much of the focus of this literature is the degree of rent dissipation through rent seeking. That is, determining what percent of the prize is exerted in effort to obtain such prize. Here the analysis is often concerned with how efficient a contest is- e.g., politicians competing for election.

Another strand has to do with casting a contest as an all-pay auction, with the war of attrition as an example (see Bulow & Klemperer (1999)). Here we find we can analyze contest outcomes and participant behavior by drawing on the rich auction literature. However, of note, is it is almost always assumed effort costs are linear, as in auctions a bidder's cost of a bid is most often linear. Nonetheless, in practice, and in many economic applications the firm or individual's cost function is assumed to be convex. An important exception of assumed cost linearity (though only under complete-information) is the recent contribution of Siegel (2009). He is concerned with player behavior and equilibrium payoffs, and shows a (complete-information) multiple prize all-pay auction a special case of the "all-pay contest."

The third strand has to do with designing contests, moving from the focus on participant behavior to how to design an optimal contest. That is, from the perspective of a contest designer, deciding how much to allocate between multiple prizes or deciding between single or multiple stage contests to maximize contest revenue or effort. Moldovanu and Sela (2001) is eponymous of this work. As mentioned previously, Moldovanu and Sela make a seminal contribution in this literature of allowing participant costs to be convex. See also Moldovanu and Sela (2006) and Moldovanu et al. (2007) for more examples of contest design.

The early work of Lazear and Rosen (1981) can also be thought of as from the perspective of a contest designer. They analyze when having output rank order payments is preferred over piece rate pay based on output. We now first generalize MS.

3 The Model

Our general model consists of k agents that commonly value $n < k$ prizes at $V_1, V_2, \cdots V_n$. However, in contrast to past literature, we do not require any ordering on the value of prizes. In addition, each agent has private information of their cost of effort. In particular, their cost of effort level e is assumed to be $c\gamma(e)$, where c is drawn from some F with lower support $\underline{c}$ bounded away from zero (to assume away costless or negative cost of effort) and upper support $\bar{c}$. Our cost function $\gamma(e)$ is assumed endowed with $\gamma'(e) > 0$, $\gamma''(e) \geq 0$, and $\gamma(0) = 0$. Hence, the objective function of each agent is:

$$\max_e P_1(e, \mathbf{e}_{-i}) \times V_1 + ... + P_n(e, \mathbf{e}_{-i}) \times V_n - c\gamma(e)$$

Each $P_i(e, \mathbf{e}_{-i})$ is then the probability effort level e induces for winning the ith prize given the strategy of all the other players. However, using the revelation principle we can rewrite the agent's problem as simply choosing a type x to declare himself, yielding:

$$\max_x F_1(x) \times V_1 + ... + F_n(x) \times V_n - c\gamma(b(x))$$

Here we have $b(x)$ as the equilibrium bidding function and $F_i(x)$ the probability of placing i given her declaration of being type x. Now since we assume each agent's cost type is also unknown to the contest designer, the designer has the following problem, as he wants to maximize total expected agent effort:

$$\max_{(V_1,...,V_n)} k \int h\left(b(c, V_1, ..., V_n)\right) F'(c) dc$$

That is, the designer is simply choosing the values of the prizes $V_{1,...,} V_n$ such that the expected revenue is maximized. For example, if $h(x) \equiv x$ then this is simply the expected effort of any particular agent multiplied by k, the number of agents[1].

[1] At first blush, the notion of choosing different prize mass distributions for optimal effort output seems similar to so called handicapping. That is, under handicapping the designer forces certain participant(s) to essentially get partial credit for their effort, thus causing different outcomes. This

We thus generally say expected revenue over effort since $h(\cdot)$ may not be linear, and in particular we will sometimes assume it is concave to allow for different designer goals. Based on our setup, we can now find the equilibrium bidding function $b(\cdot)$. Note we will use the term bidding and effort function interchangeably, as we can think of the optimal effort of a particular agent as their bid for the given prize.

4 The Effort Function

We will mostly use the notation, as well as several important results, found in MS. We now focus our analysis on two prizes, which will be sufficient to provide our results and intuition. Further, in considering two prizes we can use some previous results from MS.

As do MS, we denote the value of 2nd prize as α and $1-\alpha$ the value of the first prize, giving a normalized total prize mass of 1. However, we will relax their constraint of $\alpha \in [0, \frac{1}{2}]$, instead allowing for any distribution of first and second prize: $\alpha \in [0,1]$. That is, we now will solve for the optimal mechanism given the designer's choice of *any* prize distribution over two prizes and the designer's observation of rank order contestant output.

The inverse of our contestant cost (of effort) function $\gamma(x)$ is $g(x)$. As outlined above, we assume $\gamma(x)$ is convex. First assuming $V_1 \geq V_{2,}$ It is then routine to find the bidding function of each participant by integrating "down" the first order condition of contestants (i.e., their differential equations) with the initial condition of the highest cost type providing zero effort. However, MS provide their bidding function in a particular helpful form, defining a participant's bid as a convex combination of two objects based on the distribution of prize mass:

$$b(c) = g(A(c)(1-\alpha) + B(c)(\alpha))$$

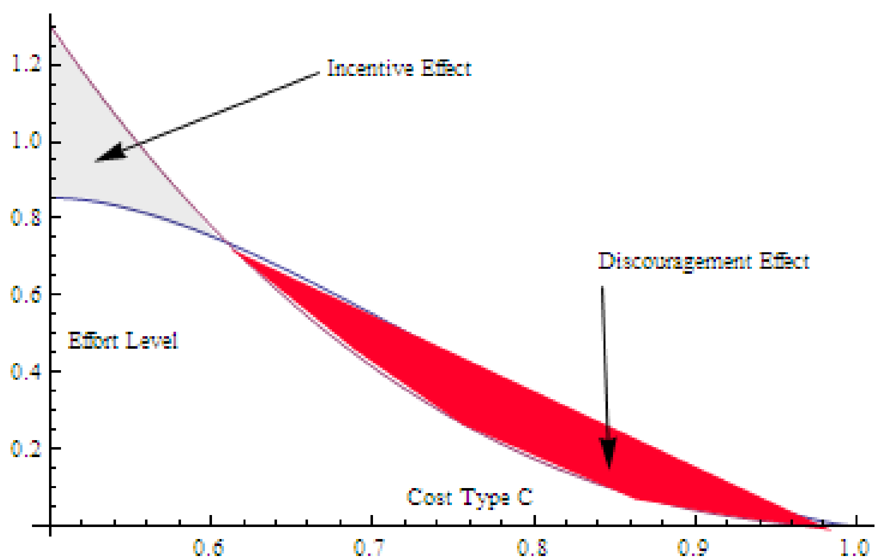
These two objects $A(c)$ and $B(c)$ represent the optimal bid by a cost type c with linear costs of effort under the case of there only being a first prize and second prize, respectively. They are defined thus:

can be sensible under a complete information setting, where much of handicapping literature resides. However, in our setting, where a designer does not precisely each participant's type, handicapping is not practical; the designer does not know who has a particular cost type. Consequently, handicapping in an incomplete information world means the designer arbitrarily designates one contestant (s) with a handicap. However, doing so necessarily means *less* revenue for the designer, and thus we do not consider this scenario, as we are studying contest design from a revenue maximization perspective.

$$A(c) \equiv (k-1)\int_{c}^{\bar{c}} \frac{1}{a}(1-F(a))^{k-2} \times F'(a)da$$

$$B(c) \equiv (k-1)\int_{c}^{\bar{c}} \frac{1}{a}(1-F(a))^{k-3} \times [(k-1)F(a)-1] \times F'(a)da$$

Below we now consider the optimal effort as a function of contestant type in the face of a single prize of \$1 and two equal prizes, each worth \$.50. The red curve, representing effort under a single prize, is the highest for the lowest cost (i.e., most able types), but then is lower for the top 80% of cost types compared with the blue effort curve, which is effort under equal prizes. Thus, we see equal prizes elicit less effort from the most able, but more from all the rest. In fact, there is a crossing point at about 20% of the most able population (i.e., the cost type $c \approx .62$). Hence, to the left of this point, a single prize incentivizes greater effort from the most able, thus we label this the "Incentive Effect"–i.e., all the area between the blue and red curves for the most able type. However, to the right of this cost type point we lose effort from all participants and thus label this area of difference as the "Discouragement Effect"–i.e., incentivizing the most able means discouraging over 80% of the population, resulting in their reduced effort. The intuition is since the top 80% cost types now have to be best rather than just second best to get a prize, they start giving up, as their chances for such achievement is dismal.



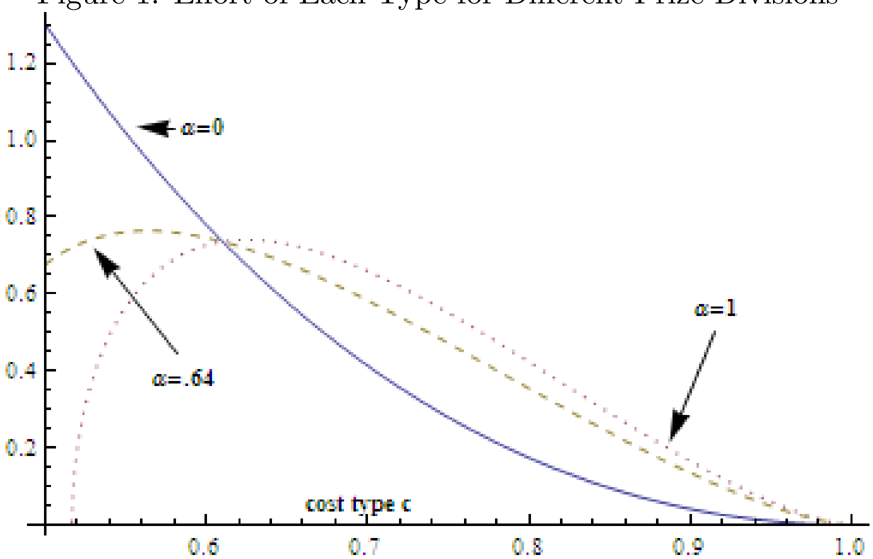
Now to explore all the possible incentive structures of two prizes, we want to consider what would happen if we actually offer a *larger* second than first prize. That is, in our notation, we want to be able to explore allowing $\alpha > .5$. When we do allow $\alpha > .5$, we run into the problem (for an incomplete contest setting) that the bidding function then becomes non-monotonic, as we prove in our next lemma. Thus, we will need to provide a mechanism to correct for this.

Lemma 1 *If $\alpha > .5$, the contestant bidding function becomes single peaked with a maximum at $\widehat{c}$ such that $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$*

Proof: see appendix.

We now turn to another example of the effect of increasing the value of the second prize compared with the first prize, as well as now what happens with a larger second than first prize. Here, as we assumed above, we have $\gamma(x) = x^2$, $c \in U[.5, 1]$,and $k = 5$ participants:

Figure 1: Effort of Each Type for Different Prize Divisions



Our horizontal axis on the above figure represents cost type c. The vertical axis is optimal effort for a given type. The blue (solid) line is the bidding function assuming $\alpha = 0$ (i.e., only a 1st prize is offered) and the red (dotted) shows bidding under $\alpha = 1$ (i.e., only a second prize is offered). These two lines show the trade off between offering more of a first versus second prize. The 1st prize always increases the effort of the lowest cost types until about type .61. However as the cost becomes greater for a given type, then it is the second prize that creates more effort. Hence, offering more of a second prize increases the effort of the roughly top 80% of cost types, but reduces the effort of the bottom 20% of cost types. Thus, where the marginal revenue increase of raising the 2nd prize equals the marginal loss in reducing the 1st prize, we find our optimal α^*.

Finally, the yellow (dashed) line traces the bidding function of $\alpha \approx .64$, which is

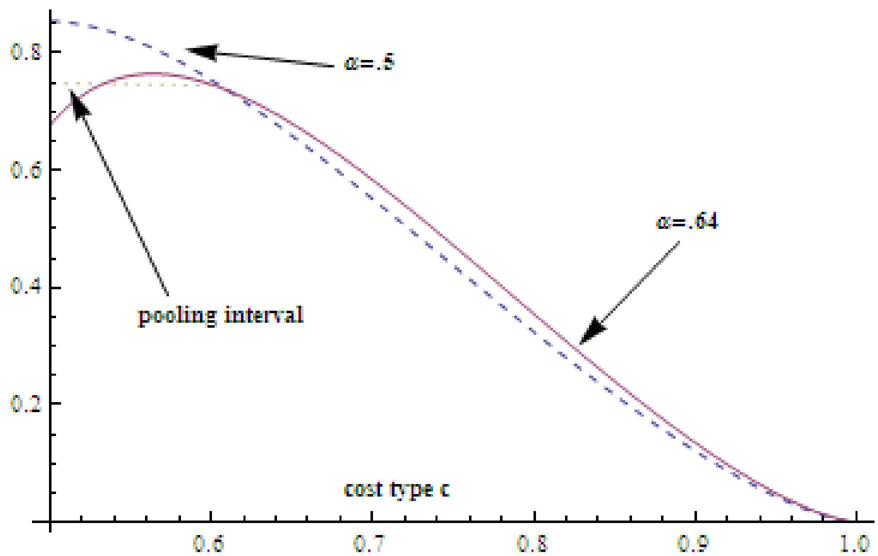
the optimal α for this example. Of course, even though $\alpha \approx .64$ yields the highest total expected effort, it is not feasible[2] due to its non-monotonicity. We now turn to making such payoff feasible.

