Sequential Disappearance of Reputations in Two-Sided Incomplete-Information Games

Ayça Özdoğan *
TOBB - Economy and Technology University
aozdogan@etu.edu.tr

INCOMPLETE

December 31, 2010

Abstract

This paper studies long-run sustainability of false reputations in a class of games with imperfect public monitoring and two long-lived players, both of whom have private information about their own type and uncertainty over the types of the other player. This class, namely *reputation games with one-sided binding moral hazard*, can capture a wide range of economic interactions between two parties that involve hidden-information (e.g. between a regulator and a regulatee) or hidden-action (e.g between an employer and an employee). Extending the techniques of Cripps, Mailath, and Samuelson (2004), I find that neither strategic player can sustain a false reputation permanently for playing a noncredible behavior in reputation games with one-sided binding moral hazard. Moreover, false reputations disappear *sequentially*. Hence, in this class, the true types of both players will be revealed eventually, one after the other, in all Nash equilibria and the asymmetric information does not affect equilibrium analysis in the long-run.

Journal of Economic Literature Classification Numbers: C70; C73.
Keywords: Reputation; Two-Sided Incomplete Information; Moral hazard.

*This paper is revised version of Chapter 1 of my dissertation at the University of Minnesota. I am grateful to my advisor Aldo Rustichini for his invaluable guidance. I thank Chris Phelan, Jan Werner and Rajesh Aggarwal for their constant support. I am grateful for the conversations with Beth Allen, Ioanna Grypari, Semih Koray, David Rahman, Gina Pieters, Ismail Sağlam, Itai Sher, Yuichiro Waki, Randall Wright, and especially Larry Samuelson. I also thank the participants of Mathematical Economics Workshop at the University of Minnesota, UECE Lisbon Meetings on Game Theory and Applications, and Stony Brook Game Theory Festival. Finally, I thank the seminar participants at University of Iowa, Bogazici University, Middle East Technical University, Bilkent University and TOBB-ETU for their comments and suggestions. Financial support from Graduate Research Partnership Program Fellowship is gratefully acknowledged. All remaining errors are mine.

1 Introduction

This paper studies long-run sustainability of false reputations in a class of games with imperfect public monitoring and two long-lived players, both of whom have private information about their own type and uncertainty over the types of the other player. Players may be either a *strategic* type who maximizes expected utility or a simple *commitment* type who finds it optimal to play a prespecified action every period. The strategic players may gain from opponent player's uncertainty about their types, by trying to convince the opponent that they are non-strategic. As in standard models, the reputation of a strategic type of player for being commitment type is established by mimicking the behavior of the commitment type. The distinct feature of my model is that, *both* players aim to establish a false reputation for being the commitment type in order to induce the opponent player behave in a specific way. I believe that wanting to establish a reputation is a key concern for *all* parties involved in several economic interactions. Specifically, in the economic applications that can be explained by the class of games I study (that is to be discussed in the subsequent sections and will be called *reputation games with one-sided binding moral hazard*), the reputation concerns of both parties are apparent.

Extending the techniques of Cripps, Mailath, and Samuelson (2004) (hereafter CMS), I find that neither strategic player can sustain a reputation permanently for playing a noncredible behavior in reputation games with one-sided binding moral hazard. Furthermore, the disappearance of false reputations is *sequential* in this class, i.e. the true types of the players will be revealed eventually one after the other in *all* Nash equilibria. As a result of the disappearance of false reputations; after an arbitrarily long history, any equilibrium of the continuation game is an equilibrium of the complete-information game. Thus, the asymmetric information does not affect equilibrium analysis in the long-run, which makes the reputation effect a short-run phenomena. The main contribution of this paper to the literature is that the well-known result of disappearance of reputation (when there is one player whose type is uncertain) is robust introducing uncertainty over the types of the second player in reputation games with one-sided binding moral hazard. Best to my knowledge, this is the first paper that attempts to answer what happens to reputations in the long-run in two-sided incomplete-information games. Moreover, I believe that this is the only class of two-sided incomplete-information games with simple commitment types and imperfect public monitoring, where reputations for noncredible behavior disappears in the long-run, in *all* Nash equilibria. To do so, I provide an example where reputations for noncredible behavior may be sustained in a Nash equilibrium in a game that does not belong to this class.

Reputation games with one-sided binding moral hazard feature the following property: There is exactly one player who wants to deviate from the commitment action profile in the corresponding complete-information game; i.e. the commitment action is suboptimal for only one of the strategic players who believes that either the opponent is of the commitment type or plays the commitment action in the stage game. This class of games encompass a wide range of economic applications between two parties that involve hidden-information (e.g. between a regulator and regulatee) or hidden-action (e.g. between an employer and employee). The common feature of these economic interactions is that one party-the principal- prefers that the other party-the agent- play in a certain way and use costly auditing to enforce this behavior. Suppose that the principal can choose either to be lazy or diligent in auditing the agent, which results in different probabilities of detecting an undesirable behavior of the agent. [1] Moreover, the agent believes that the principal could be a *tough type* who commits to play diligent every

[1] A detailed example is provided in Section 1.2.

period with some probability. Suppose also that the principal believes that the agent could be a *virtuous* type who acts properly every period. Then, the strategic principal may aim to establish a reputation for being the tough type, by choosing diligence every period, in order to enforce the agent to choose the proper behavior. Similarly, the agent aims to establish a reputation for being the virtuous type to induce the principal to be lazy. The fact that the actions of these two parties are not observable to each other prevents them learning each other's true type. The commitment action profile in this principal-agent setting is the (proper behavior, diligence). Suppose that the payoffs are constructed such that the only player who has an incentive to deviate from this profile is the principal, i.e. the principal wants to deviate to be lazy given that the agent chooses to behave properly. For the agent, the best reply against a diligent principal is to behave properly. With this construction of payoffs, this game fits into the class of reputation games with one-sided binding moral hazard and two-sided incomplete-information.

To obtain the results, I impose a monitoring structure that ensures that the public signals are statistically informative about each player's actions; and also, that allows players to infer the opponent's beliefs about their own type (thus makes the beliefs of each player *public*). The assumptions I make on monitoring structure enable players to identify any fixed stage game action of the other player from frequencies of signals after sufficiently many observations and compute the other player's posterior belief about their own types. In this setting, I show that reputations of both of the players disappear sequentially in all Nash equilibria if the commitment actions are noncredible, i.e. not part of an equilibrium for the strategic types in the stage game. More precisely; if for both of the players, their best response to the best response of the opponent to their commitment strategy is not their commitment strategy, then in any Nash equilibrium of the incomplete-information game, the true types will be revealed (almost surely).

The techniques of the proofs are borrowed from CMS, who study games with imperfect public monitoring in which only one of the players has uncertainty about the types of the other player. In our setting, i.e. two-sided incomplete-information reputation games with one-sided binding moral hazard, the long-run behavior of the reputation of the player who is subject to binding moral hazard (e.g. the principal) disappears eventually, independent of the behavior of the reputation of the player who is not (e.g. the agent). On the other hand, this condition necessitates the long-run behavior of the reputation of the player who is *not* subject to binding moral hazard depend on the behavior of the reputation of the player who is subject to binding moral hazard (e.g. the principal). Let player 2 be the one who is subject to binding moral hazard at the commitment action profile. I first show that player 2's type is (almost) revealed in the long-run. Then, I show that player 1's type is revealed eventually if and only if player 2's type is revealed. Hence, the one-sided binding moral hazard condition allows us to break the analysis of the long-run behavior of the reputations of the two players into two stages. These results also imply that the asymmetric information about the types of players does not interfere in the long-run equilibrium behavior, in the sense that continuation play in every Nash equilibrium of the incomplete-information game converges to an equilibrium of the complete-information game.

The paper is organized as follows: The following section 1.1 is about the related literature. In section 1.2, I provide an example of the class of games considered in this paper. Section 2 describes the model, Section 3 states the main results of the paper and Section **??** provides the proofs of the results.

1.1 Related Literature

Most of the early literature on games with reputation concerns focus on settings in which a long-lived player faces a sequence of short-lived players, each of whom plays only once but observes the previous play. In this environment, Fudenberg and Levine (1989) and Fudenberg and Levine (1992) provide a lower bound on the long-lived player's average payoff, namely the stage game Stackelberg payoff,[2] given that she is sufficiently patient. Following that tradition, Schmidt (1993) and Celentani, Fudenberg, Levine, and Pesendorfer (1996) show that such reputation effects arise in settings that involve two long-lived players and can even be stronger.[3]

The main concern of my paper is to understand what happens to false reputations in the long-run, rather than the payoff implications of reputation effects. In that regard, CMS show that a long-lived player can maintain a permanent reputation for playing a commitment strategy in a game with imperfect monitoring only if that strategy plays an equilibrium of the corresponding complete-information stage game. Thus, the powerful results about the lower bounds on the long-lived informed player's average payoff are short-run reputation effects, where the long-lived informed player's payoff is calculated at the beginning of the game. Cripps, Mailath, and Samuelson (2007) extend their earlier result to games with two long-lived players where the uninformed long-lived player has private beliefs over the types of informed long-lived player. Another important result on the long-run properties of reputations is by Benabou and Laroque (1992), who study a game with a long-lived player who can be one of two types, honest or opportunist, and a continuum of myopic players in asset markets. They focus on the Markov perfect equilibrium of this game where the actions of the long-lived player is not observable. They show that the long-lived player reveals her type in any Markov perfect equilibrium. These studies focus on games where the uncertainty is over the types of one of the players. I show that this result extends to two-sided incomplete-information reputation games with one-sided binding moral hazard.

1.2 Example

Consider the repeated interaction between a regulator (he) and regulatee (she) where the possible actions for the regulatee are to be truthful or untruthful about some noisy information she gets regarding the state of nature that is realized at the end of the period; and those for the regulator are to be lazy or diligent in auditing the regulatee.[4] For instance, the regulatee could be a bank which gets a noisy

[2] Stackelberg payoff is the payoff players receive in the stage game when they play their Stackelberg action (i.e. the action players would like to commit given that such a commitment induces a best response from the opponent player) and the opponent best responds to it. Stackelberg action is the action players would like to choose in an extensive-form game when they move the first.

[3] See Cripps, Dekel, and Pesendorfer (2005), Chan (2000), Atakan and Ekmekçi (2009a), Atakan and Ekmekçi (2009b), and Atakan and Ekmekçi (2009c) among others to see the reputation effects on *payoffs*. But, it is also well established that such reputation results may fail when both players are long-lived. I refer the reader to Schmidt (1993), Celentani, Fudenberg, Levine, and Pesendorfer (1996), Cripps and Thomas (1997) for further discussion about lower bounds on the payoffs.

[4] This game can be considered as a variant of *inspection games* extensively studied in the literature. I refer the reader Rudolf, Bernhard, and Zamir (2002) for a discussion on inspection games.

information about its own financial health and the regulator could be a government official.[5,6]

The actions of the regulatee, i.e. to be truthful or untruthful, are not observable to the regulator. However, the regulator observes if the message sent by the regulatee matches the state of nature that is realized at the end of the period. Since the regulatee's information about the state of nature is noisy; an incorrect message can come from a truthful behavior, as well as a correct message can come from an untruthful behavior. Similarly, the actions of the regulator, being lazy or diligent that induce different probability of audit, are not observable to the regulatee.[7] However, she observes if there is an audit or not at the end of the period, which may result after a lazy or a diligent behavior. The probability of correct message is higher if the regulatee is truthful; similarly, the probability of an audit is higher if the regulator is diligent. The regulator prefers the regulatee to be truthful and the regulatee prefers the regulator to be lazy. The expected (ex-ante) payoffs of the players are given by Table 1.2. Row player is the regulatee who chooses to be truthful (T) or untruthful (U) and the column player is the regulator who chooses to be diligent (D) or lazy (L).

Table 1: Expected Payoff Matrix

	L	D
T	x, y	$x - l_1, y - c$
U	$x + g, z$	$x - l_2, z - c + d$

where $y, z, g, c, d > 0$, $l_2 > l_1 \geq 0$ and $y > y - c > z - c + d > z$.

The regulatee's best response against the choice of being lazy is to be untruthful, since she has an expected gain of g. However, the regulatee's best response when the regulator is diligent in auditing is to be truthful, since the expected loss from untruthfulness when the regulator is diligent l_2 is higher than l_1. For the regulator, the best response when the regulatee is truthful is to be lazy, since the diligence in auditing has a cost of c. On the other hand, the regulator's best response is to be diligent when the regulatee is untruthful, since there is a expected gain d from possible detection of the untruthful behavior. The regulator gets his highest payoff when he is lazy and the regulatee is truthful; whereas the regulatee gets her highest payoff when the regulator is lazy and she is untruthful. Thus, the regulator wants to convince the regulatee that he is diligent to enforce truthfulness. However, the best response of the regulator if the regulatee is truthful is to be lazy. On the other hand, the regulatee prefers the regulator to be lazy, and thus she wants to convince the regulator that she is truthful to enforce laziness in auditing. However, the regulatee's best response once she thinks that the regulator is lazy is to be untruthful.

[5]The other possible applications for which these games can be used to model include analyzing the interaction between an employer and employee; tax evasion through the interaction between a tax payer and tax collecting agency; or the asset market manipulation via strategic announcements of an insider in the presence of a regulator.