5 Generalized 2nd Prize Contest

We propose the generalized second prize contest (GSPC), which then fixes the non-monotonicity of the bidding function for $\alpha > .5$. We "iron" out the non-monotonic part of the bidding function by creating a pooling interval. In particular, we find some maximal effort level e^* at which pooling will occur endogenously by participants. Any exerting effort below this level will be ranked by effort, as before, to determine prize allocation. However, any contestants at e^* will be pooled. If there is only one such contestant, they receive 1st prize. The next highest effort contestant with effort below e^* will get second prize. If there are two or more contestants in the effort pooling interval, first and second prize will be randomly allocated with equal chance among contestants along the pooling interval. For example, if 3 people pool, each of them has a separate 1/3 chance of getting 1st or 2nd prize. Hence, there is a 1/9 chance a contestant receives both first and second prize. This allocation is then the same as the auction literature that typically assumes a "tie" is broken by equal random allocation among tied bidders.

To see an example of the GSPC mechanism, we continue our last bidding function example and add the location of the pooling interval:

[2] Here we mean not feasible in the contract theory sense. That is, since types are private information to each contestant, we must have each type's local IC met, which is violated with a non-monotonic bidding function.



Here the blue (dashed) line represents effort if we instead set $\alpha = .5$,whereas the red (solid) line shows $\alpha \approx .64$. The yellow (dotted) line then shows the effort level of the pooling interval. Note if the area between the blue and red line but below the yellow line is greater than the area above the yellow line and below the blue line, then the GSPC generates more total effort than a contest constraining $\alpha = .5$.

It turns out we can always use a generalized second prize contest mechanism, finding a symmetric equilibrium, as our next proposition gives:

Proposition 1 *The generalized 2nd prize contest mechanism exists, meets all incentive compatibility constraints, and induces a (weakly) monotonic bidding function*

Proof:
See Appendix.

The idea of the proof is we can find a unique contestant type that is indifferent between pooling and participating under the non-pooling contests. We then show that everyone in the pooling interval (i.e., all cost types lower than the indifferent cost type) prefers not to deviate up or down. Next we see everyone not pooling (i.e., everyone with greater cost than the indifferent cost type) strictly prefers to remain as they are. Finally, we then show the pooling interval always arises beyond the single peak of the non-modified "bidding" function, ensuring the new mechanism induces a weekly monotonic bidding structure. We also note if $\alpha \leq .5$, then the GSPC collapses to no pooling since the bidding function is then monotonic and thus the pooling interval has mass zero. That is, this mechanism is then a generalization of the constrained contest mechanism found in past literature (i.e., requiring $\alpha \leq .5$). Additionally, we now see our generalization not only allows any allocation of second and first prize, but, in particular, it also allows us to offer *only* a second prize.

6 GSPC With Divisible Prizes

First note In the case of divisible prizes it is then natural to solve for an optimal α^* such that we maximize expected contest revenue. We are using the term revenue over effort to accommodate that under concave designer benefit functions, it is total revenue and not effort *per se* that we are maximizing. That is, we solve under linear designer benefits:

$$\max_{\alpha \in [0,1]} R(\alpha) = k \int_{\underline{c}}^{\overline{c}} g(A(c) + \alpha(B(c) - A(c))) \times F'(c) dc$$

However, if we allow the designer's benefit function to be non-linear, we then have:

$$\max_{\alpha \in [0,1]} R(\alpha) = k \int_{\underline{c}}^{\overline{c}} h(g(A(c) + \alpha(B(c) - A(c)))) \times F'(c) dc$$

Again we have $h(\cdot)$ denoting the designer's benefit function, whereas in the linear designer benefit case we obviously have $h(x) = x$.

We can find α^* by taking the first order condition and solving for α^*:

$$R'(\alpha) = k \int_{\underline{c}}^{\overline{c}} g'(A(c) + \alpha(B(c) - A(c)))(B(c) - A(c)) \times F'(c) dc \equiv 0$$

Unfortunately, α^* must generally be solved for numerically, thus requiring a given set of parameters for a given problem. Here are some examples of the optimal α given the total participants k and pair (w, z), where the contest designer's benefit function is y^w and the contestant's *total* cost function is $c \cdot x^z$. Throughout our examples we assume c is distributed uniform such that $c \in U[.5, 1]$:

Optimal Second Prize $\alpha^* \in [0,1]$				
benefit/cost	k participants			
(w,z)	3	4	5	10
$(1,2)$	.38	.54	.64	.84
$(1,3)$	.50	.65	.74	.89
$(.5,2)$	.54	.69	.77	.91

For example, with $k=3$ participants, functional form of participant cost $c\gamma(x) = cx^2$ and designer concave benefit of $y^{.5}$, the optimal prize allocation is 54% to second and 46% to first. This gives us a ratio of prizes of $\frac{.54}{.46}$,or about a 17% greater second prize.

Thus, with a bit more convexity of cost, concavity of designer benefit or more participants, we can quickly get the optimal allocation being great than 50% to second prize. However, these optimal α^* are based on an unfeasible bidding function due to its non-monotonicity in the face of incomplete information. We now show with the GSPC, whenever we have $\alpha^* > .5$ (i.e., we would like to offer a larger second prize), not only are we able to do so as the previous Proposition shows, but the GSPC also provides more total revenue than restricting $\alpha \leq .5$.

Proposition 2 *It is optimal to offer a larger second prize than first prize through our GSPC if our sufficient condition is met:*

$$k\int_{\underline{c}}^{\overline{c}} h'\left(g(\frac{1}{2}\left(A(c)+B(c)\right))\right) \times g'(\frac{1}{2}\left(A(c)+B(c)\right))(B(c)-A(c)) \times F'(c)dc > 0$$

Proof: see appendix.

Note this sufficient condition is only an assumption on the primitives: the convexity of the participant cost function (which determines its inverse $g(\cdot)$), the concavity of the designer benefit function $h(\cdot)$, number of participants k, and the distribution $F(\cdot)$ of cost types c. If these four factors are combined in a sufficient manner, then our above condition is met. The above table shows a variety of simple examples where indeed this condition is met (i.e., whenever $\alpha^* > .5$).

Corollary 1 *The GSPC always yields (weakly) more total revenue than the constrained (MS) contest.*

This corollary, which follows immediately from our previous Proposition, then gives that whenever we would like to offer a larger second prize under the constrained contest (i.e., as in the case of MS), the GSPC will provide more total effort with a larger second prize. When the optimal $\alpha < .5$, then the two mechanisms agree, providing the same revenue. Thus, in short, the GSPC dominates the constrained mechanism of MS.

Now many prizes in practice aren't readily divided up into equal (or even multiple prizes). For example, take the position of CEO. A firm would not (likely) want to divide this into 10 smaller equal positions due to (presumed) synergy of the CEO multi-tasking. Also, we could think of certain prizes costing the designer in terms of both a fixed and variable cost for each prize unit offered. With sufficient fixed costs, the designer will want to limit the number of prizes, maybe even only offering a single prize. We now explore the question *when is it better under an indivisible prize to offer it to second place over first place.*

7 GSPC With an Indivisible Prize

From our previous analysis of divisible prizes, we have an obvious condition for offering only a second prize versus first prize being optimal if $\widetilde{R}(1) > R(0)$. However, this is more than is needed. There may be some $0 < \alpha < 1$ such that $\widetilde{R}(\alpha) > \widetilde{R}(0)$ and yet will still have $\widetilde{R}(1) > \widetilde{R}(0)$. Indeed, as long as $\alpha < 1$ is great enough, we will still have $\widetilde{R}(1) > \widetilde{R}(0)$ (i.e., even though we may have $\widetilde{R}'(1) < 0$). Thus, in the spirit of our divisible prize results, we can make similar assumptions on the primitives to assure a sole second prize is a preferred to a first prize.

We now consider some examples comparing the revenue of offering only a first prize versus only a second prize. We as before assume total cost is cx^z for effort x and cost type $c \in U[.5, 1]$. The designer's benefit function is simply y^w, where y is a contestant's total effort. We report the increased revenue in the table below.

Strikingly, total revenue is increased some 10% to 20% once we have four or five contestants by offering a prize only to second place over first place. The intuition is again although shifting the prize from first to second causes us to lose some effort from the lowest cost types, we enjoy increased effort from all the "average" and "high" cost types, which more than offset the reduced effort of the low types. These "average" types exert more effort simply because they now have a better chance of winning a now larger second prize.

$\frac{\text{Revenue of } \alpha=1}{\text{Revenue of } \alpha=0} - 1$				
benefit/cost	k participants			
(w, z)	3	4	5	10
$(1, 2)$	1.9%	6.4%	9.4%	16.4%
$(1, 3)$	6.4%	11.3%	14.5%	21.2%
$(.5, 2)$	7.2%	11.9%	14.9%	21.3%

It certainly seems a bit strange to think of only offering a second prize and no first prize. However, recall with the GSPC our pooling interval: as long as two or more participants pool, these lowest (cost) types will have a chance of receiving the sole second prize. We now turn to considering a designer with different goals.

8 Designers with Different Effort Goals

One can imagine a designer might not value effort over contestants on a simple substitution basis. Consider the professor that has a student moving from a 98% to a 100% grade and another student from 68% to 70%. The latter 2% change is likely more valued than the former 2% change. This suggests a class of contests where the designer does not value effort as perfect substitutes: proficiency. If a teacher is educating students with the primary goal of helping them all reach a level of proficiency, then the designer has diminishing valuation of effort across participants. Similarly, consider the regulator that wants to move firms to a certain standard of environmental care. As yet another example of contests consider contestants that have complementary effort inputs. One could even argue this case is more the rule than the exception across workers. If this is so, then again the designer values effort across any particular worker in a diminishing manner. Mathematically, each of these classes are such that the designer has concave benefit over the effort of contestants. Returning to our revenue function, we write it thus:

$$\max_{\alpha\in[0,1]} R(\alpha) = k\int_{\underline{c}}^{\overline{c}} h(g(A(c) + \alpha(B(c) - A(c)))) \times F'(c)dc$$

Again we have $h(\cdot)$ denoting the designer's benefit function, whereas in the linear designer benefit case we have $h(x) = x$. First note if a contestant carries a linear cost function, we get:

$$\max_{\alpha \in [0,1]} R(\alpha) = k \int_{\underline{c}}^{\overline{c}} h(A(c) + \alpha(B(c) - A(c))) \times F'(c) dc$$

Thus, if the designer now has concave $h(\cdot)$, this is *mathematically equivalent* to a designer with linear benefits and a contestant with convex costs. If instead, we have both convex contestant costs and concave designer benefit, it is clear they amplify one another, even more readily pushing the prize mass down from the top performer to second place. It is also then more likely to call for offering a larger second prize to generate maximal total revenue. We use the term revenue now over effort, as the designer can now value effort in a non-linear manner. We next turn to a new class of cost functions to provide some additional insights and crisp empirical predictions.

9 Effort Capacity Constraints

The reason for using this class of cost functions is their tractability allows for analytical solutions, as well as provides the intuition of why for general convex cost functions sufficient convexity of participant cost and number of participants assures the optimality of offering more of a second prize. Additionally, the notion of capacity constraints on effort is more the rule than the exception in practice, whether we consider a maximum of 24 hours in a day or simply maximal physical strength.

Note we could instead have analyzed this problem as linear costs with a budget constraint that binds for at least the most able. With a few modifications of the below proofs, this alternative interpretation provides the same results to our assuming an effort capacity constraint; however, analyzing the latter is more intuitive and straight forward in the proofs.

Consider the class of linear cost functions with cost type c, effort e, and capacity constraint $\widehat{e}$[3]:

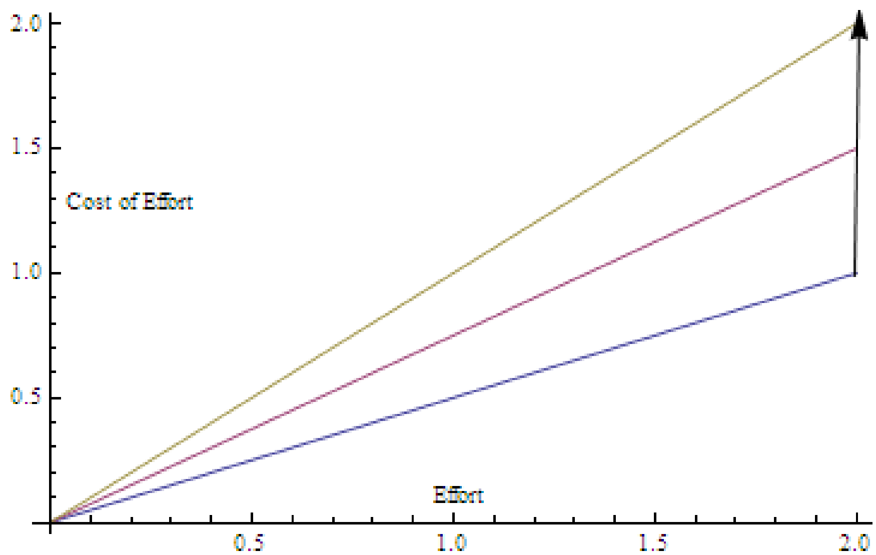
$$c\gamma(e) = ce \text{ if } 0 \leq e < \widehat{e}$$

$$c\gamma(e) = +\infty \text{ if } e \geq \widehat{e}$$

We will call the capacity threshold $\widehat{e}$ *maximal effort*.