[6]Benabou and Laroque (1992) provide a model of repeated strategic communication that analyzes manipulation in asset markets, where they extend Sobel (1985)'s model to the case in which the sender (insider) has noisy private information about the value of an asset. The sender can deceive public and distort the asset price through strategic announcements. However, their model is missing a strategic receiver who can audit the sender. Ozdogan (2010) incorporates a strategic receiver to that model.

[7]One can interpret this as the regulator chooses between two mixed strategies; or allocates some resources or time to auditing among its other tasks.

Suppose that the regulator believes that the regulatee is a *virtuous* type, who is truthful in every period, with some probability. The regulatee wants to use regulator's uncertainty over her types and pretend to be the virtuous type (by acting like the virtuous type) to enforce the regulator to be lazy in the continuation play. On the other hand, the regulatee believes that the regulator is a *tough* type, who is diligent in every period, with some probability.[8] Then the regulator may find it worthwhile to exploit regulatee's uncertainty over his type by pretending to be the tough type to induce truthfulness. Since the actions of the players are not observable to each other, they can't learn each other's true types for sure.

My question is to find out what happens to the reputations (for being tough and virtuous) of the regulator and the regulatee in the long-run in order to understand long-run equilibrium (steady-state) behavior and payoffs. For instance, if there were to be a Nash equilibrium where both of the reputations are sustainable, this means that each player should be seeing similar public signals on average from both types of the opponent player, since they cannot distinguish between the types. But, this is only possible if both types of the players act the same on average in the limit. Then the regulator should be diligent and the regulatee should be truthful on average indefinitely.[9] I show this is not the case: The reputation of being tough for the regulator disappears in the long-run (regardless of the long-run behavior of the regulatee's reputation of being virtuous) since the regulator is the player who is subject to binding moral hazard at the commitment profile. It is shown that after his true type is almost known, the regulatee starts to take advantage of regulator's uncertainty over her type and regulatee's reputation of being virtuous disappears eventually as well. More precisely, unless regulator's type is revealed, the regulatee's type won't be revealed. Intuitively, the regulatee waits for the revelation of the regulator's type, before revealing her type. These results suggest that if there is a possibility that the regulator is replaced every period so that the uncertainty about the regulator's type renewed every period, then the regulatee is never fully convinced that the regulator is tough. In this situation, the regulator keeps playing *diligent* and will not have incentive to deviate since the regulatee is not convinced.[10] So, one way to sustain the reputations is to introduce the possibility that keeps the uncertainty over the type of regulator every period.[11]

2 Model

This section first defines the complete-information game, the game without uncertainty over the types of players (i.e. the game when both players are *strategic* types). Then I present the incomplete-information game by adding commitment types of players to the model.

[8]These types are also players' Stackelberg types.

[9]Note that this wouldn't be efficient since this profile is Pareto inferior to always playing the profile $(Truthful, Lazy)$.

[10]Mailath and Samuelson (2001) and Phelan (2006) provide models where the long-lived informed player's type is governed by a stochastic process that has long-run implications.

[11]However, this will create an equilibrium which is not efficient in this game. That is why the replacements should be strategically scheduled in this setting.

2.1 Complete-information game

This is an infinitely repeated game with imperfect public monitoring. The stage game is a two-player finite simultaneous-move game. Player 1 ("she") chooses an action $i \in I \equiv \{1, ..., \bar{I}\}$ and player 2 ("he") chooses an action $j \in J \equiv \{1, ..., \bar{J}\}$. The public signal y is drawn from the finite set Y. The probability that the public signal is realized under the action profile (i, j) is given by ρ_{ij}^y. The ex post (realized) stage game payoff to player 1 (resp., 2) from action i (resp., j) and signal y is given by $u_1(i, y)$ (resp. $u_2(j, y)$). The ex ante (expected) stage game payoffs are $\pi_1(i, j) = \sum_y u_1(i, y)\rho_{ij}^y$ and $\pi_2(i, j) = \sum_y u_2(j, y)\rho_{ij}^y$.

Both players are long-lived with discount factor $\delta_1 < 1$ for player 1 and $\delta_2 < 1$ for player 2. [12] The set of histories is $h_t^f \equiv ((i_0, j_0, y_0), ..., (i_{t-1}, j_{t-1}, y_{t-1})) \in H_t^f \equiv (I \times J \times Y)^t$. Each player observes the realization of the public signal and his or her own past actions. Player 1's private history is denoted by $h_{1t} \equiv ((i_0, y_0), ..., (i_{t-1}, y_{t-1})) \in H_{1t} \equiv (I \times Y)^t$. Similarly, player 2's private history is denoted by $h_{2t} \equiv ((j_0, y_0), ..., (j_{t-1}, y_{t-1})) \in H_{2t} \equiv (J \times Y)^t$. And, the public history observed by both players is $h_t \equiv (y_0, ..., y_{t-1}) \in H_t \equiv Y^t$. The filtration on $(I \times J \times Y)^\infty$ induced by the private histories of player $m = 1, 2$ is denoted by $\{\mathcal{H}_{mt}\}_{t=0}^\infty$, while the filtration induced by the public histories is denoted by $\{\mathcal{H}_t\}_{t=0}^\infty$. Player 1's strategy, $\sigma \equiv \{\sigma_t\}_{t=0}^\infty$, is a sequence of maps $\sigma_t : H_{1t} \to \Delta(I)$. Similarly, Player 2's strategy, $\tau \equiv \{\tau_t\}_{t=0}^\infty$, is a sequence of maps $\tau_t : H_{2t} \to \Delta(J)$. The payoffs in the infinitely repeated game are normalized discounted sum of stage game payoffs, $(1 - \delta_m) \sum_{s=0}^\infty \delta_m^s \pi_m(i_s, j_s)$ for player $m = 1, 2$. The average discounted payoffs in period t is denoted by $\pi_{mt} \equiv (1-\delta_m) \sum_{s=t}^\infty \delta_m^{s-t} \pi_m(i_s, j_s)$.

It is assumed that the public signals have full support (Assumption 1). So, every public signal is possible after any action profile and players can not infer the actions chosen by the other player perfectly after a signal. Full support assumption prevents perfect inference of actions after any signal. We also assume "individual full rank" conditions, so that after sufficiently many observations, any fixed stage game action of either player can be identified from the frequencies of the signals (Assumptions 2 and 3).

Assumption 1 *(Full support) For all* $(i, j) \in I \times J$ *and* $y \in Y$, $\rho_{ij}^y > 0$.

Assumption 2 *(Individual 1 full rank) For all* $j \in J$*, the* I *columns in the matrix* $(\rho_{ij}^y)_{y \in Y, i \in I}$ *are linearly independent.*

Assumption 3 *(Individual 2 full rank) For all* $i \in I$*, the* J *columns in the matrix* $(\rho_{ij}^y)_{y \in Y, j \in J}$ *are linearly independent.*

Assumption 2 and 3 ensure that, for each player, the distribution of signals generated by any (possibly mixed) action is statistically distinguishable from any other for any given action of the other player. Note that these conditions require that $|Y| \geq \max\{|I|, |J|\}$.

A strategy profile (σ, τ) induces a probability distribution $P_{(\sigma,\tau)}$ over $H_\infty^f \equiv (I \times J \times Y)^\infty$. We denote the expectation with respect to this distribution by $E_{(\sigma,\tau)}$.

Definition 1 *A Nash equilibrium of the complete information game is a strategy profile* (σ^*, τ^*) *such that* $E_{(\sigma^*,\tau^*)}[\pi_{10}] \geq E_{(\sigma',\tau^*)}[\pi_{10}]$ *for all* σ' *and* $E_{(\sigma^*,\tau^*)}[\pi_{20}] \geq E_{(\sigma^*,\tau')}[\pi_{20}]$ *for all* τ'*.*

[12] The discount factors are allowed to be different.

This definition implies that under the equilibrium strategy profile, player m's strategy maximizes continuation expected utility after any history that occurs with positive probability. Note that, with the full support assumption, all public histories occur with positive probability. Hence, any Nash equilibrium outcome is also the outcome of a perfect Bayesian equilibrium.

2.2 Incomplete-information game

The uncertainty about players' preferences is modeled with Harsanyi (1967)'s notion of games with incomplete information by introducing a *commitment* type for each player. At time $t = -1$, before the game starts, nature selects a type for both players and tells each player her or his own type privately. With probability $1 - \mu_0 > 0$, player 1 is a "strategic" type, denoted by n, with the preferences described above and with probability $\mu_0 > 0$, she is a "commitment" type, denoted by c, who finds it optimal to play the action $s_1 \in \Delta(I)$ in each period regardless of the history. Similarly, with probability $1 - \gamma_0 > 0$, player 2 is a "strategic" type, denoted by n, whose preferences are described above and with probability $\gamma_0 > 0$, he is a "commitment" type, denoted by c, who plays the action $s_2 \in \Delta(J)$ in each period independent of the history. [13]

Let the best responses against the commitment action of the opponent be denoted by $r_1 \equiv BR_1(s_2)$ and $r_2 \equiv BR_2(s_1)$, respectively.

Definition 2 *Player $m = 1, 2$ is subject to binding moral hazard at the commitment profile (s_m, s_{-m}) if $r_m \neq s_m$.*

Since Assumption 1 ensures that deviations by players are not unambiguously detectable, if player m is subject to binding moral hazard at strategy profile (s_m, s_{-m}), then he has strict incentive to deviate from the profile (s_m, s_{-m}).

Definition 3 *A game is called a reputation game with one-sided binding moral hazard if there is exactly one player who is subject to binding moral hazard at the commitment strategy profile (s_1, s_2).*

The key restriction on the commitment action profiles and the payoff in the stage game is given by the following assumption.

Assumption 4 *The stage game satisfies the following:*

1. *The stage game is a reputation game with one-sided binding moral hazard.*

2. *Each player has a unique best reply r_m to s_{-m} and (s_1, r_2) and (r_1, s_2) are not stage game Nash equilibria.*

[13] Instead of modeling the incomplete-information by behavioral types, I could have modeled the commitment types as agents whose payoffs are different from those of the strategic ones, as in Koren (1992). Then players would know their own payoffs and have uncertainty over the payoffs of the other player.

Assumption 4 necessitates $s_1 = r_1$ a pure action. Although there is no need for such a restriction on s_2, for the sake of easing exposition in the proofs, I will assume s_2 to be a pure action as well.

An example of a stage game with one-sided binding moral hazard is given in Table 2.2. [14] Suppose that the commitment action profile is (T, D). [15] This game has one-sided binding moral hazard, since only player 2 is subject to binding moral hazard at (T, D).

Table 2: One-sided binding moral hazard

	L	D
T	2, 3	0, 2
U	3, 0	-1, 1

The first requirement of Assumption 4 already implies that either (s_1, r_2) or (r_1, s_2) is not Nash equilibrium, depending on the player who is subject to binding moral hazard at (s_1, s_2). Let player 2 be the one who is subject to binding moral hazard, then $r_1 = s_1$ and (r_1, s_2) is not a Nash equilibrium. Moreover, the second condition necessitates s_1 to be a pure action, whereas the commitment action of player 2 could be a mixed-action. The unique Nash of the above stage game is $\alpha_1(T) = \frac{1}{2}$ and $\alpha_2(D) = \frac{1}{2}$, where $\alpha_1 \in \Delta(I)$ and $\alpha_2 \in \Delta(J)$; providing a payoff vector of (1, 1.5). The minmax value for player 1 is 0 (the action that minmaxes player 1 is D) and the minmax value for player 2 is 1 (the action that minmaxes player 2 is U).

Let $\hat{\sigma}$ denote the repeated game strategy of playing s_1 in each period independent of history and $\hat{\tau}$ denote the repeated game strategy of playing s_2 in each period independent of history. Since r_1 is unique best response to s_2, by Assumption 4, the best response of player 1 in the repeated game, i.e. $BR_1(\hat{\tau})$, is a singleton and prescribes playing r_1 in each period for every history. Similarly, the best response of player 2 against the commitment strategy of player 1, $BR_2(\hat{\sigma})$, is a singleton and prescribes playing r_2 in each period for every history. Since (s_1, r_2) and (r_1, s_2) are not stage game Nash equilibrium, $(\hat{\sigma}, BR_2(\hat{\sigma}))$ and $(BR_1(\hat{\tau}), \hat{\tau})$ are not Nash equilibrium of the complete-information infinitely repeated game. The unique stage game best responses guarantee that there are no multiple best responses to the commitment strategies in the infinitely repeated game. Since $(\hat{\sigma}, BR_2(\hat{\sigma}))$ and $(BR_1(\hat{\tau}), \hat{\tau})$ are not Nash equilibrium in the complete-information infinitely repeated game, each player has an incentive to deviate to a strategy other than the repeated game commitment strategy, given that opponent is best responding to the commitment strategy.

Let $K = \{c, n\}$ and $L = \{c, n\}$ be the type spaces for player 1 and player 2, respectively. The repeated game strategy for player 1, σ, is a sequence of maps $\sigma_t : H_{1t} \times K \to \Delta(I)$. For player 2, the repeated game strategy is denoted by τ is a sequence of maps $\tau_t : H_{2t} \times L \to \Delta(J)$. Let σ be $\sigma \equiv (\hat{\sigma}, \tilde{\sigma})$ where $\hat{\sigma}$ is the strategy of the c type of player 1 that and $\tilde{\sigma}$ is the repeated game strategy of n type of player 1. Similarly, $\tau \equiv (\hat{\tau}, \tilde{\tau})$, where $\hat{\tau}$ is the repeated game strategy of c type of player 2 and $\tilde{\tau}$ is the strategy of n type of player 2. Then, a state of the world in the incomplete information game, ω, is a type for player 1, a type for player 2, and a sequence of actions and public signals. The

[14]This is a numerical example of payoff matrix of the games discussed in section 1.2 between a regulator and regulatee.

[15]Note that these are players Stackelberg actions. So, the commitment types in the game between the regulatee and regulator, i.e. *virtuous* and *tough*, are indeed their Stackelberg types.

set of states is $\Omega \equiv K \times L \times H^f_\infty$, where $H^f_\infty = (I \times J \times Y)^\infty$. The priors (μ_0, γ_0), the strategies $\sigma \equiv (\hat{\sigma}, \tilde{\sigma})$ of player 1, and the strategies $\tau \equiv (\hat{\tau}, \tilde{\tau})$ of player 2 jointly induce a probability measure $Q_{(\sigma,\tau;\mu,\gamma)}$ on $(\Omega, \mathcal{F}) \equiv (K \times L \times H^f_\infty, 2^K \otimes 2^L \otimes \mathcal{H}^f_\infty)$. The probability measure $Q_{(\sigma,\tau;\mu,\gamma)}$ describes how an uninformed observer of the game expects the play to evolve. I denote the expectation with respect to $Q_{(\sigma,\tau;\mu,\gamma)}$ by $E_{(\sigma,\tau)}$. Let $E_{(\sigma,\tau)}[.|\mathcal{H}_{1t}]$ and $E_{(\sigma,\tau)}[.|\mathcal{H}_{2t}]$ denote players expectations with respect to $Q_{(\sigma,\tau)}$ conditional on the filtration induced by the private histories, $\mathcal{H}_{1t}$ and $\mathcal{H}_{2t}$, respectively.

The strategy profile $(\hat{\sigma}, \tau)$ and $(\tilde{\sigma}, \tau)$, where $\tau = (\hat{\tau}, \tilde{\tau})$, induce probability measure $Q^{c\cdot}$ and $Q^{n\cdot}$, which describes how the play evolves when player 1 is the commitment and strategic type, respectively. The probability measure $Q^{k\cdot} \equiv Q_{(\sigma_k,\tau)}$, where σ_k is the strategy of the k type of player 1, describes how the game evolves if player 1 is of type k. The associated expectation is denoted by $E^{k\cdot} \equiv E_{(\sigma_k,\tau)}$. Similarly, the strategy profile $(\sigma, \hat{\tau})$ and $(\sigma, \tilde{\tau})$, where $\sigma = (\hat{\sigma}, \tilde{\sigma})$, induce probability measure $Q^{\cdot c}$ and $Q^{\cdot n}$, which describe how the play evolves when player 2 is the commitment and strategic type, respectively. So, the probability measure $Q^{\cdot l} \equiv Q_{(\sigma,\tau_l)}$, where τ_l is the strategy of the l type of player 2, describes how the game evolves if player 2 is of type l, and the associated expectation is $E^{\cdot l} \equiv E_{(\sigma,\tau_l)}$.

I will denote $Q_{(\sigma,\tau)}$, $E_{(\sigma,\tau)}$, $Q_{(\sigma_k,\tau)}$, $E_{(\sigma_k,\tau)}$, $Q_{(\sigma,\tau_l)}$ and $E_{(\sigma,\tau_l)}$ by Q, E, $Q^{k\cdot}$, $E^{k\cdot}$, $Q^{\cdot l}$ and $E^{\cdot l}$, respectively. Players' payoffs in the repeated game is then

$$\begin{aligned} E^{k\cdot}[\pi_{10}] &= E^{k\cdot}\big[(1-\delta_1)\sum_{t=0}^{\infty} \delta_1^t \pi_1(i_t, j_t)\big] \\ E^{\cdot l}[\pi_{20}] &= E^{\cdot l}\big[(1-\delta_2)\sum_{t=0}^{\infty} \delta_2^s \pi_2(i_t, j_t)\big] \end{aligned}$$

We assume that the players are indeed "strategic." Then,

Definition 4 *A Nash equilibrium of the incomplete information game is a strategy profile $(\tilde{\sigma}, \tilde{\tau})$ such that*

$$\begin{aligned} E^{n\cdot} \equiv E_{\tilde{\sigma},\tau}[\pi_{10}] \geq E_{\sigma',\tau}[\pi_{10}], \quad \forall \sigma' \\ E^{\cdot n} \equiv E_{\sigma,\tilde{\tau}}[\pi_{20}] \geq E_{\sigma,\tau'}[\pi_{20}], \quad \forall \tau' \end{aligned}$$

2.3 Beliefs and Inference

Player 1's posterior belief in period t about player 2's type is given by $\mathcal{H}_{1t}$ - measurable random variable

$$\gamma_t \equiv Q^{n\cdot}(c \mid \mathcal{H}_{1t}) : \Omega \to [0,1],$$

and player 2's posterior belief in period t about player 1's type is given by $\mathcal{H}_{2t}$ - measurable random variable

$$\mu_t \equiv Q^{\cdot n}(c \mid \mathcal{H}_{2t}) : \Omega \to [0,1].$$

The main theorem establishes that the reputations for being the commitment types cannot be sustainable indefinitely; in other words (false) reputations disappear and the true types will be revealed eventually in almost all histories, i.e. $\mu_t \to 0$ and $\gamma_t \to 0$ almost surely (with respect to the probability distribution induced by the strategies of the strategic type of the players).

With this specification, the reputations are *private* since the players' beliefs about each other's type is private. This means that the players do not know the beliefs of the other player about their own types perfectly. I impose a condition on the public monitoring structure that rules out the dependence of beliefs about the opponent on player's own past actions, and thus enables players to infer opponent's beliefs about their own types. I assume that the monitoring structure is such that the informativeness of the public signal about a player's action is independent of the other player's action (Assumption 5), and as a consequence, the reputations become *public*.

Let Prob$(i \mid y, j, \alpha_1)$ be the posterior probability of "player 1 having chosen pure action i", given mixed α_1 and given that player 2 observed signal y after playing action j, and Prob$(j \mid y, i, \alpha_2)$ is the corresponding posterior probability of player 2's action.

Assumption 5 *(Independence) For any $\alpha_1 \in \Delta(I)$ and $\alpha_2 \in \Delta(J)$, and any signal $y \in Y$,*

$$\begin{aligned} \textit{Prob}(i \mid y, j, \alpha_1) &= \textit{Prob}(i \mid y, j', \alpha_1), \quad \textit{for all} \quad j, j' \\ \textit{Prob}(j \mid y, i, \alpha_2) &= \textit{Prob}(j \mid y, i', \alpha_2), \quad \textit{for all} \quad i, i'. \end{aligned}$$

Assumption 5 implies that for all $\alpha_1 \in \Delta(I)$ and $j, j' \in J$,

$$\frac{\text{Prob}(y \mid i, j)\alpha_1(i)}{\sum_{i \in I} \alpha_1(i)\text{Prob}(y \mid i, j)} = \frac{\text{Prob}(y \mid i, j')\alpha_1(i)}{\sum_{i \in I} \alpha_1(i)\text{Prob}(y \mid i, j')} \tag{1}$$

Similarly, for all $\alpha_2 \in \Delta(J)$ and $i, i' \in I$,

$$\frac{\text{Prob}(y \mid i, j)\alpha_2(j)}{\sum_{j \in J} \alpha_2(j)\text{Prob}(y \mid i, j)} = \frac{\text{Prob}(y \mid i', j)\alpha_2(j)}{\sum_{j \in J} \alpha_2(j)\text{Prob}(y \mid i', j)} \tag{2}$$

Assumption 5 ensures that public signals allow players to infer the other player's beliefs about their type since the information that public signal provides about the player's action is independent of the opponent's behavior. In other words, this monitoring structure allows players to calculate opponent's inference about their reputation without knowing the opponent's action, thus reputations of both players becomes *public* and beliefs are common knowledge.

Assumption 5 holds if the monitoring has the *product structure*, i.e. each player has individual specific signals. [16,17] The public signal y is such that $y = (y_1, y_2) \in Y = Y_1 \times Y_2$ where y_1 is a signal of player 1's action and y_2 is a signal of player 2's, with

$$\rho_{ij}^{y} = \rho_i^{y_1} \rho_j^{y_2}, \quad \forall i, j, y.$$

For games with product structure, every sequential equilibrium payoff in the complete-information infinitely repeated game (equilibrium when private histories are used as beliefs) is also a public perfect

[16] In the motivating example game presented in section 1.2, each player had separate public signal, distribution of each depended only on player's own action, independent of the other players action.

[17] For games with product structure, pure-action profiles satisfy pairwise identifiability condition. Thus Fudenberg, Levine, and Maskin (1994) Folk theorem for games with imperfect public monitoring holds. Hence, any feasible and Pareto efficient payoff dominating a Nash equilibrium payoff of the stage game can be attained as an equilibrium payoff of the repeated (complete-information) game if players are sufficiently patient.

equilibrium payoff. [18] Thus, there is no loss of generality if the attention is restricted to public strategies. A public strategy $\sigma \equiv \{\sigma_t\}_{t=0}^{\infty}$ for the strategic player 1 is a sequence of maps $\sigma_t : H_t \times K \to \Delta(I)$ and that for player 2, $\tau \equiv \{\tau_t\}_{t=0}^{\infty}$, is a sequence of maps $\tau_t : H_t \times L \to \Delta(J)$. The strategy profile (σ, τ) induces a probability distribution $Q_{(\sigma,\tau)}$ over $H_{\infty}^{f} \equiv (I \times J \times Y)^{\infty}$. Let $E_{(\sigma,\tau)}[. \mid \mathcal{H}_t]$ denote players expectations with respect to $Q_{(\sigma,\tau)}$ conditional on the filtration induced by the public history, $\mathcal{H}_t$.

Due to Assumption 5 (i.e. under a monitoring structure such as the product structure), γ_t and μ_t can be viewed as $\mathcal{H}_t$ - measurable random variable $Q(c \mid \mathcal{H}_t)$ on Ω. This property enables both players to compute the opponent player's beliefs about themselves. So, in period t, strategic type of player 1 is maximizing $E_{\tilde{\sigma},\tau}[\pi_{1t} \mid \mathcal{H}_t]$, and a strategic player 2 is maximizing $E_{\sigma,\tilde{\tau}}[\pi_{2t} \mid \mathcal{H}_t]$, that depend on the information sets generated by public histories.

At any Nash equilibrium of the incomplete information game, γ_t is a bounded martingale with respect to the measure Q and filtration $\{\mathcal{H}_{1t}\}_t$ (and also with respect to filtration $\{\mathcal{H}_t\}_t$ by Assumption 5). Therefore, γ_t converges Q-almost surely (and also $Q^{\cdot n}$ - almost surely and Q^{nn} - almost surely, since $Q^{\cdot n}$ and Q^{nn} are absolutely continuous with respect to Q) to a random variable γ_{∞} on Ω. Similarly, at any Nash equilibrium of the incomplete information game, μ_t is a bounded martingale with respect to the measure Q and filtration $\{\mathcal{H}_{2t}\}_t$ (and also with respect to filtration $\{\mathcal{H}_t\}_t$), and thus converges Q-almost surely (and hence $Q^{n\cdot}$ - almost surely and Q^{nn} - almost surely, since $Q^{n\cdot}$ and Q^{nn} are absolutely continuous with respect to Q) to a random variable μ_{∞} on Ω.

3 Results

3.1 Reputations in the long-run

The main result of this paper is that neither player can sustain a reputation for playing a strategy that is not part of a Nash equilibrium of the complete-information stage game for reputation games with one-sided binding moral hazard under imperfect public monitoring. Moreover, this disappearance of reputations is sequential. The first proposition argues that reputation of player 2, who is subject to binding moral hazard, disappears uniformly in any Nash equilibrium of the incomplete-information game, i.e. $\gamma_t \to 0, \quad Q^{\cdot n} -$ almost surely and convergence is uniform across all Nash equilibria. The second proposition argues that if player 2's reputation disappears uniformly, then player 1's reputation disappears uniformly in any Nash equilibrium as well. Hence, the one-sided binding moral hazard condition allows us to break the analysis of the long-run behavior of the reputations of the two players into two stages. [19]

Proposition 1 *Suppose the monitoring technology satisfies Assumptions 1, 3 and 5, and the stage game is a reputation game with one-sided binding moral hazard. Then, in any Nash equilibrium of the incomplete-information game, reputation of player 2, who is subject to binding moral hazard, cannot be sustained indefinitely:*

$$\gamma_t \to 0, \quad Q^{\cdot n} - \textit{almost surely} \quad (\textit{and } Q^{nn} \textit{ - a.s.})$$

[18] See Mailath and Samuelson (2006) (p.330) and Fudenberg and Levine (1994) (Theorem 5.2) for further discussion.

[19] The proofs are presented in the subsequent sections.