[3]See Megidish and Sela (2009) for example of linear costs and floor constraints. That is, contestants must put in a minimal amount of effort.

Thus the total cost function for type $c \sim U(.5, 1)$,taking the lowest, highest, and mean cost type, and $\widehat{x} = 2$ is:



Hence, we can think of this function as a simple form of convex cost functions. When we below say this function becomes more convex we simply mean the threshold $\widehat{e}$ is lower.

We also assume the designer has linear benefits (i.e., $h(x) = x$). We could instead imagine a capacity constrained analog for the designer where after a given level of effort for any particular contestant she receives no further utility. This is an extreme version of the proficiency goal of a designer mentioned earlier. Imagine a trainer that is only paid based on her students reaching a level of proficiency. In this case, it is just as if her total benefit has a cap for each contestant at the proficiency level. For the balance of our analysis, for the sake of brevity we are going to assume linear benefits for the designer. However, it will be clear in most proofs how to extend them to include this class of settings as well.

From the reasoning in the proofs below, it should also be clear that although we will prove given enough capacity constraints and participants the optimal prize structure actually inverts—it is best to offer a larger second over first prize—all of the comparative statics still hold for lesser degrees. That is, given enough participants or capacity constraint, it is best to offer some of a second prize over a winner-takes-all, and with even more participants and/ or greater capacity constraints, it is best to offer equal prizes.

Now recall under linear costs, we have our bidding function thus:

$$b(\alpha, c) = (1-\alpha)A(c) + \alpha B(c)$$

Recall also (see appendix) that there exists some unique c^{**} such that $A(c^{**}) = B(c^{**})$, where $A(c) > B(c)$ when $c < c^{**}$, and $A(c) < B(c)$ when $c > c^{**}$.

Hence, increasing the share of second prize α increases effort for all cost types $c > c^{**}$ and reduces effort for all types $c < c^{**}$. Define the effort of the type c^{**} : $b(\alpha, c^{**}) = e^{**}$.

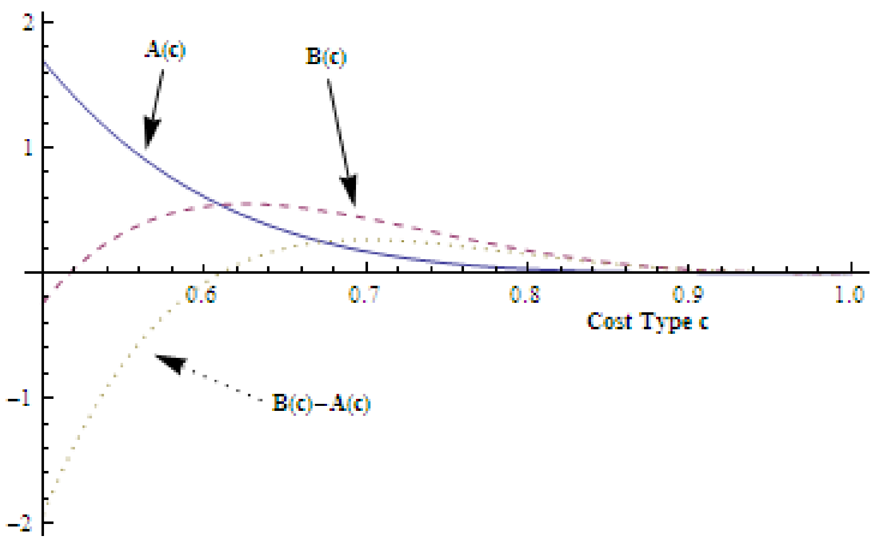
Our general strategy of proof will be to show given enough capacity constraint or enough participants numbering k, we will have the maximal effort $\widehat{e} < e^{**}$, where again e^{**} is the effort of the cost type c^{**} assuming there was no capacity constraint. That is, the marginal type c^{**} is bound by the capacity constraint $\widehat{e}$.

Having $\widehat{e} < e^{**}$ will then assure us we can increase α to garner more effort from all types $c > \widehat{c}$, where $b(\alpha, \widehat{c}) = \widehat{e}$. This is because all types $c < c^{**}$ are already providing maximal effort $\widehat{e}$, and without the $\widehat{e}$ constraint they would be providing even more effort. Hence, when their incentive for effort is slightly reduced by increasing α,they want to provide a bit less effort, but such (capacity unconstrained) effort level is still above the maximal effort $\widehat{e}$,so they still provide maximal effort $\widehat{e}$. Meanwhile, all cost types $c > c^{**}$ want to provide more effort with increased α. Hence, these higher cost types increase their effort while the lower cost types still provide maximal effort $\widehat{e}$, yielding a net overall increase of effort.

This nicely parallels the intuition of a general convex function causing the designer to want to offer a larger second prize. If the designer offered a single first prize, the most able, low cost, types would provide a high level of effort that would not increase much beyond a certain value of first prize due to the convexity of their cost functions. That is, they are "far up" their convex (enough) cost curve and are thus less sensitive than the high cost types to changes in expected benefit (i.e., the prize value). In particular, a slight reduction in the marginal benefit via reduced first prize does not cause much reduced effort for the lowest cost since the marginal cost is already so high. Hence, by shifting a bit of the first prize to second prize, the designer does not much reduce the effort of the most able, but significantly increases the effort of everyone else since they were so "low down" on their convex cost curves.

We now conduct our formal proof in two parts. We first show once we fix our number of participants, as we increase the convexity of the cost function, we will want to offer a larger second prize. We will secondly show the more complicated, portion of the proof where we instead fix the convexity of the cost function, and then show given enough participants k we will also then want to offer a larger second prize.

To see what is going on, below is a graph of actual values for $k = 5$,linear costs (with no capacity constraint), and $c \sim U[.5, 1]$. We can see it is roughly the top 80% of cost types (i.e., .6 to 1.0) that increase effort as we shift prize mass from the first to second prize (i.e., $c^{**} \approx .6$). However, the most able 20% significantly reduce effort as prize mass is shifted from first to second prize.



Effort as a Function of Cost Type and Prize Type

Now we can prove our proposition:

Proposition 3 *Assuming linear participant cost functions with (sufficient) capacity constraints we know for any distribution of ability F with positive support:*

1) Fix the number of participants $k \geq 3$. Given sufficient effort capacity constraint, it is optimal to offer a larger second over first prize

2) Fix the degree of capacity constraint. Given sufficient number of participants, it is optimal to offer a larger second over first prize.

Proof:

1) For the first part of the proposition, fix some distribution F and the number of participants k. Assume participant costs are linear with capacity constraint effort $\widehat{e}$. Now we only need to show given a low enough capacity constraint, we increase total effort by having a larger second than first prize.

Thus, to denote equal first and second prizes, set $\alpha = .5$. We know then our bidding function is strictly decreasing in cost type c. Now assume the capacity constraint is sufficient such that $e^{**} - \left(\underline{B(c) - A(c)}\right) \times \varepsilon > \widehat{e}$ for some small $\varepsilon > 0$. That is, we can then simply choose an $\widehat{e}$ low enough such that this is true since neither e^{**} nor $\underline{B(c) - A(c)}$ are affected by the degree of capacity constraint. Recall again e^{**} is the optimal effort of the type c^{**}, assuming no capacity constraint. Hence, with $e^{**} > \widehat{e}$, the *implemented* effort of c^{**} is $\widehat{e}$.

But this means we can then increase α by an ε and the lowest cost type $\underline{c}$ will still be providing maximal effort since such type's reduction of effort by $\left(\underline{B(c) - A(c)}\right) \times \varepsilon$ still leaves her subject to the capacity constraint level of effort $\widehat{e}$. This then also means all types $c < c^{**}$, which were previously providing maximal effort, will continue to provide maximal effort, since they reduce the optimal, unconstrained effort by even less than the lowest cost type– thus they are also still bound by the capacity constraint since they too previously would have provided effort $e > e^{**}$, save the capacity constraint. Additionally, those cost types $c^{**} < c < \widehat{c}$ will want to provide more effort, but they are already subject to a capacity constraint, thus they maintain their previous $\widehat{e}$ of effort. Finally, increasing α by an ε will then increase the effort of all cost types $c > \widehat{c}$. Therefore, we have now increased total effort by increasing α by an ε. However, this then means it is better to offer $\alpha = .5 + \varepsilon > .5$,or a larger second over first prize. It should be obvious that this total effort is also greater than any $\alpha \in [0, \frac{1}{2}]$. That is, reducing α causes all the lower cost types $c < c^{**}$ will want to provide even more effort once we reduce α,meaning the continue to have their capacity constraint bind, thus providing same effort as before. However, now all the higher cost types $c > c^{**}$ will be *disincentivized* by lowering α, which means they will all provide weakly less effort (weakly because some of the lower cost of the higher cost types might still be capacity constrained at their new lower desired level of effort). $\square$

2) We want to show once we fix the capacity constraint, we need only garner enough participants to assure us offering a larger second prize over first is optimal.

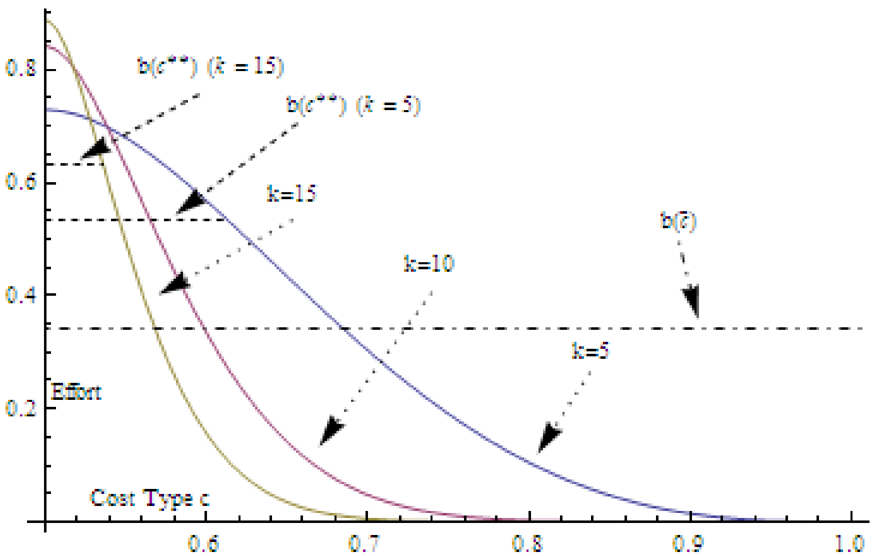
Fix F and $\alpha = .5$. Fix effort capacity with $\frac{1}{2\bar{c}} - \left(\underline{B(c) - A(c)}\right) \times \varepsilon > \widehat{e}$, where $\frac{1}{2\bar{c}}$ is again the minimal bid by the lowest cost type with linear costs (see appendix for lower bound of bid by the lowest cost type). As long as $\widehat{e}$ satisfies this equation, we will say the there is "sufficient capacity constraint." This assumption simply says there is enough capacity constraint that once we change α by an ε, it still binds for at least the lowest cost type. Otherwise, we have the trivial case where there is no capacity constraint for any participants for any increase in α.

We will use the same trick as in (1) of showing we will ultimately have $e^{**} > \widehat{e}$

and thus offering a larger second prize will be optimal. However, we now need to show we get this inequality simply by increasing k enough since the location of $\widehat{e}$ is fixed. That is, we will show given enough k, we will have e^{**} become greater than $\widehat{e}$,or equivalently that $c^{**} < \widehat{c}$.

When we increase k there are two effects on the bid, or effort, of the type c^{**}. The first effect is we know as k increases, holding all else constant, c^{**} is becoming smaller, which means the bid of the c^{**} (i.e., e^{**}) is becoming greater (since bid is strictly decreasing in cost type). In fact, we know from our estimates found in our appendix, the type c^{**} approaches $\underline{c}$ in the limit. In particular, we know $\frac{1}{k} < F(c^{**}) < \frac{2}{k}$ for all k.

The second effect is the lower cost types *increase* their bid as k increases (and all the higher cost types decrease their bid as k increases). Here is a graph that illustrates this effect (and this effect can readily be verified via simply taking the derivative of the bidding function with respect to k) :



This means as k becomes large enough, not only is $b\left(c^{**}\right)$ increasing by moving "up" the bidding curve, but also when k becomes large enough, the portion of the curve $b\left(c^{**}\right)$ resides is moving up since the lowest cost types bids are increasing in k, and in the limit $c^{**} \rightarrow \underline{c}$. Hence, since $b(\widehat{c})$ is fixed, we know there is some k such that we finally have $\widehat{e} = b(\widehat{c}) < b\left(c^{**}\right) = e^{**}$. That is, ultimately the marginal type c^{**} is bound by the capacity constraint $\widehat{e}$. We are assured of ultimately meeting this inequality by requiring $b(\widehat{c})$ to be set such that $\frac{1}{2\bar{c}} - \left(\underline{B(c) - A(c)}\right) \times \varepsilon > \widehat{e}$. This inequality dictates the lowest cost type (and in the limit $c^{**} \rightarrow \underline{c}$), increasing α

from .5 to $.5+\varepsilon$ will still result in the lowest cost type providing a maximal effort level, and thus all other previously providing a maximal bid will continue to do so. However, now all the other types that were not providing a maximal bid reside above the cost type c^{**} and thus will provide (weakly) increased effort as we shift ε of prize mass from first to second prize. That is, we then again have increased total effort by having a larger second prize. $\square$

The first portion of the proposition shows us once we fix our other primitives, we can then increase the convexity enough to assure we optimally offer a larger second prize. The intuition, as given earlier, applies to more general convex cost functions: with sufficiently convex cost curves, taking a bit away from the lowest cost types via reducing the first prize does not much reduce effort since they are "far up" their convex cost curve. Meanwhile, adding a bit to the higher cost types expected profit by increasing second prize adds more to increased effort since they are "low down" their convex cost curve. When the latter positive effect overcomes the former negative one, net effort is increased.