Moreover, the disappearance of player 2's reputation is uniform, that is for all $\varepsilon > 0$, there exists T, such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$ of the incomplete-information game,

$$Q^{\cdot n}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma, \tilde{\tau}) < \varepsilon, \forall t > T) > 1 - \varepsilon,$$

where $Q^{\cdot n}_{\sigma=(\hat{\sigma},\tilde{\sigma}),\tilde{\tau}}$ is the probability measure induced on Ω by $(\sigma, \tilde{\tau})$ and $\gamma_t(\sigma, \tilde{\tau})$ is the associated reputation of player 2. [20]

The disappearance of player 2's reputation is independent of the asymptotic behavior of player 1's reputation. So, player 2's reputation of being the commitment type converges to zero $Q^{\cdot n}$-almost surely, even if there were to be a Nash equilibrium that induces a positive measure histories where 1 is sustained, i.e. player 1's type is not revealed. One immediate implication is that the set of histories induced by any Nash equilibrium, on which the reputation of player 2 is sustained assuming player 1's is sustained, has measure zero.

Corollary 1 *Suppose the monitoring technology satisfies Assumptions 1, 3 and 5, and the stage game is a reputation game with one-sided binding moral hazard. Suppose there exists a Nash equilibrium $(\tilde{\sigma}, \tilde{\tau})$ that induces a set of histories on which the reputation of player 2 (who is subject to binding moral hazard) does not disappear, given that the reputation of player 1 is sustained on these histories, i.e. suppose that there exists $A \in \Omega$ satisfying $\gamma_t(\omega) \to \gamma_\infty > \eta$ for some $\eta > 0$, given that $\mu_t(\omega) \to \mu_\infty > \epsilon$ for some $\epsilon > 0$ for all $\omega \in A$. Then $Q(A) = 0$ (and thus $Q^{\cdot n}(A) = 0$).*

Also, the other immediate implication of Proposition 1 is that there is no positive measure histories where player 1's reputation disappears in the long-run, but player 2's not.

Corollary 2 *Suppose the monitoring technology satisfies Assumptions 1, 3 and 5, and the stage game satisfies one-sided moral hazard at the commitment profile. Suppose there exists a Nash equilibrium $(\tilde{\sigma}, \tilde{\tau})$ that induces a set of histories on which player 1's reputation disappears but not player 2's (who is subject to binding moral hazard), i.e. there exists $A \in \Omega$ such that $\mu_t(\omega) \to 0$, but $\gamma_t(\omega) \to \gamma_\infty > \eta$ for some $\eta > 0$ and for all $\omega \in A$. Then $Q^{\cdot n}(A) = 0$, where $Q^{\cdot n}$ denotes $Q_{(\sigma,\tilde{\tau})}$.*

Having established that player 2 reveals his true type eventually regardless of the asymptotic behavior of player 1's reputation, the game can be considered to be the one with "almost" one-sided incomplete-information where the uncertainty is about the types of player 1 only. The next proposition gives the sufficient conditions for the disappearance of player 1's reputation.

Proposition 2 *Suppose the monitoring technology satisfies Assumptions 1, 2 and 5, and player 2's reputation γ_t converges uniformly to zero $Q^{\cdot n}$-almost surely in any equilibrium.*[21] *Suppose also (s_1, r_2) is*

[20] *Also, uniform convergence holds under the measure Q^{nn}.*
[21] *Thus, γ_t converges uniformly to zero Q^{nn}-a. s.*

not a Nash equilibrium of the stage game. Then, in any Nash equilibrium of the incomplete-information game, player 1*'s reputation cannot be sustained indefinitely:*

$$\mu_t \to 0, \quad Q^{nn} - \textit{almost surely}$$

where the convergence is uniform, i.e. for all $\varepsilon > 0$*, there exists* T*, such that for all Nash equilibria* $(\tilde{\sigma}, \tilde{\tau})$*,*

$$Q^{nn}_{\tilde{\sigma},\tau}(\mu_t(\tilde{\sigma}, \tau) < \varepsilon, \forall t > T) > 1 - \varepsilon,$$

where $Q^{n\cdot}_{\tilde{\sigma},\tau=(\hat{\tau},\tilde{\tau})}$ *is the probability measure induced on* Ω *by* $(\tilde{\sigma}, \tau)$ *and* $\mu_t(\tilde{\sigma}, \tau)$ *is the associated reputation of player 1.*

Proposition 2 implies that the sustainability of player 1's reputation depends on that of player 2's reputation. If player 2's reputation disappears, player 1's reputation disappears eventually, given that (s_1, r_2) is not a Nash equilibrium of the stage game (and thus $(\hat{\sigma}, BR_2(\hat{\sigma}))$ is not a Nash equilibrium of the repeated complete-information game).

Corollary 3 *Suppose the monitoring technology satisfies Assumptions 1, 2 and 5. Suppose there exists a Nash equilibrium that induces a set of histories* $A \in \Omega$ *with* $Q(A) > 0$ *on which the reputation of player 2 does not disappear, i.e.* $\gamma_t(\omega) \to \gamma_\infty > \eta$ *for some* $\eta > 0$ *and for all* $\omega \in A$*. Suppose also the stage game best reply of player 1 against* s_2 *is the same as her commitment action, i.e.* $r_1 = s_1$*. Then,* $\mu_t(\omega) \to \mu_\infty > \epsilon$ *for some* $\epsilon > 0$ *and* $Q^{n\cdot}$*-almost surely in* A*.*

Corollary 3 says that if the uncertainty over player 2's type persists, the uncertainty over player 1's type persists as well, since then player 1 expects to see s_2 on average in the long-run and gives a best response to it ($r_1 = s_1$). However, by Proposition 1, the uncertainty over player 1's type can persist only if either (s_1, s_2) a Nash equilibrium of the stage game or there is a mechanism that replenish the uncertainty over player 2's type. One such mechanism can be introducing a possibility for replacing the type of player 2 every period. With such a mechanism, player 2 need to mimic the commitment type always to convince player 1, since player 1 is never fully convinced because of the replacement possibility. Hence, player 2's type will not be revealed. As player 2's type is not revealed, player 1's type will not be revealed as well.

The results stated by the Propositions can be summarized in Theorem 1.

Theorem 1 *Suppose Assumptions 1-5 hold. In any Nash equilibrium of the incomplete-information game, reputations of players cannot be sustained indefinitely:*

$$\mu_t \to 0, \quad Q^{nn} - \textit{almost surely},$$
$$\gamma_t \to 0, \quad Q^{nn} - \textit{almost surely}.$$

Moreover, the convergence is uniform. [22]

[22] *Note that* $\gamma_t \to 0 \quad Q^{nn} - $ *almost surely, since* Q^{nn} *is absolutely continuous with respect to* $Q^{\cdot n}$*.*

The proof of Theorem 1 is immediate by Proposition 1 and 2. Section **??** is devoted to the proofs of Proposition 1 and 2.

The implication of these results for the regulatee-regulator game presented in Section 1.2 is that the reputation of being tough for the regulator disappears in the long-run since regulator is the player who is subject to binding moral hazard at the commitment profile (by Proposition 1). After his true type is almost known, the regulatee starts to take advantage of regulator's uncertainty over her type and regulatee's reputation of being virtuous disappears eventually as well (by Proposition 2). Furthermore, the set of histories where the regulatee's true type is almost known, but regulator's true type is not revealed has measure zero. One way to make both reputations sustainable is to introduce the possibility that the type of the regulator changes every period (with some probability). Then, the regulator can never convince the regulatee perfectly that he is tough, so he needs to be diligent every period. Both reputations can be made permanent this way. However, from the welfare point of view, this is inefficient. In order to get the efficient stage game outcome played (frequently), one needs a mechanism that allows for some deterioration for the reputation of the regulator up to a lower bound, so that the regulatee will not start exploiting this deterioration. By strategically scheduled replacement periods, there are periods of $(Truthful, Lazy)$ which Pareto dominates $(Truhful, Diligent)$. There is an optimal schedule for replacement periods that depends on the parameter values of the payoffs, as well as the values for the prior beliefs and discount factors.

3.2 Equilibrium behavior

After establishing that the true types will be (almost) known in the long-run and the information structure of the game approaches to that of the complete-information game, one expects to see such a convergence result holds for the equilibrium behavior. I show that any Nash equilibrium of the incomplete-information game converges to a public perfect equilibrium of the complete-information game, following the definitions and the methods provided by CMS for one long-lived and a sequence of short-lived players. I would like to point out that by our assumptions on the monitoring technology (Assumptions 2, 3 and 5), the imperfect public monitoring Folk theorem holds for the complete-information game.[23] Hence, any feasible and individually rational payoff vector of stage game in the complete-information game can be attained as an equilibrium payoff of the complete-information repeated game, if players are sufficiently patient. However, Theorem 2 neither constrain the possible set of equilibrium payoffs, nor answers if any particular Nash equilibrium strategy profile (or payoff) of the complete-information game can be achieved as a limit of a Nash equilibrium of the incomplete-information game.

Now let $t' = 0, 1, ...$ denote the time periods of the continuation play of the game that starts at some period t. A pure public (continuation game) strategy ς_1 for player 1 is a sequence of maps $\varsigma_{1t'} : H_{t'} \to I$ for $t' = 0, 1, ...$ Similarly, a pure public strategy for player 2 in the continuation game t' is ς_2, a sequence of maps $\varsigma_{2t'} : H_{t'} \to J$ for $t' = 0, 1, ...$ Let $S_1 = I^{\bigcup_{t'=0}^{\infty} Y^{t'}}$ and $S_2 = J^{\bigcup_{t'=0}^{\infty} Y^{t'}}$ be the set of pure strategies for player 1 and player 2, respectively.[24] Note that S_1 and S_2 include the pure

[23] See Fudenberg, Levine, and Maskin (1994).

[24] The sets S_1 and S_2 are countable products of finite sets I and J. Define σ-algebras for each set that are generated by cylinder sets and denote by $\mathcal{S}_m$, $m = 1, 2$. $(S_m, \mathcal{S}_m)$ is equipped with the product topology.

strategies in the original game as well. Player m's payoff is given by,[25]

$$U_m(\varsigma_1, \varsigma_2) = E_{(\varsigma_1, \varsigma_2)}\left[(1-\delta_m)\sum_{t'=0}^{\infty}\delta_m^{t'}\pi_m(i_{t'}, j_{t'})\right]$$

The mixed strategies (of the repeated continuation game) are the probability distributions over the set of pure strategies, i.e. ϑ_m be probability measures on $(S_m, \mathcal{S}_m)$.[26] Let Θ_m denote the set of all probability measures ϑ_m, $m = 1, 2$. Note that Θ_m is sequentially compact with respect to the product topology. Since players' payoffs are discounted, the utility function $U_m : \Theta_1 \times \Theta_2 \to \Re$ is continuous for each $m = 1, 2$ with this topology.[27]

A sequence of measures ϑ_1^n converges to $\hat{\vartheta}_1$ if the following holds: For every $T \geq 0$,

$$\vartheta_1^n|_{I^{Y^T}} \to \hat{\vartheta}_1|_{I^{Y^T}}$$

and, similarly, ϑ_2^n converges to $\hat{\vartheta}_2$ if for every $T \geq 0$,

$$\vartheta_2^n|_{J^{Y^T}} \to \hat{\vartheta}_2|_{J^{Y^T}}$$

Pick an equilibrium $(\tilde{\sigma}, \tilde{\tau})$ of the incomplete-information game and a public history h_t. These strategies specifies behavior strategies in the continuation game, $\tilde{\sigma}_{h_t}$ and $\tilde{\tau}_{h_t}$, which are realization equivalent to the mixed strategies $\tilde{\vartheta}_1^{h_t}$ and $\tilde{\vartheta}_2^{h_t}$ (for the continuation game), by Kuhn's Theorem. The following theorem states that the limit of every convergent subsequence of $(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})$ is a Nash equilibrium of the complete-information game. Similar convergence result about the asymptotic equilibrium behavior can be found CMS for one long-lived and a sequence of short-lived players. The appropriate modifications of their proofs for our model is given below.

Theorem 2 *Suppose Assumptions 1-5 are satisfied. For any Nash equilibrium of the incomplete-information game and for almost all sequences of public histories $\{h_t\}_t$ (with respect to measure Q^{nn}), the limit of every convergent subsequence of continuation equilibrium profiles $(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})$ is a public perfect equilibrium of the complete-information game (game with strategic types of players).*

Proof. We modify the proof of CMS for two long-lived player with uncertainty over the types of both players. Since $(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})$ are continuation equilibrium profile, for each public history h_t and pure strategies $\varsigma_1' \in S_1$ and $\varsigma_2' \in S_2$, the continuation expected payoffs should satisfy:

$$E_{(\tilde{\vartheta}_1^{h_t}, \gamma_t\hat{\vartheta}_2^{h_t}+(1-\gamma_t)\tilde{\vartheta}_2^{h_t})}[U_1(\varsigma_1, \varsigma_2)] \geq E_{\gamma_t\hat{\vartheta}_2^{h_t}+(1-\gamma_t)\tilde{\vartheta}_2^{h_t}}[U_1(\varsigma_1', \varsigma_2)] \tag{3}$$

$$E_{(\mu_t\hat{\vartheta}_1^{h_t}+(1-\mu_t)\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})}[U_2(\varsigma_1, \varsigma_2)] \geq E_{\mu_t\hat{\vartheta}_1^{h_t}+(1-\mu_t)\tilde{\vartheta}_1^{h_t}}[U_1(\varsigma_1, \varsigma_2')] \tag{4}$$

where $\hat{\vartheta}_1^{h_t}$ and $\hat{\vartheta}_2^{h_t}$ are the commitment mixed strategies corresponding to commitment behavior strategies $\hat{\sigma}_{h_t}$ and $\hat{\tau}_{h_t}$ in the continuation game. By Theorem 1, $\mu_t \to 0 \quad Q^{n\cdot}$-almost surely and $\gamma_t \to 0$

[25]Even though the strategies are pure, the payoffs are random because of imperfect public monitoring.

[26]Note that by Kuhn's theorem, one can replace mixed strategies by behavior strategies for games with perfect recall.

[27]Reader is referred to Fudenberg and Tirole (1991) for a detailed discussion and proof of the continuity of the utility function (due to discounting $\delta_1, \delta_2 < 1$).

$Q^{\cdot n}$-almost surely which imply $\gamma_t \to 0$ and $\mu_t \to 0 \quad Q^{nn}$- almost surely, by absolute continuity of Q^{nn} with respect to $Q^{n\cdot}$ and $Q^{\cdot n}$. Suppose $\{h_t\}_t$ is a sequence of public histories on which $\gamma_t, \mu_t \to 0$ and $\{(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})\}_{t=1}^{\infty} \to (\tilde{\vartheta}_1^*, \tilde{\vartheta}_2^*)$ on this sequence. We need to show $(\tilde{\vartheta}_1^*, \tilde{\vartheta}_2^*)$ satisfies (3) and (4), which suffices to show expectation $E_{(\vartheta_1,\vartheta_2)}$ is continuous in $(\vartheta_1, \vartheta_2)$. The continuity of this expectation is given by Theorem 4.4 of Fudenberg and Tirole (1991) and it is due to discounting (since $\delta_1, \delta_2 < 1$). ■

3.3 Discussion on reputations games with one-sided binding moral hazard

The condition of one-sided binding moral hazard at the commitment profile (s_1, s_2) is crucial for the results and proofs.[28] If none of the players has an incentive to deviate at (s_1, s_2) in the complete-information stage game, it means (s_1, s_2) is a Nash equilibrium of the stage game and thus repetition of (s_1, s_2) every period independent of history is a Nash equilibrium of the repeated complete-information game. Hence, the reputations can be sustained in that situation as the commitment strategies $(\hat{\sigma}, \hat{\tau})$ is a Nash equilibrium for the strategic types. If, on the other hand, the stage game has two-sided binding moral hazard at the commitment action profile, i.e. both players have an incentive to deviate at (s_1, s_2) in the stage game, then the results are not clear. I believe that one can construct a Nash equilibrium where the reputations do not necessarily disappear.

Consider the following extended Prisoner's dilemma game and suppose that there are $cooperative$ types for both players that play C every period. The strategic types have an incentive to deviate from the commitment action profile (C, C). Let $\mu_1 = \mu_2 = \mu_0$ be the prior belief that the players are $cooperative$ type.

Table 3: Two-sided binding moral hazard at (C, C)

	C	D_1	D_2
C	4, 4	0, 0	0, 5
D_1	5, 0	1, 3	0, 0
D_2	0, 0	0, 0	1, 3

Let $Y_1 = \{H, L\}$ and $Y_2 = \{h, l\}$ be the public signal spaces for player 1 and 2, $y \in Y_1 \times Y_2$, and the probability distributions over signals are given as below:

$$prob(H|C) = p, \quad prob(H|D_1) = r, \quad prob(H|D_2) = q$$
$$prob(h|C) = p, \quad prob(h|D_1) = q, \quad prob(h|D_2) = r$$

where $p > 1/2 > q > r$.

The strategy profile represented by the following automaton is a public perfect equilibrium for the complete-information infinitely repeated game. The states are $W = \{w_{CC}, w_1, w_2\}$ and the initial state

[28] Note that this is not only a restriction on the stage game payoff set, but also a restriction on the simple commitment types that are allowed.

is w_{CC}, and players choose the following action profiles corresponding to each state:

$$\begin{aligned} f(w_{CC}) &= CC, \\ f(w_1) &= D_1 D_1, \\ f(w_2) &= D_2 D_2. \end{aligned}$$

and the transition is

$$t(y) = \begin{cases} w_{CC} & \text{if} \quad y = Hh \quad \text{or} \quad Ll, \\ w_1 & \text{if} \quad y = Lh, \\ w_2 & \text{if} \quad y = Hl, \end{cases}$$

Note that w_1 is the punishment state for player 1 and w_2 is the punishment state for player 2. The equilibrium path can be described by an ergodic Markov chain on the state space with stationary distribution putting more weight on w_{CC}. This strategy profile is also an equilibrium profile for low enough μ_0, and the types will not be revealed because of frequent play of CC.

However, one can also construct Nash equilibria on which the uncertainty over the types of both players is going to be revealed eventually. In fact, if one of the player's type is revealed, the other's type is going to be revealed as well.

Lastly, I'd like to point out that CMS show that if there is uncertainty over the types of only one of the players, the true type (which is strategic) of this player will be revealed eventually in any Nash equilibria in games with imperfect monitoring. Adding uncertainty over the types of the second player may change this result for some Nash equilibria in games other than one-sided moral hazard.

4 Proofs

4.1 Player 2's reputation disappears uniformly (Proposition 1)

I first establish that either the true type of player 2 is revealed; or if not, since player 1 cannot distinguish the true type of player 2, her expectation of the strategy played by the strategic player 2 is in the limit the same as the strategy played by the commitment type, given that the public signals are statistically informative about player 2's behavior (Lemma 1). In other words, if player 1 is not eventually convinced that player 2 is strategic, she must be convinced that player 2 is mimicking the commitment type on average in the long-run. This is the standard merging of beliefs argument modified for imperfect public monitoring games.[29] Then, it is shown that if there is a set of histories with positive measure in which player 2's reputation does not disappear, in those histories, player 1 must be convinced that player 2 will play the commitment strategy in the continuation play; and moreover, reputations being public implies that player 2 knows about player 1's beliefs about his behavior (Lemma 2). Hence, player 2 believes that the strategic type of player 1 should be best responding to the commitment strategy of player 2 (Lemma 3), which also coincide with the strategy of commitment type of player 1 (since player 1 is *not* subject to binding moral hazard at the commitment type, and thus $r_1 = s_1$). Then player 2 has an

[29]See Sorin (1999) and CMS.

incentive to deviate from his commitment strategy (as player 2 is subject to binding moral hazard at the commitment profile), knowing that these deviations will not be detected due to imperfect monitoring [30] and player 1's beliefs have nearly converged, and thus the effect of deviations on player 1's beliefs will be arbitrarily small. However, the long-run effect of many such deviations, which generate different distributions over the public signals (Assumption 3), reveals that player 2 plays a strategy different than the commitment strategy. This provides the ground for the desired contradiction to the hypothesis of having a positive measure set of histories in which player 1 is convinced that player 2 is playing the commitment strategy on average in the long run.

4.1.1 Player 1's posterior beliefs about player 2

The following Lemma and Corollary establish that either player 1's expectation of the strategy played by the strategic type of player 2 is in the limit the same as the strategy played by the commitment type of player 2, or player 1's posterior probability that player 2 is the commitment type converges to zero (given that player 2 is indeed strategic). The proof is as the one provided by CMS.

Lemma 1 *Suppose Assumptions 1, 3 and 5 are satisfied. In any Nash equilibrium of the incomplete-information game,*

$$\lim_{t\to\infty} \gamma_t(1-\gamma_t)\|\hat{\tau}_t - E^{\cdot n}[\tilde{\tau}_t|\mathcal{H}_t]\| = 0, \quad Q-a.s. \tag{5}$$

Note that since $\hat{\tau}_t$ is a simple strategy, it can be replaced by s_2.

Proof. Let $\gamma_{t+1}(h_t; i_t, y_t)$ denote player 1's belief that player 2 is the commitment type in period $t+1$ after playing i_t and observing public signal y_t in period t, and given public history h_t. By Bayes' rule,

$$\gamma_{t+1}(h_t; i_t, y_t) = \frac{\gamma_t \text{prob}[y_t \mid h_t, i_t, c]}{\gamma_t \text{prob}[y_t \mid h_t, i_t, c] + (1-\gamma_t)\text{prob}[y_t \mid h_t, i_t, n]}$$

Since the probability of observing the signal y_t from the commitment type of player 2 is $\sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}$, and from the strategic type is $E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{i_t j}^{y_t} \mid h_t]$, one can rewrite the above expression as,

$$\begin{aligned}
\gamma_{t+1}(h_t; i_t, y_t) &= \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t} + (1-\gamma_t) E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{i_t j}^{y_t} \mid h_t]} \\
&= \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{\sum_{j\in J} \rho_{i_t j}^{y_t} \left(\gamma_t s_2^j + (1-\gamma_t) E^{\cdot n}[\tilde{\tau}_t^j \mid h_t] \right)}
\end{aligned}$$

[30] Player 2's incentive to deviate from the commitment strategy is stronger in two-sided incomplete-information game compared to one-sided incomplete information game where there is uncertainty only over the types of player 2.

The difference between $\gamma_{t+1}(h_t; i_t, y_t)$ and $\gamma_t(h_t)$ gives,

$$\begin{aligned}
|\gamma_{t+1}(h_t; i_t, y_t) - \gamma_t(h_t)| &= \left| \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{\sum_{j\in J} \rho_{i_t j}^{y_t}(\gamma_t s_2^j + (1-\gamma_t)E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])} - \gamma_t \right| \\
&= \left| \frac{\gamma_t(1-\gamma_t)\sum_{j\in J} s_2^j \rho_{i_t j}^{y_t} - \gamma_t(1-\gamma_t)\sum_{j\in J} \rho_{i_t j}^{y_t} E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])}{\sum_{j\in J} \rho_{i_t j}^{y_t}(\gamma_t s_2^j + (1-\gamma_t)E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])} \right| \\
&= \frac{\gamma_t(1-\gamma_t)\left| \sum_{j\in J} \rho_{i_t j}^{y_t}(s_2^j - E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) \right|}{\sum_{j\in J} \rho_{i_t j}^{y_t}(\gamma_t s_2^j + (1-\gamma_t)E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])}
\end{aligned}$$

Note that the denominator $\sum_{j\in J} \rho_{i_t j}^{y_t}(\gamma_t s_2^j + (1-\gamma_t)E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) < \max_{j\in J} \rho_{i_t j}^{y_t} < 1$ (by Assumption 1). Thus,

$$|\gamma_{t+1}(h_t; i_t, y_t) - \gamma_t(h_t)| \geq \gamma_t(1-\gamma_t)\left| \sum_{j\in J} \rho_{i_t j}^{y_t}(s_2^j - E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) \right|$$

for all y_t, which implies

$$\max_{y\in Y} |\gamma_{t+1}(h_t; i_t, y_t) - \gamma_t(h_t)| \geq \gamma_t(1-\gamma_t)\left| \sum_{j\in J} \rho_{i_t j}^{y_t}(s_2^j - E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) \right|$$

Since γ_t is a martingale on $[0,1]$ (with respect to Q and filtration $\{\mathcal{H}_t\}_t$) and bounded martingales converge almost surely, $|\gamma_{t+1} - \gamma_t| \to 0$ Q-almost surely. This implies, for any $y \in Y$,

$$\gamma_t(1-\gamma_t)\left| \sum_{j\in J} \rho_{i_t j}^{y_t}(s_2^j - E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) \right| \to 0, \qquad Q-\text{a.s.} \tag{6}$$

Since (6) holds for all y, it can be restated as

$$\gamma_t(1-\gamma_t)\left\| \Pi_{i_t}(s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t]) \right\| \to 0, \qquad Q-\text{a.s.}$$

where Π_{i_t} is a $|Y| \times |J|$ matrix that contains the values for $\rho_{i_t j}^{y_t}$. Since for all i_t and $y \in Y$, $\rho_{i_t j}^{y_t} > 0$ by Assumption 1 and J columns are linearly independent by Assumption 3, the unique solution to $\Pi_{i_t} x = 0$ is $x = 0$ and there exists a strictly positive constant $k = \inf_{i\in I, x\neq 0} \frac{\|\Pi_i x\|}{\|x\|}$. Thus, $\|\Pi_i x\| \geq k\|x\|$, which implies

$$\gamma_t(1-\gamma_t)\left\| \Pi_{i_t}(s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t]) \right\| \geq \gamma_t(1-\gamma_t)k\left\| (s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t]) \right\| \to 0, \quad Q-\text{a.s.}$$

This implies (5). ■

Note that Lemma 1 holds also $Q^{n\cdot}$ almost surely, since γ_t is also a bounded martingale with respect to $Q^{n\cdot}$, which is the probability measure that describes how the game evolves from the perspective of the strategic type of player 1. Note that any statement that holds Q almost surely, also holds $Q^{n\cdot}$ almost surely.

The immediate implication of Lemma 1 is Corollary 4:

Corollary 4 *At any Nash equilibrium of the incomplete-information game satisfying Assumptions 1, 3 and 5,*

$$\gamma_\infty = \lim_{t\to\infty} \gamma_t < 1, \qquad Q^{\cdot n} - a.s.$$

$$and$$

$$\lim_{t\to\infty} \gamma_t \left\| s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t] \right\| \to 0, \qquad Q^{\cdot n} - a.s.$$

Corollary 4 says that if player 2 is indeed strategic and the game evolves according to the play of strategic type of player 2, either his reputation for being the commitment type disappears, i.e. $\gamma_t \to 0$, $Q^{\cdot n}$ - almost surely, or he is expected to play the commitment action in the limit.