Our second portion of our proposition also provides intuition how with a generic convex cost function we only need have enough participants to optimally offer a larger second prize. In particular, as we increase k, we again witness the twin effects of the type c^{**} becoming a lower cost type since $c^{**} \to \underline{c}$, with k and also since the lowest cost types increase their effort as k increases. Thus, once we fix the convexity of the cost curve, with enough k, it is only a very few that are disincentivized by increasing the second prize. Further, since these few low cost types are "far up" their convex cost curve, there are not very disincentivized by a small reduction in first prize value, since they already requiring much increased expected prize for any additional unit of effort. Meanwhile, all the other higher cost types, being "lower down" their convex cost curve have more sensitivity to increasing their effort with an increased second prize value. This increase of all the higher types then offsets the nominal decrease of the lowest of cost types.

Note also the complementary of the degree of convexity of the number of types. If we begin with a more convex cost function, it will not take as many participants to assure offering a larger second over first prize is optimal. Similarly, beginning with a larger of number of participants means the cost function will need not be as convex to assure we ought to offer a larger second over first prize.

Finally, it should be clear we can allow for different levels of capacity constraint across contestants and still maintain our above results. Taking the natural assumption that cost type is negatively correlating to capacity level, we then just take $\widehat{e}$ to be the capacity constraint of the lowest cost type.

Therefore, we can then conclude if we live in a world with no capacity constraints and linear costs, we should just load all the reward to the best performer. However, once capacity constraints binding this can be a costly strategy. If people are enough time constrained or enough physically constrained the optimal prescription becomes spreading the reward to more than just the most able.

10 Heterogeneity of Skill and Optimal Prizes

The distribution of skill can vary widely across different settings of competition. In one case we might have many participants that are of similar high skill. On the other hand we might have a distribution of workers that have enormous skill differences.

We lastly consider the role of cost type heterogeneity on determining the optimal prize distribution. In particular, we will find the *less* heterogenous are types, the more likely we want to offer a larger second prize.

First we note some results helpful for estimating linear bids (these are derived in our appendix):

$$\begin{aligned}
\frac{1}{2}(\underline{A(c)+B(c)}) &= \frac{1}{2\overline{c}}\left((F(c)\times(k-2)+1)\cdot(1-F(c))^{k-2}\right) \\
\frac{1}{2}(\overline{A(c)+B(c)}) &= \frac{1}{2\underline{c}}\left((F(c)\times(k-2)+1)\cdot(1-F(c))^{k-2}\right)
\end{aligned}$$

$$\underline{A}(c) = \frac{1}{\overline{c}}(1-F(c))^{k-1}$$

$$\overline{A}(c) = \frac{1}{\underline{c}}(1-F(c))^{k-1}$$

Now we calculate the expected bid under a winner takes all (WTA) under incomplete information as $\underline{c}$ and $\overline{c}$, the bounds of our distribution of types, collapse to the mean of $\widetilde{c}$ of our type distribution. We then get:

$$\frac{1}{\widetilde{c}}\int(1-F(c))^{k-1}f(c)dc = \frac{1}{k}\times\frac{1}{\widetilde{c}}$$

Hence, as usually assumed in the complete information case, with $\widetilde{c}=1$, we simply get the expected bid is

$\frac{1}{k}$ and thus expected revenue is $\frac{1}{k} \times k = 1$,which means 100% rent dissipation.

In other words, as we approach (perfect) homogeneity, the expected bid in a WTA contest is simply $\frac{1}{k}$ per each contestant, which is precisely the same as the *expected bid* under a complete information case (and symmetric equilibrium) with mixed strategies. Note, however, under incomplete information the strategies are unique pure strategies as opposed to the mixed strategies of the complete information case. That is, we only need to introduce an ε of uncertainty over types and we get a unique pure strategy equilibria over the complete information case of multiple equilibria (i.e., which can have both symmetric and asymmetric equilibria). This provides our next result.

We consider the expected bids under incomplete information and two equal prizes (i.e., $\alpha = .5$). We first use integration by parts to note:

$$
\begin{aligned}
& \int_{c}^{\bar{c}} (1 - F(a))^{k-3} \times F(a) \times F'(a) da \\
= \quad & F(c) \times \frac{1}{k-2}(1 - F(c))^{k-2} + \frac{(1 - F(c))^{k-1}}{(k-2)(k-1)} \\
= \quad & \frac{(k-2)F(c) + 1}{(k-2)(k-1)} \times (1 - F(c))^{k-2}
\end{aligned}
$$

$$
\Rightarrow \int_{c}^{\bar{c}} (1 - F(a))^{k-2} \times F(a) \times F'(a) da = \frac{(k-1) \cdot F(c) + 1}{(k-1)(k)} \times (1 - F(c))^{k-1}
$$

We can then use this to solve for the expected value of our linear bid with $\alpha = .5$ (i.e., $\frac{1}{2}(A(c) + B(c))$) as the (incomplete info) contest approaches homogeneity:

$$
\begin{aligned}
& \int_{\underline{c}}^{\bar{c}} \frac{1}{2}(A(c) + B(c)) \times F'(c) dc \\
= \quad & \int_{\underline{c}}^{\bar{c}} \frac{1}{2} \left((F(c) \times (k-2) + 1) \cdot (1 - F(c))^{k-2} \right) \times F'(c) dc
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{(k-2)(k-1)F(\underline{c})+(k-2)}{(k-1)(k)} \times (1-F(\underline{c}))^{k-1} + \frac{1}{k-1}\right) \\
&= \frac{1}{2}\left(\frac{(k-2)}{(k-1)(k)} + \frac{1}{k-1}\right) = \frac{1}{2}\left(\frac{2(k-1)}{(k-1)(k)}\right) = \frac{1}{k}
\end{aligned}
$$

With cost type mean $\widetilde{c} = 1$, we get the same result for equal and WTA and also for complete information. This means as we approach homogeneity both in the case of complete information and incomplete information we get the same expected revenue of 1 regardless if we have a WTA or equal prizes (as well as the same expected bids per player under each contest (assuming the symmetric equilibrium under the complete information case). An easy extension to our above analysis shows this relationship between the complete and incomplete information contest is true for any $\alpha \in [0, 1]$. This then provides our next Lemma:

Lemma 2 ***Relationship between Complete and Incomplete Information Contests***

Fix $\alpha \in [0, 1]$. The expected (pure strategy) bid of the incomplete information contest equals the expected bid in the mixed strategy (homogeneous) complete information contest under a symmetric equilibrium. Consequently, the expected revenue of an incomplete information contest is equal to the expected revenue of a (homogeneous) complete information contest (where the homogeneous type is the mean type of the incomplete information type distribution)

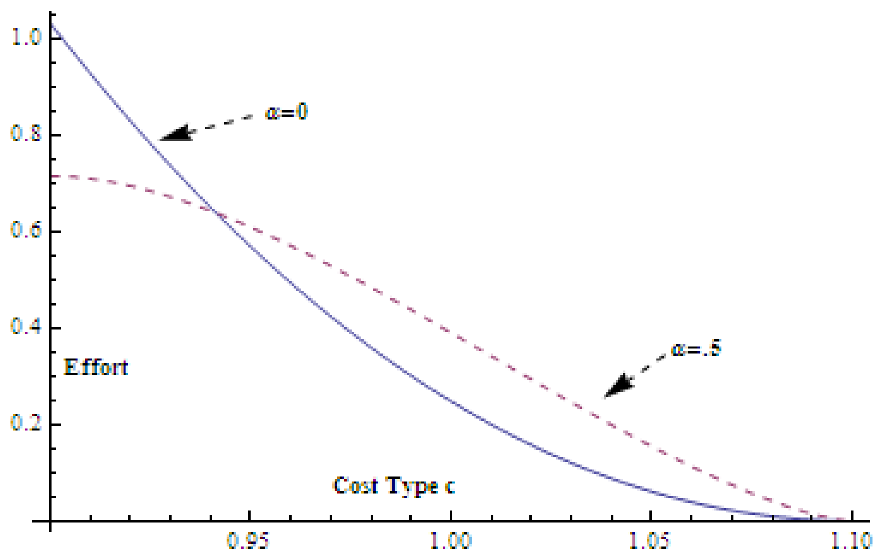
In the spirit of the purification theorem (i.e., Harsanyi (1973)), we have a found a sequence of pure strategies (i.e., the sequence of unique bids for each type from a distribution that converges to its mean type) under the incomplete information game that converges to the (symmetric) mixed strategy equilibrium of the same game under complete information. That is, we could consider our found relationship as a refinement of the multiple equilibria of a complete information contests– only the symmetric equilibria survive. And we also see it only takes an ε of uncertainty over the type space and we move from a (non-unique) mixed strategy equilibrium to a unique strict pure strategy equilibrium. We will now use this relationship to consider another proposition:

Proposition 4 *If an effort capacity constraint binds for some measure of the type space, given enough homogeneity of types, an equal prize contest provides more revenue than a WTA contest*

Proof: Now we will show as we approach homogeneity under an incomplete information case, two equal prizes provide strictly more revenue than a WTA contest (again assuming linear costs).

We will do this by first showing as we approach homogeneity in the incomplete information contest that $\int_{\underline{c}}^{\overline{c}} (B(c) - A(c)) \times F'(c)dc = 0$. This then means the expected revenue from a sole first prize and sole second prize is the same (assuming no capacity constraints). But this means any convex combination of these two (i.e., $\alpha \in [0,1]$) will also yield the same expected revenue, and in particular, then, the expected revenue of a WTA and equal prize contest provide the same expected revenue in the limit of homogeneity. Next, we will show the bid of the lowest cost type is always higher under a WTA than an equal prize contests. This will then imply having just a small measure of the type space capacity constrained causes an equal prize contest to generate greater revenue than a WTA contest. This follows since such constraint will reduce revenue from the WTA contest but not affect (i.e., not be binding in) the equal prize contest, which means the equal prize contest will dominate the WTA contest.

Here is a graph to understand what is happening with $c \in [.9, 1.1]$:



Imagine now adding an effort capacity constraint at $e = .8$. If $\int_{\underline{c}}^{\overline{c}} (B(c) - A(c)) \times$

$F'(c)dc = 0$, the area under the curve (i.e., the expected revenue) of both $\alpha = 0$ and $\alpha = .5$ is the same, which then clearly means setting a capacity constraint of $e = .8$ results in the equal prize contest providing more revenue than the WTA contest. That is, doing so reduces the area of the former without affecting the area of the latter. Now it remains to show as the contest approaches homogeneity, we have $\int_{\underline{c}}^{\bar{c}} (B(c) - A(c)) \times F'(c)dc = 0$ and also the bid of the lowest type is strictly greater under the WTA over equal prize contest.

We use some additional estimates from our appendix for our upper and lower bound estimates of $B(c) - A(c)$:

$$\overline{B(c) - A(c)} = \frac{1}{F^{-1}\left(\frac{2}{k}\right)} \left(kF(c) - 1\right) \times (1 - F(c))^{k-2}\right)$$

$$\begin{aligned}
\underline{B(c) - A(c)} &= \frac{1}{\bar{c}} \left((kF(c) - 1) \times (1 - F(c)^{k-2}\right) \text{ when } c \geq F^{-1}\left(\frac{2}{k}\right) \\
\underline{B(c) - A(c)} &= \frac{1}{\bar{c}}(\frac{k-2}{k})^{k-2} + \frac{1}{\underline{c}}\left((kF(c) - 1)\,(1 - F(c))^{k-2} - (\frac{k-2}{k})^{k-2}\right) \text{ when } c < F^{-1}\left(\frac{2}{k}\right)
\end{aligned}$$

Thus as $\underline{c}$ and $\bar{c}$ approach the mean cost type $\widetilde{c}$, we get:

$$B(c) - A(c) \approx \frac{1}{\widetilde{c}} \left((kF(c) - 1) \times (1 - F(c)^{k-2}\right)$$

Again, taking $\widetilde{c} = 1$ as commonly assumed in the complete information case, we get simply:

$$\left((kF(c) - 1) \times (1 - F(c)^{k-2}\right)$$

Integrating over our type space then gives:

$$
\begin{aligned}
& \int\limits_{\underline{c}}^{\bar{c}} \left((kF(c)-1)\times(1-F(c)^{k-2}\right)\times F'(c)dc \\
= \quad & \int\limits_{\underline{c}}^{\bar{c}} \left((kF(c))\times(1-F(c)^{k-2}\right)\times F'(c)dc - \int\limits_{\underline{c}}^{\bar{c}} (1-F(c)^{k-2}\times F'(c)dc \\
= \quad & \frac{k(k-1)F(\underline{c})+k}{(k-1)(k)}\times(1-F(\underline{c}))^{k-1}-\frac{1}{k-1} \\
= \quad & \frac{1}{k-1}-\frac{1}{k-1}=0
\end{aligned}
$$

In other words, as the contest approaches homogeneity, the expected value of $B(c)-A(c)$ is zero, which means the expected value of revenue from any convex combination of first and second prize is the same.

Finally, to show the lowest cost type always bids more under a WTA over equal prize contest, first note as the contest approaches homogeneity the lowest cost type bids:

$$\frac{1}{\widetilde{c}}\times(1-F(\underline{c}))^{k-1}=\frac{1}{\widetilde{c}}$$

With $\widetilde{c}=1$, we then simply have a bid of 1. Now with equal prizes we get in the limit the lowest cost type bidding:

$$\frac{1}{2\widetilde{c}}\left((F(\underline{c})\times(k-2)+1)\,(1-F(\underline{c}))^{k-2}\right)=\frac{1}{2\widetilde{c}}$$

With $\widetilde{c}=1$,we simply have $\frac{1}{2}$. Thus, with equal prizes the lowest cost type bids just half the lowest cost type versus under a WTA. Since both contests are monotonically decreasing in the bid as a function of type, we are finished. That is, by placing the capacity constraint such that $e\in(\frac{1}{2\widetilde{c}},\frac{1}{\widetilde{c}})$, it is then trivial the equal prize contest provides greater total revenue over the WTA contest since they both provided equal revenue before such constraint. $\square$

Thus, we see as we approach type homogeneity, revenue is all but the same for any prize distribution. However, since the greatest bids are under a WTA contest, adding an ε of effort capacity constraint reduces the effort of the most able types. However, with equal prizes, the most able type has a maximal bid of just half the same type under a WTA contest; thus, there is no effort constraint under equal prizes. Consequently, now equal prize contests provide more revenue.

This proof also provides nice intuition of why having convex costs assures us offering more of a second prize is better once types become homogeneous enough: as types are sufficiently homogeneous, any combination of first and second prize provide roughly the same revenue. However, under a WTA contest, the lowest types are providing much greater effort than under an equal prize. Hence, introducing convex costs distorts downward these greatest effort levels more than the lower effort levels. The net result is less expected net revenue from a WTA over equal prize contest scheme. We also have a corollary from the above results that offering a larger second prize is also superior to a WTA contest given enough homogeneity:

Corollary 2 *If an effort capacity constraint binds for some measure of the type space, given enough homogeneity of types, offering a larger second prize through the GSPC provides more revenue than offering a weakly larger first prize*

Proof: Since revenue is the same for any distribution of prizes as we approach homogeneous types, we have $R'(\frac{1}{2}) \to 0$. However, per Lemma 4 (in appendix), we know $\widetilde{R}'(\frac{1}{2}) > R'(\frac{1}{2}) \to 0$. $\square$

11 Empirical Predictions

Ideally, we would like to take our predictions to the data. However, the results of MS and our earlier results that generalize theirs are difficult to test. This is because we rely on the curvature of the contestant cost function. Cost functions are difficult to observe and measure in practice, yet alone their degree of curvature.

Fortunately, our last class of contests—linear costs with capacity constraints—provide some crisp empirical predictions that do not rely on measuring the curvature of cost functions. Indeed, it should be clear if we now introduce convex costs coupled with capacity constraints, this only strengthens the above comparative statics. Hence, regardless of the curvature of the cost function, the comparative statics still hold in the face of capacity constraints.

Thus, we can predict as follows:

When participants or firms have a substantive limitation on their input or effort level, we can say the following regarding optimal incentive structure:

1) As the range of ability *decreases*, incentives flatten

2) As the designer values effort in a complementary manner or has the purpose of incentivizing agents to reach a given standard or proficiency, incentives flatten

3) As the number of competitors increase, incentives flatten

4) As capacity limitations become more severe, incentives flatten

If we are able to observe output, then we can replace all the above predictions' statement that "incentives flatten" with "revenue increases with flatter incentives."

We now turn to the English auction for not only further intuition by means of order statistics, but also to show this interaction effect of the "incentive" and "discouragement" effect applies to more settings than contests.

12 Intuition via the English Auction

We consider linear participant cost contests with one versus two equal prizes and link these to an English Auction, showing their revenue equivalence. To correlate the two forms, we assume participants in the English auction only want, or are able, to acquire one unit. For the former, using some results from Moldovanu et al (2008) and some further analysis, we write the revenue of an all-pay auction (or equivalently a contest with linear participant costs) as the following, normalizing the total prize mass to 1:

$$R_{AP}(\alpha) = (1 - 2 \cdot \alpha) \times E(k-1, k) + 2 \cdot \alpha \times E(k-2, k)$$

We then see the revenue from an all-pay auction is a convex combination of the 2nd and 3rd order statistics. Thus, with a WTA auction (i.e., $\alpha = 0$) this then collapses to $E(k-1, k)$, the expected value of the second most valuing type. Through the revenue equivalence theorem, we know this is also the same as the English auction with a single prize being auctioned off (with linear costs).

Now with two equal prizes, we get $R_{AP}(\frac{1}{2}) = E(k-2, k)$,the third most valuing type. We could appeal to Krishna (2002), for a multi-unit revenue equivalence theorem to show we then obtain the same revenue from a (generalized) English auction (i.e., he proves the expected payoff for players is the same up to an additive constant for each type of auction). However, it is more instructive to explicitly find the revenue from the (generalized) English auction. We first note once the third to last participant drops out of the auction, both remaining contestants will immediately drop out; since first and second prize are the same, there is no value in further waiting. Hence, ex-ante, for the auctioneer, the expected revenue for two equal prizes is simply:

$$R_{EA}\left(\frac{1}{2}\right) = \frac{1}{2} \times E(k-2,k) + \frac{1}{2} \times E(k-2,k) = E(k-2,k)$$

That is, the first and second most valuing type drop out immediately after the third most valuing type drops out, which for her is when her net expected return is zero (since we assume the net return to dropping out is zero): $\frac{1}{2} \times A_i - b_i = 0 \Rightarrow b_i = \frac{1}{2} \times A_i$, where A_i is the ith order statistic. Taking the expectation then gives the third most valuing type dropping out at the bid $\frac{1}{2} \times E(k-2,k)$. Since there are two remaining contestants after she drops out, total revenue is then $2 \times \frac{1}{2} \times E(k-2,k) = E(k-2,k)$, just the same as an AP auction.

Now when we introduce convex participant costs in the auction (or contest), the revenue equivalence theorem fails since convex costs are equivalent to assuming risk averse bidders. That is, we have $\frac{1}{2}A_i - \gamma\left(b_i\right) = 0$ has the same dropout expected value as $g\left(\frac{1}{2} \times A_i\right) - b_i = 0$, where $\gamma(\cdot)^{-1} \equiv g(\cdot)$,the former being the convex cost setup and the latter the concave benefit or risk aversion.

Despite revenue equivalence failure, we can still find the intuition of the generalized English Auction in the face of convex costs helpful. Again, with this auction format, the bidding strategy is simple in either a single or equal prizes: remain in the auction until your value is reached.

Thus, for the single prize, we get $A_i - \gamma\left(b_i\right) = 0$, which means the second most valuing type drops out. The expected ex-ante revenue from this is then:

$$R_{EA}\left(0\right) = \int g\left(a\right) \times f_2^k(a) \times da < g\left(\int a \times f_2^k(a) \times da\right) = g\left(E(k-1,k)\right)$$

$f_2^k(a)$ is the pdf of the distribution of the second order statistic and integration is over the support of types with k total contestants. The inequality follows from Jensen's inequality.

Similarly, we then find for the equal prize expected revenue is:

$$R_{EA}\left(\frac{1}{2}\right) = 2 \times \int g\left(\frac{1}{2} \times a\right) \times f_3^k(a) \times da$$

We can now compare the two prize distribution revenues to determine which garners more total (expected) effort than the other. First, note as before, when costs are linear we get $g(x) = x$:

$$R_{EA}\left(0\right) = E\left(k-1,k\right) > E\left(k-2,k\right) = R_{EA}\left(\frac{1}{2}\right)$$

Thus, it is again always best to only offer a single 1st prize when costs are linear. However, now consider what happens as convexity increases. Here we mean convexity of the cost function $\gamma(\cdot)$ increases and thus the concavity of its inverse $g(\cdot)$ increases in the Arrow-Pratt sense: $\frac{-g(\cdot)''}{g(\cdot)'} \rightarrow z$. This then means, in the limit, we get $g(\cdot) \rightarrow c$, some constant c. Hence, we have equal prizes producing revenue in the limit of:

$$\begin{aligned}
& \lim_{\frac{-g(\cdot)''}{g(\cdot)'} \rightarrow z} R_{EA}\left(\frac{1}{2}\right) \\
= \;& 2 \times \int \lim_{\frac{-g(\cdot)''}{g(\cdot)'} \rightarrow z} \left[g\left(\frac{1}{2} \times a\right)\right] \times f_3^k(a) \times da \\
= \;& 2 \times \int c \times f_3^k(a) \times da = 2 \times c \times 1 = 2 \times c
\end{aligned}$$

.

Thus, we have:

$$\begin{aligned}
& \lim_{\frac{-g(\cdot)''}{g(\cdot)'} \rightarrow z} R_{EA}\left(\frac{1}{2}\right) \\
= \;& 2 \times c > c = \lim_{\frac{-g(\cdot)''}{g(\cdot)'} \rightarrow z} R_{EA}\left(0\right)
\end{aligned}$$

Thus, there exists some degree of convexity such that an equal prize English auction provides more revenue than a single prize auction, whereas with linear costs the single prize auction provides strictly more revenue.

The intuition of how convexity causes equal prizes to dominate a single prize is simple: with increased convexity, the differential in revenue garnered from offering a \$.50 versus \$1 prize becomes increasingly small. However, under equal prizes we are getting two participants paying this revenue rather than just one under a WTA. In other words, convexity starts putting the "breaks" on how much more of a bid a larger prize elicits. We then reach a crossing point where although the bid is less for a \$.5 over \$1 prize, it is not less than *half* the greater bid offered for the \$1 prize. Thus, under an English auction with (sufficient) participant convex costs, if an auctioneer could divide an object into two equal parts, auctioning them off simultaneously would provide greater expected revenue than auctioning it off as a single object. This intuition then also follows for a contest setting: increased convexity starts limiting the value of having a larger first prize, thus allowing the *two* slightly less incentivizing prizes to garner more *total* contestant effort. Hence, with enough convexity, an equal prize contests yields more total revenue than a WTA contest.

The English auction also provides intuition how increasing the number of participants makes an equal prize auction more likely to provide more revenue than a single prize auction. Recall under linear costs we have the revenue of allocating the prize mass as:

$$R_{AP}(\alpha) = (1 - 2 \cdot \alpha) \times E(k-1, k) + 2 \cdot \alpha \times E(k-2, k)$$

Thus, as $k \to \infty$, we have $E(k-1, k) \to E(k-2, k)$. This means in the limit of a large contest, we receive the same revenue regardless of prize distribution. Nonetheless, in a finite population it is still always best to only auction a single first prize with linear participant costs. However, when we introduce *strict* participant cost convexity, we are assured there exists some finite k such that offering equal prizes yields more revenue than offering only a first. This is immediate from our above analysis of strict convexity causing the expected revenue garnered from a \$.5 prize to be strictly greater than revenue garnered from a \$1 prize. That is, given large enough k, $E(k-1, k)$ and $E(k-2, k)$ become sufficiently similarly valued to provide greater total revenue from auctioning equal prizes over a single prize. Thus the size of the auction and the degree of cost convexity amplify one another: more of one requires then less of the other to still be assured we optimally offer two equal over a single prize for auction.

Finally, we can also now see the role of heterogeneity of prize valuation. As the support of the distribution of types approaches a single type, the 1st and 2nd order statistic converge to one another—i.e., $E(k-1, k) \to E(k-2, k)$. Hence, following our argument above for increasing the number of participants, once we fix the convexity of the cost function and number of participants, *decreasing* the heterogeneity of valuation will also result in equal prizes providing more revenue than a single prize to the auctioneer.

We can now write our proposition:

Proposition 5 *Fix the size (number of participants k), convexity of participant bid costs, and degree of heterogeneity of valuation in an English auction.*

1) There exists some $k^* \geq k$ such that for all $k \geq k^*$ offering two equal prizes each worth V provides greater total revenue than offering a single prize worth $2V$

2) There exists some degree of convexity of bid cost such that $c^* \equiv \frac{-h(\cdot)''}{h(\cdot)'} \geq \frac{-g(\cdot)''}{g(\cdot)'}$ *yields for all* $\frac{-h(\cdot)''}{h(\cdot)'} \geq c^*$ *offering two equal prizes each worth* V *provides greater total revenue than offering a single prize worth* $2V$ *(where our cost function* $\gamma^{-1}(\cdot) = g(\cdot)$*).*

3) There exists some decreased level of heterogeneity of valuation such that offering two equal prizes each worth V *provides greater total revenue than offering a single prize worth* $2V$

This then means we are given three levers to assure more revenue from auctioning equal prizes over a single prize: the size of the auction, the convexity of participant costs, and the degree of valuation homogeneity. We only need to increase one to provide our result. However, the larger the other, the less the other needs to be to arrive at the same result. Increasing the size of the auction or the homogeneity of valuation is reducing the benefit from having a first over second prize by making the magnitude of their incentive effects more similar. And just the same, increasing convexity of bidding costs, is also make the magnitude of a first or second rank prize incentive similar. Hence, once sufficiency is reached, offering the two but lesser prizes wins the day.