Proof. I first show that $\frac{\gamma_t}{(1-\gamma_t)}$ is a $Q^{\cdot n}$-martingale (with respect to filtration $\{\mathcal{H}_t\}$ due to Assumption 5). For all h_{t+1}, i_t and for all i,

$$\begin{aligned}
E^{\cdot n}[\frac{\gamma_{t+1}}{(1-\gamma_{t+1})}|\mathcal{H}_{t+1}] &= \sum_{y\in Y} \text{prob}[y_t|h_t, n].\frac{\gamma_{t+1}(h_t, i_t, y_t)}{1-\gamma_{t+1}(h_t, i_t, y_t)} \\
&= \sum_{y\in Y} E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{ij}^{y_t} \mid h_t] \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{(1-\gamma_t) E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{i_t j}^{y_t} \mid h_t]} \\
&= \sum_{y\in Y} \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{ij}^{y_t}}{(1-\gamma_t)} \\
&= \frac{\gamma_t \sum_{j\in J} s_2^j \sum_{y\in Y} \rho_{ij}^{y_t}}{(1-\gamma_t)} \\
&= \frac{\gamma_t}{(1-\gamma_t)}
\end{aligned}$$

The third equation is due to Assumption 5 and the last step is by Assumption 1. Thus, for all t,

$$E^{\cdot n}[\frac{\gamma_t}{(1-\gamma_t)}] = \frac{\gamma_0}{(1-\gamma_0)} \tag{7}$$

Since γ_t converging to some random variable Q - a.s. implies that γ_t converges $Q^{\cdot n}$ - a.s. (since $Q^{\cdot n}$ is absolutely continuous with respect to Q). Since $\frac{\gamma_0}{(1-\gamma_0)}$ is finite, $\lim_{t\to\infty}\gamma_t < 1 \quad Q^{\cdot n}$ - a.s. (Suppose on the contrary, there is a set $D \in \Omega$ with $Q^{\cdot n}(D) > 0$ such that $\gamma_t(\omega) \to 1$ for all $\omega \in D$. Then, $\frac{\gamma_t}{(1-\gamma_t)} \to \infty$ on D, which contradicts to (7)). ■

Note also that $\lim_{t\to\infty} \gamma_t \| s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t] \| \to 0$, also $Q^{nn} -$ a.s.

4.1.2 Player 2's beliefs about player 1's beliefs

After showing that if player 1 does not eventually learn that player 2 is strategic (when player 2 is strategic and the histories are induced by the play of strategic player 2), then player 1 must think that strategic type of player 2's strategy should be close to that of commitment type of player 2's; now, I want to show that player 2 will know that player 1 eventually expects to see the commitment action of player 2 in the continuation game on these histories where the true type is not revealed (due to beliefs being public by Assumption 5).

Lemma 2 *Suppose Assumptions 1, 3 and 5 hold and suppose there exists $A \in \Omega$ such that $Q^{\cdot n}(A) > 0$ and $\gamma_\infty(\omega) > 0$ for all $\omega \in A$, i.e. there exists a set of events with strictly positive measure in which reputation of player 2 does not necessarily disappear. Then, there exists $\eta > 0$ and $F \subset A$ with $Q^{\cdot n}(F) > 0$ (and $Q(F) > 0$) such that, for any $\xi > 0$, there exist T for which,*

$$\gamma_t > \eta, \quad \forall t \geq T,$$

$$E\left[\sup_{s\geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| \middle| \mathcal{H}_t\right] < \xi, \quad \forall t \geq T \tag{8}$$

for all $\omega \in F$; and for all $\psi > 0$

$$Q\left(\sup_{s\geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \psi \mid \mathcal{H}_t\right) \to 1 \tag{9}$$

where the convergence is uniform on F. [31]

Proof. First observe that on set A, $0 < \lim_{t\to\infty} \gamma_t(\omega) < 1$ by Corollary 4. Since $Q^{\cdot n}(A) > 0$ and $\gamma_\infty(\omega) > 0$ for all $\omega \in A$, there exist sufficiently small $\nu > 0$ and $\eta > 0$ such that $Q^{\cdot n}(D) > 2\nu$, where $D := \{\omega \in A : 2\eta < \lim_{t\to\infty} \gamma_t(\omega) < 1\}$. Note that D has positive measure under Q, i.e. there exists ν such that $Q(D) > 2\nu$ (since $Q^{\cdot n}$ is absolutely continuous with respect to Q.) Then, by Lemma 1, $\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\|$ converge Q - almost surely to zero on D. [32] So, the random variables $Z_t := \sup_{s\geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\|$ also converge Q - almost surely (also $Q^{\cdot n}$ - almost surely) to zero on D. Thus, on D, by an extension of Hart (1985) Lemma 4.24, provided by Mailath and Samuelson (2006), [33]

$$E[Z_t|\mathcal{H}_t] \to 0, \quad Q - \text{a. s.} \quad (\text{and} \quad Q^{\cdot n} - \text{a. s.} \quad \text{by absolute continuity})$$

and also $E^{\cdot n}[Z_t|\mathcal{H}_t] \to 0, \quad Q^{\cdot n} - \text{a. s.}$.

Now, one needs to show that this convergence is uniform. Egorov's Theorem (Chung (1974)) [34] then implies that there exists an $F \subset D$ such that $Q^{\cdot n}(F) > \nu$ (note that $Q(F) > 0$) on which the convergence of γ_t and $E[Z_t|\mathcal{H}_t]$ (and $E^{\cdot n}[Z_t|\mathcal{H}_t]$) is uniform. The uniform convergence of $E[Z_t|\mathcal{H}_t]$ on F implies that, for any $\xi > 0$, there exist a T such that on F, for all $t > T$, $\gamma_t > \eta$ and

$$E[Z_t|\mathcal{H}_t] = E\left[\sup_{s\geq t} \|s_2 - E[\tilde{\tau}_s \mid \mathcal{H}_s]\| \middle| \mathcal{H}_t\right] < \xi \tag{10}$$

[31] *The claims of Lemma 2 can also be stated for $E^{\cdot n}$ and $Q^{\cdot n}$. In subsequent sections, both versions are going to be used.*

[32] Note that $\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\|$ also converge $Q^{\cdot n}$ and $Q^{\cdot c}$ - almost surely to zero on D.

[33] This lemma states that if $\{X_n\}_{n=1}^\infty$ is a bounded sequence of real random variables on some $(\Omega, \mathcal{F}, P)$, converging to 0 as $n \to \infty$ and $\{\mathcal{F}_n\}_{n=1}^\infty$ is a nondecreasing sequence of σ - fields, then $E[X_n \mid \mathcal{F}_n] \to 0$ P-a.s.

[34] Egorov's Theorem states that if $\{X_n\}$ converges on the set C, then for any $\epsilon > 0$, there exists $C_0 \subset C$ with measure $\mathcal{P}(C \backslash C_0) < \epsilon$ such that X_n converges uniformly in C_0.

In order to show (9), fix $\psi > 0$. Then, for all $\xi' > 0$ such that $\xi = \xi'\psi$, (10) holds. Hence,

$$\begin{aligned} E[Z_t|\mathcal{H}_t] &= E[Z_t|Z_t < \psi, \mathcal{H}_t].Q(Z_t < \psi \mid \mathcal{H}_t) \\ &+ E[Z_t|Z_t \geq \psi, \mathcal{H}_t].Q(Z_t \geq \psi \mid \mathcal{H}_t) < \xi'\psi. \end{aligned}$$

Since the first expression is greater and equal to 0 and $E[Z_t|Z_t \geq \psi, \mathcal{H}_t] \geq \psi$,

$$Q(Z_t \geq \psi \mid \mathcal{H}_t) < \xi',$$

or $Q(Z_t < \psi \mid \mathcal{H}_t) > 1 - \xi'$ for all $t > T$ on F. This implies (9) and completes the proof.[35] ■

4.1.3 Player 1's best response to player 2

If player 1 were to be short-lived, as long as she thinks that she is facing a commitment strategy, she gives the myopic best reply to the commitment strategy of the opponent, which is s_1 in my setting (because of Assumption 4, i.e. unique best reply). This may not be true if player 1 is long-lived. She may have an incentive to play something other than the best response to the commitment action of player 2. In this case, since player 1 discounts future, she would play something other than myopic best reply if any losses from not playing a current best response should be recovered within a finite period of time. However, if player 1 is convinced that the commitment action will be played not only now, but also in the future, there will be no opportunity to accumulate subsequent gains, and hence she might as well play the stage-game best response. The next lemma, which follows from Lemma 4 of CMS, uses this intuition. It shows that if the commitment type and strategic type of player 2 play sufficiently similar from some time on, strategic player 1 will be best responding to the commitment strategy of the opponent for arbitrarily many periods.

Lemma 3 *Suppose $\hat{\tau}$ be a simple pure public strategy and $\hat{\sigma} \equiv BR_1(\hat{\tau})$ is the best reply of strategic player 1 to $\hat{\tau}$.*[36] *Let $(\tilde{\sigma}, \tilde{\tau})$ be Nash equilibrium strategies in the incomplete-information game. If $\tilde{\sigma}$ is a pure strategy, then for all $T > 0$, there exists $\psi > 0$ such that if the strategic player 1 observes a public history h_t so that*

$$Q\left(\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \psi \mid h_t \right) > 1 - \psi \tag{11}$$

then the continuation strategy of $\tilde{\sigma}$ after the history h_t agrees with $\hat{\sigma}$ for the next T periods.

Proof. Fix $T > 0$ and a public history h_t'. Let $\hat{\tau}(h_s) = s_2$ denote the continuation play of committed player 2 after the public history h_s, where h_t' is the initial segment of h_s.

Since player 1 is discounting, there exist $T' \geq T$ and $\epsilon > 0$ such that for $s = t, ..., t + T'$ and for all h_s with initial segment h_t'

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid h_s]\| < \epsilon, \tag{12}$$

[35] The other way to show this: $Q(Z_t \geq \psi \mid \mathcal{H}_t) \leq \frac{E[Z_t|\mathcal{H}_t]}{\psi} < \frac{\xi}{\psi}$ by Chebyshev-Markov inequality since Z_t has a finite mean and $Z_t \geq 0$. Since $\psi > 0$ and $\xi = \xi'\psi$, one gets $Q(Z_t \geq \psi \mid \mathcal{H}_t) < \xi'$ for all $\xi' > 0$.

[36] *Remember that $\hat{\tau}$ assigns s_2, which is a pure action, in each period independent of history and the repeated strategy best response $\hat{\sigma} \equiv BR_1(\hat{\tau})$ is a singleton, which assigns r_1 in every period after any history.*

is satisfied, then the continuation strategy of $\tilde{\sigma}$ after the history h'_t agrees with $\hat{\sigma} \in BR_1(\hat{\tau})$, for the next T periods.

Now, one needs to show (12) holds for all h_s with initial segment h'_t ($s = t, ..., t + T'$). Suppose not, i.e. there exist h_s, for some $s = t, ..., t + T'$ such that

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid h_s]\| \geq \epsilon,$$

For a contradiction, define $\bar{\rho} \equiv \min_{y,i,j} \rho^y_{ij}$ and $\psi = \frac{1}{2}\min\{\epsilon, \bar{\rho}^{T'}\}$. Since player 1 is playing a pure strategy, the probability of the continuation history h_s, conditional on the history h'_t, is at least $\bar{\rho}^{T'}$. [37] Thus,

$$Q\Bigg(\|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s])\| \geq \epsilon \mid h'_t\Bigg) \geq \bar{\rho}^{T'},$$

Since $\psi < \epsilon$,

$$Q\Bigg(\sup_{s \geq t} \|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s])\| \geq \psi \mid h'_t\Bigg) \geq \bar{\rho}^{T'}$$

contradicting (11), since $\bar{\rho}^{T'} > \psi$. ■

4.1.4 Proof of disappearance of player 2's reputation

The intuition of the proof is as follows: If $\gamma_t \nrightarrow 0$ on a positive measure set of histories, then on a subset of such states F (in Lemma 2) the strategic type of player 1 believes that she should be playing a best response to the commitment strategy of player 2 in the continuation games, which is the same as her commitment strategy. So, the strategic player 2, knowing what player 1 thinks about his future behavior and how she is going to respond to that on those histories, will best respond to the commitment strategy of player 1 with a high probability. Since, player 2's best response to player 1's strategy is different than his commitment strategy, the strategic and the commitment type of player 2 are expected to play differently. Thus player 1's second order beliefs about the future behavior of player 2 contradicts with her first order beliefs, leading to a contradicting to $\gamma_t \nrightarrow 0$ on F.

Specifically, one needs show that player 1 assigns a probability more than $1-\zeta$ to player 2 believing with probability at least $1 - \eta$ that player 1 thinks player 2's strategy is within ξ of the commitment strategy when the probability measure over the histories are induced by the play of the strategic type of player 2, i.e.

$$Q^{\cdot n}\Bigg(Q^{\cdot n}\Bigg(\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\Bigg) > 1 - \eta \mid \mathcal{H}_t\Bigg) > 1 - \zeta \tag{13}$$

Picking ξ, η and ζ such that $\zeta < 1$ and $\xi < \min\{\psi, 1 - \zeta\}$ will enable to get the desired contradiction.