13 Conclusion

Whether it be business, politics, or even academics, much is actually a contest. As such, an important task is to consider how to best design a contest. Central to this problem is accounting for the interaction of the "incentive effect" and "discouragement effect." This interaction arises with the combination of competition and heterogeneous ability. With only one of these factors, there is no tradeoff. Similarly, If only living in a world of linear costs and benefits and no capacity constraints, this interaction means little. Though it is convenient to study only one of these factors in isolation, seldom does this characterize the real world. Instead, the presence of these dual forces is more the norm than the exception.

And in a world with both forces, once we face contestants with capacity constraints or convex costs, larger contests, contestants of similar ability, or designers with marginal decreasing benefits over effort, this interaction can become severe: the "discouragement" effect dominates the "incentive" effect calling for optimal incentives to be flat or possibly even inverted—second prize should be larger than first prize.

However, bidding under an inverted incentive scheme becomes non-monotonic. For such a problem we designed a mechanism we dubbed the generalized second prize contest (GSPC) mechanism, which nests in it the constrained contest that restricts a weakly greater first prize. We then found the GSPC (weakly) dominates the constrained contest in terms of total revenue generated for the contest designer.

In addition, we studied a new class of contests—contestants with linear costs and capacity constraints—that has the characteristic of providing sharp and measurable empirical predictions that can now be taken to the data.

We do note we only considered the case of two prizes. It would be interesting to expand our analysis and consider the case when we can offer n prizes with $n < k$ contestants; which prize should be largest? What about if prizes are indivisible-which place should receive the sole prize? We suspect we will find a k prize analog of our results.

14 Appendix

Lemma 1: *If $\alpha > .5$, the contestant bidding function becomes single peaked with a maximum at $\widehat{c}$ such that $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$*

Proof: We first write the bidding function as $b(\alpha, c) = g((1-\alpha)A(c) + \alpha B(c))$, where $g(\cdot)^{-1} = \gamma(\cdot)$ (i.e., $g(\cdot)$ is the inverse of the cost function). Now note $\frac{d}{dc}g((1-\alpha)A(c)+\alpha B(c)) = \underbrace{g'((1-\alpha)A(c)+\alpha B(c))}_{>0}[(1-\alpha)A'(c)+\alpha B'(c)]$. The former term is always positive for $c \in [\underline{c}, \overline{c})$ since $g(\cdot)$ is strictly increasing and $(1-\alpha)A(c)+\alpha B(c)$ is always positive. The latter term, we will see, is single peaked, thus making our entire expression single peaked. Expanding $(1-\alpha)A'(c) + \alpha B'(c)$, we get:

$$
\begin{aligned}
&(1-\alpha)(-(k-1)\frac{1}{c}(1-F(c))^{k-2} \times F'(c) \\
&+\alpha((k-1)\frac{1}{c}(1-F(c))^{k-3} \times [(1-(k-1)F(c)] \times F'(c)
\end{aligned}
$$

Rearranging terms then yields:

$$
\underbrace{(k-1)\frac{1}{c}(1-F(c))^{k-3} \times F'(c) \times}_{>0}[(F(c)-1)(1-\alpha) + \alpha(1-(k-1)F(c)]
$$

The former term is always positive so we only focus on the latter term, which further rearranging gives:

$$
\begin{aligned}
& F(c) - \alpha F(c) - 1 + \alpha + \alpha - k\alpha F(c) + \alpha F(c) \\
= \; & -k\alpha F(c) - (1-F(c)) + 2\alpha
\end{aligned}
$$

First note at the lowest cost type $\underline{c}$,we get simply $2\alpha - 1$, which is always positive for $\alpha > .5$ at $c = \underline{c}$. That is, our bidding function is increasing at the lowest cost type. Similarly, with the highest cost type, we get $-k\alpha + 2\alpha$, which is always negative for $k \geq 3$. Thus, our bidding function is decreasing at the highest cost type.

Now solving the above for a unique zero gives:

$$\begin{aligned} -k\alpha F(c) - (1 - F(c)) + 2\alpha &\equiv 0 \\ \Rightarrow & \quad (k\alpha - 1)F(c) = 2\alpha - 1 \Rightarrow F(c) = \frac{2\alpha - 1}{k\alpha - 1} \end{aligned}$$

.

Define then $\widehat{c}$ such that $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$.Now when $c \in [\underline{c}, \widehat{c})$ we have $a \equiv F(c)$:

$$\underbrace{(1 - k\alpha)}_{<0} a + 2\alpha - 1$$

Once we fix α and k, we see the above expression, which then determines the sign of the derivative of the bidding function, is strictly *decreasing* in a. At $a = \frac{2\alpha-1}{k\alpha-1}$, the above expression equals zero. Meanwhile, with $a \in [\underline{c}, \widehat{c})$ the expression is positive and with $a \in (\widehat{c}, \overline{c}]$ the expression is negative. Hence, the bidding function is single peaked at $\widehat{c}$. Thus, our type $\underline{c} < \widehat{c} < \overline{c}$ provides the highest effort over all types. $\square$.

Proposition 1 *The generalized 2nd prize contest mechanism exists, meets all incentive compatibility constraints, and induces a (weakly) monotonic bidding function*

Proof: We first show the mechanism meets all incentive compatibility constraints.

For contestants who would optimally provide effort below e^*, this problem is just as before so their bidding function remains as under a contest with no pooling, which we will call *no pool* bidding or *no pool* contest, depending on the context. Call this cutoff c^* such that for all $c \in [c^*, \overline{c}]$ these participants provide their effort below e^* as under *no pool*. Now all that have costs of effort $c \in [\underline{c}, c^*]$ are to be in the pooling effort interval. For this pooling group we now must make sure such effort interval is incentive compatible against the deviation of exerting more or less effort than e^*, make sure such c^* is beyond $\widehat{c}$ (i.e., the peak of the non-modified bidding function) to induce a (weakly) monotonic bidding function, and also make sure at c^* such participant is indifferent between optimal effort solved under *no pool* and the pooling interval payoff. We check each of these necessary conditions in turn.

Let the total prize be worth 1, as before. Let $\alpha > \frac{1}{2}$ be the second place share with the remainder being the first place share. Suppose there are k players. Let p be the measure of types in the pooling interval. Then, the expected payoff from bidding in the pooling interval is

$$\pi_{pool} = \underbrace{\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} p^i (1-p)^{k-1-i}}_{\text{1 or more other contestants pool}} + \underbrace{(1-p)^{k-1}(1-\alpha)}_{\text{No other contestant pool}}$$

We first need to verify the contestant c^* at the end of the pooling interval (i.e., type c^* where $p = F(c^*)$) is indifferent between pooling or exerting the identical effort e^* under *no pool* bidding. When we set $p \equiv F(c^*)$, the payoff for c^* under a *no pool* contest is as follows:

$$\pi_{no\ \ pool} = (k-1)\alpha p\left(1-p\right)^{k-2} + \left(1-\alpha\right)\left(1-p\right)^{k-1}$$

Hence, we require $\pi_{pool} = \pi_{no\ \ pool}$, thus we solve for the indifferent value of p

$$\begin{aligned} & \sum_{i=1}^{k-1}\binom{k-1}{i}\frac{1}{i+1}p^i\left(1-p\right)^{k-1-i} + \left(1-\alpha\right)\left(1-p\right)^{k-1} \\ = \quad & (k-1)\alpha p\left(1-p\right)^{k-2} + \left(1-\alpha\right)\left(1-p\right)^{k-1} \Longleftrightarrow \end{aligned}$$

$$\sum_{i=1}^{k-1}\binom{k-1}{i}\frac{1}{i+1}p^i\left(1-p\right)^{k-1-i} = (k-1)\alpha p\left(1-p\right)^{k-2}$$

Now divide by $\left(1-p\right)^{k-1}$ to obtain

$$\sum_{i=1}^{k-1}\binom{k-1}{i}\frac{1}{i+1}\left(\frac{p}{1-p}\right)^i = (k-1)\alpha\frac{p}{1-p}$$

Now, with a change of variable, let $z = \frac{p}{1-p}$ to obtain

$$\sum_{i=1}^{k-1}\binom{k-1}{i}\frac{1}{i+1}\left(z\right)^{i-1} = (k-1)\alpha$$

Now fix α and k. Then the LHS of the equality is strictly increasing in z,which is strictly increasing in p. Further at the limit as $p \to 0 \Rightarrow z \longrightarrow 0$, the LHS converges to $\frac{k-1}{2}$, only the $i = 1$ term remains[4]. The RHS is then a greater finite number $(k-1)\alpha > \frac{k-1}{2}$ with $\alpha > .5$ as $p \to 0$.

Oppositely, as $p \longrightarrow 1$ the LHS approaches $+\infty$,whereas the RHS is again a finite number. Hence, there exists a unique $p^* \in (0,1)$ that solves the above equation. Solving for p^* then determines both e^* and c^*. Hence, once we fix α and k, we

[4]This can also be seen by taking the limit of the generating function of $\sum_{i=1}^{k-1}\binom{k-1}{i}\frac{1}{i+1}\left(z\right)^{i-1} \equiv \frac{(1+z)^k-(k)z-1}{(k)z^2}$ as $z \to 0$.

can always find our needed c^* *uniquely.* Further, by meeting the above equality we have actually also met the IC constraint, which we call IC_{down}, for preventing pooling types from deviating down; thus, we see IC_{down} binds. Note also p^* is increasing in α. However, p^* can be either increasing or decreasing in k depending on the parameterization, as k affects both α and p (holding α constant) in a complex way.

Once we have e^* and c^* we already know any $c \in [c^*, \overline{c}]$ does not want to deviate, as they are already choosing their optimal effort per the *no pool* bidding structure. Meanwhile, any $c \in [\underline{c}, c^*)$ will not want to deviate by providing less effort than the pooling effort level because if it was not worth it for the c^* type to do so, then it certainly is not worth it for the lower cost types. That is, in considering whether to exert less effort, the c^* type trades off the saved cost of less effort with a reduced expected gross benefit. Thus, if the c^* type's cost savings did not justify less effort, it certainly will not be justified for those with lower cost (savings) facing the same reduced expected benefit.

Now we need to check that a participant in the pooling interval does not want to deviate up, as doing so would guarantee a first prize. The payoff from deviating up is thus:

$$\pi_{up} = 1 - \alpha$$

Hence, we require that

$$\pi_{pool} - \pi_{up} \geq 0$$

Substituting.

$$\pi_{in} - \pi_{up} = \sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} p^i (1-p)^{k-1-i} - \left(1 - (1-p)^{k-1}\right)(1-\alpha)$$

Now, recall by the binomial theorem:

$$1 - (1-p)^{k-1} = \sum_{i=1}^{k-1} \binom{k-1}{i} p^i (1-p)^{k-1-i}$$

Hence

$$\pi_{in} - \pi_{up} = \sum_{i=1}^{k-1} \binom{k-1}{i} \left(\frac{1}{i+1} - (1-\alpha)\right) p^i (1-p)^{k-1-i}$$

Clearly, if $\frac{1}{k} > (1-\alpha) \iff \alpha \geq 1 - \frac{1}{k}$, IC_{up} is met, as it means all sums being added above are positive. This requirement simply says the second prize share α

needs to be weakly greater than 1 minus the inverse the number of participants. Thus, this requirement increases in k; however, the optimal α^* is also increasing in k. It is meanwhile trivial if $\alpha = 1$ (i.e., there is only a second prize), IC_{up} is met. However, this sufficient condition is obviously more than needed.

The precise requirement is readily found by solving the generating function of $\sum_{i=1}^{k-1} \binom{k-1}{i} \left(\frac{1}{i+1} - (1-\alpha)\right) p^i (1-p)^{k-1-i}$:

$$\frac{(1-p)^{k-1}(p-1-kp\alpha + (\frac{1}{1-p})^{k-1}(1+kp(\alpha-1))}{kp}$$

The term of interest is $(p-1-kp\alpha+(\frac{1}{1-p})^{k-1}(1+kp(\alpha-1))$, as this determines if the entire equation is (weakly) positive and thus IC_{up} is met. We can then solve for when this term is (weakly) greater than zero:

$$(p-1-kp\alpha+(\frac{1}{1-p})^{k-1}(1+kp(\alpha-1)) \geq 0 \Rightarrow$$

$$kp\alpha((\frac{1}{1-p})^{k-1}-1) \geq 1-p+(kp-1)(\frac{1}{1-p})^{k-1} \Rightarrow$$

$$1 \geq \alpha \geq \frac{1-p}{kp((\frac{1}{1-p})^{k-1}-1)} + \frac{(kp-1)(\frac{1}{1-p})^{k-1}}{kp((\frac{1}{1-p})^{k-1}-1)}$$

The middle term is our first sufficient condition. Since the designer gets to choose α, this condition can regardless always be met since any $\alpha \in [\frac{1-p}{kp((\frac{1}{1-p})^{k-1}-1)} + \frac{(kp-1)(\frac{1}{1-p})^{k-1}}{kp((\frac{1}{1-p})^{k-1}-1)}, 1]$ will satisfy IC_{up}. Additionally, we will consider indivisible prizes, which means we again have $\alpha = 1$ or $\alpha = 0$. Finally, when we do allow for divisible prizes, we could allow that the designer to simply declare any observed effort greater than e^* is still counted as e^*. Since effort is costly, no player would ever exert greater than e^*.

Lastly, we also need to check the type $c^* \geq \widehat{c}$. That is, we need to make sure the indifference point from where we end the pooling interval is *after* the single peak of the *no pool* bidding function; otherwise, we still have not solved the non-monotonicity problem. Recall our IC_{down} condition was the following being (weakly) positive:

$$\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\alpha$$

We then substitute in $z = \frac{F(\widehat{c})}{1-F(\widehat{c})}$, where $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$ as found in our first Lemma. If the expression is negative, it means $c^* > \widehat{c}$, since IC_{down} is not yet met at $\widehat{c}$. That is, we need to choose a larger $p^* > F(\widehat{c})$ (since the above is strictly increasing in z, which is strictly increasing in p) to meet IC_{down}. But this then means we get $c^* > \widehat{c}$.

To see we always have $c^* > \widehat{c}$, first note $\frac{dF(\widehat{c})}{d\alpha} = \frac{\partial}{\partial\alpha} \frac{2\alpha-1}{k\alpha-1} = \frac{k(1-2\alpha)-1}{(k\alpha-1)^2} < 0$ (for $\alpha \geq .5$). This then means $\frac{\partial z}{\partial \alpha} < 0$ when evaluated at $c = \widehat{c}$ since z is strictly increasing in $p \equiv F(c)$. Now we take the derivative of our IC_{down} condition with respect to α and consider its value when evaluated at $c = \widehat{c}$:

$$\begin{aligned} & \frac{d}{d\alpha}\left(\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\alpha\right) \\ = \; & \underbrace{\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (i-1)(z)^{i-2} \underbrace{\frac{\partial z}{\partial \alpha}}_{<0}}_{\leq 0} + \underbrace{-(k-1)}_{<0} < 0 \end{aligned}$$

Hence, our IC_{down} condition is strictly decreasing in α. This means if we can show such expression is non positive at $\alpha = .5$, we are done. Recall, as we already showed, when $\alpha = .5$, we get $z = 0 \Rightarrow \sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} \to \frac{k-1}{2}$. But this then means $\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\frac{1}{2} = \frac{k-1}{2} - \frac{k}{2} + \frac{1}{2} = 0$ with $\alpha = .5$. Hence, since $\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\alpha$ is strictly decreasing in α, it has to be the case for any $\alpha > .5$ we get $c^* > \widehat{c}$.$\square$

Lemma 3 *R and $\widetilde{R}$ are concave in α for $\alpha \in [0, \frac{1}{2}]$ and $\alpha \in [0, \frac{1}{2})$, respectively*

Proof: Recall $R(\alpha) = k \int_{\underline{c}}^{\overline{c}} g(A(c) + \alpha(B(c) - A(c))) \times F'(c)dc$.

Taking the first derivative with respect to α yields:

$$R'(\alpha) = k\int_{\underline{c}}^{\bar{c}} g'(A(c) + \alpha(B(c) - A(c))) \times (B(c) - A(c))) \times F'(c)dc$$

Taking the derivative again with respect to α yields:

$$R''(\alpha) = k\int_{\underline{c}}^{\bar{c}} g''(A(c) + \alpha(B(c) - A(c))) \times (B(c) - A(c)))^2 \times F'(c)dc$$

Since $g''(\cdot) < 0$ (i.e., because $g(\cdot)$ is concave), we get $R''(\alpha) < 0$, as desired. Extending this to the GSPC, note $\widetilde{R}(\alpha)$ is the same as $R(\alpha)$ for $\alpha \in [0, \frac{1}{2})$. $\square$

Lemma 4 $\widetilde{R}'(\frac{1}{2}) > R'(\frac{1}{2})$

First note $\widetilde{R}'(\alpha) = R'(\alpha)$ for all $\alpha \in [0, \frac{1}{2})$. Also recall for all $\alpha \in [0, \frac{1}{2}]$ we have $\widetilde{R}(\frac{1}{2}) = R(\frac{1}{2})$, the revenue is the same (and the functions precisely the same) since there is no pooling interval until $\alpha > \frac{1}{2}$. It would then be tempting to immediately assert $\widetilde{R}'(\frac{1}{2}) = R'(\frac{1}{2})$. However, once we increase α by an ε a pooling interval develops, and thus we must account for this to determine $\widetilde{R}'(\frac{1}{2})$. As we do increase α by an ε, all we do is shift the bottom support of the integral comprising $R'(\frac{1}{2})$ to some $c > \underline{c}$ just greater than $\underline{c}$. Thus, we want to show $\frac{d}{dc}R(\frac{1}{2}) > 0$, and we will be done:

$$\frac{d}{dt}\bigg|_{\underline{c}} \left[k\int_{t}^{\bar{c}} g'(\frac{1}{2}(A(c) + B(c))) \times (B(c) - A(c))) \times F'(c)dc \right] > 0$$

Hence, we get

$$\frac{d}{dt}\left[k\int_{t}^{\bar{c}} g'(\frac{1}{2}(A(c) + B(c))) \times (B(c) - A(c))) \times F'(c)dc \right]$$
$$= -k \cdot g'(\frac{1}{2}(A(t) + B(t))) \times (B(t) - A(t)) \times F'(t)$$

However, we know $(B(t) - A(t)) < 0$ when $t = \underline{c}$. Thus, since we always have $g'(\cdot) > 0$ and $F'(t) > 0$, the entire expression is then strictly positive. In addition, we have now added a pooling interval that induces positive total revenue value once $\alpha > \frac{1}{2}$ that is in addition to the revenue related to the above expression.

But both these facts then mean $\widetilde{R}'(\frac{1}{2}) > R'(\frac{1}{2})$,as desired. $\square$

Proposition 5 It is optimal to offer a larger second prize than first prize through

our GSPC if our sufficient condition is met:

$$k\int_{\underline{c}}^{\overline{c}} h'\left(g(\frac{1}{2}(A(c)+B(c)))\right) \times g'(\frac{1}{2}(A(c)+B(c)))(B(c)-A(c)) \times F'(c)dc > 0$$

Proof:

First take the derivative of our revenue function with respect to α :

$$\frac{d}{d\alpha}R(\alpha) = \frac{d}{d\alpha}k\int_{\underline{c}}^{\overline{c}} h(g(A(c)+\alpha(B(c)-A(c)))) \times F'(c)dc$$

$$= k\int_{\underline{c}}^{\overline{c}} h'(g(A(c)+\alpha(B(c)-A(c)))) \times g'(A(c) + \alpha(B(c)-A(c)))) \times (B(c)-A(c)) \times F'(c)dc$$

No we evaluate this expression at $\alpha = 5$:

$$\frac{d}{d\alpha}R(.5) = k\int_{\underline{c}}^{\overline{c}} h'\left(g(\frac{1}{2}(A(c)+B(c)))\right) \times g'(\frac{1}{2}(A(c)+B(c)))(B(c)-A(c)) \times F'(c)dc$$

Now if $\frac{d}{d\alpha}R(.5) > 0$, we know then that $\widetilde{R}'(\frac{1}{2}) > R(.5) > 0$, per the previous Lemma . But this then means it is optimal to offer a larger second prize via the GSPC. $\square$

14.1 Estimating Some Convex Combinations of $A(c)$ and $B(c)$

To solve this analytically, we need to find some upper and lower bounds for our various terms.

We first find upper and lower bounds of $A(c)$.

First recall $A(c) \equiv (k-1)\int_c^{\bar{c}} \frac{1}{a}(1-F(a))^{k-2} \times F'(a)da$.

Hence,

$$\begin{aligned}
(k-1)\int_c^{\bar{c}} \frac{1}{a}(1-F(a))^{k-2} \times F'(a)da &> \frac{1}{\bar{c}}\int_c^{\bar{c}} (k-1)(1-F(a))^{k-2} \times F'(a)da \\
&= \frac{1}{\bar{c}}\left[-(1-F(a))^{k-1}\right]_c^{\bar{c}} \\
&= \frac{1}{\bar{c}}(1-F(c))^{k-1} = \underline{A}(c).
\end{aligned}$$

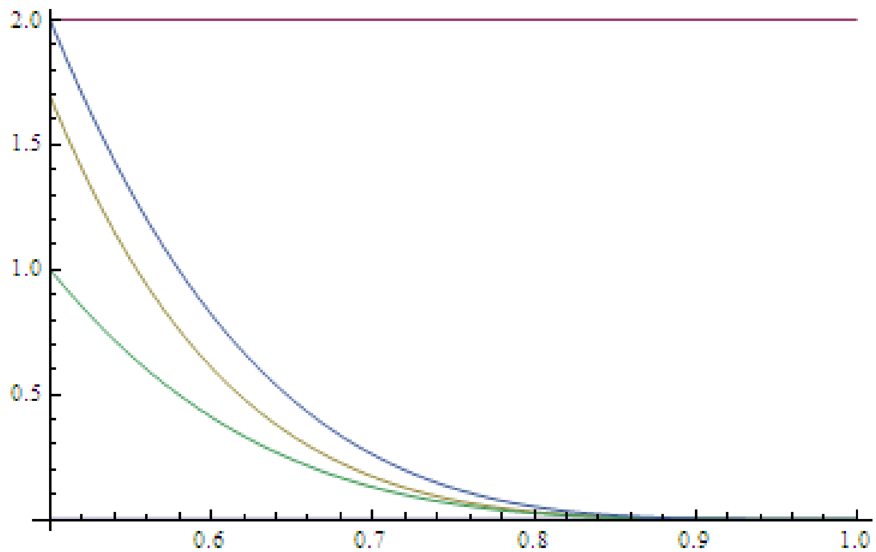
Thus we have:

$$\underline{A}(c) = \frac{1}{\bar{c}}(1-F(c))^{k-1}$$

$$\overline{A}(c) = \frac{1}{\underline{c}}(1-F(c))^{k-1}$$

That is, we have $\underline{A}(c) < A(c) < \overline{A}(c)$.

Here is a plot of our estimates with $k=5$, quadratic costs, and $c \in U[.5,1]$:



We next solve for some values of the above expression individually.

By integration of parts we have

$$\int_{c}^{\bar{c}} (1 - F(a))^{k-3} \times F(a) \times F'(a)da$$

$$= [F(a) \times -\frac{1}{k-2}(1-F(a))^{k-2}]_{c}^{\bar{c}} - \int_{c}^{\bar{c}} -\frac{1}{k-2}(1-F(a))^{k-2} \times F'(a)da$$

$$\begin{aligned} &= F(c) \times \frac{1}{k-2}(1-F(c))^{k-2} - \left[\frac{1}{(k-2)(k-1)} \times (1-F(a))^{k-1}\right]_{c}^{\bar{c}} \\ &= F(c) \times \frac{1}{k-2}(1-F(c))^{k-2} + \frac{(1-F(c))^{k-1}}{(k-2)(k-1)} \end{aligned}$$

.

That is, we get:

$$
\begin{aligned}
&\int\limits_{c}^{\bar{c}}(1-F(a))^{k-3}\times F(a)\times F'(a)da\\
=\ & F(c)\times\frac{1}{k-2}(1-F(c))^{k-2}+\frac{(1-F(c))^{k-1}}{(k-2)(k-1)}\\
=\ & \frac{(k-2)\cdot F(c)+1}{(k-2)(k-1)}\times(1-F(c))^{k-2}
\end{aligned}
$$

Secondly, we note through similar analysis:

$$
\int\limits_{c}^{\bar{c}}(1-F(a))^{k-3}\times F'(a)da=(1-F(c))^{k-2}\times\frac{1}{k-2}
$$

Now we estimate $\frac{1}{2}(A(c)+B(c))$.
Again after some basic calculations we have

$$
\frac{1}{2}(A(c)+B(c))=\frac{1}{2}(k-1)(k-2)\int_{c}^{\bar{c}}\frac{(1-F(a))^{k-3}}{a}\times F(a)\times F'(a)da
$$

We then get using our results from above:

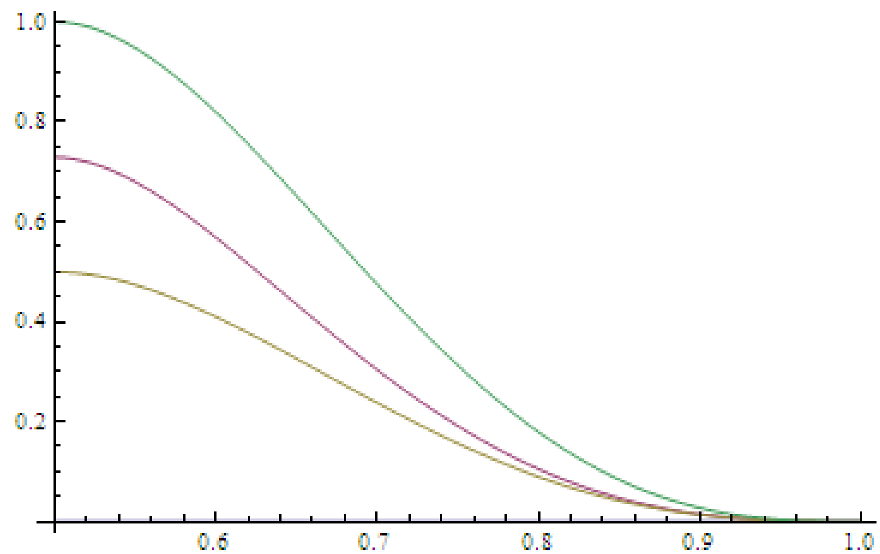
$$
\begin{aligned}
&\frac{1}{2}(k-1)(k-2)\left(\int_{c}^{\bar{c}}\frac{(1-F(a))^{k-3}}{a}\times F(a)\times F'(a)da\right)\\
>\ & \frac{1}{2\bar{c}}(k-1)(k-2)\left(\frac{(k-2)\cdot F(c)+1}{(k-2)(k-1)}\times(1-F(c))^{k-2}\right)\\
=\ & \frac{1}{2\bar{c}}\left((F(c)\times(k-2)+1)\cdot(1-F(c))^{k-2}\right)
\end{aligned}
$$

Hence, we have:

$$
\begin{aligned}
\frac{1}{2}(\underline{A(c)+B(c)}) &= \frac{1}{2\bar{c}}\left((F(c)\times(k-2)+1)\cdot(1-F(c))^{k-2}\right)\\
\frac{1}{2}(\overline{A(c)+B(c)}) &= \frac{1}{2\underline{c}}\left((F(c)\times(k-2)+1)\cdot(1-F(c))^{k-2}\right)
\end{aligned}
$$

This gives us $0<\frac{1}{2}(\underline{A(c)+B(c)})<\frac{1}{2}(A(c)+B(c))<\frac{1}{2}(\overline{A(c)+B(c)})$.