I want to show that $\gamma_t \to 0$, $Q^{\cdot n}$ -almost surely. Suppose for a contradiction that there is a set of states A with $Q^{\cdot n}(A) > 0$ and $\gamma_\infty(\omega) > 0$ for all $\omega \in A$ (note that $\gamma_\infty(\omega) < 1$ on A by Corollary 4). Then, by Lemma 2, there is a set $F \subset A$ with $Q^{\cdot n}(F) > 0$ such that for any $\xi > 0$, there exists a T such that for any $t > T$ and $\omega \in F$,

$$Q\Bigg(\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t\Bigg) \to 1 \tag{14}$$

[37] If $\tilde{\sigma}$ is not pure, one could assume that there exists $k > 0$ such that for all h_t, if $\tilde{\sigma}^i(h_t) > 0$, then $\tilde{\sigma}^i(h_t) > k$. Then, one needs to define $\psi = \frac{1}{2}\min\{\epsilon, (k\bar{\rho})^{T'}\}$.

Hence, there exists a subset $G \subset F$ with $Q^{\cdot n}(G) > 0$ such that on G,

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\| < \xi \quad Q - \text{a.s.} \tag{15}$$

Note that (15) implies, for any $\xi > 0$ and any $t > T$, on G,

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\| < \xi \quad Q^{\cdot n} - \text{a.s.} \tag{16}$$

Also, by following the same reasoning in Lemma 2, one can say that for some η and ζ, on G,

$$Q^{\cdot n}\left(\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t\right) > 1 - \eta\zeta. \tag{17}$$

This shows that with a high probability $(1 - \eta\zeta)$, player 2 believes that player 1 assigns player 2's strategy to be ξ close to the commitment strategy in the continuation game of any $t > T$ (on histories induced by the play of strategic player 2).

Define

$$g_t := Q^{\cdot n}\left(\sup_{s \geq t} \|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t\right)$$
$$\kappa_t := Q^{\cdot n}(g_t > 1 - \eta | \mathcal{H}_t)$$

I want to show $\kappa_t > 1 - \zeta$ (to get 13). Since, $E^{\cdot n}[g_t \mid \mathcal{H}_t] > 1 - \eta\zeta$ by condition (17), and

$$\begin{aligned} E^{\cdot n}[g_t \mid \mathcal{H}_t] &= E^{\cdot n}[g_t \mid g_t \leq 1 - \eta, \mathcal{H}_t](1 - \kappa_t) + E^{\cdot n}[g_t \mid g_t > 1 - \eta, \mathcal{H}_t]\kappa_t \\ &\leq (1 - \eta)(1 - \kappa_t) + \kappa_t \end{aligned}$$

Thus,

$$1 - \eta\zeta < (1 - \eta)(1 - \kappa_t) + \kappa_t$$

which implies $\kappa_t > 1 - \zeta$ on G. So,

$$Q^{\cdot n}\left(Q^{\cdot n}\left(\sup_{s \geq t} \|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) > 1 - \eta \mid \mathcal{H}_t\right) > 1 - \zeta$$

This says that player 1 assigns a probability of at least $1 - \zeta$ (after observing histories generated by the play of strategic type of player 2) to strategic type player 2 believing with probability at least $1 - \eta$ that player 1 believes player 2's strategy is within ξ of the commitment strategy in the continuation game for every t after T. By Lemma 3, for any $T' > 0$, there exists $\psi > 0$ such that if player 1 observes a public history h_t so that

$$Q\left(\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \psi \mid h_t\right) > 1 - \psi,$$

then the continuation strategy of $\tilde{\sigma}$ after the history h_t agrees with $\hat{\sigma} \in BR_1(\hat{\tau})$ for the next T' periods, where $\hat{\sigma} = \{r_1\}_{t=0}^{\infty}$. [38] Since ξ is picked to be $\xi < \psi$, strategic player 1's strategy agrees with the best

[38] Remember that r_1 is the myopic best reply of strategic player 1 to s_2 and also the action of the commitment type of player 1.

reply to s_2 for T' periods after t for every $t > T$. That is why, player 2 believes that for all $t > T$, both types of player 1 is expected to play r_1 thereafter, there won't be any revision $\mu_t \equiv Q(c \mid \mathcal{H}_t)$, posterior about player 1's type and hence $\mu_{t>T} = \mu_T$.

Since s_2 is not a best response to r_1 for strategic player 2, there exists $\eta_\mu > 0$ such that for any repeated game strategy of the strategic player 1 that attaches probability at least $1 - \eta_\mu$ to $\hat{\sigma}$, s_2 is suboptimal for strategic player 2 (by the upper-hemicontinuity of the best response correspondence) in period 0 (current period). Let $\bar{\eta} \equiv \sup_{\mu \in (0,1)} \eta_\mu$ such that s_2 is suboptimal for strategic player 2 in period 0 if strategic player 1 attaches $1 - \bar{\eta}$ to $\hat{\sigma}$, regardless of player 2's belief about player 1's type.[39] Define $\bar{\rho} := \min_{y,i,j} \rho_{ij}^{y}$ (> 0 by Assumption 1). So, if player 2 assigns probability at least $1 - \bar{\rho}\bar{\eta}$ (and pick η such that $1 - \eta \equiv 1 - \bar{\rho}\bar{\eta}$) to $\hat{\sigma}$ (at the beginning of the current period), then he assigns at least probability $1 - \bar{\eta}$ to $\hat{\sigma}$ after any deviation that leaves the probability of $\hat{\sigma}$ (conditional on any signal) unchanged at the beginning of the subsequent period. Thus, strategic player 1 expects to see a deviation by strategic player 2 in the subsequent period whenever she believes that player 2 attaches $1 - \eta$ to $\hat{\sigma}$ in the current period.

Hence, in any period $t > T$, player 1 assigns a probability of at least $1 - \zeta$ to player 2 believing that player 1's subsequent play is r_1 thereafter with at least probability $1 - \eta$. Thus, player 1 assigns at least $1 - \zeta$ to player 2's play in period t being a best response to $\hat{\sigma}$, knowing that player 2 believes that his deviation will leave $\hat{\sigma}$ unaltered in the subsequent period. Since s_2 is pure, it specifies an action $\hat{j}$ with probability 1. Hence, player 1 must believe that that action is played with no more than ζ probability in period t. But since $1 - \zeta > \xi$, this contradicts (16). Player 1's second order beliefs about strategic player 2's behavior (after observing the relevant game has been evolving and histories have been generated by the play of strategic player 2) contradicts with her first order beliefs. This completes the proof of the first part of Proposition 1, i.e. $\gamma_t \to 0 \quad Q^{\cdot n}$ - almost surely, which implies $\gamma_t \to 0 \quad Q^{nn}$ - almost surely.

4.1.5 Uniform disappearance of player 2's reputation

Uniform convergence of $\gamma_t \to 0$ means that there exists some period T after which reputation converges to zero across all Nash equilibria. Suppose, on the contrary, there is a Nash equilibrium for each T after which the reputation of player 2 is sustained. Then the sequence of these Nash equilibria where the reputation lasts beyond T converges to a limiting Nash equilibrium with a sustainable reputation, which contradicts to disappearance of reputation result for any Nash equilibria. Specifically, one needs to show that for all $\varepsilon > 0$, there exists T, such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$ of the incomplete-information game,

$$Q^{\cdot n}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma, \tilde{\tau}) < \varepsilon, \forall t > T) > 1 - \varepsilon,$$

where $Q^{\cdot n}_{\sigma=(\hat{\sigma},\tilde{\sigma}),\tilde{\tau}}$ is the probability measure induced on Ω by $(\sigma, \tilde{\tau})$ and $\gamma_t(\sigma, \tilde{\tau})$ is the associated reputation of player 2. The uniform disappearance of player 2's reputation, can be proved as the proof of Theorem 3 of Cripps, Mailath, and Samuelson (2007) with minor modifications.

[39] Note that with this specification s_2 is suboptimal for any belief μ, in particular $\mu_{t>T} = \mu_T$.

4.2 Player 1's reputation disappears uniformly (Proposition 2)

Suppose that player 2's reputation disappears uniformly in any Nash equilibrium of the two-sided incomplete-information game, i.e. for all $\varepsilon > 0$, there exists T_2 such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$,

$$Q^{nn}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma, \tilde{\tau}) < \varepsilon, \forall t > T_2) > 1 - \varepsilon.$$

So, after T_2 on, player 1 attaches a very high probability to be facing the strategic type of player 2, i.e. facing the commitment type with a probability more than ε at some time t after T_2 is given a probability less than ε,

$$Q^{\cdot n}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma, \tilde{\tau}) \geq \varepsilon, \quad \text{for some} \quad t > T_2) \leq \varepsilon,$$

Player 1 thinks she will be seeing a strategy by the strategic player 2 for all t after T_2 with a very high probability. This at hand, I proceed with a similar reasoning used for the proof of Proposition 1. The counterparts of Lemma 1, 2 and 3 hold for player 1. Using these results, it is shown that player 1's reputation disappears (uniformly) as well.

The following Lemma argues that either player 2's expectation of the strategy played by the strategic type of player 1 is in the limit the same as the strategy played by the commitment type of player 1, or player 2's posterior probability that player 1 is the commitment type converges to zero (given that player 1 is indeed strategic). The key idea is the same: Strictly positive beliefs about player 1's types can exist in the long-run only if both types of player 1 play identically in the limit provided that the public signals are statistically informative about player 1's actions.

Lemma 4 (Player 2's beliefs about player 1) *Suppose Assumptions 1, 2 and 5 are satisfied. In any Nash equilibrium of the incomplete-information game,*

$$\lim_{t\to\infty} \mu_t(1-\mu_t)\|\hat{\sigma}_t - E^{n\cdot}[\tilde{\sigma}_t|\mathcal{H}_t]\| = 0, \quad Q - a.s. \tag{18}$$

Note that since $\hat{\sigma}_t$ is a simple commitment strategy, it can be replaced by s_1. The proof is the same as the one given for Lemma 1.

Corollary 5 *At any Nash equilibrium of the incomplete-information game satisfying Assumptions 1, 2 and 5,*

$$\lim_{t\to\infty} \mu_t\|s_1 - E^{n\cdot}[\tilde{\sigma}_t|\mathcal{H}_t]\| = 0, \quad Q^{n\cdot} - a.s.$$

Note that $\lim_{t\to\infty} \mu_t\|s_1 - E^{n\cdot}[\tilde{\sigma}_t|\mathcal{H}_t]\| = 0$, also Q^{nn} - a.s.

Corollary 5 says that if player 2 does not eventually learn that player 1 is strategic, then player 2 must think that strategic type 1's strategy should be close to that of commitment type since the distributions of public signals induced by the two types are not distinguishable. Strategic player 1 will know that player 2 believes this, since reputations are public by Assumption 5.

Lemma 5 (Player 1's beliefs about player 2's beliefs) *Suppose Assumptions 1, 2 and 5 hold and suppose there exists $A \in \Omega$ such that $Q^{nn}(A) > 0$ and $\mu_\infty(\omega) > 0$ for all $\omega \in A$, i.e. there exists a set of events with strictly positive measure in which reputation of player 1 does not necessarily disappear. Then, there exists $\eta > 0$ and $F \subset A$, with $Q^{nn}(F) > 0$, such that, for any $\xi > 0$, there exist T_1 for which,*

$$\mu_t > \eta, \quad \forall t \geq T_1,$$

$$E\left[\sup_{s \geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| \middle| \mathcal{H}_t\right] < \xi, \quad \forall t \geq T_1 \tag{19}$$

for all $\omega \in F$; and for all $\psi > 0$

$$Q\left(\sup_{s \geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \psi \mid \mathcal{H}_t\right) \to 1 \tag{20}$$

where the convergence is uniform on F.[40]

The next lemma shows that if the commitment type and strategic type of player 1 play sufficiently similar not only now but in the continuation game, strategic player 2 will be best responding to the commitment type's strategy for arbitrarily many periods.

Lemma 6 (Player 2's best response to player 1) *Suppose $\hat{\sigma}$ be a simple pure public strategy and $\hat{\tau} \equiv BR_2(\hat{\sigma})$ is the best reply of strategic player 2 to $\hat{\sigma}$.*[41] *Let $(\tilde{\sigma}, \tilde{\tau})$ be Nash equilibrium strategies in the incomplete-information game. If $\tilde{\tau}$ is a pure strategy, then for all $T > 0$, there exists $\psi > 0$ such that if player 2 observes a public history h_t so that*

$$Q\left(\sup_{s \geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \psi \mid h_t\right) > 1 - \psi \tag{21}$$

then for $\hat{\tau} \in BR_2(\hat{\sigma})$, the continuation strategy of $\tilde{\tau}$ after the history h_t agrees with $\hat{\tau}$ for the next T periods.