Here is a plot of these upper and lower bounds around the true value plotted as the red curve:



Now we solve for upper and lower bounds of $B(c) - A(c)$.
First, we have by definition and some trivial calculations:

$$B(c) - A(c) = (k-1)\int_{c}^{\bar{c}} \frac{(1-F(a))^{k-3}}{a} \times (k \cdot F(a) - 2)\, F'(a)da$$

This means we know

$$\begin{aligned} & B(c) - A(c) \\ < \; & (k-1)\int_{F^{-1}\left(\frac{2}{k}\right)}^{\bar{c}} \frac{(1-F(a))^{k-3}}{F^{-1}\left(\frac{2}{k}\right)} \times (kF(a) - 2)\, F'(a)da \\ & +(k-1)\int_{\underline{c}}^{F^{-1}\left(\frac{2}{k}\right)} \frac{(1-F(a))^{k-3}}{F^{-1}\left(\frac{2}{k}\right)} \times (kF(a) - 2)\, F'(a)da \end{aligned}$$

Thus, we have

$$\begin{aligned} & \overline{B(c) - A(c)} \\ = \; & \frac{(k-1)}{F^{-1}\left(\frac{2}{k}\right)} \left(k \times \frac{(k-2)F(c)+1}{(k-2)(k-1)} \times (1-F(c))^{k-2} - (1-F(c))^{k-2} \times \frac{2}{(k-2)} \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{F^{-1}\left(\frac{2}{k}\right)}\left(k \times \frac{(k-2)\cdot F(c)+1}{(k-2)} \times (1-F(c))^{k-2} - (1-F(c))^{k-2} \times \frac{2(k-1)}{(k-2)}\right) \\
&= \frac{1}{F^{-1}\left(\frac{2}{k}\right)} \frac{k(k-2)F(c)+k-2(k-1)}{(k-2)} \times (1-F(c))^{k-2}
\end{aligned}
$$

$$
\Longrightarrow \overline{B(c)-A(c)} = \frac{1}{F^{-1}\left(\frac{2}{k}\right)}\left(k\cdot F(c)-1\right) \times (1-F(c))^{k-2}\Big)
$$

This then implies we have $F(c^{**}) > \frac{1}{k}$, where again $A(c^{**}) = B(c^{**})$. This follows from noting $\overline{B(c)-A(c)} = 0$ when $c = F^{-1}(\frac{1}{k})$. However, since $\overline{B(c)-A(c)}$ is an upper bound, we know we have $B(c)-A(c) < 0$ at $c = F^{-1}(\frac{1}{k})$ (since $B(c)-A(c) < 0$ for all $c < c^{**}$). But this then also means it must be that $F(c^{**}) > \frac{1}{k}$. Thus, we now know:

$$
\frac{1}{k} < F(c^{**}) < \frac{2}{k}
$$

Next, we solve for $\underline{B(c)-A(c)}$.

We want

$$
\begin{aligned}
B(c)-A(c) \quad > \quad & \\
& (k-1)\int_{F^{-1}\left(\frac{2}{k}\right)}^{\overline{c}} \frac{(1-F(a))^{k-3}}{\overline{c}} \times \left(k\cdot F(a)-2\right)F'(a)da \\
& +(k-1)\int_{c}^{F^{-1}\left(\frac{2}{k}\right)} \frac{(1-F(a))^{k-3}}{\underline{c}} \times \left(k\cdot F(a)-2\right)F'(a)da
\end{aligned}
$$

First assuming $c \geq F^{-1}\left(\frac{2}{k}\right)$, we can solve the above sum as simply the analog to $\overline{B(c)-A(c)}$, replacing $\underline{c}$ with $\overline{c}$:

$$
\begin{aligned}
& \frac{(k-1)}{\overline{c}}\int_{c}^{\overline{c}} (1-F(a))^{k-3} \times \left(k\cdot F(a)-2\right)F'(a)da \\
= \; & \frac{1}{\overline{c}}\left(k\cdot F(c)-1\right) \times (1-F(c)^{k-2}\Big)
\end{aligned}
$$

Now when $c < F^{-1}\left(\frac{2}{k}\right)$ we must solve for the latter part of the sum, which we find as:

$$\begin{aligned}
&(k-1)\int_{\underline{c}}^{F^{-1}\left(\frac{2}{k}\right)} \frac{(1-F(a))^{k-3}}{\underline{c}} \times (k\cdot F(a)-2)\,F'(a)da \\
=\ & \frac{1}{\underline{c}}\left(-(\frac{k-2}{k})^{k-2} + (k\cdot F(c)-1)\times(1-F(c))^{k-2}\right)
\end{aligned}$$

Thus, combining both the sums we have

$$\begin{aligned}
&\underline{B(c)-A(c)} \\
=\ & \frac{1}{\overline{c}}\left(k\cdot\left(\frac{2}{k}\right)-1)\times(1-\left(\frac{2}{k}\right))^{k-2}\right) + \frac{1}{\underline{c}}\left(-(\frac{k-2}{k})^{k-2} + (k\cdot F(c)-1)\times(1-F(c))^{k-2}\right) \\
=\ & \frac{1}{\overline{c}}(\frac{k-2}{k})^{k-2} + \frac{1}{\underline{c}}\left(-(\frac{k-2}{k})^{k-2} + (kF(c)-1)\times(1-F(c))^{k-2}\right)
\end{aligned}$$

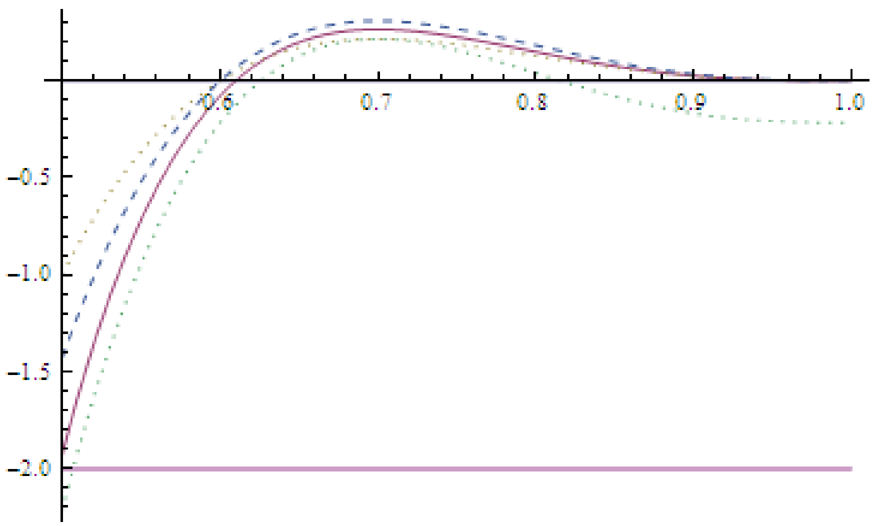
Thus, in total we have:

$$\begin{aligned}
\underline{B(c)-A(c)} &= \frac{1}{\overline{c}}\left((k\cdot F(c)-1)\times(1-F(c)^{k-2}\right) \\
\text{when } c &\geq F^{-1}\left(\frac{2}{k}\right) \\
\underline{B(c)-A(c)} &= \frac{1}{\overline{c}}(\frac{k-2}{k})^{k-2} + \frac{1}{\underline{c}}\left((kF(c)-1)\times(1-F(c))^{k-2} - (\frac{k-2}{k})^{k-2}\right) \\
\text{when } c &\leq F^{-1}\left(\frac{2}{k}\right)
\end{aligned}$$

Note also at $F^{-1}(\frac{2}{k})$, we have

$$\begin{aligned}
&\frac{1}{\overline{c}}\left((kF(c)-1)\times(1-F(c)^{k-2}\right) \\
=\ & \frac{1}{\overline{c}}(\frac{k-2}{k})^{k-2} + \frac{1}{\underline{c}}\left((kF(c)-1)\times(1-F(c))^{k-2} - (\frac{k-2}{k})^{k-2}\right)
\end{aligned}$$

Here is a graph of our upper and lower bounds- red is the actual values (for upper bound we have blue and for the lower bound we switch plots from yellow to green at $c = .7$):



Lemma 5 *For any distribution F with positive support, we have:*

1) A lower bound for $B(\underline{c}) - A(\underline{c})$ is $\frac{1}{3} \times \left(\frac{1}{\overline{c}} - \frac{1}{\underline{c}}\right) - \frac{1}{\underline{c}}$, where $\underline{c}$ is the lowest cost type and $\overline{c}$ the highest

2) A lower bound for the bid of the lowest cost type with equal first and second prizes (i.e., $\frac{1}{2}\left(B(\underline{c}) - A(\underline{c})\right)$) is $\frac{1}{2\overline{c}}$

We can estimate this with the following lower bound:

$$\underline{B(\underline{c}) - A(\underline{c})} = \frac{1}{\overline{c}}(\frac{k-2}{k})^{k-2} + \frac{1}{\underline{c}}\left((k \cdot F(\underline{c}) - 1) \times (1 - F(\underline{c}))^{k-2} - (\frac{k-2}{k})^{k-2}\right)$$

$$= \frac{1}{\overline{c}}(\frac{k-2}{k})^{k-2} + \frac{1}{\underline{c}}\left((-1) \times (1)^{k-2} - (\frac{k-2}{k})^{k-2}\right) = \left(\frac{1}{\overline{c}} - \frac{1}{\underline{c}}\right)(\frac{k-2}{k})^{k-2} - \frac{1}{\underline{c}}$$

As can be readily verified, this increases in k to a finite value in the limit. Thus with $k = 3$ (we assume at least 3 participants), $\underline{B(\underline{c}) - A(\underline{c})}$ has a minimal lower bound of $\left(\frac{1}{\overline{c}} - \frac{1}{\underline{c}}\right) \cdot (\frac{3-2}{3})^{3-2} - \frac{1}{\underline{c}} = \frac{1}{3} \times \left(\frac{1}{\overline{c}} - \frac{1}{\underline{c}}\right) - \frac{1}{\underline{c}} \equiv \underline{B(c) - A(c)}$. That is, recall also $B(c) - A(c)$ is most negative at $B(\underline{c}) - A(\underline{c})$; thus, $\frac{1}{3} \times \left(\frac{1}{\overline{c}} - \frac{1}{\underline{c}}\right) - \frac{1}{\underline{c}}$is the absolute lower bound of $B(c) - A(c)$. For example, with $c \sim U(.5, 1)$, we get $\underline{B(c) - A(c)}=$ $-\frac{7}{3}$,regardless of the number of participants.

Now we find a lower bound for the lowest cost type's bid when we have equal first and second prizes (i.e., $\alpha = .5$) and linear cost functions. Previous analysis shows a lower bound for $\frac{1}{2}(\underline{A(c) + B(c)}) = \frac{1}{2\overline{c}}\left((F(c) \times (k-2) + 1)(1 - F(c))^{k-2}\right)$. For the lowest cost type $\underline{c}$, this becomes:

$$\frac{1}{2\overline{c}}\left((F(\underline{c}) \times (k-2) + 1)(1 - F(\underline{c}))^{k-2}\right) = \frac{1}{2\overline{c}}$$

For example, with $c \sim U(.5, 1)$, we get the lower bound of the lowest cost type's bid to be $\frac{1}{2\overline{c}} = \frac{1}{2}$, for any number of contestants . $\square$

Thus, for our first portion of our Lemma, we have a found a simple lower bound on $B(\underline{c}) - A(\underline{c})$, the *maximal dis-incentivizing* effect of increasing the second prize for the lowest cost type $\underline{c}$, which is again also the most disincentivized type. In particular, if we shift ε of the prize mass from first to second prize, the lowest cost type will reduce effort by no more than $\left(\frac{1}{3} \times \left(\frac{1}{\overline{c}} - \frac{1}{\underline{c}}\right) - \frac{1}{\underline{c}}\right) \times \varepsilon$ (and all other low cost types $c < c^{**}$ will reduce effort by a lesser amount).

The second part of the Lemma is simply finding a lower bound on the bid of the lowest cost type, regardless of the number of participants. That is, we know the lowest cost type $\underline{c}$ will always bid at least $\frac{1}{2\overline{c}}$.

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