The sketch of the proof is as follows: Suppose for a contradiction that there is a set of states with positive measure that is induced by the play of the strategic types of players on which $\mu_t \nrightarrow 0$. Then, on a subset of states F (Lemma 5), strategic player 2 believes that player 1's strategy is very close to her commitment strategy in the continuation game for every t after T_1 and thus he should be playing a best response to the commitment strategy of player 1. Since the both players can compute what the other player believes about themselves and their future play, strategic player 1 knows what player 2 thinks of her future behavior is going to be and act accordingly. Since by Proposition 1, the reputation

[40] *The claims of this lemma can also be stated with $E^{n\cdot}$, $Q^{n\cdot}$ and E^{nn}, Q^{nn}.*

[41] *Note that $\hat{\sigma}$ assigns s_1 in every period independent of history. The repeated game best response of player 2 in the complete-information game is $BR_2(\hat{\sigma})$ is a singleton that assigns r_2 in each period.*

of player 2 will disappear in all Nash equilibria after T_2, player 1 expects to see a best reply to her commitment strategy from the *strategic* player 2 with a very high probability in the continuation play for every t after $T = \max\{T_1, T_2\}$. Thus, strategic player 1 best responds to the strategic player 2 (who gives a best reply to the commitment strategy of player 1 which is different than commitment type of player 2's strategy) with high probability. However, since best reply of strategic player 1 to player 2's strategy is different than her commitment strategy (by assumption), strategic and commitment types of player 1 are expected to play differently, which will provide the contradiction to $\mu_t \nrightarrow 0$ on F. More specifically, I am going to show that player 2 assigns a probability $1-\zeta$ to player 1 believing with probability at least $1-\eta$ that player 2 thinks player 1's strategy is within ξ of the commitment strategy when the probability measure over the histories are induced by the play of the strategic type of player 1. Picking ξ such that $\xi < \min\{\psi, 1-\zeta\}$ enables to arrive the desired contradiction.

$$Q^{nn}\left(Q^{nn}\left(\sup_{s\geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) > 1-\eta \mid \mathcal{H}_t\right) > 1-\zeta.$$

Suppose that there is a set of states A with $Q^{nn}(A) > 0$ and $\mu_\infty(\omega) > 0$ for all $\omega \in A$. Then, by Lemma 5, there is a set $F \subset A$ with $Q^{nn}(F) > 0$ (also $Q(F) > 0$) such that for any $\xi > 0$, there exists T_1 such that for any $t > T_1$ and $\omega \in F$,

$$Q\left(\sup_{s\geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) \to 1 \tag{22}$$

Then, there exists a subset $G \subset F$ with $Q^{nn}(G) > 0$ such that for any $t > T_1$, on G,

$$\|s_1 - E^{n\cdot}[\tilde{\sigma}_t \mid \mathcal{H}_t]\| < \xi, \quad Q - \text{a.s.} \tag{23}$$

Note that (23) implies $\|s_1 - E^{n\cdot}[\tilde{\sigma}_t \mid \mathcal{H}_t]\| < \xi \quad Q^{n\cdot}$- a.s., Q^{nn}-a.s. and $Q^{\cdot n}$-a.s. Also, again using the same argument of Lemma 5, one can conclude that for some η and ζ,

$$Q^{nn}\left(\sup_{s\geq t} \|s_1 - E^{\cdot n}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t\right) > 1-\eta\zeta. \tag{24}$$

Define,

$$g_t := Q^{nn}\left(\sup_{s\geq t} \|(s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t\right)$$
$$\kappa_t := Q^{nn}(g_t > 1-\eta | \mathcal{H}_t)$$

I want to show $\kappa_t > 1-\zeta$. Since, $E^{nn}[g_t \mid \mathcal{H}_t] > 1-\eta\zeta$ by condition (24), and

$$\begin{aligned} E^{nn}[g_t \mid \mathcal{H}_t] &= E^{nn}[g_t \mid g_t \leq 1-\eta, \mathcal{H}_t](1-\kappa_t) + E^{nn}[g_t \mid g_t > 1-\eta, \mathcal{H}_t]\kappa_t \\ &\leq (1-\eta)(1-\kappa_t) + \kappa_t \\ 1-\eta\zeta &< (1-\eta)(1-\kappa_t) + \kappa_t \end{aligned}$$

which implies $\kappa_t > 1-\zeta$ on F. So,

$$Q^{nn}\left(Q^{nn}\left(\sup_{s\geq t} \|(s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) > 1-\eta \mid \mathcal{H}_t\right) > 1-\zeta$$

This says that player 2 assigns a probability of at least $1-\zeta$ (after observing histories generated by the play of the strategic types of players) to player 1 believing with probability at least $1-\eta$ that player 2 believes player 1's strategy is within ξ of the commitment strategy.

Note that, by Lemma 6, for all $T' > 0$, there exists $\psi > 0$ such that if player 2 observes a (public) history h_t so that

$$Q\left(\sup_{s\geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \psi \mid h_t\right) > 1-\psi$$

then for $\hat{\tau} \equiv BR_2(\hat{\sigma})$, the continuation strategy of $\tilde{\tau}$ after the history h_t agrees with $\hat{\tau}$ for the next T' periods. Since $\xi < \psi$ is picked, strategic player 2 best responds to the commitment strategy of player 1 for the next T' periods for every t after T_1.

Also, for all $\varepsilon > 0$, there exists T_2 such that for all Nash equilibria,

$$Q^{nn}(\gamma_t(\sigma,\tilde{\tau}) < \varepsilon, \forall t > T_2) > 1-\varepsilon,$$

So, after T_2 on, player 1 attaches a very high probability to be facing the strategic type of player 2, i.e. facing the commitment type with a probability more than ε at some time t after T_2 is given a probability less than ε i.e.

$$Q^{nn}(\gamma_t(\sigma,\tilde{\tau}) \geq \varepsilon, \quad \text{for some} \quad t > T_2) \leq \varepsilon,$$

Define $T := \max\{T_1, T_2\}$. Note that after time $t > T$, by uniform disappearance of player 2's reputation, player 1 believes that he is facing the strategic player 2 with at least $1-\varepsilon$ probability, who thinks player 1's continuation strategy is ξ close to his commitment strategy and will give a best response to s_1, that is $\hat{\tau}$ (repetition of the myopic best reply r_2).

Since s_1 is not a best response to $r_2 \neq s_2$, there exists $\bar{\eta}_\varepsilon > 0$ such that for any strategy of strategic player 2 (who is expected to be faced with probability $1-\varepsilon$) that attaches probability at least $1-\bar{\eta}_\varepsilon$ to $\hat{\tau}$, s_1 is suboptimal for strategic player 1 (by the upper-hemicontinuity of the best response correspondence and the continuity of the expected utility) in the current period. Define $\bar{\rho} := \min_{y,i,j} \rho^y_{ij}$. So, if player 2 assigns probability at least $1-\eta \equiv 1-\bar{\rho}\bar{\eta}_\varepsilon$ to $\hat{\tau}$, then he assigns at least probability $1-\bar{\eta}_\varepsilon$ to $\hat{\tau}$ after any deviation that leaves the probability of $\hat{\tau}$ conditional on any signal unchanged.

Since $\xi < \psi$, for all $t > T$, strategic type of player 2 chooses to play r_2, the unique best response to the commitment action thereafter, whenever he believes that player 1's strategy is within ξ of the commitment strategy. Hence, in any period $t > T$, player 2 assigns a probability of at least $1-\zeta$ to player 1 believing that player 2's subsequent play is r_2 thereafter with at least probability $1-\eta$. Thus, player 2 assigns probability at least $1-\zeta$ to player 1's play in period t being a best response to $\hat{\tau}$. Since s_1 is pure, it specifies an action $\hat{i}$ with probability 1. However, player 2 must believe that that action is played with no more than ζ probability in period t. But since, $1-\zeta > \xi$, this contradicts (23). Player 2's second order beliefs about strategic player 1's behavior (after observing the relevant game has been evolving and histories have been generated by the play of strategic player 1) contradicts with his first order beliefs.

The uniform disappearance of player 2's reputation follows the same argument as in Section 4.1.5. Hence, for all $\varepsilon > 0$, there exists T, such that for all Nash equilibria $(\tilde{\sigma},\tilde{\tau})$ of the incomplete-information game,

$$Q^{nn}_{\tilde{\sigma},\tilde{\tau}}(\mu_t(\tilde{\sigma},\tilde{\tau}) < \varepsilon, \forall t > T) > 1-\varepsilon,$$

where $Q^{nn}_{\tilde{\sigma},\tilde{\tau}}$ is the probability measure induced on Ω by $(\tilde{\sigma}, \tilde{\tau})$ and the strategic types of players and $\mu_t(\tilde{\sigma}, \tilde{\tau})$ is the associated reputation of player 1.

5 Concluding Remarks

The main result of this paper is that the reputations of players for playing a strategy that is not part of an equilibrium of the stage game can not be sustainable in the long-run for reputation games with one-sided binding moral hazard under imperfect public monitoring. The way I prove our result is by first showing that the reputation of the player who is subject to binding moral hazard disappears (uniformly) and then after the type of that player is almost known, the reputation of the other player should disappear as well. Moreover, the continuation equilibrium of the incomplete-information game converges to an equilibrium of the complete-information game in the limit.

There are some interesting related questions left for future research such as how the rate of disappearance (convergence) is affected by different priors. For instance, I believe that in the regulatee-regulator game, the existence of a tough regulator "postpones" the revelation of the true type of the regulatee; whereas the existence of a virtuous regulatee "speeds up" the revelation of the type of the regulator. So, a regulator whose goal is to understand the type of the regulatee should not pretend to be the tough type.

The other important observation one could make about the regulatee-regulator game is that the reputations are sustainable for more complicated commitment strategies that are equilibria of the repeated complete-information game. For instance, if there is a grim trigger type for the regulator; I believe that the reputations would be sustainable and the equilibrium would be almost efficient, in the sense that players achieve the highest total payoff (very close to the efficient frontier of the feasible and individually rational payoff set). Hence, if a regulator could choose to establish a reputation for a type in the presence of a grim trigger and a tough type, he should choose to mimic the grim trigger type.

References

ATAKAN, A. E., AND M. EKMEKÇI (2009a): "Reputation in Long-Run Relationships," *Working paper*.

——— (2009b): "Reputation with Long Run Players and Imperfect Observation," *Working paper*.

——— (2009c): "A Two-sided Reputation Result with Long Run Players," *Working paper*.

BENABOU, R., AND G. LAROQUE (1992): "Using Privileged Information to Manipulate Markets: Insiders, Gurus and Credibility," *Quarterly Journal of Economics*, 107(3), 921–958.

CELENTANI, M., D. FUDENBERG, D. K. LEVINE, AND W. PESENDORFER (1996): "Maintaining A Reputation Against A Long-Lived Opponent," *Econometrica*, 64(3), 691–704.

CHAN, J. (2000): "On the non-existence of reputation effects in two-person infinitely repeated games," *Working paper*.

CHUNG, K. L. (1974): *A Course in Probability Theory*. New York: Academic Press.

CRIPPS, M., E. DEKEL, AND W. PESENDORFER (2005): "Reputation with equal discounting in repeated games with strictly conflicting interests," *Journal of Economic Theory*, 121, 259–272.

CRIPPS, M., AND J. THOMAS (1997): "Reputation and perfection in repeated common interest games," *Games and Economic Behavior*, 18, 141–158.

CRIPPS, M. W., G. J. MAILATH, AND L. SAMUELSON (2004): "Imperfect monitoring and impermanent reputations," *Econometrica*, 72(2), 407–432.

——— (2007): "Disappering private reputations in long-run relationships," *Journal of Economic Theory*, 134, 287–316.

FUDENBERG, D., AND D. K. LEVINE (1989): "Reputation and Equilibrium Selection in Games with a Patient Player," *Econometrica*, 57(4), 759–778.

——— (1992): "Maintaining a Reputation when Strategies are Imperfectly Observed," *Review of Economic Studies*, 59, 561–579.

——— (1994): "Efficiency and Observability with Long-Run and Short-Run Players," *Journal of Economic Theory*, 62(1), 103–135.

FUDENBERG, D., D. K. LEVINE, AND E. MASKIN (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica*, 62(5), 997–1039.

FUDENBERG, D., AND J. TIROLE (1991): *Game Theory*. Cambridge, MA: MIT Press.

HARSANYI, J. C. (1967): "Games of incomplete information played by Bayesian players, Parts I," *Manag. Sci.*, 14, 159–182.

HART, S. (1985): "Nonzero-sum two person repeated games with incomplete information," *Mathematics of Operations Research*, 10, 117–153.

KOREN, G. (1992): "Two-Person Repeated Games where Players Know Their Own Payoffs," *Courant Institute, preprint*.

MAILATH, G. J., AND L. SAMUELSON (2001): "Who Wants a Good Reputation?," *Review of Economic Studies*, 68, 415–441.

MAILATH, G. J., AND L. SAMUELSON (2006): *Repeated Games and Reputations*. Oxford University Press.

OZDOGAN, A. (2010): "Strategic Communication vs. Strategic Auditing with Reputation Concerns on Both Sides," *work in progress*.

PHELAN, C. (2006): "Public trust and government betrayal," *Journal of Economic Theory*, 130, 27–43.

RUDOLF, A., V. S. BERNHARD, AND S. ZAMIR (2002): "Chapter 51: Inspection games," in *Handbook of Game Theory with Economic Applications*, ed. by R. Aumann, and S. Hart, vol. 3. Elsevier.

SCHMIDT, K. M. (1993): "Reputatition and Equilibrium Characterization in Repeated Games of Conflicting Interests," *Econometrica*, 61(2), 325–351.

SOBEL, J. (1985): "A Theory of Credibility," *Review of Economic Studies*, pp. 557–573.

SORIN, S. (1999): "Merging, Reputation, and Repeated Games with Incomplete Information," *Games and Economic Behavior*, 29, 274 – 308.